Brief Soln for Homework #1
Introduction to Algorithms/Algorithms 1
600.433/633
Spring 2018

Due on: Tuesday, February 6th, 5pm
Late submissions: will NOT be accepted
Format: Please start each problem on a new page.
Where to submit: Gradescope.
Otherwise, please bring your solutions to the lecture.

February 25, 2018

1 Problem 1 (25 points)

1.1 (15 points)

For each statement below explain if it is true or false and prove your answer. Be as precise as you can. The base of log is 2 unless stated otherwise.

1. $2^{n^2} = \Theta(3^{n+\sqrt{n}})$

   False
   Since $\lim_{n\to\infty} \frac{2^{n^2}}{3^{n+\sqrt{n}}} = \lim_{n\to\infty} 2^{n^2 - \log 3(n+\sqrt{n})} = \infty$, we know that there exists a constant C (say $C = 1$) and some large n_0 (say $n_0 = 100$), such that $2^{n^2} > C \cdot 3^{n+\sqrt{n}}$ for $n \geq n_0$. Thus $2^{n^2} = \omega(3^{n+\sqrt{n}})$ and $2^{n^2} \neq O(3^{n+\sqrt{n}})$.

2. $n^{n^2} = \Omega\left(e^{n^3}\right)$

   False
   Since $\lim_{n\to\infty} \frac{n^{n^2}}{e^{n^3}} = \lim_{n\to\infty} \frac{e^{n^2 \ln n}}{e^{n^3}} = 0$, we can know that there exists a constant C and some sufficiently large n_0, such that $n^{n^2} < C \cdot e^{n^3}$ for $n \geq n_0$. Thus $n^{n^2} = o\left(e^{n^3}\right)$.

3. $2^{2n} = \Theta\left(2^{n+2}\right)$

   False
   Since $\lim_{n\to\infty} \frac{2^{2n}}{2^{n+2}} = \lim_{n\to\infty} 2^{n-1} = \infty$, we can know that there exists a constant C and some sufficiently large n_0, such that $2^{2n} > C \cdot 2^{n+2}$ for $n \geq n_0$. Thus $2^{2n} = \omega\left(2^{n+2}\right)$ and $2^{2n} \neq \Theta\left(2^{n+2}\right)$.

4. $\log\log n = O(\log \frac{\sqrt{n}}{3\log n} + \log(n - \log n))$

   True
   Let's take $C = 1$ and $n_0 = 256$, than by definition of $O(\cdot)$ notation:

   $$\log \frac{\sqrt{n}}{3\log n} + \log\left(n - \log n\right) = \frac{1}{2}\log n - \log\log n - \log 3 + \log(n - \log n)$$

   Where for $n > n_0 = 256$ we have $-\log 3 + \log(n - \log n) > 0$: Thus we only need to show that

   $$\frac{1}{2}\log n - \log\log n > \log\log n$$

   $$\log n > 4\log\log n$$

   $$n > 2^4 \log n$$

   Which also holds for all $n > n_0 = 256$.

5. $n^{n\log n} = \Omega(e^{n^2 - n\log n})$

   False
   Since $\lim_{n\to\infty} \frac{n^{n\log n}}{e^{n^2-n\log n}} = \lim_{n\to\infty} \frac{e^{n\log n \ln n}}{e^{n^2 - n\log n}} = 0$, we can know that there exists a constant C and some sufficiently large n_0, such that $n^{n\log n} < C \cdot e^{n^2 - n\log n}$ for $n \geq n_0$. Thus $n^{n\log n} = o(e^{n^2-n\log n})$.

6. $n! = \omega(2^n)$

   True
   Since $\lim_{n\to\infty} \frac{n!}{2^n} \geq \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot 4^{n-3}}{2^n} = \infty$, we can know that there exists a constant C and some sufficiently large n_0, such that $n! > C \cdot 2^n$ for $n \geq n_0$. Thus $n! = \omega(2^n)$.

7. Let f, g be positive functions with $g(n) = o(f(n))$. Then $f(n)g(n) = o(f(n))$.

False
Show by counterexample. Let $g(n) = n^2$ and $f(n) = n^4$. Thus $f(n)g(n) = n^6 \neq o(f(n))$.

8. Let f, g be positive functions. If $f(n) + g(n) = \Omega(f(n))$ then $g(n) = O((f(n))^2)$.

   False
   Show by counterexample. Let $f(n) = n^2$ and $g(n) = n^6$. Thus $g(n) \neq O((f(n))^2)$.

9. Let f, g, h be positive functions. If $g(n) = o(f(n))$ and $f(n) = O(h(n))$, then $g(n) = o(h(n))$.

   True
   Since $g(n) = o(f(n))$, $g(n) < c_1 \cdot f(n)$ for some constant c_1 and for all n larger than some n_1, by definition. Since $f(n) = O(h(n))$, by definition, $f(n) \leq c_2 \cdot h(n)$ for some constant c_2 and for all n larger than some n_2. Thus, $g(n) < c_1 c_2 \cdot h(n)$ for all n larger than some $n_0 = max\{n_1, n_2\}$.

10. Let f be positive function. Then $f(n) = O((f(n))^2)$.

    False
    Let's consider decreasing function $f(n) = 1/n^2$ and thus $f(n)$ is asymptotically faster than $1/n^4$.

1.2 (10 points)

1. Prove that
$$\sum_{i=1}^{n} \frac{1}{i} = O(\log n).$$

   Answer: Since $\sum_{i=1}^{n} \frac{1}{i} \leq 1 + \int_1^n \frac{1}{x} dx = ln(n) + 1$. So there exists a positive constant c and a positive n_0 to suffice $ln(n) + 1 < c \cdot \log(n)$ for all $n > n_0$.

2 Problem 2(25 Points)

2.1 (9 points)

Prove by induction that $\sum_{i=0}^{n-1} \binom{i}{k} = \binom{n}{k+1}$ for $n \geq 1, 0 \leq k < n$.
(Hint: Pascal's rule states that for $1 \leq k \leq n$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.)
Answer:
We can prove by induction with n.

- Base case: when $n = 1$, the equation holds.

$$\binom{0}{0} = \binom{1}{1} = 1$$

- Inductive step: assume that the equation holds for $n = n_0$ where $n_0 > 1$, we need to prove if the equation still holds for $n = n_0 + 1$. Given the assumption,

$$\sum_{i=0}^{n_0-1} \binom{i}{k} = \binom{n_0}{k+1}$$

When $n = n_0 + 1$, by Pascal's rule,

$$\sum_{i=0}^{n_0} \binom{i}{k} = \sum_{i=0}^{n_0-1} \binom{i}{k} + \binom{n_0}{k} = \binom{n_0}{k+1} + \binom{n_0}{k} = \binom{n_0+1}{k+1}$$

Thus, for $n \geq 1, 0 \leq k < n$, $\sum_{i=0}^{n-1} \binom{i}{k} = \binom{n}{k+1}$, QED.

2.2 (16 points)

1. Let A, B, C be sets. Prove that

   (a) $A \setminus (A \setminus B) = A \cap B$

   Answer: To prove it we will show that:

   i. $x \in A \setminus (A \setminus B) \Rightarrow x \in A \cap B$
   ii. $x \in A \cap B \Rightarrow x \in A \setminus (A \setminus B)$

   Suppose $x \in A \setminus (A \setminus B)$ then $x \in A$ and $x \notin A \setminus B$, thus $x \in B$, therefore $x \in A \cap B$.
   Suppose $x \in A \cap B$, thus $x \in A$ and $x \in B$, then $x \notin A \setminus B$, therefore $x \in A \setminus (A \setminus B)$.

(b) $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$

Answer: We will use same approach.
Suppose $x \in A \cup B$, then
if $x \in A$ and $x \notin B$ then $x \in A \setminus B$,
if $x \notin A$ and $x \in B$ then $x \in B \setminus A$,
if $x \in A$ and $x \in B$ then $x \in A \cap B$.
In all cases $x \in (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$.
Suppose now that $x \in (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$, then:
if $x \in A \setminus B$ then $x \in A \setminus B \cup B = A \cup B$,
if $x \in B \setminus A$ then $x \in B \setminus A \cup A = A \cup B$,
if $x \in A \cap B$ then $x \in A$ and $x \in B$, thus $x \in A \cup B$.
In all cases we conclude that $x \in A \cup B$.

(c) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Answer: Will use same technique.
Suppose $x \in A \setminus (B \setminus C)$, then $x \in A$ and $x \notin B \setminus C$. Thus one of next three cases holds:
$x \in A, x \notin B, x \notin C \Rightarrow x \in A \setminus B$
$x \in A, x \notin B, x \in C \Rightarrow x \in A \cap C$
$x \in A, x \in B, x \in C \Rightarrow x \in A \cap C$
In all cases $x \in (A \setminus B) \cup (A \cap C)$.
Suppose $x \in (A \setminus B) \cup (A \cap C$, then one of next three cases holds:
$x \in A \setminus B$ and $x \in A \cap C$, then $x \in A, x \notin B, x \in C$
$x \in A \setminus B$ and $x \notin A \cap C$, then $x \in A, x \notin B, x \notin C$
$x \notin A \setminus B$ and $x \in A \cap C$, then $x \in A, x \in B, x \in C$.
In all cases we can conclude that $x \in A$ and $x \notin B \setminus C$. Thus for all of them we can conclude that $x \in A \setminus (B \setminus C)$.

2. Suppose that a random machine outputs each number from 1 to $x - 1$ with equal probability. What is probability that the output is coprime with x, where $x = 3^n 5^m 7^k$ and n, m, k are positive integers.

Anwser: For simplicity of calculations we considered x to be co-prime to itself. If consider range from 1 to $x - 1$, you will get next answer, like:

$$1 - \frac{x/3 - 1}{x - 1} + \frac{x/5 - 1}{x - 1} + \frac{x/7 - 1}{x - 1} - \frac{x/15 - 1}{x - 1} - \frac{x/21 - 1}{x - 1} - \frac{x/35 - 1}{x - 1} + \frac{x/105 - 1}{x - 1} =$$

$$= 1 - \frac{x/3 - 1 + x/5 - 1 + x/7 - 1 - x/15 + 1 - x/21 + 1 - x/35 + 1 + x/105 - 1}{x - 1} =$$

$$= 1 - \frac{19x/35 - 1}{x - 1} = \frac{x - 1 - 19x/35 + 1}{x - 1} = \frac{16x}{35(x - 1)}$$

3. We have n balls. Each ball, independently and randomly, is placed into one of n bins. What is the probability that there are no empty bins at the end of our experiment?

   Answer: assume balls are labelled, there are $n!$ ways that leave no empty bin in the end and there are in total n^n ways to distribute all balls. So the probability is $\frac{n!}{n^n}$.

4. Prove the pigeon-hole principle, which states that the maximum of a set of numbers is greater than or equal to the arithmetic mean of the set. (Hint: assume the contrary and derive a contradiction.)

   Answer: assume that for some set of numbers $x_1, x_2, \ldots, x_n$, the maximum of the set (call it x_i) is less than the arithmetic mean of the set. That is, suppose that (i) $x_i \geq x_j$ for all $1 \leq i, j \leq n$ and that (ii) $x_i < \frac{1}{n} \sum_{k=1}^{n} x_k$. We have, by assumption that x_i is the maximum, $\sum_{k=1}^{n} x_k \leq n x_i$. Combining this with assumption (ii) that x_i is strictly less than the arithmetic mean, we see that $x_i < x_i$, which is a contradiction.