Dependent Randomization in Parallel Binary Decision Fusion

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Supplement file, Part C

C-1 Proof: $P_d^{C'}$ is the maximum probability of detection when either (i) $A' \in \Omega^A$ or (ii) $B' \in \Omega^B$ or both ($A' \in \Omega^A$ and $B' \in \Omega^B$) (Section V-C)

C-2 An efficient way to find A' from Ω^A if $A' \in \Omega^A$ and B' from Ω^B if $B' \in \Omega^B$ (Section V-D)

In this file, the index of each Table/Figure/Equation has a prefix "C-"; the index of each Table/Figure/Equation that appears in the main paper has a prefix "P-".

SUPPLEMENT FILE, PART C

When the DFC only synchronizes with the m LDs in Y, the system operates at some point on *ROC curve A* with probability p and some point on *ROC curve B* with probability $1-p$. *ROC curve A* can be drawn by connecting points in $\Omega^A = \{w_1^A = (0,0), w_2^A, \ldots, w_{mA-1}^A, w_{mA}^A = (1,1)\}$ sequentially and *ROC curve B* can be drawn by connecting points in $\Omega^B = \{w_1^B = (0,0), w_2^B, \ldots, w_{mB-1}^B, w_{mB}^B = (1,1)\}$ sequentially. We want to find a specific point on *ROC curve A*, denoted as A', and a specific point on *ROC curve B*, denoted as B' that allow the system to maximize the probability of detection while satisfying the probability of false alarm constraint. The optimal resulting system operating point $C' = (P_f^{C'}, P_d^{C'})$ is on the line segment connecting A' and B'.

Part C-1 proves a result used in Section V-C of the paper: $P_d^{C'}$, the probability of detection at the redesigned operating point, is the maximum probability of detection when either (i) $A' \in \Omega^A$ or (ii) $B' \in \Omega^B$ or both ($A' \in \Omega^A$ and $B' \in \Omega^B$).

Part C-2 provides an efficient way to locate A' if $A' \in \Omega^A$ and B' if $B' \in \Omega^B$ (per Section V-D of the paper).

C-1. PROOF: $P_d^{C'}$ ((P-29) IN THE PAPER) IS THE MAXIMUM PROBABILITY OF DETECTION WHEN EITHER (I) $A' \in \Omega^A$ OR (II) $B' \in \Omega^B$ OR BOTH ($A' \in \Omega^A$ AND $B' \in \Omega^B$)

We are going to show that for $P_d^{C'}$ in (P-29) to be the maximum probability of detection either (i) $A' \in \Omega^A$ or (ii) $B' \in \Omega^B$ or both ($A' \in \Omega^A$ and $B' \in \Omega^B$).

Point $a = (P_f^a, P_d^a)$ is on *ROC curve A*, which is a concave linear ROC curve. P_d^a can be expressed as

$$P_d^a = f_A(P_f^a)P_f^a + b_A(P_f^a). \tag{C-1}$$

$f_A(P_f^a)$ and $b_A(P_f^a)$ are respectively the slope and the P_d-axis intercept of the line segment on *ROC curve A* that passes through point a. $f_A(P_f^a)$ is a decreasing piecewise-constant function of P_f^a and $b_A(P_f^a)$ is an increasing piecewise-constant function of P_f^a.

Similarly, point $b = (P_f^b, P_d^b)$ is on *ROC curve B*, which is a concave linear ROC curve. P_d^b can be expressed as

$$P_d^b = f_B(P_f^b)P_f^b + b_B(P_f^b). \qquad \text{(C-2)}$$

$f_B(P_f^b)$ and $b_B(P_f^b)$ are respectively the slope and the P_d-axis intercept of the line segment on *ROC curve B* that passes through point b. $f_B(P_f^b)$ is a decreasing piecewise-constant function of P_f^b and $b_B(P_f^b)$ is an increasing piecewise-constant function of P_f^b.

Figure C-1 shows the relation between $a = (P_f^a, P_d^a)$ (cyan circle) on *ROC curve A*, $b = (P_f^b, P_d^b)$ (purple triangle) on *ROC curve B*, and the resulting operating point $c = (P_f^c, P_d^c) = (pP_f^a + (1-p)P_f^b = \alpha, pP_d^a + (1-p)P_d^b)$ (purple square), calculated by (P-26) and (P-27), which is the intersection of $P_f = \alpha$ and line ab.

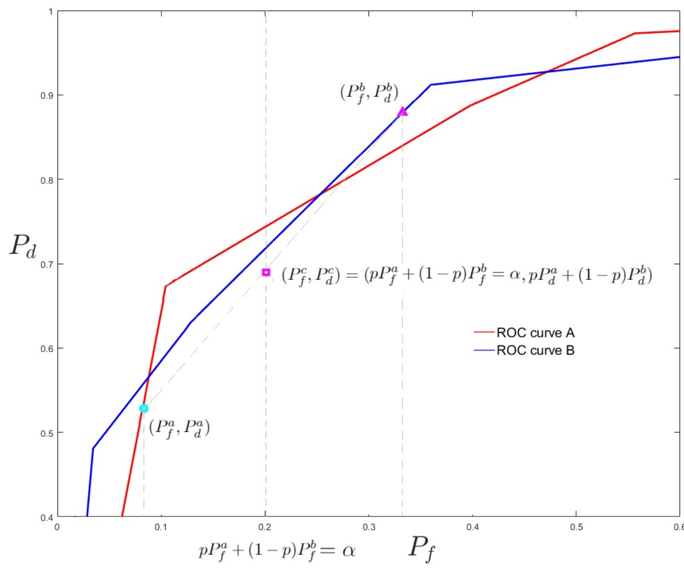

Figure C-1: The system operates at $a = (P_f^a, P_d^a)$ (cyan circle) with probability p. The system operates at $b = (P_f^b, P_d^b)$ (purple triangle) with probability $1-p$. The resulting operating point is $c = (P_f^c, P_d^c) = (pP_f^a + (1-p)P_f^b = \alpha, pP_d^a + (1-p)P_d^b)$, shown by the purple square.

From (P-26), when the probability of false alarm constraint is met, P_f^a is a decreasing function of P_f^b (and P_f^b is a decreasing function of P_f^a):

$$P_f^a = \frac{\alpha - (1-p)P_f^b}{p}, \text{ and,} \tag{C-3}$$

$$P_f^b = \frac{\alpha - pP_f^a}{1-p}. \tag{C-4}$$

Combining (P-27), (C-1), and (C-2), we have:

$$\begin{aligned} P_d^c &\overset{\text{from }(P-27)}{=} pP_d^a + (1-p)P_d^b \\ &\overset{\text{from }(C-1),(C-2)}{=} p(f_A(P_f^a)P_f^a + b_A(P_f^a)) + (1-p)(f_B(P_f^b)P_f^b + b_B(P_f^b)). \end{aligned} \tag{C-5}$$

From (C-4), P_f^b is a decreasing function of P_f^a (from (C-3), P_f^a is also a decreasing function of

P_f^b). Substitute (C-4) into (C-5), P_d^c can be expressed as a function of P_f^a [3]:

$$\begin{aligned} P_d^c &= \boldsymbol{s}P_f^a + \boldsymbol{l}, \text{where} \\ \boldsymbol{s} &= p[f_A(P_f^a) - f_B(P_f^b)], \\ &\overset{\text{from } (C-4)}{=} p[f_A(P_f^a) - f_B(\frac{\alpha - pP_f^a}{1-p})], \text{and} \\ \boldsymbol{l} &= f_B(P_f^b)\alpha + pb_A(P_f^a) + (1-p)b_B(\frac{\alpha - pP_f^a}{1-p}) \end{aligned} \tag{C-6}$$

which have the following properties:

Property 1:*P_d^c is a continuous function of P_f^a.*

P_d^a is a continuous function of P_f^a and P_d^b is a continuous function of P_f^b (since a and b are points on continuous ROC curves). Meanwhile, since P_f^b is a continuous function of P_f^a (from (C-4)), P_d^b is a continuous function of P_f^a. P_d^c is a weighted sum of P_d^a and P_d^b (from P-27), therefore it is a continuous function of P_f^a.

Property 2: *P_d^c is a piecewise-linear function of P_f^a.*

Since both *ROC curve A* and *ROC curve B* are composed of finite line segments, in (C-6), $f_A(P_f^a), f_B(\frac{\alpha-pP_f^a}{1-p}), b_A(P_f^a),$ and $b_B(\frac{\alpha-pP_f^a}{1-p})$ are piecewise-constant functions of P_f^a. Therefore, P_d^c is a piecewise-linear function of P_f^a. The graph of P_d^c consists of finite number of line segments on the $P_f^a - P_d^c$ plane. The slope of each line segment is $s = f_A(P_f^a) - f_B(\frac{\alpha-pP_f^a}{1-p})$ and the P_d^c-axis intercept of each line segment is $l = f_B(P_f^b)\alpha + pb_A(P_f^a) + (1-p)b_B(\frac{\alpha-pP_f^a}{1-p})$.

Property 3: *P_d^c is a concave function of P_f^a.*

From properties 1 and 2, P_d^c is a continuous piecewise-linear function of P_f^a. The slope of each line segment is $s = f_A(P_f^a) - f_B(P_f^b)$. $f_A(P_f^a)$ is a decreasing function of P_f^a since *ROC curve A* is piecewise-linear concave. $f_B(P_f^b) = f_B(\frac{\alpha-pP_f^a}{1-p})$ is an increasing function of P_f^a since $f_B(P_f^b)$ is a decreasing function of P_f^b and P_f^b is a decreasing function of P_f^a. In (C-6), $s = f_A(P_f^a) - f_B(P_f^b)$

[3]Similarly, if we substitute (C-3) into (C-5), then P_d^c can be expressed as a function of P_f^b.

is a decreasing function of P_f^a. In this circumstance, P_d^c is a concave function of P_f^a.

Property 4: The range of P_f^a is $P_f^a \in [max(0, \frac{\alpha+p-1}{p}), min(1, \frac{\alpha}{p})]$ when the probability of false alarm constraint is satisfied.

Since points a and b are on ROC curves, we have $P_f^a \in [0,1]$ and $P_f^b \in [0,1]$. From (C-4), when the probability of false alarm constraint is satisfied, $P_f^b = 0$ indicates $P_f^a = \frac{\alpha}{p}$ and $P_f^b = 1$ indicates $P_f^a = \frac{\alpha+p-1}{p}$. Therefore, $P_f^b \in [0,1]$ indicates that $P_f^a \in [\frac{\alpha+p-1}{p}, \frac{\alpha}{p}]$. Therefore, the range of P_f^a is $P_f^a \in [max(0, \frac{\alpha+p-1}{p}), min(1, \frac{\alpha}{p})]$.

From Properties 1-4, P_d^c is a piecewise-linear concave function of P_f^a and its domain satisfies $P_f^a \in [max(0, \frac{\alpha+p-1}{p}), min(1, \frac{\alpha}{p})]$. Note that a piecewise-linear concave function sometimes can be a monotonic linear function. We discuss three different cases about finding the maximum of P_d^c: (a) P_d^c is a non-decreasing linear function of P_f^a; (b) P_d^c is a non-increasing linear function of P_f^a; (c) P_d^c is first non-decreasing function and then a non-increasing function of P_f^a. Figure C-2 shows a graphical illustration of these three cases.

Case (a): $s \geq 0$ when $P_f^a = max(0, \frac{\alpha+p-1}{p})$ and when $P_f^a = min(1, \frac{\alpha}{p})$. In this case, P_d^c is a non-decreasing function of P_f^a and the maximum value of P_d^c is achieved at $P_f^a = min(1, \frac{\alpha}{p})$. If $P_f^a = min(1, \frac{\alpha}{p}) = 1$, since point $a = (P_f^a, P_d^a)$ is on *ROC curve A*, when $P_f^a = 1$, $P_d^a = 1$. Therefore, $A' = (1,1) = \omega_{mA}^A \in \Omega^A$. If $P_f^a = min(1, \frac{\alpha}{p}) = \frac{\alpha}{p} = \frac{\alpha-(1-p)0}{p} = \frac{\alpha-(1-p)P_f^{\omega_1^B}}{p} \overset{\text{from } (C-3)}{=} \frac{\alpha-(1-p)P_f^b}{p}$, $P_f^b = 0$. Since point $b = (P_f^b, P_d^b)$ is on *ROC curve B*, when $P_f^b = 0$, $P_d^b = 0$. Therefore, $B' = (0,0) = \omega_1^B \in \Omega^B$

Case (b): $s < 0$ when $P_f^a = max(0, \frac{\alpha+p-1}{p})$ and when $P_f^a = min(1, \frac{\alpha}{p})$. In this case, P_d^c is a non-increasing function of P_f^a and the maximum value of P_d^c is achieved at $P_f^a = max(0, \frac{\alpha+p-1}{p})$. If $P_f^a = max(0, \frac{\alpha+p-1}{p}) = 0$, we have $A' = (0,0) = \omega_1^A \in \Omega^A$. If $P_f^a = max(0, \frac{\alpha+p-1}{p}) = \frac{\alpha+p-1}{p} = \frac{\alpha-(1-p)1}{p} = \frac{\alpha-(1-p)P_f^{\omega_{mB}^B}}{p} \overset{\text{from } (C-3)}{=} \frac{\alpha-(1-p)P_f^b}{p}$, $P_f^b = 1$. We have $B' = (1,1) = \omega_{mB}^B \in \Omega^B$.

Case (c): $s \geq 0$ when $P_f^a = max(0, \frac{\alpha+p-1}{p})$ and $s < 0$ when $P_f^a = min(1, \frac{\alpha}{p})$. In this case, when P_f^a increases from $max(0, \frac{\alpha+p-1}{p})$ to $min(1, \frac{\alpha}{p})$, P_d^c is first a non-decreasing function and then a non-increasing function of P_f^a. We define the intersection of two line segments on a piecewise-linear ROC curve as a *corner point* of that ROC curve. The maximum value of P_d^c is achieved at a *corner point* on the graph of P_d^c where the slope of the left line segment at that *corner point* and the slope of the right line segment at that *corner point* have different sign (the sign of s changes at that *corner point*).

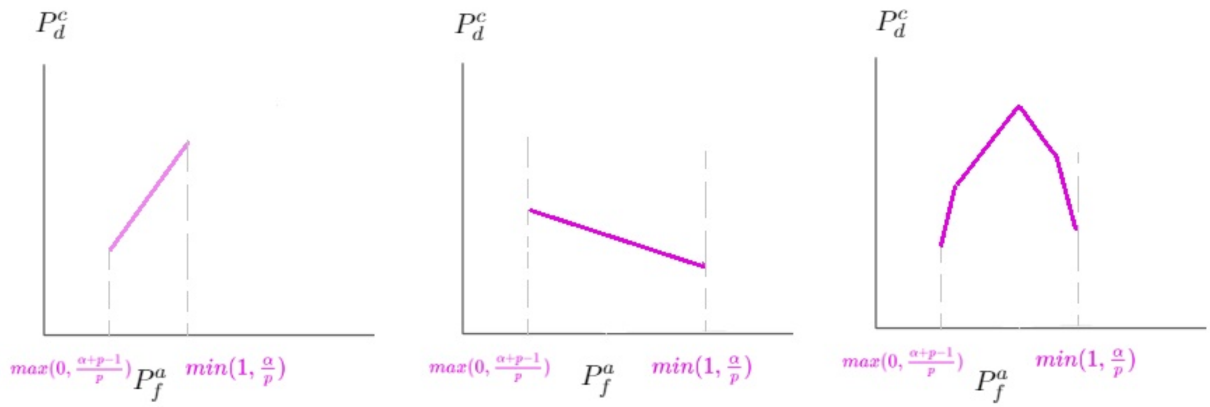

Figure C-2: A graphical illustration of cases (a), (b), and (c) (from left to right).

In cases (a) and (b), P_d^c is a monotonic function of P_f^a, its maximum is achieved when one of the points a and b is at $(0,0)$ or $(1,1)$. Since $\omega_1^A = \omega_1^B = (0,0)$ and $\omega_{mA}^A = \omega_{mA}^B = (1,1)$, either $A' \in \Omega^A$ or $B' \in \Omega^B$ (or both). In case (c), P_d^c is maximized when the sign of s changes from positive to negative. In the expression of s, $p \in (0,1)$, $f_A(.)$ changes only when a is a *corner point* of *ROC curve A* ($A' \in \{\omega_2^A \ldots \omega_{mA-1}^A\} \subset \Omega^A$), $f_B(.)$ changes only when b is a *corner point* of *ROC curve B* ($B' \in \{\omega_2^B \ldots \omega_{mB-1}^B\} \subset \Omega^B$). Therefore, $P_d^{C'}$ is the maximum probability of detection when either (i) $A' \in \Omega^A$ or (ii) $B' \in \Omega^B$ or both ($A' \in \Omega^A$ and $B' \in \Omega^B$).

C-2. Finding A' from Ω^A if $A' \in \Omega^A$ and B' from Ω^B if $B' \in \Omega^B$ (improved version)

In Section V of the paper, we proposed a 2-step procedure to find A' and B': Step 1 - Finding A' from Ω^A if $A' \in \Omega^A$ and B' from Ω^B if $B' \in \Omega^B$ (Section V-D); Step 2 - Finding A' if $A' \notin \Omega^A$ and B' if $B' \notin \Omega^B$ (Section V-E). Step 1 requires examining all the points in Ω^A and Ω^B ($mA + mB$ points). In this section, we propose a more efficient way to realize Step 1.

In the previous section, we discussed three different cases about finding the maximum of P_d^c, $P_d^{C'}$. We need to first calculate the value of s in (C-6) when $P_f^a = max(0, \frac{\alpha+p-1}{p})$ and when $P_f^a = min(1, \frac{\alpha}{p})$ to determine which case we are encountering.

1) Case (a): P_d^c is a non-decreasing function of P_f^a and the maximum of P_d^c achieved when $P_f^a = min(1, \frac{\alpha}{p})$, shown as the first graph in Figure C-2. When $P_d^{C'}$ is the maximum of probability of detection, we can find either $A' = (P_f^{A'}, P_d^{A'}) = (1, 1)$ or $B' = (P_f^{B'}, P_d^{B'}) = (0, 0)$ (or both).

2) Case (b): P_d^c is non-increasing function of P_f^a and the maximum of P_d^c is achieved at $P_f^a = max(0, \frac{\alpha+p-1}{p})$, shown as the second graph in Figure C-2. When $P_d^{C'}$ is the maximum of probability of detection, we can find either $A' = (P_f^{A'}, P_d^{A'}) = (0, 0)$ or $B' = (P_f^{B'}, P_d^{B'}) = (1, 1)$ (or both).

3) Case (c): P_d^c is first a non-decreasing function and then a non-increasing function of P_f^a. The slope of each line segment of P_d^c is $s = p[f_A(P_f^a) - f_B(P_f^b)]$ (from (C-6)). Our target is to find $a = A' \in \Omega^A$ or $b = B' \in \Omega^B$ such that the sign of $f_A(P_f^a) - f_B(P_f^b)$ changes from positive to negative.

Let $F^A = \{P_f^{\omega_1^A}, .., P_f^{\omega_{mA}^A}\}$ be the probabilities of false alarm of the operating points in $\Omega^A = \{\omega_1^A \dots \omega_1^{mA}\}$. $A' \in \Omega^A$ when $P_f^{A'} \in F^A$.

Let $F^B = \{P_f^{\omega_1^B}, .., P_f^{\omega_{mB}^B}\}$ be the probabilities of false alarm of the operating points in $\Omega^B = \{\omega_1^B \dots \omega_1^{mB}\}$. $B' \in \Omega^B$ when $P_f^{B'} \in F^B$.

In order to meet the probability of false alarm constraint α, from (C-4), when $P_f^{B'} \in F^B =$

$\{P_f^{\omega_1^B}, .., P_f^{\omega_{mB}^B}\}$, the probability of false alarm of point A' satisfies $P_f^{A'} \in G^A = \{\frac{\alpha+(p-1)P_f^{\omega_1^B}}{p}, \dots \frac{\alpha+(p-1)P_f^{\omega_{mB}^B}}{p}\}$.

Let $H = F^A \bigcup G^A$, when $A' \in \Omega^A$ or $B' \in \Omega^B$ (or both), $P_f^{A'} \in H$. Therefore, our target now is to find $P_f^{A'} \in H$ such that the sign of $f_A(P_f^a) - f_B(P_f^b)$ changes from positive to negative at $P_f^a = P_f^{A'} \in H$.

Recall that $f_a(P_f^a)$ represents the slopes of all straight line segments $w_1^A w_2^A, w_2^A w_3^A, \dots, w_{mA-1}^A w_{mA}^A$ composing *ROC curve A*. $f_a(P_f^a)$ can be expressed as a decreasing piecewise-constant function of P_f^a:

$$f_A(P_f^a) = \begin{cases} \dfrac{P_d^{\omega_{j+1}^A} - P_d^{\omega_j^A}}{P_f^{\omega_{j+1}^A} - P_f^{\omega_j^A}}, P_f^a \in [P_f^{\omega_j^A}, P_f^{\omega_{j+1}^A}), j = 1, \dots, m_A - 1 \\ \dfrac{P_d^{\omega_{m_A}^A} - P_d^{\omega_{m_A-1}^A}}{P_f^{\omega_{m_A}^A} - P_f^{\omega_{m_A-1}^A}}, P_f^a = P_f^{\omega_{m_A}^A} \end{cases}. \tag{C-7}$$

Similarly, $f_b(P_f^b)$ represents the slopes of all straight line segments $w_1^B w_2^B, w_2^B w_3^B, \dots, w_{mB-1}^B w_{mB}^B$ composing *ROC curve B*. $f_b(P_f^b)$ can be expressed as a decreasing piecewise-constant function of P_f^b:

$$f_B(P_f^b) = \begin{cases} \dfrac{P_d^{\omega_{j+1}^B} - P_d^{\omega_j^B}}{P_f^{\omega_{j+1}^B} - P_f^{\omega_j^B}}, P_f^b \in (P_f^{\omega_j^B}, P_f^{\omega_{j+1}^B}], j = 1, \dots, m_B - 1 \\ \dfrac{P_d^{\omega_2^B} - P_d^{\omega_1^B}}{P_f^{\omega_2^B} - P_f^{\omega_1^B}}, P_f^b = P_f^{\omega_1^B} \end{cases}. \tag{C-8}$$

From (C-4), when the probability of false alarm constraint is met $P_f^c = \alpha$, $P_f^b = \frac{\alpha - pP_f^a}{1-p}$. We can rewrite $f_B(P_f^b) = f_B(\frac{\alpha - pP_f^a}{1-p}) = g_A(P_f^a)$ as a function of P_f^a. From (C-3), $P_f^a = \frac{\alpha-(1-p)P_f^b}{p}$. When $P_f^b \in (P_f^{\omega_j^B}, P_f^{\omega_{j+1}^B}]$, $P_f^a \in [\frac{\alpha+(p-1)P_f^{\omega_{j+1}^B}}{p}, \frac{\alpha+(p-1)P_f^{\omega_j^B}}{p})$. $g_A(.)$ can be expressed as:

$$g_A(P_f^a) = f_B(\frac{\alpha - pP_f^a}{1-p}) = \begin{cases} \dfrac{P_d^{\omega_{j+1}^B} - P_d^{\omega_j^B}}{P_f^{\omega_{j+1}^B} - P_f^{\omega_j^B}}, P_f^a \in [\dfrac{\alpha + (p-1)P_f^{\omega_{j+1}^B}}{p}, \dfrac{\alpha + (p-1)P_f^{\omega_j^B}}{p}), j = m_B - 1, \dots, 1 \\ \dfrac{P_d^{\omega_2^B} - P_d^{\omega_1^B}}{P_f^{\omega_2^B} - P_f^{\omega_1^B}}, P_f^a = \dfrac{\alpha + (p-1)P_f^{\omega_1^B}}{p} \end{cases}. \tag{C-9}$$

$g_A(P_f^a)$ calculates the slope of the line segment on *ROC curve B* intersecting the vertical line $P_f = P_f^b = \frac{\alpha - pP_f^a}{1-p}$ (the line segment on *ROC curve B* passing through point $b = (P_f^b = \frac{\alpha - pP_f^a}{1-p}, P_d^b)$). $g_A(P_f^a)$ is a piecewise increasing constant function of P_f^a.

Therefore, in case (c), our target becomes finding $P_f^{A'} \in H$ such that the sign of $f_A(P_f^a) - g_A(P_f^a)$ changes from positive to negative at $P_f^a = P_f^{A'}$.

$f_A(P_f^a) - g_A(P_f^a)$ is a piecewise decreasing constant function of P_f^a and its value only changes when $P_f^{A'} \in H$. Since we defined each one of the constant functions composing $f_A(P_f^a) - g_A(P_f^a)$ on a left-closed right-open interval, when $f_A(P_f^a) - g_A(P_f^a)$ changes from positive to negative, $P_f^{A'}$ can be found as the smallest value of P_f^a in H such that $f_A(P_f^a) - g_A(P_f^a) < 0$.

One way to find $P_f^a = P_f^{A'} \in H$ is using a common binary search algorithm which contains following steps:

1) Sort the elements in H
2) Calculate $f_A(P_f^a) - g_A(P_f^a)$ for the middle element in H

   (if H has an even number of elements, use the smaller one of the middle two elements)
3) If the result is negative: eliminate the latter half of H (excluding the middle element);

   Otherwise: eliminate the former half of H (including the middle element)
4) Repeat steps 2) and 3) until H has only one element, which is $P_f^{A'}$

Since $H = F^A \bigcup G^A$ contains at most $mA + mB$ elements (mA elements in F^A and mB elements in G^A), the binary search algorithm performs at most $log(mA + mB)$ iterations.

When $P_f^{A'}$ is found, if $P_f^{A'} \in F^A$, we can use $P_f^{A'}$ to find A' in Ω^A; if $P_f^{A'} \in G^A$, we can calculate $P_f^{B'} = \frac{\alpha - pP_f^{A'}}{1-p}$ (from (C-4)) and use $P_f^{B'}$ to find B' in Ω^B.

After finding A' if $A' \in \Omega^A$ and B' if $B' \in \Omega^B$, we can use Step 2 in Section V-E of the paper to find A' if $A' \notin \Omega^A$ and B' if $B' \notin \Omega^B$.