Homework #9
Introduction to Algorithms/Algorithms 1
600.363/463
Spring 2014

Due on: Tuesday, April 8th, 5pm
Late submissions: will NOT be accepted
Format: Please start each problem on a new page.
Where to submit: On blackboard, under student assessment
Please type your answers; handwritten assignments will not be accepted.
To get full credit, your answers must be explained clearly,
with enough details and rigorous proofs.

April 1, 2014

1 Problem 1 (20 points)

Let $G = (V, E)$ be a flow network with source node s and sink node t and non-negative integer capacities, so that $c(a, b) \in \{0, 1, 2, \ldots\}$ for all $(a, b) \in E$. Assume that G has a unique minimum cut. Suppose that we have already found the optimal flow on graph G and that this flow is an integral flow (i.e., for all $(a, b) \in E$, $f(a, b)$ is an integer). Call this flow f. Suppose now that we are informed that one capacity in this graph needs to be changed. In particular, we are going to change the capacity of edge $(u, v) \in E$ so that its new capacity is $c(u, v) - 1$, while all other edges retain their old capacities. Edge (u, v) is such that (u, v) crosses the unique minimum cut in graph G, and our flow f uses edge (u, v) to its full capacity, i.e., $f(u, v) = c(u, v)$. One might think that computing the flow with this updated capacity would require recomputing the flow from scratch (e.g., by running Ford-Fulkerson all over again), but it turns out that we can compute the new optimal flow in $O(|V| + |E|)$ time.

Give an algorithm that

(i) Accepts a flow network $G = (V, E)$ (with non-negative integer capacities that is guaranteed to have a unique minimum cut), a maximum flow f for that flow network, and an edge $(u, v) \in E$ for which $f(u, v) = c(u, v)$.

(ii) Returns a new flow for the same graph with new capacity function

$$c'(a, b) = \begin{cases} c(a, b) - 1 & \text{if } a = u, b = v \\ c(a, b) & \text{otherwise.} \end{cases}$$

(iii) Runs in $O(|V| + |E|)$ time.

Prove the correctness of your algorithm and prove its runtime.

2 Problem 2 (20 points)

1. **(10 points)** Prove or disprove the following claim: If f is a maximum flow on a graph G with source node s and sink node t, then $f(a, t) = c(a, t)$ for all edges $(a, t) \in E$.

2. **(10 points)** Suppose you are given a flow network $G = (V, E)$ with source node $s \in V$ and sink node $t \in V$ with unit capacities (i.e., $c(a, b) = 1$ for all $(a, b) \in E$). Consider the following problem: given some non-negative integer k, we would like to construct a new flow network $G' = (V, E')$, constructed by removing k edges from G (so $E' \subset E$ with $|E'| = |E| - k$), so that the maximum s–t flow in G' is as small as possible.

   Give an algorithm which constructs such a G' in $O(|E| f^*)$ time, where f^* denotes the value of the maximum flow on G. Prove the correctness of your algorithm and its runtime.

Optional exercises

Solve the following problems and exercises from CLRS: 26.2-7, 26.2-11, 26.3-2, 26.3-5