Math 425 Applied & Comput. Lin. Algebra Fall 2024 Lecture

Transposes and Symmetric Matrices

Definition 1. If A is an $m \times n$ matrix, then its *transpose*, denoted by A^T, is the $n \times m$ matrix whose (i, j) entry equals the (j, i) entry of A.

Example 1.

$$A = \begin{pmatrix} 0 & 1 & -2 & -2 \\ 5 & 3 & -1 & 9 \\ 1 & 1 & 1 & -1 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 5 & 1 \\ 1 & 3 & 1 \\ -2 & -1 & 1 \\ -2 & 9 & -1 \end{pmatrix}$$

We observe that the rows of A become the columns of A^T, and the columns of A become the rows of A^T. In particular, the transpose of a column vector is a row vector, and the transpose of a row vector is a column vector:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \quad \mathbf{v}^T = \begin{pmatrix} 1 & 3 & -2 \end{pmatrix}.$$

The transpose of a scalar (as a 1×1 matrix) is itself: $c^T = c$. And we make the following immediate observations:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$; in particular if $\mathbf{v}$ and $\mathbf{w}$ are two column vectors with the same number of entries then $\mathbf{v}^T\mathbf{w} = (\mathbf{v}^T\mathbf{w})^T = \mathbf{w}^T\mathbf{v}$

Lemma 1. If A is a nonsingular matrix, so is A^T, and its inverse is denoted $A^{-T} = (A^T)^{-1} = (A^{-1})^T$.

Proof. We check that $(A^{-1})^T$ is the inverse of A^T:

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

This means A^T is invertible and its inverse is $(A^{-1})^T$. $\square$

Symmetric Matrices.

Definition 2. A square matrix is called symmetric if it equals its own transpose: $A = A^T$.

Example 2. Symmetric 4×4 matrices look like this

$$A = \begin{pmatrix} * & a & b & c \\ a & * & d & e \\ b & d & * & f \\ c & e & f & * \end{pmatrix}.$$