My notes on STAT 330 – Mathematical Statistics

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Preliminaries

Sample space The set of all possible outcomes of an experiment, S.

σ-field Let S be a sample space with power set $\mathbb{P}(S)$. A collection of sets $\mathcal{B} \subseteq \mathbb{P}(S)$ is a σ-field / σ-algebra on S if

1. $\emptyset, S \in \mathcal{B}$
2. $\mathcal{B}$ closed under complementation
3. $\mathcal{B}$ closed under countable unions

Measurable space Let S be a sample space and $\mathcal{B}$ be a σ-algebra. Then the pair $(S, \mathcal{B})$ is a measurable space.

Measure Let X be a set and Σ be a σ-algebra over X. A function

$$\mu : X \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

is called a measure if it satisfies:

- Non-negativity: For all $E \in \Sigma$, $\mu(E) \geq 0$
- Null empty set: $\mu(\emptyset) = 0$
- Countable additivity: For all countable collections $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets[1] in Σ,

$$\mu\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Definition from wikipedia

Probability Measure Let S be a sample space for a random experiment. Let $\mathcal{B} = \{A_1, A_2, \cdots\}$ be a σ-field on S. A probability measure is a function $\Pr : \mathcal{B} \to [0, 1]$ that satisfies:

- $\Pr(A) \geq 0$ for all $A \in \mathcal{B}$.
- $\Pr(S) = 1$
- If $A_1, A_2, \cdots \in \mathcal{B}$ are disjoint events then $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$.

Measure space A measure space is a measurable space equipped with a measure

Probability space A probability space is a measureable space equipped with a probability measure.

Theorem: Boole's Inequality If $A_1, A_2, \cdots$ is a sequence of events, then $\Pr\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \Pr(A_i)$.

Proof: In the case where each A_i are independent, the result follows by the definition of a probability measure. Suppose any A_i, A_j are not independent. Then we have that $\Pr(A_i \cup A_j) = \Pr(A_i) + \Pr(A_j) - \Pr(A_i \cap A_j)$ which is certainly less than the case where they are independent. The proof follows by induction.

[1] For any two elements in the set, them being disjoint is pairwise disjoint.

Theorem: Bonferroni's Inequality If $A_1, \cdots, A_k$ are events, then $\Pr\left(\cap_{i=1}^k A_i\right) \geq 1 - \sum_{i=1}^k \Pr(A_i^C)$.

Proof: Recall De Morgan's Laws, which states that $(A \cup B)^C = A^C \cap B^C$, and also that $(A \cap B)^C = A^C \cup B^C$. Then

$$\begin{aligned}
\Pr\left(\cap_{i=1}^k A_i\right) &= \Pr\left(\cap_{i=1}^k (A_i^C)^C\right) \\
&= \Pr\left(\cup_{i=1}^k A_i^C\right) \\
&\leq \sum_{i=1}^k \Pr(A_i^C) && \text{Boole's inequality} \\
&= 1 - \sum_{i=1}^k \Pr(A_i)
\end{aligned}$$

The result has been proven.

Theorem: Continuity property If $A_1 \subset A_2 \subset \cdots$ is a sequence where $A = \cup_{i=1}^\infty A_i$ then $\lim_{n\to\infty} \Pr(\cup_{i=1}^n A_i) = \Pr(A)$.

Proof: We note that for any A_i, A_{i+1} we have that $\Pr(A_i \cap A_{i+1}) = \Pr(A_{i+1})$. Consider some finite n. Then we have that $\Pr(\cap_{i=1}^n A_i) = A_n$. Assuming that $\lim_{n\to\infty} A_n = A$, then $\lim_{n\to\infty} \Pr(\cup_{i=1}^n A_i) = \lim_{n\to\infty} A_n = A$.

Random Variable Consider a probability space $(S, \mathcal{B}, \Pr)$. The function $X : S \to \mathbb{R}$ is called a *Random Variable* if $\Pr(X \leq x) = \Pr(\{\omega \in S \mid X(\omega) \leq x\})$ is valid for all $x \in \mathbb{R}$.

Cumulative Distribution Function The *cumulative distribution function* (CDF) of a random variable X is defined to be

$$F(x) = \Pr(X \leq x)$$

for all $x \in \mathbb{R}$.

Furthermore, it has the following properties:

- F is non-decreasing (since if $A_i \subset A_{i+1}$ then $\Pr(A_i) \leq \Pr(A_{i+1})$).
- $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.
- F is right-continuous, by convention. I.e., $\lim_{x\to a^+} F(x) = F(a)$.