Homework #8
Introduction to Algorithms/Algorithms 1
600.363/463
Spring 2014

Due on: Tuesday, April 1st, 5pm
Late submissions: will NOT be accepted
Format: Please start each problem on a new page.
Where to submit: On blackboard, under student assessment
Please type your answers; handwritten assignments will not be accepted.
To get full credit, your answers must be explained clearly,
with enough details and rigorous proofs.

March 25, 2014

Problem 1 (20 points)

Let $G = (V, E)$ be an undirected graph with distinct non-negative edge weights. Consider a problem similar to the single-source shortest paths problem, but where we define path cost differently. We will define the cost of a simple s–t path $P_{s,t} = \{e_1, e_2, \ldots, e_k\}$ to be

$$c(P_{s,t}) = \max_{e \in P} w_e.$$

That is, the cost of a path is now just the largest weight on that path, rather than the sum of the weights on the path. Give an algorithm that takes as input an undirected graph $G = (V, E)$ with non-negative edge weights, and a vertex $s \in V$, and computes the path from s to every other node in G with the least cost under cost function $c(\cdot)$. That is, for each $t \in V$, find a simple path connecting s to t, $P_{s,t} = \{(s, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, t)\}$, such that, letting $\mathcal{S}$ denote the set of all simple paths connecting s and t, $c(P_{s,t}) = \min_{P \in \mathcal{S}} c(P)$. Your algorithm should run in $O(|E| + |V| \log |V|)$ time. Prove the correctness of your algorithm and its runtime. Hint: note the similarities between Dijkstra's algorithm and Prim's algorithm.

Problem 2 (20 points)

Let $G = (V, E)$ be a connected undirected graph with unit weights. We will define two similar but distinct *graph statistics*, numbers that summarize information about graph G. The first we'll call the *diameter* of G,

$$\mathrm{diam}(G) = \max_{u,v \in V} \delta(u, v),$$

where $\delta(u, v)$ is the length of the shortest path connecting u and v. The second we'll call the *average distance* of G,

$$\mathrm{avedist}(G) = \frac{1}{\binom{n}{2}} \sum_{\{u,v\} \subseteq V} \delta(u, v).$$

In general, it need not be the case that $\mathrm{diam}(G) = \mathrm{avedist}(G)$. For example, consider an unweighted graph H on three nodes a, b, c, with two edges (a, b) and (b, c). This graph has $\mathrm{diam}(H) = \delta(a, c) = 2$, while $\mathrm{avedist}(H) = \frac{\delta(a,b)+\delta(b,c)+\delta(a,c)}{3} = 4/3$. Prove or disprove the following claim: there exists a positive constant c such that for all connected, undirected graphs G, we have

$$\frac{\mathrm{diam}(G)}{\mathrm{avedist}(G)} \leq c.$$

Hint: start with a fully connected graph, and connect a "path" to it.

Optional exercises

Solve the following problems and exercises from CLRS: 26-4, 26-5, 26-6.