Sequence Labeling with Hidden Markov Models

Kevin Duh

Intro to NLP, Fall 2019

Sequence Labeling/Tagging Problem

- Input: a sequence of **T** words (e.g. a sentence)
- Output: a sequence of **T** labels/tags, one for each word

- In contrast, text classification:
  - Input: a sequence of **T** words
  - Output: **1** label

A naive implementation

- Treat each **T** word as independent
- Apply a classifier to each input word independently
- But this ignores sequence structure
  - e.g. maybe some labels are more likely to follow others

Part-of-Speech Tagging

- Input: The grand jury commented on a number of topics
- Ouput: DT JJ NN VBD IN DT NN IN NNS

Named Entity Recognition

- Task: Find text spans that refer to a proper name and label its type, e.g.
  - [George Washington PERSON] was the first president.
  - [Washington ORGANIZATION] won the World Series
- Input: George Washington was the first president
- Output: B-PER I-PER O O O O

Running Example: Ice Cream & Global Warming (courtesy of Jason Eisner)

Running Example: Ice Cream & Global Warming (courtesy of Jason Eisner)

- It's 2799. You're a climatologist studying the history of global warming. You have Jason Eisner's diary from 2007, which lists how many ice creams he ate every day.

Running Example: Ice Cream & Global Warming (courtesy of Jason Eisner)

- It’s 2799. You’re a climatologist studying the history of global warming. You have Jason Eisner’s diary from 2007, which lists how many ice creams he ate every day.
    - Assume there are two kinds of days: Cold (C) and Hot (H)

Running Example: Ice Cream & Global Warming (courtesy of Jason Eisner)

- It's 2799. You're a climatologist studying the history of global warming. You have Jason Eisner's diary from 2007, which lists how many ice creams he ate every day.
    - Assume there are two kinds of days: Cold (C) and Hot (H)
- Task: given a sequence of diary observations O, figure out the correct hidden sequence Q of weather states.

Running Example: Ice Cream & Global Warming (courtesy of Jason Eisner)

- It's 2799. You're a climatologist studying the history of global warming. You have Jason Eisner's diary from 2007, which lists how many ice creams he ate every day.
  - Assume there are two kinds of days: Cold (C) and Hot (H)
- Task: given a sequence of diary observations O, figure out the correct hidden sequence Q of weather states.
  - e.g. Jason ate 3 icecreams on Day 1, 1 icecream on Day 2, and 3 icecreams on Day 3. What's the weather on those three days?

This lecture

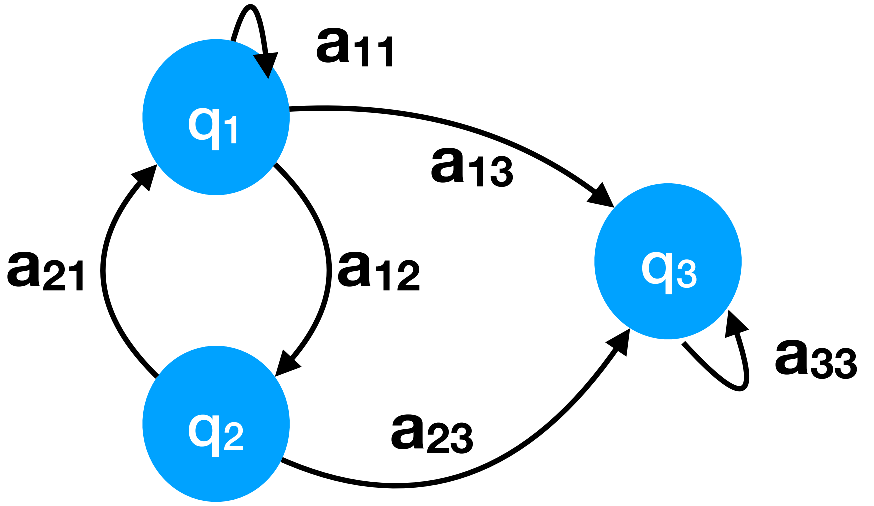
- We’ll discuss in-depth **Hidden Markov Models (HMM)**
  - Later in the course, we will also cover Conditional Random Fields (CRF), etc.
- HMM is a very useful pedagogical tool to illustrate:
  - How to **probabilistically model** the sequence labeling problem
  - How to efficiently compute with **Dynamic Programming**
  - How a model’s parameters can be learned by either **supervised** and **unsupervised** learning (for the latter, we’ll focus on **Expectation-Maximization**)

Outline

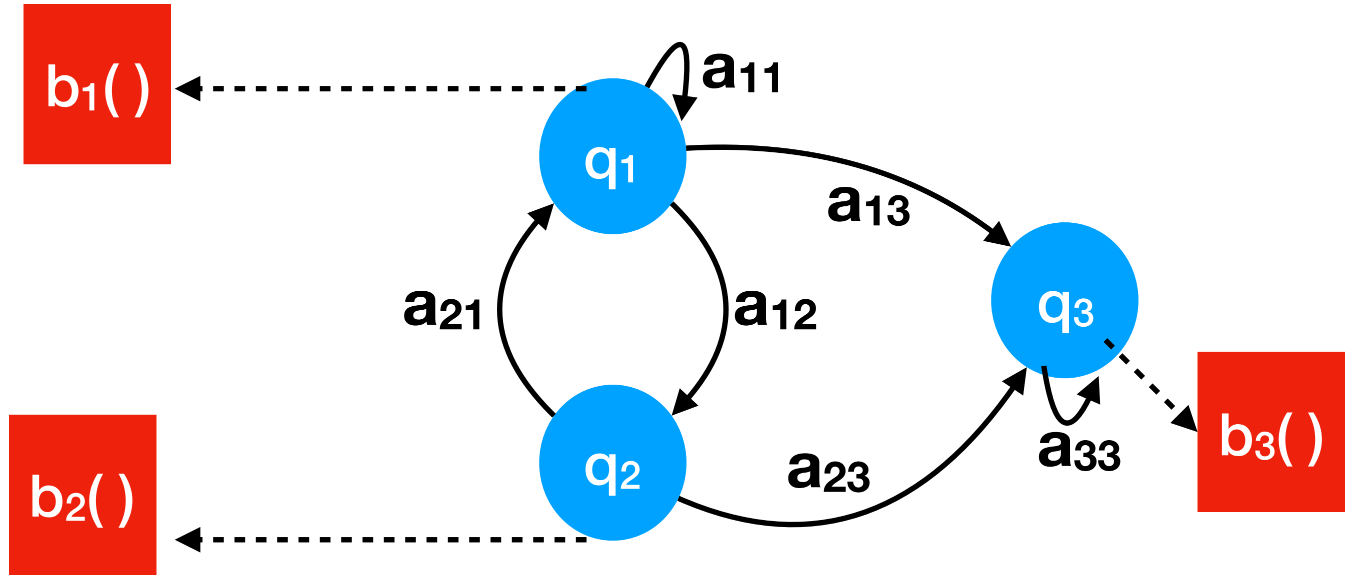
- Motivation/Examples
- HMM: Basic Definition & Three Problems
- Problem 1: Likelihood
- Problem 2: Decoding
- Problem 3: Learning
    - Supervised
    - Unsupervised: EM

Markov Chains

- A Markov Chain is specified by:
  - $Q = q_1 q_2 \ldots q_N$: a set of N states
  - $A = a_{11} a_{12} \ldots a_{ij} \ldots a_{NN}$: a transition probability matrix
  - $\pi = \pi_1 \pi_2 \ldots \pi_N$: initial probability distribution
- Markov Assumption: $P(q_i | q_1 \ldots q_{i-1}) = P(q_i | q_{i-1})$



- A **Hidden Markov Model (HMM)** is specified by:
  - $Q = q_1 q_2 \ldots q_N$: a set of N states
  - $A = a_{11} a_{12} \ldots a_{ij} \ldots a_{NN}$: a transition probability matrix
  - $\pi = \pi_1 \pi_2 \ldots \pi_N$: initial probability distribution
  - $B = b_i(o_t)$: emission probabilities
  - $O = o_1 o_2 \ldots o_T$: a sequence of T observations
- Markov Assumption: $P(q_i | q_1 \ldots q_{i-1}) = P(q_i | q_{i-1})$
- Output Independence: $P(o_i | q_1 \ldots q_T, o_1 \ldots o_T) = P(o_i | q_i)$



Three Problems for HMM

	Problem 1: Likelihood	**Problem 2: Decoding**	**Problem 3: Learning**
Given	HMM parameters λ = (A,B) Observation O	HMM parameters λ = (A,B) Observation O	Supervised: O and Q Unsupervised: O
Goal	Likelihood P(O|λ)	Most likely hidden sequence Q	HMM parameters λ = (A,B) that maximize likelihood
Method	Forward Algorithm	Viterbi Algorithm	Supervised: Count Unsupervised: Forward-Backward Algorithm

Running Example: Ice Cream

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)
- We have a sequence of integer observations, e.g.

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)
- We have a sequence of integer observations, e.g.
  - **3 1 3** (3 icecream on day 1, 1 icecream on day 2,…)

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)
- We have a sequence of integer observations, e.g.
    - **3 1 3** (3 icecream on day 1, 1 icecream on day 2,…)
- Likelihood: What's the probability of 3 1 3 occurring under our HMM model?

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)
- We have a sequence of integer observations, e.g.
  - **3 1 3** (3 icecream on day 1, 1 icecream on day 2,…)
- Likelihood: What's the probability of 3 1 3 occurring under our HMM model?
- Decoding: What's the most likely weather sequence for 3 1 3? Hot Cold Hot?

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)
- We have a sequence of integer observations, e.g.
  - **3 1 3** (3 icecream on day 1, 1 icecream on day 2,…)
- Likelihood: What’s the probability of 3 1 3 occurring under our HMM model?
- Decoding: What’s the most likely weather sequence for 3 1 3? Hot Cold Hot?
- Learning: How to estimate the HMM parameters?

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)
- We have a sequence of integer observations, e.g.
  - **3 1 3** (3 icecream on day 1, 1 icecream on day 2,…)
- Likelihood: What’s the probability of 3 1 3 occurring under our HMM model?
- Decoding: What’s the most likely weather sequence for 3 1 3? Hot Cold Hot?
- Learning: How to estimate the HMM parameters?
  - Supervised: given 3 1 3 and H C H

Running Example: Ice Cream

- HMM has two states: Cold (C) and Hot (H)
- We have a sequence of integer observations, e.g.
  - **3 1 3** (3 icecream on day 1, 1 icecream on day 2,…)
- Likelihood: What's the probability of 3 1 3 occurring under our HMM model?
- Decoding: What's the most likely weather sequence for 3 1 3? Hot Cold Hot?
- Learning: How to estimate the HMM parameters?
  - Supervised: given 3 1 3 and H C H
  - Unsupervised: given 3 1 3 only

Outline

- Motivation/Examples
- HMM: Basic Definition & Three Problems
- Problem 1: Likelihood
- Problem 2: Decoding
- Problem 3: Learning
  - Supervised
  - Unsupervised: EM

Likelihood: P(O|λ = (A,B))

- Joint probability is easy:

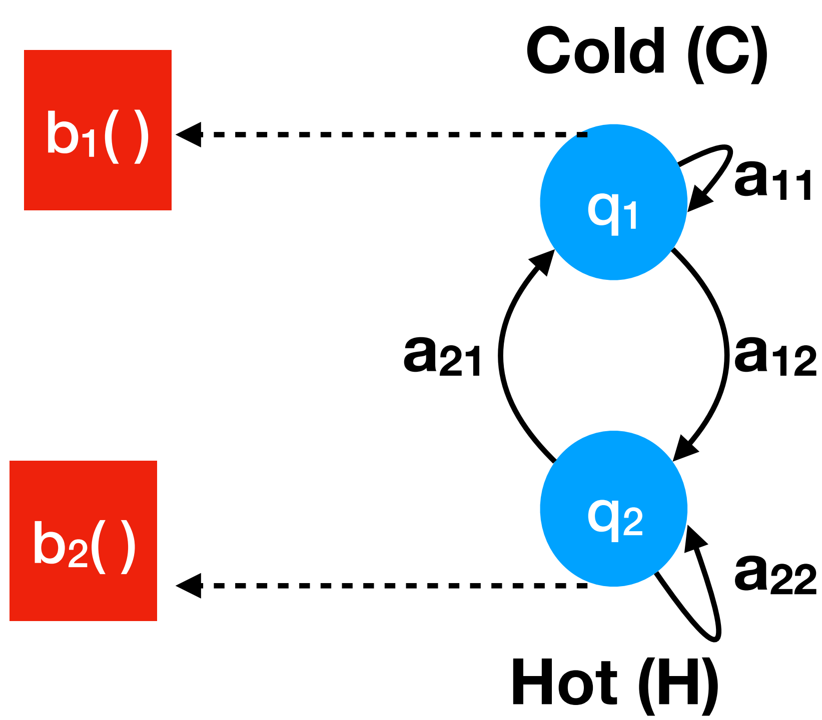
$$P(O,Q) = P(O|Q)P(Q) = \prod_{t=1}^{T} P(o_t|q_t) \times \prod_{t=1}^{T} P(q_t|q_{t-1})$$

Note: λ = (A,B) is implicit; dropped for notational simplicity

- For example: Q = H H C, O = 3 1 3

Then P(O,Q) = P(3|H)P(1|H)P(3|C) x P(H|start)P(H|H)P(C|H)

$= b_2(3)\ b_2(1)\ b_1(3) \times \pi_2\ a_{22}\ a_{21}$



Likelihood: P(O|λ = (A,B))

- Joint probability is easy:

$$P(O,Q) = P(O|Q)P(Q) = \prod_{t=1}^{T} P(o_t|q_t) \times \prod_{t=1}^{T} P(q_t|q_{t-1})$$

Note: λ = (A,B) is implicit; dropped for notational simplicity

- Since we don't know Q, sum over it

$$P(O) = \sum_{allQ} P(O,Q)$$

e.g. P(O=313) = P(O=313,Q=HHH)+P(O=313,Q=CCC)+…

For N states and T-length sequence, there are N^T hidden sequences!

Efficient enumeration by Dynamic Programming: Forward Algorithm

- Prepare a trellis data structure
- Each cell represents probability of being in state j after seeing the first t observations:

$$\alpha_t(j) = P(o_1, o_2, \ldots, o_t, q_t = j|\lambda)$$

- DP subproblem recursion:

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t)$$

$$\alpha_t(j) = P(o_1, o_2, \ldots, o_t, q_t = j|\lambda)$$

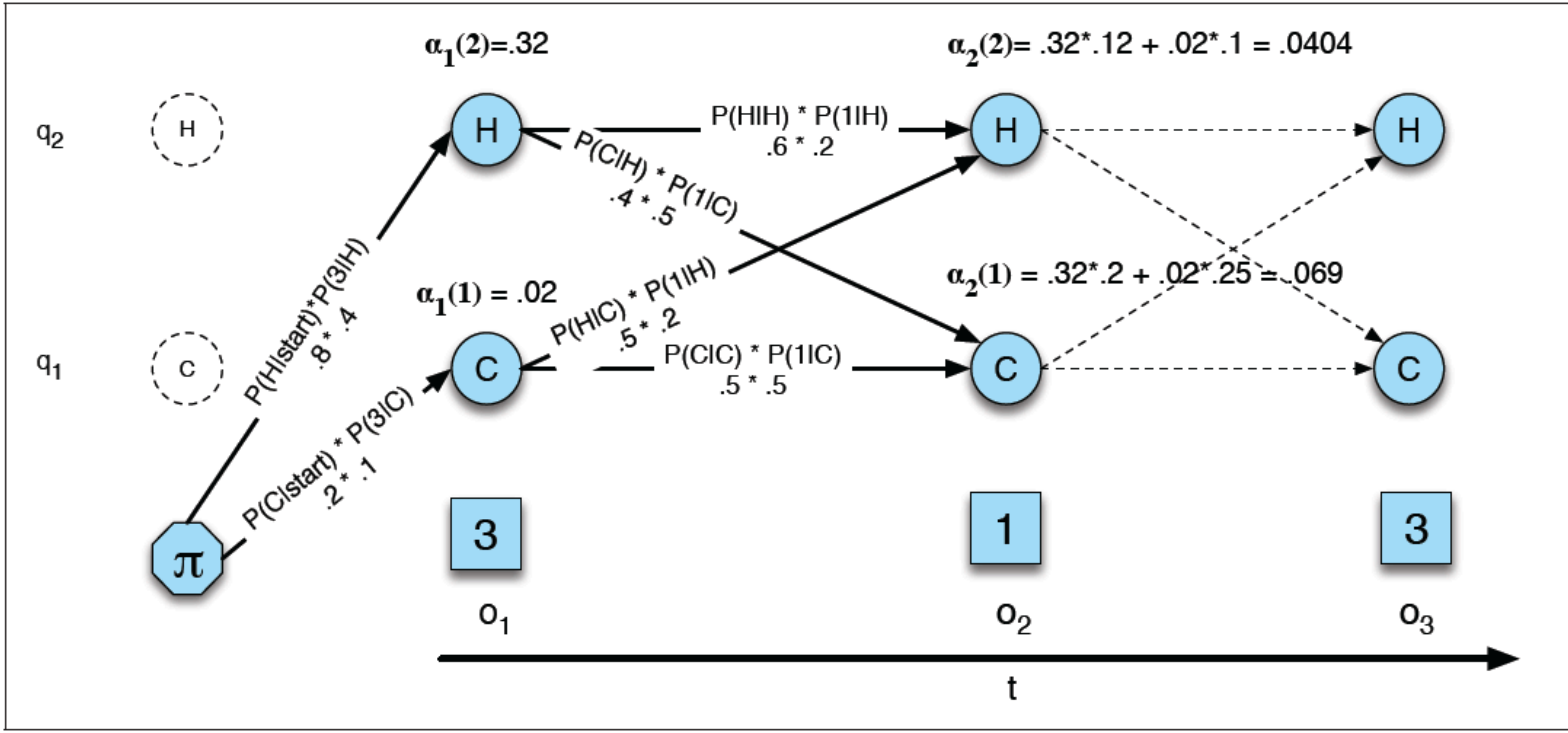
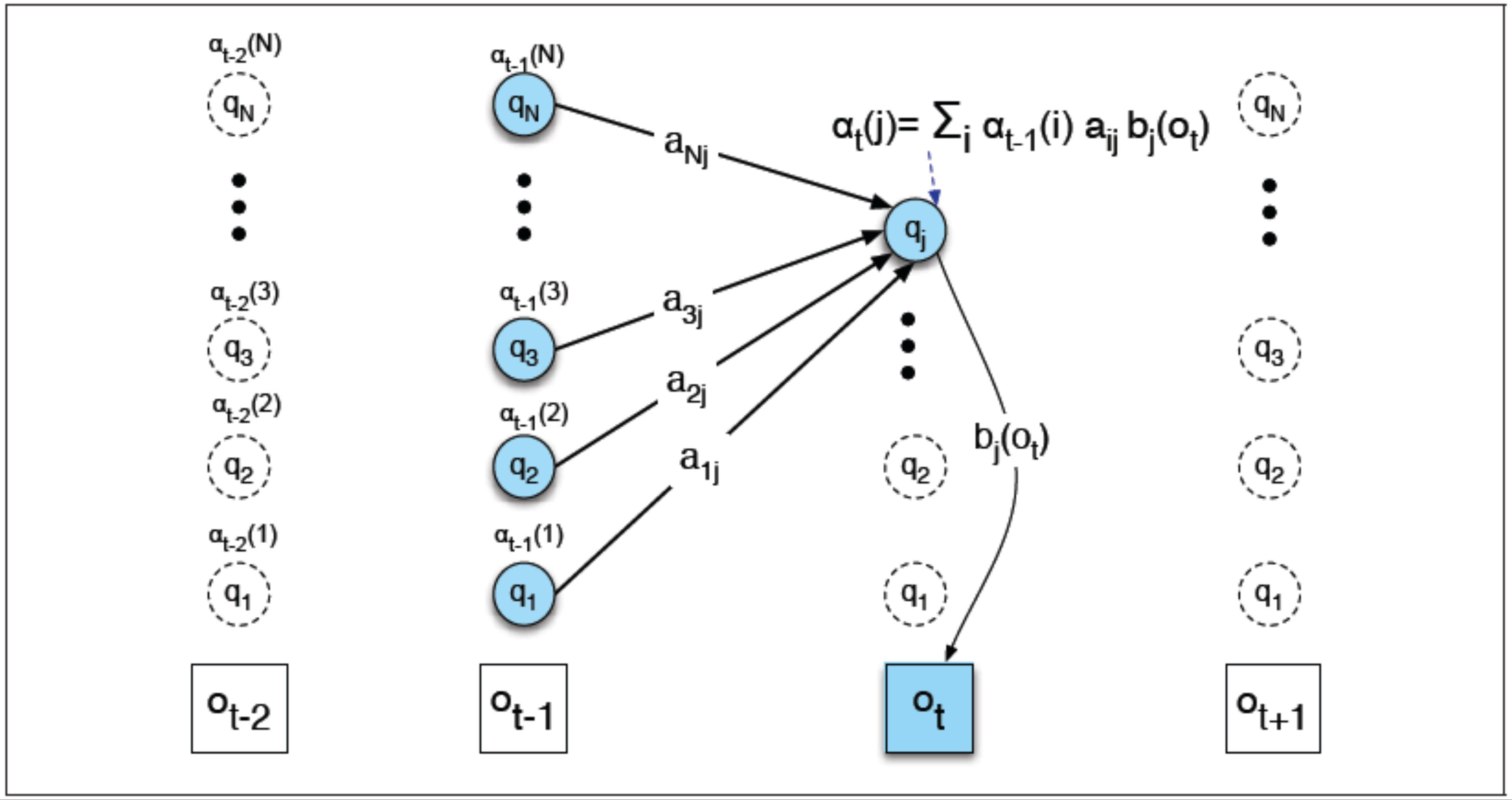


Figure A.5 The forward trellis for computing the total observation likelihood for the ice-cream events *3 1 3*. Hidden states are in circles, observations in squares. The figure shows the computation of $\alpha_t(j)$ for two states at two time steps. The computation in each cell follows Eq. A.12: $\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t)$. The resulting probability expressed in each cell is Eq. A.11: $\alpha_t(j) = P(o_1, o_2 \ldots o_t, q_t = j|\lambda)$.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf



function FORWARD(*observations* of len T, *state-graph* of len N) **returns** *forward-prob*

create a probability matrix *forward[N,T]*
for each state s **from** 1 **to** N **do** ; initialization step
$\quad forward[s,1] \leftarrow \pi_s * b_s(o_1)$
for each time step t **from** 2 **to** T **do** ; recursion step
$\quad$ **for** each state s **from** 1 **to** N **do**

$$forward[s,t] \leftarrow \sum_{s'=1}^{N} forward[s',t-1] * a_{s',s} * b_s(o_t)$$

$$forwardprob \leftarrow \sum_{s=1}^{N} forward[s,T] \qquad \text{; termination step}$$

return *forwardprob*

Figure A.7 The forward algorithm, where $forward[s,t]$ represents $\alpha_t(s)$.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

Outline

- Motivation/Examples
- HMM: Basic Definition & Three Problems
- Problem 1: Likelihood
- Problem 2: Decoding
- Problem 3: Learning
  - Supervised
  - Unsupervised: EM

Decoding (Viterbi Algorithm): very similar to Likelihood Computation (Forward Algo.)

- Likelihood: Each cell in trellis represents probability of being in state j after seeing the first t observations:

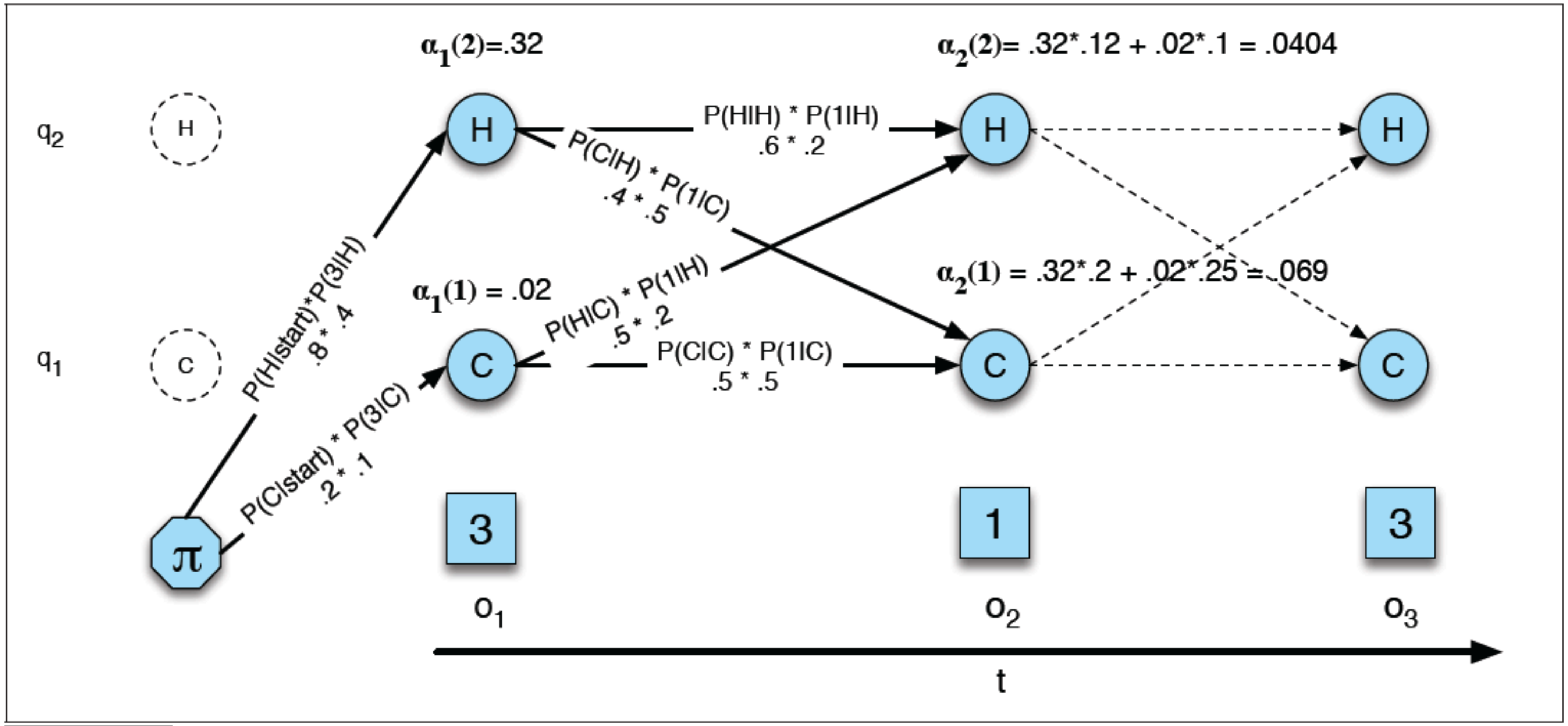
$$\alpha_t(j) = P(o_1, o_2, \ldots, o_t, q_t = j|\lambda)$$

- Decoding: Each cell in trellis represents probability of being in state j after seeing the first t observations *and* passing through the most probable state sequence $q_1..q_{t-1}$

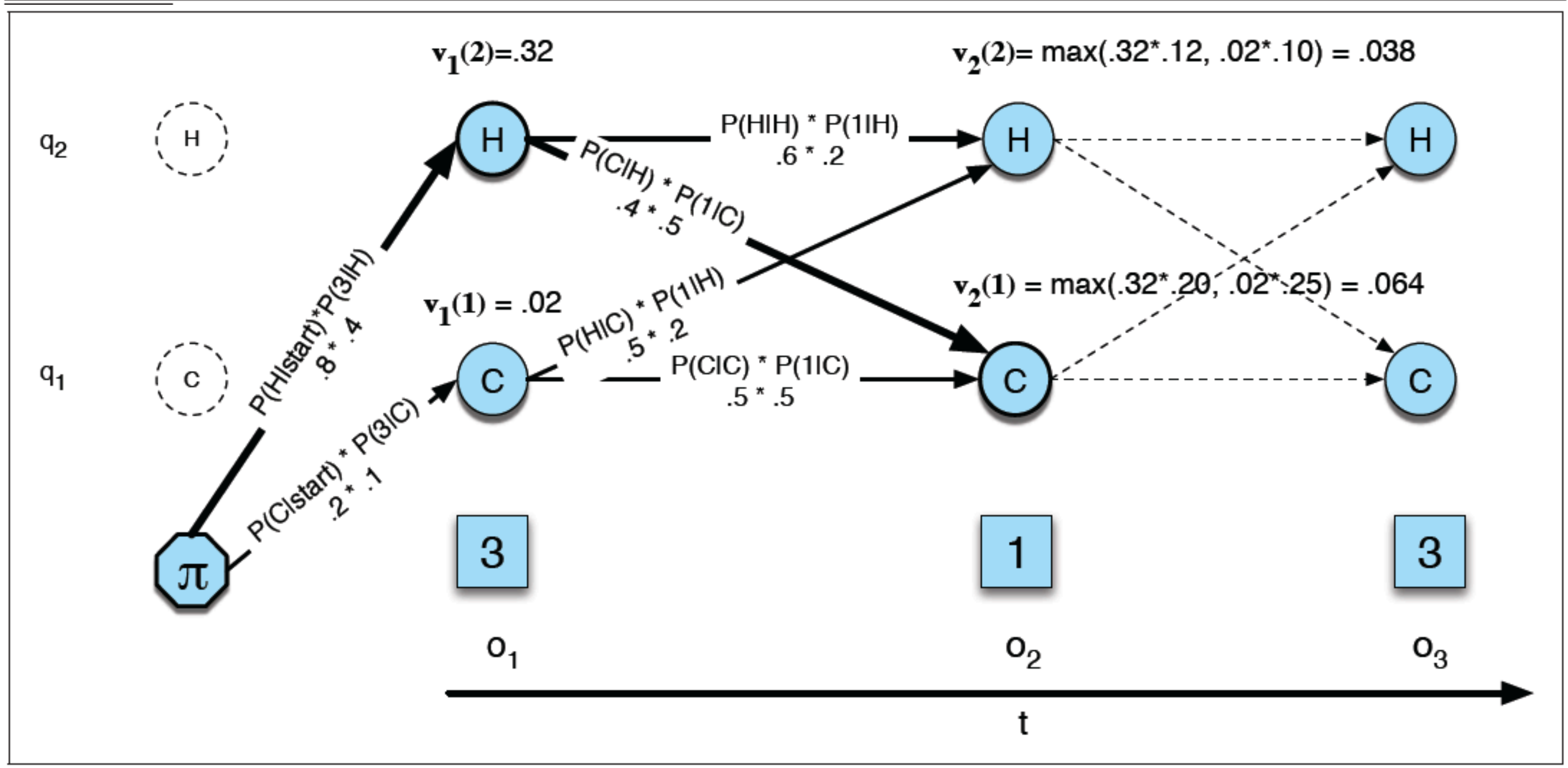
$$v_t(j) = \max_{q_1,\ldots,q_{t-1}} P(o_1, o_2, \ldots, o_t, q_1, \ldots, q_{t-1}, q_t = j, |\lambda)$$

- DP subproblem recursion: $v_t(j) = \max_{i=1,\ldots,N} v_{t-1}(i) a_{ij} b_j(o_t)$

Forward Algo:
Computes likelihood of observation

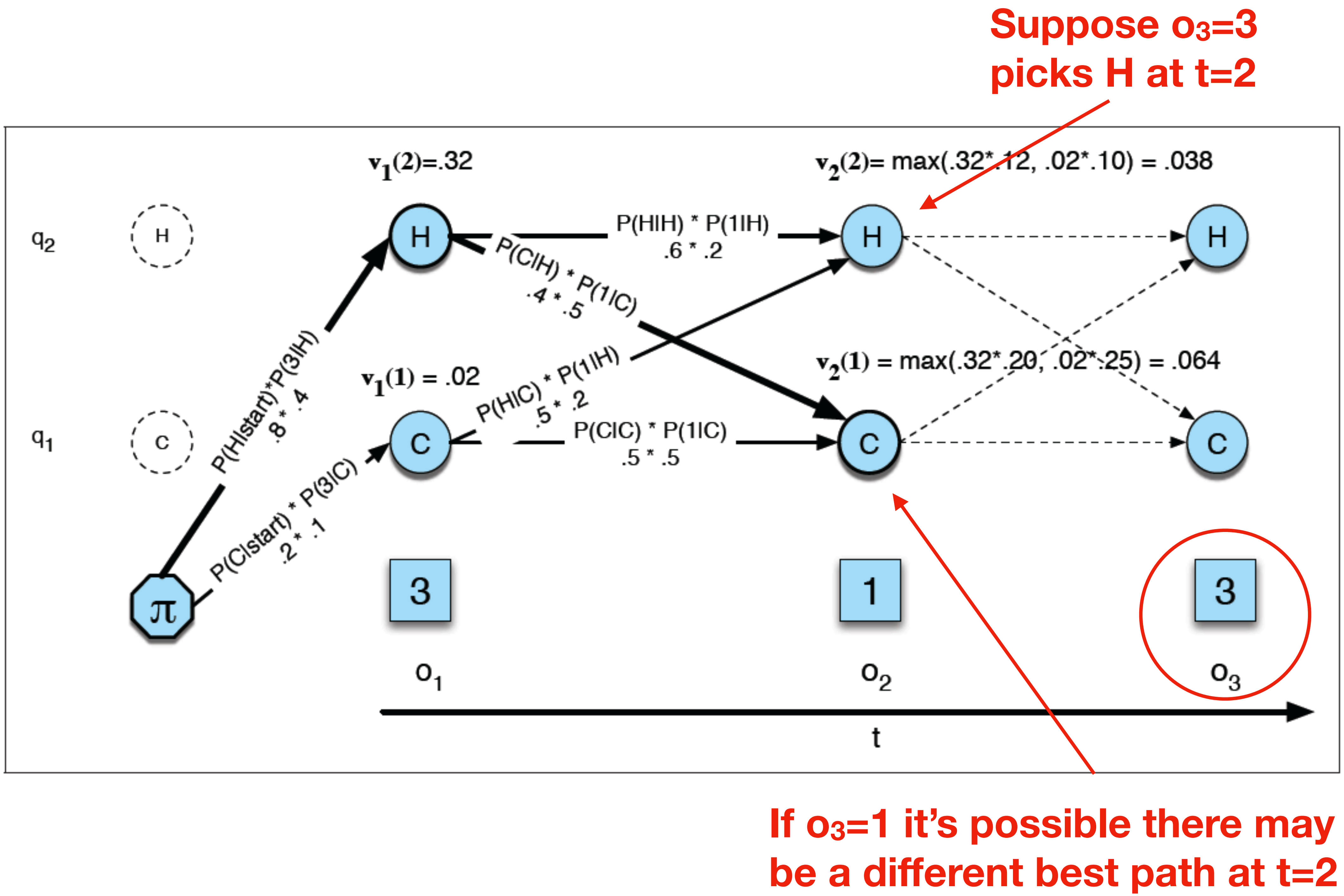


Viterbi Algo:
Decode most probable states given observation



Note: we're just changing sum to max. In later lectures we'll discuss general structures & semirings

Note: even though the HMM equation with $P(q_t|q_{t-1})$ seems unidirectional, it actually models the whole sequence and both future/past observations matter



function VITERBI(*observations* of len T,*state-graph* of len N) **returns** *best-path*, *path-prob*

create a path probability matrix *viterbi[N,T]*
for each state s **from** 1 **to** N **do** ; initialization step
$viterbi[s,1] \leftarrow \pi_s * b_s(o_1)$
$backpointer[s,1] \leftarrow 0$
for each time step t **from** 2 **to** T **do** ; recursion step
for each state s **from** 1 **to** N **do**
$$viterbi[s,t] \leftarrow \max_{s'=1}^{N} \; viterbi[s',t-1] * a_{s',s} * b_s(o_t)$$
$$backpointer[s,t] \leftarrow \operatorname{argmax}_{s'=1}^{N} \; viterbi[s',t-1] * a_{s',s} * b_s(o_t)$$
$$bestpathprob \leftarrow \max_{s=1}^{N} \; viterbi[s,T] \quad \text{; termination step}$$
$$bestpathpointer \leftarrow \operatorname{argmax}_{s=1}^{N} \; viterbi[s,T] \quad \text{; termination step}$$
bestpath $\leftarrow$ the path starting at state *bestpathpointer*, that follows backpointer[] to states back in time
return *bestpath*, *bestpathprob*

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

Outline

- Motivation/Examples
- HMM: Basic Definition & Three Problems
- Problem 1: Likelihood
- Problem 2: Decoding
- Problem 3: Learning
  - Supervised
  - Unsupervised: EM

Supervised Learning

- Suppose we're given a dataset with both O and Q.

3	3	2	1	1	2	1	2	3
hot	hot	cold	cold	cold	cold	cold	hot	hot

- It's easy to estimate HMM paramaters by counting:

e.g.
P(hot) = Cnt(hot)/3
P(3|hot)=Cnt(3,hot)/Cnt(hot)

Supervised Learning

- Suppose we're given a dataset with both O and Q.

3	3	2	1	1	2	1	2	3
hot	hot	cold	cold	cold	cold	cold	hot	hot

- It's easy to estimate HMM paramaters by counting:

e.g.
P(hot) = Cnt(hot)/3
P(3|hot)=Cnt(3,hot)/Cnt(hot)

Initial Probability:

$$\pi_h = 1/3$$
$$\pi_c = 2/3$$

Supervised Learning

- Suppose we're given a dataset with both O and Q.

3	3	2	1	1	2	1	2	3
hot	hot	cold	cold	cold	cold	cold	hot	hot

- It's easy to estimate HMM paramaters by counting:

e.g.
P(hot) = Cnt(hot)/3
P(3|hot)=Cnt(3,hot)/Cnt(hot)

Emission Probability (B):

$$P(1|hot) = 0/4 = 0 \quad p(1|cold) = 3/5 = .6$$
$$P(2|hot) = 1/4 = .25 \quad p(2|cold = 2/5 = .4$$
$$P(3|hot) = 3/4 = .75 \quad p(3|cold) = 0$$

Initial Probability:

$$\pi_h = 1/3$$
$$\pi_c = 2/3$$

Supervised Learning

- Suppose we're given a dataset with both O and Q.

3	3	2	1	1	2	1	2	3
hot	hot	cold	cold	cold	cold	cold	hot	hot

- It's easy to estimate HMM paramaters by counting:

e.g.
P(hot) = Cnt(hot)/3
P(3|hot)=Cnt(3,hot)/Cnt(hot)

Emission Probability (B):

$$P(1|hot) = 0/4 = 0 \quad p(1|cold) = 3/5 = .6$$
$$P(2|hot) = 1/4 = .25 \quad p(2|cold = 2/5 = .4$$
$$P(3|hot) = 3/4 = .75 \quad p(3|cold) = 0$$

Initial Probability:

$$\pi_h = 1/3$$
$$\pi_c = 2/3$$

Transition Probability (A):

$$p(hot|hot) = 2/3 \quad p(cold|hot) = 1/3$$
$$p(cold|cold) = 2/3 \quad p(hot|cold) = 1/3$$

Unsupervised Learning

- We're only given observations O.

3 3 2 1 1 2 1 2 3

Unsupervised Learning

- We're only given observations O.

3 3 2 1 1 2 1 2 3

- How to estimate HMM parameters? Don't know hidden states associated with observation, can't get counts.

Unsupervised Learning

- We're only given observations O.

3 3 2 1 1 2 1 2 3

- How to estimate HMM parameters? Don't know hidden states associated with observation, can't get counts.

- The Baum-Welch / Forward-Backward Algorithm does this by iteratively estimating the counts, and using this to re-derive initial/transition/emission probabilities.

Unsupervised Learning

- We're only given observations O.

3 3 2 1 1 2 1 2 3

- How to estimate HMM parameters? Don't know hidden states associated with observation, can't get counts.

- The Baum-Welch / Forward-Backward Algorithm does this by iteratively estimating the counts, and using this to re-derive initial/transition/emission probabilities.

- This is an instance of Expecation-Maximization (EM) Algo.

- Forward Probability

$$\alpha_t(j) = P(o_1, o_2, \ldots, o_t, q_t = j|\lambda)$$

- Backward Probability

$$\beta_t(i) = P(o_{t+1}, o_{t+2}, \ldots, o_T|q_t = i, \lambda)$$

- Recursion for Backward Probability

$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(o_{t+1}) \beta_{t+1}(j) \qquad P(O|\lambda) = \sum_{j=1}^{N} \pi_j b_j(o_1) \beta_1(j)$$

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j | O, \lambda)$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j | O, \lambda)$$

$$\text{not-quite-}\xi_t(i,j) = P(q_t = i, q_{t+1} = j, O | \lambda)$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j | O, \lambda)$$

$$\text{not-quite-}\xi_t(i,j) = P(q_t = i, q_{t+1} = j, O | \lambda) = \alpha_t(i)\, a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j|O,\lambda)$$

$$\text{not-quite-}\xi_t(i,j) = P(q_t = i, q_{t+1} = j, O|\lambda) \; = \alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)$$

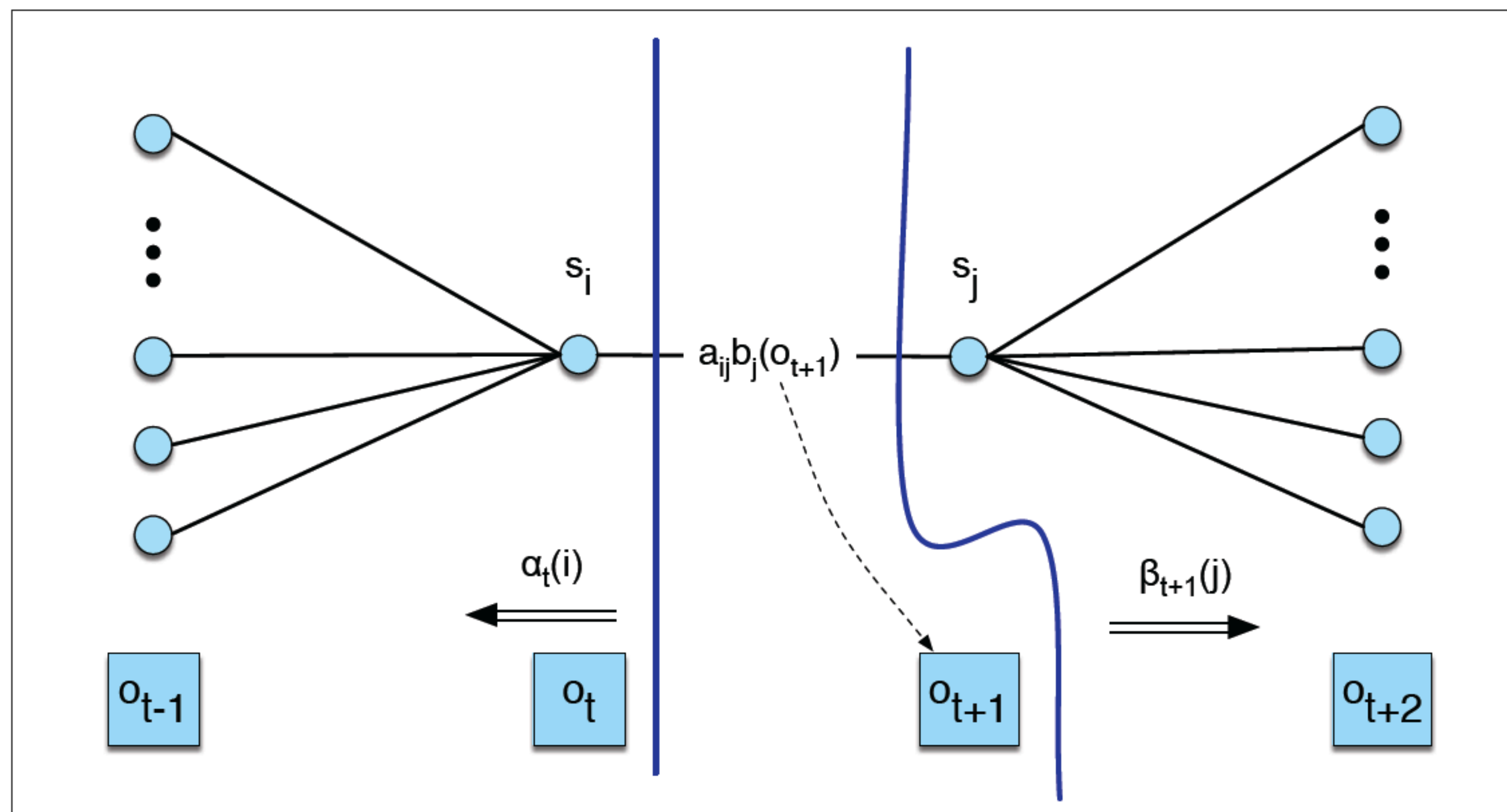


Figure A.12 Computation of the joint probability of being in state i at time t and state j at time $t+1$. The figure shows the various probabilities that need to be combined to produce $P(q_t = i, q_{t+1} = j, O|\lambda)$: the α and β probabilities, the transition probability a_{ij} and the observation probability $b_j(o_{t+1})$. After Rabiner (1989) which is ©1989 IEEE.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j | O, \lambda) \;= \frac{\alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$

$$\text{not-quite-}\xi_t(i,j) = P(q_t = i, q_{t+1} = j, O | \lambda) \;= \alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)$$

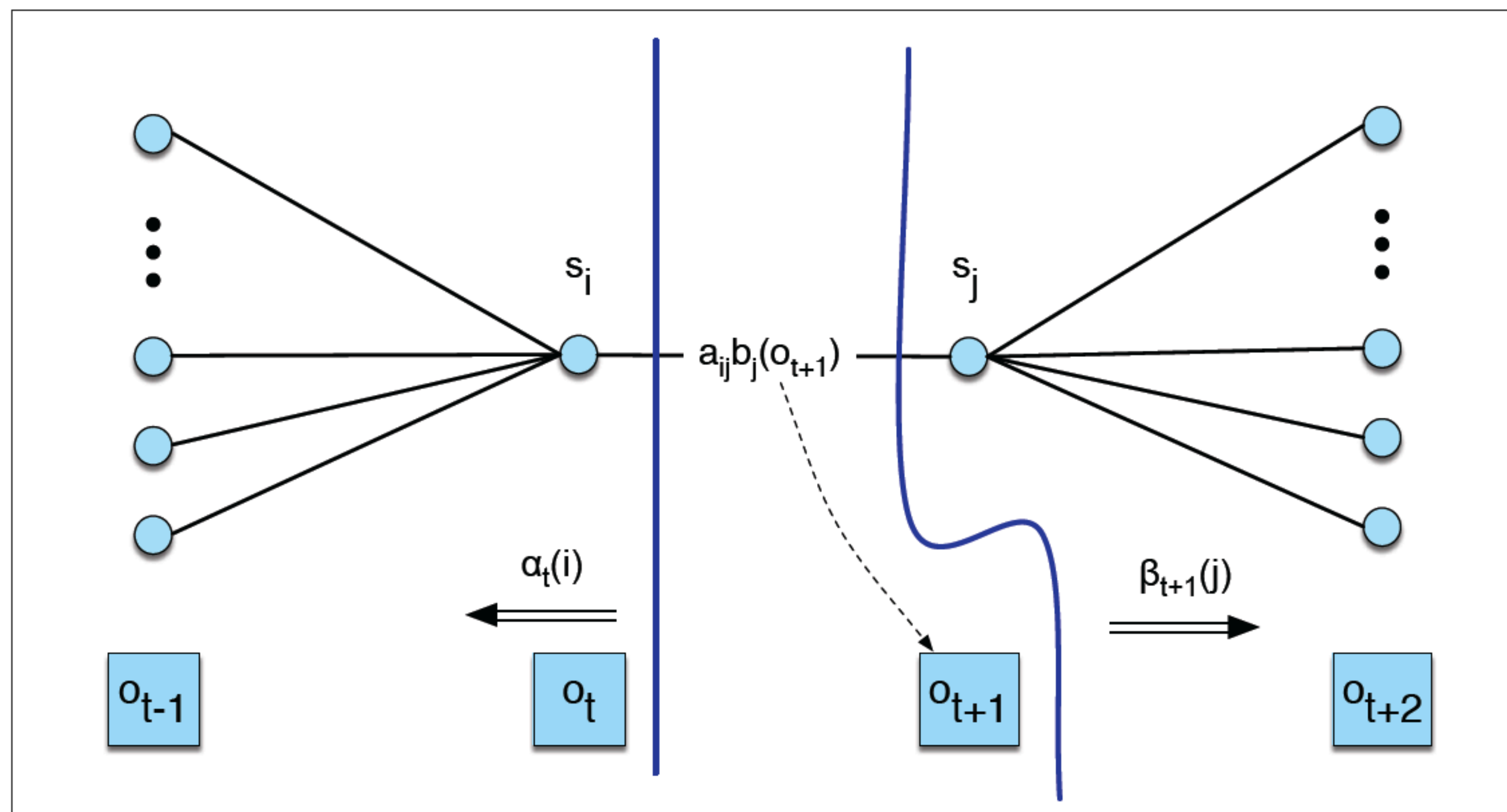


Figure A.12 Computation of the joint probability of being in state i at time t and state j at time $t+1$. The figure shows the various probabilities that need to be combined to produce $P(q_t = i, q_{t+1} = j, O|\lambda)$: the α and β probabilities, the transition probability a_{ij} and the observation probability $b_j(o_{t+1})$. After Rabiner (1989) which is ©1989 IEEE.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$P(O|\lambda) = \sum_{j=1}^{N} \alpha_t(j)\beta_t(j)$$

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j|O,\lambda) \; = \frac{\alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$

$$\text{not-quite-}\xi_t(i,j) = P(q_t = i, q_{t+1} = j, O|\lambda) \; = \alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)$$

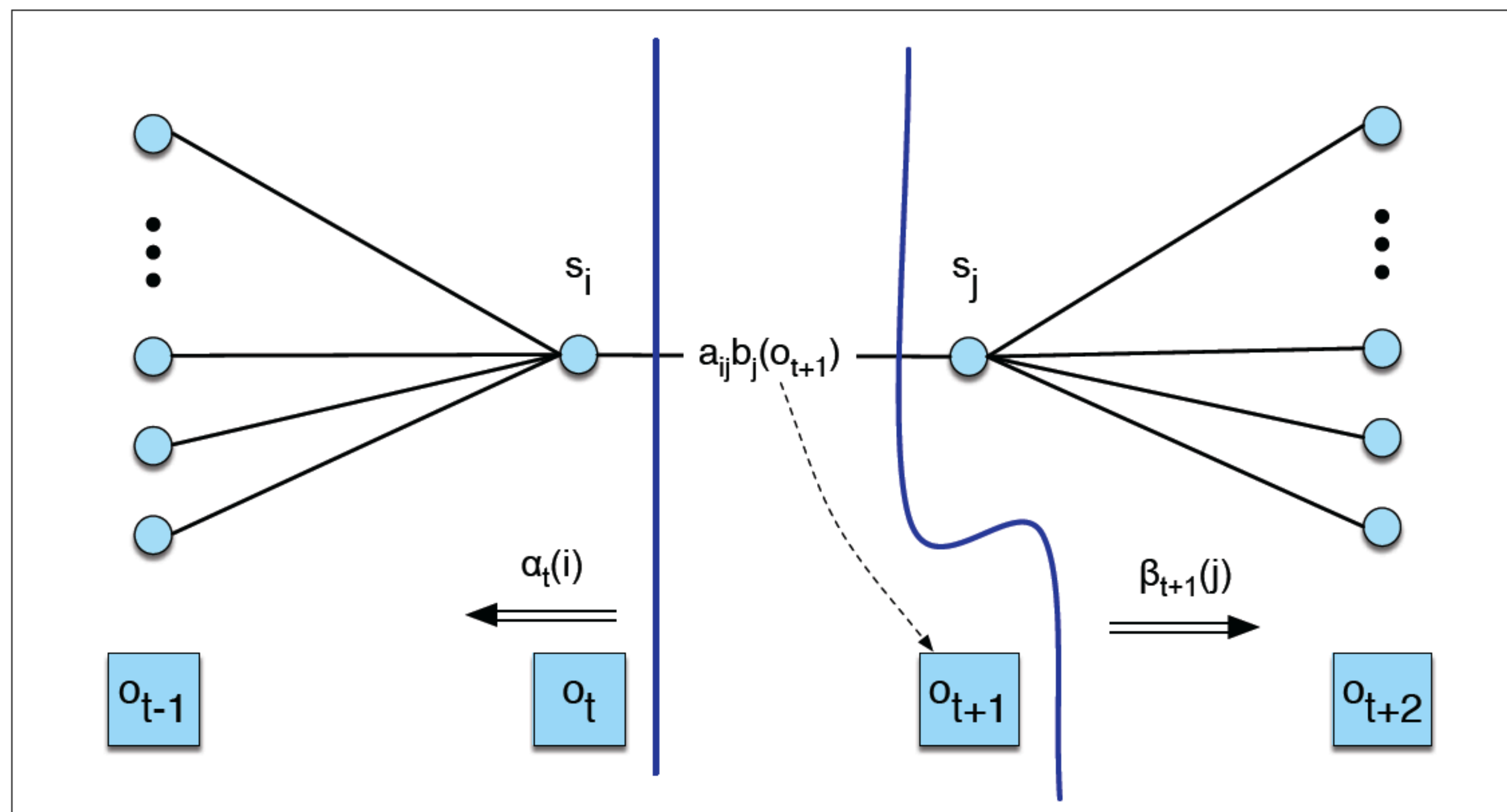


Figure A.12 Computation of the joint probability of being in state i at time t and state j at time $t+1$. The figure shows the various probabilities that need to be combined to produce $P(q_t = i, q_{t+1} = j, O|\lambda)$: the α and β probabilities, the transition probability a_{ij} and the observation probability $b_j(o_{t+1})$. After Rabiner (1989) which is ©1989 IEEE.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$P(O|\lambda) = \sum_{j=1}^{N} \alpha_t(j)\beta_t(j)$$

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j|O,\lambda) \ = \frac{\alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$

$$\text{not-quite-}\xi_t(i,j) = P(q_t = i, q_{t+1} = j, O|\lambda) \ = \alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)$$

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \sum_{k=1}^{N} \xi_t(i,k)}$$

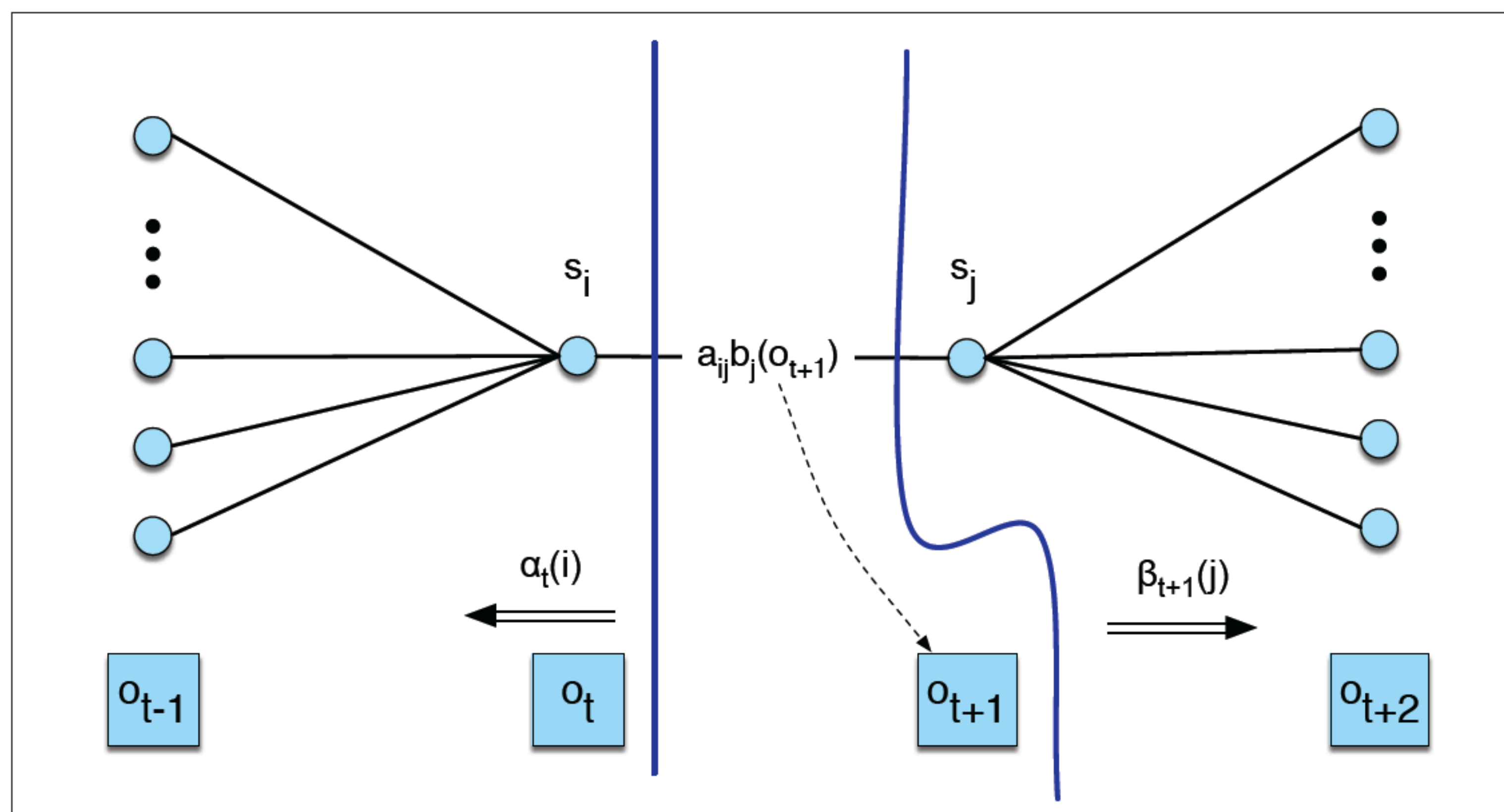


Figure A.12 Computation of the joint probability of being in state i at time t and state j at time $t+1$. The figure shows the various probabilities that need to be combined to produce $P(q_t = i, q_{t+1} = j, O|\lambda)$: the α and β probabilities, the transition probability a_{ij} and the observation probability $b_j(o_{t+1})$. After Rabiner (1989) which is ©1989 IEEE.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{b}_j(v_k) = \frac{\text{expected number of times in state } j \text{ and observing symbol } v_k}{\text{expected number of times in state } j}$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{b}_j(v_k) = \frac{\text{expected number of times in state } j \text{ and observing symbol } v_k}{\text{expected number of times in state } j}$$

$$\gamma_t(j) = P(q_t = j | O, \lambda)$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{b}_j(v_k) = \frac{\text{expected number of times in state } j \text{ and observing symbol } v_k}{\text{expected number of times in state } j}$$

$$\gamma_t(j) = P(q_t = j|O,\lambda) = \frac{P(q_t = j, O|\lambda)}{P(O|\lambda)}$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{b}_j(v_k) = \frac{\text{expected number of times in state } j \text{ and observing symbol } v_k}{\text{expected number of times in state } j}$$

$$\gamma_t(j) = P(q_t = j|O,\lambda) = \frac{P(q_t = j, O|\lambda)}{P(O|\lambda)} = \frac{\alpha_t(j)\beta_t(j)}{P(O|\lambda)}$$

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{b}_j(v_k) = \frac{\text{expected number of times in state } j \text{ and observing symbol } v_k}{\text{expected number of times in state } j}$$

$$\gamma_t(j) = P(q_t = j|O,\lambda) = \frac{P(q_t = j, O|\lambda)}{P(O|\lambda)} = \frac{\alpha_t(j)\beta_t(j)}{P(O|\lambda)}$$

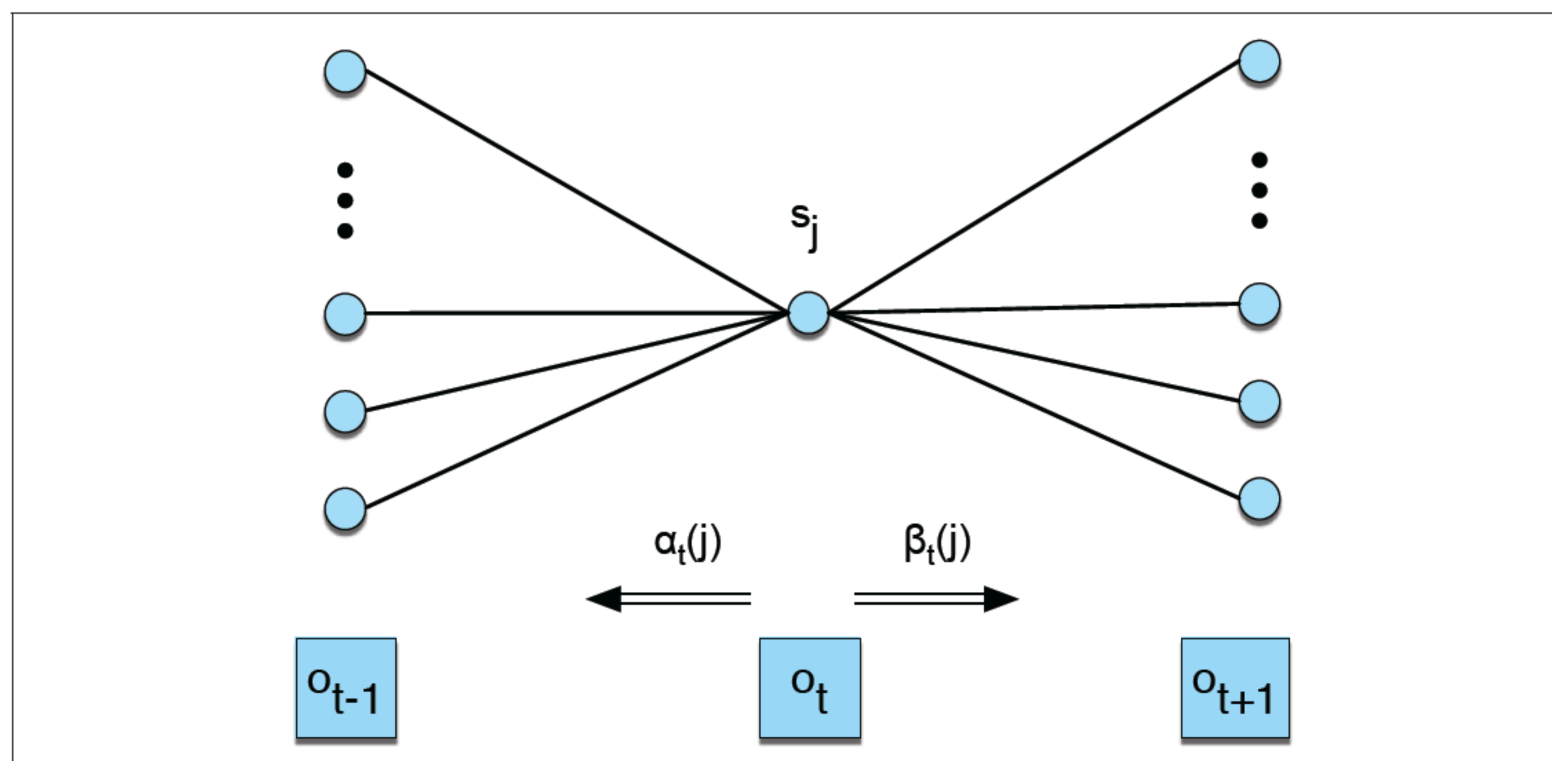


Figure A.13 The computation of $\gamma_t(j)$, the probability of being in state j at time t. Note that γ is really a degenerate case of ξ and hence this figure is like a version of Fig. A.12 with state i collapsed with state j. After Rabiner (1989) which is ©1989 IEEE.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

$$\hat{b}_j(v_k) = \frac{\text{expected number of times in state } j \text{ and observing symbol } v_k}{\text{expected number of times in state } j}$$

$$\gamma_t(j) = P(q_t = j|O,\lambda) = \frac{P(q_t = j, O|\lambda)}{P(O|\lambda)} = \frac{\alpha_t(j)\beta_t(j)}{P(O|\lambda)}$$

$$\hat{b}_j(v_k) = \frac{\sum_{t=1\ s.t. O_t=v_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$$

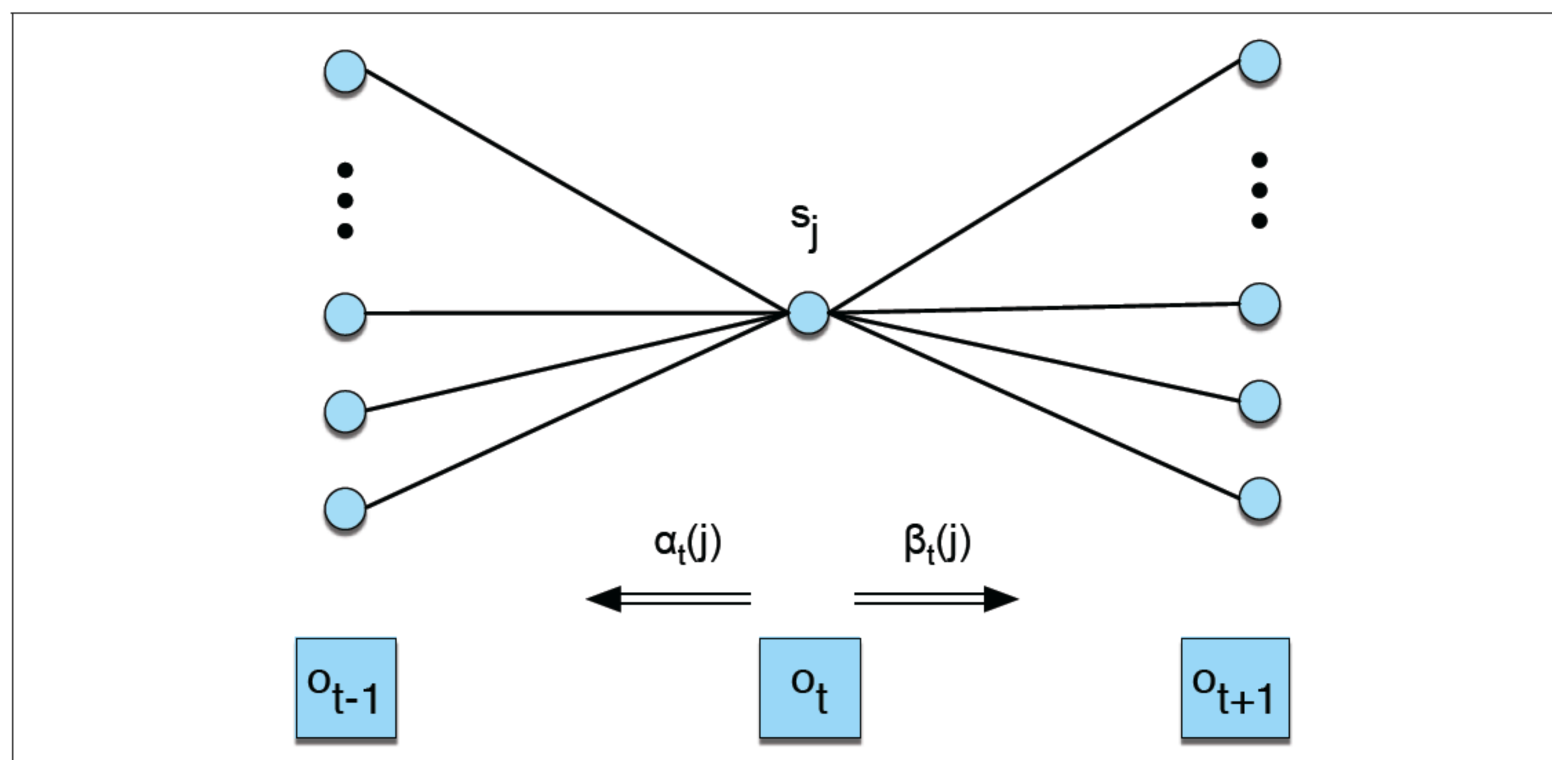


Figure A.13 The computation of $\gamma_t(j)$, the probability of being in state j at time t. Note that γ is really a degenerate case of ξ and hence this figure is like a version of Fig. A.12 with state i collapsed with state j. After Rabiner (1989) which is ©1989 IEEE.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

function FORWARD-BACKWARD(*observations* of len T, *output vocabulary* V, *hidden state set* Q) **returns** $HMM=(A,B)$

initialize A and B
iterate until convergence
E-step

$$\gamma_t(j) = \frac{\alpha_t(j)\beta_t(j)}{\alpha_T(q_F)} \ \forall\, t \text{ and } j$$

$$\xi_t(i,j) = \frac{\alpha_t(i)\,a_{ij}b_j(o_{t+1})\beta_{t+1}(j)}{\alpha_T(q_F)} \ \forall\, t,\ i, \text{ and } j$$

M-step

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1}\xi_t(i,j)}{\sum_{t=1}^{T-1}\sum_{k=1}^{N}\xi_t(i,k)}$$

$$\hat{b}_j(v_k) = \frac{\sum_{t=1\, s.t.\, O_t=v_k}^{T}\gamma_t(j)}{\sum_{t=1}^{T}\gamma_t(j)}$$

return A, B

- Data likelihood is non-decreasing at each iteration
- In practice, initialization may be important

Figure A.14 The forward-backward algorithm.

Jurafsky & Martin (2019) Speech & Language Processing, 3rd ed.
https://web.stanford.edu/~jurafsky/slp3/A.pdf

Expectation-Maximization (EM) Algorithm

- Why does the algorithm on the previous slide work? Baum-Welch/Forward-Backward in an instance of EM
- EM is a general method for finding maximum-likelihood estimate of parameters when the data has missing values
- In HMM unsupervised learning, we want to find λ that maximize likelihood $P(O,Q|\lambda)$ but we only observe O not Q

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define <u>complete-data log-likelihood</u>:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$
  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$
  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize
  - Note O and λ^{t-1} are constants. λ is a variable we wish to adjust, and Q is a random variable governed by distribution $p(Q|O, \lambda^{t-1})$.

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$
  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize
  - Note O and λ^{t-1} are constants. λ is a variable we wish to adjust, and Q is a random variable governed by distribution $p(Q|O, \lambda^{t-1})$.
  - So $L(\lambda,\lambda^{t-1})$ is a deterministic function of λ. We can optimize it.

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q

- Define complete-data log-likelihood:

  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$

  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize

  - Note O and λ^{t-1} are constants. λ is a variable we wish to adjust, and Q is a random variable governed by distribution $p(Q|O, \lambda^{t-1})$.

  - So $L(\lambda,\lambda^{t-1})$ is a deterministic function of λ. We can optimize it.

- E-step corresponds to computing this expectation

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$
  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize
  - Note O and λ^{t-1} are constants. λ is a variable we wish to adjust, and Q is a random variable governed by distribution $p(Q|O, \lambda^{t-1})$.
  - So $L(\lambda,\lambda^{t-1})$ is a deterministic function of λ. We can optimize it.
- E-step corresponds to computing this expectation

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$
  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize
  - Note O and λ^{t-1} are constants. λ is a variable we wish to adjust, and Q is a random variable governed by distribution $p(Q|O, \lambda^{t-1})$.
  - So $L(\lambda,\lambda^{t-1})$ is a deterministic function of λ. We can optimize it.
- E-step corresponds to computing this expectation
- M-step corresponds to maximizing this expectation: $\mathrm{argmax}_\lambda L(\lambda,\lambda^{t-1})$

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$
  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize
  - Note O and λ^{t-1} are constants. λ is a variable we wish to adjust, and Q is a random variable governed by distribution $p(Q|O, \lambda^{t-1})$.
  - So $L(\lambda,\lambda^{t-1})$ is a deterministic function of λ. We can optimize it.
- E-step corresponds to computing this expectation
- M-step corresponds to maximizing this expectation: $\mathrm{argmax}_{\lambda}\ L(\lambda,\lambda^{t-1})$
- Each iter won't degrade log-likelihood; we're guaranteed local optimum

- We want λ that maximize log $P(O,Q|\lambda)$ but we're missing Q
- Define complete-data log-likelihood:
  - $L(\lambda,\lambda^{t-1}) = E[\log p(O,Q|\lambda)|O,\lambda^{t-1}]$
  - λ^{t-1} are current parameter estimate to compute expectation. λ are new parameters we wish to optimize
  - Note O and λ^{t-1} are constants. λ is a variable we wish to adjust, and Q is a random variable governed by distribution $p(Q|O, \lambda^{t-1})$.
  - So $L(\lambda,\lambda^{t-1})$ is a deterministic function of λ. We can optimize it.
- E-step corresponds to computing this expectation

$$\gamma_t(j) = P(q_t = j|O,\lambda) \quad \xi_t(i,j) = P(q_t = i, q_{t+1} = j|O,\lambda)$$

- M-step corresponds to maximizing this expectation: $\text{argmax}_\lambda L(\lambda,\lambda^{t-1})$
- Each iter won't degrade log-likelihood; we're guaranteed local optimum

Summary

- Motivation/Examples
- HMM: Basic Definition & Three Problems
- Problem 1: Likelihood
- Problem 2: Decoding
- Problem 3: Learning
  - Supervised
  - Unsupervised: EM