The Gaussian Nonlinearity's Output Power Spectral Density Due to a Sinusoid Plus Band-Limited Noise

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Abstract—A sinusoid accompanied by stationary, additive, nonzero-mean, band-limited, white Gaussian noise is passed through a Gaussian nonlinear characteristic, and the output power spectral density is evaluated by using known results due to Rice and Atherton. Several characteristics of the Gaussian nonlinearity are revealed in the process, and input tuning is shown to contribute to output noise reduction. The results are applicable in the analysis of tracking systems employing sinusoidal dithers.

I. Introduction

The output spectra of nonlinear input–output characteristics with well-characterized inputs are of practical importance in the analysis and design of nonlinear systems [1]–[5]. In this correspondence we examine the Gaussian nonlinearity which, unlike many nonlinearities that have been widely investigated in this context, is symmetrical and is concave around its apex. The input of interest is a pure sinusoid accompanied by biased Gaussian noise. This configuration finds application in conical scan laser tracking systems [6].

We study the output power spectral density in order to use it for postprocessing (e.g., filtering). Some of the parameters of the input signal and the nonlinearity may not be known or may need to be controlled. In some tracking problems the input bias represents the tracked parameter, which the system estimates from the nonlinearity's output and then controls. In this correspondence, however, no feedback effects are considered, and the power spectral density is observed in an open loop.

II. Statement of the Problem

We consider the Gaussian input–output relationship

$$z[y(t)] = I_0 \exp\left[-(y(t)-y_0)^2/2\sigma_g^2\right] \tag{1}$$

where

- $y(t)$ input (not containing a dc term),
- z output,
- I_0 output scale factor,
- σ_g width parameter of the nonlinearity,
- y_0 constant shift parameter representing a possible dc term in the input.

The input waveform applied to this nonlinearity is

$$y(t) = a\sin\omega_d t + n(t) \tag{2}$$

where a and ω_d are the known input-signal amplitude and frequency, respectively, and $n(t)$ is a band-limited white stationary Gaussian noise with zero mean and variance σ_n^2. This formulation is similar to the one obtained when $z = \exp(-y^2/2\sigma_g^2)$, and the noise $n(t)$ has a nonzero mean $m_n = -y_0$ (Fig. 1). We want to obtain the power spectral density of z in response to this input.

III. The General Form of the Output Autocorrelation and Power Density Spectrum

The response of a general memoryless nonlinear system to an input in the form of (2) has been investigated [1, part IV], [2, ch. 13], and the output autocorrelation and power density spectrum were shown to be

$$R_z(\tau) = h_{00}^2 + 2\sum_{m=1}^{\infty} h_{m0}^2 \cos m\omega_d\tau + \sum_{k=1}^{\infty}\frac{h_{0k}^2}{k!}R_n^k(\tau) + 2\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{h_{mk}^2}{k!}R_n^k(\tau)\cos m\omega_d\tau \tag{3}$$

and

$$S_z(f) = h_{00}^2\delta(f) + \sum_{m=1}^{\infty} h_{m0}^2\left[\delta(f+mf_d)+\delta(f-mf_d)\right] + \sum_{k=1}^{\infty}\frac{h_{0k}^2}{k!}S_n(f)^{*k} + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{h_{mk}^2}{k!}\cdot\left[S_n(f+mf_d)^{*k}+S_n(f-mf_d)^{*k}\right], \tag{4}$$

respectively, where

- h_{mk} coefficients depending on the nonlinear characteristic and the input parameters,
- $R_n(\tau)$ autocorrelation of $n(t)$,
- $S_n(f)$ power density spectrum of $n(t)$,
- $S_n(f)^{*k}$ $(k-1)$-fold convolution of $S_n(f)$ with itself.

The terms in (3) and (4) have the following interpretation:

- The first terms in both equations represent dc.
- The second terms represent an interaction of the signal with itself ($S \times S$).
- The third terms are the result of an interaction of the noise with itself ($N \times N$, additive noise terms).
- The fourth terms represent multiplicative noise resulting from an interaction of the noise with the signal ($S \times N$).

The input spectrum of the band-limited white noise (which is shifted and convolved with itself according to (4)) is

$$S_n(f) = \Pi\left(\frac{f+f_d}{W}\right) + \Pi\left(\frac{f-f_d}{W}\right) \tag{5}$$

where

$$\Pi(x) = \begin{cases}1, & -1/2 < x < 1/2\\ 0, & \text{otherwise}\end{cases} \tag{6}$$

and W is the noise bandwidth. The $(k-1)$-fold autoconvolution of $S_n(f)$ can be derived inductively to be

$$S_n(f)^{*k} = \sum_{q=0}^{k}\binom{k}{q}\left[\Pi\left(\frac{f+f_d}{W}\right)\right]^{*(k-q)} * \left[\Pi\left(\frac{f-f_d}{W}\right)\right]^{*q} \tag{7}$$

where

$$\left[\Pi\left(\frac{f\pm f_d}{W}\right)\right]^{*k} = \sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\left(f\pm kf_d+\frac{1}{2}kW-lW\right)^{k-1}}{(k-1)!}\cdot U\left(f\pm kf_d+\frac{1}{2}kW-lW\right) \tag{8}$$

and $U(\cdot)$ is the unit step function.

In Appendix I (7) is expressed in an explicit form (not requiring convolutions). The peaks of the spectral components of $S_n(f)^{*k}$ form a Pascal's triangle weighted by the coefficient $h_{mk}^2/k!$ in (4). The shapes of the spectral components resulting from high-order autoconvolutions ($n \gg 1$) tend to be Gaussian, according to central limit considerations [7].

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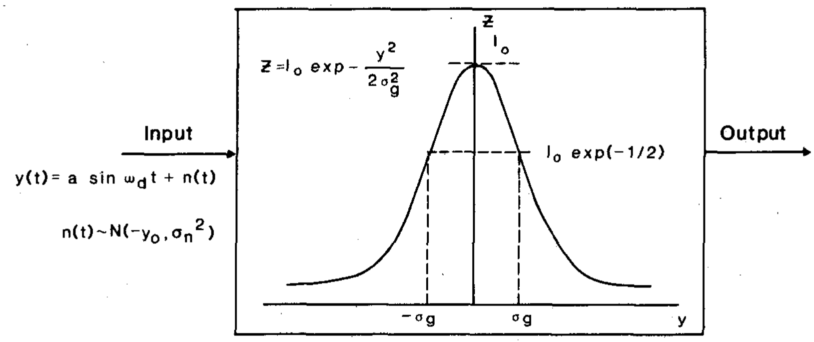

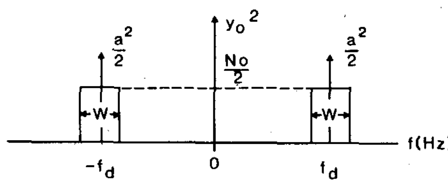

Fig. 1. Statement of problem. (a) Gaussian nonlinearity. (b) Input power spectral density.

V. Calculation of h_{mk}: Relative Amplitudes of the Spectral Components

The coefficients h_{mk} in (4) determine the relative amplitudes of the spectral components at the harmonics of the input frequency. For sinusoidal signals and Gaussian noise, these coefficients were shown by Blachman [3, pp. 128–135] to have the following physical interpretation: each term h_{mk} is the average (over the distribution of the noise) of the kth derivative (with respect to the noise) of the coefficient of the mth term in the Fourier series of the nonlinearity's output.

In the following derivation we employ Rice's characteristic-function method [1, sec. 4.8] and calculation techniques by Atherton [8], [9] based on this method. The coefficients h_{mk} in (4) take the form

$$h_{mk} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Z}(j\omega)(j\omega)^k \exp\left[-\sigma_n^2\omega^2/2\right] j^m J_m(a\omega)\, d\omega, \tag{9}$$

where

$$\hat{Z}(j\omega) = \int_{-\infty}^{\infty} z(y) e^{-j\omega y}\, dy \tag{10}$$

(the Fourier transform of $z(y)$) and $J_m(x)$ = the Bessel function of the first kind of order m. Equation (9) is valid when the number of poles of $Z(s)$ at the origin ($s = 0$) is less than $m + k$. The Gaussian nonlinearity fulfills this condition.

The coefficients h_{mk} can be decomposed into the form

$$h_{mk} = \sum_{p=0}^{\infty} A_{pm}(a) B_{pk}(\sigma_n) \tag{11}$$

where

$$A_{pm} = \begin{cases} 0, & \text{for odd } m+p, \quad m > p \\ a^p \Big/ \left[2^p \Gamma\left(\frac{2+p+m}{2}\right)\Gamma\left(\frac{2+p-m}{2}\right)\right], & \\ \text{otherwise} & \end{cases} \tag{12}$$

and

$$B_{pk}(\sigma_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)^{p+k} \hat{Z}(j\omega) \exp\left[-\sigma_n^2\omega^2/2\right] d\omega. \tag{13}$$

Γ is Euler's gamma function.

The attractiveness of (12) and (13) lies in the fact that A_{pm} depends only on the input amplitude, while $B_{pk}(\sigma_n)$ depends only on the nonlinear characteristics and the noise variance. A recursive relation exists between the B coefficients, and a method has been devised to obtain rapid estimates of the coefficients for signal-to-noise ratios larger than 6 dB [9].

For the Gaussian nonlinearity with $I_0 = 1$ we get (see Appendix II for details) for $y_0 = 0$

$$B_{pk}(\sigma_n) = 0, \quad p+k \text{ odd}$$

$$B_{pk}(\sigma_n) = \frac{1}{\sqrt{8\pi}}\sigma_g \frac{(-1)^{(p+k)/2}\Gamma\frac{(p+k+1)}{2}}{\left(\frac{\sigma_n^2+\sigma_g^2}{2}\right)^{p+k+1}}, \qquad p+k \text{ even} \tag{14}$$

and for $y_0 \neq 0$

$$B_{pk}(\sigma_n) = \frac{1}{\sqrt{8}} 2^{-(p+k)}\sigma_g \left(\frac{\sigma_n^2+\sigma_g^2}{2}\right)^{-(p+k+1)/2} \cdot \exp\left[-y_0^2/2\left(\sigma_n^2+\sigma_g^2\right)\right] H_{p+k}\left[\frac{y_0}{\sqrt{2\left(\sigma_n^2+\sigma_g^2\right)}}\right] \tag{15}$$

where $H_n(x)$ is the Hermite polynomial of order n, defined as

$$H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}\left[\exp(-x^2)\right]. \tag{16}$$

Equation (4) can now be used to calculate the output power spectral density by using the autoconvolutions (Appendix I, (A.1)–(A.3)) and the coefficients h_{mk} ((11), (12), and (15)). The infinite sums will be truncated when the contribution of higher order terms to the components in the frequency band of interest is negligible (Section VI). Some representative results of the h_{mk} calculations are shown in Figs. 2–6, where the amplitudes of some spectral components in (4) are depicted.

In Figs. 2 and 3, $y_0 = 0$, and the varying parameter is a/σ_g, the ratio of the input amplitude to the nonlinearity "width." Fig. 2 shows the $S \times S$ term-signal interaction with itself. As expected, the symmetric characteristic around $y_0 = 0$ produces only even harmonics, and the higher order harmonics become more significant as a/σ_g increases. The ratio of the second harmonic to

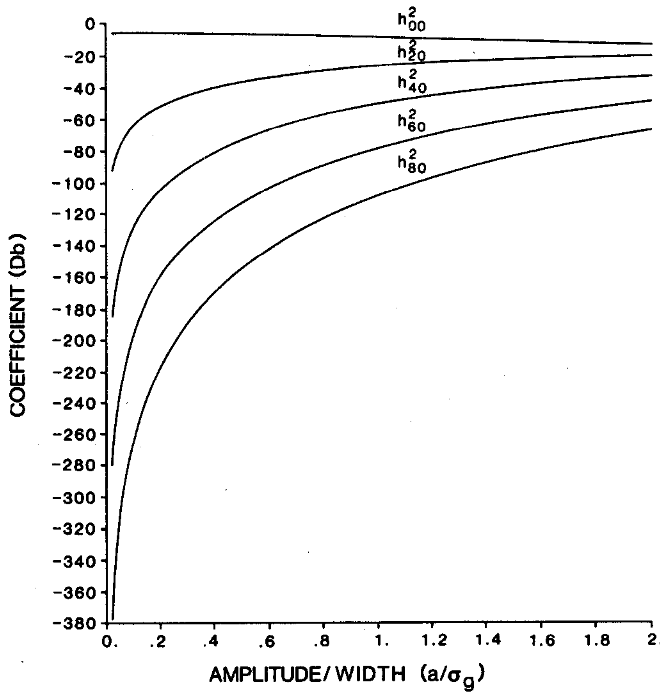

Fig. 2. $S \times S$ coefficients versus a/σ_g (orders 0–8); zero bias ($y_0 = 0$).

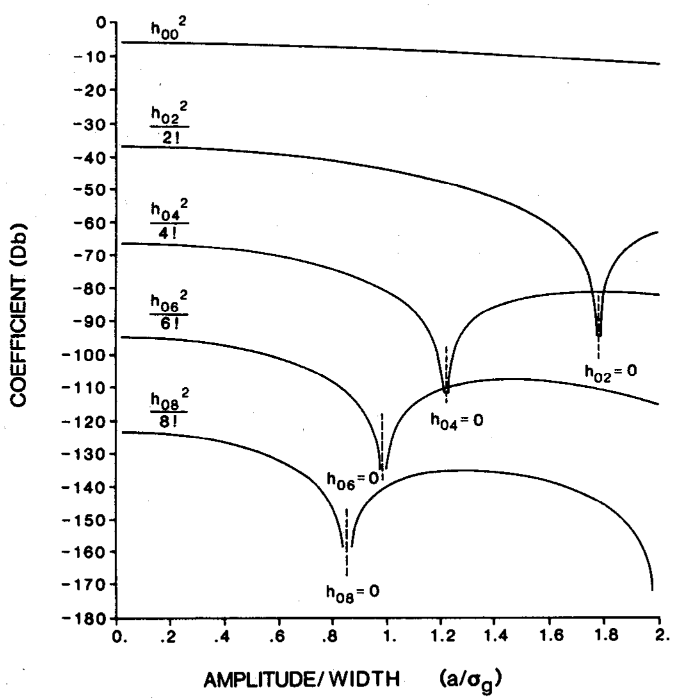

Fig. 3. $N \times N$ coefficients versus a/σ_g (orders 0–8); zero bias ($y_0 = 0$).

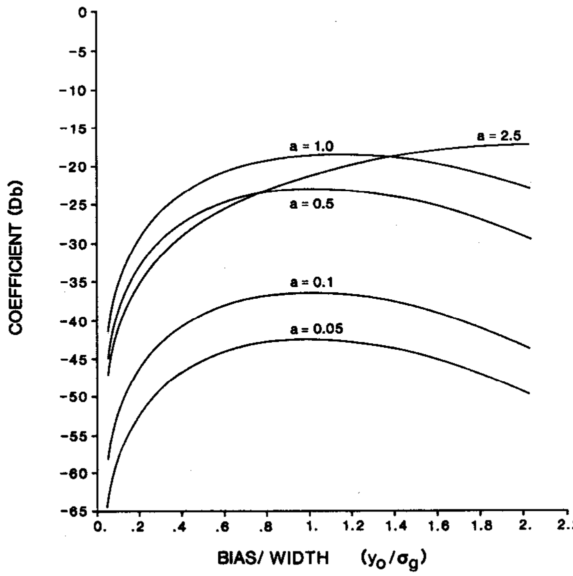

Fig. 4. Coefficient of output signal at f_d versus y_0/σ_g and a.

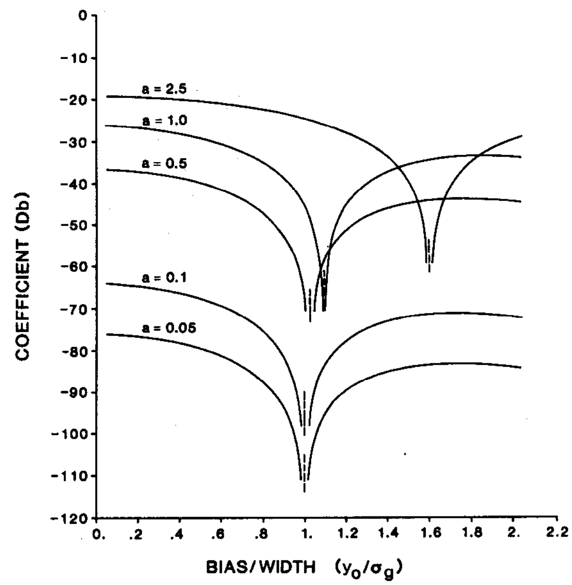

Fig. 5. Coefficient of second harmonic versus y_0/σ_g and a.

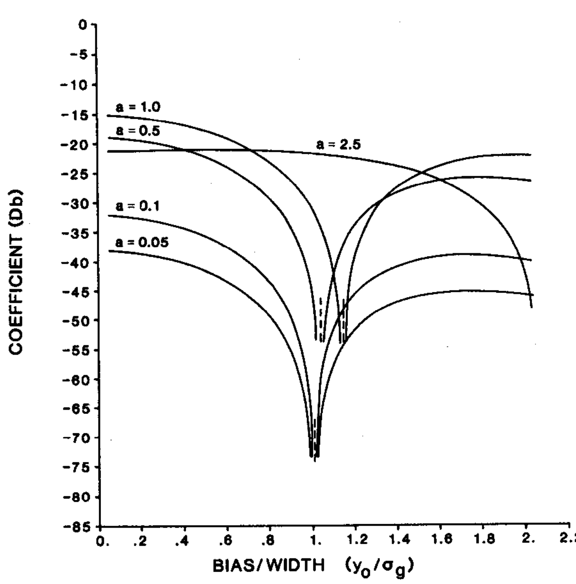

Fig. 6. $S \times N$ coefficient h_{11}^2 versus y_0/σ_g and a ($\sigma_n = 0.1$).

the fourth harmonic is reduced from about 50 dB to about 17 dB as a/σ_g increases from 0.2 to 2. Fig. 3 depicts the $N \times N$ coefficients versus a/σ_g. Because of the symmetry of the nonlinearity around $y_0 = 0$, the nonzero contributions are again the even-order coefficients ($m + k$ is even). The curves manifest deep minima (viz., $h_{0k} = 0$) at certain a/σ_g. At these input levels the noise terms vanish. This phenomenon can serve to enhance the signal-to-noise ratio at the nonlinearity's output by tuning the input sinusoid amplitude in such a way that the dominant noise component at a certain frequency practically vanishes. For example, if we choose a/σ_g to get $h_{02} \approx 0$, the largest noise contribution around $2\omega_d$ is due to the term of the order $m = 1$, $k = 1$, and it has the same rectangular shape as the input noise. The output spectrum is then

$$S(f) \approx h_{00}^2\delta(f) + h_{20}^2[\delta(f+2f_d) + \delta(f-2f_d)] + h_{11}^2[S_n(f+f_d) + S_n(f-f_d)]. \quad (17)$$

Fig. 4 shows the coefficients of the output signal at the input frequency ω_d as functions of the bias y_0. The figure supports small-signal approximations (i.e., small $\boldsymbol{a}$). In particular, around the center of symmetry $y_0 = 0$, the fundamental component at ω_d vanishes; around the point of inflection $y_0 = \sigma_g$, the output signal at the fundamental frequency ω_d is maximum.

The curves in Fig. 4 allow the assessment of the impact of the bias y_0 on the output signal at ω_d and show the changes in output behavior when the small signal assumption is no longer valid ($\boldsymbol{a} = 2.5$). This impact may be used for determination of an unknown parameter y_0 from the output power spectral density.

Fig. 5 shows the coefficients of the output second harmonic for different input amplitudes versus y_0. For a small signal we get the maximum at zero bias $y_0 = 0$ and the minimum at a σ_g-shift $y_0 = \sigma_g$. As the input amplitude increases, the minimum shifts to higher bias points.

Graphs for h_{mk} coefficients can now be constructed for all relevant orders m and k as functions of $\boldsymbol{a}$ and y_0 (as an example, see Fig. 6, which depicts the coefficient h_{11}^2). These graphs can be used in spectrum construction and in design applications: the significant coefficients of noise in a certain frequency can be minimized by a proper choice of $\boldsymbol{a}$, or information about an unknown y_0 can be extracted.

VI. Numerical Considerations

For practical calculations the infinite sums in (4) have to be truncated with the remainder having negligible impact on the output power spectral density. We assume that the band of interest is confined to $[-Mf_d, Mf_d]$, where f_d is the frequency of the input sinusoid and M is an even integer. The input noise is described by (5) and (6) with $W = 1$. It can be shown inductively that contributions to the spectral content around rf_d ($r = 0, \pm 1, \pm 2, \cdots$) from the autoconvolutions $S_n(f + mf_d)^{*k}$ are due to the orders

$$k = |r + m| + 2p, \qquad p = 0,1,2 \cdots. \quad (18)$$

Hence the spectrum at $f = rf_d$ is

$$S_r(f) = h_{r0}^2\delta(f - rf_d) + \sum_{k \in D_0} \frac{h_{0k}^2}{k!} S_n(f)^{*k} \Pi\left(\frac{f - rf_d}{kf_d}\right) + \sum_{m=1}^{\infty} \sum_{k \in D_m} \frac{h_{mk}^2}{k!}\left[S_n(f + mf_d)^{*k} + S_n(f - mf_d)^{*k}\right] \cdot \Pi\left(\frac{f - rf_d}{kf_d}\right)$$

where $D_m = \{|r + m|, |r + m| + 2, \cdots\}$.

The peaks of the terms contributed to the harmonics of f_d from the autoconvolutions form a Pascal's triangle. The peak of the contribution due to $S_n(f + mf_d)^{*k}$ at rf_d is the binomial coefficient

$$P_{mk,r} = \binom{k}{\dfrac{k + |r + m|}{2}} \quad (19)$$

where for $k = 2S$ (even)

$$r + m = -2S, -2S + 2, \cdots, -2, 0, 2, \cdots, 2S - 2, 2S,$$

and for $k = 2S - 1$ (odd)

$$r + m = -2S + 1, -2S + 3, \cdots, -3, -1, 1, 3, \cdots, 2S - 3, 2S - 1.$$

These peaks are further weighted by $h_{mk}^2/k!$.

Let S_{r0} be the peak of the component $S_r(f)$. We shall require that the worst calculation error in the determination of $S_r(f)$ be ϵS_{r0}. This requirement is fulfilled if L terms are added to determine $S_r(f)$, and each term has a calculation error which does not exceed $\epsilon' S_{r0}$ where $\epsilon' = \epsilon/L$. Since L is not known before $m_{\max}$ and $k_{\max}$ have been calculated, ϵ' may have to be determined by trial and error. For a small input sinusoid ($a \ll \sigma_g$) we get the following procedure.

a) Set an accuracy requirement (ϵ) and choose ϵ' (typically, $\epsilon' \approx \epsilon/100$).

b) For the single (third) sum in (4) add terms until $k \geq k_{\max}$ where

$$k_{\max} = \left(\min k | k \text{ is even}, \frac{2}{(k/2)!} < \epsilon'\right). \quad (20)$$

c) For the double sum in (4) add terms until

$$k \geq k_{\max} \qquad m \geq m_{\max} = M + k_{\max}. \quad (21)$$

d) Calculate L, the maximum number of elements which are added at each harmonic of f_d. If $L\epsilon' > \epsilon$, reduce ϵ' and return to b). If $L\epsilon' \ll \epsilon$, increase ϵ' and return to b).

Example: To obtain the $k_{\max}$ and $m_{\max}$ required to get $\epsilon = 10^{-2}$ for the power spectral density calculation in the band $(-5f_d, 5f_d)$, we choose $M = 6$ (even) and $\epsilon' = 10^{-4}$. This choice will be justified later. From (20) and (21) we get $k_{\max} = 16$ and $m_{\max} = 22$.

The number of terms needed to approximate the output spectrum to within $0.01S_{r0}$ at rf_d is 1 dc component + 22 $S \times S$ elements + 16 $N \times N$ elements + 352 $S \times N$ elements = 391 terms. At each harmonic we get fewer than 100 nonzero added terms; hence the required accuracy is achieved with a moderate number of terms, and the choice of ϵ' is justified.

When the input amplitude $\boldsymbol{a}$ is not much smaller than the nonlinearity width, (20) and (21) should be modified to take into account the effects of deep minima in the coefficients h_{mk} (e.g., Fig. 3). Equation (20) is replaced by

$$k_{\max} = \left(\min k | k \text{ is even}, \frac{2}{(l/2)!} < \epsilon'; \; l = k, k-2, k-4\right). \quad (22)$$

This modification is based on the fact that minima in h_{mk} do not appear at the same $\boldsymbol{a}/\sigma_g$ ratio for different $m + k$.

VII. Conclusion

An exact expression was obtained for the output power spectral density of a Gaussian nonlinearity in response to an input sinusoid accompanied by band-limited noise. This expression, in the form of an infinite sum, is valid for a large class of input parameters. Simple rules were established for the truncation of the sum to obtain a practical approximation for the output spectrum.

Some general properties of the Gaussian nonlinearity are revealed through the use of the truncated expression. In particular, the output noise level can be minimized at a certain frequency by a judicious choice of the input amplitude. Also, the bias parameter (which in tracking problems represents the pointing error) may possibly be estimated from the relative magnitude of the output components at multiples of the input frequency.

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