Functions
Discrete Mathematics

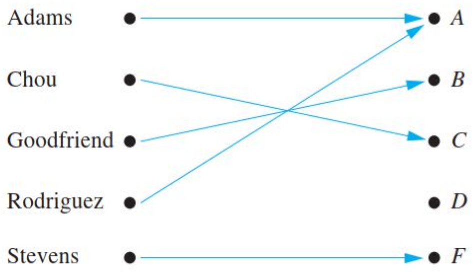


FIGURE 1 **Assignment of Grades in a Discrete Mathematics Class.**

In many instances we assign to each element of a set a particular element of a second set. For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A, B, C, D, F }. This assignment of grades is illustrated in Figure 1. This assignment is an example of a function.

Functions are specified in many different ways. Sometimes we explicitly state the assignments, as in Figure 1. Often we give a formula, such as

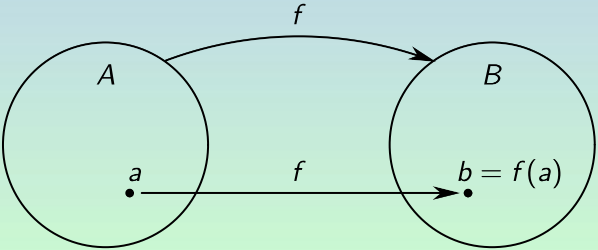
$f(x) = x^2$. In this function, the function $f(x)$ takes the value of “x” and then squares it. For instance,
if $x = 3$, then $f(3) = 9$.

A few more examples of functions are:
$f(x) = x^2 + 3$
$f(x) = 1/x$
$f(x) = 2x + 3$

Definition: Function

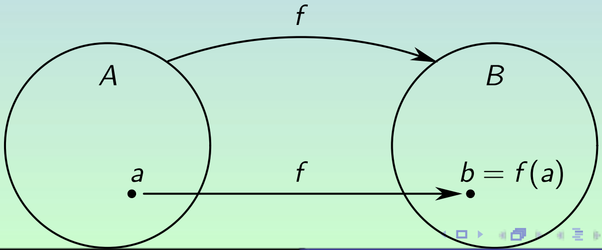
Definition

Let A and B be non empty sets. A **function** f from A to B is an assignment of *exactly one* element of B to each element of A. We write $f(a) = b$ if b is the *unique* element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

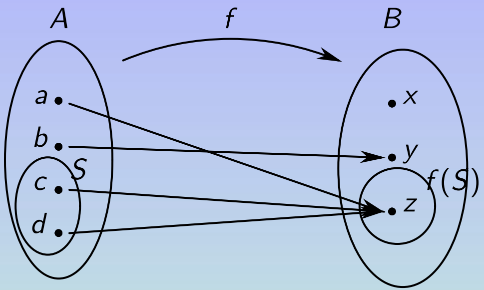


Definitions: Domain, Codomain, Image, Preimage and Range

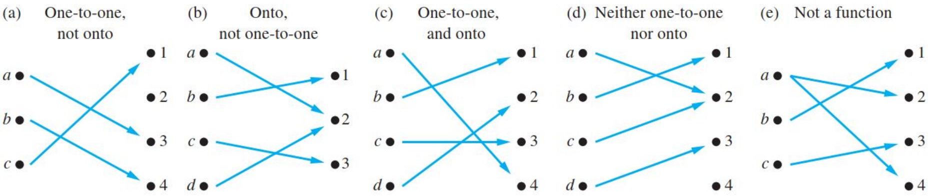
If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f.
If $f(a) = b$, we say that b is the **image** of a and a is the **preimage** of b.
The **range** of f is the set of all images of elements of A.
Also, If f is a function from A to B, we say that f **maps** A to B.



Example



- The domain of f is $A = \{a, b, c, d\}$.
- The codomain of f is $B = \{x, y, z\}$.
- $f(a) = z$.
- The image of a is z.
- The preimages of z are a, c and d.
- The range of f is $f(A) = \{y, z\} \subseteq B$.
- The image of the subset $S = \{c, d\} \subseteq A$ is $f(S) = \{z\} \subseteq B$.


(a) One-to-one, not onto
(b) Onto, not one-to-one
(c) One-to-one, and onto
(d) Neither one-to-one nor onto
(e) Not a function
a
b
c
d
1
2
3
4

Definition: One-To-One (Injective) Function

Definition

A function f from A to B is said to be **one-to-one**, or **injective**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain A. A function is said to be an **injection** if it is injective.

By taking the contrapositive of the implication in this definition, a function is injective if and only if $a \neq b$ implies $f(a) \neq f(b)$.

Another way to understand it, a function is injective means that if an element of the codomain has a preimage, then it is a unique preimage.

Definition: Onto (Surjective) Function

Definition

A function f from A to B is called **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a **surjection** if it is surjective.

Another way to understand it, a function is surjective means that each element of the codomain has at least one preimage.

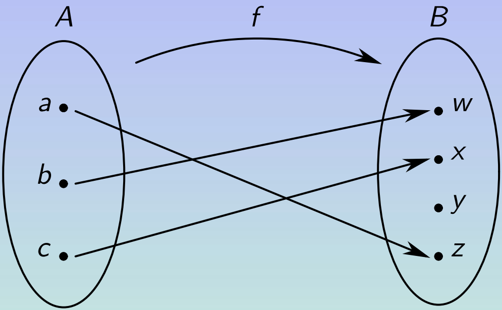
Definition: One-To-One Correspondence (Bijective) Function

Definition

The function f is a **one-to-one correspondence** if it is both one-to-one and onto.

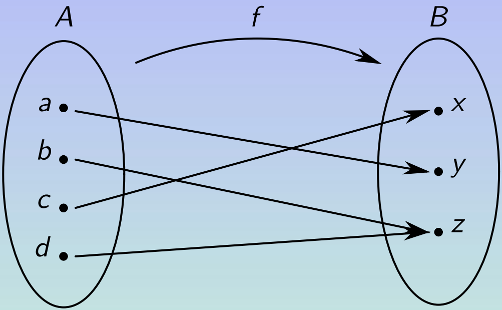
The function f is is said to be **bijective** if it is both injective and surjective. A function is said to be a **bijection** if it is bijective.

Example 1



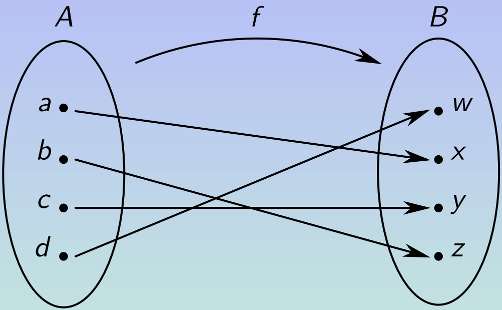
Is f injective?
Is f surjective?
Is f bijective?

Example 2



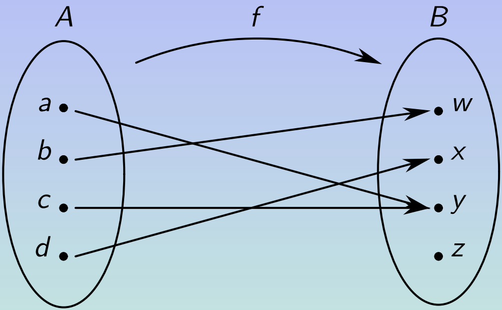
Is f injective?
Is f surjective?
Is f bijective?

Example 3



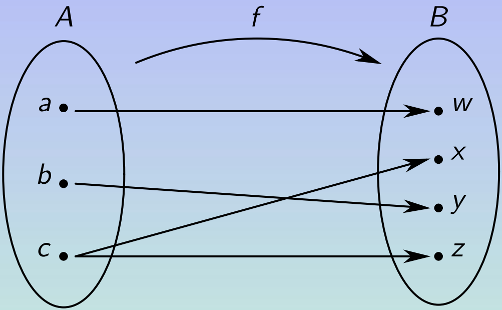
Is f injective?
Is f surjective?
Is f bijective?

Example 4



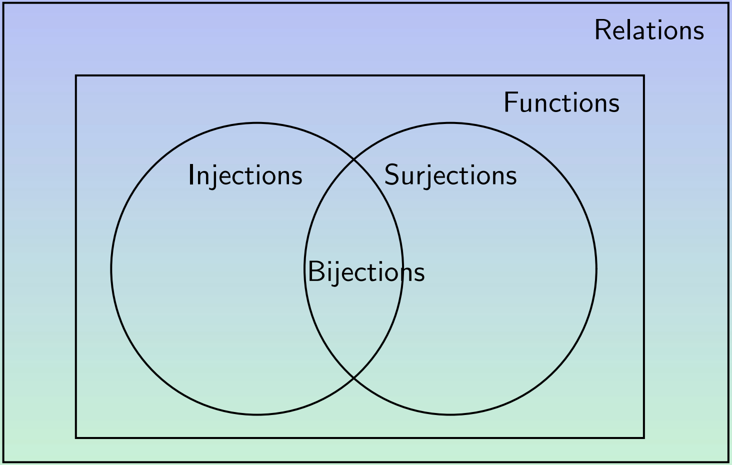
Is f injective?
Is f surjective?
Is f bijective?

Example 5



Is f injective?
Is f surjective?
Is f bijective?

Venn Diagram of Function Classification



Definition

Let f_1 and f_2 be functions from A to . Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Definition: Composition of Functions

Definition

Let g be a function from the set A to the set B, and let f be a function from the set B to the set C. The **composition of the functions** f **and** g, denoted by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

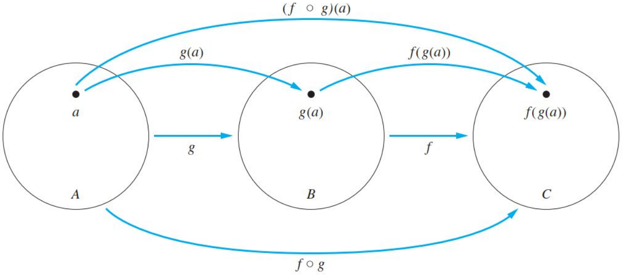


FIGURE 7 The Composition of the Functions f and g.

Definition: Inverse Function

Definition

Let f be a bijection from the set A to the set B. The **inverse function** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$. The inverse function is also a bijection.

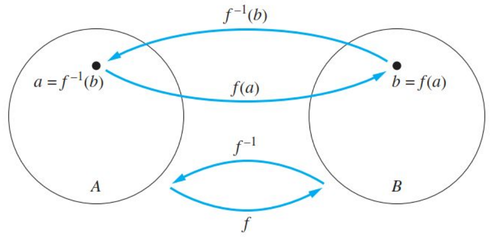


FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

Identity Function

Definition

Identity function (also called identity mapping): The identity mapping ${}_X : X \to X$ is the function with domain and codomain X defined by

$${}_X(x) = x, \quad \forall x \in X.$$

Left and Right Inverse

Definition

Let $f : X \to Y$ be a fonction with domain X and codomain Y , and $g : Y \to X$ be a fonction with domain Y and codomain X.

The function g is a **left inverse** of f if $g \circ f = {}_X$.

The function g is a **right inverse** of f if $f \circ g = {}_Y$.

The function g is an **inverse** of f if g is both a left and right inverse of f. When f has an inverse, it is often written f^{-1}.

Left and Right Inverse

Theorem

A function is **injective** *if and only if it has a* **left inverse***.*

A function is **surjective** *if and only if it has a* **right inverse***.*

A function is **bijective** *if and only if it has an* **inverse***.*

If a function has an inverse, then this **inverse is unique***.*

Note: The left and right inverses are not necessarily unique.