Homework #1
Introduction to Algorithms/Algorithms 1
600.363/463
Spring 2015

Due on: Tuesday, February 3rd, 5pm
Late submissions: will NOT be accepted
Format: Please start each problem on a new page.
Where to submit: On blackboard, under student assessment.
Otherwise, please bring your solutions to the lecture.

February 2, 2015

1 Problem 1 (20 points)

1.1 (10 points)

For each statement below explain if it is true or false and prove your answer. Be as precise as you can. The base of $\log$ is 2 unless stated otherwise.

1. $\frac{n^4}{\log^2 n} = \Theta(n^2)$
   False. $\frac{n^4}{\log^2 n} = \Omega(n^2)$ but $\frac{n^4}{\log^2 n} \neq O(n^2)$. This is because n dominates on $\log n$ and there is no positive constant c and n_0 to let $\frac{n^4}{\log^2 n} \leq c \cdot n^2$ for any $n > n_0$.

2. $2^n = O(3^n)$
   True. There exists a positive constant c and a n_0, let $2^n \leq c * 3^n$ for all $n \geq n_0$

3. $2^n = \Theta(3^n)$
   False. Although $2^n = O(3^n)$, $2^n \neq \Omega(3^n)$. (provide simple proof here).

4. $e^n = \Theta(2^{(n+3)})$
   False. Since $e > 2$ and thene^n dominates on 2^n, $e^n \neq O(2^{(n+3)})$

5. $\log\log n = O(\log(\frac{\sqrt{n}}{3\log n}) + \log(n-3))$
True. $\log(\frac{\sqrt{n}}{3\log n}) + \log(n-3) = 1/2\log(n) - \log(3) - \log\log n + \log(n-3) \geq 1/2\log(n) - \log(3) > c \cdot \log\log n$ can hold for some constant c (e.g. $1/4$) and n is sufficiently large.

6. Let f, g, h be positive functions. Then $h(n)(f(n)+g(n)) = O(h(n)f(n)g(n))$
False. If functions f, g are some decreasing functions, e.g. $f(n) = 1/n$ and $g(n) = 1/n$, $f(n) + g(n) \leq c * f(n)g(n)$ does not always hold for some positive constant c when $n > n_0$.

7. Let f, g be positive functions. Then $f(n) + g(n) = \Theta(\max(f(n), g(n)))$
True. No matter whether$f \geq g$ or not, we can find some positive constants c_1, c_2 to satisfy $f(n) + g(n) \leq c_1 \cdot \max(f(n), g(n)))$ and $f(n) + g(n) \geq c_2 \cdot \max(f(n), g(n)))$

8. $n^{n^2} = \Omega(\left(e^{n^3}\right))$
False. Let $f(n) = n^{n^2}$ and $g(n) = e^{n^3}$. Clearly, $\log(f(n)) = n^2 \log n$ and $\log(g(n)) = n^3 \log e$. Then $\lim_{n\to\infty} \frac{\log(f(n))}{\log(g(n))} = 0$, since n increases much faster than $\log n$. That is, when n is increasing, $g(n)$ will increase much faster than $f(n)$. So we cannot find any positive constant c and some n_0 suffice to $n^{n^2} \geq c \cdot \left(e^{n^3}\right)$ for all $n \geq n_0$. Therefore,$n^{n^2} \neq \Omega(\left(e^{n^3}\right))$

9. Let f and g be positive functions. If $f(n) + g(n) = \Omega(f(n))$ then $g(n) = O((f(n))^2)$.
False. Let's consider a counter example here. Let $f(n) = n$ and $g(n) = n^3$. Apparently, $f(n) + g(n) = \Omega(f(n))$ but $n^3 \neq O(n^2)$.

1.2 (10 points)

1. Prove that

$$\sum_{i=1}^{n} \frac{1}{i} = O(\log n).$$

Proof. Since $\sum_{i=1}^{n} \frac{1}{i} \leq 1 + \int_1^n \frac{1}{x}dx = \ln(n) + 1$. So there exists a positive constant c and a positive n_0 to suffice $\ln(n) + 1 < c \cdot \log(n)$ for all $n > n_0$. $\square$

2 Problem 2(20 Points)

2.1 (10 points)

Prove by induction that $n! > 2^n$ for all $n >= 4$.

Proof. To prove the above inequation by induction,

1. Let $n = 4$, $4! > 2^4$.

2. Let $n = k$ and $k > 4$, assume that $n! > 2^n$ is true. So $k! > 2^k$. Then we need to prove $n! > 2^n$ is still true when $n = k + 1$. From $k! > 2^k$, we know that $(k + 1)k! > (k + 1)2^k$. Since $k > 4$, then $(k + 1)2^k > 2 \cdot 2^k$. So $(k + 1)! > 2^(k + 1)$. $\square$

2.2 (10 points)

1. Let A, B, C be sets. Prove that

$$A \setminus (B \cup C) = (A \setminus C) \cap (A \setminus B).$$

Proof. To prove this statement, let us first assume that $x \in A \setminus (B \cup C)$. Then by definition, we have $x \in A$ and $x \notin B \cup C$. $x \notin B \cup C$ means that x is an element of neither B nor C, i.e., $x \notin B$ and $x \notin C$. Thus, we have $x \in A \setminus B$ and $x \in A \setminus C$, which by definition means that $x \in (A \setminus B) \cap (A \setminus C)$. Thus $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.

Conversely, suppose $x \in (A \setminus B) \cap (A \setminus C)$. By definition, this means that $x \in A \setminus B$ and $x \in A \setminus C$. By definition of $A \setminus B$ and $A \setminus C$, we have that $x \in A$, $x \notin B$ and $x \notin C$. $x \notin B$ and $x \notin C$ jointly imply that $x \notin B \cup C$. Thus, we have $x \in A$ and $x \notin B \cup C$, so by definition we have $x \in A \setminus (B \cup C)$. Since x was an arbitrary element in $(A \setminus B) \cap (A \setminus C)$, we have that $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$. Combining this fact with the one proved in the previous paragraph yields the desired result.

2. There are m books on the shelf. What is the number of different ways to divide the books between Alice, Bob and John? What is the answer if we request that each person gets at least $k < m/3$ books?
Answer:

If books are indistinguishable, the first problem is equivalent to the problem of counting the the number of distinct 3-tuples of non-negative integers whose sum is m. By the Stars and Bars theorem from combinatorics (given by binomial coefficient), there are $\binom{m+3-1}{3-1} = \binom{m+2}{m}$. If each person needs to get at least k books, we can also consider an equivalent problem of counting the ways of distributing $m - 3k$ books into 3 different people. So the problem becomes counting the the number of distinct 3-tuples of non-negative integers whose sum is $m - 3k$. By applying the result above, we can get $\binom{m-3k+2}{2}$.

3. We have n balls. Each ball, independently and randomly, is placed into one of n bins. What is the probability that there are no empty bins at the end of our experiment?
Answer: If we need to have no empty bins at the end of our experiment, each bin has exactly 1 ball in it. So there are $n \cdot (n-1) \cdot (n-2), \ldots, \cdot 1$ number of ways to place the balls. In total we have n^n ways to place the balls. Therefore, the probability is $\frac{n!}{n^n}$.

4. There are n students in the class. How many different handshakes are possible?
Answer: Each handshake happens between a pair of students. So there are in total $\binom{n}{2}$ pairs of students in the class, and hence $\frac{n^2-n}{2}$ possible handshakes.