DP Coin Change
DP 01 Knapsack

Week-06, Lecture-02

Course Code: CSE221

Course Title: Algorithms

Program: B.Sc. in CSE

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Knapsack Problem

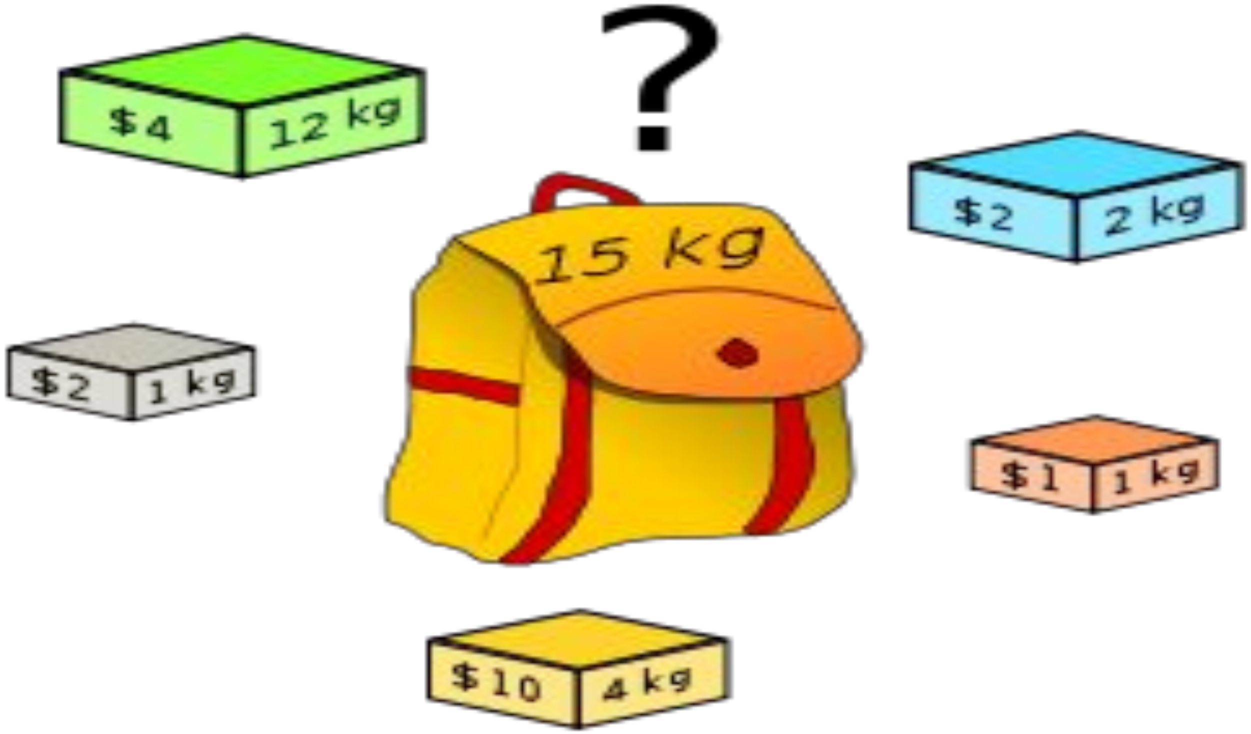
Knapsack problem

A 1998 study of the Stony Brook University Algorithm Repository showed that, out of 75 algorithmic problems, the knapsack problem was the 18th most popular and the 4th most needed after kd-trees, suffix trees, and the bin packing problem

Knapsack Problem

- Given a set of items, each with a mass and a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.
- It derives its name from the problem faced by someone who is constrained by a fixed-size knapsack and must fill it with the most valuable items.

Knapsack Problem



which boxes should be chosen to maximize the amount of money while still keeping the overall weight under or equal to 15 kg?

Answer: **3 yellow boxes** and **3 grey boxes**

Dynamic Programming

Recursive formula for sub problems:

$$V[k, s] = \begin{cases} V[k-1, s] & if\, s_k > \\ \max\{V[k-1, s], V[k-1, s-s_k] + v_k\} & els \end{cases}$$

Dynamic Programming

for s = 0 to S

V[0, s] = 0

for i = 1 to n

V[i, 0] = 0

Dynamic Programming

- for i = 1 to n

if $s_i \le s$

  if $v_i + V[i-1, s-s_i] > V[i-1, s]$

    $\mathrm{V[i,s]} = v_i + V[i-1, s-s_i]$

  else

    $\mathrm{V[i,s] = V[i-1,s]}$

else $\mathrm{V[i,s] = V[i-1,s]}$

Example

- Given:

n = 4 (# of elements)

S = 5 (maximum size)

Elements (size, value) =

{ (2, 3), (3, 4), (4, 5), (5, 6) }

Example

i\s	0	1	2	3	4	5
0						
1						
2						
3						
4						

Example

- for s = 0 to S

  V[0, s] = 0

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

Example

- for i = 0 to n

  V[i, 0] = 0

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
4	0					

Example

Items (s_i, v_i)

$\boxed{1: (2, 3)}$

2: (3, 4)

3: (4, 5)

4: (5, 6)

if $s_i \leq s$

if $v_i + V[i-1, s-s_i] > V[i-1, s]$

$$\mathrm{V[i,s]} = v_i + V[i-1, s-s_i]$$

else

$$\mathrm{V[i,s] = V[i-1,s]}$$

else V[i, s] = V[i − 1, s]

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0				
2	0					
3	0					
4	0					

Example

- if $s_i \leq s$
  - if $v_i + V[i-1, s-s_i] > V[i-1, s]$

$$\mathbf{V[i,s]} = \boldsymbol{v_i + V[i-1, s-s_i]}$$

  - else

$$\mathrm{V[i,s] = V[i-1,s]}$$

- else $V[i,s] = V[i-1,s]$

Items (s_i, v_i)

$\boxed{\text{1: (2, 3)}}$

2: (3, 4)

3: (4, 5)

4: (5, 6)

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3			
2	0					
3	0					
4	0					

Example

- if $s_i \leq s$
  - if $v_i + V[i-1, s-s_i] > V[i-1, s]$

$$\mathbf{V[i, s] = v_i + V[i-1, s-s_i]}$$

  - else

$$V[i, s] = V[i-1, s]$$

- else $V[i, s] = V[i-1, s]$

Items (s_i, v_i)

1: (2, 3)

2: (3, 4)

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0	0	0	0	0	0	0
1	0	0	3	3		
2	0					
3	0					
4	0					

Example

Items (s_i, v_i)

$\boxed{1: (2, 3)}$

2: (3, 4)

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4: (5, 6)

- if $s_i \leq s$

  if $v_i + V[i-1, s-s_i] > V[i-1, s]$

  $$\mathbf{V[i, s]} = \boldsymbol{v_i + V[i-1, s-s_i]}$$

  else

  $$\mathrm{V[i, s] = V[i-1, s]}$$

  else $V[i, s] = V[i-1, s]$

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	
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Example

if $s_i \leq s$

  if $v_i + V[i-1, s-s_i] > V[i-1, s]$

$$\mathbf{V[i,s] = v_i + V[i-1, s-s_i]}$$

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Items (s_i, v_i)

$\boxed{1: (2, 3)}$

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i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
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3	0					
4	0					

Example

Items (s_i, v_i)

1: (2, 3)
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- if $s_i \leq s$
  - if $v_i + V[i-1, s-s_i] > V[i-1, s]$
    - $\mathrm{V[i,s]} = v_i + V[i-1, s-s_i]$
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    - $\mathrm{V[i,s]} = \mathrm{V[i-1,s]}$

else $\mathbf{V[i, \boldsymbol{s}] = V[i-1, s]}$

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
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Example

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Items (s_i, v_i)

1: (2, 3)
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i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
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Example

Items (s_i, v_i)

1: (2, 3)
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- if $s_i \leq s$
  - if $v_i + V[i-1, s-s_i] > V[i-1, s]$

$$\mathbf{V[i, s] = v_i + V[i-1, s-s_i]}$$

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Example

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  - if $v_i + V[i-1, s-s_i] > V[i-1, s]$

    $$\mathbf{V[i, s] = v_i + V[i-1, s-s_i]}$$

  - else

    $$V[i, s] = V[i-1, s]$$

- else $V[i, s] = V[i-1, s]$

Items (s_i, v_i)

$\boxed{\begin{array}{l} 1: (2, 3) \\ 2: (3, 4) \end{array}}$

3: (4, 5)

4: (5, 6)

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	
3	0					
4	0					

Example

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  if $v_i + V[i-1, s-s_i] > V[i-1, s]$
  $$\mathbf{V[i, s] = v_i + V[i-1, s-s_i]}$$
  else
  $$\mathrm{V[i, s] = V[i-1, s]}$$
  else $V[i, s] = V[i-1, s]$

Items (s_i, v_i)

1: (2, 3)
2: (3, 4)
3: (4, 5)
4: (5, 6)

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0					
4	0					

Example

- if $s_i \leq s$

  if $v_i + V[i-1, s-s_i] > V[i-1, s]$

  $\mathrm{V[i, s]} = v_i + V[i-1, s-s_i]$

  else

  $\mathrm{V[i, s] = V[i-1, s]}$

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Items (s_i, v_i)

$\boxed{\begin{array}{l} 1\text{: }(2, 3) \\ 2\text{: }(3, 4) \\ 3\text{: }(4, 5) \end{array}}$

4: (5, 6)

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4		
4	0					

Example

- if $s_i \leq s$

  if $v_i + V[i-1, s-s_i] > V[i-1, s]$

  $$\mathbf{V[i, s] = v_i + V[i-1, s-s_i]}$$

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Example

if $s_i \leq s$

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Example

Items (s_i, v_i)

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i\s	0	1	2	3	4	5
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Example

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Example

- $i = n, k = S$

while $i, k > 0$

if $V[i,k] \neq V[1-i,k]$

$i = i - 1, k = k - s_i$

else

$i = i - 1$

Items (s_i, v_i)

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Example

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else
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1	0	0	3	3	3	3
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3	0	0	3	4	5	7
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Example

Items (s_i, v_i)

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i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
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Example

$i = n, k = S$
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0	0	0	0	0	0	0
1	0	0	3	3	3	3
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3	0	0	3	4	5	7
4	0	0	3	4	5	7

Example

The optimal knapsack should contain {1,2} = 7

Items (s_i, v_i)

1: (2, 3)
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i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
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3	0	0	3	4	5	7
4	0	0	3	4	5	7

Exercise

Solve the Unbounded Knapsack Problem using dynamic programming for the following instance:

- Weights: [1, 3, 4, 5]
- Values: [1, 4, 5, 7]
- Capacity: 7

Textbooks & Web References

- Reference book iii (Chapter 19)
- www.geeksforgeeks.org
- www.codeforces.com
- www.topcoder.com

Thank you

&

Any question?