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MATH 330 Homework #6

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Exercise 5.8. Show that A_{10} contains an element of order 15.

Proof. Consider $\sigma \in A_{10} \subset S_{10}$ given by $\sigma = (12345)(678)$, the product of two disjoint cycles. Then $\sigma \in A_{10}$ as σ is the product of two even permutations and Theorem 5.7 states A_{10} is a sub-group of S_{10}, therefore closed. Notice $(12345)^{-1} = (12345)^4$ and $(678)^{-1} = (678)^2$, and clearly $(12345)^{-1} \neq (678)^n$ and $(678)^{-1} \neq (12345)^m$ for any $(n, m) \in \mathbb{Z}^2$. Because $|A_{10}|= \frac{10!}{2}$ is finite, $|\sigma|\neq \infty$ as $\sigma^n \in A_{10}$ for all $n \in \mathbb{Z}$ by Theorem 5.7 that A_n is a sub-group of S_n. Then

$$\begin{aligned}
\sigma^{15} &= [(12345)(678)]^{15} \\
&= (12345)^{15}(678)^{15} \quad \text{by Proposition 5.2 (that disjoint cycles commute) and Theorem 3.8.3} \\
&= \left[(12345)^5\right]^3 \left[(678)^3\right]^5 \quad \text{by Theorem 3.8.2} \\
&= (id)^3\,(id)^5 \\
&= id
\end{aligned}$$

So $\sigma = (12345)(678) \in A_{10}$ is an element of A_{10} of order 15. □

Exercise 5.13. Let $\sigma = \sigma_1 \cdots \sigma_m \in S_n$ be the product of disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of the cycles $\sigma_1, \ldots, \sigma_m$.

Proof. Let $|\sigma|= k$. So

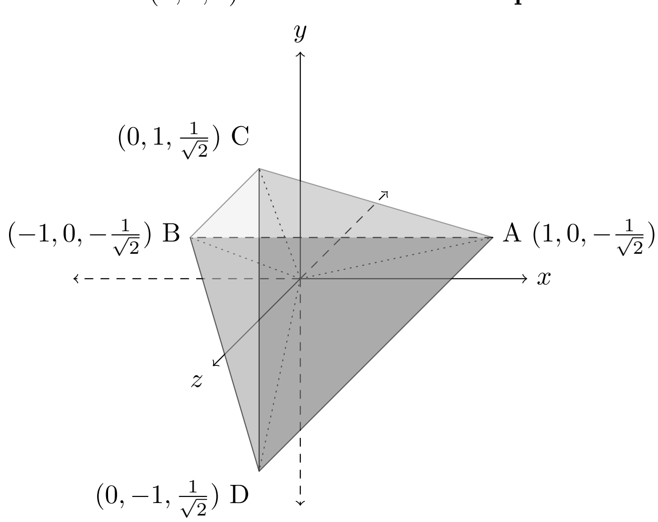
$$\begin{aligned}
\sigma^k &= (\sigma_1 \cdots \sigma_m)^k \\
&= \sigma_1^k \cdots \sigma_m^k \quad \text{by Proposition 5.2 (that disjoint cycles commute) and Theorem 3.8.3} \\
&= id \quad \text{because } \sigma^k = id \text{ as } |\sigma|= k
\end{aligned}$$

So $\sigma_i^k = id$ for $i \in ([1, m] \cap \mathbb{Z})$. So if So $\sigma_i^k = id$ then k must be a common multiple of the length of each σ_i. So the smallest k (that is, the order of σ) must be equal to the least common divisor of lengths of $\sigma_1, \ldots \sigma_m$ by definition of least common multiple. □

Exercise 5.16. Find all group of rigid motions of a tetrahedron. Show that this is the same group as A_4.

Proof.

A regular tetrahedron centered at $(0, 0, 0)$ with each face an equilateral triangle of side length $\frac{\sqrt{6}}{2}$



Consider the position of face $ACD \to A'C'D'$ for each rigid motion of the tetrahedron. The point A may assume 4 distinct locations. Once A is fixed, C may assume one of 3 remaining distinct locations. Once A and C are chosen, D may assume only 1 distinct location. So the order is $4 \times 3 \times 1 = 12$. The group of rigid rotations is given by $\rho_A = \left(\begin{smallmatrix} A & B & C & D \\ A & D & B & C \end{smallmatrix}\right)$, $\rho_A^2 = \left(\begin{smallmatrix} A & B & C & D \\ A & C & D & B \end{smallmatrix}\right)$, $\rho_B = \left(\begin{smallmatrix} A & B & C & D \\ C & B & D & A \end{smallmatrix}\right)$, $\rho_B^2 = \left(\begin{smallmatrix} A & B & C & D \\ D & B & A & C \end{smallmatrix}\right)$, $\rho_C = \left(\begin{smallmatrix} A & B & C & D \\ B & D & C & A \end{smallmatrix}\right)$, $\rho_C^2 = \left(\begin{smallmatrix} A & B & C & D \\ D & A & C & B \end{smallmatrix}\right)$, $\rho_D = \left(\begin{smallmatrix} A & B & C & D \\ C & A & B & D \end{smallmatrix}\right)$, $\rho_D^2 = \left(\begin{smallmatrix} A & B & C & D \\ B & C & A & D \end{smallmatrix}\right)$, $\rho_{AB,BC} = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$, $\rho_{AC,BD} = \left(\begin{smallmatrix} A & B & C & D \\ C & D & A & B \end{smallmatrix}\right)$, $\rho_{AD,BC} = \left(\begin{smallmatrix} A & B & C & D \\ D & C & B & A \end{smallmatrix}\right)$, and $id = \left(\begin{smallmatrix} A & B & C & D \\ A & B & C & D \end{smallmatrix}\right)$. By Proposition 5.8, $A_4 \subset S_4$ is of order $\frac{4!}{2} = 12$ and A_4 is given by

$$\begin{aligned} A_4 &= \{id, (12)(13), (12)(14), (12)(34), (13)(12), (13)(14), (13)(24), (14)(12), (14)(13), (14)(23), (23)(24), (24)(23)\} \quad \text{by definition} \\ &= \{(24)(23), (23)(24), (14)(13), (13)(14), (14)(12), (12)(14), (13)(12), (12)(13), (12)(34), (13)(34), (14)(23), id\} \quad \text{by reordering} \\ &= \{(234), (243), (134), (143), (124), (142), (123), (132), (12)(34), (13)(24), (14)(23), id\} \quad \text{as } (a_i a_k)(a_i a_j) = (a_i a_k a_j). \end{aligned}$$

Notice this matches the set A_4 as listed in Chapter 5, **Example 8.**

Since the order of the rigid motions of a tetrahedron equals the order of A_4, to show that the two groups are equivalent we must show that every rigid motion of a tetrahedron is the product even number of permutations. Label A, B, C, D as $1, 2, 3, 4$ respectively. Then

- (234) corresponds to ρ_A ((243) to ρ_A^2)
- (134) corresponds to ρ_B ((143) to ρ_B^2)
- (124) corresponds to ρ_C ((142) to ρ_C^2)
- (123) corresponds to ρ_D ((132) to ρ_D^2)
- $(12)(34)$ corresponds to $\rho_{AB,BC}$
- $(13)(24)$ corresponds to $\rho_{AC,BD}$
- $(14)(23)$ corresponds to $\rho_{AD,BC}$
- The identity id corresponds to itself

Notice every rigid motion of is the product of an even number of permutations as for each $x \in$ {group of rigid motions of a tetrahedron}, $x \in A_4$. So the group of rigid motions of a tetrahedron is the same as A_4 as

$$\begin{aligned} \{\text{group of rigid motions of a tetrahedron}\} &= \left\{\rho_A, \rho_A^2, \rho_B, \rho_B^2, \rho_C, \rho_C^2, \rho_D, \rho_D^2, \rho_{((AB)(BC))}, \rho_{((AC)(BD))}, \rho_{((AD)(BC))}\right\} \\ &\quad \text{from way above} \\ &= \{(234), (243), (134), (143), (124), (142), (123), (132), (12)(34), (13)(24), (14)(23), id\} \\ &\quad \text{from above} \\ &= \{id, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\} \\ &\quad \text{by reordering} \\ &= A_4 \quad \text{as listed in Chapter 5, } \textbf{Example 8.} \text{ and above} \end{aligned}$$

□

Exercise 5.19. Prove that D_n is non-abelian for $n \geq 3$.

Proof. By Theorem 5.10, we know that the group D_n consists of all products of the two elements r and s satisfying the relations

$$\begin{aligned} r^n &= id \\ s^2 &= id \\ srs &= r^{-1} \end{aligned}$$

for $n \geq 3$.

Let $n \geq 3$ and label r, s such that $r^n = id$ and $s^2 = id$, which is certainly possible by Theorem 5.10. Now assume for a contradiction of D_n is abelian. Then

$$\begin{aligned} srs = (sr)s &= (rs)s \quad \text{by assumption that } D_3 \text{ is abelian} \\ &= r(ss) = rs^2 \quad \text{by associativity of elements in } D_n \text{ as } D_n \text{ is a sub-group of } S_n \text{ by Theorem 5.9} \\ &= r(id) \quad \text{by Theorem 5.15 and chose of } s \in D_n \\ &= r \end{aligned}$$

However $srs = r$ is a contradiction to Theorem 5.10 that $srs = r^{-1}$ as this would imply

$$\begin{aligned} rr^{-1} = id &\implies r(srs) = id \quad \text{by Theorem 5.10} \\ &\implies r(r) = id \quad \text{by calculation above that } srs = r \\ &\implies r^2 = id \end{aligned}$$

and $r^2 = id$ cannot happen for $r \in D_n$ for $n \geq 3$ as r is necessarily of order n and $2 < n$. So our original assumption that D_n is abelian must be false, so D_n must be non-abelian for $n \geq 3$. ☐

Exercise 5.23. If σ is a cycle of odd length, prove that σ^2 is also a cycle.

Proof. Let $\sigma = (\sigma_1, \ldots, \sigma_k)$ for some odd integer k. Then σ may be written as $\sigma = (\sigma_1\sigma_k)(\sigma_1\sigma_{k-1})\cdots(\sigma_1\sigma_3)(\sigma_1\sigma_2)$, a finite product of transpositions. Then

$$\begin{aligned} \sigma^2 &= ((\sigma_1\sigma_k)(\sigma_1\sigma_{k-1})\cdots(\sigma_1\sigma_3)(\sigma_1\sigma_2))^2 \\ &= [(\sigma_1\sigma_k)(\sigma_1\sigma_{k-1})\cdots(\sigma_1\sigma_3)(\sigma_1\sigma_2)]\,[(\sigma_1\sigma_k)(\sigma_1\sigma_{k-1})\cdots(\sigma_1\sigma_3)(\sigma_1\sigma_2)] \quad \text{by definition of exponentiation} \end{aligned}$$

Then σ^2 is given by $\sigma^2(\sigma_\ell) = \sigma(\sigma(\sigma_\ell)) = \sigma(\sigma_{\ell+1}) = \sigma_{\ell+2}$ for $\ell = 1, 2, \ldots, k-2$. So $\sigma^2 : \sigma_1 \mapsto \sigma_3$, and $\sigma^2 : \sigma_3 \mapsto \sigma_5$, and eventually we will arrive at $\sigma^2 : \sigma_{k-2} \mapsto \sigma_k$ as k is an odd number. Then $\sigma^2(\sigma_k) = \sigma(\sigma_1) = \sigma_2$. $\sigma^2(\sigma_\ell) = \sigma(\sigma(\sigma_\ell)) = \sigma(\sigma_{\ell+1}) = \sigma_{\ell+2}$ for $\ell = 1, 2, \ldots, k-2$. So $\sigma^2 : \sigma_2 \mapsto \sigma_4$, and $\sigma^2 : \sigma_4 \mapsto \sigma_6$, and eventually we will arrive at $\sigma^2 : \sigma_{k-3} \mapsto \sigma_{k-1}$ as k is an odd number so $k-3$ is even. Then $\sigma_{k-1} \mapsto \sigma_1$ as $\sigma_1 \mapsto \sigma_3$ as before. So $\sigma^2 = (\sigma_3, \sigma_5, \ldots, \sigma_{k-2}, \sigma_k, \sigma_2, \sigma_4, \ldots, \sigma_{k-1}, \sigma_1)$ is a cycle. ☐

Exercise 5.26. Prove that any element in S_n can be written as a finite product of the following permutations.

(a) $(12), (13), \ldots, (1n)$ (b) $(12), (23), \ldots, (n-1, n)$ (c) $(12), (12 \ldots n)$

Proof. Let $\sigma \in S_n$.

(a) Then by Theorem 5.3, σ can be written as the product of disjoint cycles $\sigma = a_1 a_2 \cdots a_k$. For $i = 1, \ldots k$, let $a_i = (\alpha_{i_1}, \ldots, \alpha_{i_\ell})$. Then $a_i : \alpha_{i_m} \mapsto \alpha_{i_{m+1}}$ for $m = 1, \ldots, \ell - 1$ and $a_i : \alpha_{i_\ell} \mapsto \alpha_1$. Consider $a_i' = (\alpha_{i_1} a_{i_\ell})(a_{i_1} a_{i_{\ell-1}})\cdots(a_{i_1} a_{i_3})(a_{i_1} a_{i_2})$, which is a product of $(12), (13), \ldots, (1n)$. Then $a_i' : \alpha_{i_m} \mapsto \alpha_{i_{m+1}}$ for $m = 1, \ldots, \ell - 1$ and $a_i' : \alpha_{i_\ell} \mapsto \alpha_1$. So $a_i = a_i'$ for all i. So

$$\begin{aligned} \sigma &= a_1 a_2 \cdots a_k \\ &= a_1' a_2' \cdots a_k' \quad \text{as } a_i = a_i' \\ &= ((a_{1_1} a_{1_\ell})\cdots(a_{1_1} a_{1_2}))\,((a_{2_1} a_{2_\ell})\cdots(a_{2_1} a_{2_2}))\cdots((a_{k_1} a_{k_\ell})\cdots(a_{k_1} a_{k_2})) \end{aligned}$$

which is a product of $(12), (13), \ldots, (1n)$ ☐

Exercise 6.1. Suppose that G is a finite group with an element g of order 5 and an element h of order 7. Why must $|G| \geq 35$?

Proof. Let $g = (g_1 g_2 g_3 g_4 g_5)$ and $h = (h_1 h_2 h_3 h_4 h_5 h_6 h_7)$. By Corollary 6.6, the orders of g and h, (5 and 7 respectively) must divide the number of elements in G, so $|G|$ is 35 at least, or larger. ☐

Exercise 6.3. Prove or disprove: Every sub-group of the integers has finite index.

Proof. This is false. Let $H = \{1\}$. Then H is a sub-group of $\mathbb{Z}$ and $[\mathbb{Z} : H] = \#\mathcal{L}_H = \#\{g \cdot 1 : g \in \mathbb{Z}\} = \infty$ ☐

Exercise 6.5. List the left and right co-sets of the sub-groups in each of the following.

(a) $\langle 8\rangle$ in $\mathbb{Z}_{24}$ (b) $\langle 3\rangle$ in $U(8)$ (d) A_4 in S_4 (f) D_4 in S_4

Solution.

(a) The left and right co-sets of $\langle 8\rangle$ in $\mathbb{Z}_{24}$ are the same as addition is commutative in $\mathbb{Z}_{24}$. So the left and right co-set are

$$\begin{aligned}
0+\langle 8\rangle &= 8+\langle 8\rangle = 16+\langle 8\rangle = \{0,8,16\}\\
1+\langle 8\rangle &= 9+\langle 8\rangle = 17+\langle 8\rangle = \{1,9,17\}\\
2+\langle 8\rangle &= 10+\langle 8\rangle = 18+\langle 8\rangle = \{2,10,18\}\\
3+\langle 8\rangle &= 11+\langle 8\rangle = 19+\langle 8\rangle = \{3,11,19\}\\
4+\langle 8\rangle &= 12+\langle 8\rangle = 20+\langle 8\rangle = \{4,12,20\}\\
5+\langle 8\rangle &= 13+\langle 8\rangle = 21+\langle 8\rangle = \{5,13,21\}\\
6+\langle 8\rangle &= 14+\langle 8\rangle = 22+\langle 8\rangle = \{6,14,22\}\\
7+\langle 8\rangle &= 15+\langle 8\rangle = 23+\langle 8\rangle = \{6,14,22\}
\end{aligned}$$

(b) The left and right co-sets of $\langle 3\rangle$ in $U(8)$ are the same as multiplication is commutative in $\mathbb{Z}_8$. $U(8) = \{1,3,5,7\}$ and $\langle 3\rangle = \{1,3\}$, so the left and right co-sets are:

$$\begin{aligned}
1\cdot\{3,1\} &= \{3,1\}\\
3\cdot\{3,1\} &= \{1,3\}\\
5\cdot\{3,1\} &= \{7,5\}\\
7\cdot\{3,1\} &= \{5,7\}
\end{aligned}$$

(d) The order of A_4 in S_4 is 2, so the left co-sets equals the right co-sets. So the left and right co-sets of

$$A_4 = \{(234),(243),(134),(143),(124),(142),(123),(132),(12)(34),(13)(24),(14)(23),id\}$$

are

$$\begin{aligned}
A_4 &= \{(234),(243),(134),(143),(124),(142),(123),(132),(12)(34),(13)(24),(14)(23),id\}\\
(12)A_4 &= \{(1234),(1243),(1342),(1432),(24),(14),(23),(34),(1324),(1423),(12)\}
\end{aligned}$$

(f) From Chapter 5 **Example 9.**, $D_4 = \{(1234),(13)(24),(1432),id,(24),(13),(12)(34),(14)(32)\}$. So the left co-sets are

$$\begin{aligned}
D_4 &= \{(1234),(13)(24),(1432),id,(24),(13),(12)(34),(14)(32)\}\\
(12)D_4 &= \{(12),(234),(2413),(143),(34),(1423),(132),(124)\}\\
(14)D_4 &= \{(14),(123),(1342),(243),(1243),(23),(134),(142)\}
\end{aligned}$$

□