Homework #1
Introduction to Algorithms/Algorithms 1
600.363/463
Spring 2017

Due on: Tuesday, February 7th, 5pm
Late submissions: will NOT be accepted
Format: Please start each problem on a new page.
Where to submit: On blackboard, under student assessment.
Otherwise, please bring your solutions to the lecture.

February 9, 2017

1 Problem 1 (20 points)

1.1 (5 points)

Prove that, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = k$, where k is a possible constant, then $f(n) = \Theta(g(n))$.

Proof. By the definition of limit, when n goes to large enough and beyond, $\frac{f(n)}{g(n)}$ will be k or some constant k_0 that nearly reaches k. So when $n \geq n_0$ for some large enough n_0, $f(n) \leq k_0 \times g(n)$ and thus $f(n) = O(g(n))$. Similarly, by the property of limit, since k is a positive constant, $\forall m \in (0, k)$, there exists some $n_0 > 0$, for any $n > n_0$, $\frac{f(n)}{g(n)} > m$. So $f(n) \geq m \times g(n)$ when $n > n_0$, $f(n) = \Omega(g(n))$. Thus, $f(n) = \Theta(g(n))$. $\square$

1.2 (5 points)

For each statement below explain if it is true or false and prove your answer. Be as precise as you can. The base of $\log$ is 2 unless stated otherwise.

1. $\frac{n^2}{\log n} = \Theta(n)$. Answer: False.

Proof. Let $f(n) = \frac{n^2}{\log n}$ and $g(n) = n$. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n}{\log n} = \infty$. So by the definition of limit, we cannot find a n_0 and a positive constant c, such that $\forall n > n_0$, $f(n) \le cg(n)$. Thus, $f(n) \ne O(g(n))$. $\square$

2. $2^n = O(3^n)$. Answer: True.

   Proof. Let $f(n) = 2^n$ and $g(n) = 3^n$. For any $n \ge 0$, $f(n) \le g(n)$. Thus $f(n) = O(g(n))$. $\square$

3. $\sqrt{n} = \Theta(2\log n^2)$. Answer: False.

   Proof. Let $f(n) = \sqrt{n}$ and $g(n) = \Theta(2\log n^2)$. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \infty$. Thus we cannot find a n_0 and a positive constant c, such that $\forall n > n_0$, $f(n) \le cg(n)$. Thus, $f(n) \ne O(g(n))$. $\square$

4. $3n\log n + n = O(\frac{n^2-n}{2})$. Answer: True.

   Proof. Let $f(n) = 3n\log n + n$ and $g(n) = \frac{n^2-n}{2}$. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log n}{n} = 0$. Thus we find have a n_0 and a positive constant c, such that $\forall n > n_0$, $f(n) \le cg(n)$. Thus, $f(n) = O(g(n))$. $\square$

5. Let f and g be positive functions. If $f(n) + g(n) = \Omega(f(n))$ then $g(n) = O((f(n))^2)$. Answer: False.

   Proof. Since $f(n) + g(n) = \Omega(f(n))$, $g(n)$ is lower bounded by $f(n)$ but $g(n)$ has no upper bound. Let $g(n) = (f(n))^4$. $g(n)$ meets the above condition but $g(n) \ne O((f(n))^2)$. $\square$

1.3 (10 points)

1. Prove that

$$\sum_{i=1}^{n} \log i = O(n\log n).$$

   Proof. Since $\log i$ is an increasing function, $\log n > \log i$ when $i = 1 \ldots n-1$. Then $n\log n \ge \log 1 + \log 2 \cdots + \log n$. Thus $\sum_{i=1}^{n} \log i = O(n\log n)$. $\square$

2 Problem 2 (20 Points)

2.1 (10 points)

1. Prove by induction that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \geq 1$.

   Proof. • When $n = 1$, $\frac{1}{2} = \frac{1}{2}$. True.

   • Let's assume when $n = k$, $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. We need to show that when $n = k + 1$, the statement still holds.
   So $(\sum_{i=1}^{k} \frac{1}{i(i+1)}) + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

   $\square$

2. Alice wants to distribute three movie tickets to ten friends. If each friend could get up to one ticket, how many ways can Alice distribute these tickets?

   Answer: If we assume tickets are identical, there are $\binom{10}{3} = \frac{10*9*8}{3*2*1} = 120$ ways to distribute.

   If we assume tickets are different, for the first ticket, there are 10 friends to give; for the second ticket, there are 9 friends to give. Thus in total, there are $10 * 9 * 8 = 720$ ways to distribute the tickets.

3. We have n books. Each book, independently and randomly, is placed into one of n shelves. What is the probability that there are no empty shelves at the end of our experiment?

   Answer: If we assume books are identical, there is only one way that leaves no empty shelves at the end — every shelf has exactly one book. In total, we have n shelves, by the Stars and Bars theorem, there are $\left(\binom{n+1}{n-1}\right) = \binom{2n-1}{n-1}$.
   So the probability is $\frac{1}{\binom{2n-1}{n-1}} = \frac{(n-1)!n!}{(2n-1)!}$.
   If we assume books are different, there is $n!$ ways that leaves no empty shelves at the end. In total, there are, n^n ways. So the probability is $\frac{n!}{n^n}$.

2.2 (10 points)

You are given $n = 2^k$ compact discs, all of which look identical. However, there is one defective disc among all discs — it weighs different than the rest. You are also given an equivalence tester, which has two compartments. You could place any set of objects in each compartment, and the tester tells you whether or not the two sets weigh the same. Note that the tester doesn't tell you which side is heavier and which one lighter.

Your goal is to use the tester at most k times to determine which of the discs is defective. You may assume that $k > 1$. Can you describe your method and briefly explain why it works? (Note that you are allowed to use the tester only k times, not $O(k)$. You will receive partial credit for slightly worse solutions.)
Answer:

- Starting with 2^k discs, split them into 4 groups s_1, s_2, s_3, s_4 and each group has 2^{k-2} discs.

- Test s_1 with s_2; and test s_1 with s_3. // If s_1 and s_2 weight different, the defective disc must be in s_1 or s_2. By testing s_1 with s_3, you will know the defective is whether in s_1 or not. If s_1 and s_2 weight the same, the defective disc must be in s_3 or s_4. By testing s_1 with s_3, you will know the defective is whether in s_3 or not.

- Find the group s_i with the defective disc inside. Repeat the above steps (split into 4 groups and test twice) until find the defective disc. // In total we need repeat $\log_4 2^k = \frac{k}{2}$ times.

  Since each split step requires 2 times of testing to find the correct group that has that defective disc, we test $\frac{k}{2} * 2 = k$ times.