of eleven possible doors were successfully identified. The reduction rules are used to combine multiple returns from the same source in the environment.

Doors are not identified by any single set of sonar data from a fixed position, but from a series of data collected from a minimum of four different positions. With this methodology, door identification does not require a diffuse environment with perfect sonar returns, but is made possible by the fact that there is specularity in the environment. The long specular returns in the absence of door frames is as important to door identification as are the short diffuse returns due to the corner geometry of door frames.

IV. Conclusion

Using humans as MRS's to develop autonomous performances is more than just an approach to robotics, it is a philosophy. This philosophy is based on the belief that the best way to solve a problem is from the inside rather than the outside; to take the perspective of having to solve the problem oneself, rather than seeing it as just a problem the robot must solve. The four step methodology discussed in this paper combines experimentation with the development of autonomy, and puts the human in the loop by building an interface designed to suppress the human's reality and enhance the robot's reality. Using a MRS reveals the capabilities and limitations of the robotic system, and through the process of knowledge extraction, provides the means for the development of autonomous behavior.

The proof lies in the development of the door traversal, and door identification performances described in this paper. These performances use ultrasonic sensors in ways that seem totally unintuitive from the outsider's perspective; identifying doors by using sonars not facing the doors, and traversing doors by moving at 45° angles to the doorways. They seem unintuitive only when seen from the human's perspective and not from the robot's perspective.

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Robot Motion Planning on N-Dimensional Star Worlds Among Moving Obstacles

Robert A. Conn and Moshe Kam

***Abstract*— Inspired by an idea of Rimon and Koditschek [1], we develop a motion planning algorithm for a point robot traveling among moving obstacles in an N-dimensional space. The navigating point must meet a goal point at a fixed time T, while avoiding several translating, nonrotating, nonintersecting obstacles on its way. All obstacles, the goal point, and the navigating point are confined to the interior of a star-shaped set in $\mathbb{R}^N$ over the time interval $[0, T]$. Full *a priori* knowledge of the goal's location and of the obstacle's trajectories is assumed. We observe that the topology of the obstacle-free space is invariant in the time interval $[0, T]$ as long as the obstacles are nonintersecting and as long as they do not cover the goal point at any time during $[0, T]$. Using this fact we reduce the problem, for any fixed time $t_0 \in [0, T]$, to a stationary-obstacle problem, which is then solved using the method of Rimon and Koditschek [1]. The fact that the obstacle-free space is topologically invariant allows a solution to the moving-obstacle problem over $[0, T]$ through a continuous deformation of the stationary-obstacle solution obtained at time t_0. We construct a vector field whose flow is in fact one such deformation. We believe that ours is the first *global* solution to the moving-obstacle path-planning problem which uses vector fields.**

***Index Terms*—Motion planning, moving obstacles, vector fields.**

I. Introduction

No global solution is known to the general problem of planning a path for a point robot traveling among arbitrarily-shaped moving obstacles rotating and translating in a space of arbitrary dimension, even when the obstacle motions are known *a priori*. (By *global* we mean that the algorithm always finds a solution when one exists and reports failure if no solution exists).[1] However, several sub-problems of this general problem have been addressed in the literature. When the obstacle shapes are semi-algebraic, the obstacle motion is nonrotational, the robot speed is bounded, and the dimension is $N = 2$, we know that the problem is NP-hard (Canny and Reif [3]). With the same restrictions but with $N = 3$, Reif and Sharir [4] have shown that the problem is PSPACE-hard; if we drop the bounded speed restriction then they have shown that the three-dimensional (3-D) case is NP-hard. When the obstacles are represented by semi-algebraic sets and the obstacle trajectories are expressed in algebraic form, Latombe [5] gives a global algorithm based on an exact cell decomposition. The problem thus solved is still very general; however, as the author notes about the algorithm, "its computational complexity makes it impractical, even for small values of [the spatial dimension]." Fujimura and Samet [6] have solved the problem when the obstacle shapes are polygonal, the obstacle motion is restricted to constant-velocity translation, the robot speed is bounded, and the spatial dimension is $N = 2$.

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[1] This usage corresponds to the combination of *exact* and *global* in Hwang and Ahuja [2].

In this paper we address a different case. In our case, the obstacles are star-shaped sets with analytic boundaries, the obstacles are non-rotating and translate along analytic trajectories, and the dimension N is arbitrary ($N \geq 2$). We provide a global solution to this fairly general problem.

II. Statement of the Problem

Definition 1 [1, p. 508]: A *star-shaped* set $\mathcal{B}_i$ in $\mathbf{R}^N$ is characterized by possession of a distinguished "center point," $\mathbf{x}_i \in \mathcal{B}_i$, from which all rays cross its boundary once and only once. Star-shaped sets are more general than (they include all) the convex sets.

Definition 2: The boundary of a set A will be denoted ∂A.

Problem Statement: Given

I) A closed, star-shaped world set $\mathcal{B}_0$, with $\partial\mathcal{B}_0$ analytic;

II) at time $t = 0$

a. the initial position $\mathbf{x}_0 \in \mathcal{B}_0 - \partial\mathcal{B}_0$ of a navigating point;

b. n closed, star-shaped obstacle sets $\mathcal{B}_i(0)$, $i \in \{1, \cdots, n\}$, with $\partial\mathcal{B}_i(0)$ analytic, and a goal point, $\mathcal{B}_{n+1}(0)$, with $\mathbf{x}_i(0)$, $i \in \{1, \cdots, n\}$ the center of $\mathcal{B}_i(0)$, and $\mathbf{x}_{n+1}(0) = \mathcal{B}_{n+1}(0)$, and which satisfy

$$\mathcal{B}_i(0) \subset \mathcal{B}_0 - \partial\mathcal{B}_0, \qquad i \in \{1, \cdots, n+1\} \tag{1a}$$

and

$$\mathcal{B}_i(0) \cap \mathcal{B}_j(0) = \emptyset, \qquad i, j \in \{1, \cdots, n+1\}, i \neq j \tag{1b}$$

and

$$\mathbf{x}_0 \notin \mathcal{B}_i(0), \qquad i \in \{1, \cdots, n\} \tag{1c}$$

III) a fixed arrival time T;

IV) on the time interval $[0, T]$, and with the ith obstacle set at time t, $\mathcal{B}_i(t)$, obtained by translating every element of $\mathcal{B}_i(0)$ by $[\mathbf{x}_i(t) - \mathbf{x}_i(0)]$, $n+1$ analytic functions $\mathbf{x}_i(t)$, $i \in \{1, \cdots, n+1\}$, $t \in [0, T]$ which describe the motion of the centers of the n star-shaped obstacles and the goal point, with the $\mathbf{x}_i(t)$ specified such that

$$\mathcal{B}_i(t) \subset \mathcal{B}_0 - \partial\mathcal{B}_0, \qquad i \in \{1, \cdots, n+1\}, \forall t \in [0, T] \tag{1d}$$

and

$$\mathcal{B}_i(t) \cap \mathcal{B}_j(t) = \emptyset, \qquad i, j \in \{1, \cdots, n+1\}, \; i \neq j, \forall t \in [0, T] \tag{1e}$$

find

a continuous curve $\mathbf{p}(t)$, $t \in [0, T]$ such that

$$\mathbf{p}(0) = \mathbf{x}_0 \tag{1f}$$

$$\mathbf{p}(T) = \mathbf{x}_{n+1}(T) \tag{1g}$$

$$\mathbf{p}(t) \in \mathcal{B}_0 - \partial\mathcal{B}_0, \qquad \forall t \in [0, T] \tag{1h}$$

$$\mathbf{p}(t) \notin \mathcal{B}_i(t), \qquad i \in \{1, \cdots, n\}, \forall t \in [0, T]. \tag{1i}$$

III. Construction of a Flow Which Smoothly Deforms the Free Space to Accommodate the Movement of Obstacles

Definition 3: The *free space* $F(t)$ at time t is $\mathcal{B}_0 - \partial\mathcal{B}_0 - \cup_{i=1}^{n+1} \mathcal{B}_i(t)$.

Since the obstacles never intersect each other or leave $\mathcal{B}_0 - \partial\mathcal{B}_0$, we know that the topology of the free space is invariant with time, i.e., that $F(t_1)$ is topologically equivalent (smoothly deformable) to $F(t_2)$ for any $t_1, t_2 \in [0, T]$. Therefore, we know that there exists at least one diffeomorphism relating $F(t_1)$ and $F(t_2)$. We construct a flow which embeds such a diffeomorphism.

Definition 4 [1, p. 507]: The *obstacle boundary function* $\beta_i(\mathbf{x}, t)$, $i \in \{1, \cdots, n\}$, is a real-valued mapping of $\mathcal{B}_0 \times [0, T]$ which satisfies the following conditions:

$$\beta_i(\mathbf{x}, t) \text{ is analytic}$$
$$\beta_i(\mathbf{x}, t) < 0 \Rightarrow \mathbf{x} \in \text{interior}[\mathcal{B}_i(t)]$$
$$\beta_i(\mathbf{x}, t) = 0 \Rightarrow \mathbf{x} \in \partial\mathcal{B}_i(t)$$

and

$$\beta_i(\mathbf{x}, t) > 0 \Rightarrow \mathbf{x} \in \mathcal{B}_0 - \mathcal{B}_i(t).$$

An example of a possible obstacle boundary function for a spherical obstacle of radius ρ_i is

$$\beta_i(\mathbf{x}, t) = \|\mathbf{x} - \mathbf{x}_i(t)\|^2 - \rho_i^2. \tag{2}$$

Definition 5 [1, p. 507]: The *world obstacle function* $\beta_0(\mathbf{x}, t)$ is a real-valued mapping of $\mathbf{R}^N \times [0, T]$ which satisfies the following conditions:

$$\beta_0(\mathbf{x}, t) \text{ is analytic}$$
$$\beta_0(\mathbf{x}, t) < 0 \Rightarrow \mathbf{x} \notin \mathcal{B}_0$$
$$\beta_0(\mathbf{x}, t) = 0 \Rightarrow \mathbf{x} \in \partial\mathcal{B}_0$$

and

$$\beta_0(\mathbf{x}, t) > 0 \Rightarrow \mathbf{x} \in \mathcal{B}_0 - \partial\mathcal{B}_0.$$

An example of a possible world obstacle function for a sphere world of radius 1 is

$$\beta_0(\mathbf{x}, t) = 1 - \|\mathbf{x}\|^2. \tag{3}$$

Note that the world obstacle function is "inverted" from the obstacle boundary functions of Definition 4 in the sense that $\beta_0(\mathbf{x}, t)$ is positive everywhere *inside* $\mathcal{B}_0$. Note also that the world obstacle function is analytic, and does not depend explicitly on the argument t. In the sequel, the set for which $\beta_0(\mathbf{x}, t)$ is nonpositive, i.e., $\mathbf{R}^N - (\mathcal{B}_0 - \partial\mathcal{B}_0)$, will be referred to as the 0th obstacle.

Definition 6: The *goal obstacle function* $\beta_{n+1}(\mathbf{x}, t)$ is a real-valued mapping of $\mathcal{B}_0 \times [0, T]$ given by

$$\beta_{n+1}(\mathbf{x}, t) = \|\mathbf{x} - \mathbf{x}_{n+1}(t)\|^2. \tag{4}$$

Note that the goal obstacle function is analytic. Since the goal is a point, it has no interior, and the goal obstacle function is nonnegative. In the sequel, the goal point $\mathcal{B}_{n+1}(t)$ will be referred to as the $n+1$st obstacle.

Definition 7 [1, p. 509]: The *omitted product of the obstacle functions,* $\overline{\beta}_i(\mathbf{x}, t)$, is a real-valued mapping of $\mathcal{B}_0 \times [0, T]$ given by

$$\overline{\beta}_i(\mathbf{x}, t) = \prod_{j=0,\, j\neq i}^{n+1} \beta_j(\mathbf{x}, t). \tag{5}$$

The omitted product $\overline{\beta}_i(\mathbf{x}, t)$ is zero on $\partial\mathcal{B}_j(t)$, $j \in \{0, \cdots, n+1\}$, $j \neq i$, $t \in [0, T]$.

Definition 8 [1, p. 509]: The *analytic switches* $\sigma_i(\mathbf{x}, t)$, $i \in \{1, \cdots, n+1\}$ are real-valued mappings of $\mathcal{B}_0 \times [0, T]$ given by

$$\sigma_i(\mathbf{x}, t) = \frac{\overline{\beta}_i(\mathbf{x}, t)}{\overline{\beta}_i(\mathbf{x}, t) + \lambda_i \beta_i(\mathbf{x}, t)}, \qquad \lambda_i > 0. \tag{6}$$

The analytic switch $\sigma_i(\mathbf{x}, t)$ is an analytic function [1, p. 509] which for each time $t \in [0, T]$ maps the $\partial\mathcal{B}_i(t)$ to 1, the $\partial\mathcal{B}_j(t)$, $j \neq i$, to 0, and the free space $F(t)$ to the open interval (0, 1). The constant λ_i affects the behavior of $\sigma_i(\mathbf{x}, t)$ near $\partial\mathcal{B}_i(t)$.

Definition 9: The *deformation vector field* $\mathbf{X}(\mathbf{x}, t)$ is a mapping from $\mathcal{B}_0 \times [0, T]$ into $\mathbf{R}^N$. The mapping is given by

$$\mathbf{X}(\mathbf{x}, t) = \sum_{i=1}^{n+1} \sigma_i(\mathbf{x}, t)\dot{\mathbf{x}}_i(t). \tag{7}$$

The vector field $\mathbf{X}(\mathbf{x}, t)$ is smooth. The reason is as follows. Since the obstacle trajectories $\mathbf{x}_i(t)$ are analytic, we know that the vector fields $\dot{\mathbf{x}}_i(t)$ are smooth. Further, since the switches $\sigma_i(\mathbf{x}, t)$ are analytic, the scaled vector fields $\sigma_i(\mathbf{x}, t)\dot{\mathbf{x}}_i(t)$ are smooth. Finally, since the sum of smooth fields is smooth, we see that the deformation vector field $\mathbf{X}(\mathbf{x}, t)$ is smooth.

Definition 10: Given any initial time $t_0 \in [0, T]$ and any initial point $\mathbf{x}(t_0) \in \mathcal{B}_0$, denote the *deformation flow* generated by the deformation vector field $\mathbf{X}$ as $\mathbf{x}[t; t_0, \mathbf{x}(t_0)]$, $t \in [0, T]$. Since $\mathbf{X}$ is smooth, the flow $\mathbf{x}[t; t_0, \mathbf{x}(t_0)]$ exists and is unique.

Lemma 1: Given any initial time $t_0 \in [0, T]$, if $\mathbf{x}(t_0)$ is on $\partial F(t_0)$, the boundary of the free space at time t_0, then for all $t \in [0, T]$, the point $\mathbf{x}[t; t_0, \mathbf{x}(t_0)]$ of the deformation flow lies on $\partial F(t)$, the boundary of the free space at time t. For example

$$\text{given any } t_0 \in [0, T],\ \mathbf{x}(t_0) \in \partial F(t_0) \ \Rightarrow\ \mathbf{x}[t; t_0, \mathbf{x}(t_0)] \in \partial F(t) \qquad \forall t \in [0, T].$$

Proof: If $\mathbf{x}(t_0) \in \partial F(t_0)$, then $\mathbf{x}(t_0) \in \partial\mathcal{B}_i(t_0)$ for some $i \in \{0, \cdots, n+1\}$.

Let $i = 0$. From Definitions 8 and 9, $\mathbf{X}(\mathbf{x}, t) \equiv 0$ for $\mathbf{x} \in \partial\mathcal{B}_0(t)\ \forall t \in [0, T]$. Therefore, $\mathbf{x}[t; t_0, \mathbf{x}(t_0)] \equiv \mathbf{x}(t_0) \in \partial F(t_0) \quad \forall t \in [0, T]$. So the lemma holds if $i = 0$.

Let $i \in \{1, \cdots, n+1\}$. Then $\mathbf{x}(t_0) = \mathbf{b}(t_0) \in \partial\mathcal{B}_i(t_0)$. From Section II (Problem Statement, IV) the motion of $\mathbf{b}$ due to the motion of $\mathcal{B}_i$ is given by $\mathbf{b}(t) = \mathbf{b}(t_0) + [\mathbf{x}_i(t) - \mathbf{x}_i(t_0)]$ and $\mathbf{b}(t) \in \partial\mathcal{B}_i(t)\ \forall t \in [0, T]$. From Definition 3, $\partial\mathcal{B}_i(t) \subset \partial F(t) \quad \forall t \in [0, T]$, and so $\mathbf{b}(t) \in \partial F(t) \quad \forall t \in [0, T]$. We also know that the flow from $\mathbf{x}(t_0)$ due to $\mathbf{X}$ satisfies $\dot{\mathbf{x}}[t; t_0, \mathbf{x}(t_0)] = \mathbf{X}\{\mathbf{x}[t; t_0\mathbf{x}(t_o)], t\}$. Substituting $\mathbf{x}[t; t_0, \mathbf{x}(t_0)] = \mathbf{b}(t)$ yields $\dot{\mathbf{x}}_i(t) = \dot{\mathbf{x}}_i(t)$, i.e., $\mathbf{b}(t)$ satisfies the same differential equation as the flow $\mathbf{x}[t; t_0, \mathbf{x}(t_0)]$ and has the same initial condition $[\mathbf{b}(t_0) = \mathbf{x}(t_0)]$. Since $\mathbf{x}[t; t_0, \mathbf{x}(t_0)]$ is unique, the flow $\mathbf{x}[t; t_0, \mathbf{x}(t_0)]$ must equal $\mathbf{b}(t)$. Therefore, $\mathbf{x}[t; t_0, \mathbf{x}(t_0)] \in \partial F(t)\ \forall t \in [0, T]$, and the lemma holds for $i \in \{1, \cdots, n+1\}$.

Theorem 1: Given any initial time $t_0 \in [0, T]$, if $\mathbf{x}(t_0)$ is in $F(t_0)$, the free space at time t_0, then for all $t \in [0, T]$, the point $\mathbf{x}[t; t_0, \mathbf{x}(t_0)]$ of the deformation flow lies in $F(t)$, the free space at time t. Namely

$$\text{given any } t_0 \in [0, T],\ \mathbf{x}(t_0) \in F(t_0) \ \Rightarrow\ \mathbf{x}[t; t_0, \mathbf{x}(t_0)] \in F(t) \qquad \forall t \in [0, T].$$

Proof: The proof follows from Lemma 1 and the uniqueness theorem of ordinary differential equations. Assume $\mathbf{x}(t_0) \in F(t_0)$, and furthermore there exists some $t_b \in [0, T]$, such that $\mathbf{x}[t_b; t_0, \mathbf{x}(t_0)] \in \partial F(t_b)$. Define $\mathbf{y}(t_b) = \mathbf{x}[t_b; t_0, \mathbf{x}(t_0)]$. By Lemma 1, $\mathbf{y}(t_b) \in \partial F(t_b)$ implies that $\mathbf{y}(t_0) = \mathbf{x}[t_0; t_b, \mathbf{y}(t_b)] \in \partial F(t_0)$. By the uniqueness theorem, $\mathbf{y}(t_b) = \mathbf{x}[t_b; t_0, \mathbf{y}(t_0)]$. Since $\mathbf{x}(t_0) \in F(t_0)$ and $\mathbf{y}(t_0) \in \partial F(t_0)$ and since $F(t_0) \cap \partial F(t_0) = \emptyset$, we know that $\mathbf{y}(t_0) \neq \mathbf{x}(t_0)$. So we have $\mathbf{x}[t_b; t_0, \mathbf{x}(t_0)] = \mathbf{y}(t_b) = \mathbf{x}[t_b; t_0, \mathbf{y}(t_0)]$, and $\mathbf{y}(t_0) \neq \mathbf{x}(t_0)$. $\Rightarrow \Leftarrow$ (uniqueness). Therefore, no such $t_b \in [0, T]$ exists, and the theorem is proved.

Theorem 1 tells us that for any initial condition in the free space, the flow remains in the free space over the entire time interval $[0, T]$. To provide a solution to the moving-obstacle problem, we quote first the stationary-obstacle problem which was solved by Rimon and Koditschek [1].

IV. The Stationary-Obstacle Problem for a Fixed Time t_0

Problem Statement: Given

I) the initial position $\mathbf{x}_{\text{init}} = \mathbf{x}(t_0; 0, \mathbf{x}_0) \in \mathcal{B}_0 - \partial\mathcal{B}_0$ of the navigating point, and

II) n closed star-shaped obstacle sets $\mathcal{B}_i(t_0)$, $i \in \{1, \cdots, n\}$ with $\partial B_i(t_0)$ analytic, and a goal point, $\mathcal{B}_{n+1}(t_0) = \mathbf{x}_{n+1}(t_0)$, which satisfy

$$\mathcal{B}_i(t_0) \subset \mathcal{B}_0 - \partial\mathcal{B}_0, \qquad i \in \{1, \cdots, n+1\} \tag{8a}$$

$$\mathcal{B}_i(t_0) \cap \mathcal{B}_j(t_0) = \emptyset, \qquad i, j \in \{1, \cdots, n+1\},\ i \neq j \tag{8b}$$

and

$$\mathbf{x}_{\text{init}} \notin \mathcal{B}_i(t_0), \qquad i \in [1, \cdots, n] \tag{8c}$$

find

a smooth curve $\mathbf{z}(s; t_0)$ and a real number $s_F > 0$ such that

$$\mathbf{z}(0; t_0) = \mathbf{x}_{\text{init}} \tag{8d}$$

$$\mathbf{z}(s_F; t_0) = \mathbf{x}_{n+1}(t_0) \tag{8e}$$

and

$$\mathbf{z}(s; t_0) \in F(t_0), \qquad \forall s \in [0, s_F]. \tag{8f}$$

Clearly, for a fixed $t_0 \in [0, T]$, the data assumed provided in the statement of the moving obstacle problem in Section II and the construction of the deformation flow in Section III supply all the information necessary to set up the stationary-obstacle problem. The stationary-obstacle problem as defined above is solved in Rimon and Koditschek [1]. We assume in the sequel that the solution $\mathbf{z}(s; t_0)$, $s \in [0, s_F]$ and the real number s_F are available for specified t_0.

V. Solution of the Moving-Obstacle Problem

Theorem 2: Choose any $t_0 \in [0, T]$. Then

$$\mathbf{p}(t; t_0) = \mathbf{x}\{t; t_0, \mathbf{z}[\upsilon(t); t_0]\} \tag{9}$$

where ν: $[0, T] \rightarrow [0, s_F]$: $t \rightarrow s$ is <u>any</u> smooth monotonically increasing function such that

$$\nu(0) = 0 \tag{10a}$$

and

$$\nu(T) = s_F \tag{10b}$$

is the solution to the moving obstacle problem.

Proof: $\mathbf{p}(0; t_0) = \mathbf{x}\{0; t_0, \mathbf{z}[\nu(0); t_0]\} = \mathbf{x}[0; t_0, \mathbf{z}(0; t_0)] = \mathbf{x}(0; t_0, \mathbf{x}_{\text{init}}) = \mathbf{x}[0; t_0, \mathbf{x}(t_0; 0, \mathbf{x}_0)] = \mathbf{x}_0$ by uniqueness.

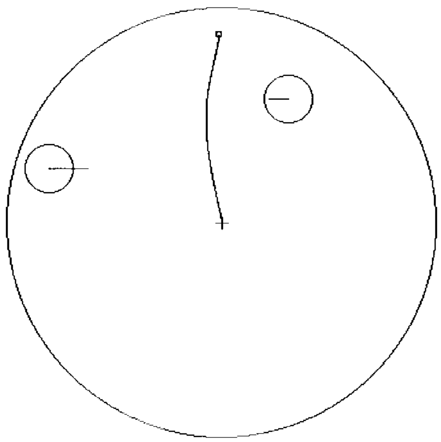

Fig. 1. Configuration at t_0 and stationary-obstacle solution $\mathbf{z}(s; t_0)$, $s \in [0, s_{\mathrm{F}}]$.

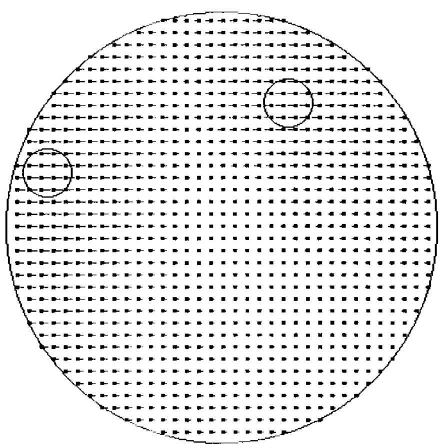

Fig. 2. $\mathbf{X}(\mathbf{x}, t_0)$ (direction and log-magnitude).

$\mathbf{p}(T; t_0) = \mathbf{x}\{T; t_0, \mathbf{z}[\nu(T); t_0]\} = \mathbf{x}[T; t_0, \mathbf{z}(s_{\mathrm{F}}; t_0)] = \mathbf{x}[T; t_0, \mathbf{x}_{n+1}(t_0)] = \mathbf{x}_{n+1}(T)$ by proof of Lemma 1.

$\mathbf{p}(t; t_0) = \mathbf{x}\{t; t_0, \mathbf{z}[\nu(t); t_0]\} \in F(t)\ \forall t \in [0, T]$. This follows from Theorem 1 and the fact that by construction $\mathbf{z}[\nu(t); t_0] \in F(t_0)\ \ \forall \nu(t) \in [0, s_{\mathrm{F}}]$, i.e., $\forall t \in [0, T]$. By Definition 3, $\mathbf{p}(t; t_0) \in F(t)\ \ \forall t \in [0, T]$ implies $\mathbf{p}(t; t_0) \in S^{N-1}\ \forall t \in [0, T]$ and $\mathbf{p}(t; t_0) \notin \mathcal{B}_i(t)$, $i \in \{1, \cdots, n\}$, $\forall t \in [0, T]$.

We interpret Theorem 2 in the following way. Given t_0, $\mathbf{z}(s; t_0)$, and a suitable ν, obtain the point $\mathbf{p}(t; t_0)$ on the solution of the moving-obstacle problem by selecting the point $\mathbf{z}[\nu(t); t_0]$ on the stationary-obstacle solution and operating on it with the deformation flow from t_0 to t.

VI. Example

We consider a case with $N = 2$, and $n = 2$. Fig. 1 shows the initial configuration of the obstacles at $t = 0$. The small square is the initial position of the navigating point ($\mathbf{x}_0$). The crosshair at the origin represents the goal point, which is fixed. The two moving obstacles are the small circles, and their initial velocities are indicated by the vectors emanating from their centers. The two obstacles move with constant velocity.

For $t_0 = 0$, Fig. 1 shows the solution $\mathbf{z}(s; t_0)$, $s \in [0, s_F]$ of the stationary-obstacle problem which connects $\mathbf{x}_0$ with the origin. For this example, $s_{\mathrm{F}} = 0.88$. Fig. 2 shows the deformation vector field $\mathbf{X}(\mathbf{x}, t_0)$ (direction and log-magnitude). To obtain the moving obstacle solution, we set $T = s_{\mathrm{F}}$ and use the identity mapping for ν. The results are shown in Fig. 3(a) ($t = 0.2$), Fig. 3(b) ($t = 0.4$), Fig. 3(c) ($t = 0.6$), and Fig. 3(d) ($t = 0.88$), along with the stationary-obstacle solution up to these times.

VII. Topological Considerations

Consider the case of a fixed goal point $\{\dot{\mathbf{x}}_{n+1}(t) \equiv 0\ \forall t \in [t_0, T]\}$. In the plane ($N = 2$), the topological class of $\mathbf{z}(s; t_0)$ (the solution of the stationary-obstacle problem at time t_0) determines the topological class of $\mathbf{p}(t; t_0)$ (the solution of the moving-obstacle problem). The constraint that $\mathbf{z}(s; t_0)$ and $\mathbf{p}(t; t_0)$ be of the same topological class for $N = 2$ is imposed by the topology of the plane, not by our particular choice of the deformation flow. To see the effect of this constraint, examine the example of Section VI again. Since the solution to the stationary problem at time t_0 (shown in Fig. 1) went to the left of the upper obstacle and to the right of the lower obstacle, the solution to the moving-obstacle problem (shown in Fig. 3) is constrained to also pass to the left of the upper obstacle and to the right of the lower obstacle. Note that as the speed of the obstacles increases, the amount of deformation increases as well.

For $N > 2$, there is only one topological class of curve in the free space (remember that the world boundary is a sphere, and all of the obstacles are star-shaped). In other words, for $N > 2$, the free space has the property that every possible solution to the stationary-obstacle problem can be continuously deformed into every possible solution to the moving-obstacle problem. To visualize the additional freedom this property confers, consider the example of Section VI as a 3-D problem, where the obstacles are spheres and all figures are shown from a top view. Let the solution to the stationary-obstacle problem be that shown in Fig. 1, where the starting point, the goal point, and the entire stationary-obstacle solution are co-planar. Instead of deforming the solution to the stationary-obstacle problem to account for the motion of the obstacles in that same plane, we could simply deform that solution to move up (out of the page) and over the entire path of the obstacles. By deforming the solution to the stationary-obstacle problem in this way, we make the solution to the moving obstacle problem invariant to the speed with which the obstacles move.

VIII. Construction of New Deformation Flow

We have stated that for $N > 2$, there exists additional topological freedom in the choice of a diffeomorphism relating the solution of the stationary-obstacle problem to the solution of the moving-obstacle problem. The deformation vector field of Definition 9 does not use this additional freedom. We next construct an auxiliary deformation vector field which, when added to the field of Definition 9, generates a new deformation flow which does use the additional freedom.

Definition 11: The *interpolation point* $\mathbf{a}_i(\mathbf{x}, t)$ on the ith obstacle satisfies the following properties

$$\mathbf{a}_i(\mathbf{x}, t) \in \partial\mathcal{B}_i(t) \tag{11a}$$

$$\mathbf{a}_i(\mathbf{x}, t) = \mathbf{x}_i(t) + \psi_i(\mathbf{x}, t)[\mathbf{x} - \mathbf{x}_i(t)] \quad \text{for some } \psi_i(\mathbf{x}, t) \geq 0. \tag{11b}$$

The interpolation point is the unique point determined by intersecting the ray from $\mathbf{x}_i(t)$ to $\mathbf{x}$ with the boundary $\partial\mathcal{B}_i(t)$ of the ith obstacle at time t. The uniqueness of the interpolation point follows from Definition 1. This point is unlabeled but easy to identify in Fig. 4.

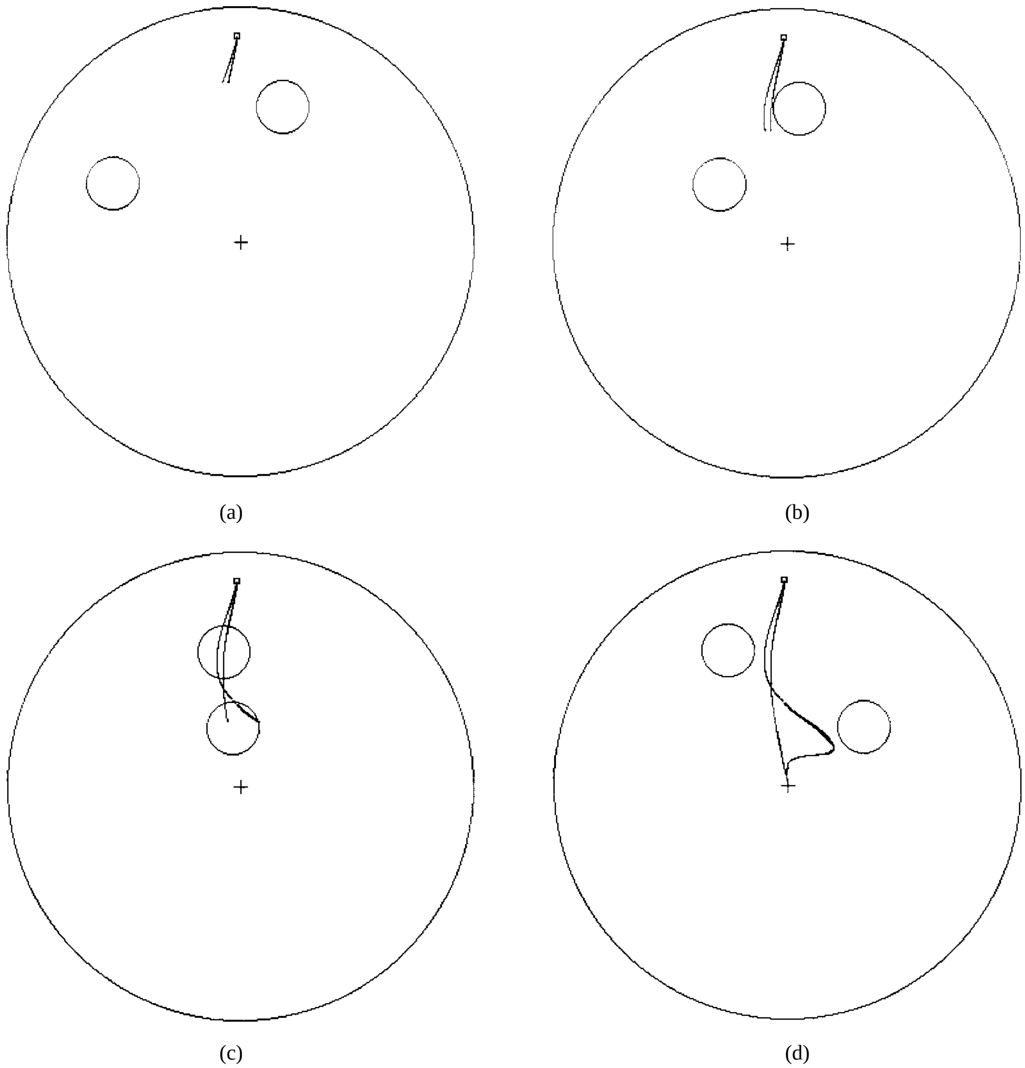

Fig. 3. Moving-obstacle solution $\mathbf{p}(t;\, t_0)$ [trajectory on left in (a)] and stationary-obstacle solution $\mathbf{z}[\nu(t); t_0]$ [trajectory on right in (a)]. (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, and (d) $t = 0.88$.

Definition 12: The normal to the surface of the ith obstacle at the interpolation point is given by

$$\mathbf{n}_i(\mathbf{x},\, t) = \nabla\beta_i[\mathbf{a}_i(\mathbf{x},\, t),\, t]. \tag{12}$$

This construction follows from Definition 4. The normal is labeled in Fig. 4.

Definition 13: The negative velocity vector of the ith obstacle projected onto the ray from the center of the ith obstacle to the robot location is given by

$$\mathbf{z}_{i,\mathrm{N}}(\mathbf{x},\, t) = -\dot{\mathbf{x}}_i(t) \bullet [\mathbf{x}(t) - \mathbf{x}_i(t)]\, \frac{\mathbf{x}(t) - \mathbf{x}_i(t)}{\|\mathbf{x}(t) - \mathbf{x}_i(t)\|}. \tag{13}$$

For a robot position in "front" of the obstacle, $\mathbf{z}_{i,\,\mathrm{N}}(\mathbf{x},\, t)$ points back along the ray from the robot position to the center of the obstacle. See Fig. 4.

Definition 14: The negative velocity vector of the ith obstacle projected onto the space orthogonal to $\mathbf{z}_{i,\,\mathrm{N}}(\mathbf{x},\, t)$ is given by

$$\mathbf{z}_{i,\,\mathrm{T}}(\mathbf{x},\, t) = -\dot{\mathbf{x}}_i(t) - \mathbf{z}_{i,\mathrm{N}}(\mathbf{x},\, t). \tag{14}$$

For almost all robot positions in "front" of the obstacle, $\mathbf{z}_{i,\,\mathrm{T}}(\mathbf{x},\, t)$ points away from the line of motion of the obstacle. See Fig. 4.

Definition 15: The projection of $\mathbf{z}_{i,\,\mathrm{T}}(\mathbf{x},\, t)$ onto the space orthogonal to $\mathbf{n}_i(\mathbf{x},\, t)$ is given by

$$\mathbf{w}_i(\mathbf{x},\, t) = \mathbf{z}_{i,\,\mathrm{T}}(\mathbf{x},\, t) - [\mathbf{z}_{i,\,\mathrm{T}}(\mathbf{x},\, t) \bullet \mathbf{n}_i(\mathbf{x},\, t)]\, \frac{\mathbf{n}_i(\mathbf{x},\, t)}{\|\mathbf{n}_i(\mathbf{x},\, t)\|}. \tag{15}$$

The vector $\mathbf{w}_i(\mathbf{x},\, t)$ is always tangent to the surface of the obstacle. For almost all robot positions in "front" of the obstacle, $\mathbf{w}_i(\mathbf{x},\, t)$ points away from the line of motion of the obstacle (see Fig. 4). The exceptions are those positions along the instantaneous line of motion of the obstacle, for which $\mathbf{w}_i(\mathbf{x},\, t) \equiv 0$.

Note that for spherical obstacles, $\mathbf{z}_{i,\,\mathrm{T}} \equiv \mathbf{w}_i$.

Definition 16: The *new deformation vector field* $\hat{\mathbf{X}}(\mathbf{x},\, t)$is a mapping from $\mathcal{B}_0 \times [0,\, T]$ into $\mathbf{R}^N$. The mapping is given by

$$\hat{\mathbf{X}}(\mathbf{x},\, t) = \sum_{i=1}^{n+1} \sigma_i(\mathbf{x},\, t)\, [\dot{\mathbf{x}}_i(t) + \gamma_i\, \mathbf{w}_i(\mathbf{x},\, t)] \tag{16}$$

where the γ_i are positive real numbers.

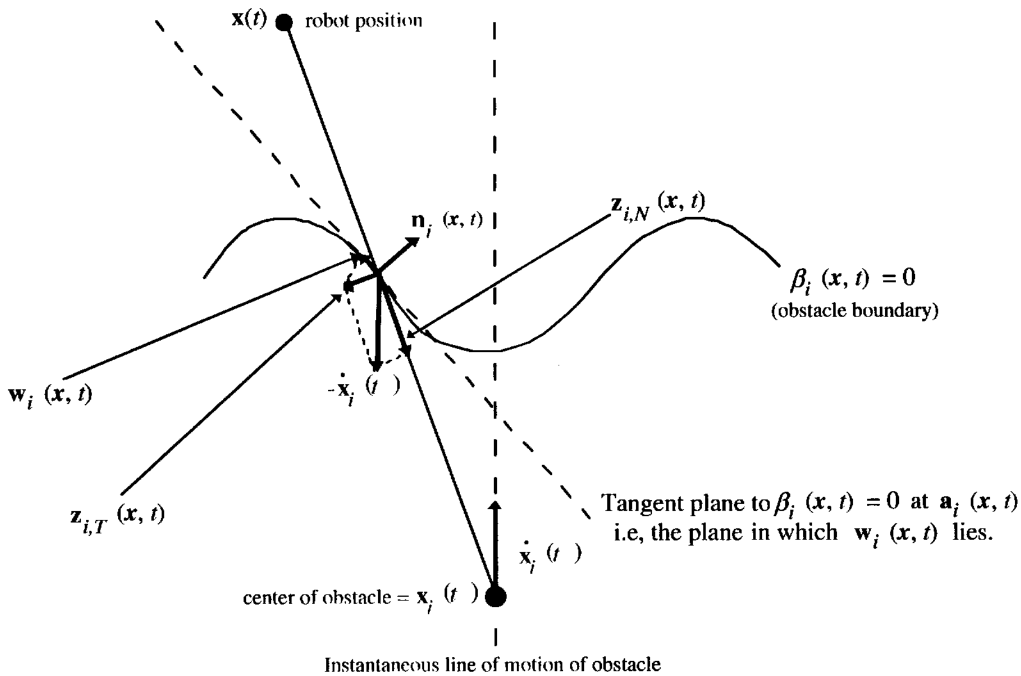

Fig. 4. Construction of $\mathbf{w}$ for the new deformation flow. Consider the example of Section VI in three dimensions (obstacles are spheres). Let the instantaneous line of motion in this figure correspond to the obstacles' plane of motion viewed edge-on. Then for any initial position not exactly in the plane of motion, the effect of $\mathbf{w}$ will be to drive the path away from the plane of motion and "over" the obstacles.

Definition 17: Given any initial time $t_0 \in [0, T]$ and any initial point $\mathbf{x}(t_0) \in \mathcal{B}_0$, denote the *new deformation flow* generated by the new deformation vector field $\hat{\mathbf{X}}$ as $\hat{\mathbf{x}}[t;\, t_0,\, \mathbf{x}(t_0)]$, $t \in [0, T]$.

The new deformation flow satisfies Lemma 1, and hence Theorems 1 and 2. Consider an initial condition on the surface of an obstacle. The portion of the deformation vector field from Definition 9 generates motion with the obstacle set, while the new portion from Definition 15 generates motion *along the boundary* of the obstacle. Thus every point on the boundary of an obstacle remains on the boundary of that obstacle, but the density of the points on the boundary of the obstacle changes with time.

For ease of visualization, one may consider that the new deformation flow acts similarly to a hydrodynamic flow where the free space is composed of fluid while the obstacles are solid bodies. The new deformation flow given in Definition 16 takes advantage of the additional topological freedom granted by spatial dimensions of 3 and higher ($N > 2$).

IX. Future Work

1) The role of the mapping ν in controlling path shape and designing a velocity profile along the path should be investigated.
2) Consider $\mathbf{p}(t;\, t_0)$ as a mapping of $[0, T]$ into $\mathcal{B}_0$. The mapping $\mathbf{p}(t;\, t_0)$ is causal for $t > t_0$. This fact should allow our solution to the moving-obstacle problem to be extended to obstacle movements which are sensed (the dynamic problem) rather than known *a priori* (the static problem).

X. Conclusion

We have provided a global solution to the problem of a point robot navigating among moving obstacles in N-dimensional space. The solution accommodates a moving goal point and an arbitrary number of nonintersecting moving obstacles, and yields a continuous trajectory which terminates at the goal point at a given time T. The solution is obtained by continuously deforming the stationary-obstacle solution of Rimon and Koditschek obtained at time $t_0 \in [0, T]$.

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