

Question 1

If Q is an $n \times n$ orthogonal matrix, then $Q^\top Q = QQ^\top = I_n$ where I_n is the $n \times n$ identity matrix.

Part A

Let Q_1 and Q_2 be orthogonal matrices. Then,

$$\begin{aligned}(Q_1 Q_2)^\top (Q_1 Q_2) &= Q_2^\top Q_1^\top Q_1 Q_2 = Q_2^\top Q_2 = I_n \\ &= (Q_1 Q_2)(Q_1 Q_2)^\top = Q_1 Q_2 Q_2^\top Q_1^\top = Q_1 Q_1^\top = I_n.\end{aligned}$$

Hence, $Q_1 Q_2$ must be orthogonal.

Part B

Suppose Q^\top is an orthogonal matrix. Then,

$$\begin{aligned}(Q^\top)^\top Q^\top &= QQ^\top = I_n \\ &= Q^\top (Q^\top)^\top = Q^\top Q = I_n.\end{aligned}$$

As such, Q^\top must be orthogonal. Subsequently, if Q is an orthogonal matrix (where its column form an orthonormal basis), then its rows must also form an orthonormal basis.

Proof: Suppose Q is made up of column vectors \vec{q}_i for each column $i = 1, \dots, n$. If Q is an $n \times n$ orthogonal matrix, then

$$Q^\top Q = \begin{pmatrix} - & \vec{q}_1^\top & - \\ & \vdots & \\ - & \vec{q}_n^\top & - \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix} = I_n.$$

Since the columns \vec{q}_i form an orthonormal basis of \mathbb{R}^n , the value of the diagonal entries (i, i) is $\langle \vec{q}_i^\top, \vec{q}_i \rangle = 1$ and the value of all other entries (i, j) where $i \neq j$ is $\langle \vec{q}_i^\top, \vec{q}_j \rangle = 0$. Thus, $\|\vec{q}_i\| = \|\vec{q}_i^\top\| = 1$ for all rows i .

Part C

Let $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. Then, $Q^\top = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. As such,

$$\begin{aligned} QQ^\top &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \sin \theta \cos \theta & \sin^2 \theta + \cos^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= Q^\top Q \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & -\cos \theta \sin \theta + \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \sin \theta \cos \theta & \sin^2 \theta + \cos^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I_2. \end{aligned}$$

Thus, $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is an orthogonal matrix. This special matrix is known as the rotational matrix. Let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be a vector on the Cartesian plane. Applying Q to a vector \vec{v} will rotate it counterclockwise by θ .

$$Q\vec{v} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix}$$

For example, let $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be a vector on an xy -plane. We can flip the vector across y -axis by rotating the \vec{v} by $\theta = \pi$.

$$-\vec{v} = \begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1(\cos \pi) - 0(\sin \pi) \\ 1(\sin \pi) + 0(\cos \pi) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Part D

By definition, the norm of a vector $\vec{x} \in \mathbb{R}^n$, is given by $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\vec{x}^\top \vec{x}}$. If Q is an orthogonal matrix, then:

$$\begin{aligned} \|Q\vec{x}\| &= \sqrt{\langle Q\vec{x}, Q\vec{x} \rangle} \\ &= \sqrt{(Q\vec{x})^\top (Q\vec{x})} \\ &= \sqrt{\vec{x}^\top Q^\top Q \vec{x}} \\ &= \sqrt{\langle \vec{x}, \vec{x} \rangle} \\ &= \|\vec{x}\|. \end{aligned}$$