Stabilizing Control with Structured Norm-bounded Uncertainty for a Lower-Limb Exoskeleton

By Mason Smith

Objectives

- Provide a gait rehabilitation background
- Define the system for the exoskeleton
- Formulate the LFT with uncertainty blocks
- Construct a controller
    - H_∞ Optimal State-Feedback Controller with Structured Norm-Bounded Uncertainty
- Calculate structured singular value

Background

Patient Stroke

- My Research:
    - Machine learning to model therapist-patient interactions during gait rehabilitation for stroke patients
- Hemiparesis
    - Difficulty standing
    - Difficulty walking
    - Unusual sensations in the affected side of the body
    - Strain on the unaffected side of the body caused by overcompensation
- Gait Rehabilitation
    - Gravity Compensation
    - Partial Compensation of Swing Leg Weight
    - Feedforward Movement Assistance during Swing
    - Compensatory Gait Correction

Lower-Limb Exoskeleton

- Exoskeleton Motivation
  - Extended therapist ergonomics
  - Quantitative feedback
  - Improvement to patient care
  - Used cooperatively
- Elastic Actuator:
  - Compliable mechanism
  - Allows for torque measurement by measuring displacement between actuator and shank

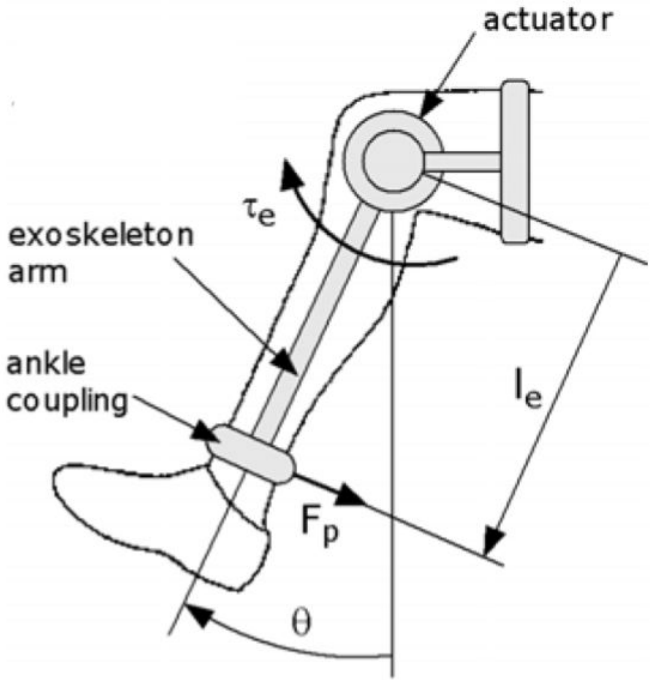


Figure 1: Knee Exoskeleton Example

System Definition

Freebody

- Assumptions
    - Static point at the hip
    - Pivot point at the knee
    - F_g actuates at the foot in +y
    - Exoskeleton and human mass combined
    - Negligible damping from knee (low angular velocities)
    - Small angle approximation sin(θ)=θ is valid since θ [0^0,30^0]

$$\tau_h = I\ddot{\theta}_k(t) = \tau_g - \tau_e$$

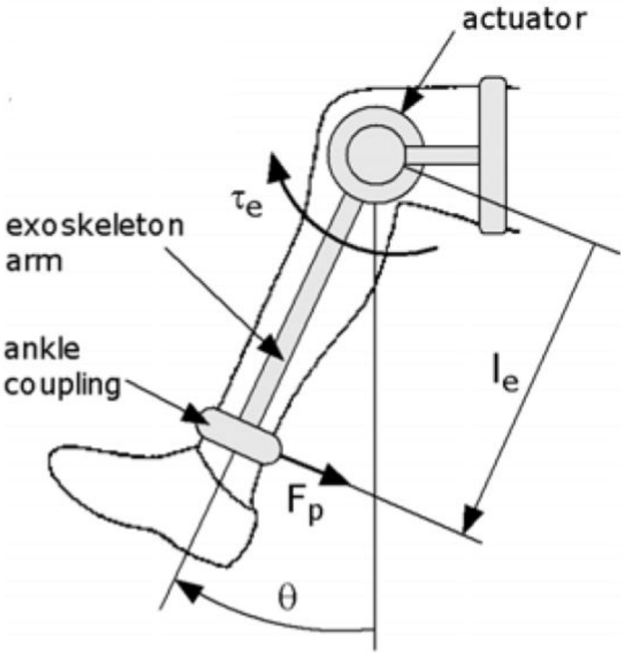


Figure 1: Knee Exoskeleton Example

Formulating Transfer Function

Variable Definitions:

θ_k: angle of knee displacement
θ_e: angle of exoskeleton displacement
τ_h: torque applied by human
τ_g: torque caused by gravity
τ_e: torque caused by exoskeleton
I : Inertia of lower leg + exoskeleton
l_h : length from ankle to knee
l_s : length from applied F_shank to knee
m: mass of human+exoskeleton
k: rotary spring constant

Uncertain Parameters:

$m = m_0(1 + \eta_m\delta_m)$
$k = k_0(1 + \eta_k\delta_k)$

let:

$x_1(t) = \theta_k(t)$
$x_2(t) = \dot{x}_1(t)$
$\dot{x}_2(t) = \ddot{\theta}_k(t)$
$u(t) = \theta_e(t)$

$\tau_h = I\ddot{\theta}_k(t) = \tau_g - \tau_e$
$I\ddot{\theta}_k(t) = l_h gmsin(\theta_k) - l_s k(\theta_e - \theta_k)$

Small angle approximation $sin(\theta_k) = \theta$

$$I\ddot{\theta}_k(t) = l_h gm\theta_k - l_s k(\theta_e - \theta_k)$$
$$I\ddot{\theta}_k(t) - l_h gm\theta_k - l_s k\theta_k = -l_s k\theta_e$$
$$I\ddot{\theta}_k(t) - (l_h gm + l_s k)\theta_k = -l_s k\theta_e$$

Laplace Transform

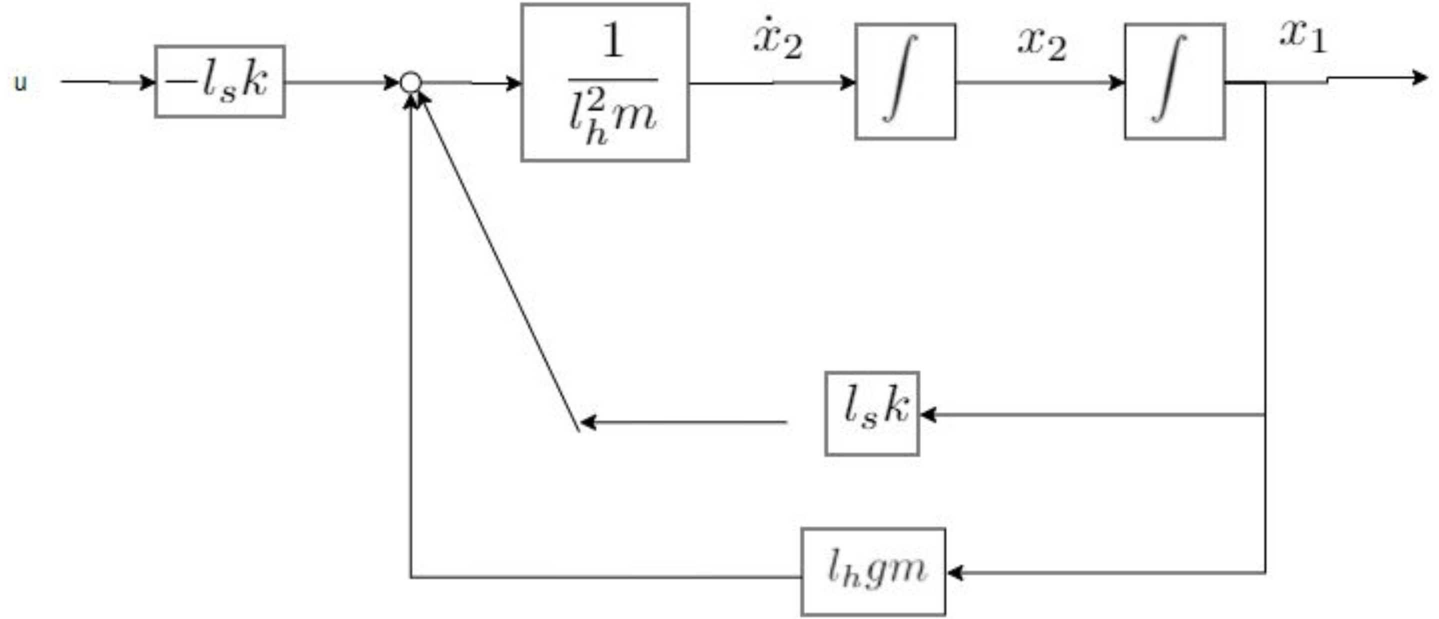
$$I\theta_k(s)s^2 - (l_h gm + l_s k)\theta_k(s) = -l_s k\theta_e(s)$$
$$\theta_k(s) = \frac{-l_s k}{(Is^2 - (l_h gm + l_s k))}\theta_e(s)$$

Letting $I = ml_h^2$

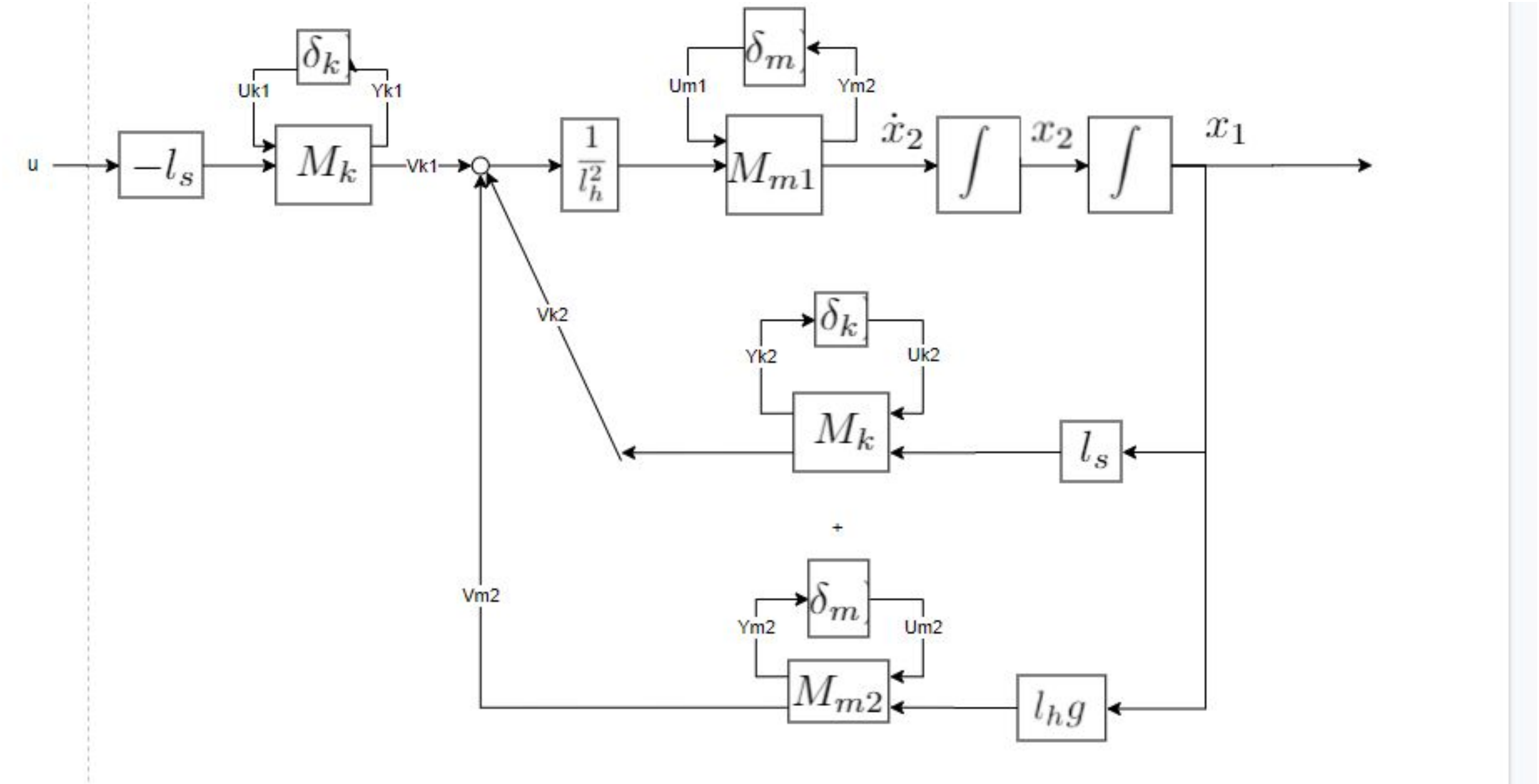
$$x(s) = \frac{-l_s k}{ml_h^2 s^2 - (l_h gm + l_s k)}u(s)$$
$$x(s) = \frac{-l_s k_0(1+\eta_k\delta_k)}{m_0(1+\eta_m\delta_m)l_h^2 s^2 - (l_h gm_0(1+\eta_m\delta_m) + l_s k_0(1+\eta_k\delta_k))}u(s)$$

Formulating the LFT

Formulate Block Diagram



Isolate Uncertainty Blocks



Rewrite Uncertainty Blocks as LFTs

for $\frac{1}{m_0(1+\eta_m\delta_m}$) term

$$\frac{1}{m} = \frac{1}{m_0(1+\eta_m\delta_m)} = \frac{1}{m_0} - \frac{\eta_m}{m_0}(1+\eta_m\delta_m)^{-1} = \bar{S}(M_{m1}, \delta_m)$$

$$M_{m1} = \begin{bmatrix} -\eta_m & \frac{1}{m_0} \\ -\eta_m & \frac{1}{m_0} \end{bmatrix}$$

for $m_0(1+\eta_m\delta_m)$ term

$$m = m_0(1+\eta_m\delta_m) = \bar{S}(M_{m1}, \delta_m)$$

$$M_{m2} = \begin{bmatrix} 0 & m_0 \\ \eta_m & m_0 \end{bmatrix}$$

for $k_0(1+\eta_k\delta_k)$ term

$$k = k_0(1+\eta_k\delta_k) = \bar{S}(M_k, \delta_k)$$

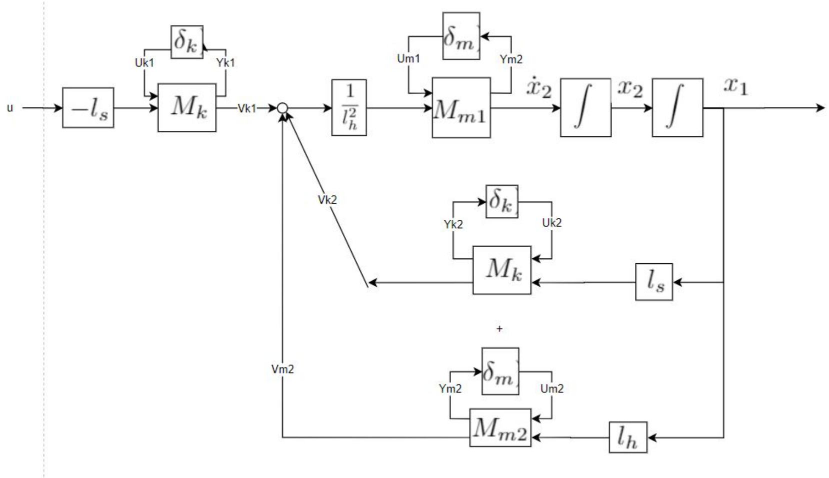
$$M_k = \begin{bmatrix} 0 & k_0 \\ \eta_k & k_0 \end{bmatrix}$$

Define Equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\eta_m u_{m1} + \frac{1}{l_h^2 m_0}(v_{k1} + v_{k2} + v_{m2})$$
$$y_{m1} = -\eta_m u_{m1} + \frac{1}{l_h^2 m_0}(v_{k1} + v_{k2} + v_{m2})$$
$$y_{m2} = l_h g m_0 x_1$$
$$y_{k1} = -l_s k_0 w$$
$$y_{k2} = l_s k_0 x_1$$
$$z = x_1$$

$$v_{m2} = \eta_m u_{m2} + m_0 l_h g x_1$$
$$v_{k1} = \eta_k u_{k1} - k_0 l_s w$$
$$v_{k2} = \eta_k u_{k2} + k_0 l_s x_1$$

$$u_{m1} = \delta_m y_{m1}$$
$$u_{m1} = \delta_m y_{m1}$$
$$u_{k1} = \delta_k y_{k1}$$
$$u_{k2} = \delta_k y_{k2}$$



Solving and Substituting for Variables:

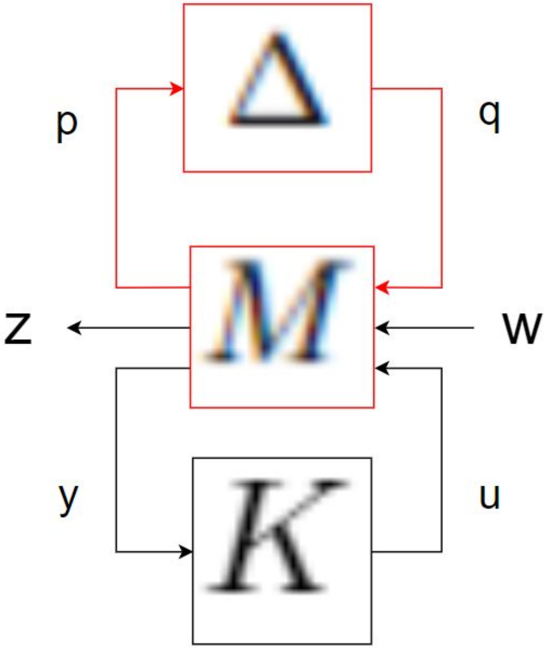
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{k_0 l_s + m_0 l_h g}{l_h^2 m_0} x_1 + 0x_2 - \eta_m u_{m1} + \frac{\eta_m}{l_h^2 m_0} u_{m2} + \frac{\eta_k}{l_h^2 m_0} u_{k1} + \frac{\eta_k}{l_h^2 m_0} u_{k2} - \frac{k_0 l_s}{l_h^2 m_0} w$$
$$y_{m1} = -\frac{k_0 l_s + m_0 l_h g}{l_h^2 m_0} x_1 + 0x_2 - \eta_m u_{m1} + \frac{\eta_m}{l_h^2 m_0} u_{m2} + \frac{\eta_k}{l_h^2 m_0} u_{k1} + \frac{\eta_k}{l_h^2 m_0} u_{k2} - \frac{k_0 l_s}{l_h^2 m_0} w$$
$$y_{m2} = l_h g m_0 x_1$$
$$y_{k1} = -l_s k_0 w$$
$$y_{k2} = l_s k_0 x_1$$
$$z = x_1$$

9- Matrix Representation

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{k_0 l_s + m_0 l_h g}{l_h^2 m_0} x_1 + 0 x_2 - \eta_m u_{m1} + \frac{\eta_m}{l_h^2 m_0} u_{m2} + \frac{\eta_k}{l_h^2 m_0} u_{k1} + \frac{\eta_k}{l_h^2 m_0} u_{k2} - \frac{k_0 l_s}{l_h^2 m_0} w \\
y_{m1} &= -\frac{k_0 l_s + m_0 l_h g}{l_h^2 m_0} x_1 + 0 x_2 - \eta_m u_{m1} + \frac{\eta_m}{l_h^2 m_0} u_{m2} + \frac{\eta_k}{l_h^2 m_0} u_{k1} + \frac{\eta_k}{l_h^2 m_0} u_{k2} - \frac{k_0 l_s}{l_h^2 m_0} w \\
y_{m2} &= l_h g m_0 x_1 \\
y_{k1} &= -l_s k_0 w \\
y_{k2} &= l_s k_0 x_1 \\
z &= x_1
\end{aligned}
$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y_{m_1} \\ y_{m_2} \\ y_{k_1} \\ y_{k_2} \\ z \end{bmatrix}
=
\left[\begin{array}{cc|cccc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{k_0 l_s + m_0 l_h g}{l_h^2 m_0} & 0 & -\eta_m & \frac{\eta_m}{l_h^2 m_0} & \frac{\eta_k}{l_h^2 m_0} & \frac{\eta_k}{l_h^2 m_0} & -\frac{k_0 l_s}{l_h^2 m_0} \\
\hline
\frac{k_0 l_s + m_0 l_h g}{l_h^2 m_0} & 0 & -\eta_m & \frac{\eta_m}{l_h^2 m_0} & \frac{\eta_k}{l_h^2 m_0} & \frac{\eta_k}{l_h^2 m_0} & -\frac{k_0 l_s}{l_h^2 m_0} \\
l_h g m_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -l_s k_0 \\
l_s k_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} x_1 \\ x_2 \\ u_{m_1} \\ u_{m_2} \\ u_{k_1} \\ u_{k_2} \\ w \end{bmatrix}
\qquad
q = \begin{bmatrix} \delta_m & 0 & 0 & 0 \\ 0 & \delta_m & 0 & 0 \\ 0 & 0 & \delta_k & 0 \\ 0 & 0 & 0 & \delta_k \end{bmatrix} p
$$

Data for the LMI Controller Synthesis LMI



Completing the Upper Feedback Interconnection

$$\bar{S}(P,\Delta) = (P_{22} + P_{21}(I - \Delta P_{11})^{-1}\Delta P_{12})\begin{bmatrix} x \\ F \end{bmatrix}$$

And substituting values of:

$$g = 9.8 m/s^2 \quad l_h = 1 \text{ meters} \quad l_s = 0.5 \text{ meters}$$
$$m_0 = 60 \text{ kg} \quad \eta_m = 0.2$$
$$k_0 = 15 \text{ Nm/rad} \quad \eta_k = 0.1$$

Gives Matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 598 & 0 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ -0.0017 \end{bmatrix} \; B2 = \begin{bmatrix} 0 \\ -0.0017 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; D12 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \; D22 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.2000 & 3.4708e-05 & 1.7354e-05 & 1.7354e-05 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.1055 & 0 \\ 588 & 0 \\ 0 & 0 \\ 10 & 0 \end{bmatrix} \; Q = 0$$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mp(t) + B_2 w(t)$$
$$q(t) = Nx(t) + D_{12}u(t)$$
$$y(t) = Cx(t) + D_{22}u(t)$$

$$p(t) = \Delta(t)q(t)$$
$$\Delta \in \boldsymbol{\Delta}, ||\Delta|| \leq 1$$

Controller Synthesis

H_∞ Optimal State-Feedback Controller with Structured Norm-Bounded Uncertainty

There exists K such that the system with u(t) = Kx(t)

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Mp(t) + B_2 w(t) \\
q(t) &= Nx(t) + D_{12}u(t) \\
y(t) &= Cx(t) + D_{22}u(t)
\end{aligned}$$

$$\begin{aligned}
p(t) &= \Delta(t)q(t) \\
\Delta &\in \mathbf{\Delta}, ||\Delta|| \leq 1
\end{aligned}$$

satisfies $||y||_{L_2} \leq \gamma ||u||_{L_2}$ if there exists some $\Theta \in \mathbf{P\Theta}$, Z and $P > 0$ such that:

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + M\Theta M^T & (CP + D_{22}Z)^T & PN^T + Z^T D_{12}^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ NP + D_{12}Z & 0 & -\Theta \end{bmatrix} < 0$$

where the recovered controller is $K = ZP^{-1}$

Matlab Results

$$Z = \begin{bmatrix} -0.6121 & 2.2434e+04 \end{bmatrix}$$

$$\Theta = \begin{bmatrix} 126.2961 & 0 & 0 & 0 \\ 0 & 126.2961 & 0 & 0 \\ 0 & 0 & 4.3233e+06 & 0 \\ 0 & 0 & 0 & 4.3233e+06 \end{bmatrix}$$

$$X = \begin{bmatrix} 0.0184 & -0.4633 \\ -0.4633 & 12.3419 \end{bmatrix}$$

$$K = \begin{bmatrix} 8.4048e+05 & 3.3369e+04 \end{bmatrix}$$

$$\gamma = 3.5941e+03$$

```
%===================================================
% Define Scalings Variables
gamma=sdpvar(1);
th1=sdpvar(1);
th2=sdpvar(1);
Th=diag([th1;th1;th2;th2],0);

%===================================================
% Define Constraints
F=[];
F=[F;Th>=0];
P=sdpvar(n);
Z=sdpvar(nc,n,'full');

F=[F;P>=eta*eye(n)];
MAT=[
    (A*P+B*Z+P*A'+Z'*B'+B2*B2'+M*Th*M') (C*X+D22*Z)'     (P*N'+Z'*D12');
    (C*P+D22*Z)             (-gamma*eye(nro))   (zeros(nro,np));
    (N*P+D12*Z)              (zeros(np,nro))  (-Th)];
F=[F;MAT<=0];

%===================================================
% Run optimization
objective=gamma;
sol=optimize(F,objective,options);

%===================================================
% Return feasable solutions
Zn=value(Z);
Xn=value(P);
K=Zn*inv(Xn);
Thn=value(Th);
gamman=sqrt(value(gamma));
```

Structured Singular Value

The LMI

Suppose the system M has a transfer function $\hat{M}(s) = C(sI - A)^{-1}B + D$ with $\hat{M} \in H_\infty$. The following are equivalent

1. There exists $\Theta \in \mathbf{\Theta}$ such that $||\Theta M \Theta^{-1}||^2 < \gamma$
2. There exists $\Theta \in \mathbf{P\Theta}$ and $X > 0$ such that

$$\begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} \\ B_{cl}^T X & -\Theta \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_{cl}^T \\ D_{cl}^T \end{bmatrix} \Theta \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0$$

Close the loop with controller K

$$A_{cl} = A + B_{cl}K$$
$$C_{cl} = C_1 + D_{cl}K$$
$$B_{cl} = B$$
$$D_{cl} = D_{12}$$

Matlab Results

Using bisection on γ

$$\gamma = 0.3174$$

Since

1. $\mu(M, \mathbf{\Delta}) = \inf_{\Theta \in \mathbf{\Theta}} ||\Theta M \Theta^{-1}||$
2. $||\Theta M \Theta^{-1}||^2 < \gamma$

$$\mu < 0.5634$$

Interpretation:

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf\limits_{\Delta \in \mathbf{\Delta}, I - M\Delta \text{ is singular}} ||\Delta||}$$

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf ||\text{destabilizing perturbations in } \mathbf{\Delta}||}$$

```
% ==========================================
% Declare Scalings
gamma=sdpvar(1);
th1=sdpvar(1);
th2=sdpvar(1);
Th=diag([th1;th1;th2;th2],0);

% ==========================================
% Declare sdp Vars
X=sdpvar(n);

% ==========================================
% Constraints
F=[];
F =[F;X>=eta*eye(n)];
MAT=[Acl'*X+X*Acl X*Bcl;Bcl'*X -Th]+1/gam_new/gam_new*[Clcl Dcl]'*Th*[Clcl Dcl];
Ftemp=[F;MAT<=0];
sol=optimize(Ftemp);
if sol.problem==0
    gam_feasable=gam_new;
else
    gam_lnfeasable=gam_new;
end
gam_new=(gam_feasable+gam_lnfeasable)/2;
err=gam_feasable-gam_lnfeasable;
```

Future Work

Future Work

- More accurate uncertainty parameters
  - Measured using our prototype
- Validate the system
  - Theoretical vs Prototype Response
- Reduce assumptions
  - Model exoskeleton & human separately
  - Account for small angle approximation
  - Damping

Questions & Feedback