Math 425 Applied & Comput. Lin. Algebra Fall 2024 Handout

Handout

(1) The rank of an $m \times n$ matrix A is equal to the number of nonzero rows in its row reduced echelon form (rref). The `MATLAB` command to compute the rref of A is "`rref(A)`".

Compute the rank of the following matrices using this command.

$$\begin{pmatrix} 3 & 5 & -2 \\ 4 & 1 & 1 \\ -4 & -4 & 5 \end{pmatrix}, \begin{pmatrix} -16 & 28 & 20 & -4 \\ 9 & -12 & -12 & 12 \\ -15 & 25 & 19 & -7 \end{pmatrix}, \begin{pmatrix} 6 & -3 & 3 & -6 \\ 2 & -1 & 1 & -2 \\ 4 & -2 & 2 & -4 \end{pmatrix}$$

(2) Check your answers by using the `MATLAB` command `rank`.

(3) If A is an $m \times n$ matrix, what is the relationship of rank(A) to m and n ?

(4) Consider an $m \times n$ matrix A. The system of linear equations $Ax = 0$ always has a solution, namely, $x = 0$. But it could have more solutions. What should rank(A) be to guarantee that $x = 0$ is the only solution ? Give a condition on m and n so that the system has more than the trivial solution $x = 0$.

(5) The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^m$ are called *linearly dependent* if there exists scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Vectors that are not linearly dependent are called *linearly independent*. In other words, we say $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ implies that $c_1 = c_2 = \cdots = c_k = 0$.

Let's take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$. How would you decide whether these vectors are linearly dependent or independent?

(6) More generally, suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are in $\mathbb{R}^m$. Describe a general method that decides whether these vectors are linearly dependent or independent.

(7) Continuing with the above question: under what conditions on k and m must these vectors be linearly dependent?