Final Exam
Introduction to Algorithms
601.433/633
Spring 2020

Due: Friday, May 8th, 10am

Ethics Statement

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Name

Signature   Date

1 Problem #1: YES/NO QUESTIONS (100 points) (10 questions, 10 points per question)

For every question answer yes or no.

1. Let T be an MST in graph $G = (V, E)$ and let $a, b \in V$ be two vertices. Let P be the path in T from a to b. P is always a shortest path from a to b.

   yes no

2. Prim and Kruskal algorithm may yield different spanning trees.

   yes no

3. There exist two strictly positive functions f, g such that $f(x) = o(g(x))$ and $(g(x))^2 = o((f(x))^3)$.

   yes no

4. If $f(n) = \Theta(g(n))$ then $2^{f(n)} = \Theta\left(2^{g(n)}\right)$.

   yes no

5. It is always possible to sort an array A of n real numbers in linear time if $\max_{i=1}^{n} |A[i]| \leq 5$.

   yes no

6. Let $G = (V, E)$ be a directed weighted graph and let $\delta(a, b)$ be the length of a shortest path from a to b. Let $a, b, c \in V$. If $\delta(a, b) + \delta(b, c) \leq \delta(a, c)$

then b belongs to a shortest path from a to c.

yes no

7. There is some fixed input X for which Randomized QuickSort always runs in $\Theta(n^2)$ time.

yes no

8. For all undirected, weighted graphs, if we increase all weights by exactly one then all shortest paths do not change.

yes no

9. An algorithm whose running time satisfies the recurrence

$$P(n) = 10P(n/2) + O(n^{10})$$

is asymptotically faster than an algorithm whose running time satisfies the recurrence

$$E(n) = 1.1E(n - 1000) + O(1).$$

yes no

10. Let G be undirected, weighted graph. If MST is unique then all weights are distinct.

yes no

2 Problem #2: Basic (60 points)

You are given array A of n distinct integers. It is known that A has been sorted and then cyclically shifted (to the right) by k elements where k is unknown to you. For example, array $A = \{5, 6, 1, 2, 3, 4\}$ is sorted and cyclically shifted by $k = 2$ elements.

Design an algorithm for finding the median value of the elements in the array A. Prove the correctness of your algorithm and give the running time. Your answers must be explained clearly, and with enough details. The full credit will be given if the running time of your algorithm is $o(\sqrt{n})$ (little o).

3 Problem #3: Network Flow (60 points)

You are given a flow network G, and G has edges entering the source s. Let f be an integer flow in G in which one of the edges (v, s) entering the source has $f(v, s) = 2$. Prove formally that there must exist another flow f' with $f'(v, s) = 0$ such that $|f| = |f'|$.

Hint: Maybe you can use a result from your homework?

4 Problem #4: Dynamic Programming (60 points)

You are given an $n \times n$ binary matrix (each entry of the matrix is either "0" or "1"), design a dynamic programming algorithm to find the largest rectangle containing only 1's and return its area, prove correctness and provide running time analysis. Full score will given for the solution with the running time $O(n^2)$.

For example, given the following matrix,

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

the largest rectangle containing only 1's is the following and its area is 6.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

5 Problem #5: MST (60 points)

Let $G = (V, E)$ be an undirected, connected and weighted graph with the weight function w. Assume that there exist an edge e_0 and two numbers X, Y such that $X < Y$, $w(e_0) = Y$ and $w(e) = X$ for all $e \in E, e \neq e_0$. Design an efficient algorithm that finds an MST in G. Prove the correctness of your algorithm and give the running time. Your answers must be explained clearly, and with enough details. The full credit will be given if the running time of your algorithm is $o(|E| \log(|V|))$.

6 Problem #6: Median (60 points)

For n distinct numbers $x_1, x_2, \ldots, x_n$ with distinct positive weights $w_1, w_2, \ldots, w_n$ such that $\sum_{i=1}^{n} w_i = 1$,

the **weighted (lower) median** is the number x_k satisfying

$$\sum_{x_i < x_k} w_i < \frac{1}{2}$$

and

$$\sum_{x_i > x_k} w_i \leq \frac{1}{2}$$

For example, if numbers are $0.1, 0.35, 0.05, 0.1, 0.15, 0.05, 0.2$ and each number equals its weight (that is $w_i = x_i$ for $i = 1, 2, \ldots, 7$), then the median is 0.1, but the weighted median is 0.2.

Devise an algorithm that computes the weighted median in $\Theta(n)$ worst-case time. Prove correctness and provide running time analysis. 30 points will be given for the algorithm with running time $O(n \log n)$.