Program : **B.Tech**

Subject Name: **Mathematics-I**

Subject Code: **BT-102**

Semester: **1st**

UNIT-I

Syllabus: Role's theorem, Mean value theorem, Expansion of functions by McLaren's and Taylor's for one and two variables, Partial Differentiation, Maxima & minima, Method of Lagrange's

Maclaurin's and Taylor's Expansion-

1.Use Maclaurin's theorem to expand f(x) from first principles

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

For Taylor's
$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots$$

If cannot show expressions involving derivatives, you should still continue the question to find the expansion!

2.Use standard expansions

a) Replacement method to obtain expansions like sin 2*x*, e^{-x}, etc.Some useful manipulations like $\ln(2+x) = \ln\left(2(1+\frac{x}{2})\right) = \ln 2 + \ln(1+\frac{x}{2})$, $\cos\left(2x + \frac{\pi}{4}\right)$ using compound angle formula,

$e^{\frac{\pi}{2}-x} = e^{\frac{\pi}{2}} e^{-x}$ .Use standard series to expand cos(ln(1+*x*) like in Q3.

b) Differentiating one expansion to obtain the expansion of another function.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots$$ Differentiate wrt *x*, we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots$$

If we differentiate the expansion of e^x, LHS and RHS stays the same! Interesting! Try it.We can also obtain the expansion of $\frac{1}{1+x}$ by differentiating the expansion of $\ln(1+x)$.

c) We can use the standard series to check the correctness of the terms too like in tutorial Q3, 7.

d) When *x* is small such that x^3 and higher powers are neglected.

sin x = x, cos x = $1-\frac{x^2}{2!}$ (From formula list)

tan x = x (this one can deduced conveniently from sin x / cos x = x / 1 = x)

Find expansions of such like in Q9 and 10. Useful technique: $\frac{1}{2-\theta}=(2-\theta)^{-1}$

3. Approximation gets better when there are more terms.

When more terms are used, the graph of its expansion will resemble more and more like the graph of the function near the region of x = 0.

4. Suppose $y=f(x)=1-\frac{1}{2}x+\frac{1}{8}x^2+\frac{3}{16}x^3+\ldots$

a) Find the linear approximation to the curve $y=f(x)$.

b) Find the tangent to the curve $y=f(x)$ at x = 0.

Both answers are $y=1-\frac{1}{2}x$ by obtaining it from the expansion above!

MAXIMUM/MINIMUM PROBLEMS

The following problems are maximum/minimum optimization problems. They illustrate one of the most important applications of the first derivative. Many students find these problems intimidating because they are "word" problems, and because there does not appear to be a pattern to these problems. However, if you are patient you can minimize your anxiety and maximize your success with these problems by following these guidelines :

GUIDELINES FOR SOLVING MAX./MIN. PROBLEMS

1. Read each problem slowly and carefully. Read the problem at least three times before trying to solve it. Sometimes words can be ambiguous. It is imperative to know exactly what the problem is asking. If you misread the problem or hurry through it, you have NO chance of solving it correctly.

2. If appropriate, draw a sketch or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out your thoughts.

3. Define variables to be used and carefully label your picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.

4. Write down all equations which are related to your problem or diagram. Clearly denote that equation which you are asked to maximize or minimize. Experience will show you that MOST optimization problems will begin with two equations. One equation is a "constraint" equation and the other is the "optimization" equation. The "constraint" equation is used to solve for one of the variables. This is then substituted into the "optimization" equation before differentiation occurs. Some problems may have NO constraint equation. Some problems may have two or more constraint equations.

5. Before differentiating, make sure that the optimization equation is a function of only one variable. Then differentiate using the well-known rules of differentiation.

6. Verify that your result is a maximum or minimum value using the first or second derivative test for extrema.

The following problems range in difficulty from average to challenging.

SOLUTIONS TO MAXIMUM/MINIMUM PROBLEMS

SOLUTION 1 : Let variables x and y represent two nonnegative numbers. The sum of the two numbers is given to be

$9 = x + y$, so that $y = 9 - x$.

We wish to MAXIMIZE the PRODUCT $P = x y^2$. However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting

$$P = x y^2$$

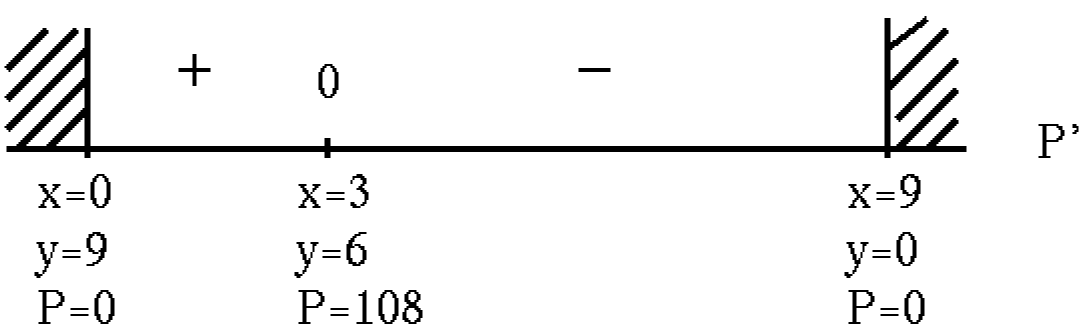
$$= x (9-x)^2 .$$

Now differentiate this equation using the product rule and chain rule, getting

$P' = x (2) (9-x)(-1) + (1) (9-x)^2 = (9-x) [-2x + (9-x)] = (9-x) [9-3x] = (9-x) (3)[3-x] = 0$

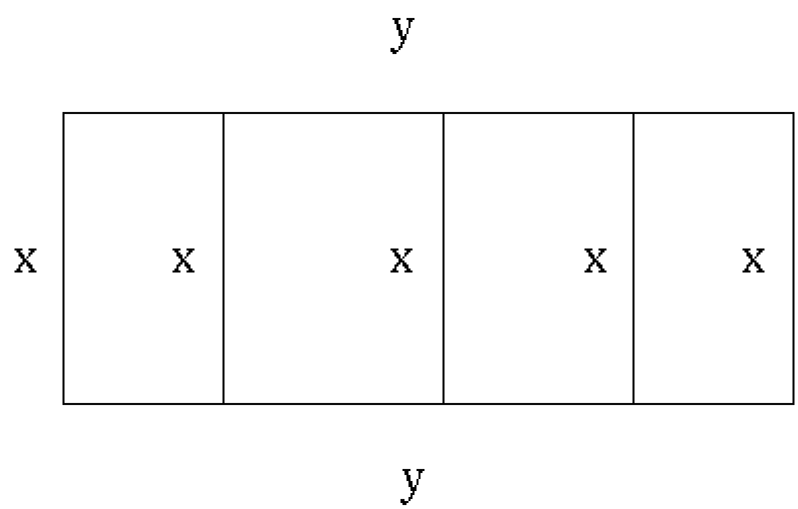
for x=9 or x=3 .

Note that since both x and y are nonnegative numbers and their sum is 9, it follows that $0 \le x \le 9$. See the adjoining sign chart for P' .



If x=3 and y=6 ,then P= 108 is the largest possible product.

SOLUTION 2 : Let variable x be the width of the pen and variable y the length of the pen.



The total amount of fencing is given to be 500 = 5 (width) + 2 (length) = $5x + 2y$,

so that $2y = 500 - 5x$

or $y = 250 - (5/2)x$.

We wish to MAXIMIZE the total AREA of the pen A = (width) (length) = $x\,y$.

However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting

$$A = x\,y$$

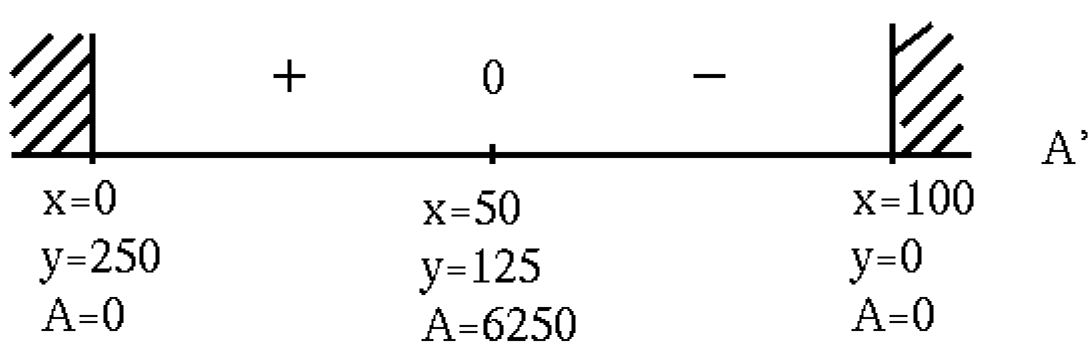
$$= x\,(\,250 - (5/2)x)$$

$$= 250x - (5/2)x^2\,.$$

Now differentiate this equation, getting $A' = 250 - (5/2)\,2x = 250 - 5x = 5\,(50 - x\,) = 0$

for x=50 .

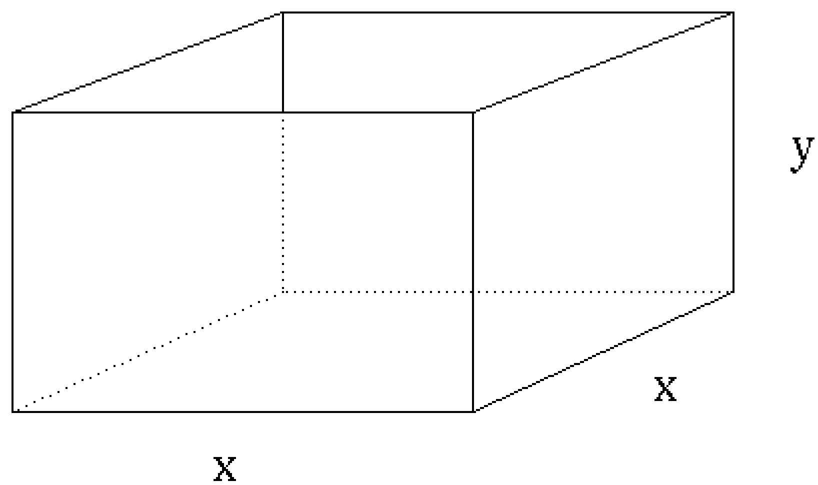
Note that since there are 5 lengths of x in this construction and 500 feet of fencing, it follows that $0 \le x \le 100$. See the adjoining sign chart for A' .



If x=50 ft. and y=125 ft. ,

then A = 6250 ft.^{2} is the largest possible area of the pen.

SOLUTION 3 : Let variable x be the length of one edge of the square base and variable y the height of the box.



The total surface area of the box is given to be 48 = (area of base) + 4 (area of one side) = $x^2 + 4(xy)$,

so that $4xy = 48 - x^2$

$$y = \frac{48 - x^2}{4x}$$

or

$$= \frac{48}{4x} - \frac{x^2}{4x}$$

$$= \frac{12}{x} - (1/4)x$$

.

We wish to MAXIMIZE the total VOLUME of the box V = (length) (width) (height) = $(x)(x)(y) = x^2 y$. However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting

$$V = x^2 y$$

$$= x^2 \left(\frac{12}{x} - (1/4)x\right)$$

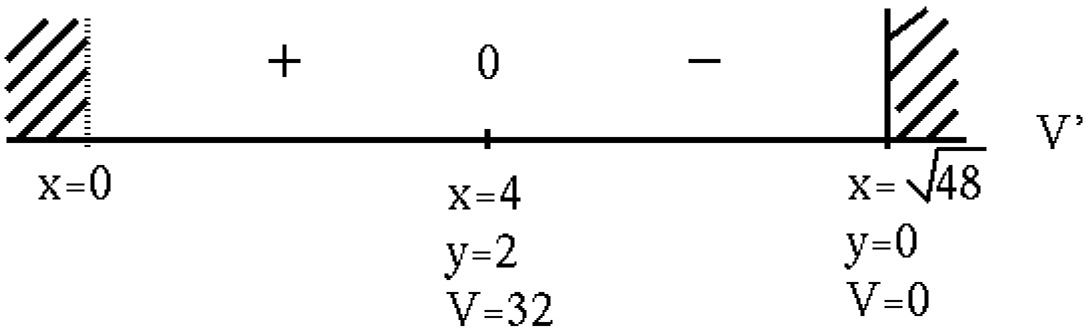
$$= 12x - (1/4)x^3 .$$

Now differentiate this equation, getting

$$V' = 12 - (1/4)3x^2 = 12 - (3/4)x^2 = (3/4)(16 - x^2) = (3/4)(4 - x)(4 + x) = 0$$

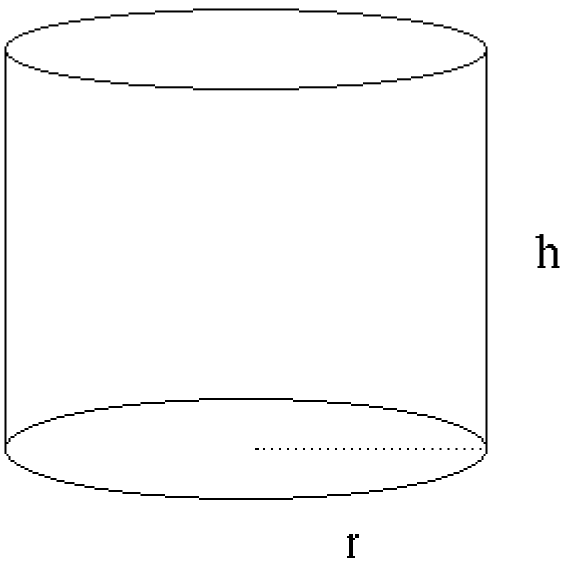
for x=4 or x=-4 .

But $x \neq -4$ since variable *x* measures a distance and *x* > 0 . Since the base of the box is square and there are 48 ft.^{2} of material, it follows that $0 < x \leq \sqrt{48}$. See the adjoining sign chart for *V*' .



If *x*=4 ft. and *y*=2 ft. ,then *V* = 32 ft.^{3} is the largest possible volume of the box.

SOLUTION 4 : Let variable *r* be the radius of the circular base and variable *h* the height of the cylinder.



The total surface area of the cylinder is given to be

$3\pi =$ (area of base) + (area of the curved side)

$= \pi r^2 + (2\pi r)h$,

so that

$2\pi r h = 3\pi - \pi r^2$

or

$$h = \frac{3\pi - \pi r^2}{2\pi r}$$

$$= \frac{3}{2r} - (1/2)r$$

.

We wish to MAXIMIZE the total VOLUME of the cylinder *V* = (area of base) (height) $= \pi r^2 h$.

However, before we differentiate the right-hand side, we will write it as a function of *r* only. Substitute for *h* getting $V = \pi r^2 h$

$$= \pi r^2 \left(\frac{3}{2r} - (1/2)r\right)$$

$$= (3/2)\pi r - (1/2)\pi r^3$$

.

Now differentiate this equation, getting

$$V' = (3/2)\pi - (1/2)\pi 3r^2$$

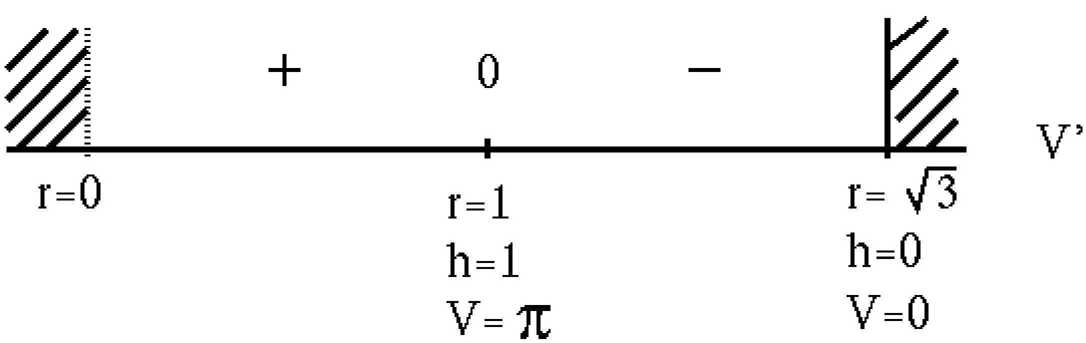
$$= (3/2)\pi(1 - r^2)$$

$$= (3/2)\pi(1 - r)(1 + r)$$

= 0

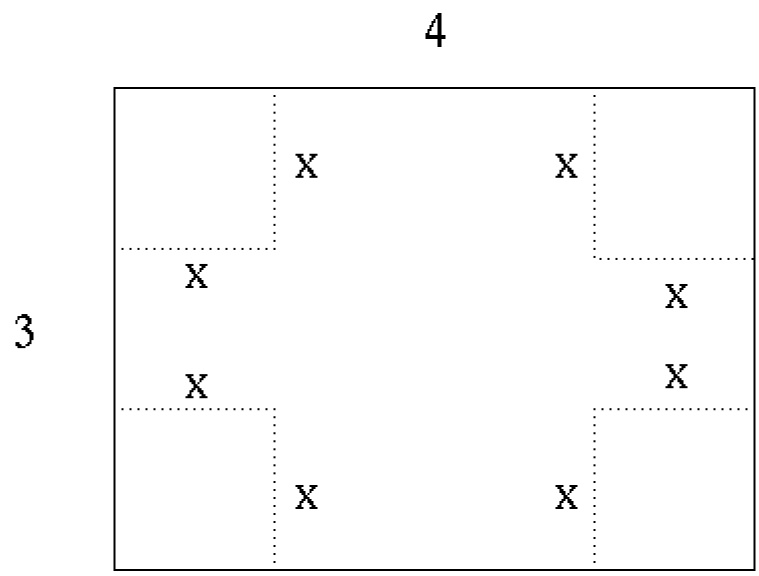
for *r*=1 or *r*=-1 .

But $r \neq -1$ since variable *r* measures a distance and *r* > 0 . Since the base of the box is a circle and there are 3π ft.^{2} of material, it follows that $0 < r \leq \sqrt{3}$. See the adjoining sign chart for *V'* .

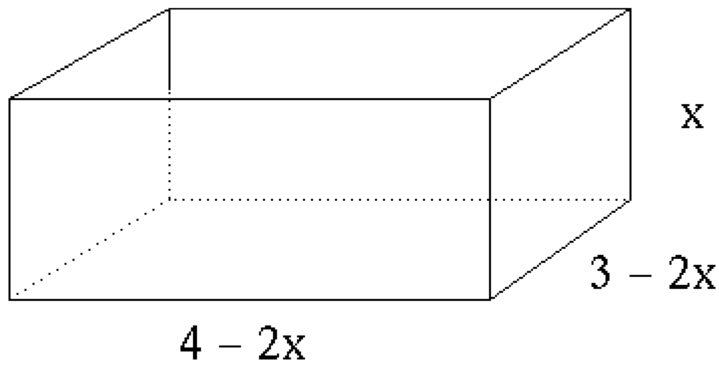


If *r*=1 ft. and *h*=1 ft. , then $V = \pi$ ft.^{3} is the largest possible volume of the cylinder.

SOLUTION 5 : Let variable x be the length of one edge of the square cut from each corner of the sheet of cardboard.



After removing the corners and folding up the flaps, we have an ordinary rectangular box.



We wish to MAXIMIZE the total VOLUME of the box

$$V = \text{(length) (width) (height)} = (4-2x)(3-2x)(x) .$$

Now differentiate this equation using the triple product rule, getting

$$V' = (-2)(3-2x)(x) + (4-2x)(-2)(x) + (4-2x)(3-2x)(1)$$

$$= -6x + 4x^2 - 8x + 4x^2 + 4x^2 - 14x + 12$$

$$= 12x^2 - 28x + 12$$

$$= 4(3x^2 - 7x + 3)$$

$$= 0$$

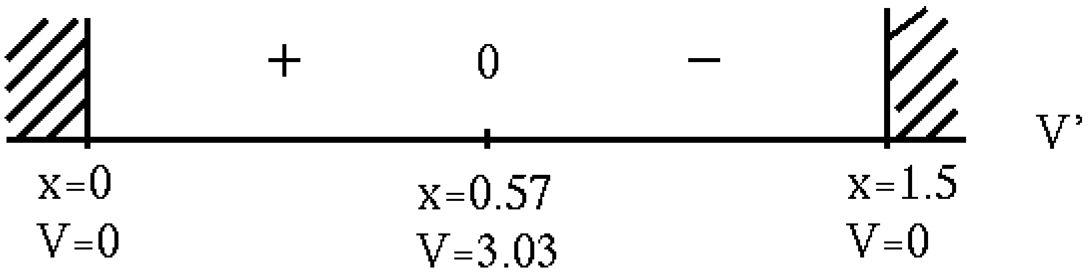
for (Use the quadratic formula.)

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(3)}}{2(3)} = \frac{7 \pm \sqrt{13}}{6}$$,

i.e., for

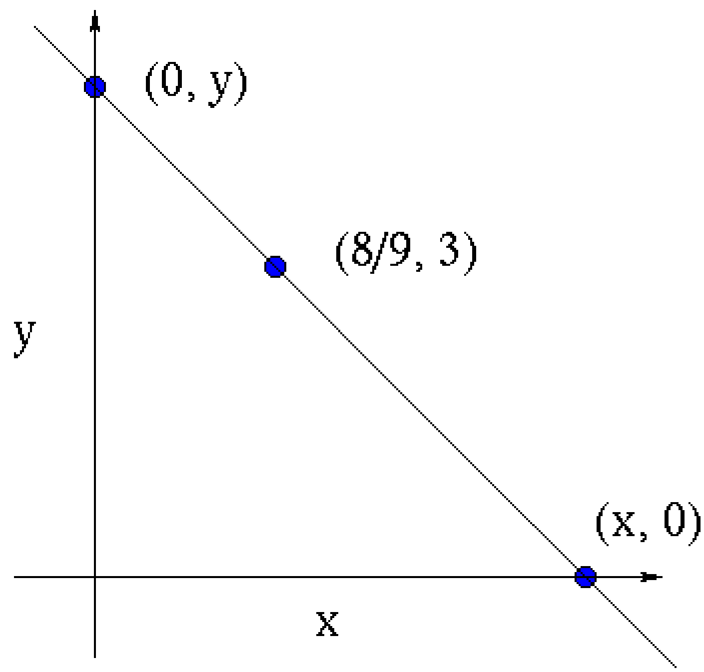
$x \approx 0.57$ or $x \approx 1.77$.

But $x \neq 1.77$ since variable *x* measures a distance. In addition, the short edge of the cardboard is 3 ft., so it follows that $0 \leq x \leq 1.50$. See the adjoining sign chart for *V*' .

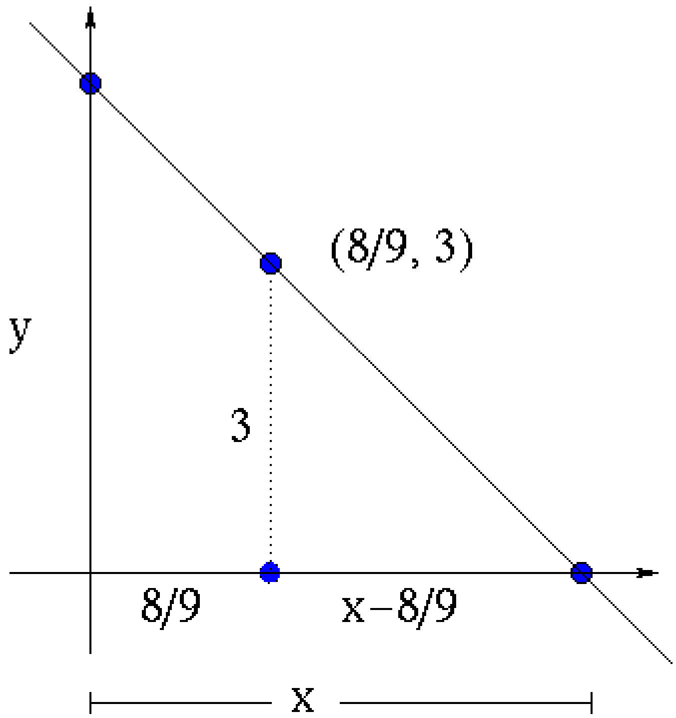


If $x \approx 0.57$ ft. , then $V \approx 3.03$ ft.^{3} is largest possible volume of the box.

SOLUTION 6 : Let variable *x* be the x-intercept and variable *y* the y-intercept of the line passing throught the point (8/9, 3) .



Set up a relationship between *x* and *y* using similar triangles.



One relationship is

$$\frac{y}{x} = \frac{3}{x - 8/9},$$

so that

$$y = \frac{3x}{x - 8/9}.$$

We wish to MINIMIZE the length of the HYPOTENUSE of the triangle

$$H = \sqrt{x^2 + y^2}.$$

However, before we differentiate the right-hand side, we will write it as a function of *x* only. Substitute for *y* getting

$$H = \sqrt{x^2 + y^2}$$

$$= \sqrt{x^2 + \left(\frac{3x}{x - 8/9}\right)^2}.$$

Now differentiate this equation using the chain rule and quotient rule, getting

1

$$H' = (1/2)\left(x^2 + \left(\frac{3x}{x-8/9}\right)^2\right)^{-1/2}\left\{2x + 2\left(\frac{3x}{x-8/9}\right)\frac{(x-8/9)(3)-(3x)(1)}{(x-8/9)^2}\right\}$$

(Factor a 2 out of the big brackets and simplify.)

$$= (1/2)\left(x^2 + \left(\frac{3x}{x-8/9}\right)^2\right)^{-1/2}(2)\left\{x + \frac{3x}{(x-8/9)}\frac{-8/3}{(x-8/9)^2}\right\}$$

$$= \frac{x - \dfrac{8x}{(x-8/9)^3}}{\sqrt{x^2 + \left(\dfrac{3x}{x-8/9}\right)^2}}$$

= 0 ,

so that (If $\frac{A}{B} = 0$, then A=0 .)

$$x - \frac{8x}{(x-8/9)^3} = 0 .$$

By factoring out x , it follows that

$$x\left\{1 - \frac{8}{(x-8/9)^3}\right\} = 0$$

, so that (If AB= 0 , then A=0 or B=0 .)

x=0

(Impossible, since x> 8/9. Why ?) or

$$1 - \frac{8}{(x-8/9)^3} = 0 .$$

Then

$$1 = \frac{8}{(x-8/9)^3} ,$$

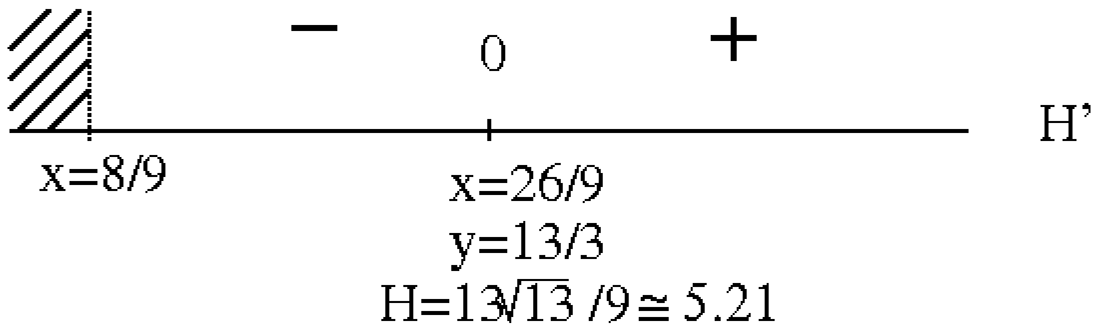
so that

$$(x-8/9)^3 = 8 ,$$

$$x-8/9 = 2 ,$$

and

$x = 26/9$. See the adjoining sign chart for H' .



If

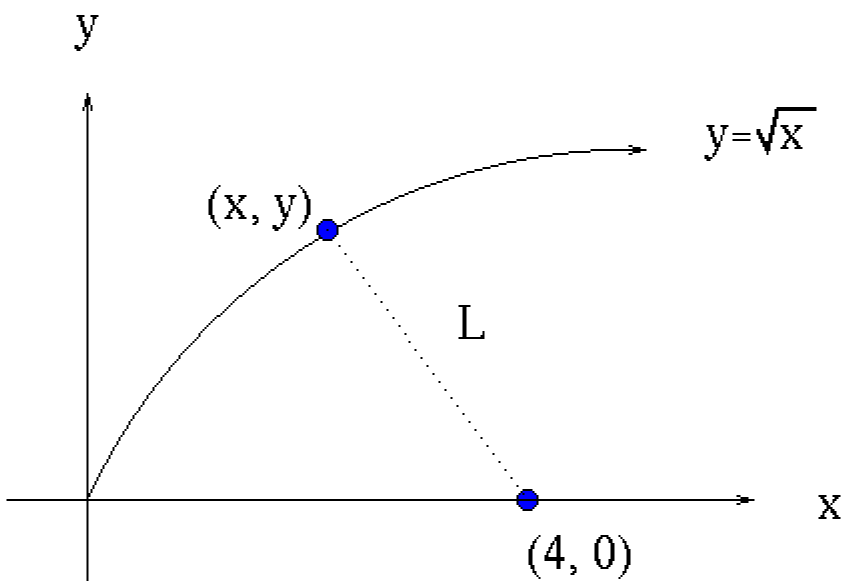
$x = 26/9$ and $y=13/3$,

then

$$H = \frac{13\sqrt{13}}{9} \approx 5.21$$

is the shortest possible hypotenuse.

SOLUTION 7 : Let (*x*, *y*) represent a randomly chosen point on the graph of $y = \sqrt{x}$.



We wish to MINIMIZE the DISTANCE between points (*x*, *y*) and (4, 0) ,

$$L = \sqrt{(x-4)^2 + (y-0)^2}$$

$$= \sqrt{(x-4)^2 + y^2}.$$

However, before we differentiate the right-hand side, we will write it as a function of *x* only. Substitute for *y* getting

$$L = \sqrt{(x-4)^2 + y^2}$$

$$= \sqrt{(x-4)^2 + (\sqrt{x})^2}$$

$$= \sqrt{(x-4)^2 + x}.$$

Now differentiate this equation using the chain rule, getting

$$L' = (1/2)\Big((x-4)^2 + x\Big)^{-1/2}\{2(x-4)+1\}$$

$$= \frac{2x-7}{2\sqrt{(x-4)^2+y^2}}$$

= 0 ,

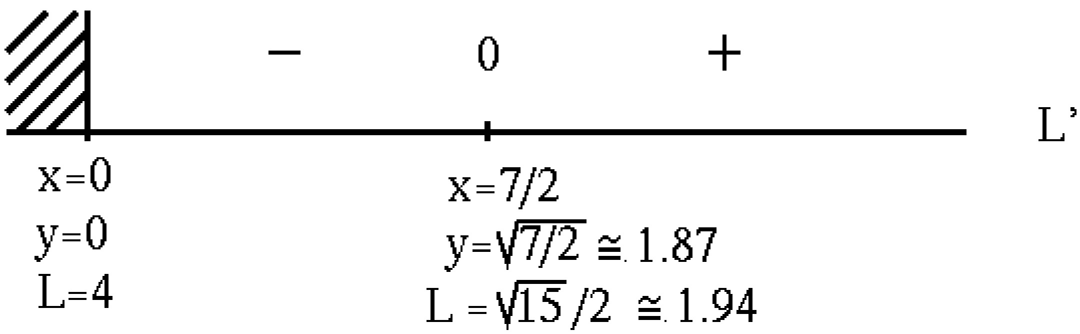
so that (If $\frac{A}{B} = 0$, then A=0 .)

$$2x - 7 = 0 ,$$

or

$$x = 7/2 .$$

See the adjoining sign chart for L' .



If

x = 7/2 and $y = \sqrt{7/2} \approx 1.87$,

then

$$L = \frac{\sqrt{15}}{2} \approx 1.94$$

is the shortest possible distance from (4, 0) to the graph of $y = \sqrt{x}$.

That relatively obvious statement is the Mean Value Theorem as it applies to a particular trip. It may seem strange that such a simple statement would be important or useful to anyone, but the Mean Value Theorem is important and some of its consequences are very useful for people in a variety of areas. Many of the results in the rest of this chapter depend on the Mean Value Theorem, and one of the corollaries of the Mean Value Theorem will be used every time we calculate an "integral" in later chapters. A truly delightful aspect of mathematics is that an idea as simple and obvious as the Mean Value Theorem can be so powerful.

Before we state and prove the Mean Value Theorem and examine some of its consequences, we will consider a simplified version called Rolle's Theorem.

Rolle's Theorem

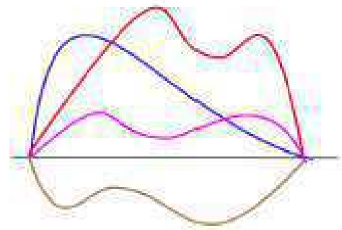


Fig. 1

Suppose we pick any two points on the x–axis and think about all of the differentiable functions which go through those two points (Fig. 1). Since our functions are differentiable, they must be continuous and their graphs can not have any holes or breaks. Also, since these functions are differentiable, their derivatives are defined everywhere between our two points and their graphs can not have any "corners" or vertical tangents. The graphs of the functions in Fig. 1 can still have all sorts of shapes, and it may seem unlikely that they have any common properties other than the ones we have stated, but Michel Rolle (1652–1719) found one. He noticed that every one of these functions has one or more points where the tangent line is horizontal (Fig. 2), and this result is named after him.

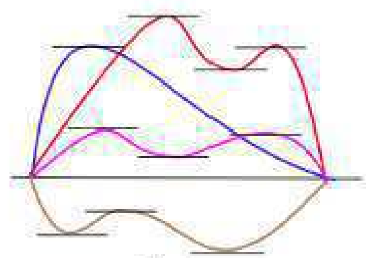


Fig. 2

Rolle's Theorem:

If $f(a) = f(b)$, and $f(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$,

then there is at least one number c, between a and b, so that $f'(c) = 0$.

Proof: We consider two cases: when $f(x) = f(a)$ for all x in (a,b) and when $f(x) \ne f(a)$ for some x in (a,b).

Case I, $f(x) = f(a)$ for all x in (a,b): If $f(x) = f(a)$ for all x between a and b, then f is a horizontal line segment and $f'(c) = 0$ for all values of c strictly between a and b.

Case II, $f(x) \ne f(a)$ for some x in (a,b): Since f is continuous on the closed interval $[a,b]$, we know from the Extreme Value Theorem that f must have a maximum value in the closed interval $[a,b]$ and a minimum value in the interval.

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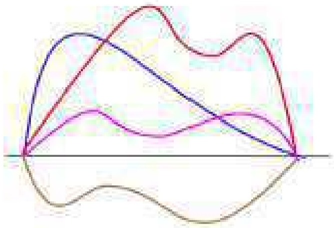

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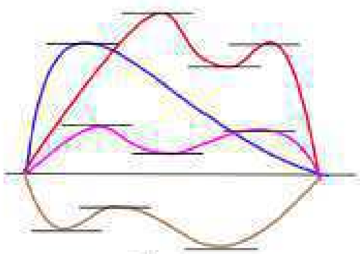

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The Mean Value Theorem and Its Consequences

> If you averaged 30 miles per hour during a trip, then at some instant during the trip you were traveling exactly 30 miles per hour.

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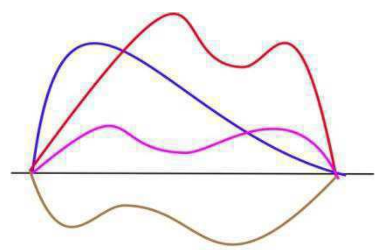

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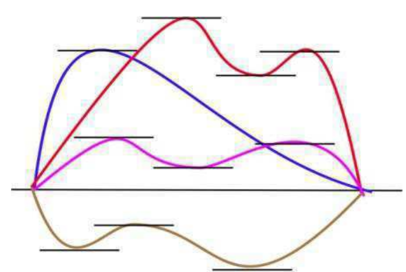


Fig. 2

where the tangent line is horizontal (Fig. 2), and this result is named after him.

Rolle's Theorem:

If f(a) = f(b), and f(x) is continuous for a $\leq$ x $\leq$ b and differentiable for a < x < b, then there is at least one number c, between a and b , so that f '(c) = 0.

Proof: We consider two cases: when f(x) = f(a) for all x in (a,b) and when f(x) $\neq$ f(a) for some x in (a,b).

Case I, f(x) = f(a) for all x in (a,b) : If f(x) = f(a) for all x between a and b, then f is a horizontal line segment and f '(c) = 0 for all values of c strictly between a and b.

Case II, f(x) $\neq$ f(a) for some x in (a,b): Since f is continuous on the closed interval [a,b], we know from the Extreme Value Theorem that f must have a maximum value in the closed interval [a,b] and a minimum value in the interval.

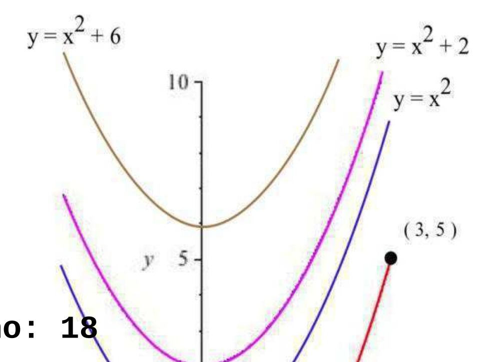
Mean Value Theorem:

If f(x) is continuous for a ≤ x ≤ b and differentiable for a < x < b,

then there is at least one number c, between a and b, so the tangent line at c is parallel to the secant line through the points (a, f(a)) and (b,f(b)): $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Example : (a) Find **all** functions whose derivatives equal 2x .

(b) Find a function g(x) with g '(x) = 2x and g(3) = 5.



Solution: (a) We can recognize that if $f(x) = x^2$ then $f'(x) = 2x$ so one function with the derivative we want is $f(x) = x^2$. Corollary 2 guarantees that every function g whose derivative is 2x has the form $g(x) = f(x) + K = x^2 + K$. The only functions with derivative 2x have the form $x^2 + K$.

(b) Since g '(x) = 2x, we know that g must have the form $g(x) = x^2 + K$, but this is a whole "family" of functions (Fig. 8), and we want to find one member of the family . We know that g(3) = 5 so we want to find the member of the family which goes through the point (3,5). All we need to do is replace the g(x) with 5 and the x with 3 in the formula $g(x) = x^2 + K$, and then solve for the value of K: $5 = g(3) = (3)^2 + K$ so K = – 4 . The function we want is $g(x) = x^2 - 4$.

Partial Differentiation:

Now we enter new territory. Having spent the semester studying functions of several variables, and having worked through the concept of a partial derivative, we are in position to generalize the concept of a differential equation to include equations that involve partial derivatives, not just ordinary ones. Solutions to such equations will involve functions not just of one variable, but of several variables. Such equations arise naturally, for example, when one is working with situations that involve positions in space that vary over time. To model such a situation, one needs to use functions that have several variables to keep track of the spatial dimensions, and an additional variable for time.

Examples of some important PDEs:

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional wave equation*

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional heat equation*

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ *Two-dimensional Laplace equation*

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ *Two-dimensional Poisson equation*

Note that for PDEs one typically uses some other function letter such as *u* instead of *y*, which now quite often shows up as one of the variables involved in the multivariable function.

In general we can use the same terminology to describe PDEs as in the case of ODEs. For starters, we will call any equation involving one or more partial derivatives of a multivariable function a ***partial differential equation***. The ***order*** of such an equation is the highest order partial derivative that shows up in the equation. In addition, the equation is called ***linear*** if it is of the first degree in the unknown function *u*, and its partial derivatives, u_x, u_{xx}, u_y, etc. (this means that the highest power of the function, *u*, and its derivatives is just equal to one in each term in the equation, and that only one of them appears in each term). If each term in the equation involves either *u*, or one of its partial derivatives, then the function is classified as ***homogeneous***.

Take a look at the list of PDEs above. Try to classify each one using the terminology given above. Note that the *f(x,y)* function in the Poisson equation is just a function of the variables *x* and *y,* it has nothing to do with *u(x,y)*.

Answers: all of these PDEs are second order, and are linear. All are also homogeneous except for the fourth one, the Poisson equation, as the *f(x,y)* term on the right hand side doesn't involve *u* or any of its derivatives.

The reason for defining the classifications *linear* and *homogeneous* for PDEs is to bring up the ***principle of superposition***. This excellent principle (which also shows up in the study of linear homogeneous ODEs) is useful exactly whenever one considers solutions to linear homogeneous PDEs. The idea is that if one has two functions, u_1 and u_2 that satisfy a linear homogeneous differential equation, then since taking the derivative of a sum of functions is the same as taking the sum of their derivatives, then as long as the highest powers of derivatives involved in the equation are one (i.e., that it's *linear*), and that each term has a derivative in it (i.e. that it's *homogeneous*), then it's a straightforward exercise to see that the sum of u_1 and u_2 will also be a solution to the differential equation. In fact, so will any linear combination, $au_1 + bu_2$, where *a* and *b* are constants.
For instance, the two functions $\cos(xy)$ and $\sin(xy)$ are both solutions for the first-order linear homogeneous PDE:

(5) $$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

It's a simple exercise to check that $\cos(xy) + \sin(xy)$ and $3\cos(xy) - 2\sin(xy)$ are also solutions to the same PDE (as will be any linear combination of $\cos(xy)$ and $\sin(xy)$)

This principle is extremely important, as it enables us to build up particular solutions out of infinite families of solutions through the use of Fourier series. This trick of superposition is examined in great detail at the end of math 21b, and although we will mention it during the classes on PDEs, we won't have time in 21a to go into any specifics about the use of Fourier series in this way (so come back for more in Math 21b!)

Solving PDEs

Solving PDEs is considerably more difficult in general than solving ODEs, as the level of complexity involved can be great. For instance the following seemingly completely unrelated functions are all solutions to the two-dimensional Laplace equation:

(1) $$x^2 - y^2, \quad e^x\cos(y) \text{ and } \ln(x^2 + y^2)$$

You should check to see that these are all in fact solutions to the Laplace equation by doing the same thing you would do for an ODE solution, namely, calculate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, substitute them into the PDE equation and see if the two sides of the equation are identical.

Now, there are certain types of PDEs for which finding the solutions is not too hard. For instance, consider the first-order PDE

(2) $$\frac{\partial u}{\partial x} = 3x^2 + xy^2$$

where *u* is assumed to be a two-variable function depending on *x* and *y*. How could you solve this PDE? Think about it, is there any reason that we couldn't just undo the partial derivative of *u* with respect to *x* by integrating with respect to *x*? No, so try it out! Here, note that we are given information about just one of the partial derivatives, so when we find a solution, there will be an unknown factor that's not necessarily just an arbitrary *constant*, but in fact is a completely arbitrary *function* depending on *y*.

To solve (2), then, integrate both sides of the equation with respect to *x*, as mentioned. Thus

$$\int \frac{\partial u}{\partial x}\,dx = \int \left(3x^2 + xy^2\right) dx \tag{3}$$

so that $u(x,y) = x^3 + \frac{1}{2}x^2y^2 + F$. What is *F*? Note that it could be any function such that when one takes its partial derivative with respect to *x*, the result is 0. This means that in the case of PDEs, the arbitrary constants that we ran into during the course of solving ODEs are now taking the form of whole functions. Here *F*, is in fact any function, *F(y),*of *y* alone. To check that this is indeed a solution to the original PDE, it is easy enough to take the partial derivative of this $u(x,y)$ function and see that it indeed satisfies the PDE in (2).

Now consider a second-order PDE such as

$$\frac{\partial^2 u}{\partial x\,\partial y} = 5x + y^2 \tag{4}$$

where *u* is again a two-variable function depending on *x* and *y*. We can solve this PDE by integrating first with respect to *x*, to get to an intermediate PDE,

$$\frac{\partial u}{\partial y} = \frac{5}{2}x^2 + xy^2 + F(y) \tag{5}$$

where *F(y)* is a function of *y* alone. Now, integrating both sides with respect to *y* yields

$$u(x,y) = \frac{5}{2}x^2y + \frac{1}{3}xy^3 + F(y) + G(x) \tag{6}$$

where now *G(x)* is a function of *x* alone (Note that we could have integrated with respect to *y* first, then *x* and we would have ended up with the same result). Thus, whereas in the ODE world, general solutions typically end up with as many arbitrary *constants* as the order of the original ODE, here in the PDE world, one typically ends up with as many arbitrary *functions* in the general solutions.

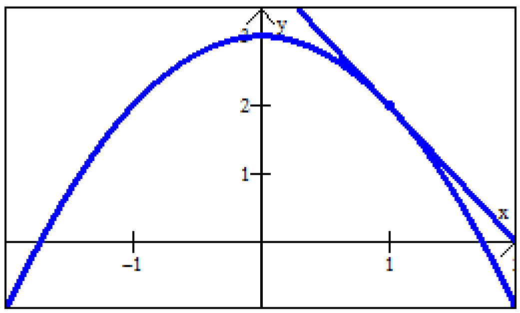
To end up with a specific solution, then, we will need to be given extra conditions that indicate what these arbitrary functions are. Thus the initial conditions for PDEs will typically involve knowing whole functions, not just constant values. We will also see that the initial conditions that appeared in specific ODE situations have slightly more involved analogs in the PDE world, namely there are often so-called *boundary* conditions as well as initial conditions to take into consideration.

Partial Derivatives

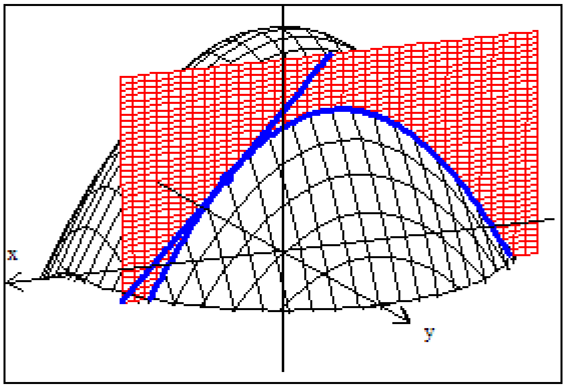
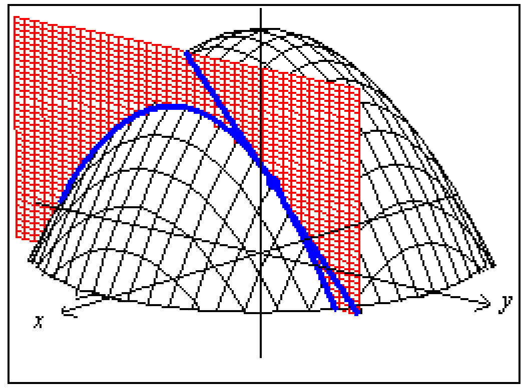
Big idea: The notion of the derivative of a single-variable function can be extended to a multivariate function if a derivative is taken with respect to one variable while holding the value(s) of the other variable(s) constant. This is called a partial derivative.

Big skill: You should be able to compute first-order and higher-order partial derivatives.

The derivative $f'(a)$ of a univariate function $f(x)$ tells us the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. The partial derivative $\frac{\partial f}{\partial x}(a, b)$ tells us the slope of the tangent line to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ in the plane $y = b$. The partial derivative $\frac{\partial f}{\partial y}(a, b)$ tells us the slope of the tangent line to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ in the plane $x = a$.



A graph of $y = f(x) = 3 - x^2$ and the tangent line to the curve at (1, 2). Note that $f'(x) = -2x$ and the equation of the tangent line is $y = -2x + 4$.

	
A graph of $z = f(x, y) = 4 - x^2 - y^2$ and the plane *y* = 1. The plane intersects the surface along a curve specified by $\begin{cases} z = 3 - x^2 \\ y = 1 \end{cases}$. The tangent line to the surface at (1, 1, 2) in the plane *y* = 1 is also shown. Note that in this plane, $z'(x) = -2x$ and the equation of the tangent line is $\begin{cases} x = t \\ y = 1 \\ z = -2t + 4 \end{cases}$ For the function $f(x, y) = 4 - x^2 - y^2$, $\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}\left[4 - x^2 - y^2\right] = 0 - 2x - 0 = -2x$.	A graph of $z = f(x, y) = 4 - x^2 - y^2$ and the plane *x* = 1. The plane intersects the surface along a curve specified by $\begin{cases} z = 3 - y^2 \\ x = 1 \end{cases}$. The tangent line to the surface at (1, 1, 2) in the plane *x* = 1 is also shown. Note that in this plane, $z'(y) = -2y$ and the equation of the tangent line is $\begin{cases} x = 1 \\ y = t \\ z = -2t + 4 \end{cases}$ For the function $f(x, y) = 4 - x^2 - y^2$, $\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}\left[4 - x^2 - y^2\right] = 0 - 0 - 2y = -2y$.

Definition 3.1: Partial Derivative

The **partial derivative of *f*(*x*, *y*) with respect to *x***, written as $\frac{\partial f}{\partial x}$, is defined by

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

for any values of *x* and *y* for which the limit exists.

The **partial derivative of *f*(*x*, *y*) with respect to *y***, written as $\frac{\partial f}{\partial y}$, is defined by

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

for any values of *x* and *y* for which the limit exists.

Note: To compute a partial derivative in practice, just treat all independent variables as constants except for the variable with respect to which the derivative is being taken.

Various notations for partial derivatives of $z = f(x, y)$:

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \frac{\partial z}{\partial x}(x, y) = \frac{\partial}{\partial x}\left[f(x, y)\right]$$

(Likewise for partial derivatives with respect to *y*)

The expression $\frac{\partial f}{\partial x}$ is a partial differential operator; it indicates to take the partial derivative with respect to *x* of whatever expression follows it.

Practice:

1. Compute $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ for $f(x,y)=5x^2-2y$ using the limit definition of the derivative.

   Compute $\frac{\partial f}{\partial x}(x,y)$ for $f(x,y)=3x^2+4y^2-2x^2y+7xy^2-9$

2. Compute $\frac{\partial f}{\partial y}(x,y)$ for $f(x,y)=\cos(x)\sin(y)-e^{xy}$

Higher order partial derivatives are partial derivatives of partial derivatives.

For a function of two variables, there are four different second-order partial derivatives:

- The partial derivative with respect to *x* of $\frac{\partial f}{\partial x}$ is $\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$.

  Alternative notations: $\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial^2 f}{\partial x^2}=f_{xx}$

- The partial derivative with respect to *y* of $\frac{\partial f}{\partial y}$ is $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$.

  Alternative notations: $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial^2 f}{\partial y^2}=f_{yy}$

- The partial derivative with respect to *x* of $\frac{\partial f}{\partial y}$ is $\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$.

  Alternative notations: $\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial^2 f}{\partial x\,\partial y}=\left(f_y\right)_x=f_{yx}$. This is a **mixed second-order partial derivative**.

- The partial derivative with respect to *y* of $\frac{\partial f}{\partial x}$ is $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$.

- Alternative notations: $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial^2 f}{\partial y\,\partial x}=\left(f_x\right)_y=f_{xy}$. This is also a **mixed second-order partial derivative**.

Practice:

1. Compute all four second-order partial derivatives for $f(x,y)=3x^2+4y^2-2x^2y+7xy^2-9$.
2. Compute all four second-order partial derivatives for $f(x,y)=\cos(x)\sin(y)-e^{xy}$.

Notice that the mixed second-order partial derivatives are equal in both cases above… this is usually the case …

Theorem 3.1 (Equality of Mixed Second-Order Partial Derivatives)
If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous on an open set containing (a, b), then $f_{xy}(x, y) = f_{yx}(x, y)$.

Note: order of differentiation doesn't matter for higher than second-order partial derivatives as well.
Practice:

1. Compute f_{xyx} and f_{xyxy} for $f(x,y)=\cos(x)\sin(y)-e^{xy}$

2. Compute f_x, f_{xy}, and f_{xyz} for $f\left(x,y,z\right)=\sqrt{xyz}+x^2y^3z^4$

3. The sag *S* in a beam of length *L*, width *w*, and height *h* is given by $S\left(L,w,h\right)=c\dfrac{L^4}{wh^3}$. Write all three first-order partial derivatives in terms of S and one other variable to determine which variable has the greatest proportional effect on the sag.

1. DERIVATIVES

The derivative of a function f is the function f' whose value at point x is given by,

$$f'(x)=\lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}$$

provided the limit exists. For $y=f\left(x\right)$, various notations for derivates are;

$$f'(x)=y'=\frac{dy}{dx}=\frac{d}{dx}f\left(x\right)=D_x f\left(x\right)=D_x y\ .$$

Rules:

a) $y=f\left(x\right)=x^n$

$$y'=\frac{dy}{dx}=nx^{n-1}$$

b) $y=f\left(x\right)\pm g\left(x\right)$

$$y'=\frac{dy}{dx}=\frac{d}{dx}f\left(x\right)\pm\frac{d}{dx}g\left(x\right)$$

c) Product Rule

$y=f\left(x\right)*g\left(x\right)$

$$y'=\frac{dy}{dx}=g\left(x\right)\frac{d}{dx}f\left(x\right)+f\left(x\right)\frac{d}{dx}g\left(x\right)$$

d) $y=\dfrac{f\left(x\right)}{g\left(x\right)}$

$$y'=\frac{dy}{dx}=\frac{g\left(x\right)\frac{d}{dx}f\left(x\right)-f\left(x\right)\frac{d}{dx}g\left(x\right)}{\left[g\left(x\right)\right]^2}$$

e) Chain Rule

$y=f\left(z\right)$ where $z=g\left(x\right)$

$$y=f\left(g\left(x\right)\right)$$

$$y'=\frac{dy}{dx}=\frac{dy}{dz}*\frac{dz}{dx}$$

Examples:

$y = z^3$ where $z = 8x^2$

By substitution;

$y = \left(8x^2\right)^3$ and take $\frac{dy}{dx}$

$$y' = 8^3 * x^6 = \left(8^3 * 6\right) * x^5 = 3072x^5$$

with chain rule;

$$\frac{dy}{dx} = \frac{dy}{dz} * \frac{dz}{dx} = 3z^2 * 16x = 3\left(8x^2\right)^2 * 16x = 3072x^5$$

f) Derivative of ln x and e^x (transcendental functions)

$y = \log_a x$ means $a^y = x$ Ex: $5^2 = 25 \Rightarrow \log_5 25 = 2$

$$y = \log_a g(x) \Rightarrow \frac{dy}{dx} = \frac{g'(x)}{g(x).\ln a}$$

The natural logarithm $\ln$;

$u = \ln a$ implies $e^u = a$ with base being the irrational number $e = 2.7182818459....$

Then;

$$y = \ln f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} * \frac{d}{dx} f(x) = \frac{f'(x)}{f(x)}$$

Examples:

$$y = \ln x \Rightarrow y' = \frac{1}{x}$$

$$y = \ln 3x \Rightarrow y' = \frac{1}{3x} * 3 = \frac{1}{x}$$

$$y = \ln 6x^3 \Rightarrow y' = \frac{1}{f(x)} \frac{df(x)}{dx} = \frac{1}{6x^3} * 18x^2 = \frac{1}{3}x$$

The exponential function e^x;

$$y = e^x \Rightarrow y' = \frac{dy}{dx} = e^x$$

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)$$

$y = e^{6x^3} \Rightarrow y' = e^{6x^3} * 18x^2$

2. MULTIVARIATE FUNCTIONS

Given the multivariate function $y = f(x_1, x_2, x_3, \ldots\ldots x_n)$

a) Partial derivative: The effect of a change in an independent variable x_i, on the dependent variable y holding other variables constant. (For example: the statement that 'the marginal utility of some good x_i is positive' means that if x_i is increased by some amount Δx_i, holding the other goods (x_i's) constant, the resulting change in total utility will be positive. This is nothing but taking the partial derivative of marginal utility with respect to x_i, holding the other variables constant.)

$y = f(x_1, x_2)$

From now on we will use the notation

$f_1 = \frac{\partial f}{\partial x_1}$; (holding x_2 constant) and

$f_2 = \frac{\partial f}{\partial x_2}$; (holding x_1 constant)

Example:

$f(x, y) = ax^2 + bxy + cy^2$

$f_1 = \frac{\partial f}{\partial x} = 2ax + by$ and $f_2 = \frac{\partial f}{\partial y} = bx + 2cy$

Second Derivatives:

$f_{12} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y} = b$ and $f_{21} = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x} = b$

by Young's Theorem $f_{12} = f_{21}$

Example: (Silberberg and Suen)

If the consumer's utility function is given by $U(x_1, x_2) = x_1 \log x_2$ the marginal utility derived from consuming good x_2 is;

$U_2 = \frac{\partial U}{\partial x_2} = \frac{x_1}{x_2}$, (the partial derivative w.r.t. x_2, holding x_1 constant), and

$U_{22} = \frac{\partial U_2}{\partial x_2} = -\frac{x_1}{x_2^2}$, implying diminishing marginal utility.

b) Total Derivative: change in the dependent variable due to an infinitesimally small change in one of the independent variable when all the independent variables are allowed to change.

Consider the function:

$y = f(x_1, x_2)$, (so there are three variables, one dependent and two independent)

$f(x_1, x_2) = y_0$, where y_0 is a pre-assigned constant value, represents the level curves of the function. (level curves; two dimensional graphs of three dimensional functions). This equation can be solved for one of the unknowns in terms of the other as,

$x_2 = x_2(x_1)$ and hence f functions becomes, $f(x_1, x_2(x_1)) = y_0$.

Then since we defined x_2 explicitly as a function of x_1, **the slope of any level curve,** $\frac{dx_2}{dx}$ makes sence. Then, total derivative of $y = f(x_1, x_2)$ is defined as;

$$\frac{df}{dx_1}\frac{dx_1}{dx_1} + \frac{df}{dx_2}\frac{dx_2}{dx_1} \equiv \frac{dy_0}{dx_1} \equiv 0$$

$$\Rightarrow f_1 + f_2 \frac{dx_2}{dx_1} \equiv 0$$

Then the slope of the level curve is,

$$\Rightarrow \frac{dx_2}{dx_1} \equiv -\frac{f_1}{f_2}, \text{ assuming } f_2 \neq 0$$

For $y = f(x, z)$,

$$\frac{dy}{dx} = \frac{\partial f(x,z)}{\partial x}\frac{dx}{dx} + \frac{\partial f(x,z)}{\partial z}\frac{dz}{dx} = \frac{\partial f(x,z)}{\partial x} + \frac{\partial f(x,z)}{\partial z}\frac{dz}{dx}$$

Then the **'Total Differential',** the total change in y is defined as

$$\begin{aligned} dy &= \frac{\partial f(x,z)}{\partial x}dx + \frac{\partial f(x,z)}{\partial z}dz \\ &= f_1 dx + f_2 dz \end{aligned}$$

Example: $y = x^2 z^3 \quad where \quad z = 2x$ compute the total derivative of y with respect to x.

$\frac{dy}{dx} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial z}\frac{dz}{dx}$ ===> total derivative

$$= 2xz^3 + 3x^2z^2\frac{dz}{dx} = 2xz^3 + 6x^2z^2$$

Example:

If the level curve is the production function $y = F(L, K)$, the slope is,

$\frac{dK}{dL} = -\frac{f_L}{f_K}$, showing that the slope measures the willingness of firms to substitute labor for capital, because it measures the benefits of additional labor f_L, to the output loss due to using less capital f_K. Also, if the marginal products of each factor input are positive, then the isoquants will have a negative slope, so, downward sloping.

Similarly, the slope of an indifference curve expresses that the willingness of a consumer to make exchanges is based on the ratio of perceived gains and losses from such an exchange.To show the **convexity** of the indifference curve we need to show $\frac{d^2 x_2}{dx_1^2} > 0$ as well. Following the necessary steps explained in the textbook (Silberberg and Suen, p52) we derive that;

$$\frac{d^2 x_2}{dx_1^2} = (-f_2^2 f_{11} + 2 f_1 f_2 f_{12} - f_1^2 f_{22}) \frac{1}{f_2^3}$$

So convexity of the indifference curve does not imply or implied by "diminishing marginal utility, (that is $f_{11} < 0$ and $f_{22} < 0$). The cross effect f_{12} must be considered as well.

Example: Monopolist Output

$$Q = 1000 - 10P$$

Monopolist ==> only provider ==> as quantity changes price changes too. So in a monopolist market price is a function of the quantity.

$$P = 100 - \frac{1}{10} Q$$

$TR = P * Q$, a function of two variables.

Computing MR necessitates taking the total derivative of TR

$$MR = \frac{dTR}{dQ} = \frac{\partial TR}{\partial Q} + \frac{\partial TR}{\partial P} * \frac{dP}{dQ} = P + Q\left(-\frac{1}{10}\right)$$ and substitute P

$$= 100 - \frac{1}{10} Q - \frac{1}{10} Q = 100 - \frac{1}{5} Q$$

c) Application of Chain Rule to Multivariate Functions

$y = f(x_1, x_2)$ where $x_1 = x_1(t)$ & $x_2 = x_2(t)$ ====> x_1, x_2 are functions of t.

As t changes, x_1 and x_2 change and hence y changes. So y is a function of t.

$$y = f(x_1(t), x_2(t)) = y(t) \Rightarrow y' = ?$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt}$$

Example:

$$y = \log(x_1 + x_2)$$
$$x_1 = x_1(t) = t$$
$$x_2 = x_2(t) = t^2$$

$$y' = \frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt}$$
$$= \frac{1}{x_1 + x_2} 1 + \frac{1}{x_1 + x_2} 2t$$
$$= \frac{1}{x_1 + x_2}(1 + 2t)$$

Substitute x_1 and x_2 ====> $\frac{1}{t + t^2}(1 + 2t)$

Example:

The function $y = F(x_1, \ldots x_n)$ is an **explicit function** of the independent variables.

The function $G(x_1, x_2, \ldots, x_n, y) = 0$ defines the dependent variable y as an **implicit function** of the independent variables $x_1, \ldots, x_n$.

$$f(x, y) = 0 \Rightarrow implicit\ \ function$$

$$f(x, y) = x^2 + y^2 - 1 = 0 \Rightarrow y' = ?$$
$$\frac{dx^2}{dx} + \frac{dy^2}{dx} - \frac{d(1)}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} - 0 = 0 \ldots\ldots\ldots \left(\frac{dy^2}{dx} = \frac{dy^2}{dy}\frac{dy}{dx}\right)$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Example: $4xy^3 - x^2y + x^3 - 5x + 6 = 0 \Rightarrow y' = \frac{dy}{dx} = ?$

$$\Rightarrow 4y^3 + 12xy^2\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} + 3x^2 - 5 = 0$$
$$\Rightarrow (12xy^2 - x^2)\frac{dy}{dx} = 5 - 4y^3 + 2y - 3x^2$$
$$\frac{dy}{dx} = \frac{5 - 4y^3 + 2y - 3x^2}{12xy^2 - x^2}$$

3. MONOTONIC TRANSFORMATIONS

Utility function $U = U(x_1, x_2)$, shows an ordinal ranking of the preferences.

The following utility function V conveys the same information as U and preserves the same ordinal ranking;

$$V(x_1, x_2) = F(U) = F(U(x_1, x_2)), \quad \text{where } F'(U) > 0.$$

Then,

- U and V move in the same direction
- V is a **monotonically increasing** function of U.
  (and if $F'(U) < 0$ V is a monotonically decreasing function of U)
- Here F relabels the level curves of U giving them new numbers

Terminology: Monotonically increasing $\leftrightarrow$ Monotonic

Example: $U(x_1, x_2)$ is a utility function

$V = \log^{U(x_1, x_2)}$ is a monotonic transformation of U.

$V = e^{U(x_1, x_2)}$ is another monotonic transformation of U.

First Partial Derivatives of Monotonic Functions

How can the partial derivatives of $V(x_1, x_2) = F(U) = F(U(x_1, x_2))$ be interpreted in terms of partial derivatives of U ?

$$V_1 = F'(U)U_1$$

$$V_2 = F'(U)U_2$$

Then the slope of the level curve;

$$\frac{dx_2}{dx_1} = -\frac{V_1}{V_2} = -\frac{F'U_1}{F'U_2} = -\frac{U_1}{U_2}$$

is unaffected by this relabeling of the indifference curve. (So the MRS is preserved under monotonic transformations.However, diminishing marginal utility is not preserved under monotonic transformations, since 'diminishing marginal utility' has no meaning in the context of ordinal utility.

$$V_{11} = F'U_{11} + F''U_1^2$$

$$V_{22} = F'U_{22} + F''U_2^2$$

$V_{12} = V_{21}$ by Young's Theorem.

Obviously $U_{11} < 0$ and $U_{22} < 0$, U_{11} and V_{11} do not necessarily have the same sign, (WHY?)

4. HOMOGENOUS FUNCTIONS AND EULER'S THEOREM

A function $f(x_1, x_2, \ldots, x_n)$ is said to be homogenous of degree "r" if

$$f(tx_1, tx_2, \ldots, tx_n) = t^r f(x_1, x_2, \ldots, x_n)$$

Example:

$$Y = L^{\alpha}K^{1-\alpha} = f(L,K)$$
$$f(tL,tK) = (tL)^{\alpha}(tK)^{1-\alpha}$$
$$= t^{\alpha}L^{\alpha}t^{1-\alpha}K^{1-\alpha} = t$$
$$= tL^{\alpha}K^{1-\alpha} = tf(L,K)$$

homogenous of degree 1

Theorem 1: (p59 Silberberg and Suen)

If $f(x_1, x_2, \ldots, x_n)$ is homogenous of degree r, then, the first partial $f_1, f_2, \ldots, f_n$ are homogenous of degree r-1.

Euler's Theorem

Suppose $f(x_1, x_2, \ldots, x_n)$ is homogenous of degree r, then,

$$\frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2 + \frac{\partial f}{\partial x_3}x_3 + \ldots + \frac{\partial f}{\partial x_n}x_n \equiv rf(x_1, x_2, x_3, \ldots, x_n)$$

Example Page 65 Q1 a)

$f(x_1, x_2) = x_1 x_2^2$ show that the function is homogenous and verify Euler's Theorem.

$f(tx_1, tx_2) = tx_1(tx_2)^2 = t^3 x_1 x_2^2 = t^3 f(x_1, x_2^2) \Rightarrow$ homogenous of degree 3.

Euler's Theorem

$$\frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2 = 3f(x_1, x_2)$$
$$\frac{\partial f}{\partial x_1} = x_2^2,$$
$$\frac{\partial f}{\partial x_2} = 2x_1x_2$$
$$then \quad \Rightarrow x_2^2 x_1 + 2x_1x_2^2 = 3x_1x_2^2 = 3f(x_1x_2)$$

Homothetic Functions

A homothetic function is a monotonic transformation of a function that is homogenous of degree 1.

(Homothetic functions are functions whose MRTS (slope of the level curve) is homogeneous of degree zero. Therefore along any ray through the origin, level curves have the same slope, so that theyare radial blow-ups of each other.

5. OPTIMIZATION

Unconstrained Optimization

For $f(x_1, x_2)$, conditions for

<u>Relative Max</u> <u>Relative Min</u>

$f_1, f_2 = 0$
$f_{11}, f_{22} < 0$
$f_{11}f_{22} - f_{12}^2 > 0$

$f_1, f_2 = 0$
$f_{11}, f_{22} > 0$
$f_{11}f_{22} - f_{12}^2 > 0$

Comments

1. $f_{11}f_{22} - f_{12}^2 < 0$ ==> if f_{11} and f_{22} both have the same sign ==> infection point.
   ==>if f_{11} and f_{22} have different sign ==> saddle point.

2. $f_{11}f_{22} - f_{12}^2 = 0$ ==>test is inconclusive

Constrained Optimization

Problem:

maximize $f(x, y)$ (the objective function)

subject to $g(x, y) = k$ (the constraint)

Solution is done through the Lagrange function;

L$= f(x, y) + \lambda(k - g(x, y))$

The choice variables $(x*, y*)$ and the Lagrange multiplier $\lambda*$ are derived through solving the first order conditions (f.o.c.). Substituting solution in the objective function yields the 'indirect objective function' $f*(x*, y*)$. It shows the maximum values of the objective function f at point $(x*, y*)$.

Meaning of Lagrange Multiplier (λ)

λ approximates the effect of a small change in constraint, on the optimum of objective function.

Case (Silberberg p 167): Suppose that the problem is output maximization subject to a resource (say labor), constrained at level L. If the constrained resource increases by an additional increment, ΔL, then output will increase by $\Delta Q \cong \lambda^* \Delta L$. So it appears that λ^* is the marginal value of that resource. In a competitive economy, firms would be willing to pay λ^* for each increment in the resource. Then, λ^* is the shadow price of that resource.

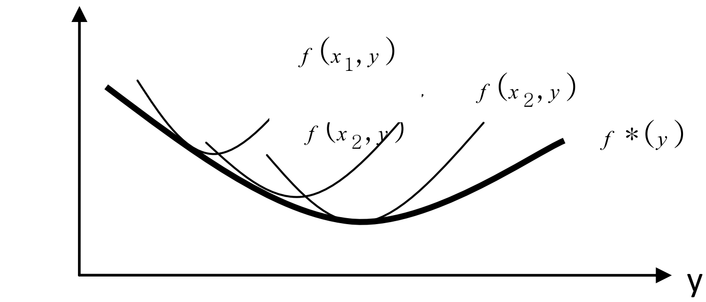
For a cost minimization problem λ^* measures the change in the total cost if input changes. So it is marginal cost.

If the problem is utility maximization subject to the budget constraint, then **λ** becomes the marginal utility of income.

6. ENVELOPE THEOREM

The Envelope Theorem is about creating a new function out of a set of functions by choosing the optimum value of every function in the set.

$$f^*(y) = \min_x f(x, y)$$



The theorem says that the slope of the envelope at any point is the same as the slope of every single function it touches.Then how does an indirect objective function f^* vary (as compared to objective function f) when an exogenous variable changes?The theorem enables us to measure the effect of a change in an exogenous variable on the optimal value of the objective function, by taking derivative of the Lagrange function and evaluating the derivative at the value of the optimal solution.

This is a very useful tool. Below envelope theorem is proved for unconstrained and constrained optimization.

-Unconstrained Model

maximize $y = f(x_1, x_2; \alpha)$, two variables and a parameter α .

Solution gives each choice variable $x_i = x_i(\alpha)$, in terms of the parameter.

Substituting in the objective function we get

$$y^*(\alpha) = f\left(x_1^*(\alpha), x_2^*(\alpha), \alpha\right) \Longrightarrow indirect\ object\ function$$

Then to see the effect of a change in α on the maximum value of the objective function, we take the derivative of the indirect objective function's derivative w.r.t. α.

$$y_\alpha^*(\alpha) = \frac{\partial f}{\partial x_1}\frac{\partial x_1^*}{\partial \alpha} + \frac{\partial f}{\partial x_2}\frac{\partial x_2^*}{\partial \alpha} + \frac{\partial f}{\partial \alpha} = f_1\frac{\partial x_1^*}{\partial \alpha} + f_2\frac{\partial x_2^*}{\partial \alpha} + f_\alpha$$

But since $f_1 = f_2 = 0$ by the f.o.c. given above,

$y_\alpha^*(\alpha) = f_\alpha$ is the rate of change of f as α varies (holding x_1, x_2 constant). (1)

In other words the change in the indirect objective function when one parameter of the problem is changed, is equal to the change in the objective function. But what does the change in the indirect objective function mean??? It is the change in the objective function when x is chosen optimally. So the change in the objective function when we change α and we adjust x optimally, is the same as the change when we do not adjust x optimally.

-Constrained Model

maximize $f(x_1, x_2, \ldots, x_n; \alpha)$

subject to $g(x_1, x_2, \ldots, x_n; \alpha) = y$

$L = f + \lambda(g - y)$

f.o.c.

$$L_{x} = fx_1 + \lambda g_{x_i} = f_i + \lambda g_i = 0$$
$$L_\alpha = f_\alpha + \lambda g_\alpha = 0$$
$$L_\lambda = g = 0$$
$$\Rightarrow x_i = x_i^*(\alpha)$$
$$\lambda = \lambda_i^*(\alpha)$$

Then, substitute this into the object function and get y^*, **<u>the indirect objective function</u>;**

$$y^*(\alpha) = f\left(x_1^*(\alpha), \ldots, x_n^*(\alpha); \alpha\right)$$

then, how does $y*(\alpha)$ change as α changes?

==> $$\frac{\partial y*}{\partial \alpha} = \frac{\partial f}{\partial x_1}\frac{\partial x_1^*}{\partial \alpha} + \ldots\ldots\ldots\ldots\ldots\ldots\frac{\partial f}{\partial x_n}\frac{\partial x_n^*}{\partial \alpha} + \frac{\partial f}{\partial \alpha} \qquad (2)$$

$$\sum_i f_i \frac{\partial x_i^*}{\partial \alpha} + f_\alpha$$

Pay attention that $f_i \neq 0$

Take the derivative of the constraint with respect to α

==> $$\sum_i g_i \frac{\partial x_i^*}{\partial \alpha} + g_\alpha = 0 \qquad (3)$$

(2) + (3) (multiply (3) by λ)

$$\frac{dy*}{d\alpha} = \sum_i f_i \frac{\partial x_i^*}{\partial \alpha} + f_\alpha + \sum_i \lambda g_i \frac{\partial x_i^*}{\partial \alpha} + \lambda g_\alpha = \sum\left(f_i + \lambda g_i\right)\frac{\partial x_i^*}{\partial \alpha} + f_\alpha + \lambda g_\alpha = L_\alpha \qquad (4)$$

So the Envelope Theorem tells us that

- for the unconstrained model y_α^* is the same as f_α, (1)
- for the constrained model $y_\alpha^* = L_\alpha$. (4)

Lagrange Multipliers

In Example 3 of the previous section, we maximized the volume $V(l, w, h) = lwh$ subject to the constraint that $A(l, w, h) = 2hl + 2hw + lw = 12$. We solved that problem by substituting the constraint into the function we wished to maximize and then found critical points.

In this section, we present a second method for dealing with these problems known as Lagrange's method for maximizing (or minimizing) a general function $f(x, y)$ subject to a constraint $g(x, y) = c$. This applies to a more general setting than the method we used in the previous section. (Below, we present the theory for two variables, but the method applies to functions of three or more variables, like Example 3 from the last section.)

Suppose we want to find the extreme values of $f(x, y)$ subject to a constant $g(x, y) = c$. That is, we want to find the extreme values of $f(x, y)$ when the point (x, y) is restricted to the level curve $g(x, y) = c$. Figure 1 below shows some level curves of $f(x, y)$ (in blue) as well as the constraint $g(x, y) = c$ (in red).

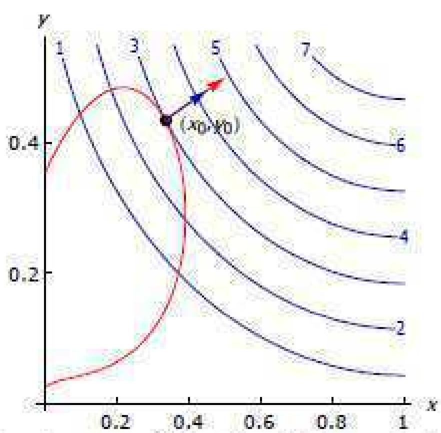


Figure 1: Level curves of $f(x, y)$ and the constraint $g(x, y) = c$

To maximize $f(x, y)$ subject to $g(x, y) = c$ is the same as finding the largest c value such that the level curve $f(x, y) = k$ intersects $g(x, y) = c$. In the above figure, this happens at the point (x_0, y_0). Notice that at this point, the curves just touch each other. That is, they have a common tangent line. (Otherwise, the value of k could be increased further.)

But if the two curves have a common tangent line, this means that their gradient vectors must be parallel. (See the colored arrows above.) This gives us the relationship $\nabla f(x, y) = \lambda \nabla g(x, y)$, for some scalar λ.

Using this relationship, we can describe the method of Lagrange multipliers.

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = c$ (assuming these extreme values exist)

1. Find all values of x, y and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = c$$

2. Evaluate $f(x, y)$ at all of the points found in (1). The largest of these values is the maximum value of $f(x, y)$ and the smallest value is the minimum value of $f(x, y)$.

