1(a)

Assume that $X = Z - Y$, since $Y \subseteq Z$, we got $X \cap Y = \emptyset$, $Z = X \cup Y$

then $p(Z) = p(X \cup Y) = p(X) + p(Y) \geq p(Y)$

1(b)

$p(X \mid Z) = \frac{p(X \cap Z)}{p(Z)}$

Since $p(X \cap Z) \subseteq p(Z)$, then $p(x \cap Z) \leq p(Z)$, $\frac{p(X \cap Z)}{p(Z)} \leq 1$

then we got $p(X \mid Z) = \frac{p(X \cap Z)}{p(Z)} \leq 1$

On the other hand, Since $\emptyset \subseteq X \cap Z$, $0 = p(\emptyset) \leq p(X \cap Z)$;

Since $\emptyset \subseteq Z$, $0 = p(\emptyset) \leq p(Z)$

Therefore, $p(x \mid y) = \frac{p(X \cap Z)}{p(Z)} \geq 0$

To sum up, $p(X \mid Z)$ always fall in the range [0, 1]

1(c)

Since $E \cap \emptyset = \emptyset$, $p(\mathrm{E} \cup \emptyset) = p(\mathrm{E}) + p(\emptyset)$

$p(\mathrm{E}) = 1, p(\mathrm{E} \cup \emptyset) = p(E) = 1$

$p(\emptyset) = 0$

1(d)

Since $X \cap \bar{X} = \emptyset$,

We got $P(\bar{X} \cup X) = P(\bar{X}) + P(X)$

Then $P(X) = P(X \cup \bar{X}) - P(\bar{X})$

$= P(E) - P(\bar{X})$

$= 1 - P(\bar{X})$

1(e)

Assume $A = singing, B = rainy$

Then $p(singing\ AND\ rainy \mid rainy) = P(A \cap B \mid B)$

$= \frac{P((A \cap B) \cap B)}{P(B)}$

$= \frac{P(A \cap B \cap B)}{P(B)}$

$= \frac{P(A \cap B)}{P(B)}$

$= P(A \mid B)$

$= P(singing \mid rainy)$

1(f)

$$\begin{aligned} p(X \mid Y) + p(\bar{X} \mid Y) &= \frac{p(X \cap Y)}{p(Y)} + \frac{p(\bar{X} \cap Y)}{p(Y)} \\ &= \frac{p(X \cap Y) + p(\bar{X} + Y)}{p(Y)} \end{aligned}$$

Since

$$(X \cap Y) \ \cap \ (\bar{X} \cap Y) = X \cap Y \cap X \cap \bar{Y} = \emptyset$$

Then

$$p((X \cap Y) \cup (\bar{X} \cap Y)) = p(X \cup Y) + p(\bar{X} \cap Y)$$

Since

$$\begin{aligned} (X \cap Y) \ \cup \ (\bar{X} \cap Y) &= ((X \cap Y) \cup \bar{X}) \cap ((X \cap Y) \cup Y) \\ &= ((X \cup \bar{X}) \cap (Y \cup \bar{X})) \cap ((X \cup Y) \cap (Y \cup Y)) \\ &= (E \cap (Y \cup \bar{X})) \cap ((X \cup Y) \cap Y) \\ &= (Y \cup \bar{X}) \cap (Y \cup X) \\ &= (Y \cap (X \cup \bar{X})) \\ &= Y \end{aligned}$$

then the term above is equal to

$$\begin{aligned} p(X \mid Y) + p(\bar{X} \mid Y) &= \frac{p(X \cap Y) + p(\bar{X} + Y)}{p(Y)} \\ &= \frac{p((X \cap Y) \ \cup \ (\bar{X} \cap Y))}{p(Y)} \\ &= \frac{p(Y)}{p(Y)} = 1 \end{aligned}$$

which means $p(X \mid Y) = 1 - p(\bar{X} \mid Y)$

1(g)

$$\begin{aligned} Originalterm &= (p(X \mid Y) \cdot p(Y) + p(X \mid \bar{Y}) \cdot p(\bar{Y})) \cdot p(\bar{Z}|X)/p(\bar{Z}) \\ &= (\frac{p(X \cap Y)}{p(Y)} \cdot p(Y) + \frac{p(X \cap \bar{Y})}{p(\bar{Y})} \cdot p(\bar{Y})) \cdot \frac{p(\bar{Z} \cap X)}{p(X) \cdot p(\bar{Z})} \\ &= (p(X \cap Y) + p(X \cap \bar{Y})) \cdot \frac{p(\bar{Z} \cap X)}{p(X) \cdot p(\bar{Z})} \end{aligned}$$

Since $(X \cap Y) \cap (X \cap \bar{Y}) = X \cap Y \cap X \cap \bar{Y} = \emptyset$

We get $p((X \cap Y) \cup (X \cap \bar{Y})) = p(X \cap Y) + p(X \cap \bar{Y})$

As proved above, $(X \cap Y) \cup (X \cap \bar{Y}) = X$

Then the term above

$$\begin{aligned}(p(X \cap Y) + p(X \cap \bar{Y})) \cdot \frac{p(\bar{Z} \cap X)}{p(X) \cdot p(\bar{Z})} &= p((X \cap Y) \cup (X \cap \bar{Y})) \cdot \frac{p(\bar{Z} \cap X)}{p(X) \cdot p(\bar{Z})} \\ &= p(X) \cdot \frac{p(\bar{Z} \cap X)}{p(X) \cdot p(\bar{Z})} \\ &= \frac{p(X \cap \bar{Z})}{p(\bar{Z})} = p(X \mid \bar{Z})\end{aligned}$$

1(h)

Under the condition that singing and rainy are mutually exclusive

It is because that $p(singing \cup rainy) = p(singing) + p(rainy) - p(singing \cap rainy)$

If you want $p(singing \cup rainy) = p(singing) + p(rainy)$, then $p(singing \cap rainy)$ must be 0.

Only when singing and rainy are mutually exclusive(which means $singing \cap rainy = \emptyset$), $p(singing \cap rainy) = 0$. Then $p(singing \cup rainy) = p(singing) + p(rainy)$

1(i)

Under the condition that singing and rainy are independant

It is because that $p(singing \cap rainy) = p(singing \mid rainy) \cdot p(rainy) = p(rainy \mid singing) \cdot p(singing)$

If you want $p(singing \cap rainy) = p(singing) \cdot p(rainy)$, then $p(singing) = p(singing \mid rainy)\ and\ p(rainy) = p(rainy \mid singsing)$,which means that singing and rainy are independent.

1(j)

Since $p(X \mid Y) = \frac{p(X \cap Y)}{p(Y)} = 0$

we get $p(X \cap Y) = 0$

Since $X \cap Y \cap Z \subseteq X \cap Y$, $p(X \cap Y \cap Z) \leq p(X \cap Y) = 0$, so $p(X \cap Y \cap Z) = 0$

Then $p(X \mid Y, Z) = \frac{p(X \cap Y \cap Z)}{p(Y \cap Z)} = 0$

1(k)?????

Since $p(W \mid Y) = \frac{p(W \cap Y)}{p(Y)} = 1$

we get $p(W \cap Y) = p(Y)$

2(a)

Since

$$p(Actual = blue|Claimed = blue) = \frac{p(Actual = blur \cap Claimed = blue)}{p(Claimed = blue)}$$
$$p(Claimed = blue|Actual = blue) = \frac{p(Claimed = blue \cap Actual = blue)}{p(Actual = blue)}$$

We get:

$$\frac{p(Actual = bule \mid Claimed = blue)}{p(Claimed = blue \mid Actual = blue)} = \frac{p(Actual = blue)}{p(Claimed = blue)}$$

2(b)

prior probility is $p(Actual = blue)$

likelihood of the evidence is $p(Claimed = blue \mid Actual = blue)$

posterior probability is $p(Actual = blue \mid Claimed = blue)$

2(c)

The judge should care about the posterior probability, which is $p(Actual = blue \mid Claimed = blue)$

It is because that Bayes Theorem tells us

$p(A \mid B) = \frac{p(B|A)p(A)}{p(B)}$

in which p(A | B) is the posterrior probablity, the final result of all calculation.

2(d)

$$\begin{aligned}\frac{p(B \mid A, Y) \cdot p(A \mid Y)}{p(B \mid Y)} &= \frac{p(B \cap A \cap Y) \cdot \frac{1}{p(A \cap Y)} \cdot p(A \cap Y) \cdot \frac{1}{p(Y)}}{\frac{p(B \cap Y)}{p(Y)}} \\ &= \frac{p(B \cap A \cap Y)}{p(B \cap Y)} \\ &= p(A \mid B, Y)\end{aligned}$$

2(e)

$$
\begin{aligned}
&p(B \mid A, Y) \cdot p(A \mid Y) + p(B \mid \bar{A}, Y) \cdot p(\bar{A} \mid Y) \\
&= \frac{p(B \cap A \cap Y)}{p(A \cap Y)} \cdot \frac{p(A \cap Y)}{p(Y)} + \frac{p(B \cap \bar{A} \cap Y)}{p(\bar{A} \cap Y)} \cdot \frac{p(\bar{A} \cap Y)}{p(Y)} \\
&= \frac{p(B \cap A \cap Y)}{p(Y)} + \frac{p(B \cap \bar{A} \cap Y)}{p(Y)} \\
&= \frac{p(B \cap Y)}{p(Y)} \\
&= p(B \mid Y)
\end{aligned}
$$

So,

$$
\begin{aligned}
p(A \mid B, Y) &= \frac{p(B \mid A, Y) \cdot p(A \mid Y)}{p(B \mid Y)} \\
&= \frac{p(B \mid A, Y) \cdot p(A \mid Y)}{p(B \mid A, Y) \cdot p(A \mid Y) + p(B \mid \bar{A}, Y) \cdot p(\bar{A} \mid Y)}
\end{aligned}
$$

2(f)?????

A = "Actual = Blue", $p(A) = 0.1$

B= "Claimed = Blue", $p(B \mid A) = 0.8$

Y = "city = Baltimore", p(Y) = 1

Then

$$
p(A \mid B, Y) = \frac{0.8 \times 0.1}{0.8 \times 0.1 + 0.2 * 0.9} = 0.3077
$$

3(a)

$$
\begin{aligned}
\sum_{i=1}^{3} p(cry_i \mid situation = Predator!) = 1 \\
\sum_{i=1}^{3} p(cry_i \mid situation = Timber!) = 1 \\
\sum_{i=1}^{3} p(cry_i \mid situation = Ineedhelp!) = 1
\end{aligned}
$$

3(b)

p(cry, situation)	Predator!	Timber!	I need help!	Total
bwa	0	0	0.64	0.64
bwee	0	0	0.08	0.08
Kiki	0.2	0	0.08	0.28
Total	0.2	0	0.8	1

3(c)

i. $p(situation = Predator! \mid cry = kiki)$

i. $\frac{p(situation=Predator! \cap cry=kiki)}{p(cry=kiki)}$

iii. $\frac{0.2}{0.28} = 0.7143$

iv. $\frac{p(Predator!) \cdot p(kiki)}{p(Predator!) \cdot p(kiki) + p(Timber!) \cdot p(kiki) + p(Ineedhelp!) \cdot p(kiki))}$

v. $\frac{0.2}{0.2+0+0.08} = 0.7243$
