Modal Logic and Gödel's Theorems

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1 Introduction

Modal logic once a subject of philosophy, has now become a branch of mathematical logic. Nowadays computer scientists use modal logic to reason about knowledge, time, beliefs and proof theory. The simplicity, elegance and power of modal logic can be best seen in its application in reasoning about proofs, where it throws light on Gödel's theorems.

Modal logic is an extension of propositional logic by introducing modalities on propositions: instead of a proposition being merely just true or false, it may in addition be necessarily true or possibly true. This investigation was largely confined to the domain of philosophical logic. However in the 1950s Kripke gave precise mathematical meanings to the notions of modality in terms of possible world models and brought it in the domain of mathematics. Besides having precise syntax and semantics, the logic was shown to have several applications by appropriately interpreting the modalities. The two modalities *necessity* and and its dual *possibility* respectively are traditionally interpreted as *always* and *sometimes* in temporal logic, *knowledge* and *belief* in logic of knowledge, *provable* and *consistent* in proof theory.

In philosophical logic, the notion of absolute truth of a proposition was replaced by the notion of truth with respect to the *world* where it was uttered. It envisaged a set of worlds related to each other. A proposition was necessarily true in a world if it was true in every world related to that world, and possibly true in it if it was true in at least one world related to it.

2 Formulae

The language consists of logical and nonlogical symbols. The nonlogical symbols are a set of propositions which assert facts about the world. The logical symbols are the truth constants: true ($\mathcal{T}$), false ($\mathcal{F}$); the usual propositional connectives: conjunction (*and*), disjunction (*or*), implication ($\rightarrow$), if and only (*iff*), negation (*not*); along with the modal operators necessarily ($\Box$) and possibly ($\Diamond$).

The set of *formulae* is given by applying the following rules any number of times: Every proposition is a *formula*. If ϕ and ψ are *formulae* then so are the following: ϕ *and* ψ, ϕ *or* ψ, $\phi \rightarrow \psi$, *not* ϕ, $\Box\phi$, and $\Diamond\phi$.

3 Truth of formulae

Every formula must be either true or false in any given situation (*model*). Moreover the truth value is meaningfully assigned: that is, if both ϕ and ψ are assigned $\mathcal{T}$, then (ϕ *and* ψ) will also be assigned $\mathcal{T}$. Using possible world as our basic model, truth value can be assigned to every formula of modal logic. A possible world model consist of a set *worlds*, a binary relation between the worlds, and truth assignments to the propositions in every world. The formula $\Box\phi$ is true in a world w, if and only if ϕ

is true in all worlds related to w. Whereas its dual formula $\Diamond\phi$ is true in a world w, if and only if ϕ is true in some (at least one) world related to w.

First we notice that all possible world models (also called Kripke models) satisfy all formulae of the form:

$$(\Box(\phi \to \psi)\ \textit{and}\ \Box\phi) \to \Box\psi$$

We therefore take formulae of this form to be axioms of modal logics based on possible worlds. This axiom is called the Kripke (K) or the normality axiom, and will hold for all modal logics discussed here.

Other axioms characterize certain type of relations between worlds in the Kripke model. For example the axiom T:

$$\Box\phi \to \phi$$

describes all kripke models where the worlds are related to themselves. Next we look at relation between truth and proof.

4 Truth and Proof

The truth of a formulae can be assigned in two ways:

1. First, if it is assigned $\mathcal{T}$ in every model. Such formulae are called *valid*.
2. Second by giving a procedure which generates (proves) the set of true formulae. The syntactic approach describes a logic by a set of *axioms*, and *rules of inference*. The axioms, taken as apriori truths, describe classes of models with certain properties. Finding axioms to describe a certain class of models is a challenging job for a mathematician. A rule of inference consist of hypothesis and conclusion; whenever its hypothesis is true, then we infer that its conclusion also holds (is true). The axioms and rules of inference together generate the set of formulae called the *theorems* (theory) of the logic. A formula ϕ is assigned false by the system, if and only if its negation viz. *not* ϕ is proved.

The proof system is *sound* if the rule of inference generates only valid formulae from previously proved formulae and is *complete* when every valid formula is provable. It is *decidable* if every formula can either be proved or disproved (that is, by proving the negation of the formula).

5 Some Modal Logics

We usually present a logic by a set of axioms and the following two rules of inference:

Modus Ponens From A and $A \to B$, conclude B.

Necessitation If A can be proved in every world, conclude $\Box A$.

We will strive for one that is sound and complete. Now a *Normal* modal logic is incomplete, in the sense that it has several completions: Witness that either Axiom T ($\Box\phi \to \phi$) or its negation can be assigned $\mathcal{T}$ in the logic. In our search for axioms we will be guided by the desire that the modal operators and true formulae, are in agreement with our interpretation. More over we want a minimal set of axioms, that is no axiom should be provable from the remaining ones, in such a case the axioms are called *independent*.

6 Reasoning about Proofs

A modal logic only gives us a set of theorems and nontheorems and a way of generating them. We are free to interpret the $\Box$ and $\Diamond$ in any way we chose.

In reasoning about proofs, the modalities $\Box\phi$ and $\Diamond\psi$ respectively mean: ϕ is provable and ψ is consistent (that is: *not* ψ is not provable). We can use a modal logic to look into another logic system whose proof system satisfies the axioms of the modal logic. Then the theorems of the modal logic can be interpreted as properties of the proof system. The rule of inference Modus Ponens translates as

$$(\ \mathit{provable}\,\phi\ \ \mathit{and}\ \ \mathit{provable}\,(\phi \to \psi)) \to\ \ \mathit{provable}\,\psi$$

that is, such proof systems are normal.

A modal logic may even be self referential, in the sense, that it discusses its own proof theory. An example of such a modal logic is Löb's logic. Löb's logic (KW) is an extension of normal modal logic by the axiom W. The first axiom (K) says that if you can prove ϕ, and if you can prove that ϕ implies ψ, then you can also prove ψ:

$$(\ \mathit{provable}\,(\phi \to \psi)\ \ \mathit{and}\ \ \mathit{provable}\,\phi) \to\ \ \mathit{provable}\,\psi$$

The next axiom (W) describes possible world models where no world is related to itself, and no world has an infinite chain of ancestors:

$$\mathit{provable}\,((\ \mathit{provable}\,\phi) \to \phi) \to\ \ \mathit{provable}\,\phi$$

It was with considerable effort that the following theorem was proved from the axioms K and W. It claims that the relation between the worlds is transitive:

$$\mathit{provable}\,\phi \to\ \ \mathit{provable}\ \ \mathit{provable}\,\phi$$

This logic exactly captures the properties of proofs in number theory. Number theory is the set of theorems about numbers provable from the Peano's axiom. It is called *consistent* if it does not contain $\mathcal{F}$ as a theorem. Hence, the theorems of this logic will reflect properties on the proofs in number theory. The next sections look at some of these theorems.

7 A Paradox

In Löb's logic there is a formula G such that the following is a theorem:

$$G\ \mathit{iff}\ \ \mathit{not}\ \ \mathit{provable}\,G$$

The formula G seems to says "I am not provable". Now G is either true or false. We reason as follows:

- If *not* G is provable, that means *not* G is true, that is equivalent to: G is provable, contradicting the soundness of our system. Hence *not* G is not provable.

- If G is provable, then by soundness of our logic G must be true, and G being true says that it is not provable, which again is a contradiction.

we therefore conclude that for some formula in such a system, neither it nor its negation can be proved.

Such a system is called *incomplete*. Hence a system which is so powerful so as to talk about itself must remain incomplete. Negative self reference often lies at the heart of many paradoxes.

8 Gödel's Theorems

The above argument was used by Gödel to show that number theory is so powerful so as to be incomplete. Gödel by an ingenious scheme called *Gödel numbering* showed how every formula of number theory can itself be represented by a number. Not only formulae, but even axioms, proofs, sets and sequences of formulae were given Gödel numbers. Now he used number theory to talk about its own proof system and showed that the system is incomplete.

The following theorems of Löb's logic give some insight into number theory:

- Gödel's second incompleteness theorem says "if the consistency of number theory can be proved (in number theory), then number theory is inconsistent":

$$\textit{provable}\ \ \textit{consistent}\ \mathcal{T} \rightarrow\ \textit{provable}\ \mathcal{F}$$

- This theorem says "number theory is consistent iff its consistency is not provable".

$$(\ \textit{consistent}\ \mathcal{T})\ \textit{iff}\ (\ \textit{not}\ \ \textit{provable}\ \ \textit{consistent}\ \mathcal{T})$$

- It can be proved in number theory that "if the inconsistency of number theory is unprovable (in number theory) then its consistency is undecidable (in number theory)".

$$\begin{aligned}&\textit{not}\ \ \textit{provable}\ \ \textit{provable}\ \mathcal{F}\\ &\ \ \rightarrow\quad \textit{not}\ \ \textit{provable}\ \ \textit{consistent}\ \mathcal{T}\quad \textit{and}\\ &\qquad\ \ \textit{not}\ \ \textit{provable}\ \ \textit{not}\ \ \textit{consistent}\ \mathcal{T}\end{aligned}$$

9 Further Reading

You can read more about this subject in Raymond Smullyan's book: *Forever Undecided*, OUP, 1986. The book introduces you to modal logic through a series of puzzles about beliefs of reasoners. Those interested in details can look up the following books in the central library: *A Mathematical Introduction to Logic* by Enderton, *Introduction to Modal Logic* by Chellas, *The Unprovability of Consistency* by Boolos. It may open up a world of possibilities for you as it has done for many computer scientists and mathematicians.