Question 1

Let A be a matrix composed of the column vectors, such that $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & -2 & 3 \\ 0 & -1 & 2 \end{pmatrix}$. We first note that the system contains no solution:

$$\operatorname{rref}\left(A \mid \vec{\mathbf{b}}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Using `A\b`, we find that the least-squares solution is $\vec{\mathbf{x}}^* = \texttt{A\textbackslash b} = \begin{pmatrix} 1/4 \\ 0 \\ 1/4 \end{pmatrix}$. Hence, the closest point from $\vec{\mathbf{b}}$ is $A\vec{\mathbf{x}}^* = \texttt{A * (A\textbackslash b)} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$.

Question 2

Refer to section `%% Question 2` in the `math425hw5.m` file for the relevant code.

Method one

This method involves using the QR decomposition. Where $A = QR = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & -2 & 3 \\ 0 & -1 & 2 \end{pmatrix}$, we have that

$$\begin{aligned} &A\vec{\mathbf{x}} = \vec{\mathbf{b}} \\ &QR\vec{\mathbf{x}} = \vec{\mathbf{b}} \\ &\cancel{Q^\top Q}R\vec{\mathbf{x}} = Q^\top\vec{\mathbf{b}} \\ &R\vec{\mathbf{x}} = Q^\top\vec{\mathbf{b}}. \end{aligned}$$

Since Q is orthogonal, $Q^\top Q = I$. And since R is upper triangular, we can simply solve for $\vec{\mathbf{x}}$ by performing backward substitution. With the QR factorization of A and the magic of MATLAB, the least-squares solution is

$$\vec{\mathbf{x}} = \texttt{fixed.backwardSubstitute(R, Q' * b)} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Method two

This method involves using the Cholesky decomposition. Since $A^\top A$ is symmetric, Using MATLAB, we can find the Cholesky factorization using `chol(A'A)`. Where R is an upper triangular matrix such that $R^\top R = A^\top A$, we have that

$$\begin{aligned} A^\top A\vec{\mathbf{x}} = A^\top \vec{\mathbf{b}} &\iff R^\top R\vec{\mathbf{x}} = A^\top \vec{\mathbf{b}} \\ &\iff R^\top \vec{\mathbf{y}} = A^\top \vec{\mathbf{b}} \end{aligned}$$

where $\vec{\mathbf{y}} = R\vec{\mathbf{x}}$.

As such, we can compute the least-squares solution in two steps. First, solve the lower triangular system for $\vec{\mathbf{y}}$:

$$\vec{\mathbf{y}} = \texttt{fixed.forwardSubstitute(R', A' * b)} = \begin{pmatrix} -5.4272... \\ 6.0554... \\ 7.6731... \end{pmatrix}$$

Then, solve the upper triangular system, giving us the least-squares solution

$$\vec{\mathbf{x}} = \texttt{fixed.backwardSubstitute(R, y)} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Question 3

Refer to section `%% Question 3` in the `math425hw5.m` file for the relevant code.

From the table, we construct a system of equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where

$$\begin{cases} \alpha_1 + 1989\beta_1 &= 86.4 \\ \alpha_2 + 1990\beta_2 &= 89.8 \\ \alpha_3 + 1991\beta_3 &= 92.8 \\ & \vdots \\ \alpha_{11} + 1999\beta_{11} &= 129.5 \end{cases} \iff \underset{\overset{\parallel}{A}}{\begin{pmatrix} 1 & 1989 \\ 1 & 1990 \\ 1 & 1991 \\ \vdots & \vdots \\ 1 & 1999 \end{pmatrix}} \underset{\overset{\parallel}{\vec{\mathbf{x}}}}{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} = \begin{pmatrix} 86.4 \\ 89.8 \\ 92.8 \\ \vdots \\ 129.5 \end{pmatrix} = \vec{\mathbf{b}}.$$

Then, applying $A^\top$ to both sides and solving for $\vec{\mathbf{x}}$ yields

$$A^\top A\vec{\mathbf{x}} = A^\top \vec{\mathbf{b}} \implies \vec{\mathbf{x}}^* = \texttt{A' * A \textbackslash A' * b} \approx \begin{pmatrix} -54024/7 \\ 863/220 \end{pmatrix},$$

producing a line of best fit that estimates the median price (in thousand dollars)

$$y \approx -\frac{54024}{7}x + \frac{863}{220} \approx -7717.70909x + 3.92273$$

for the year x. Hence, for the years $x = 2005$ and $x = 2010$, the estimated median price are approximately \$147359 and \$166973, respectively.