Question 1

Part A

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ for an eigenvector $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$. $c\lambda + d$ is an eigenvalue of $B = cA + dI$ if $B\vec{\mathbf{v}} = (c\lambda + d)\vec{\mathbf{v}}$.

Since

$$\begin{aligned} B\vec{\mathbf{v}} &= (cA + dI)\vec{\mathbf{v}} \\ &= cA\vec{\mathbf{v}} + dI\vec{\mathbf{v}} \\ &= c\lambda\vec{\mathbf{v}} + d\vec{\mathbf{v}} && \because A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} \\ &= (c\lambda + d)\vec{\mathbf{v}}, \end{aligned}$$

$(c\lambda + d)\vec{\mathbf{v}}$ is an eigenvalue of B.

Part B

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. For A^k, consider $k = 2$:

$$A^2\vec{\mathbf{v}} = A(A\vec{\mathbf{v}}) = A(\lambda\vec{\mathbf{v}}) = \lambda(A\vec{\mathbf{v}}) = \lambda(\lambda\vec{\mathbf{v}}) = \lambda^2\vec{\mathbf{v}}.$$

Then, for any positive integer k, we have that:

$$\begin{aligned} A^k\vec{\mathbf{v}} = (\underbrace{A \cdot A \cdots A}_{k \text{ terms}})\vec{\mathbf{v}} &= (\underbrace{A \cdot A \cdots A}_{k-1 \text{ terms}})(A\vec{\mathbf{v}}) \\ &= (\underbrace{A \cdot A \cdots A}_{k-1 \text{ terms}})(\lambda\vec{\mathbf{v}}) \\ &= \lambda(\underbrace{A \cdot A \cdots A}_{k-2 \text{ terms}})(A\vec{\mathbf{v}}) \\ &= \lambda(\underbrace{A \cdot A \cdots A}_{k-2 \text{ terms}})(\lambda\vec{\mathbf{v}}) \\ &= \lambda^2(\underbrace{A \cdot A \cdots A}_{k-3 \text{ terms}})(A\vec{\mathbf{v}}) \end{aligned}$$

Hence, after each successive applications of k, we have that $A^k\vec{\mathbf{v}} = \lambda^k\vec{\mathbf{v}}$.

Part C

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. For $\lambda = 0$, we have that $A\vec{\mathbf{v}} = 0\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

If A is singular, then A has a nontrivial kernel i.e., $\ker(A) \neq \{\vec{\mathbf{0}}\}$. As such, there exists a nonzero vector $\vec{\mathbf{v}} \in \ker(A)$. This means that the subspace containing eigenvectors (hereinafter, *eigenspace*) corresponding to the eigenvalue $\lambda = 0$ is precisely $\ker(A)$ and thus have the same dimension.

Conversely, A must be singular for $\lambda = 0$ to be its eigenvalue. If A is nonsingular, then A has a trivial kernel. This means that there exists no nonzero vector $\vec{\mathbf{v}}$ such that $A\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Part D

Let A be an $n \times n$ matrix where each entry of A is equal to one:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Obviously, we first note that $\operatorname{rank}(A) = 1$ and that A has a nontrivial kernel. Thus A is singular and by (c), we know that $\lambda = 0$ is an eigenvalue of A.

By the rank–nullity theorem, the nullity of A is $n - \operatorname{rank}(A) = n - 1$. From (c), we found that the dimension of eigenspace corresponding to the eigenvalue $\lambda = 0$ is the nullity of A. As such $\lambda = 0$ is an eigenvalue of A with multiplicity $n - 1$. This means there exists one other nonzero eigenvalue.

Take $\vec{\mathbf{v}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\top$, an $n \times 1$ vector whose entries consists of ones. Since the sum of each rows of A is n, we have that

$$A\vec{\mathbf{v}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n\vec{\mathbf{v}}.$$

As such, $\lambda = n$ is an eigenvalue of A.

So, the eigenvalues of A are $\lambda = 0$ with the corresponding eigenvectors forming the subspace $\{\vec{\mathbf{v}} \in \mathbb{R}^n \mid A\vec{\mathbf{v}} = \vec{\mathbf{0}}\}$ and $\lambda = n$ with the corresponding eigenvector $\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\top$.

Part E

If λ is an eigenvalue of A, then $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. If A is nonsingular, then applying A^{-1} yields:

$$\begin{aligned} A^{-1}A\vec{\mathbf{v}} &= A^{-1}\lambda\vec{\mathbf{v}} \\ \vec{\mathbf{v}} &= A^{-1}\lambda\vec{\mathbf{v}} \\ \frac{1}{\lambda}\vec{\mathbf{v}} &= A^{-1}\vec{\mathbf{v}} \end{aligned}$$

Thus, λ^{-1} is an eigenvalue of A^{-1}.

Question 2

Part A

Let $\vec{\mathbf{u}} \in \mathbb{R}^n$ be a unit vector and let $A = \vec{\mathbf{u}}\vec{\mathbf{u}}^\top$. Since we construct A in such a way that the columns are spanned by $\vec{\mathbf{u}}$, A has a rank of one.

From 1(c) and 1(d), we found that such a matrix (a rank-one, singular matrix) contains an eigenvalue of $\lambda = 0$ with multiplicity $n-1$ and exactly one other nonzero eigenvalue.

To find the nonzero eigenvalue, take $\vec{\mathbf{v}} = \vec{\mathbf{u}}$. Then, $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}} \implies \vec{\mathbf{u}}\vec{\mathbf{u}}^\top\vec{\mathbf{u}} = \lambda\vec{\mathbf{u}}$. Since $||\vec{\mathbf{u}}|| = 1$ and $\vec{\mathbf{u}}^\top\vec{\mathbf{u}} = \langle\vec{\mathbf{u}}, \vec{\mathbf{u}}\rangle = 1$, we have that $\vec{\mathbf{u}} = \lambda\vec{\mathbf{u}}$. Thus, $\lambda = 1$ is an eigenvalue of A and the corresponding eigenvector is $\vec{\mathbf{u}}$.

As such, the eigenvalues of A are $\lambda = 0$ with the corresponding eigenvectors forming the subspace $\{\vec{\mathbf{v}} \in \mathbb{R}^n \mid A\vec{\mathbf{v}} = \vec{\mathbf{u}}\vec{\mathbf{u}}^\top\vec{\mathbf{v}} = \vec{\mathbf{0}}\}$ and $\lambda = 1$ with the corresponding eigenvector $\vec{\mathbf{u}}$.

Part B

If λ is an eigenvalue of H, then $H\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. If $H = I - 2\vec{\mathbf{u}}\vec{\mathbf{u}}^\top$ is a Householder matrix where I is the identity matrix and $\vec{\mathbf{u}}$ are unit vector.

First, consider the case where the vector lies in the span of the unit vector $\vec{\mathbf{u}}$. Take $\vec{\mathbf{v}} = \vec{\mathbf{u}}$. Then,

$$\begin{aligned}
H\vec{\mathbf{u}} &= \lambda\vec{\mathbf{u}} \\
&= (I - 2\vec{\mathbf{u}}\vec{\mathbf{u}}^\top)\vec{\mathbf{u}} \\
&= \vec{\mathbf{u}} - 2(\vec{\mathbf{u}}\vec{\mathbf{u}}^\top)\vec{\mathbf{u}} \\
&= \vec{\mathbf{u}} - 2\vec{\mathbf{u}} \\
&= -\vec{\mathbf{u}} & \therefore \lambda = -1.
\end{aligned}$$

Thus, $\lambda = -1$ is an eigenvector of H.

Then consider the case where the vector is orthogonal to $\vec{\mathbf{u}}$. That is, any vector $\vec{\mathbf{v}}$ such that $\langle\vec{\mathbf{v}}, \vec{\mathbf{u}}\rangle = \vec{\mathbf{u}}^\top\vec{\mathbf{v}} = 0$. We have that:

$$\begin{aligned}
H\vec{\mathbf{v}} &= (I - 2\vec{\mathbf{u}}\vec{\mathbf{u}}^\top)\vec{\mathbf{v}} \\
&= \vec{\mathbf{v}} - 2(\vec{\mathbf{u}}\vec{\mathbf{u}}^\top)\vec{\mathbf{v}} \\
&= \vec{\mathbf{v}} - 0 \\
&= \lambda\vec{\mathbf{v}} & \therefore \lambda = 1.
\end{aligned}$$

As such, the eigenvalues of $H = I - 2\vec{\mathbf{u}}\vec{\mathbf{u}}^\top$ are $\lambda = -1$ and $\lambda = -1$.

Part C

If λ is an eigenvalue of P, then $P\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. Since $P^2 = P$, then:

$$\begin{aligned}
P^2\vec{\mathbf{v}} &= P\vec{\mathbf{v}} \\
P(P\vec{\mathbf{v}}) &= \lambda\vec{\mathbf{v}} \\
\lambda(\lambda\vec{\mathbf{v}}) &= \lambda\vec{\mathbf{v}} \\
\lambda^2 &= \lambda \\
\lambda^2 - \lambda &= 0
\end{aligned}$$

As such, the eigenvalues of P are $\lambda = 0$ and $\lambda = 1$.

Question 3

Let $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$.

Then,

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{pmatrix} && = 0 \\
&= (-\lambda) \begin{vmatrix} -\lambda & a \\ -a & -\lambda \end{vmatrix} - c \begin{vmatrix} -c & a \\ b & -\lambda \end{vmatrix} - b \begin{vmatrix} -c & -\lambda \\ b & -a \end{vmatrix} && = 0 \\
&= \text{careful calculations...} && = 0 \\
&= (-\lambda)(\lambda^2 - a^2) - c(c\lambda - ab) - b(ac + \lambda b) && = 0 \\
&= -\lambda^3 + \lambda a^2 - c^2\lambda + abc - b^2\lambda - abc && = 0 \\
&= -\lambda^3 - (c^2 + b^2)\lambda + a^2\lambda && = 0 \\
&= -\lambda \left(\lambda^2 - (a^2 + b^2 + c^2) \right) && = 0.
\end{aligned}$$

And so, the eigenvalues are $\lambda = 0$, $\lambda = -\sqrt{a^2 + b^2 + c^2}$, and $\lambda = \sqrt{a^2 + b^2 + c^2}$. Since we have three distinct eigenvalues, the three corresponding eigenvectors will be linearly independent and form a basis. Thus, A is diagonalizable.