

## Question 1

### Part A

If  $\lambda$  is an eigenvalue of  $A$ , then  $A\vec{v} = \lambda\vec{v}$  for an eigenvector  $\vec{v} \neq \vec{0}$ .  $c\lambda + d$  is an eigenvalue of  $B = cA + dI$  if  $B\vec{v} = (c\lambda + d)\vec{v}$ .

Since

$$\begin{aligned} B\vec{v} &= (cA + dI)\vec{v} \\ &= cA\vec{v} + dI\vec{v} \\ &= c\lambda\vec{v} + d\vec{v} && \because A\vec{v} = \lambda\vec{v} \\ &= (c\lambda + d)\vec{v}, \end{aligned}$$

$(c\lambda + d)\vec{v}$  is an eigenvalue of  $B$ .

### Part B

If  $\lambda$  is an eigenvalue of  $A$ , then  $A\vec{v} = \lambda\vec{v}$ . For  $A^k$ , consider  $k = 2$ :

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}.$$

Then, for any positive integer  $k$ , we have that:

$$\begin{aligned} A^k\vec{v} &= \underbrace{(A \cdot A \cdots A)}_{k \text{ terms}}\vec{v} = \underbrace{(A \cdot A \cdots A)}_{k-1 \text{ terms}}(A\vec{v}) \\ &= \underbrace{(A \cdot A \cdots A)}_{k-1 \text{ terms}}(\lambda\vec{v}) \\ &= \lambda \underbrace{(A \cdot A \cdots A)}_{k-2 \text{ terms}}(A\vec{v}) \\ &= \lambda \underbrace{(A \cdot A \cdots A)}_{k-2 \text{ terms}}(\lambda\vec{v}) \\ &= \lambda^2 \underbrace{(A \cdot A \cdots A)}_{k-3 \text{ terms}}(A\vec{v}) \end{aligned}$$

Hence, after each successive applications of  $k$ , we have that  $A^k\vec{v} = \lambda^k\vec{v}$ .

### Part C

If  $\lambda$  is an eigenvalue of  $A$ , then  $A\vec{v} = \lambda\vec{v}$ . For  $\lambda = 0$ , we have that  $A\vec{v} = 0\vec{v} = \vec{0}$ .

If  $A$  is singular, then  $A$  has a nontrivial kernel i.e.,  $\ker(A) \neq \{\vec{0}\}$ . As such, there exists a nonzero vector  $\vec{v} \in \ker(A)$ . This means that the subspace containing eigenvectors (hereinafter, *eigenspace*) corresponding to the eigenvalue  $\lambda = 0$  is precisely  $\ker(A)$  and thus have the same dimension.

Conversely,  $A$  must be singular for  $\lambda = 0$  to be its eigenvalue. If  $A$  is nonsingular, then  $A$  has a trivial kernel. This means that there exists no nonzero vector  $\vec{v}$  such that  $A\vec{v} = \vec{0}$ .

## Part D

Let  $A$  be an  $n \times n$  matrix where each entry of  $A$  is equal to one:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Obviously, we first note that  $\text{rank}(A) = 1$  and that  $A$  has a nontrivial kernel. Thus  $A$  is singular and by (c), we know that  $\lambda = 0$  is an eigenvalue of  $A$ .

By the rank-nullity theorem, the nullity of  $A$  is  $n - \text{rank}(A) = n - 1$ . From (c), we found that the dimension of eigenspace corresponding to the eigenvalue  $\lambda = 0$  is the nullity of  $A$ . As such  $\lambda = 0$  is an eigenvalue of  $A$  with multiplicity  $n - 1$ . This means there exists one other nonzero eigenvalue.

Take  $\vec{v} = (1 \ 1 \ \cdots \ 1)^\top$ , an  $n \times 1$  vector whose entries consists of ones. Since the sum of each rows of  $A$  is  $n$ , we have that

$$A\vec{v} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n\vec{v}.$$

As such,  $\lambda = n$  is an eigenvalue of  $A$ .

So, the eigenvalues of  $A$  are  $\lambda = 0$  with the corresponding eigenvectors forming the subspace  $\{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{0}\}$  and  $\lambda = n$  with the corresponding eigenvector  $(1 \ 1 \ \cdots \ 1)^\top$ .

## Part E

If  $\lambda$  is an eigenvalue of  $A$ , then  $A\vec{v} = \lambda\vec{v}$ . If  $A$  is nonsingular, then applying  $A^{-1}$  yields:

$$\begin{aligned} A^{-1}A\vec{v} &= A^{-1}\lambda\vec{v} \\ \vec{v} &= A^{-1}\lambda\vec{v} \\ \frac{1}{\lambda}\vec{v} &= A^{-1}\vec{v} \end{aligned}$$

Thus,  $\lambda^{-1}$  is an eigenvalue of  $A^{-1}$ .

## Question 2

### Part A

Let  $\vec{u} \in \mathbb{R}^n$  be a unit vector and let  $A = \vec{u}\vec{u}^\top$ . Since we construct  $A$  in such a way that the columns are spanned by  $\vec{u}$ ,  $A$  has a rank of one.

From 1(c) and 1(d), we found that such a matrix (a rank-one, singular matrix) contains an eigenvalue of  $\lambda = 0$  with multiplicity  $n - 1$  and exactly one other nonzero eigenvalue.

To find the nonzero eigenvalue, take  $\vec{v} = \vec{u}$ . Then,  $A\vec{v} = \lambda\vec{v} \implies \vec{u}\vec{u}^\top\vec{u} = \lambda\vec{u}$ . Since  $\|\vec{u}\| = 1$  and  $\vec{u}^\top\vec{u} = \langle\vec{u}, \vec{u}\rangle = 1$ , we have that  $\vec{u} = \lambda\vec{u}$ . Thus,  $\lambda = 1$  is an eigenvalue of  $A$  and the corresponding eigenvector is  $\vec{u}$ .

As such, the eigenvalues of  $A$  are  $\lambda = 0$  with the corresponding eigenvectors forming the subspace  $\{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{u}\vec{u}^\top\vec{v} = \vec{0}\}$  and  $\lambda = 1$  with the corresponding eigenvector  $\vec{u}$ .

## Part B

If  $\lambda$  is an eigenvalue of  $H$ , then  $H\vec{v} = \lambda\vec{v}$ . If  $H = I - 2\vec{u}\vec{u}^\top$  is a Householder matrix where  $I$  is the identity matrix and  $\vec{u}$  are unit vector.

First, consider the case where the vector lies in the span of the unit vector  $\vec{u}$ . Take  $\vec{v} = \vec{u}$ . Then,

$$\begin{aligned} H\vec{u} &= \lambda\vec{u} \\ &= (I - 2\vec{u}\vec{u}^\top)\vec{u} \\ &= \vec{u} - 2(\vec{u}\vec{u}^\top)\vec{u} \\ &= \vec{u} - 2\vec{u} \\ &= -\vec{u} \end{aligned} \quad \therefore \lambda = -1.$$

Thus,  $\lambda = -1$  is an eigenvector of  $H$ .

Then consider the case where the vector is orthogonal to  $\vec{u}$ . That is, any vector  $\vec{v}$  such that  $\langle\vec{v}, \vec{u}\rangle = \vec{u}^\top\vec{v} = 0$ . We have that:

$$\begin{aligned} H\vec{v} &= (I - 2\vec{u}\vec{u}^\top)\vec{v} \\ &= \vec{v} - 2(\vec{u}\vec{u}^\top)\vec{v} \\ &= \vec{v} - 0 \\ &= \lambda\vec{v} \end{aligned} \quad \therefore \lambda = 1.$$

As such, the eigenvalues of  $H = I - 2\vec{u}\vec{u}^\top$  are  $\lambda = -1$  and  $\lambda = 1$ .

## Part C

If  $\lambda$  is an eigenvalue of  $P$ , then  $P\vec{v} = \lambda\vec{v}$ . Since  $P^2 = P$ , then:

$$\begin{aligned} P^2\vec{v} &= P\vec{v} \\ P(P\vec{v}) &= \lambda\vec{v} \\ \lambda(\lambda\vec{v}) &= \lambda\vec{v} \\ \lambda^2 &= \lambda \\ \lambda^2 - \lambda &= 0 \end{aligned}$$

As such, the eigenvalues of  $P$  are  $\lambda = 0$  and  $\lambda = 1$ .

### Question 3

Let  $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ .

Then,

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{pmatrix} &&= 0 \\ &= (-\lambda) \begin{vmatrix} -\lambda & a \\ -a & -\lambda \end{vmatrix} - c \begin{vmatrix} -c & a \\ b & -\lambda \end{vmatrix} - b \begin{vmatrix} -c & -\lambda \\ b & -a \end{vmatrix} &&= 0 \\ &= \text{careful calculations...} &&= 0 \\ &= (-\lambda)(\lambda^2 - a^2) - c(c\lambda - ab) - b(ac + \lambda b) &&= 0 \\ &= -\lambda^3 + \lambda a^2 - c^2\lambda + abc - b^2\lambda - abc &&= 0 \\ &= -\lambda^3 - (c^2 + b^2)\lambda + a^2\lambda &&= 0 \\ &= -\lambda(\lambda^2 - (a^2 + b^2 + c^2)) &&= 0. \end{aligned}$$

And so, the eigenvalues are  $\lambda = 0$ ,  $\lambda = -\sqrt{a^2 + b^2 + c^2}$ , and  $\lambda = \sqrt{a^2 + b^2 + c^2}$ . Since we have three distinct eigenvalues, the three corresponding eigenvectors will be linearly independent and form a basis. Thus,  $A$  is diagonalizable.

### Question 4

Let  $S = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$  and  $\Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ . Using the power of technology, we find

$$S^{-1} = \begin{pmatrix} 2 & 3 & -1 \\ 3/2 & 3/2 & -1/2 \\ -1/2 & -1/2 & 1/2 \end{pmatrix}.$$

Then,  $S\Lambda S^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 3/2 & 3/2 & -1/2 \\ -1/2 & -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -2 \\ -2 & -2 & 0 \\ 6 & 6 & -4 \end{pmatrix}$  is a

matrix with eigenvalues 0, 2, and  $-2$  with the corresponding eigenvectors  $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ .

### Question 5

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## Question 6

Refer to Section `%% Question 6` in the `math425hw6.m` file for the relevant code.