# MAT211: Engineering Mathematics


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## *Differential Equations*

### Formation of Differential Equations

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**Definition:** DE, Classification, ODE, PDE, degree, order, solution, General, particular, singular solution, Formation of Differential equations.

**Problems:**

1. Write down the order and degree of $x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^4 = 0$ .
2. Is the ODE $\sqrt[4]{\left(\frac{d^3y}{dx^3}\right)^4 - 5x\frac{d^2y}{dx^2} + y} = \sqrt[5]{\left(\frac{dy}{dx}\right)^2 + y^2 - x}$ linear?
3. Show that the ODE of any straight line is $\frac{d^2y}{dx^2} = 0$ .
4. Eliminate the constant 'a' from $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ .
5. Derive the differential equation of which $c(y+c)^2 = x^3$.
6. Derive the differential equation of which $y^2 = A(B+x)(B-x)$.
7. Find the differential equation of the family of curves $y = e^{-x}(Acosx + Bsinx)$ where A and B are arbitrary constants.
8. Obtain the ODE associated with the primitive $y = ax^2 + bx$ .
9. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$ , for different values of A and B.
10. Find the differential equation of the family of curves $y = Ae^{-3x} + Be^{2x}$ , for different values of A and B.
11. Find the differential equation of the family of curves $y = ae^{2x} + be^{-3x} + ce^{x}$
12. From the differential equation of which $y = ae^{x} + be^{-x} + c\cos x + d\sin x$ is a solution.
13. Find the differential equation of the family of curves $y = e^{x}(Acosx + Bsinx)$ where A and B are arbitrary constants.
14. Find the differential equation of the solution $xy = Ae^{x} + Be^{-x} + x^2$ .

# First order First degree ODE

**Definition:** Separation of variable, integrating factor, linear ODE, Bernoulli's Equation, Homogeneous and Exact ODE.

**Problems:**

**Variable Separable form:**

1. Solve the differential equation $\left(4+y^2\right)dx+\left(4+x^2\right)dy=0$ by variable separable Method.
2. Solve the differential equation $\frac{dy}{dx}=\sqrt{1-x^2}\,\sqrt{1-y^2}$ by variable separable Method.
3. Solve the ODE $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$ .
4. Solve $\frac{dy}{dx}=e^{x+y}+x^2e^{x^3+y}$ .

**Reducible to Variable Separable form:** $\frac{dy}{dx}=f\left(ax+by+c\right)$ Hints: Let $ax+by+c=v$

5. Solve the differential equation $\frac{dy}{dx}=\left(x+y\right)^2$.
6. Solve the differential equation $\frac{dy}{dx}=\sin\left(x+y\right)$.
7. Solve the differential equation $\frac{dy}{dx}=\sin\left(x+y\right)+\cos\left(x+y\right)$.
8. Solve the differential equation $\frac{dy}{dx}=\tan\left(x+y+6\right)$ by choosing appropriate transformation.
9. Solve $\frac{dy}{dx}=1-\sqrt{x+y}$ .

**Homogeneous ODE:** $\frac{dy}{dx}=\frac{f_1\left(x,y\right)}{f_2\left(x,y\right)}$ Hints: Let $y=vx$

10. Solve $\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$.
11. Solve $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$ .
12. Solve $\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$
13. Solve $x\frac{dy}{dx}=y+\sqrt{x^2+y^2}$

**Non-homogeneous ODE:** $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ Hints: Let $x=x'+h, y=y'+k$ and

$$a_1h+b_1k+c_1=0, a_2h+b_2k+c_2=0$$

14. Solve $(6x-5y+4)dy+(y-2x-1)dx=0$.

15. Solve the differential equation $(3x-2y+1)\frac{dy}{dx}=6x-4y+3$

**Linear ODE:** $\frac{dy}{dx}+Py=Q$ Hints: I.F = $e^{\int p\,dx}$

1. Solve $\frac{dy}{dx}+xy=x$.

2. Solve $x\frac{dy}{dx}+2y=x^2\log x$.

3. Solve the differential equation $\frac{dy}{dx}-2y\cos x=-2\sin 2x$.

**Bernoulli's ODE:** $\frac{dy}{dx}+Py=Qy^n, n\neq 0\ ,1$ Hints: Let $y^{1-n}=v$

4. Solve the differential equation $\frac{dy}{dx}=x^3y^3-xy$.

5. Solve the differential equation $\frac{dy}{dx}+\frac{y}{x}=y^2$.

**Exact ODE:**

6. Solve $(2x+y-1)dx+(x-2y+5)dy=0$

7. Solve $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$

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# Linear ODE with first degree and higher order with constant coefficients

Higher Order ODE $f(D)y=X$ : Solution : $y=y_c+y_p$ Hints: $D\equiv\frac{d}{dx}$

**Definition:** Auxiliary Equation, Particular integral, complementary function and formula for finding particular integral.

**Problems:**

1. Solve $\frac{d^2y}{dx^2}-4y=x^2+2x+1$

2. Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}$

3. Solve $\frac{d^2y}{dx^2}+4y=\sin 2x$

4. Solve $\frac{d^2y}{dx^2}-y=e^x\sin x$

5. Solve $(D^2+4)^2y=\sin 2x$

6. Solve $\frac{d^3y}{dx^3}-3\frac{dy}{dx}+2y=0$

7. Solve $D^4y-4D^2y+4y=e^x$

8. Solve $D^4y-81y=\sin 2x$

Formulas: $y_c$

1. $y_c=c_1e^{m_1x}+c_2e^{m_2x}$
2. $y_c=(c_1+c_2x)e^{mx}$
3. $y_c=e^{\alpha x}(c_1\cos\beta+c_2\sin\beta)$
4. $y_c=e^{\alpha x}\{(c_1+c_2x)\cos\beta+(c_3+c_4x)\sin\beta\}$

Formulas: $y_p$

1. $y_p=\frac{1}{f(D)}p(x)=[f(D)]^{-1}p(x)$
2. $y_p=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}=x\frac{1}{f'(a)}e^{ax}$ if $f(a)=0$
3. $y_p=\frac{1}{f(D)}\sin ax=\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax$
4. $y_p=\frac{1}{f(D)}e^{ax}.V=e^{ax}\frac{1}{f(D+a)}V$

9. Solve $D^2y+5Dy+4y=3-2x$.

10. Solve $D^2y+4Dy+4y=e^{2x}+\cos 2x$

11. Solve $D^2y-8Dy+16y=\cos 3x$

12. Solve $D^3y-2Dy+4y=e^x\cos x$.

13. Solve $\left(D^2+D\right)y=e^x\sin x$.

14. Solve $\left(D^4-7D^2-18\right)y=\cos x$.

15. Solve $\left(D^5+5D^4-2D^3-10D^2+D+5\right)y=e^x$

16. Solve $\left(D^3+D^2+D+1\right)y=\cos 3x\sin x$

17. Solve $\left(D^3-6D^2+11D-6\right)y=x^4$

18. Solve $\left(D^3-D^2-4\right)y=e^{2x}$

19. Solve $\left(D^3-6D^2+12D-8\right)y=x^2$

20. Solve $\left(D^4-2D^3+D^2\right)y=x^3$

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# Linear ODE with first degree and higher order with variable coefficients

Higher Order ODE Convert $f(D)y=X$ : Solution : $y=y_c+y_p$ Hints:

$x=e^z \Leftrightarrow z=\ln x$ , $D\equiv\dfrac{d}{dz}$

**Problems:**

1. Solve $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-4y=x^4$

2. Solve $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=x+\ln x$

3. Solve $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-3y=x^2\ln x$

4. Solve $x^3\dfrac{d^3y}{dx^3}+5x^2\dfrac{d^2y}{dx^2}+7x\dfrac{dy}{dx}+8y=xe^x$

5. Solve $x^3\dfrac{d^3y}{dx^3}+3x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=x\ln x$

6. Solve $x^3\dfrac{d^3y}{dx^3}+3x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=x+\ln x$

7. Solve $x^3\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+2y=10\left(x+\dfrac{1}{x}\right)$

8. Solve $x^3\dfrac{d^3y}{dx^3}-x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-2y=x^2+3x$

**01**

# Laplace Transformation $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$

**Definition:** Introduction, Laplace transform, properties and related proofs, Hyperbolic sine & cosine functions.

**Problems:**

Derive the followings:

1. $\mathcal{L}\{1\} = \dfrac{1}{s}$
2. $\mathcal{L}\{t\} = \dfrac{1}{s^2}$
3. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}} = \dfrac{\Gamma(n+1)}{s^{n+1}}$
4. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$
5. $\mathcal{L}\{\text{sinat}\} = \dfrac{a}{s^2+a^2}$
6. $\mathcal{L}\{\text{cosat}\} = \dfrac{s}{s^2+a^2}$
7. $\mathcal{L}\{\sinh at\} = \dfrac{s}{s^2+a^2}$
8. $\mathcal{L}\{\text{coshat}\} = \dfrac{s}{s^2-a^2}$

Special formula:

1. $e^{i\theta} = \cos\theta + i\sin\theta$
2. $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$
3. $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$
4. $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$
5. $\sinh x = \dfrac{e^{x}-e^{-x}}{2}$

## Properties Related Problems :

Linearity property:

1. $\mathcal{L}\{e^{4t}+4t^3-2\sin 3t+3\cos 5t\}$
2. $\mathcal{L}\{4e^{5t}+6t^3-3sint+2cos2t\}$

First shifting property: $\mathcal{L}\{e^{-at}F(t)\} = f(s-a)$

1. $\mathcal{L}\{e^{2t}(sint+3cos3t\}$
2. $\mathcal{L}\{e^{-t}\sin^2 4t\}$

Second shifting property: $G(t) = \begin{cases} F(t-a) & ,t>a \\ 0 & ,t<a \end{cases}$ then $\mathcal{L}\{G(t)\} = e^{-as}f(s)$.

1. Find $\mathcal{L}\{G(t)\}$ when $G(t) = \begin{cases} t-2\ ^{2} & ,\ t>2 \\ 0 & ,t<2 \end{cases}$

Multiplication by $t^n$ : $\mathcal{L}\{t^n F(t)\} = (-1)^n \dfrac{d^n}{ds^n} f(s)$

1. Show that $\mathcal{L}\{t^2\sin 2t\} = ?$

3. Show that $\mathcal{L}\{t^2\cos t\} = \dfrac{2s^3-6s}{(s^2+1)^3}$

**2.** Show that $\mathcal{L}\left\{t^3 e^{5t}\right\} = ?$ 4. Show that $\mathcal{L}\left\{t\cos 2t\right\} = ?$

Division by t: $\mathcal{L}\left\{\dfrac{F(t)}{t}\right\} = \int_s^{\infty} f(u)\,du$

**1.** $\mathcal{L}\left\{\dfrac{\cos at - \cos bt}{t}\right\}$ 2. $\mathcal{L}\left\{\dfrac{e^{-t}\sin t}{t}\right\}$

**02**

# Inverse Laplace Transformation

**Definition:** Inverse Laplace transform and its properties.

**Problems:**

Linearity property:

1. $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

3. $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$

2. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4s+3}\right\}$

First Shifting Property:

1. $\mathcal{L}^{-1}\left\{\dfrac{s+5}{s^2+4s+5}\right\}$ 2. $\mathcal{L}^{-1}\left\{\dfrac{4s+3}{s^2+9s+25}\right\}$ 3. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+6s+5}\right\}$

Second Shifting Property:

1. $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+5s+9}\right\}$ 2. $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+6s+5}\right\}$

Multiplication by $t^n$ / Inverse Laplace transform of derivative

1. $\mathcal{L}^{-1}\left\{\tan^{-1}\dfrac{1}{s}\right\}$ 2. $\mathcal{L}^{-1}\left\{\cot^{-1}(s+1)\right\}$

Division by t/ Inverse Laplace transform of integrals

1. $\mathcal{L}^{-1}\left\{\int_s^{\infty}\left(\dfrac{2}{u^2+4}+\dfrac{1}{u^3}+\dfrac{3}{u^2-2}+\dfrac{u}{u^2+4}\right)du\right\}$

2. $\mathcal{L}^{-1}\left\{\int_s^{\infty}\dfrac{5}{(u+1)(u^2+5u+6)}\,du\right\}$

Convolution theorem: $\mathcal{L}^{-1}\left\{f(S).g(S)\right\} = \int_0^t F(t-u)G(u)\,du$

1. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)(s-2)}\right\}$

3. Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

2. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

4. Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

## Application:

Formula: $\mathcal{L}\{Y^n(t)\} = s^n y(s) - s^{n-1}Y(0) - s^{n-2}Y'(0) - \cdots - Y^{(n-1)}(0)$

**Problems:**

(a) Solve IVP $Y'' + Y = t, \quad Y(0) = 1, \ Y'(0) = -2$

(b) Solve IVP $Y''(t) + 4Y(t) = 0$ ; $Y(0) = 1, Y'(0) = 3.$

(c) Solve IVP $Y''(t) + 5Y'(t) + 6Y(t) = e^t$ ; $Y(0) = 1, Y'(0) = 0.$

(d) Solve IVP $Y''(t) + 5Y'(t) + 6Y(t) = \cos t$ ; $Y(0) = 1, Y'(0) = 0.$

(e) Solve the IVP $Y''(t) + 9Y = 40e^t, \ Y(0) = 5, Y'(0) = -2$

(f) Solve the IVP $Y'' + Y = t, \ Y(0) = 1, \ Y'(0) = -2$

(g) Solve the IVP $Y'' - 3Y' + 2Y = 4e^{2t}, \ Y(0) = -3, Y'(0) = 5$

(h) Solve the IVP $Y'' + Y = 8\cos t, \ Y(0) = 1, \ Y'(0) = -1$

# *Fourier Transformation*

**Definition:** Fourier transform, Fourier sine & cosine transform.

01

**Finite Fourier sine / cosine Transformation** $F_s\{F(x)\} = \int_0^l F(x)\sin\frac{n\pi x}{l}\,dx = f_s(n)$

(a) $F(x) = 2x$ , $0 < x < 4$

(b) $F(x) = e^{mx}$ , $0 < x < \pi$

(c) $F(x) = \sin mx$ , $0 < x < \pi$

(d) $F(x) = \cos mx$ , $0 < x < \pi$

**Infinite Fourier sine / cosine Transformation** $F_c\{F(x)\} = \int_0^\infty F(x)\cos nx\,dx = f_c(n)$

(a) $F(x) = e^{-ax}$ , $0 < x < \infty$

(b) $F(x) = e^{-2x} + e^{-5x}$ , $0 < x < \infty$

## Application:

Use Finite Fourier Transform to solve the boundary value problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}; U(0,t) = U(4,t) = 0, U(x,0) = 2x \text{ where } 0 < x < 4, t > 0.$$

# Fourier series:

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{n}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$ defined on the interval $(-l\ ,\ l)$ with Fourier coefficients

$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx$ and $b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx$ .

**Special formula:** 1. $\int_{-a}^{a}f(x)\,dx=\begin{cases}2\int_{0}^{a}f(x)\,dx & ,\ f(x)\ even\\ 0 & ,\ f(x)\ odd\end{cases}$ 2. $\sin n\pi=0\ ,\ \cos n\pi=(-1)^n$

3. $\sin(n\pi+\theta)=(-1)^n\sin\theta$ 4. $\sin(n\pi-\theta)=(-1)^{n+1}\sin\theta$

5. $\cos(n\pi\pm\theta)=(-1)^n\cos\theta$

01

**Definition:** Fourier series, Even, Odd functions, half range sine and cosine series.

**Problems:**

Full range Fourier Series

(a) $F(x)=x+x^2,-\pi<x<\pi$

(b) (b) $F(x)=x+\frac{x^2}{4},-\pi<x<\pi$

(c) $F(x)=\sin ax,-\pi<x<\pi$

Half range Fourier Series

(a) $F(x)=e^{ax},0<x<\pi$

(b) $F(x)=x\ ,0<x<2$

**End**