Homework #1
Introduction to Algorithms/Algorithms 1
600.363/463
Spring 2014

Due on: Tuesday, February 4th, 5pm
Late submissions: will NOT be accepted
Format: Please start each problem on a new page.
Where to submit: On blackboard, under student assessment.
Otherwise, please bring your solutions to the lecture.

January 25, 2014

1 Problem 1 (20 points)

1.1 (10 points)

For each statement below, state whether it is true (in which case you should give a proof of the truth of the statement) or false (in which case you should provide a counter-example). Be as precise as you can. The base of $\log$ is 2 unless stated otherwise.

1. $2^{31}(n^3 + n^2) = \Theta(n^2)$
2. $2^n = \Theta(e^n)$.
3. $2^{(n^2)} = \Theta(3^{n+\sqrt{n}})$
4. $\log_2 n = O(\log_b n^3)$ (b is a constant, $b > 1$)
5. $\sin(n) = O(1)$
6. $2^n = o(n!)$
7. Let f, g be positive functions. Then $f(n) + g(n) = O(\min(f(n), g(n)))$.
8. $n^{\log n} = \Omega(n^{100})$

9. Let f, g be positive functions with $g = o(f(n))$. Then $f(n)g(n) = o(f(n))$.

1.2 (10 points)

1. Prove that $\sum_{i=1}^{n} i^3 = \Theta(n^4)$.

2 Problem 2(20 Points)

2.1 (10 points)

Prove by induction on n that $\sum_{i=0}^{n-1} \binom{i}{k} = \binom{n}{k+1}$ for $n \geq 1$, $0 \leq k < n$. (Hint: Pascal's rule states that for $1 \leq k \leq n$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.)

2.2 (10 points)

1. Prove that if A, B and C are sets, then i) $A - (B \cap C) = (A - B) \cup (A - C)$ and ii) $A - (B \cup C) = (A - B) \cap (A - C)$.

2. How many different 5-card poker hands include a 4 of a kind? A standard deck contains 52 cards of 13 different ranks in 4 suits. A four-of-a-kind is a hand which contains four cards of the same rank, one in each of the four suits. Hands of cards are not ordered. That is, the 3-card hand Ace, King, 4 is considered to be the same as the hand King, 4, Ace.

3. Each of a class of 7 students has 5 coins in his or her pocket, consisting of pennies and/or nickels. We say that two students have *equivalent* pockets if they have the same number of coins of each type. Is it possible that no two of the students in the class have equivalent pockets?

4. Prove the pigeon-hole principle, which states that *the maximum of a set of numbers is greater than or equal to the arithmetic mean of the set*. (Hint: assume the contrary and derive a contradiction.)

5. How many ordered pairs (a, b) are there such that a and b are integers, $1 \leq a, b \leq 15$ and $a + b \leq 15$? Note that the ordered pairs $(5, 10)$ and $(10, 5)$ are distinct and should be counted separately.