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MATH 330 Homework #1

2 September 2016

1.1 (a) $A \cap B = \{2\}$ as 2 is the only even prime number in $\mathbb{N}$.

(b) $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10, \cdots\} = \{x : x \in \mathbb{N} \text{ and } (x \text{ is prime or } x \text{ is even})\}$.

(c) $B \cap C = \{5\}$ as 5 is the only multiple of 5 that is prime.

(d) $A \cap (B \cup C) = \{2, 10, 20, 30, \cdots\} = \{2, 10x : x \in \mathbb{N}\}$ as $B \cup C$ is a set containing all multiples of 5 and all prime numbers, so $A \cap (B \cup C)$ is the set of all *even* multiples of 5 and all *even* prime numbers.

1.17 $f : \mathbb{Q} \to \mathbb{Q}$ is a mapping if for every $a \in \mathbb{Q}$ there exists a unique $b \in \mathbb{Q}$ such that $f(a) = b$.

(a) $f(p/q) = \frac{p+1}{p-2}$ is not a mapping because $\frac{1}{2} = \frac{3}{6}$ but $f\left(\frac{1}{2}\right) = \frac{2}{-1} = -2$ and $f\left(\frac{3}{6}\right) = \frac{4}{1} = 4$.

(b) $f(p/q) = \frac{3p}{3q}$ is a mapping because for all $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$, $\frac{p}{q} = \frac{r}{s} \implies f(p/q) = f(r/s)$ as $f(p/q) = \frac{3p}{3q} = \frac{p}{q} = \frac{r}{s} = \frac{3r}{3s} = f(r/s)$. $\square$

(c) $f(p/q) = \frac{p+q}{q^2}$ is not a mapping because $\frac{1}{2} = \frac{2}{4}$ but $f\left(\frac{1}{2}\right) = \frac{3}{2^2} = \frac{3}{4}$ but $f\left(\frac{2}{4}\right) = \frac{6}{4^2} = \frac{3}{8}$.

(d) $f(p/q) = \frac{3p^2}{7q^2} - \frac{p}{q}$ is a mapping because for all $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$, $\frac{p}{q} = \frac{r}{s} \implies f(p/q) = f(r/s)$ as $f(p/q) = \frac{3p^2}{7q^2} - \frac{p}{q} = \frac{3}{7}\left(\frac{p}{q}\right)^2 - \frac{p}{q} = \frac{3}{7}\left(\frac{r}{s}\right)^2 - \frac{r}{s} = \frac{3r^2}{7s^2} - \frac{r}{s} = f(r/s)$. $\square$

1.19 By Theorem 1.4, $f : A \to B$ and $g : B \to C$ are both bijective because they are both invertible. By Theorem 1.3.4, because f and g are bijective, so $g \circ f : A \to C$ is bijective and by Theorem 1.4, it is invertible. So there exists a unique $(g \circ f)^{-1} : C \to A$ such that $(g \circ f)^{-1} \circ (g \circ f) = \mathrm{id}_A$ and $(g \circ f) \circ (g \circ f)^{-1} = \mathrm{id}_C$.

Notice that by Theorem 1.3.1, the assumption that f and g are invertible, and because $\mathrm{id}_B \circ f = f$, $\left(f^{-1} \circ g^{-1}\right) \circ (g \circ f) = f^{-1} \circ \left(g^{-1} \circ g\right) \circ f = f^{-1} \circ (\mathrm{id}_B \circ f) = f^{-1} \circ f = \mathrm{id}_A$. By the same reasoning, $(g \circ f) \circ \left(f^{-1} \circ g^{-1}\right) = g \circ \left(f \circ f^{-1}\right) \circ g^{-1} = (g \circ \mathrm{id}_B) \circ g^{-1} = g \circ g^{-1} = \mathrm{id}_C$. So $f^{-1} \circ g^{-1}$ is the unique function $(g \circ f)^{-1}$ from above and $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. $\square$

1.25 (a) $x \sim y$ in $\mathbb{R}$ if $x \geq y$ is not an equivalence because it is not symmetric. For example, $10 \sim 5$ but $5 \not\sim 10$.

(b) $m \sim n$ in $\mathbb{Z}$ if $mn > 0$ is an equivalence relation as for all $m \in \mathbb{Z}$, $m \sim m$, $m \sim n \implies n \sim m$, and $m \sim n$ and $n \sim l \implies m \sim l$. This relation describes the first and fourth quadrants of the Cartesian plane (excluding $x = 0$ and $y = 0$).

(c) $x \sim y$ in $\mathbb{R}$ if $|x - y| \leq 4$ is not an equivalence relationship because it is not transitive. For example, $1 \sim 3$ and $3 \sim 7$ but $1 \not\sim 7$.

(d) $m \sim n$ in $\mathbb{Z}$ if $m \equiv n \pmod{6}$ is an equivalence relation. It describes the equivalence class $\mathbb{Z}/(\text{mod} 6) := \{[x]_{\text{ mod } 6} : x \in \mathbb{Z}\}$ where $[x]_{\text{ mod } 6} := \{y \in \mathbb{Z} | y \equiv x \pmod{6}\}$.

2.1 $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Base case: $\mathbf{n = 1} \to \displaystyle\sum_{k=1}^{1} k^2 = 1^2 = 1$ and $\frac{1(1+1)(2*1+1)}{6} = \frac{1*2*3}{6} = 1$ ✓

Inductive step: Assume $\sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{k^2} = \frac{\mathbf{n\,(n+1)\,(2n+1)}}{\mathbf{6}}$ and show $\sum_{\mathbf{k=1}}^{\mathbf{n+1}} \mathbf{k^2} = \frac{\mathbf{(n+1)\,[(n+1)+1]\,(2\,[n+1]+1)}}{\mathbf{6}}$.

$$
\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + n^2 + 2n + 1 \qquad \text{by inductive hypothesis} \\
&= \frac{(n^2+n)(2n+1)}{6} + n^2 + 2n + 1 = \frac{2n^3+n^2+2n^2+n}{6} + n^2 + 2n + 1 \\
&= \frac{2n^3+3n^2+n}{6} + \frac{6n^2+12n+6}{6} \\
&= \frac{2n^3+9n^2+13n+6}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)\,[(n+1)+1]\,(2\,[n+1]+1)}{6}
\end{aligned}
$$

So, by induction, for all $n \in \mathbb{N}$, $\sum_{k=1}^{n} k^2 = \frac{n\,(n+1)\,(2n+1)}{6}$. $\square$

2.15 (a) 14 and 39

(i)

$$
\begin{aligned}
39 &= 14q + r &\rightarrow\quad 39 &= 14(2) + 11 \\
14 &= 11q + r &\rightarrow\quad 14 &= 11(1) + 3 \\
11 &= 3q + r &\rightarrow\quad 11 &= 3(3) + 2 \\
3 &= 2q + r &\rightarrow\quad 3 &= 2(1) + \mathbf{1} \\
2 &= 1q + r &\rightarrow\quad 2 &= 1(2) + 0
\end{aligned}
$$

So $\gcd(14, 39) = 1$.

(ii)

$$
\begin{aligned}
3 = 2(1) + 1 \quad\Longrightarrow\quad 3 - 2(1) &= 1 &\rightarrow\quad & 3 - (11 - 3\cdot 3) = 1 \\
4\cdot 3 - 11 &= 1 &\rightarrow\quad & 4(14-11) - 11 = 1 \\
4(14) - 5(11) &= 1 &\rightarrow\quad & 4(14) - 5(39 - 14\cdot 2) \\
14(\mathbf{14}) - 5(\mathbf{39}) &= 1
\end{aligned}
$$

So $(r, s) \in \mathbb{Z}^2$ such that $\gcd(14, 39) = 14r + 39s$ is $(r, s) = (14, -5)$.

(b) 234 and 165

(i)

$$
\begin{aligned}
234 &= 165q + r &\rightarrow\quad 234 &= 165(1) + 69 \\
165 &= 69q + r &\rightarrow\quad 165 &= 69(2) + 27 \\
69 &= 27q + r &\rightarrow\quad 69 &= 27(2) + 15 \\
27 &= 15q + r &\rightarrow\quad 27 &= 15(1) + 12 \\
15 &= 12q + r &\rightarrow\quad 15 &= 12(1) + \mathbf{3} \\
12 &= 3q + r &\rightarrow\quad 12 &= 3(4) + 0
\end{aligned}
$$

So $\gcd(234, 165) = 3$.

$$
\text{(ii)} \quad
\begin{array}{lccc}
15 = 12(1) + 3 \implies & 3 = 15 - 12 & \to & 3 = 15 - (27 - 15) \\
& 3 = 2(15) - 27 & \to & 3 = 2(69 - 2 \cdot 27) - 27 \\
& 3 = 2(69) - 5(27) & \to & 3 = 2(69) - 5(165 - 2 \cdot 69) \\
& 3 = 12(69) - 5(165) & \to & 3 = 12(234 - 165) - 5(165) \\
& 3 = 12\,(\mathbf{234}) - 17\,(\mathbf{165}) & &
\end{array}
$$

So $(r, s) \in \mathbb{Z}^2$ such that $\gcd(234, 165) = 234r + 165s$ is $(r, s) = (12, -17)$.

(c) 1739 and 9923

$$
\text{(i)} \quad
\begin{array}{ccc}
9923 = 1739q + r & \to & 9923 = 1739(5) + 1228 \\
1739 = 1228q + r & \to & 1739 = 1228(1) + 511 \\
1228 = 511q + r & \to & 1228 = 511(2) + 206 \\
511 = 206q + r & \to & 511 = 206(2) + 99 \\
206 = 99q + r & \to & 206 = 99(2) + 8 \\
99 = 8q + r & \to & 99 = 8(12) + 3 \\
8 = 3q + r & \to & 8 = 3(2) + 2 \\
3 = 2q + r & \to & 3 = 2(1) + \mathbf{1} \\
2 = 1q + r & \to & 2 = 1(2) + 0
\end{array}
$$

So $\gcd(1739, 9923) = 1$.

$$
\text{(ii)} \quad
\begin{array}{ccc}
1 = 3 - 2 & \to & 1 = 3 - (8 - 2 \cdot 3) \\
1 = 3 \cdot 3 - 8 & \to & 1 = 3(99 - 12 \cdot 8) - 8 \\
1 = 3 \cdot 99 - 37 \cdot 8 & \to & 1 = 3 \cdot 99 - 37(206 - 2 \cdot 99) \\
1 = 77 \cdot 99 - 37 \cdot 206 & \to & 1 = 77(511 - 2 \cdot 206) - 37 \cdot 206 \\
1 = 77 \cdot 511 - 191 \cdot 206 & \to & 1 = 77 \cdot 511 - 191(1228 - 2 \cdot 511) \\
1 = 459 \cdot 511 - 191 \cdot 1228 & \to & 1 = 459(1739 - 1228) - 191 \cdot 1228 \\
1 = 459 \cdot 1739 - 650 \cdot 1228 & \to & 1 = 459 \cdot 1739 - 650(9923 - 5 \cdot 1739) \\
1 = 3709\,(\mathbf{1739}) - 650\,(\mathbf{9923}) & &
\end{array}
$$

So $(r, s) \in \mathbb{Z}^2$ such that $\gcd(1739, 9923) = 1739r + 9923s$ is $(r, s) = (3709, -650)$.

(d) 471 and 562

$$
\text{(i)} \quad
\begin{array}{ccc}
562 = 471q + r & \to & 562 = 471(1) + 91 \\
471 = 91q + r & \to & 471 = 91(5) + 16 \\
91 = 16q + r & \to & 91 = 16(5) + 11 \\
16 = 11q + r & \to & 16 = 11(1) + 5 \\
11 = 5q + r & \to & 11 = 5(2) + \mathbf{1} \\
5 = 1q + r & \to & 5 = 1(5) + 0
\end{array}
$$

So $\gcd(471, 562) = 1$.

$$\text{(ii)}\quad \begin{array}{rcl} 1 = 11 - 2 \cdot 5 & \to & 1 = 11 - 2(16 - 11) \\ 1 = 3 \cdot 11 - 2 \cdot 16 & \to & 1 = 3(91 - 5 \cdot 16) - 2 \cdot 16 \\ 1 = 3 \cdot 91 - 17 \cdot 16 & \to & 1 = 3 \cdot 91 - 17(471 - 5 \cdot 91) \\ 1 = 88 \cdot 91 - 17 \cdot 471 & \to & 1 = 88(562 - 471) - 17 \cdot 471 \end{array}$$

$1 = 88(\mathbf{562}) - 105\,(\mathbf{471})$

So $(r, s) \in \mathbb{Z}^2$ such that $\gcd(562, 471) = 562r + 471s$ is $(r, s) = (88, -105)$.

(e) 23,771 and 19,945

$$\text{(i)}\quad \begin{array}{rcl} 23771 = 19945q + r & \to & 23771 = 19945(1) + 3826 \\ 19945 = 3826q + r & \to & 19945 = 3826(5) + 815 \\ 3826 = 815q + r & \to & 3826 = 815(4) + 566 \\ 815 = 566q + r & \to & 815 = 566(1) + 249 \\ 566 = 249q + r & \to & 566 = 249(2) + 68 \\ 249 = 68q + r & \to & 249 = 68(3) + 45 \\ 68 = 45q + r & \to & 68 = 45(1) + 23 \\ 45 = 23q + r & \to & 45 = 23(1) + 22 \\ 23 = 22q + r & \to & 23 = 22(1) + \mathbf{1} \\ 22 = 1q + r & \to & 22 = 1(22) + 0 \end{array}$$

So $\gcd(23771, 19945) = 1$.

$$\text{(ii)}\quad \begin{array}{rcl} 1 = 23 - 22 & \to & 1 = 23 - (45 - 23) \\ 1 = 2 \cdot 23 - 45 & \to & 1 = 2(68 - 45) - 45 \\ 1 = 2 \cdot 68 - 3 \cdot 45 & \to & 1 = 2 \cdot 68 - 3(249 - 3 \cdot 68) \\ 1 = 11 \cdot 68 - 3 \cdot 249 & \to & 1 = 11(566 - 2 \cdot 249) - 3 \cdot 249 \\ 1 = 11 \cdot 566 - 25 \cdot 249 & \to & 1 = 11 \cdot 566 - 25(815 - 566) \\ 1 = 36 \cdot 566 - 25 \cdot 815 & \to & 1 = 36(3826 - 4 \cdot 815) - 25 \cdot 815 \\ 1 = 36 \cdot 2836 - 169 \cdot 815 & \to & 1 = 36 \cdot 3826 - 169(19945 - 5 \cdot 3826) \\ 1 = 881 \cdot 3826 - 169 \cdot 19945 & \to & 1 = 881(23771 - 19945) - 169 \cdot 19945 \end{array}$$

$1 = 811(\mathbf{23771}) - 1050(\mathbf{19945})$

So $(r, s) \in \mathbb{Z}^2$ such that $\gcd(23771, 19945) = 23771r + 19945s$ is $(r, s) = (881, -1050)$.

(f) -4357 and 3754

(i)
$$\begin{array}{ccc}
4357 = 3754q + r & \rightarrow & 4357 = 3754(1) + 603 \\
3754 = 603q + r & \rightarrow & 3754 = 603(6) + 136 \\
603 = 136q + r & \rightarrow & 603 = 136(4) + 59 \\
136 = 59q + r & \rightarrow & 136 = 59(2) + 18 \\
59 = 18q + r & \rightarrow & 59 = 18(3) + 5 \\
18 = 5q + r & \rightarrow & 18 = 5(3) + 3 \\
5 = 3q + r & \rightarrow & 5 = 3(1) + 2 \\
3 = 2q + r & \rightarrow & 3 = 2(1) + \mathbf{1} \\
2 = 1q + r & \rightarrow & 2 = 1(2) + 0
\end{array}$$

So $\gcd(-4357, 3754) = 1$.

(ii)
$$\begin{array}{ccc}
1 = 3 - 2 & \rightarrow & 1 = 3 - (5 - 3) \\
1 = 2 \cdot 3 - 5 & \rightarrow & 1 = 2(18 - 3 \cdot 5) - 5 \\
1 = 2 \cdot 18 - 7 \cdot 5 & \rightarrow & 1 = 2 \cdot 18 - 7(59 - 3 \cdot 18) \\
1 = 23 \cdot 18 - 7 \cdot 59 & \rightarrow & 1 = 23(136 - 2 \cdot 59) - 7 \cdot 59 \\
1 = 23 \cdot 136 - 53 \cdot 59 & \rightarrow & 1 = 23 \cdot 136 - 53(603 - 4 \cdot 136) \\
1 = 235 \cdot 136 - 53 \cdot 603 & \rightarrow & 1 = 235(3754 - 6 \cdot 603) - 53 \cdot 603 \\
1 = 235 \cdot 3754 - 1463 \cdot 603 & \rightarrow & 1 = 235 \cdot 3754 - 1463(4357 - 3754) \\
1 = 1698(\mathbf{3754}) - 1463(\mathbf{4357}) & \iff & 1 = 1698(\mathbf{3754}) + 1463(-\mathbf{4357})
\end{array}$$

So $(r, s) \in \mathbb{Z}^2$ such that $\gcd(3754, -4357) = 3754r + (-4357)s$ is $(r, s) = (1698, 1463)$.

2.17 (a) Prove that $f_n < 2^n$.

Base case: $\mathbf{n = 1}$ $\rightarrow$ $f_1 = 1 < 2^1$. ✓

Base case: $\mathbf{n = 2}$ $\rightarrow$ $f_2 = 1 < 2^2$. ✓

Inductive step: Assume $\mathbf{f_{n-1} < 2^{n-1}}$ and $\mathbf{f_n < 2^n}$ and show that $\mathbf{f_{n+1} < 2^{n+1}}$.

$$\begin{aligned}
f_{n+1} &= f_{n-1} + f_n \text{ and } f_{n-1} < 2^{n-1} \text{ and } f_n < 2^n \\
&\implies f_{n-1} + f_n = f_{n+1} \\
&< 2^n + 2^{n-1} \\
&= 2^{n-1}(2 + 1) \\
&= 3 \cdot 2^{n-1} \\
&< 4 \cdot 2^{n-1} \\
&= 2^{n+1}
\end{aligned}$$

So, by induction, for all $n \in \mathbb{N}$, $f_n < 2^n$. □

(b) Prove that $f_{n+1}f_{n-1} = f_n^2 + (-1)^n, n \geq 2$.

Base case: $\mathbf{n = 2}$ $f_3 f_1 = f_2^2 + (-1)^2 \implies 2 \cdot 1 = 1 + 1 \implies 2 = 2$ ✓

Inductive step: Assume $\mathbf{f_{n+1}f_{n-1} = f_n^2 + (-1)^n}$ and show $\mathbf{f_{n+2}f_n = f_{n+1}^2 + (-1)^{n+1}}$.

$$\begin{aligned}
f_{n+1}f_{n-1} &= f_n^2 + (-1)^n \quad \text{by induction hypothesis}\\
&\implies -f_{n+1}f_{n-1} = -f_n^2 + (-1)^{n+1}\\
&\implies -f_{n+1}\left(f_{n+1} - f_n\right) = -f_n^2 + (-1)^{n+1} \quad \text{by definition of Fibonacci numbers}\\
&\implies f_nf_{n+1} - f_{n+1}^2 = -f_n^2 + (-1)^{n+1}\\
&\implies f_nf_{n+1} + f_n^2 = f_{n+1}^2 + (-1)^{n+1}\\
&\implies f_n\left(f_{n+1} + f_n\right) = f_{n+1}^2 + (-1)^{n+1}\\
&\implies f_nf_{n+2} = f_{n+1}^2 + (-1)^{n+1}
\end{aligned}$$

So for all $n \in \mathbb{N}$ such that $n \geq 2$, $f_{n+1}f_{n-1} = f_n^2 + (-1)^n$. □

(c) Prove that $f_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$.

Base case: $\mathbf{n = 1}$ $\rightarrow$ $\frac{\left(1+\sqrt{5}\right)^1 - \left(1-\sqrt{5}\right)^1}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1.$ ✓

Base case: $\mathbf{n = 2}$ $\rightarrow$ $\frac{\left(1+\sqrt{5}\right)^2 - \left(1-\sqrt{5}\right)^2}{2^2\sqrt{5}} = \frac{1+2\sqrt{5}+5-\left(1-2\sqrt{5}+5\right)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$ ✓

Inductive step: Assume $\mathbf{f_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}}$ and $\mathbf{f_{n+1} = \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}}}$ and show that $\mathbf{f_{n+2} = \frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}}}$.

$$\begin{aligned}
f_{n+2} &= f_n + f_{n+1} \quad \text{by definition of Fibonacci numbers}\\
f_n + f_{n+1} &= \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}} + \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}} \quad \text{by inductive hypothesis}\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(1 + \frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(1 + \frac{1-\sqrt{5}}{2}\right)\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{3-\sqrt{5}}{2}\right)\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left[\frac{1}{4}\left(6+2\sqrt{5}\right)\right] - \left(\frac{1-\sqrt{5}}{2}\right)^n\left[\frac{1}{4}\left(6-2\sqrt{5}\right)\right]\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right)^2\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]\\
&= \frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}} = f_{n+2}
\end{aligned}$$

(d) Show that $\lim_{n\to\infty} f_n/f_{n+1} = \frac{\sqrt{5}-1}{2}$.

Let $x = \lim_{n\to\infty}$

$$\begin{aligned}
\lim_{n\to\infty}\frac{f_n}{f_{n+1}} &= \lim_{n\to\infty}\frac{f_{n+2} - f_{n+1}}{f_{n+1}}\\
&= \lim_{n\to\infty}\frac{f_{n+2}}{f_{n+1}} - 1\\
&\implies \lim_{n\to\infty}\frac{f_n}{f_{n+1}} = \lim_{n\to\infty}\frac{f_{n+2}}{f_{n+1}} - 1
\end{aligned}$$