MR Aronna
CR(64-L)
Depertment :CSE
Discrete Mathematics
Part :02

DISCRETE MATHEMATICS

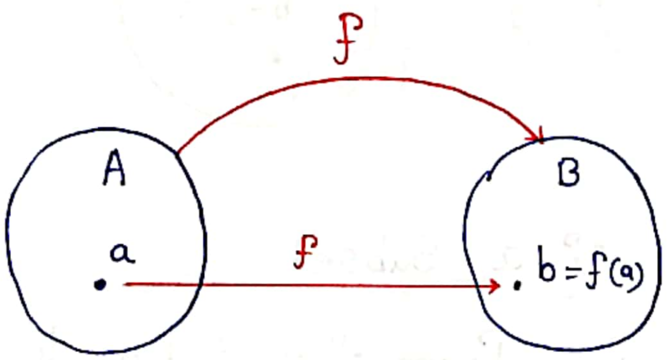
PART:02

Function

Definition:

Let A and B be non empty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B.

We write: $f : A \longrightarrow B$



Domain: Codomain, Image, Preimage and Range

⇒ If f is a function from A to B, we say that A is the domain of f and B is the codomain of f.

⇒ If $f(a) = b$, we say that b is the image of a and a is the Preimage of b.

⇒ The range of f is the set of all images of elements of A.

Also, If f is a function from A to B, we say that f maps A to B.

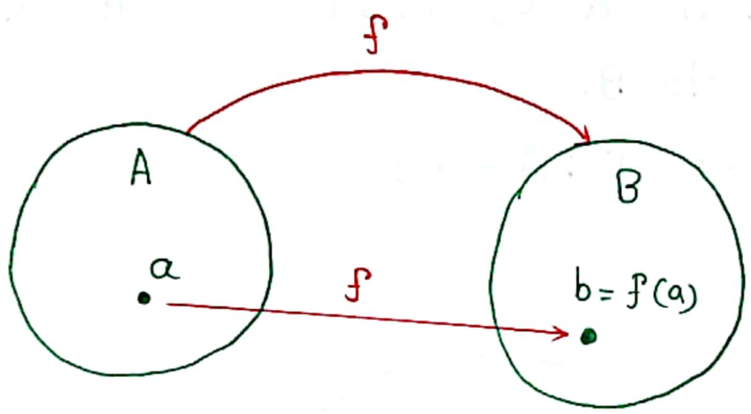


Image of a Subset

f → function from the set A to the set B.

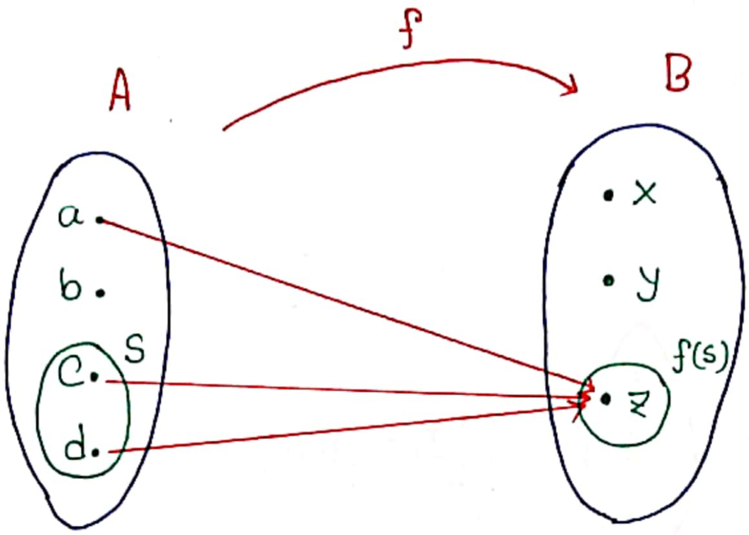
S → Subset of A.

The Image of S under the function f is the subset of B that consists of the images of the element of S. We denote by $f(S)$,

$$f(S) = \{t \in B \mid \exists s \in S \text{ with } (t = f(s))\}$$

We also use the shorthand

$$f(S) = \{f(s) \mid s \in S\}$$



$\Rightarrow$ The domain of f is $A = \{a, b, c, d\}$

$\Rightarrow$ The Codomain of f is $B = \{x, y, z\}$

$\Rightarrow$ $f(a) = z$.

$\Rightarrow$ The image of a is z.

$\Rightarrow$ The preimages of z are a, c and d.

$\Rightarrow$ The range of f is $f(A) = \{y, z\} \subseteq B$.

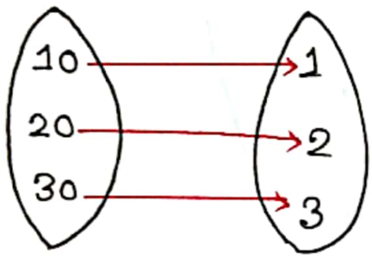
$\Rightarrow$ The image of the subset $S = \{c, d\} \subseteq A$ is

$f(S) = \{z\} \subseteq B$.

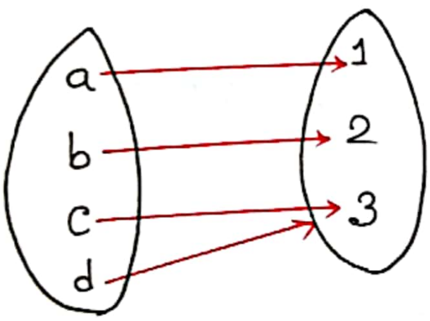
One - To - One (Injective) Function :

A function f from A to B is said to be one-to-one or injective if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain.

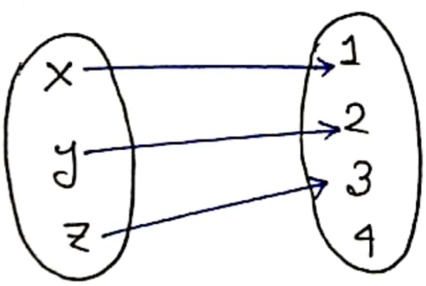
Example:



$\Rightarrow$ [Injective]



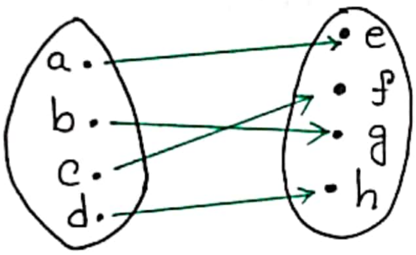
[Not Injective]



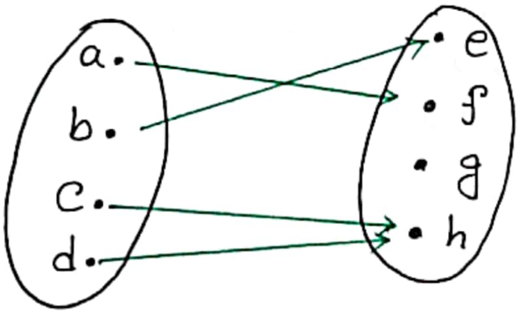
Onto (Surjective) Function:

A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$. A function f is called a surjective / surjection.

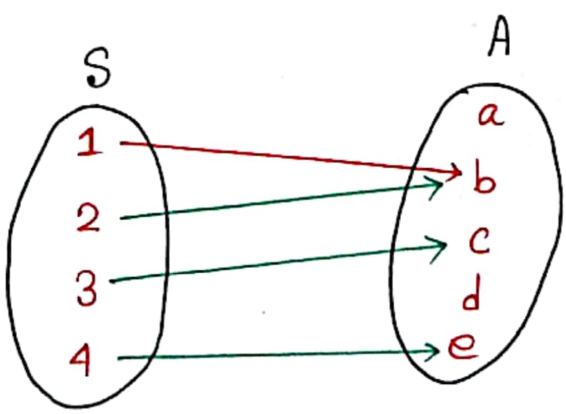
Example:



[Surjective]
[Onto]



[Not Surjective]
[not Onto]



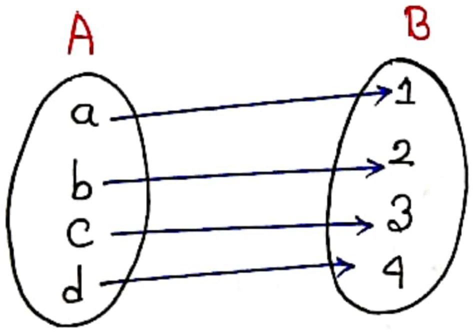
[Not Surjective]

One-To-One Correspondence [Bijective]

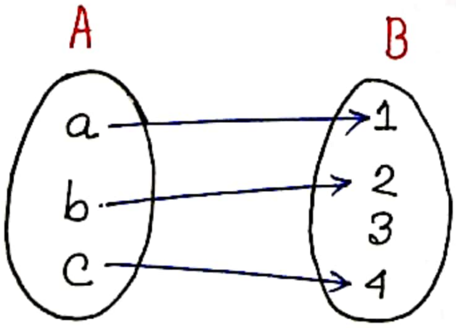
The function f is a one-to-one correspondence if it is both one-to-one and onto.

⇒ The function f is said to be bijective if it is both injective and surjective. A function is said to be a bijection.

Example

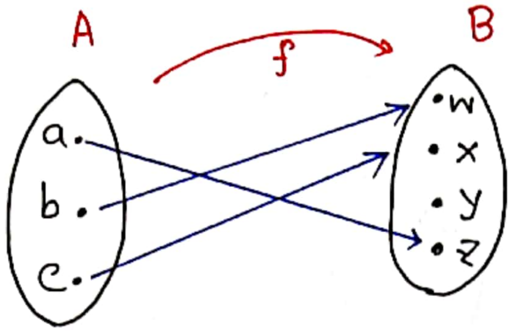


[Bijective]



[Not Bijective]

Example : 01

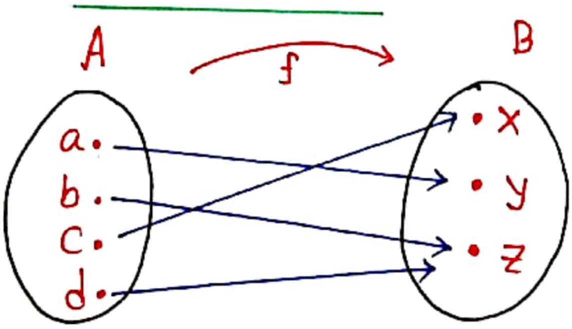


→ Is f injective ?

→ Is f Sujective ?

→ Is f bijective ?

Example : 02

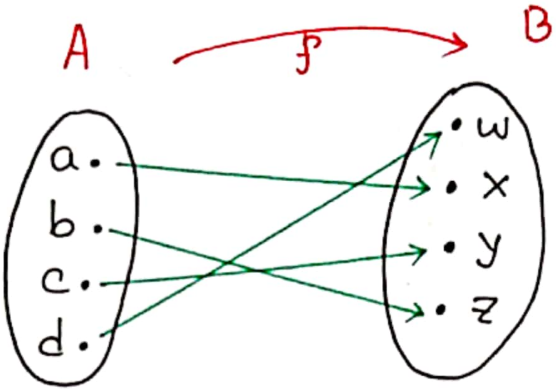


→ Is f injective ?

→ Is f Surjective ?

→ Is f bijective ?

Example: 3

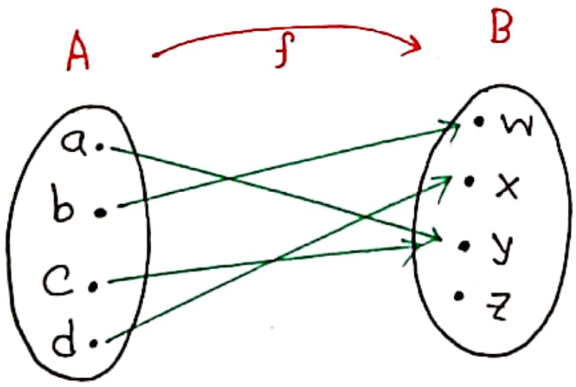


→ Is f injective ?

→ Is f Surjective ?

→ Is f bijective ?

Example: 4

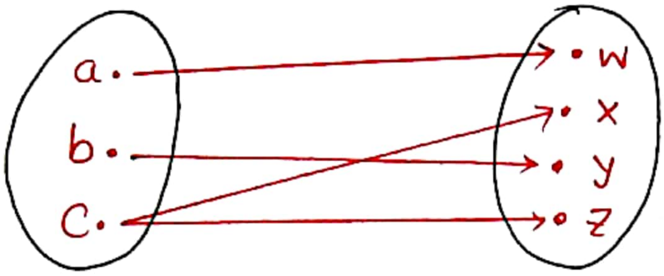


→ Is f injective ?

→ Is f Surjective ?

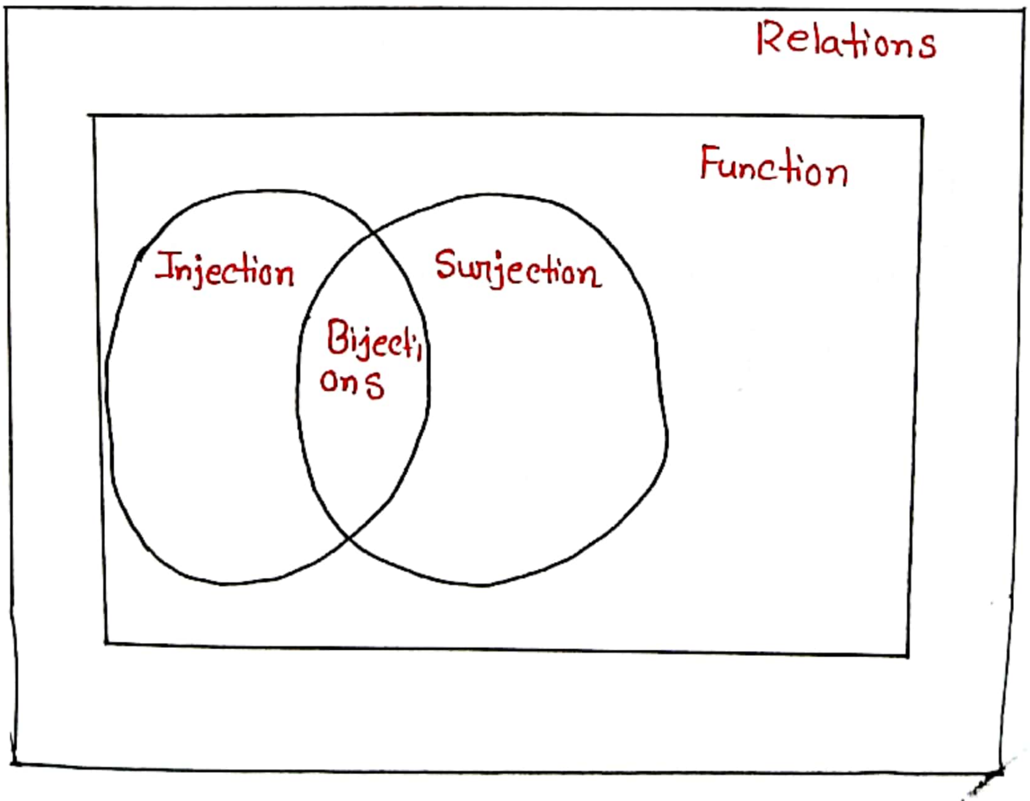
→ Is f bijective ?

Example: 5



- Is f injective ?
- Is f Surjective ?
- Is f bijective ?

Function Classification



Addition and Product of Function

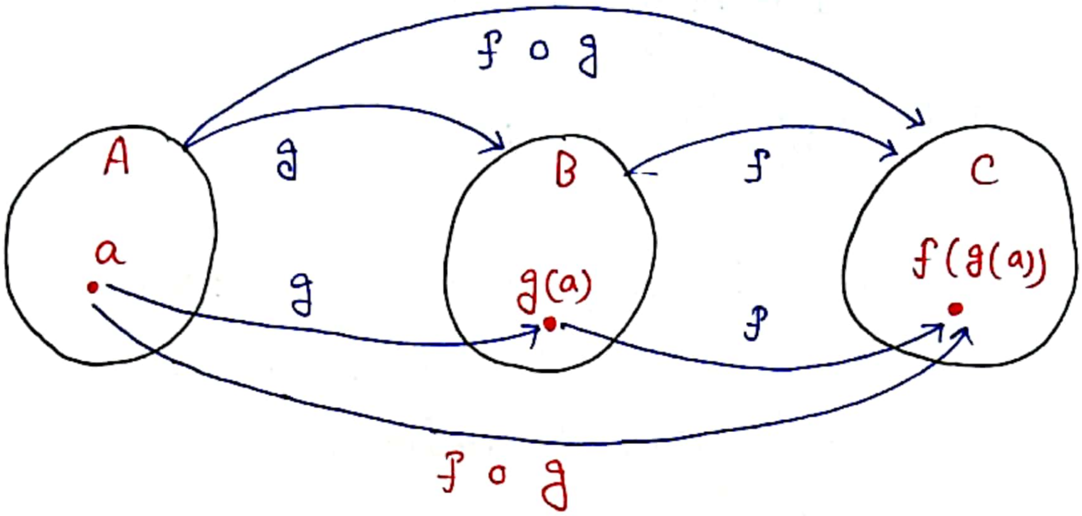
Let, f_1 and f_2 be functions from A to $\mathbb{R}$.

$$\boxed{\begin{aligned}(f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ (f_1 f_2)(x) &= f_1(x) \cdot f_2(x)\end{aligned}}$$

▣ Composition of Functions

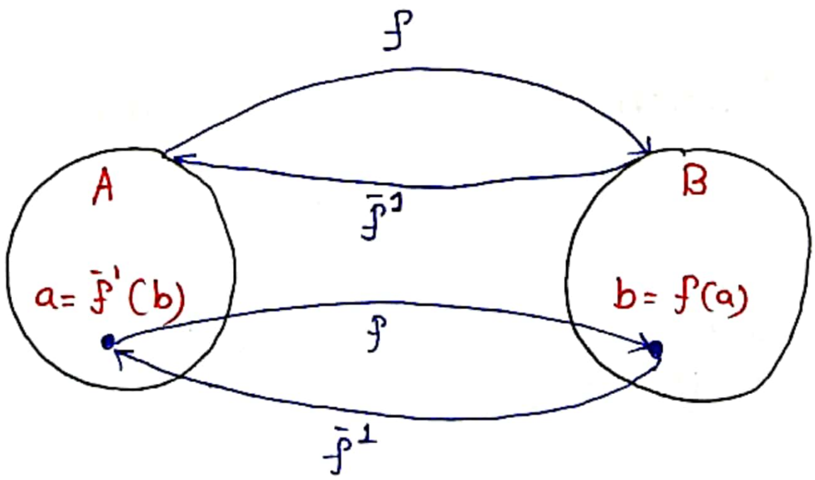
Let g be a function from the set A to the set B, and let f be a function from the set B to the set C. The composition of the functions f and g - denoted by $f \circ g$

$$(f \circ g)(a) = f(g(a))$$



Inverse Function :

Let f be a bijection from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a)=b$. The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b)=a$ when $f(a)=b$.



Identity Function

Identity function :

The identity mapping $1_X : X \to X$ is the function with domain and Codomain X defined by

$$\boxed{1_X(x) = x, \quad \forall x \in X}$$

Left and Right Inverse

- Left $f : X \to Y$ be a function with domain X and codomain Y,

and

$g : Y \to X$ be a function with domain Y and codomain X.

- The function g is a left inverse of f if $g \circ f = 1_X$.

- The function g is a Right inverse of f if $f \circ g = 1_Y$.

Theorem

- A function is injective if and only if it has a left inverse.

- A function is surjective if and only if it has a right inverse.

- A function is bijective if and only if it has an inverse.

- If a function has an inverse, then this inverse is unique.

Rules of Inference

- Motivation
- Definitions
- Rules of Inference
- Fallacies
- using Rules of Inference to Build Argumen
- Rules of Inference and Quantifiers.

Rule of Inference

Some tautologies are rules of inference. The general form of a rule of inference is.

$$(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow c$$

where,

P_i are the premises.

and,

c is the conclusion.

CamScanner

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline \therefore\ C \end{array}$$

$\therefore$ denotes "therefore".

modus ponens

The rule of inference

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

is denoted the law of detachment.

The basis of the modus ponens is the tautology.

$$((p \rightarrow q) \wedge p) \rightarrow q$$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Example

If it rains, then it is cloudy.
It rains.
Therefore, it is cloudy.

r is the proposition "it rains"
c is the proposition "it is cloudy"

$$\begin{array}{l} r \to c \\ r \\ \hline \therefore\ c \end{array}$$

modus tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore\ \neg p \end{array}$$

The basis of the modus Ponens is the tautology.

$$((p \to q) \land \neg q) \to \neg p$$

p	q	$p \to q$	$\neg q$	$(p \to q) \wedge \neg q$	$\neg p$	$((p \to q) \wedge \neg q) \to \neg p$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Example

If it rains, then it is cloudy.

It is not cloudy.

Therefore, it is not the case that it rains.

r is the proposition "it rains"

c is the proposition "it is cloudy."

$$\begin{array}{l} r \to c \\ \neg c \\ \hline \therefore \neg r \end{array}$$

Addition

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

is the rule of addition.

This rule comes from the tautology.

$$p \to (p \vee q)$$

The Simplification

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

rule of simplification.

This rules from

$$(p \wedge q) \to p$$

The Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

[hypothetical syllogism rules]

Come from

$$((p \to q) \wedge (q \to r)) \to (p \to r)$$

The Disjunctive Syllogism

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

[rule of disjunctive Syllogism]

This rule comes from

$$((p \lor q) \land \neg p) \rightarrow q$$

The Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$

Comes from

$$((p) \land (q)) \rightarrow (p \land q)$$

The Resolution

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

is the rule of resolution.

Comes from,

$$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$$

Fallacies

Fallacies are incorrect arguments.

The Fallacy of Affirming the Conclusion:

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

The basis of this fallacy is the contingency

$$(q \land (p \rightarrow q)) \rightarrow p$$

p	q	$p \rightarrow q$	$q \land (p \rightarrow q)$	$(q \land (p \rightarrow q)) \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Example:

If it rains, then it is cloudy.

It is cloudy

Therefore, it rains (wrong).

r is the proposition "it rains"

c is the proposition "it is cloudy."

$$\begin{array}{l} r \rightarrow c \\ c \\ \hline \therefore r \text{ (wrong)} \end{array}$$

The Fallacy of Denying the Hypothesis

$$\begin{array}{l} p \rightarrow q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

The Basis of this fallacy is the Contingency

$$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$$

p	q	$p \to q$	$\neg p$	$(p \to q) \wedge \neg p$	$\neg q$	$((p \to q) \wedge \neg p) \to \neg q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Example

If it rains, then it is cloudy.
It is not the case that it rains.
Therefore, it is not cloudy (wrong).

r is the proposition "it rains"
c is the proposition "it is cloudy."

$$\begin{array}{l} r \to c \\ \neg r \\ \hline \therefore \neg c \text{ (wrong)} \end{array}$$

Example

If Superman were able and willing to prevent evil, then he would so. If Supperman were unable to prevent evil, then he would be impotent; if he were unwilling to prevent evil, then he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

- w is "Superman is willing to prevent evil".
- a is "Superman is able to prevent evil"
- i is "Superman is Impotent."
- m is "Superman is malevolent"
- p is "Superman prevents evil."
- x is "Superman exists".

$h_1. \ (a \wedge w) \rightarrow p$

$h_2. \ \neg a \rightarrow i$

$h_3. \ \neg w \rightarrow m$

$h_4. \ \neg p$

$h_5. \ x \rightarrow \neg i$

$h_6. \ x \rightarrow \neg m$

Argument:

1. $\neg i \rightarrow a$ — Contrapositive of h_2.
2. $x \rightarrow a$ — h_5 and Step 1 with hyp. syll.
3. $\neg m \rightarrow w$ — Contrapositive of h_3.
4. $x \rightarrow w$ — h_6 ans Step 3 with hyp. syll.
5. $x \rightarrow (a \wedge w)$ — Step 2 and 4 with Conjunction
6. $x \rightarrow p$ — Step 5 and h_1 with hyp syll.
7. $\neg x$ — Step 6 and h_4 with modus tollens.

Rules of Inference and Quantifiers

- Universal instantiation (UI).
- Universal generalization (UG).
- Existential instantiation (EI).
- Existential generalization (EG).

Universal instantiation

$$\frac{\forall x\, P(x)}{\therefore P(c)}$$

If a propositional function is true for all element x of the universe of discource. then it is true for a particular element c of the universe of discourse.

Universal Instantiation and modus ponens

Example:

All humans have two legs. john Smith is a human. Therefore, john Smith has two legs.

⇒ $H(x)$ is "x is a human."

⇒ $L(x)$ is "x has two legs"

⇒ j is john Smith, a element of the universe of discourse.

1.	$\forall x (H(x) \rightarrow L(x))$	Premise
2.	$H(j) \rightarrow L(j)$	universal instantiation from 1.
3.	$H(j)$	Premise
	$\therefore L(j)$	Modus Ponens from 2 et 3.

$$\frac{P(c) \quad \text{for an arbitrary } c}{\therefore \forall x\, P(x)}$$

Universal Generalization

$$\frac{\exists x\, P(x)}{\therefore P(c) \quad \text{for some element } c}$$

Existential Instantiation

$$\frac{P(c) \quad \text{for some element } c}{\therefore \exists x\, P(x)}$$

Existential Generalization

Example : 1

If Batman is a superhero, then he has a superpower. If Batman has a superpower, then he can fly. Batman cannot fly. There fore, Batman is not a superhero.

Solution:

* S is "Batman is a superhero".
* P is "Batman has a superpower"
* f is "Batman can fly".

* h_1. $S \rightarrow P$
* h_2. $P \rightarrow f$
* h_3. $\neg f$

Argument:

* $\neg P \rightarrow \neg S$ Contrapositive of h_1.
* $\neg f \rightarrow \neg P$ Contrapositive of h_2.
* $\neg f \rightarrow \neg S$ Step 2 and 1 with hyp syll.
* $\neg S$ Step 3 and h_3 with modus Ponens.

Example: 2

If Alice studies hard, then she will pass the exam. If Alice passes the exam, then she will get a scholarship. Alice did not get a scholarship. Therefore, Alice did not study hard.

* h is "Alice studies hard"
* p is "Alice passes the exam"
* s is "Alice gets a scholarship".

* h_1. $h \rightarrow p$
* h_2. $p \rightarrow s$
* h_3. $\neg s$

Argument:

* $\neg p \rightarrow \neg h$ Contrapositive of h_1.
* $\neg s \rightarrow \neg p$ Contrapositive of h_2.
* $\neg s \rightarrow \neg h$ Step 2 and 1 with hyp syll.
* $\neg h$ step 3 and h_3 with modus ponens.

Example : 3

If Bob likes chocolate, then he likes cake. If Bob likes cake, then he likes ice creame. Bob does not like ice cream. There fore, Bob does not like chocolate.

Ans:

* c is "Bob likes chocolate".
* k is "Bob likes cake".
* i is "Bob likes ice cream."

* $h_1.\ c \to k$
* $h_2.\ k \to i$
* $h_3.\ \neg i$

Argument:

* $\neg k \to \neg c$ Contrapositive of h_1.
* $\neg i \to \neg k$ Contrapositive of h_2.
* $\neg i \to \neg c$ Step 2 and 1 with hyp syll.
* $\neg c$ Step 3 and h_3 with modus ponens.

Example: 4

If Eve is smart, then she is good at math. If Eve is good at math, then she is good at logic. Eve is not good at logic. Therefore, Eve is not smart.

Ans:

* m is "Eve is smart"
* a is "Eve is good at math".
* l is "Eve is good at logic".

* h_1. $m \to a$
* h_2. $a \to l$
* h_3. $\neg l$

Argument:

1. $\neg a \to \neg m$ Contrapositive of h_1.
2. $\neg l \to \neg a$ Contrapositive of h_2.
3. $\neg l \to \neg m$ Step 2 and 1 with hyp Syll.
4. $\neg m$ Step 3 and h_3 with modus Ponens.

Example: 5

If frank is honest, then he tells the truth. If frank tells the truth, then he is trust worthy. Frank is not trust worthy. Therefore, Frank is not honest.

Ans:

- o is "frank is honest"
- t is " Frank tells the truth".
- r is " Frank is trustworthy".

- h_1. $o \to t$
- h_2. $t \to r$
- h_3. $\neg r$

Argument:

- $\neg t \to \neg o$ Contrapositive of h_1.
- $\neg r \to \neg t$ Contrapositive of h_2.
- $\neg r \to \neg o$ Step 2 and 1 with hyp syll.
- $\neg o$ Step 3 and h_3 with modus ponens.

Sets

Set:

A set is an unordered collection of objects, finite or infinite that all possess the same property. membership of the set.

Object:

The objects in a set are called the elements or members, of the set.

$\boxed{a \in A}$ → is an element of set A.

$\boxed{a \notin A}$ → to denote that a is not an element of set A.

$V = \{a, e, i, o, u\}$

$C = \{1, 2, 3, \ldots\ldots, 99\}$ → Positive integers

$N = \{0, 1, 2, 3, \ldots\}$ → natural numbers.

$Z = \{\ldots, -2, -1, 0, 1, 2\}$

- The set O of all odd positive integers less than 10 can be written as.

$O = \{x \mid (x \in N) \wedge (x < 10) \wedge (x \text{ is odd})\}$

- The set Q of rational numbers can be written as

$$Q = \{a/b \mid (a \in Z) \wedge (b \in Z) \wedge (b \neq 0)\}.$$

Important Sets

- $N = \{0, 1, 2, 3 \ldots\}$ → natural Number
- $Z = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$ → integers
- $Z^+ = \{1, 2, 3, \ldots\}$ → positive integers
- $Q = \{p/r \mid (p \in Z) \wedge (r \in Z) \wedge (r \neq 0)\}$ → rational
- R = real numbers.
- C. Complex Numbers.

Equality of Sets

Two sets are equal if and only if they have the same elements.

Example:

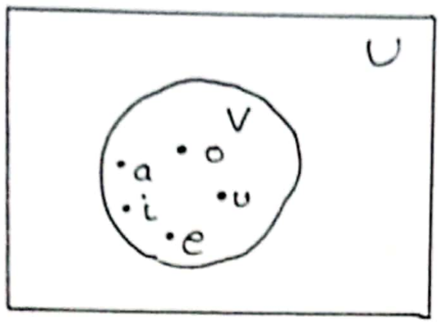
$A = B$

$A = \{1, 3, 5\}$

$B = \{3, 5, 1\}$

Venn Diagram

The universal set U, which contains all the objects under consideration is represented by a rectangle.



Empty Set:

There is a special set that has no elements.
denoted by $\emptyset$ or by $\{\}$.

Singleton:

A set that contains exactly one element is called a singleton.

Subset:

The set A is said to be a subset of the set B, denoted $A \subseteq B$, if and only if every element of A is also an element of B.

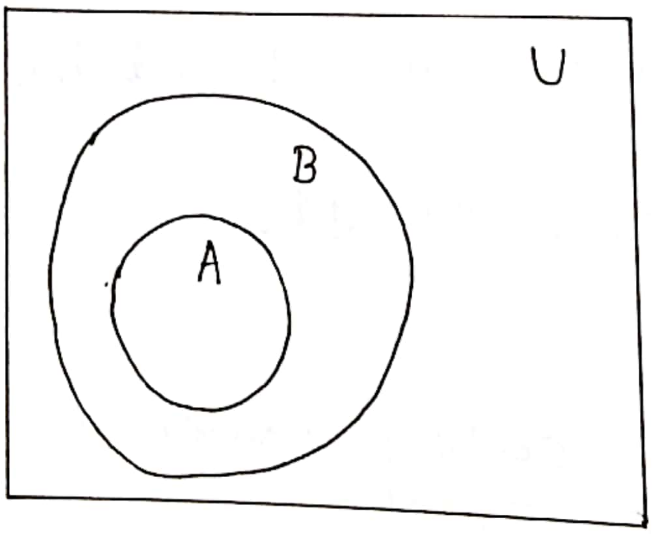
Two Subsets of a Non-Empty Set

Theorem:

The empty set is a subset of all the sets. That is, $\emptyset \subseteq S$ for any set S.

Theorem:

Every set is a subset of itself. $S \subseteq S$, for any set S.



Cardinality:

S is a finite set and that n is the cardinality of S.

Cardinality by $|S|$.

Power Set:

The power set of S is denoted by P(S).

Let,

$S = \{0, 1, 2\}$

$P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

If $|S| = n$ then, $|P(S)| = 2^n$

Order n-tuple:

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, and a_n as its nth element.

Cartesian Product:

Let, A and B be sets.

The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Example: $A = \{1, 2\}$ $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$