PM 347 [1] Sept 20, 2017

Note: • HW 1 Q5: the group is $(\mathbb{Z}_n, \oplus)$.
• Symmetric groups = permutation group.

Two important group actions:

(d) Groups acting on themselves by left multiplication.

Let G be a group. G acts on itself by $g \in G, a \in G \to g \cdot a = ga$.
by associativity of group operations.

Exercise: Multiplication on the right is NOT a group action.

We have a group corresponding permutation representation
$$G \to S_G.$$

Prop^n (P8)

$G \to S_G$ is injective.

pf: let $g, h \in G$ assume $\sigma_g = \sigma_h$. In particular, $\sigma_g(1) = \sigma_h(1)$. Since $\sigma_g(1) = g \cdot 1 = g$ and $\sigma_h(1) = h \cdot 1 = h$, we have $g = h$.

In particular, the kernel of this action is trivial. i.e., the subgroup $\{1\} \leq G$.

Def^n (Faithful)

A group action G on A is called faithful if the kernel of the action is trivial. i.e., the only group element fixing all of A pointwise is 1_G.

Def^n (Image)

Suppose $\varphi: G \to H$ is a group homomorphism. Th

Lem (L9)

Suppose $\varphi: G \to H$ is a group homomorphism.

(a) $\mathrm{Im}(\varphi) \leq H$

(b) φ injective $\Rightarrow$ φ induces isomorphism $G \xrightarrow{\simeq} \mathrm{Im}(\varphi)$.

Pf: a) Let $h_1, h_2 \in \mathrm{Im}(\varphi)$. Then $\exists$ $g_1, g_2 \in G \ni \varphi(g_1) = h_1$, $\varphi(g_2) = h_2$.

$$h_1 h_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2)$$

Since $g_1 g_2 \in G$, $h_1 h_2 \in \mathrm{Im}(\varphi)$.

Also nonempty: $\varphi(1_G) = 1_H$. Then $\mathrm{Im}(\varphi) \leq H$.

b) Let $\mathrm{Im}(\varphi)$ be a group and $\varphi: G \to \mathrm{Im}(\varphi)$ is a bijective group homomorphism. Hence $G \simeq \mathrm{Im}(\varphi)$.

Cor (10) [Cayley's Theorem].

Every group is isomorphic to a subgroup of some permutation group. Moreover, if G is finite (i.e., $|G| = n$) then G is isomorphic to a subgroup S_n.

Pf. Consider the action of g on itself by left multiplication. By P8 This gives us an injective group homomorphism $\varphi: G \to S_G$. By L9 $G \simeq \mathrm{Im}(\varphi) \leq S_G$.

Note if $|G| = n$, then $S_G \simeq S_n$ (exercise).

Examples of group actions cont'd:

(e) Groups acting on themselves by conjugation.

Def^n (conjugation).
Let G be a group with $g, h \in G$. The <u>conjugate</u> of h by g is the element ghg^{-1}.

<u>Rem</u>. if G abelian, $ghg^{-1} = h$ so conjugation does nothing.

(e): Conjugation is an action of G on itself. i.e., given $g \in G$, $a \in G$; $g \cdot a := gag^{-1}$.

Given $g, h \in G$; $g \cdot (h \cdot a) = g \cdot (hah^{-1}) = g(hah^{-1})g^{-1}$

$$= (gh)\,a(h^{-1}g^{-1}) = (gh)\,a\,(gh)^{-1}$$
$$= (gh) \cdot a$$

In particular, $1 \cdot a = 1a1^{-1} = a$.

We get another permutation representation

$$\Psi : G \to S_G.$$

coming from G acting on itself by conjugation.

$gag^{-1} = a \iff ga = ag$.

$$\mathrm{Ker}(\Psi) = \{ g \in G \mid ga = ag \text{ for all } a \in G \}.$$

If g is abelian this is the trivial action.

Def^n (centre)
For any group G,

$$Z(G) = \{ g \in G \mid gh = hg \;\; \forall h \in G \}$$

is called the <u>centre</u> of G.

<u>Rem</u>. (i) If G abelian, then $Z(G) = G$.

(ii) $Z(G) \trianglelefteq G$ since $Z(G)$ is the kernel of the action of G on itself by conjugation.