Question 1

Part A

Note that the question statement is not true if either $\vec{\mathbf{v}}$ or $\vec{\mathbf{m}}$ are zero vectors. The resulting $m \times n$ matrix would be a zero matrix, and thus its rank would be 0.

For nonzero vectors $\vec{\mathbf{v}} \in \mathbb{R}^m$ and $\vec{\mathbf{w}} \in \mathbb{R}^n$, we have that:

$$\vec{\mathbf{v}}\vec{\mathbf{w}}^\top = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix} = \begin{pmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m w_1 & v_m w_2 & \cdots & v_m w_n \end{pmatrix}$$

Intuitively, we can already see that each row is being multiplied by the first column vector $\vec{\mathbf{v}}$. So of course, the each column in $\vec{\mathbf{v}}\vec{\mathbf{w}}^\top$ is a multiple of $\vec{\mathbf{v}}$. Or, we can perform the usual Gaussian elimination step. First, we can eliminate the second row by applying the following row operation:

$$\begin{pmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m w_1 & v_m w_2 & \cdots & v_m w_n \end{pmatrix} \xrightarrow{-\frac{v_2}{v_1}R_1 + R_2 \to R_2} \begin{pmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_m w_1 & v_m w_2 & \cdots & v_m w_n \end{pmatrix}$$

As such, we can apply the row operation $-\frac{v_i}{v_1}R_1 + R_i \to R_i$ for all $i = 2, 3, \dots, m$ to eliminate all the rows below. Thus,

$$\operatorname{rank} \begin{pmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \operatorname{rank}(\vec{\mathbf{v}}\vec{\mathbf{w}}^\top) = 1.$$

Part B

Let A be an $m \times n$ matrix where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}.$$

Let R_i denote the rows $i = 1, \dots, m$ in A. For example, $R_1 = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \end{pmatrix}$. Since $\operatorname{rank}(A) = 1$, we know that all rows in A are a multiple of each other. As such, we can write the matrix A as

$$A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix} = \begin{pmatrix} c_1 R_1 \\ c_2 R_1 \\ \vdots \\ c_m R_1 \end{pmatrix}$$

for some scalars $c_1, c_2, \ldots, c_m \in \mathbb{R}$.

Then, let $\vec{\mathbf{v}} \in \mathbb{R}^m$ be a column vector formed by the scalars $c_1, c_2, \ldots, c_m \in \mathbb{R}$:

$$\vec{\mathbf{v}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

and let $\vec{\mathbf{w}} \in \mathbb{R}^n$ be a column vector comprised of the elements in the first row:

$$\vec{\mathbf{w}} = R_1^\top = \begin{pmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,n} \end{pmatrix}$$

. And so, we have that:

$$\begin{aligned} A = \vec{\mathbf{v}}\vec{\mathbf{w}}^\top &= \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} a_{1,1} a_{1,2} & \cdots & a_{1,n} \end{pmatrix} \\ &= \begin{pmatrix} c_1 a_{1,1} & c_1 a_{1,2} & \cdots & c_1 a_{1,n} \\ c_2 a_{1,1} & c_2 a_{1,2} & \cdots & c_2 a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_m a_{1,1} & c_m a_{1,2} & \cdots & c_m a_{1,n} \end{pmatrix}, \end{aligned}$$

which is precisely an $m \times n$ matrix with rank one. We can verify this by performing Gaussian elimination in the manner described in 1(a). Again, this holds for $\vec{\mathbf{v}}, \vec{\mathbf{w}} \neq \vec{\mathbf{0}}$.

Question 2

Refer to Section `%% Question 2` in the `math425hw.m` file for the relevant code.

Using MATLAB, we can simply augment the vectors into a matrix and use `rref` to find its reduced-row echelon form.

$$\text{rref} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Since the last row produces an inconsistent solution i.e., $0 = 1$, the vector $\begin{pmatrix} 3 \\ 0 \\ -1 \\ 2 \end{pmatrix}$ is not a linear combination of the other three vectors. That is, there exists no $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 2 \end{pmatrix}$.

Question 3

For this question, let

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \vec{\mathbf{v}}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \vec{\mathbf{v}}_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \text{ and } \vec{\mathbf{v}}_4 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}.$$

Part A

Using MATLAB, we can augment the vectors and use the `rank` function to ascertain the number of linearly independent vectors. To span $\mathbb{R}^3$, we need three linearly independent vectors in $\mathbb{R}^3$. Since

$$\operatorname{rank} \begin{pmatrix} | & | & | & | \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_3 & \vec{\mathbf{v}}_4 \\ | & | & | & | \end{pmatrix} = 2,$$

$\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, $\vec{\mathbf{v}}_4$ do not span $\mathbb{R}^3$.

Part B

Since $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, $\vec{\mathbf{v}}_4 \in \mathbb{R}^3$ and there are four vectors, at least one of them must be a multiple of each other. As such, they must be linearly dependent. In addition, we found in 3(a) that the matrix comprised of these column vectors is not full rank.

Part C

$\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, $\vec{\mathbf{v}}_4$ do not form a basis for $\mathbb{R}^3$ because they are linearly dependent. In 3(a), we found that a matrix comprised of these column vectors is not full rank. Specifically, its rank is 2. This means that they are only two linearly independent vectors in the set and are therefore not sufficient to span $\mathbb{R}^3$.

Part D

In 3(a), we found that a matrix comprised of these column vectors has a rank of 2. As such, a basis of subspace spanned by these vectors will contain two vectors. Hence, cardinality of the basis i.e., the *dimension* of $\operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is 2.