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Department :CSE

Course :Discrete Mathematics

Part:01

DISCRETE MATHEMATICS

PART: 01

Discrete Mathematics

Propositional Logic: (vaild logical arguments)

What is proposition?

⇒ A proposition is a declarative statement that is either true (T) or false (F), but not both.

True = T = 1
False = F = 0

Example:

$1 + 1 = 2$ ⟶ Truth value of this proposition: T

$2 + 2 = 5$ ⟶ Truth value of this proposition: F

Sylhet is the capital of Bangladesh.
⟶ Truth value of this proposition: F

Pigs can fly ⟶ Truth value of this proposition: F

Not proposition

1. What time is it? ⟶ Questions
2. Read this carefully? ⟶ Commands/Imperative
3. Do your homework.
4. $x + 1 = 2$ ⟶ Non constant values
5. Bangladesh and india ⟶ Not statements.
6. Shain is the best lecturer. ⟶ Opinion
7. He is a college student. ⟶ Person is not defined.

Propositional variables / Statement variables:

- Today is Friday → p = Today is Friday

- It is raining. → q = It is raining

  The truth value of the propositional variable can be True or False.

Compound Proposition

- Compound proposition is a proposition formed by combining two or more simple proposition.

- The logical operators that are used to form compound proposition is called connectives.

Symbol	Math name	English name
$\neg$	Negation	NOT
$\vee$	Disjunction	OR
$\wedge$	Conjunction	AND
$\oplus$	ExOR	"OR.. but not both"
$\rightarrow$	Implication	"If... then"
$\Leftrightarrow$	Equivalence	"If & only if"

Negation:

Let p be a proposition. The compound proposition "it is not the case that p", is an other proposition, called the negation of p and denoted $\neg p$.

The truth value of the negation of p is the opposite of the truth value of p. The proposition $\neg p$ is read "not p".

Truth table:

p	$\neg p$
T	F
F	T

Conjunction:

Let p and q be propositions. The compound "p and q"

denoted $p \wedge q$, is true when both p and q are true and false otherwise. This compound proposition $p \wedge q$ is called the conjunction of p and q.

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

Let p and q be proposition. The compound proposition

"p or q"

denoted $p \vee q$, is false when both p and q are false and true otherwise. This compound proposition $p \vee q$ is called the disjunction of p and q.

Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Disjunction:

Let p and q be proposition

"p exclusive or q"

denoted $p \oplus q$

True when exactly one of p and q is true and is false otherwise. This compound proposition $p \oplus q$ is called the exclusive disjunction of p and q.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication:

Let p and q be propositions. The compound proposition

"if p, then q"

denoted $p \rightarrow q$, is false when p is true and q is false, and is true otherwise.

⇒ This compound proposition $p \rightarrow q$ is called the implication (or the conditional statement) of p and q.

$p \rightarrow$ hypothesis, antecedent, premise

$q \rightarrow$ conclusion, consequence

Truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Remarks:

⇒ The implication $p \to q$ is false only when p is true and q is false.

⇒ The implication $p \to q$ is true when p is false whatever the truth value of q.

Implication $p \to q$. There are three related implications.

- Converse → $p \to q$ is the proposition $q \to p$
- Inverse → Same $\neg p \to \neg q$
- Contrapositive
  - → $p \to q$ is the proposition $\neg q \to \neg p$

Bioconditional Statement

Let p and q be propositions.

"p if and only if q"

Denoted $p \leftrightarrow q$

is true when p and q have the same truth value and is false otherwise.

This compound proposition $p \leftrightarrow q$ is called the biconditional statement (or the bi-implication) of p and q.

Biconditional Statement $p \leftrightarrow q$ is true when both implications $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise.

- "p is necessary and sufficient for q"
- "if p then q, and conversely.
- " p iff q".

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Truth table of the compound proposition:

$$(p \lor \neg q) \rightarrow (p \land q)$$

⇒ Step:01

p	q	$\neg q$
T	T	F
T	F	T
F	T	F
F	F	T

Step:02

p	q	$\neg q$	$p \lor \neg q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Step:03

p	q	¬q	p ∨ ¬q	p ∧ q
T	T	F	T	T
T	F	T	T	F
F	T	F	F	F
F	F	T	T	F

Step:04

p	q	¬q	p ∨ ¬q	p ∧ q	(p ∨ ¬q) → (p ∧ q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operator

Operator	Precedence
( )	0
¬	1
∧	2
∨	3
→	4
↔	5

Boolean Variable:

A variable is called a Boolean variable if its value is either true or false.

A bit is a symbol with possible values namely 0 and 1.

A Boolean variable can be represented using a bit.

Bit Operations:

x	y	$x \wedge y$
1	1	1
1	0	0
0	1	0
0	0	0

x	y	$x \vee y$
1	1	1
1	0	1
0	1	1
0	0	0

x	y	$x \oplus y$
1	1	0
1	0	1
0	1	1
0	0	0

Bit String:

A bit string is a sequence of zero or more bits. The length of this bit string is the number of bits in the string.

Example:

$$\begin{array}{r} 10110 \\ \wedge\ 10101 \\ \hline 10100 \end{array} \qquad \begin{array}{r} 10110 \\ \vee\ 10101 \\ \hline 10111 \end{array} \qquad \begin{array}{r} 10110 \\ \oplus\ 10101 \\ \hline 00011 \end{array}$$

Propositional Equivalences

Definition:

A compound proposition that is always true, no matter what the truth values of the occur in it, is called a tautology.

Definition:

A compound proposition that is always false, no matter what the truth values of the propositions that occur in it. is called a contradiction.

Definition:

A compound proposition that is neither a tautology nor a contradiction is called a contigency.

Example: Tautology

$p \lor \neg p$

p	$\neg p$	$p \lor \neg p$
T	F	T
F	T	T

Contradiction

$p \land \neg p$

p	$\neg p$	$p \land \neg p$
T	F	F
F	T	F

Logical Equivalence:

The compound proposition p and q are called logically equivalent.

IF $p \leftrightarrow q$ is a tautology.

The notation $p \equiv q$ denotes that p and q are logically equivalent.

$$p \Leftrightarrow q$$

Example:

$$(\neg p \lor q) \leftrightarrow (p \rightarrow q)$$

p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$	$(\neg p \lor q) \leftrightarrow (p \rightarrow q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

$$(\neg p \lor q) \equiv (p \rightarrow q)$$

Disjunctive normal form of the implication (DNFI)

De Morgan's Law – 1

$\neg(p \lor q)$ and $\neg p \land \neg q$

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$	$\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

$$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$$

De Morgan's Law 2

The compound propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

$$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$$

Logical Equivalences

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Equivalence	Name
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws

Logical Equivalences Involving Conditional Statement

$p \rightarrow q \equiv \neg p \vee q$ (DNFI)

$p \rightarrow q \equiv \neg q \rightarrow \neg p$ (Contra positive)

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Bioconditionals

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Disjunctive Normal Form

A compound proposition is said to be in disjunctive normal form if it is a disjunction of conjunction of the variables or their negations.

For example:

$$(p \land q \land r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land r)$$

Truth Table

$$(p \lor \neg q) \to (p \land q)$$

p	q	$\neg q$	$p \lor \neg q$	$p \land q$	$(p \lor \neg q) \to (p \land q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

$$(p \land q) \lor (\neg p \land q)$$

Functionally Completeness:

A collection of logical operators is called Functionally complete if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

Example: $\{\vee, \wedge, \neg\}$

Predicates and Quantifiers

Variable and Predicate:

X is greater tham 3.

→ This declarative statement is neither true nor false because the value of x is not specified. There fore this declarative statement is not a proposition.

→ Declarative Statement

* The variable x is the subject of the statement.
* Is greater than 3" is the Predicate that refers to a Property that the subject of the statement can have.

x is greater than 3.

Variable

Predicate

Propositional Function:

" x is greater than 3 "

The declarative statement $p(x)$ is said to be the value of the propositional function p at x.

Set of integers $Z = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$

$P(x)$, the propositional function " $x > 3$ "

* If no value has been assigned to x this propositional function has no truth value.

* $P(-3)$ is " $-3 > 3$ " which is a false proposition

* $P(5)$ is " $5 > 3$ " which is a true Proposition

* $P(y) \wedge \neg P(0)$ is not a proposition because the variable y has no value yet.

* $P(5) \wedge \neg P(0)$ is " $(5 > 3) \wedge \neg (0 > 3)$ which is a true proposition.

Quantification

Quantification is another method to transform a propositional function into a proposition.

- The area of logic that deals with predicates and quantifiers is called the predicate Calculus.

Quantification

Universal quantification | Existential quantification

Universe of Discourse:

Many mathematical statements assert that property is true for all values of a variable in a particular domain called universe of Discourse.

Universal Quantification:

The universal Quantification of $P(x)$ is the proposition:

"$P(x)$ for all values of x in the universe of discourse."

$\forall x\, P(x) \longrightarrow$ universal quantification of $P(x)$.

$\forall \rightarrow$ universal quantifier

$\forall x\, P(x)$ as "for all x, $P(x)$" or "for every x, $P(x)$".

$\Rightarrow$ An element in the universe of discourse for which $P(x)$ is false is called a counterexample of $\forall x\, P(x)$.

Example:

When all the elements in the universe of discourse can be listed say $x_1, x_2 \dots x_n$ it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction.

$$\boxed{P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)}$$

This conjunction is true if and only if $P(x_1), P(x_2), \dots P(x_n)$ are all true.

Practice: $U = \{1, 2, 3\}$

$$\forall x\, P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$\exists$ Existential Quantification

The existential quantification of $P(x)$ is the proposition.

"There exists an element x in the universe of discourse such that $P(x)$."

$\exists x\, P(x) \rightarrow$ existential quantification of $P(x)$.

$\exists$ → existential quantifier.

⇒ IF the universe of discourse is empty then $\exists x P(x)$ is false for any propositional function $P(x)$ because there are no elements x in the empty universe of discourse for which $P(x)$ is true.

$$P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$$

→ disjunction is true if and only if at least one of $P(x_1), P(x_2) \dots P(x_n)$ is true.

Example: $U = \{1, 2, 3\}$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

Uniqueness Quantification

"There exists a unique x in the universe of discourse such that $P(x)$."

Notation: $\exists ! x\, P(x)$

$\exists !$ → Uniqueness Quantifier

$$\exists! x\, P(x) \equiv \exists x \left(P(x) \wedge \forall y \left(P(y) \rightarrow (y = x) \right) \right)$$

$$\equiv \exists x \forall y \left(P(y) \leftrightarrow (y = x) \right)$$

Operator	Precedence
( )	-1
$\forall$, $\exists$ $\neg$	0 1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Example:

$\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$, not $\forall x\, (P(x) \vee Q(x))$.

Bound and Free Variables

⇒ If a quantifier is used on the variable x, then this variable is bound.

⇒ An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

Example:

1. Let $p(x, y)$, the propositional Function " $x + y = 0$ "

2. The logical variables x and y are free and we cannot evaluate the truth value of $p(x, y)$.

3. IF the value 3 is set to x, then X is no longer a free Variables, but $P(3, y)$ is still a propositional function because y is still a free Variable.

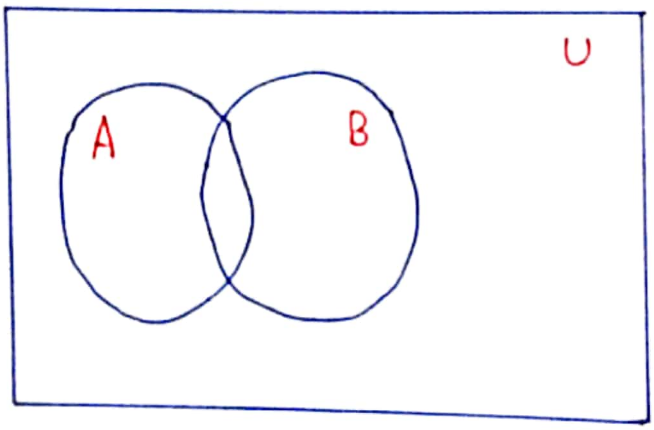
4. IF we apply the universal the quantification to the Variable y, the Propositional function $\forall y \, P(3, y)$ is now a proposition. Both variables X and y are no longer free and the truth value of the proposition is false.

Set Operations

Definition:

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B or in both.

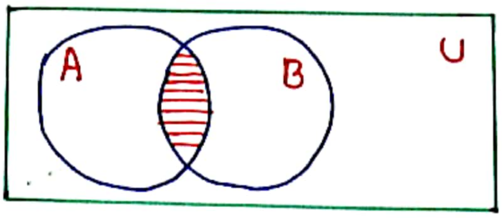
$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$



Intersection of Sets:

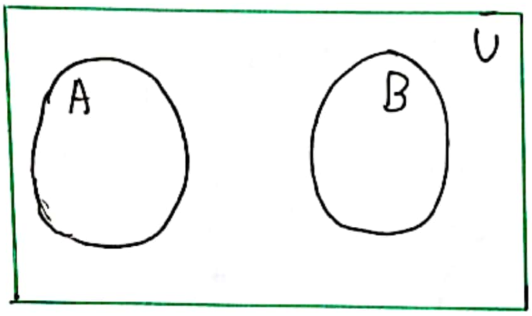
The intersection of the sets A and B denoted by $A \cap B$. is the set containing those elements in both A and B.

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$



Disjoint Sets :

Two sets are called disjoint if their intersection is the empty set.



Principle of Inclusion - Exclusion :

The number of elements in the union of two sets is equal to the number of elements in the first set plus the number of elements in the second one, minus the number of elements in the intersection of the two sets because they were counted twice.

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

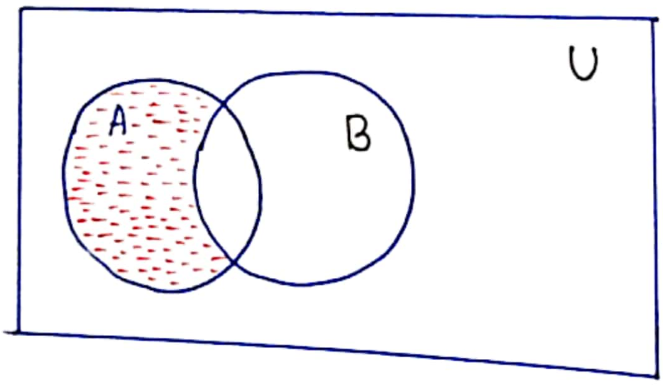
Difference of Sets :

difference of A and B

$\boxed{A-B}$

A and B is also called the complement of B with respect to A.

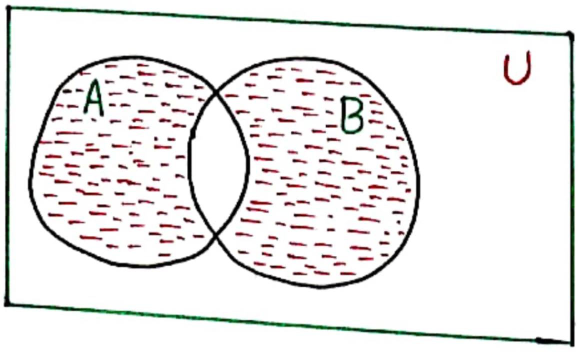
$$\boxed{A-B=\{x \mid (x\in A)\wedge(x\notin B)\}}$$



Symmetric Difference of Sets

Symmetric difference A and B. denoted by $A\oplus B$.

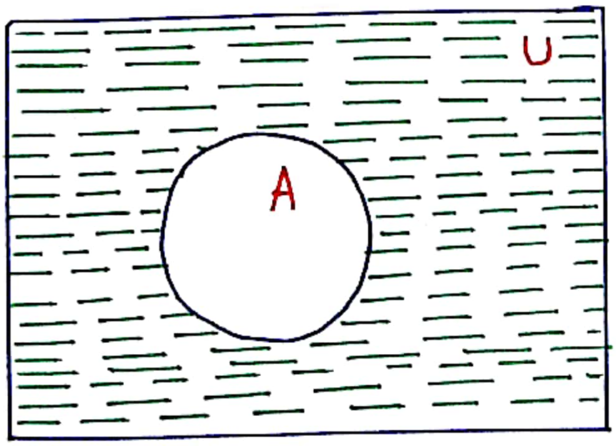
$$\boxed{A\oplus B=\{x \mid (x\in A)\oplus(x\in B)\}}$$



Complement of Sets

The complement of the set A. Denoted by $\bar{A}$ or A^c, is the set those elements that are in U but not in A. $U - A$.

$$\boxed{\bar{A} = \{x \mid x \notin A\}}$$



Set Identities

Identity	Name
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Membership Table

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

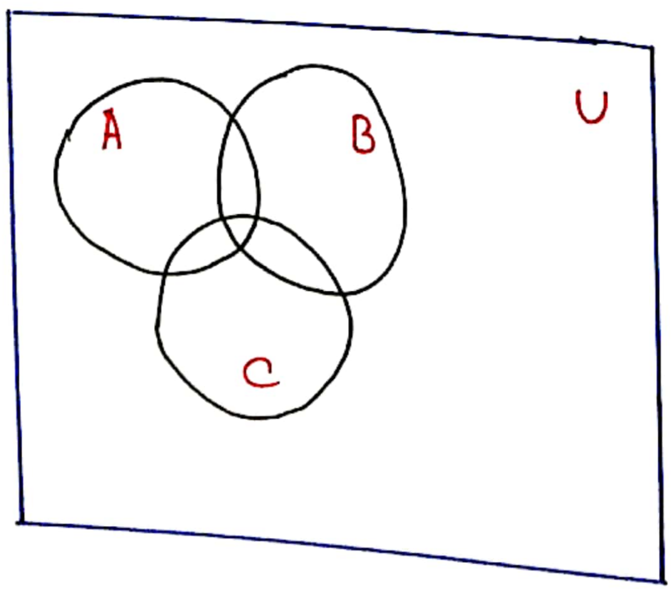
Generalized Union of Sets

The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of sets $A_1, A_2, \dots A_n$

Example:



Generalized Intersection of Sets

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets

$A_1, A_2, \dots A_n$