determinant of the matrix given by eliminating the jth column of A and substituting the kth column in its place, then crossing out the kth row, i.e.,

$$x_{j\neq k} = \frac{\sum_{\{p_i, i=1,\ldots,d-1, p_i\neq k\}} A_{p_1 1}\cdots A_{p_{j-1} j-1}(-A_{p_j k} x_k) A_{p_{j+1} j+1}\cdots A_{p_{d-1} d-1}\epsilon_{p_1\ldots p_n}}{\sum_{\{p_i, i=1,\ldots,d-1, p_i\neq k\}} A_{p_1 1}\cdots A_{p_{j-1} j-1} A_{p_j j} A_{p_{j+1} j+1}\cdots A_{p_{d-1} d-1}\epsilon_{p_1\ldots p_n}} \tag{33}$$

where we make use of the definition $\det(M) = \sum_{i,j,\ldots,z} M_{i1} M_{j2}\ldots M_{zd} \epsilon_{ij\ldots z}$ where M is of dimension d and the symbol $\epsilon_{i,j,\ldots,z}$ is the totally antisymmetric Levi–Civita alternating tensor of rank d. Accounting for the minus signs, this results in

$$x_j = x_k \frac{\text{cofactor}(A_{kj})}{\text{cofactor}(A_{kk})} \quad \forall\ (j \neq k) \tag{34}$$

which is also obviously true for $j = k$. When substituted into

$$\sum_j x_k A_{kj} x_j = 1 \tag{35}$$

the result is

$$\sum_j x_k A_{kj} x_k \frac{\text{cofactor}(A_{kj})}{\text{cofactor}(A_{kk})} = (x_k)^2 \frac{\det(\mathrm{A})}{\text{cofactor}(A_{kk})} = 1 \tag{36}$$

reducing to the simple expression

$$(x_k)^2 = \frac{\text{cofactor}(A_{kk})}{\det(A)} = A_{kk}^{-1}. \tag{37}$$

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Optimal Data Fusion of Correlated Local Decisions in Multiple Sensor Detection Systems

Recently, Chair and Varshney have solved the data fusion problem for fixed binary local detectors with statistically independent decisions. We generalize their solution by using the Bahadur–Lazarsfeld expansion of probability density functions. The optimal data fusion rule is developed for correlated local binary decisions, in terms of the conditional correlation coefficients of all orders. We show that when all these coefficients are zero, the rule coincides with the original Chair–Varshney design.

I. THE FUSION PROBLEM

We consider the following distributed-detection system: a bank of n sensors obtains observations from an environment on which two hypotheses are made: H_0 ("no target") and H_1 ("target exists"). Each local detector collects an observation $x_i \in R^n$ and transforms it, using a local mapping, to a local decision $u_i = g_i(x_i)$, $u_i \in \{0,1\}$, $i = 1,2,\ldots,n$, where

$$u_i = \begin{cases} 0, & \text{if}\quad \mathrm{H}_0 \text{ is accepted} \\ 1, & \text{if}\quad \mathrm{H}_1 \text{ is accepted} \end{cases}.$$

The local decisions (u_is) are transmitted to a data fusion center (DFC) over noiseless communication channels. Using $u_1, u_2, \ldots, u_n$, the DFC makes a global decision u ($u = 0$ for accepting H_0, $u = 1$ for accepting H_1) which minimizes the average Bayesian cost:

$$\overline{C} = \sum_{j=0}^{1}\sum_{k=0}^{1} C_{jk} P(u = j, \mathrm{H}_k) \tag{1}$$

where the costs C_{jk} are specified.

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The problem is to find the decision rule $u = f(u_1, u_2, \ldots, u_n)$, $f : \{0,1\}^n \to \{0,1\}$, which minimizes the average Bayesian cost (equation (1)). When the local decisions are conditionally statistically independent, the problem has been solved by Chair and Varshney [1]. The problem with correlated local decisions was studied in different forms by Lauer and Sandell [2], Aalo and Viswanathan [3], and Drakopoulos and Lee [4]. Here we employ the Bahadur–Lazarsfeld expansion of probability density functions [5, pp. 111–113] to design the DFC and to generalize the Chair–Varshney design.

II. OPTIMAL DECISION RULE

Let $\mathbf{U} = [u_1, u_2, \ldots, u_n]$ be the vector of the correlated local decisions, and $P(\mathbf{U})$ be the probability density function of $\mathbf{U}$. It is well known that the optimal decision rule for the DFC is

$$\lambda(\mathbf{U}) = \frac{P(\mathbf{U} \mid \mathrm{H}_1)}{P(\mathbf{U} \mid \mathrm{H}_0)} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})} \triangleq \lambda_0 \tag{2}$$

where P_0 and P_1 are the a priori probabilities of hypotheses H_0 and H_1, respectively. In order to express $P(\mathbf{U} \mid H_i)$ in a convenient form, we introduce the normalized random variables

$$\begin{aligned} z_i &= \frac{u_i - p_i}{\sqrt{p_i q_i}}, \qquad \text{where} \quad p_i = P(u_i = 1), \\ q_i &= 1 - p, \quad i = 1, 2, \ldots, n \end{aligned} \tag{3}$$

and z_i has zero mean and unit variance. The *Bahadur–Lazarsfeld polynomials* [5, pp. 111–113] are defined as

$$\varphi_i(\mathbf{U}) = \begin{cases} 1 & i = 0 \\ z_1 & i = 1 \\ z_2 & i = 2 \\ \vdots & \\ z_n & i = n \\ z_1 z_2 & i = n+1 \\ z_1 z_3 & i = n+2 \\ \vdots & \\ z_1 z_2 z_3 & i = n + 1 + \dfrac{n(n-1)}{2} \\ z_1 z_2 z_4 & i = n + 2 + \dfrac{n(n-1)}{2} \\ \vdots & \\ z_1 z_2 \ldots z_n & i = 2^n - 1 \end{cases}.$$

Facts (see [5]):

1) Let

$$P_1(\mathbf{U}) = \prod_{i=1}^{n} p_i^{u_i} q_i^{1-u_i} \tag{4}$$

then

$$\sum_{\mathbf{U}} \varphi_i(\mathbf{U}) \varphi_j(\mathbf{U}) P_1(\mathbf{U}) = \delta_{ij} \tag{5}$$

where δ_{ij} is Kronecker's delta function:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

2)

$$P(\mathbf{U}) = \sum_{i=0}^{2^n - 1} a_i \varphi_i(\mathbf{U}) \tag{6}$$

where

$$a_i = P_1(\mathbf{U}) \sum_{\mathbf{V}} \varphi_i(\mathbf{V}) P(\mathbf{V}) \tag{7}$$

and the summation is over all possible decision tuples $\mathbf{V}$.

The function $P(\mathbf{U})/P_1(\mathbf{U})$ can be expanded as

$$\frac{P(\mathbf{U})}{P_1(\mathbf{U})} = \sum_{i=0}^{2^n - 1} b_i \varphi_i(\mathbf{U}) \tag{8}$$

where

$$b_i = \sum_{\mathbf{U}} \varphi_i(\mathbf{U}) P(\mathbf{U}) = E[\varphi_i(\mathbf{U})].$$

There are 2^n b_i parameters. Clearly $b_0 = 1, b_1 = b_2 = \cdots = b_n = 0$. For $i > n$, the b_i parameters are interpreted as *correlation coefficients* (recalling that $\varphi_i(\mathbf{U})$ is a product of the normalized variables z_i). Following [5, pp. 112], we redefine the correlation coefficients of $\{z_i\}_{i=1}^{n}$ by order:

$$\left.\begin{aligned} &\gamma_{ij} = \sum_{\mathbf{U}} z_i z_j P(\mathbf{U}) \\ &\quad \text{(second-order correlation coefficient)} \\ &\gamma_{ijk} = \sum_{\mathbf{U}} z_i z_j z_k P(\mathbf{U}) \\ &\quad \text{(third-order correlation coefficient)} \\ &\vdots \\ &\gamma_{1\,2\ldots n} = \sum_{\mathbf{U}} z_1 z_2 \ldots z_n P(\mathbf{U}) \\ &\quad (n\text{th-order correlation coefficient}) \end{aligned}\right\} \tag{9}$$

The complete expansion of $P(\mathbf{U})$ becomes

$$P(\mathbf{U}) = P_1(\mathbf{U}) \left[1 + \sum_{i<j} \gamma_{ij} z_i z_j + \sum_{i<j<k} \gamma_{ijk} z_i z_j z_k + \cdots + \gamma_{1\,2\ldots n} z_1 z_2 \ldots z_n \right]. \tag{10}$$

Similarly, let the *conditional* correlation coefficients of $\{z_i\}_{i=1}^{n}$ (conditioned on hypothesis H_h) be

$$
\left.
\begin{aligned}
\gamma_{ij}^h &= \sum_{\mathbf{U}} z_i^h z_j^h P(\mathbf{U} \mid H_h) \\
\gamma_{ijk}^h &= \sum_{\mathbf{U}} z_i^h z_j^h z_k^h P(\mathbf{U} \mid H_h) \\
&\vdots \\
\gamma_{1\,2\ldots n}^h &= \sum_{\mathbf{U}} z_1^h z_2^h \ldots z_n^h P(\mathbf{U} \mid H_h)
\end{aligned}
\right\} \tag{11}
$$

where

$$
z_i^h = \frac{U_i - P(U_i = 1 \mid H_h)}{\sqrt{P(U_i = 1 \mid H_h)[1 - P(U_i = 1 \mid H_h)]}}, \qquad h = 0,1.
$$

Then $\lambda(\mathbf{U})$ of (2) can be rewritten as

$$
\lambda(\mathbf{U}) = \frac{P_1(\mathbf{U} \mid H_1)\left[1 + \sum_{i<j} \gamma_{ij}^1 z_i^1 z_j^1 + \sum_{i<j<k} \gamma_{ijk}^1 z_i^1 z_j^1 z_k^1 + \cdots + \gamma_{1\,2\ldots n}^1 z_1^1 z_2^1 \ldots z_n^1\right]}{P_1(\mathbf{U} \mid H_0)\left[1 + \sum_{i<j} \gamma_{ij}^0 z_i^0 z_j^0 + \sum_{i<j<k} \gamma_{ijk}^0 z_i^0 z_j^0 z_k^0 + \cdots + \gamma_{1\,2\ldots n}^0 z_1^0 z_2^0 \ldots z_n^0\right]}. \tag{12}
$$

Let $p_{Fi} = P(u_i = 1 \mid \mathrm{H}_0)$ be the probability of false alarm for the ith local detector, and $p_{Mi} = P(u_i = 0 \mid \mathrm{H}_1)$ be its probability of missed detection. Using this notation and (4) we can write

$$
\begin{aligned}
\frac{P_1(\mathbf{U} \mid \mathrm{H}_1)}{P_1(\mathbf{U} \mid \mathrm{H}_0)} &= \frac{\prod_{i=1}^n (1 - p_{Mi})^{u_i} p_{Mi}^{1-u_i}}{\prod_{i=1}^n (1 - p_{Fi})^{1-u_i} p_{Fi}^{u_j}} \\
&= \prod_{i=1}^n \left(\frac{1 - p_{Mi}}{p_{Fi}}\right)^{u_i} \left(\frac{p_{Mi}}{1 - p_{Fi}}\right)^{1-u_i}.
\end{aligned} \tag{13}
$$

From (12) and using (13), the log likelihood ratio test is

$$
\log \lambda(\mathbf{U}) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \log \lambda_0 \tag{14}
$$

where

$$
\begin{aligned}
\log \lambda(\mathbf{U}) &= \sum_{i=1}^n u_i \left[\log \frac{(1 - p_{Mi})(1 - p_{Fi})}{p_{Mi} p_{Fi}}\right] + \sum_{i=1}^n \log \frac{p_{Mi}}{(1 - p_{Fi})} \\
&\quad + \log \frac{1 + \sum_{i<j} \gamma_{ij}^1 z_i^1 z_j^1 + \sum_{i<j<k} \gamma_{ijk}^1 z_i^1 z_j^1 z_k^1 + \cdots + \gamma_{1\,2\ldots n}^1 z_1^1 z_2^1 \ldots z_n^1}{1 + \sum_{i<j} \gamma_{ij}^0 z_i^0 z_j^0 + \sum_{i<j<k} \gamma_{ijk}^1 z_i^0 z_j^0 z_k^0 + \cdots + \gamma_{1\,2\ldots n}^0 z_1^0 z_2^0 \ldots z_n^0}.
\end{aligned} \tag{15}
$$

Equation (15) is the data fusion rule for a distributed detection system with correlated local decisions. If the conditional correlation coefficients above a certain order can be neglected, as is the case in many practical applications, the computational burden can be reduced. If all the conditional correlation coefficients are zero under both hypotheses, then the optimal decision rule is

$$
d_0 + \sum_{i=1}^n d_i u_i \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} 0 \tag{16}
$$

where

$$
\left.
\begin{aligned}
d_0 &= -\log \lambda_0 + \sum_{i=1}^n \log \frac{p_{Mi}}{1 - p_{Fi}}, \\
d_i &= \log \frac{(1 - p_{Mi})(1 - p_{Fi})}{p_{Mi} p_{Fi}}.
\end{aligned}
\right. \tag{17}
$$

This rule is exactly the optimal fusion rule that Chair and Varshney [1] have recently developed for fixed local detectors with independent local decisions (and indeed when all the conditional correlation coefficients under both hypotheses are zero, the local decisions are statistically independent [6]).

III. EXAMPLE

We consider a system of three detectors (decision variables u_1, u_2, u_3) and a DFC (decision variable u). The DFC minimizes $\overline{C}$ (equation (1)), while the ith detector makes binary decisions about observations in normal additive noise by minimizing a local Bayesian cost

$$
\overline{C}_i = \sum_{j=0}^1 \sum_{k=0}^1 C_{jk}^{(i)} P(u_i = j, H_k) \tag{18}
$$

where the local costs $C_{jk}^{(i)}$ are specified. Corresponding to the definition of λ_0 (equation (2)), we define for the ith detector

$$
\lambda_0^{(i)} = \frac{P_0(C_{10}^{(i)} - C_{00}^{(i)})}{P_1(C_{01}^{(i)} - C_{11}^{(i)})}. \tag{19}
$$

The observation collected by the ith detector is assumed to be

$$
\text{under hypothesis } \mathrm{H}_0: \quad x_i = n_i^0 \tag{20a}
$$

$$
\text{under hypothesis } \mathrm{H}_1: \quad x_i = m + n_i^1 \tag{20b}
$$

where m is a positive constant. The noise variables n_1^0, n_2^0, n_3^0, and n_1^1, n_2^1, n_3^1 are jointly normal with zero mean and the covariance matrices

$$\sum\nolimits_0 = \sum\nolimits_{n_1^0 n_2^0 n_3^0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$\sum\nolimits_1 = \sum\nolimits_{n_1^1 n_2^1 n_3^1} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

The correlation coefficients satisfy $0 \le \rho_{ij} < 1$; $i,j = 1,2,3$. In this example the appearance of a target correlates the noise signals (for example when transmitter amplifier noise appears additively with the transmitted signal).

To minimize $\bar{C}_i$ (equation (18)), the ith detector employs locally the log likelihood ratio test

$$x_i \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \tau = \frac{1}{m}\log\lambda_0^{(i)} + \frac{m}{2}, \qquad i = 1,2,3. \tag{21}$$

For the given $\sum_0$ and $\sum_1$, all the γ^0 coefficients in (11) (which correspond to H_0) are zero; the γ^1 coefficients (which correspond to H_1) may be non-zero. There are three second-order conditional correlation coefficients (γ_{12}^1, γ_{13}^1 and γ_{23}^1) and one third-order conditional correlation coefficient (γ_{123}^1) which need to be computed.

The second-order correlation coefficients under H_1 are given by

$$\gamma_{ij}^1 = E(z_i^1 z_j^1 \mid \mathrm{H}_1) = \frac{E(u_i u_j \mid \mathrm{H}_1) - (1-p_M)^2}{p_M(1-p_M)} \tag{22}$$

where

$$p_M = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\tau-m} e^{-t^2/2}\,dt,$$

and

$$E(u_i u_j \mid \mathrm{H}_1) = P(x_i \ge \tau, x_j \ge \tau \mid \mathrm{H}_1) = \int_{\tau-m}^{\infty}\int_{\tau-m}^{\infty} \frac{1}{2\pi\sqrt{1-\rho_{ij}^2}} \times e^{-(x^2 - 2\rho_{ij}xy + y^2)/2(1-\rho_{ij}^2)}\,dx\,dy \tag{23}$$

and the third-order correlation coefficient under H_1 is given by

$$\gamma_{123}^1 = E(z_1^1 z_2^1 z_3^1 \mid \mathrm{H}_1) = E\left(\prod_{i=1}^{3} \frac{u_i - (1-p_M)}{\sqrt{p_M(1-p_M)}} \middle| \mathrm{H}_1\right) = \frac{E(u_1u_2u_3 \mid \mathrm{H}_1) + 2(1-p_M)^3 - (1-p_M)[E(u_1u_2 \mid \mathrm{H}_1) + E(u_1u_3 \mid \mathrm{H}_1) + E(u_2u_3 \mid \mathrm{H}_1)]}{p_M(1-p_M)\sqrt{p_M(1-p_M)}} \tag{24}$$

where

$$E(u_1u_2u_3 \mid \mathrm{H}_1) = P(x_1 \ge \tau, x_2 \ge \tau, x_3 \ge \tau \mid \mathrm{H}_1) = \int_{\tau-m}^{\infty}\int_{\tau-m}^{\infty}\int_{\tau-m}^{\infty} \frac{1}{\sqrt{(2\pi)^3|\sum_1|}} \times e^{-(1/2)\mathbf{x}\sum_1^{-1}\mathbf{x}^{\mathrm{T}}}\,d\mathbf{X} \tag{25}$$

with $\mathbf{X} = [x_1 x_2 x_3]^{\mathrm{T}}$.

For $m = \sqrt{2\log\lambda_0}(\lambda_0 \ge 1)$, there exist closed-form expressions for $E(u_i u_j \mid \mathrm{H}_1)$ and $E(u_1u_2u_3 \mid \mathrm{H}_1)$, namely [7]:

$$E(u_i u_j \mid \mathrm{H}_1) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho_{ij} \tag{26}$$

$$E(u_1u_2u_3 \mid \mathrm{H}_1) = \frac{1}{8} + \frac{1}{2\pi}(\sin^{-1}\rho_{12} + \sin^{-1}\rho_{13} + \sin^{-1}\rho_{23}). \tag{27}$$

Consequently $\gamma_{ij}^1 = (2/\pi)\sin^{-1}\rho_{ij}$ for $i < j$, $i,j = 1,2,3$, and $\gamma_{123}^1 = 0$. Therefore at the vicinity of $m = \sqrt{2\log\lambda_0}$, the third-order correlation coefficient can be ignored.

Fig. 1 shows (for $\rho_{ij} = \rho$) the ratio of third-order correlation coefficient to the second-order correlation coefficient versus $w = (m/2) - (1/m)\log\lambda_0$ ($\lambda_0 \ge 1$) around $w = 0$. The ratio is approximately linear in w over an extended range around $w = 0$. In fact for a small w

$$z = \frac{\text{third-order correlation coefficient}}{\text{second-order correlation coefficient}} \approx \frac{6}{\sqrt{2\pi}}\left[\frac{\sin^{-1}\dfrac{\rho}{1+\rho}}{\sin^{-1}\rho} - 1\right] w \tag{28a}$$

and if $\rho \ll 1$

$$z \approx \frac{6}{\sqrt{2\pi}}\rho w. \tag{28b}$$

Equations (28a) and (28b) allow us to determine approximately the range (in w) over which the third-order correlation coefficient can be neglected.

IV. CONCLUSIONS

We have calculated the optimal data fusion rule for n binary sensors using the Bahadur–Lazarsfeld expansion of probability density functions. In the

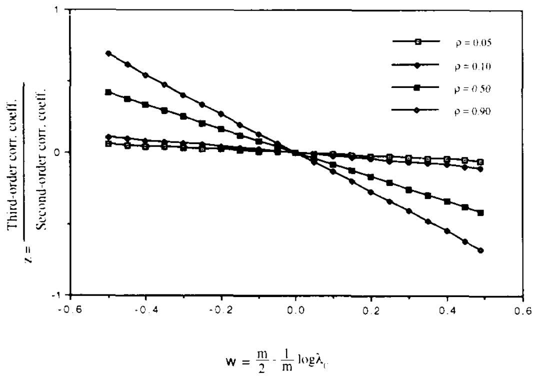

Fig. 1. Ratio of third-order to second-order correlation coefficients.

most general case we need to estimate $2(2^n - n - 1)$ correlation coefficients in order to obtain the optimal log likelihood ratio test. The computation could be greatly simplified if most correlation coefficients of the local decisions are zero, and when they are all zero, we obtain the optimal data fusion rule developed by Chair and Varshney for independent local decisions.

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Comment on "Eigenstructure Assignment for Linear Systems"

In the paper [1], an achievability subspace is given and used for eigenstructure assignment of linear systems with gain output feedback. The objective of this note is to show that the achievability subspace is not generally achievable.

Consider a linear time-invariant system described by the equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where 1) $x \in R^n$, $u \in R^m$, $y \in R^r$ are the state, control, and output, respectively; 2) A, B, C are real constant matrices of appropriate dimensions; 3) $\operatorname{rank} B = m \neq 0$, $\operatorname{rank} C = r \neq 0$; and 4) the system given is controllable and observable.

In the paper [1], gain output feedback is used to assign an eigenstructure of the above linear system, and the achievability subspace is defined as the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$ where λ_i is an eigenvalue which a designer desires to assign in the closed-loop system using gain output feedback given by

$$u(t) = Fy(t). \tag{3}$$

And it is concluded that if we choose an eigenvector v_i which lies precisely in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$, it will be achieved exactly. But this conclusion contradicts the Srinathkumar's result [1, 2]. Here is a counter example.

EXAMPLE. Consider a linear system of (1) and (2) with matrices A, B, and C given by

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad C = [1 \quad 0]. \tag{4}$$

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