Moshe Mason Rubin
MATH 330 Homework #8
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Exercise 9.9. Let $G = \mathbb{R} \setminus \{-1\}$ and define a binary operation on G by

$$a * b = a + b + ab.$$

Prove that G is a group under this operation. Show that $(G, *)$ is isomorphic to the multiplicative group of nonzero real numbers.

Proof. By **Exercise 3.7** from homework #2, $(G, *)$ is an abelian group.
The map $\phi : G \to \mathbb{R}^*$ given by $\phi(x) = 1 + x$ is clearly a bijection and well defined on each set. ϕ preserves group operations as for any $a, b \in G$,

$$\begin{aligned} \phi(a) \cdot \phi(b) &= (1+a) \cdot (1+b) \quad \text{by definition of } \phi \\ &= 1 + b + a + ab = a + b + ab + 1 \\ &= \phi(a + b + ab) \quad \text{by definition of } \phi \\ &= \phi(a * b) \quad \text{by definition of } * \end{aligned}$$

So $(G, *) \simeq (\mathbb{R}^*, \cdot)$. □

Exercise 9.12. Prove that S_4 is not isomorphic to D_{12}.

Proof. Consider $(1, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2) \in D_{12}$ which has order 12. Because every element of S_4 has order less than or equal to 4, the two groups cannot be isomorphic by Theorem from class that $\operatorname{ord}[\phi(g_1)] = \operatorname{ord}(g_1)$. □

Exercise 9.14. Show that the set of all matrices of the form $\left[\begin{smallmatrix} \pm 1 & k \\ 0 & 1 \end{smallmatrix}\right]$ is isomorphic to D_n where all entries in the matrix are in $\mathbb{Z}_n$.

Proof. Let $S = \left\{\left[\begin{smallmatrix} \pm 1 & k \\ 0 & 1 \end{smallmatrix}\right] : k \in \mathbb{Z}_n\right\}$. Notice for any $\left[\begin{smallmatrix} \pm 1 & k \\ 0 & 1 \end{smallmatrix}\right] \in S$, we have

$$\begin{bmatrix} \pm 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. So $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ generate S. Furthermore, $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ has order n, and $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ has order 2, and

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

By Theorem 5.10, D_n is generated by all the products of $r, s \in D_n$ such that $r^n = s^2 = id$ and $srs = r^{-1}$. Define $f : S \to D_n$ by $f\left(\left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]\right) = r^k$ and $f\left(\left[\begin{smallmatrix} -1 & k \\ 0 & 1 \end{smallmatrix}\right]\right) = r^k s$. To check that f preserves group operations, there are four cases to check:

1. $f\left(\left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right) = f\left(\left[\begin{smallmatrix} 1 & k+\ell \\ 0 & 1 \end{smallmatrix}\right]\right) = r^{k+\ell} = r^k r^\ell = f\left(\left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]\right) \cdot f\left(\left[\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right)$. ✓
2. $f\left(\left[\begin{smallmatrix} -1 & k \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right) = f\left(\left[\begin{smallmatrix} -1 & k-\ell \\ 0 & 1 \end{smallmatrix}\right]\right) = r^{k-\ell}s = r^k r^{-\ell} s = \cdots = \cdots = f\left(\left[\begin{smallmatrix} -1 & k \\ 0 & 1 \end{smallmatrix}\right]\right) \cdot f\left(\left[\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right)$. ✓
3. $f\left(\left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right) = f\left(\left[\begin{smallmatrix} -1 & k+\ell \\ 0 & 1 \end{smallmatrix}\right]\right) = r^{k+\ell}s = r^k r^\ell s = r^k\left(r^\ell s\right) = f\left(\left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]\right) \cdot f\left(\left[\begin{smallmatrix} -1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right)$. ✓
4. $f\left(\left[\begin{smallmatrix} -1 & k \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right) = f\left(\left[\begin{smallmatrix} 1 & k-\ell \\ 0 & 1 \end{smallmatrix}\right]\right) = r^{k-\ell} = r^k r^{-\ell} = r\,(srs)^\ell \cdots = \cdots = f\left(\left[\begin{smallmatrix} -1 & k \\ 0 & 1 \end{smallmatrix}\right]\right) \cdot f\left(\left[\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right]\right)$. ✓

□

Exercise 10.2. Find all the sub-groups of D_4. Which sub-groups are normal? What are all the factors groups of D_4 up to isomorphisms?

Proof. The sub-groups are $D_4 = \left\{id, \rho, \rho^2, \rho^3, s, \rho s, \rho^2 s, \rho^3 s\right\} = \{id, (1234), (13)(24), (1432), (24), (12)(34), (13), (14)(23)\}$ are

1. $\{id\}$
2. $\{id,\rho^2\}$
3. $\{id,s\}$
4. $\{id,\rho s\}$
5. $\{id,\rho^2 s\}$
6. $\{id,\rho^3 s\}$
7. $\{id,\rho,\rho^2,\rho^3\}$
8. $\{id,\rho^2,s,\rho^2 s\}$
9. $\{id,\rho^2,\rho s,\rho^3 s\}$
10. D_4

1. $\{id\}$ is normal.
2. $\{id,\rho^2\}$ is normal as $(1234)\{id,\rho^2\}=\{(1234),(1432)\}=\{id,\rho^2\}(1234)$ and $(24)\{id,\rho^2\}=\{(24),(13)\}=\{id,\rho^2\}(24)$ and $(12)(34)\{id,\rho^2\}=\{(12)(34),(14)(23)\}=\{id,\rho^2\}(12)(34)$. ✓
3. $\{id,s\}$ is not normal as $(1234)\{id,s\}=\{(1234),(12)(34)\}\neq\{(1234),(14)(23)\}=\{id,s\}(1234)$.
4. $\{id,\rho s\}$ is not normal as $(1234)\{id,\rho s\}=\{(1234),(13)\}\neq\{(1234),(24)\}=\{id,\rho s\}(1234)$
5. $\{id,\rho^2 s\}$ is not normal as $\rho\circ(\rho^2 s)=\rho^3 s\neq\rho s=(\rho^2 s)\circ\rho$.
6. $\{id,\rho^3 s\}$ is not normal as $\rho\circ(\rho^3 s)=s\neq\rho^2 s=(\rho^3 s)\circ\rho$.
7. $\{id,\rho,\rho^2,\rho^3\}$ is normal as $s\{id,\rho,\rho^2,\rho^3\}=\{s,\rho^3 s,\rho^2 s,\rho s\}=\{id,\rho,\rho^2,\rho^3\}s$. ✓
8. $\{id,\rho^2,s,\rho^2 s\}$ is normal as $\rho\{id,\rho^2,s,\rho^2 s\}=\{\rho,\rho^3,\rho s,\rho^3 s\}=\{id,\rho^2,s,\rho^2 s\}\rho$. ✓
9. $\{id,\rho^2,\rho s,\rho^3 s\}$ is normal as $\rho\{id,\rho^2,\rho s,\rho^3 s\}=\{\rho,\rho^3,\rho^2 s,s\}$. ✓
10. D_4 is normal.

2. The factor group $D_4/\{id,\rho^2\}=\{\{id,\rho^2\},\{\rho,\rho^3\},\{s,\rho^2 s\},\{\rho s,\rho^3 s\}\}$.
7. The factor group $D_4/\{id,\rho,\rho^2,\rho^3\}=\{\{id,\rho,\rho^2,\rho^3\},\{s,\rho^3 s,\rho^2 s,\rho s\}\}$.
8. $D_4/\{id,\rho^2,s,\rho^2 s\}=\{\{id,\rho^2,s,\rho^2 s\},\{\rho,\rho^3,\rho s,\rho^3 s\}\}$.
9. $D_4/\{id,\rho^2,\rho s,\rho^3 s\}=\{\{id,\rho^2,\rho s,\rho^3 s\},\{\rho,\rho^3,\rho^2 s,s\}\}$.

□

Exercise 10.7. Prove or disprove: If H is a normal sub-group of G such that H and G/H are abelian, then G is abelian.

Counterexample. Let $G=S_3$ and $H=A_3$. S_3 is non-abelian. By Corollary 9.4, $A_3\simeq\mathbb{Z}_3$, so A_3 *is* abelian. We must show A_3 is normal and S_3/A_3 is abelian:
A_3 is normal as for any $\sigma\in S_3$, $\sigma A_3\sigma^{-1}$ is even whether σ is even or odd. So $\sigma A_3\sigma^{-1}\subseteq A_3$, so A_3 is normal by Theorem 10.1.2. ✓
To show S_3/A_3 is abelian, notice by Lagrange's Theorem, $[S_3:A_3]=\frac{|S_3|}{|A_3|}=\frac{6}{3}=2$. By Theorem 10.2, $|S_3/A_3|=[S_3:A_3]$. By Corollary 9.4, since $|S_3/A_3|=2$ and 2 is prime, $S_3/A_3\simeq\mathbb{Z}_2$, so S_3/A_3 is abelian. ✓ □

Exercise 10.11. If a group G has exactly one sub-group H of order k, prove that H is normal in G.

Proof. By **Exercise 3.54** from homework #4, gHg^{-1} is a sub-group of G. By the assumption that H is the only sub-group of G, we have that $H=gHg^{-1}$. By Theorem 10.1.3, $H=gHg^{-1}\implies H$ is a normal subgroup of G. □

Exercise 10.12. Define the ***centralizer*** of an element g in a group G to be the set

$$C(g)=\{x\in G:xg=gx\}.$$

Show that $C(g)$ is a sub-group of G. If g generates a normal sub-group of G, prove that $C(g)$ is normal in G.

Proof. For $C(g)\subseteq G$ to be a sub-group of G, it is sufficient to show

1. For all $a,b\in C(g)$, $a\circ b\in C(g)$.
2. There exists $e\in C(g)$ such that $a\circ e=a=e\circ a$ for all $a\in C(g)$.
3. For all $a\in C(g)$ there exists $a^{-1}\in C(g)$ such that $a\circ a^{-1}=e=a^{-1}\circ a$.

1. Consider $a, b \in C(g)$. Then $a, b \in G$ as $C(g) \subseteq G$. Then

$$\begin{aligned}
(ab)x &= a(bx) && \text{by associativity of elements of } G \\
&= a(xb) && \text{by assumption that } b \in C(g) \\
&= (ax)b && \text{by associativity of elements of } G \\
&= (xa)b && \text{by assumption that } a \in C(g) \\
&= x(ab) && \text{by associativity of elements of } G
\end{aligned}$$

   So $ab \in C(g)$. ✓

2. Because $e \in G$ by definition commutes with every element of G, $e \in C(g)$. ✓

3. Consider $c \in C(g)$. Then $c \in G$ and $c^{-1} \in G$ as G is a group and $C(g) \subseteq G$. Then

$$\begin{aligned}
c \in C(g) &\implies cx = xc \\
&\implies c^{-1}cxc^{-1} = c^{-1}xcc^{-1} \quad \text{by left and right multiplying by } c^{-1} \\
&\implies xc^{-1} = c^{-1}x \quad \text{by condensing the "} c^{-1}c \text{" and "} cc^{-1} \text{" terms}
\end{aligned}$$

   So $c \in C(g) \implies c^{-1} \in C(g)$. ✓

So $C(g)$ is a sub-group of G.
Because $C(g)$ us clearly abelian, it follows that the left and right co-sets must be equal, and $C(g)$ must be normal. □

Exercise 11.2. Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

(a) $\phi : \mathbb{R}^* \to GL_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

(b) $\phi : \mathbb{R} \to GL_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

(c) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

(d) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

(e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b$$

Proof.

(a) ϕ is a homomorphism as for $a, b \in \mathbb{R}^*$,

$$\phi(a)\phi(b) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & ab \end{bmatrix} = \phi(ab)$$

   and $\ker \phi := \{x \in \mathbb{R}^* \text{ such that } \phi(x) = id\} = \{1\}$. □

(b) ϕ is not a homomorphism as

$$\phi(a)\phi(b) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix} \neq \phi(ab).$$

□

(c) ϕ is not a homomorphism as

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \phi\left(\begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}\right) = a\alpha + b\gamma + c\beta + d\delta \neq a\alpha + a\delta + d\alpha + d\delta = \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\phi\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right).$$

(d) ϕ is a homomorphism as

$$\begin{aligned}\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \phi\left(\begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}\right) &= (a\alpha + b\gamma)(c\beta + d\delta) - (a\beta + b\delta)(c\alpha + d\gamma) \\ &= ac\alpha\beta + ad\alpha\delta + bc\beta\gamma + bd\gamma\delta - ac\alpha\beta - ad\beta\gamma - bc\alpha\delta - bd\gamma\delta \\ &= ad\alpha\delta + bc\beta\gamma - ad\beta\gamma - bc\gamma\delta \\ &= (ad - bc)(\alpha\delta - \beta\gamma) = \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\phi\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right)\end{aligned}$$

and $\ker\phi := \{M \in GL_2(\mathbb{R}) \text{ such that } \phi(A) = 1\} = SL_2(\mathbb{R})$. ☐

(e) ϕ is not a homomorphism as

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \phi\left(\begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}\right) = a\beta + b\delta \neq b\beta = \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\phi\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right).$$ ☐

Exercise 11.17. If H and K are normal sub-groups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a sub-group of $G \setminus H \times G \setminus K$.

Proof. content... ☐

Homework exercises I cited:

Exercise 3.7 Let $S = \mathbb{R} \setminus \{-1\}$ and define a binary operation on S by $a * b = a + b + ab$. Prove that $(S, *)$ is an abelian group. An *abelian group* is a group G such that $a * b = b * a$ for all $a, b \in G$.

Associative For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.

$$\begin{aligned}(a * b) * c &= (a * b) + c + (a * b)c \quad \text{by definition of } a * b \\ &= (a + b + ab) + c + (a + b + ab)c \quad \text{by definition of } a * b \\ &= a + b + c + ab + ac + bc + abc \\ &= a + (b + c + bc) + a(b + c + bc) \\ &= a + (b * c) + a(b * c) \quad \text{by definition of } a * b \\ &= a * (b * c) \quad \text{by definition of } a * b\end{aligned}$$

Identity element There exists an element $e \in G$ such that for any $a \in G$, $e * a = a * e = a$.
For any a, let $b = 0$. Then $a * b = a + 0 + a(0) = a = 0 + a + 0(a) = b * a$. So $b = 0$ is the identity element such that $a * 0 = 0 * a$ for all $a \in G$.

Inverse element For each element $a \in G$ there exists an $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.
We know from above that $e = 0$. So given $a \in G$,

$$\begin{aligned}a + b + ab &= 0 \\ &\implies b(1 + a) + a = 0 \\ &\implies b = \frac{-a}{1 + a}\end{aligned}$$

which is defined for all $x \in S$. So $b = \frac{-a}{1+a}$ is the unique inverse element a^{-1} to each a such that $a * a^{-1} = a^{-1} * a = e$.

Commutative For all $a, b \in G$, $a * b = b * a$.

$$\begin{aligned} a * b &= a + b + ab \\ &= b + a + ab \quad \text{by commutative property of addition} \\ &= b + a + ba \quad \text{by commutative property of multiplication} \\ &= b * a \quad \text{by definition} \end{aligned}$$

So $(S, *)$ is an abelian group. $\square$

Exercise 3.54. Let H be a sub-group of G. If $g \in G$, show that $gHg^{-1} := \left\{g^{-1}hg : h \in H\right\}$ is also a sub-group of G.

Proof. By theorem from class, for $gHg^{-1} \subseteq G$ to be a sub-group of G, it is sufficient to show

1. For all $a, b \in gHg^{-1}$, $a \circ b \in gHg^{-1}$.
2. There exists $e \in gHg^{-1}$ such that $a \circ e = a = e \circ a$ for all $a \in gHg^{-1}$.
3. For all $a \in gHg^{-1}$ there exists $a^{-1} \in gHg^{-1}$ such that $a \circ a^{-1} = e = a^{-1} \circ a$.

Notice that gHg^{-1} is necessarily a subset of G as every element in H is contained in G (by assumption that H is a sub-group of G). So $g, h, g^{-1} \in G$. Furthermore, every element in gHg^{-1} is of the form $g^{-1}hg$, and G is closed by assumption that G is a group. So $gHg^{-1} \subseteq G$.
Let $a, b \in gHg^{-1}$. Then $a = g^{-1}h_a g$ and $b = g^{-1}h_b g$ for some $h_a, h_b \in H$.

1. Consider

$$\begin{aligned} ab &= \left(g^{-1}h_a g\right)\left(g^{-1}h_b g\right) \\ &= \left(g^{-1}h_a\right)\left(gg^{-1}\right)\left(h_b g\right) \quad \text{by associativity of elements of } G \\ &= \left(g^{-1}h_a\right)(e)\left(h_b g\right) \quad \text{by definition of } g^{-1} \\ &= \left(g^{-1}h_a\right)\left(h_b g\right), \quad \text{by definition of } e \\ &= g^{-1}\left(h_a h_b\right)g \quad \text{by associativity of elements of } G \end{aligned}$$

   and $(h_a h_b) \in H$ as H was assumed to be a sub-group, so H is closed. So $ab = g^{-1}\left(h_a h_b\right)g$ is of the form $g^{-1}hg$ for some $h \in H$. So gHg^{-1} is closed.

2. By assumption that H is a sub-group of G, $e \in H$. So $\left(g^{-1}eg\right) \in gHg^{-1}$ and

$$\begin{aligned} g^{-1}eg &= g^{-1}g \quad \text{by definition of } e \\ &= e, \quad \text{by definition of } g^{-1}. \end{aligned}$$

   So $\left(g^{-1}eg\right) \in gHg^{-1}$ and $g^{-1}eg = e$. so $e \in gHg^{-1}$.

3. By Proposition 3.4, if $a = g^{-1}h_a g$ then $a^{-1} = g^{-1}h_a^{-1}g$. So $a^{-1} \in gHg^{-1}$ if $h_a^{-1} \in H$, and h_a^{-1} is necessarily an element of H by assumption that H is a sub-group of G. So $a \in gHg^{-1} \implies a^{-1} \in gHg^{-1}$.

So this shows that gHg^{-1} is a sub-group of G. $\square$