Question 1

Let A be a nonsingular $n \times n$ matrix with real entries and $\vec{\mathbf{b}} \in \mathbb{R}^n$.

Let $A = P\Sigma Q^\top$ be the singular value decomposition (SVD) of A. Then, the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ can be written as:

$$\begin{aligned} A\vec{\mathbf{x}} = \vec{\mathbf{b}} &\iff (P\Sigma Q^\top)\vec{\mathbf{x}} = \vec{\mathbf{b}} \\ &\iff \vec{\mathbf{x}} = (P\Sigma Q^\top)^{-1}\vec{\mathbf{b}} \\ & \vec{\mathbf{x}} = (Q^\top)^{-1}\Sigma^{-1}P^{-1}\vec{\mathbf{b}} \end{aligned}$$

Note that since A is a real nonsingular matrix, the matrices $P = \begin{pmatrix} | & & | \\ \vec{\mathbf{p}}_1 & \cdots & \vec{\mathbf{p}}_n \\ | & & | \end{pmatrix}$ and $Q = \begin{pmatrix} | & & | \\ \vec{\mathbf{q}}_1 & \cdots & \vec{\mathbf{q}}_n \\ | & & | \end{pmatrix}$ are orthogonal. Thus, $Q^\top Q = QQ^\top = I \iff Q^\top = Q^{-1}$ and $P^\top P = PP^\top = I \iff P^\top = P^{-1}$. As such, the solution can be written as:

$$\begin{aligned} \vec{\mathbf{x}} &= Q\Sigma^{-1}P^\top\vec{\mathbf{b}} \\ &= \begin{pmatrix} | & & | \\ \vec{\mathbf{q}}_1 & \cdots & \vec{\mathbf{q}}_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}^{-1} \begin{pmatrix} — & \vec{\mathbf{p}}_1^\top & — \\ & \vdots & \\ — & \vec{\mathbf{p}}_n^\top & — \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \end{aligned}$$

Here, we're basically solving for $\vec{\mathbf{x}}$ by computing the inverse of A with its SVD. Thus, it is crucial that A is nonsingular. Otherwise, A would be invertible. Additionally, note that the solution requires us to compute the inverse of Σ, which we note $\Sigma^{-1} = \operatorname{diag}({}^1/_{\sigma_1}, \dots, {}^1/_{\sigma_n})$. This means that all singular values of A, $\sigma_1, \dots, \sigma_n$ must be all positive. If A is singular, then its determinant is zero, which means zero is a singular value of A, and thus Σ would also be invertible.

Question 2

From Question 1, we solved the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ by substituting the SVD of A and computing its inverse. Where $A = P\Sigma Q^\top$ is the SVD of A, the resulting expression is $\vec{\mathbf{x}} = Q\Sigma^{-1}P^\top\vec{\mathbf{b}}$. Thus, $A^{-1} = Q\Sigma^{-1}P^\top$ and as demonstrated in the previous question, the singular value of A^{-1} is simply the reciprocal of the singular values of A.

Question 5

Refer to Section `%% Question 5` in the `math425hw7.m` file for the relevant code.