Homework #1
Introduction to Algorithms/Algorithms 1
600.363/463
Spring 2013

Solutions

January 22, 2015

1 Problem 1 (20 points)

1.1 (10 points)

For each statement below, state whether it is true (in which case you should give a proof of the truth of the statement) or false (in which case you should provide a counter-example). Be as precise as you can. The base of $\log$ is 2 unless stated otherwise.

1. $2^{31}(n^3 + n^2) = \Theta(n^2)$

   False. Proof follows from the fact that $kn^3 = \omega(n^2)$ for any positive constant k.

2. $2^n = \Theta(e^n)$.

   False. $\lim_{n\to\infty} \left(\frac{e}{2}\right)^n = \infty$, which implies that for any $M \geq 0$, we can find an N such that $n > N \Rightarrow \left(\frac{e}{2}\right)^n > M$. This implies that for any $M \geq 0$, $e^n \geq M2^n$, which is the definition of $e^n = \omega(2^n)$, which implies that $2^n \neq \Omega(e^n)$, which in turn implies that $2^n \neq \Theta(e^n)$.

3. $2^{(n^2)} = \Theta(3^{n+\sqrt{n}})$

   False. Proof: We will show that $2^{(n^2)} = \omega(3^{n+\sqrt{n}})$. First note that $2^{(n^2)} = (2^n)^n$. Thus,

   $$\frac{2^{(n^2)}}{3^{n+\sqrt{n}}} = \frac{1}{3^{\sqrt{n}}}\left(\frac{2^n}{3}\right)^n.$$

   If we can show that this quantity grows without bound (i.e., that the quantity goes to infinity as $n \to \infty$), it will imply that $2^{(n^2)} = \omega(3^{n+\sqrt{n}})$. Fix some $M > 0$. We must show the existence of a number N for which

   $$n > N \Rightarrow \frac{2^{(n^2)}}{3^{n+\sqrt{n}}} > M.$$

Taking the logarithm, we have $\log \frac{2^{(n^2)}}{3^{n+\sqrt{n}}} = n^2 \log 2 - n \log 3 - \sqrt{n} \log 3$, which we know to grow as $\omega(n)$. Since the logarithm of the quantity grows without bound, the quantity itself grows without bound. We conclude that $\frac{2^{(n^2)}}{3^{n+\sqrt{n}}}$ is unbounded as n increases, as we wished to show.

4. $\log_2 n = O(\log_b n^3)$ for constant $b > 1$.

   True. Proof: $\log_b n^3 = 3 \log_b n = O(\log_b n)$. The statement follows from the change of base rule for logarithms, which states that

   $$\log_b n = \frac{\log_2 n}{\log_2 b}.$$

   Thus, $\log_b n^3 = 3 \log_2 b \log_2 n$, i.e., $\log_b n^3 = C \log_2 n$, where C is a constant.

5. $\sin(n) = O(1)$ True. Proof: we must show that there exists a positive constant that bounds $\sin(n)$ from above for all suitably large n. Any constant larger than 1 will suffice, since $|\sin(n)| \leq 1$.

6. $2^n = o(n!)$ True. Proof: $n! \equiv n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$. Fix any $M > 0$. We must show that there exists an N such that $n > N \Rightarrow n! > M2^n$, or, equivalently, we must show that $\frac{n!}{2^n}$ can be made arbitrarily large by choosing n to be suitably large. Consider the quantity $A = \frac{n!}{2^n}$ and consider the quantity $B = \frac{(n+1)!}{2^{n+1}}$.

   $$\frac{B}{A} = \frac{n+1}{2}.$$

   This implies that the quantity $\frac{n!}{2^n}$ increases without bound as n increases, by the ratio test.

7. Let f, g be positive functions. Then $f(n) + g(n) = O(\min(f(n), g(n)))$.

   False. Proof: consider the counterexample when $f(n) = n^2$ and $g(n) = 1$. Then $\min(f(n), g(n)) = 1$ for all n, and $f(n) + g(n) = n^2 + 1 = \Omega(n^2) \neq O(1) = O(\min(f(n), g(n)))$.

8. $n^{\log n} = \Omega(n^{100})$

   True. Proof: Trivially, choosing N so that $\log N = 100$ gives us $n > N \Rightarrow n^{\log n} \geq n^{100}$.

9. Let f, g be positive functions with $g(n) = o(f(n))$. Then $f(n)g(n) = o(f(n))$.

   False. Proof: Consider the counter-example $g(n) = 1$, $f(n) = n^2$.

1.2 (10 points)

1. Prove that $\sum_{i=1}^{n} i^3 = \Theta(n^4)$.

   Proof: We can show by induction that

   $$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

The result follows, since $\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = O(n^4)$ and $\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \Omega(n^4)$.

2 Problem 2(20 Points)

2.1 (10 points)

Prove by induction on n that $\sum_{i=0}^{n-1} \binom{i}{k} = \binom{n}{k+1}$ for $n \geq 1, 0 \leq k < n$. (Hint: Pascal's rule states that for $1 \leq k \leq n$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.)

Proof: The base case $n = 1$ holds because $\binom{0}{k} = 0 = \binom{1}{k+1}$ for $k \geq 1$. For $k = 0$, $\binom{0}{0} = 1 = \binom{1}{1}$. The case $n = 0$ also holds, since $\sum_{i=0}^{-1} \binom{i}{k} = 0 = \binom{-1}{k}$ for any k. For the induction step, assume that $\sum_{i=0}^{n-1} \binom{i}{k} = \binom{n}{k+1}$. Adding $\binom{n}{k}$ to both sides yields

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n}{k+1} + \binom{n}{k}. \tag{1}$$

Applying Pascal's rule to the right side of (1) yields the desired result.

2.2 (10 points)

1. Prove that if A, B and C are sets, then i) $A - (B \cap C) = (A - B) \cup (A - C)$ and ii) $A - (B \cup C) = (A - B) \cap (A - C)$.

   Proof: To see (i), suppose that element x is such that $x \in A - (B \cap C)$. Then by definition we have $x \in A$ and $x \notin (B \cap C)$. $x \notin (B \cap C)$ implies that one of the following three conditions hold:$x \in B - C$, $x \in C - B$ or $x \notin (B - C) \cup (C - B)$. If $x \in B - C$, then combining with the fact that $x \in A$, we have $x \in A - C$, since $x \notin C$ by definition of $x \in B - C$. If $x \in C - B$, then $x \in A - B$, since $x \notin B$ by similar reasoning. If $x \notin (B - C) \cup (C - B)$, then $x \in A - B$ and $x \in A - C$, since $x \notin (B - C)$ implies that $x \notin B$ and $x \notin (C - B)$ implies that $x \notin C$. Thus, combining these three facts, we have $x \in (A - B) \cup (A - C)$. Since this is true for any $x \in A - (B \cap C)$, we can conclude that $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.

   Conversely, suppose $x \in (A - B) \cup (A - C)$. If $x \in A - B$, then by definition $x \in A$ and $x \notin B$. $x \notin B$ implies $x \notin B \cap C$, since $B \cap C \subseteq B$. Thus, $x \in (A - B) \Rightarrow x \in A - (B \cap C)$. Similar reasoning for the case where $x \in (A - C)$ gives the same result, and we can conclude that $x \in (A - B) \cup (A - C) \Rightarrow x \in A - (B \cap C)$. Since x was arbitrary, we conclude that $(A - B) \cup (A - C) \subseteq A - (B \cap C)$. Combining this with the statement proved in the previous paragraph, we have that $(A - B) \cup (A - C) = A - (B \cap C)$.

   To prove statement (ii), let us first assume that $x \in A - (B \cup C)$. Then by definition, we have $x \in A$ and $x \notin B \cup C$. $x \notin B \cup C$ means that x is an element of neither B nor C, i.e., $x \notin B$ and $x \notin C$. Thus, we have $x \in A - B$ and $x \in A - C$, which by definition means that $x \in (A - B) \cap (A - C)$. Thus $A - (B \cup C) \subseteq (A - B) \cap (A - C)$.

   Conversely, suppose $x \in (A - B) \cap (A - C)$. By definition, this means that $x \in A - B$ and $x \in A - C$. By definition of $A - B$ and $A - C$, we have that $x \in A$,

$x \notin B$ and $x \notin C$. $x \notin B$ and $x \notin C$ jointly imply that $x \notin B \cup C$. Thus, we have $x \in A$ and $x \notin B \cup C$, so by definition we have $x \in A - (B \cup C)$. Since x was an arbitrary element in $(A-B) \cap (A-C)$, we have that $(A-B) \cap (A-C) \subseteq A - (B \cup C)$. Combining this fact with the one proved in the previous paragraph yields the desired result.

2. How many different 5-card poker hands include a 4 of a kind? A standard deck contains 52 cards of 13 different ranks in 4 suits. A four-of-a-kind is a hand which contains four cards of the same rank, one in each of the four suits. Hands of cards are not ordered. That is, the 3-card hand Ace, King, 4 is considered to be the same as the hand King, 4, Ace.

   There are 13 ranks from which to choose which rank will be in our four-of-a-kind. After those four cards have been taken, there are 48 remaining cards from which to choose the fifth card in the hand.. Thus there are $13 \cdot 48 = 624$ cards.

3. Each of a class of 7 students has 5 coins in his or her pocket, consisting of pennies and/or nickels. We say that two students have *equivalent* pockets if they have the same number of coins of each type. Is it possible that no two of the students in the class have equivalent pockets?

   There are 5 coins and 2 types of coin. Since the order of the coins does not matter, this is equivalent to counting the number of ways to place five coins into 2 bins, i.e., $\binom{6}{1} = 6$ ways. This can also be calculated by noting that once the number of pennies has been specified, the number of nickels is fixed. Thus, it suffices to count the different coin arrangements by simply noting that we can have $0, 1, 2, 3, 4$ or 5 pennies, or 6 unique combinations. Since there are 7 students in the class, the pigeonhole principle says that some pair of students must have equivalent pockets.

4. Prove the pigeon-hole principle, which states that *the maximum of a set of numbers is greater than or equal to the arithmetic mean of the set*. (Hint: assume the contrary and derive a contradiction.)

   Proof: assume that for some set of numbers $x_1, x_2, \ldots, x_n$, the maximum of the set (call it x_i) is less than the arithmetic mean of the set. That is, suppose that (i) $x_i \geq x_j$ for all $1 \leq i, j \leq n$ and that (ii) $x_i < \frac{1}{n} \sum_{k=1}^{n} x_k$. We have, by assumption that x_i is the maximum, $\sum_{k=1}^{n} x_k \leq n x_i$. Combining this with assumption (ii) that x_i is strictly less than the arithmetic mean, we get that $x_i < x_i$, a contradiction.

5. How many ordered pairs (a, b) are there such that a and b are integers, $1 \leq a, b \leq 15$ and $a + b = 15$? Note that the ordered pairs $(5, 10)$ and $(10, 5)$ are distinct and should be counted separately.

   We need to count the number of integer pairs (a, b) where $1 \leq a, b, 15$ and $a + b \leq 15$. For any fixed value of a, we can count the number of allowable values of b for which $a + b \leq 15$. Provided $a < 15$, these values are precisely $b = 15 - a, 15 - a - 1, \ldots, 1$, Thus, for any fixed value of $1 \leq a < 15$, there are $15 - a$ allowable values for b. Therefore there are

$$\sum_{a=1}^{14} 15 - a = 14 * 15 - \sum_{a=1}^{14} a = 210 - 14 * 15/2 = 105$$

possible combinations.