Question 1

Part A

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ for an eigenvector $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$. $c\lambda + d$ is an eigenvalue of $B = cA + dI$ if $B\vec{\mathbf{v}} = (c\lambda + d)\vec{\mathbf{v}}$.

Since

$$\begin{aligned} B\vec{\mathbf{v}} &= (cA + dI)\vec{\mathbf{v}} \\ &= cA\vec{\mathbf{v}} + dI\vec{\mathbf{v}} \\ &= c\lambda\vec{\mathbf{v}} + d\vec{\mathbf{v}} && \because A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} \\ &= (c\lambda + d)\vec{\mathbf{v}}, \end{aligned}$$

$(c\lambda + d)\vec{\mathbf{v}}$ is an eigenvalue of B.

Part B

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. For A^k, consider $k = 2$:

$$A^2\vec{\mathbf{v}} = A(A\vec{\mathbf{v}}) = A(\lambda\vec{\mathbf{v}}) = \lambda(A\vec{\mathbf{v}}) = \lambda(\lambda\vec{\mathbf{v}}) = \lambda^2\vec{\mathbf{v}}.$$

Then, for any positive integer k, we have that:

$$\begin{aligned} A^k\vec{\mathbf{v}} = (\underbrace{A \cdot A \cdots A}_{k \text{ terms}})\vec{\mathbf{v}} &= (\underbrace{A \cdot A \cdots A}_{k-1 \text{ terms}})(A\vec{\mathbf{v}}) \\ &= (\underbrace{A \cdot A \cdots A}_{k-1 \text{ terms}})(\lambda\vec{\mathbf{v}}) \\ &= \lambda(\underbrace{A \cdot A \cdots A}_{k-2 \text{ terms}})(A\vec{\mathbf{v}}) \\ &= \lambda(\underbrace{A \cdot A \cdots A}_{k-2 \text{ terms}})(\lambda\vec{\mathbf{v}}) \\ &= \lambda^2(\underbrace{A \cdot A \cdots A}_{k-3 \text{ terms}})(A\vec{\mathbf{v}}) \end{aligned}$$

Hence, after each successive applications of k, we have that $A^k\vec{\mathbf{v}} = \lambda^k\vec{\mathbf{v}}$.

Part C

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. For $\lambda = 0$, we have that $A\vec{\mathbf{v}} = 0\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

If A is singular, then A has a nontrivial kernel i.e., $\ker(A) \neq \{\vec{\mathbf{0}}\}$. As such, there exists a nonzero vector $\vec{\mathbf{v}} \in \ker(A)$. This means that the subspace containing eigenvectors (hereinafter, *eigenspace*) corresponding to the eigenvalue $\lambda = 0$ is precisely $\ker(A)$ and thus have the same dimension.

Conversely, A must be singular for $\lambda = 0$ to be its eigenvalue. If A is nonsingular, then A has a trivial kernel. This means that there exists no nonzero vector $\vec{\mathbf{v}}$ such that $A\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Part D

Let A be an $n \times n$ matrix where each entry of A is equal to one:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Obviously, we first note that $\operatorname{rank}(A) = 1$ and that A has a nontrivial kernel. Thus A is singular and by (c), we know that $\lambda = 0$ is an eigenvalue of A.

By the rank–nullity theorem, the nullity of A is $n - \operatorname{rank}(A) = n - 1$. From (c), we found that the dimension of eigenspace corresponding to the eigenvalue $\lambda = 0$ is the nullity of A. As such $\lambda = 0$ is an eigenvalue of A with multiplicity $n - 1$. This means there exists one other nonzero eigenvalue.

Take $\vec{\mathbf{v}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\top$, an $n \times 1$ vector whose entries consists of ones. Since the sum of each rows of A is n, we have that

$$A\vec{\mathbf{v}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n\vec{\mathbf{v}}.$$

As such, $\lambda = n$ is an eigenvalue of A.

So, the eigenvalues of A are $\lambda = 0$ with the corresponding eigenvectors forming the subspace $\{\vec{\mathbf{v}} \in \mathbb{R}^n \mid A\vec{\mathbf{v}} = \vec{\mathbf{0}}\}$ and $\lambda = n$ with the corresponding eigenvector $\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\top$.

Part E

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. If A is nonsingular, then applying A^{-1} yields:

$$\begin{aligned} A^{-1}A\vec{\mathbf{v}} &= A^{-1}\lambda\vec{\mathbf{v}} \\ \vec{\mathbf{v}} &= A^{-1}\lambda\vec{\mathbf{v}} \\ \frac{1}{\lambda}\vec{\mathbf{v}} &= A^{-1}\vec{\mathbf{v}} \end{aligned}$$

Thus, λ^{-1} is an eigenvalue of A^{-1}.

Question 2

Part A

Let $\vec{\mathbf{u}} \in \mathbb{R}^n$ be a unit vector and let $A = \vec{\mathbf{u}}\vec{\mathbf{u}}^\top$. Since we construct A in such a way that the columns are spanned by $\vec{\mathbf{u}}$, A has a rank of one.

From 1(c) and 1(d), we found that such a matrix (a rank-one, singular matrix) contains an eigenvalue of $\lambda = 0$ with multiplicity $n - 1$ and exactly one other nonzero eigenvalue.

To find the nonzero eigenvalue, take $\vec{\mathbf{v}} = \vec{\mathbf{u}}$. Then, $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} \implies \vec{\mathbf{u}}\vec{\mathbf{u}}^\top\vec{\mathbf{u}} = \lambda\vec{\mathbf{u}}$. Since $||\vec{\mathbf{u}}|| = 1$ and $\vec{\mathbf{u}}^\top\vec{\mathbf{u}} = \langle\vec{\mathbf{u}}, \vec{\mathbf{u}}\rangle = 1$, we have that $\vec{\mathbf{u}} = \lambda\vec{\mathbf{u}}$. Thus, $\lambda = 1$ is an eigenvalue of A and the corresponding eigenvector is $\vec{\mathbf{u}}$.

As such, the eigenvalues of A are $\lambda = 0$ with the corresponding eigenvectors forming the subspace $\{\vec{\mathbf{v}} \in \mathbb{R}^n \mid A\vec{\mathbf{v}} = \vec{\mathbf{u}}\vec{\mathbf{u}}^\top\vec{\mathbf{v}} = \vec{\mathbf{0}}\}$ and $\lambda = 1$ with the corresponding eigenvector $\vec{\mathbf{u}}$.