

Question 1

Part A

If λ is an eigenvalue of A , then $A\vec{v} = \lambda\vec{v}$ for an eigenvector $\vec{v} \neq \vec{0}$. $c\lambda + d$ is an eigenvalue of $B = cA + dI$ if $B\vec{v} = (c\lambda + d)\vec{v}$.

Since

$$\begin{aligned} B\vec{v} &= (cA + dI)\vec{v} \\ &= cA\vec{v} + dI\vec{v} \\ &= c\lambda\vec{v} + d\vec{v} && \because A\vec{v} = \lambda\vec{v} \\ &= (c\lambda + d)\vec{v}, \end{aligned}$$

$(c\lambda + d)\vec{v}$ is an eigenvalue of B .

Part B

If λ is an eigenvalue of A , then $A\vec{v} = \lambda\vec{v}$. For A^k , consider $k = 2$:

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}.$$

Then, for any positive integer k , we have that:

$$\begin{aligned} A^k\vec{v} &= (\underbrace{A \cdot A \cdots A}_{k \text{ terms}})\vec{v} = (\underbrace{A \cdot A \cdots A}_{k-1 \text{ terms}})(A\vec{v}) \\ &= (\underbrace{A \cdot A \cdots A}_{k-1 \text{ terms}})(\lambda\vec{v}) \\ &= \lambda(\underbrace{A \cdot A \cdots A}_{k-2 \text{ terms}})(A\vec{v}) \\ &= \lambda(\underbrace{A \cdot A \cdots A}_{k-2 \text{ terms}})(\lambda\vec{v}) \\ &= \lambda^2(\underbrace{A \cdot A \cdots A}_{k-3 \text{ terms}})(A\vec{v}) \end{aligned}$$

Hence, after each successive applications of k , we have that $A^k\vec{v} = \lambda^k\vec{v}$.

Part C

If λ is an eigenvalue of A , then $A\vec{v} = \lambda\vec{v}$. For $\lambda = 0$, we have that $A\vec{v} = 0\vec{v} = \vec{0}$.

If A is singular, then A has a nontrivial kernel i.e., $\ker(A) \neq \{\vec{0}\}$. As such, there exists a nonzero vector $\vec{v} \in \ker(A)$. This means that the subspace containing eigenvectors (hereinafter, *eigenspace*) corresponding to the eigenvalue $\lambda = 0$ is precisely $\ker(A)$ and thus have the same dimension.

Conversely, A must be singular for $\lambda = 0$ to be its eigenvalue. If A is nonsingular, then A has a trivial kernel. This means that there exists no nonzero vector \vec{v} such that $A\vec{v} = \vec{0}$.

Part D

Let A be an $n \times n$ matrix where each entry of A is equal to one:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Obviously, we first note that $\text{rank}(A) = 1$ and that A has a nontrivial kernel. Thus A is singular and by (c), we know that $\lambda = 0$ is an eigenvalue of A .

By the rank-nullity theorem, the nullity of A is $n - \text{rank}(A) = n - 1$. From (c), we found that the dimension of eigenspace corresponding to the eigenvalue $\lambda = 0$ is the nullity of A . As such $\lambda = 0$ is an eigenvalue of A with multiplicity $n - 1$. This means there exists one other nonzero eigenvalue.

Take $\vec{v} = (1 \ 1 \ \cdots \ 1)^\top$, an $n \times 1$ vector whose entries consists of ones. Since the sum of each rows of A is n , we have that

$$A\vec{v} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n\vec{v}.$$

As such, $\lambda = n$ is an eigenvalue of A .

So, the eigenvalues of A are $\lambda = 0$ with the corresponding eigenvectors forming the subspace $\{\vec{v} \mid A\vec{v} = \vec{0}\}$ and $\lambda = n$ with the corresponding eigenvector $(1 \ 1 \ \cdots \ 1)^\top$.

Part E

If λ is an eigenvalue of A , then $A\vec{v} = \lambda\vec{v}$. If A is nonsingular, then applying A^{-1} yields:

$$\begin{aligned} A^{-1}A\vec{v} &= A^{-1}\lambda\vec{v} \\ \vec{v} &= A^{-1}\lambda\vec{v} \\ \frac{1}{\lambda}\vec{v} &= A^{-1}\vec{v} \end{aligned}$$

Thus, λ^{-1} is an eigenvalue of A^{-1} .

Question 2

Part A

Let $\vec{u} \in \mathbb{R}^n$ be a unit vector and let $A = \vec{u}\vec{u}^\top$. Since we construct A in such a way that the columns are spanned by \vec{u} , A has a rank of one.

From 1(c) and 1(d), we found that such a matrix (a rank-one, singular matrix) contains an eigenvalue of $\lambda = 0$ with multiplicity $n - 1$ and exactly one other nonzero eigenvalue.

To find the nonzero eigenvalue, take $\vec{v} = \vec{u}$. Then, $A\vec{v} = \lambda\vec{v} \implies \vec{u}\vec{u}^T\vec{u} = \lambda\vec{u}$. Since $\|\vec{u}\| = 1$ and $\vec{u}^T\vec{u} = \langle \vec{u}, \vec{u} \rangle = 1$, we have that $\vec{u} = \lambda\vec{u}$. Thus, $\lambda = 1$ is an eigenvalue of A and the corresponding eigenvector is \vec{u} .

As such, the eigenvalues of A are $\lambda = 0$ with the corresponding eigenvectors forming the subspace $\{\vec{v} \mid A\vec{v} = \vec{u}\vec{u}^T\vec{v} = \vec{0}\}$ and $\lambda = 1$ with the corresponding eigenvector \vec{u} .