PM 347. Sept 18, 2017

Group Actions.

Def^n: A (left) group Action of a group G on a set A ~~of a set A~~ is a function $G \times A \to A$, denoted by $\cdot$ (dot), i.e., $(g,a) \mapsto g\cdot a \in A$ for ~~...~~ $g\in G$, $a \in A$. satisfying

(i) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$. multiplication in G / group action.

(ii) $1_G \cdot a = a$

$\forall\ g_1, g_2 \in G,\ a \in A$.

If ~~g~~ G acts on A, for each $g\in G$ we get

$$\sigma_g : A \to A$$
$$a \mapsto g\cdot a.$$

Lemma (L6): G acts on A, $g \in G$. Then $\sigma_g : A \to A$ is a bijection.

Pf: Injectivity: let $a, b \in A$. Suppose $\sigma_g(a) = \sigma_g(b)$. Then, $g\cdot a = g \cdot b$. Then, $g^{-1}(g\cdot a) = g^{-1}(g\cdot b)$. Then we use the "compatability with multiplication" (axiom 1 of group actions):

$(g^{-1}g)\cdot a = (g^{-1}g)\cdot b \Rightarrow 1\cdot a = 1 \cdot b$

So $a = b$, then σ_g is injective.

Surjectivity: ~~Let $b \in A$. Let $b = g^{-1}\cdot a$. Then $\sigma_g(b) = g\cdot b = g\cdot(g^{-1}\cdot a) = (gg^{-1})\cdot a$.~~

Given $b \in A$, let $a := g^{-1}\cdot b$.

$\sigma_g(a) = \sigma_g(g^{-1}\cdot b) = g\cdot(g^{-1}\cdot b) = (gg^{-1})\cdot b$
$= b$. so σ_g is surjective.

Warning: Do not confuse the action of G on A and group operation in G. i.e., we often write ga instead of $g\cdot a$ (clear from context, when A ~~...~~ is separate from group G).

Recall for any set A, $S_A :=$ group of bijections $\sigma : A \to A$ ~~where~~ under composition. If G acts on A we have just defined a function

$$\begin{aligned} G &\to S_A \\ g &\mapsto \sigma_g \end{aligned} \quad \text{by Lemma 6.}$$

prop^{n} (P7) [Permutation Representation].
The function $G \to S_A$ given by $g \mapsto \sigma_g$ is a group homomorphism.

<u>Pf</u>: Use the def^{n} of a group homomorphism. We need to check for any $g, h \in G$: $\sigma_{gh} = \sigma_g \circ \sigma_h$, (i.e., both sides are permutations of A).

Let $a \in A$ be arbitrary. $\sigma_{gh}(a) = (gh) \cdot a = \underbrace{g}_{\in G} \cdot (h \cdot a)$

but $g \cdot (h \cdot a) = g \cdot \sigma_h(a) = \sigma_g(\sigma_h(a))$ $\square$.

Exercise: Prove the converse of P7. i.e., Suppose G is a group, A is a set and $\varphi : G \to S_A$ is a group homomorphism. Then we get an action of G on A. by $g \cdot a := \varphi(g)(a) \in A$. (phi of g evaluated at a). Moreover, the associated $G \to S_A$; $g \mapsto \sigma_g$ is just φ. i.e., $\forall g \in G, a \in A$ ~~$\sigma_g(a) = \varphi(a)$~~. $\sigma_g(a) = \varphi(g)(a)$.

Examples: [group actions].
(a) Trivial action. Every group G acts on every set A by $g \cdot a = a$, so $g_1 \cdot (g \cdot a) = g_1 \cdot a = a = (g_1 g_2) \cdot a$. The associated permutation representation $G \to S_A$. is the trivial homomorphism.

Def^{n} Let G, H be groups. The <u>trivial homomorphism</u> $\varphi : G \to H$ is
$$\varphi(g) = 1_H$$
$\forall g \in G$. That is, $\ker(\varphi) = G$. i.e., $\sigma_g = id_A : A \to A$; $a \mapsto a$ $\forall g \in G$.

Def^{n}: Suppose G acts on A and $\varphi: \underset{g \mapsto \sigma_g}{G \to S_A}$ is the corresponding homomorphism.
We call $\ker(\varphi) \leq G$ the <u>kernel of the action of G on A</u>.
It is the set of elements in G that act trivially on A.

Examples [group actions]

(ii) Given any set A, S_A acts on A by $\underset{\tau \mapsto \sigma_\tau}{\sigma \cdot a := \sigma(a)}$.
The corresponding homomorphism $S_A \to S_A$ is the identity homomorphism. i.e., $\sigma_\tau = \tau$ for any $\tau \in S_A$.

(iii) Suppose V is an $\mathbb{R}$-vector space. Then scalar multiplication
$$\mathbb{R}^\times \times V \to V \qquad \text{recall } \mathbb{R}^\times \text{ nonzero reals.}$$
$$(r, v) \mapsto rv.$$
e.g., $r(sv) = (rs)v$ is a vectorspace axiom.
So $(\mathbb{R}^\times, \times)$ acts on the vector space V.
The associated homomorphism $\mathbb{R}^\times \to S_V$ is injective if V is nontrivial (exercise w/ vectorspace properties).

Aside: Bad notation ~~last~~ last lecture
$$\mathbb{Z}_5^\times = \{1,2,3,4\} \text{ with } \otimes$$
But $$\mathbb{Z}_6 \setminus \{0\} \text{ with } \otimes \text{ is not a group.}$$
($\mathbb{Z}_6^*$ is something else, see A1).