Lines

Slope of a line:

Let $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ be two distinct points. If $x_1 \neq x_2$, the slope m of the nonvertical line L containing P and Q is defined by the formula

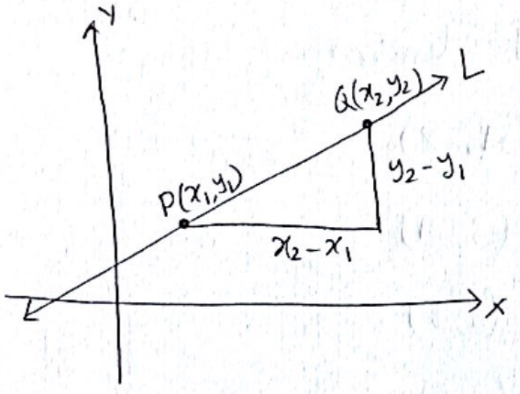
$$m = \frac{y_2-y_1}{x_2-x_1} \qquad x_1 \neq x_2$$

If $x_1 = x_2$, L is a vertical line and the slope m is undefined.

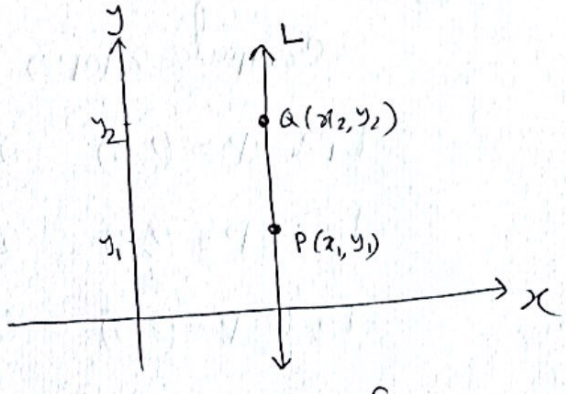
In other words, slope is the angle a line makes with x-axis.

$$m = \tan\theta$$

If a line is vertical, then the angle the line will make with x-axis will be $\pi/2$ and $\tan\frac{\pi}{2}$ = undefined. So the slope is undefined.



Slope of L is
$$m = \frac{y_2-y_1}{x_2-x_1}$$



Slope is undefined, L is vertical

$$\text{Slope, } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

The expression $\frac{\Delta y}{\Delta x}$ is called the average rate of change of y with respect to x.

Example:

Find the slope of the line containing points $(1,2)$ and $(5,-3)$

Soln. Here $P = (1,2)$ $Q = (5,-3)$

$x_1 = 1$, $y_1 = 2$, $x_2 = 5$, $y_2 = -3$

$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{5-1} = \frac{-5}{4}$$

If x increases by 4 units the y will decrease by 5 units. The average rate of change of y with respect to x is $-\frac{5}{4}$.

Example:

Compute slopes of the lines

L_1: $P = (2,3)$ $Q_1 = (-1,-2)$

L_2: $P = (2,3)$ $Q_2 = (3,-1)$

L_3: $P = (2,3)$ $Q_3 = (5,3)$

L_4: $P = (2,3)$ $Q_4 = (2,5)$

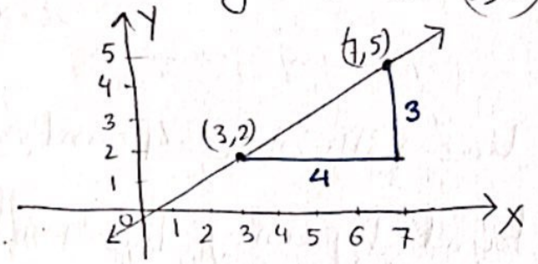
Example: 3

Draw a graph of the line that contains the point (3,2) and has a slope of (a) $\frac{3}{4}$ (b) $-\frac{4}{5}$

Soln:

(a) $\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{3}{4} = \frac{\Delta y}{\Delta x}$

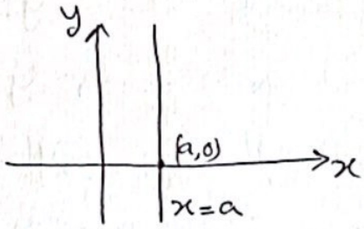
The Point given is (3,2)



Equation of a vertical line:

A vertical line is given by the equation of the form $x=a$.

where a is the x-intercept.

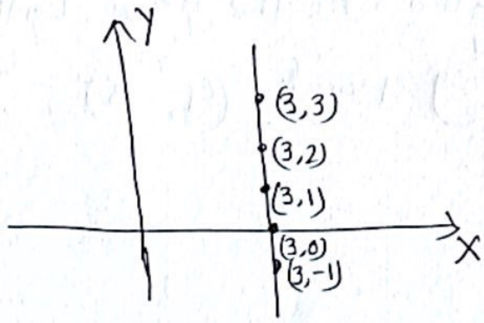


Example:

Graph the equation $x=3$

Solution:

The graph of a equation $x=3$ is a vertical line with x-intercept 3 and undefined slope.



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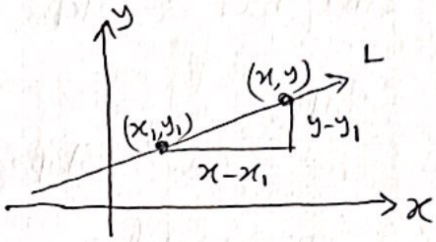
Point-slope form of a line:

An equation of a nonvertical line with slope m that contains the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$



$$m = \frac{y - y_1}{x - x_1}$$

Example:

An equation of the line with slope 4 and containing the point (1,2) can be found by using the point-slope form with $m = 4$, $x_1 = 1$, and $y_1 = 2$.

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 2 = 4(x - 1)$$
$$\Rightarrow y - 2 = 4x - 4$$
$$\Rightarrow y = 4x - 4 + 2$$
$$\Rightarrow y = 4x - 2$$

Exercise:

1. Find an equation of the line with slope 3 and containing point (-2,3).
2. Find an equation of the line with slope 2 and containing point (4,-3).

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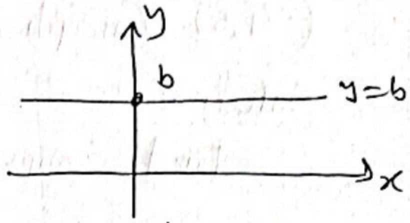
Equation of a Horizontal line:

A horizontal line is given by an equation of the form $y = b$

where b is the y-intercept.



Slope $= 0$

Example:

Find the equation of the horizontal line containing point $(3,2)$

Soln:

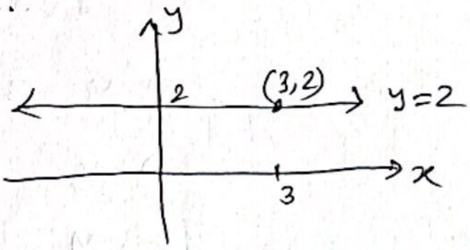
All the y-values are equal on a horizontal line, the slope of a horizontal line is 0.

Here we have $m=0,\ x_1=3,\ y_1=2$.

So $y - y_1 = m(x - x_1)$

$\Rightarrow y - 2 = 0(x-3)$

$\Rightarrow y - 2 = 0 \Rightarrow y = 2$



Exercise:

1. Find the equation of the horizontal line containing Point $(-3,2)$
2. Find the equation of the vertical line containing Point $(4,-5)$

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Find the equation of a line Given two points:

Find an equation of line containing the points (2,3) and (-4,5). Graph the line.

Solution:

First compute the slope of the line;

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Here, } x_1 = 2, y_1 = 3$$
$$x_2 = -4, y_2 = 5$$

$$\Rightarrow m = \frac{5-3}{-4-2} = \frac{2}{-6} = -\frac{1}{3}$$

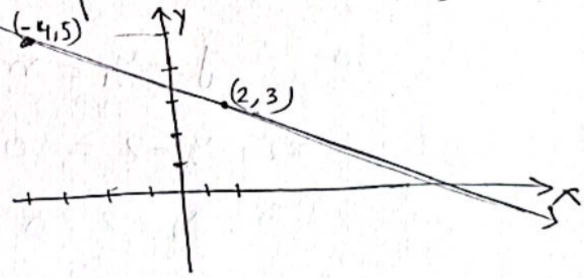
Use the point (2,3) and the slope $m = -\frac{1}{3}$ to get the point-slope form of the equation of the line,

$$y - 3 = -\frac{1}{3}(x-2) \quad \Big| \because y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 3 = -\frac{1}{3}x + \frac{2}{3}$$

$$\Rightarrow y = -\frac{1}{3}x + \frac{2}{3} + 3$$

$$\Rightarrow y = -\frac{1}{3}x + \frac{11}{3}$$

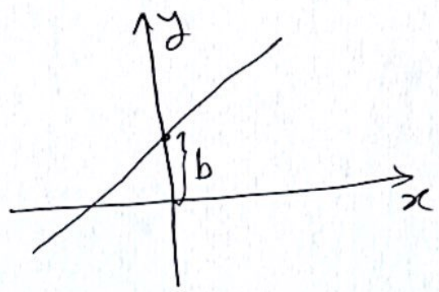


Slope-intercept form of an eqn of line:

An equation of a line with slope m and y-intercept b is

$$y = mx + b$$

Here m is the slope and b is the y-intercept.



For example:

$y = -2x + 7$, if we compare this equation to $y = mx + b$

we get $m = -2$ and $b = 7$.

so the slope of the line is -2 and its y intercept is 7.

Example:

Find the slope and y intercept b of the equation $2x + 4y = 8$. Graph the equation.

Solution:

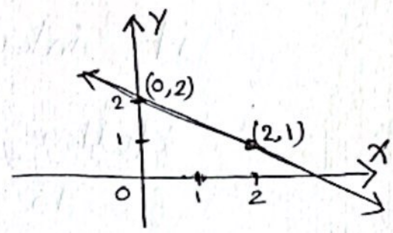
To obtain the slope and y intercept, write the equation in slope-intercept form by solving for y;

$$2x + 4y = 8$$

$$\Rightarrow \quad 4y = -2x + 8$$

$$\Rightarrow \quad y = \underbrace{-\tfrac{1}{2}}_{m} x + \underbrace{2}_{b}$$

Compare this equation with $y = mx + b$, we get the slope $m = -\frac{1}{2}$ and y-intercept $b = 2$



Exercise:

1. Find the slope and y intercept of the equation $x + 2y = 4$, $2x - 3y = 6$, $y = -3x + 4$

The equation of a line is in general form when it is written as

$$Ax + By = C$$

where A, B and C are real numbers and A and B are not both zero.

If $B=0$ and $A \neq 0$ then $Ax = C$ or $x = \frac{C}{A}$ is a equation of vertical line. Ex: $x = a \Rightarrow 1.x + 0 \cdot y = a$

If $A=0$ and $B \neq 0$ then $By = C$ or $y = \frac{C}{B}$ is a equation of horizontal line. Ex. $y = b \Rightarrow 0 \cdot x + 1 \cdot y = b$

If $B \neq 0$ and $A \neq 0$ then we can solve the equation for y and write the equation in-slope-intercept form.

Example:

Graph the equation $2x + 4y = 8$ by finding its intercepts.

Solution:

To obtain x-intercept, let $y=0$ in the eqn and solve for x.

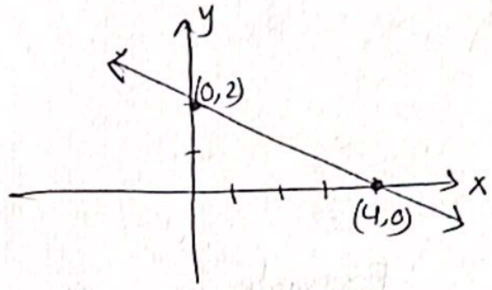
$$2x + 4y = 8$$
$$\Rightarrow 2x + 4.0 = 8$$
$$\Rightarrow 2x = 8 \Rightarrow x = 4$$

To obtain y-intercept, let $x=0$ in the eqn and solve for y,

$$2x + 4y = 8$$
$$\Rightarrow 2.0 + 4y = 8$$
$$\Rightarrow 4y = 8 \Rightarrow y = 2$$

$\therefore$ The ~~in~~ x-intercept is 4 and the point $(4,0)$ is on the graph of the equation.

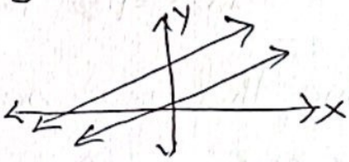
The y-intercept is 2 and the point $(0,2)$ is on the graph of the equation.



Plot the points $(4,0)$ and $(0,2)$ and draw the line through the points.

Find equations of Parallel lines:-

Two non-vertical lines are parallel if and only if their slopes are equal and they have different y-intercepts. i.e $m_1 = m_2$



Example:

Show that the lines given by the following equations are parallel.

$$L_1: 2x+3y=6 \quad ; \quad L_2: 4x+6y=0$$

Soln: To determine whether these line have equal slopes and different y-intercepts, write each equation in slope-intercept form:

L_1: $2x+3y=6$

$\Rightarrow 3y = -2x+6$

$\Rightarrow y = -\frac{2}{3}x+2$

Slope $= -\frac{2}{3}$, y-intercept $=2$

L_2: $4x+6y=0$

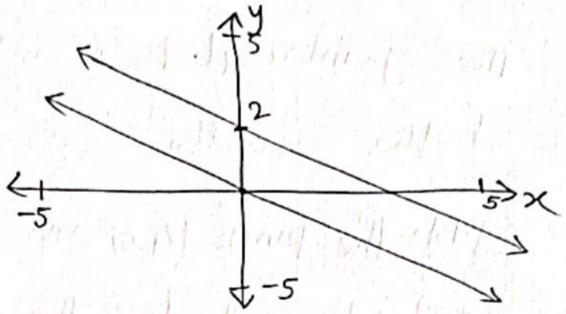
$\Rightarrow 6y = -4x$

$\Rightarrow y = -\frac{4x}{6}$

$\Rightarrow y = -\frac{2}{3}x$

Slope $= -\frac{2}{3}$, $y$intercept $=0$

Since these lines have the same slope $-\frac{2}{3}$ but different y-intercepts, the lines are parallel.



Example:

Find an equation for the line that contains the point $(2,-3)$ and is parallel to the line $2x+y=6$

Solution:

Since the two lines to be parallel, the slope of the line that we seek equals the slope of the line $2x+y=6$. Begin by writing the eqn of the line $2x+y=6$ in the slope-intercept form.

$$2x+y=6 \Rightarrow y=-2x+6$$

The slope is -2. Since the line that we seek also has slope -2 and contains the point $(2,-3)$, use the point-slope form to obtain its equation.

$$y-y_1=m(x-x_1)$$

$$\Rightarrow y-(-3)=-2(x-2)$$

$$\Rightarrow y+3=-2x+4$$

$$\Rightarrow y=-2x+1$$

$$\Rightarrow 2x+y=1$$

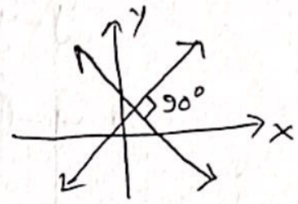
This is the line parallel to the line $2x+y=6$ and contains the point $(2,-3)$

Find Equations of perpendicular lines:

Two non-vertical lines are perpendicular if and only if the product of their slope is -1.

i.e $m_1 \cdot m_2 = -1$

or $m_1 = -\frac{1}{m_2}$ or $m_2 = -\frac{1}{m_1}$



If a line has slope $\frac{3}{2}$, any line having slope $-\frac{2}{3}$ is Perpendicular to it.

Example:

Find an equation of the line that contains the point $(1,-2)$ and is perpendicular to the line $x+3y=6$. Graph the two lines.

Solution:

First write the equation of the given line in slope intercept form to find its slope.

$$x+3y=6 \Rightarrow 3y = -x+6$$

$$\Rightarrow y = -\frac{1}{3}x+2$$

Compare this equation with $y=mx+b$ we get $m=-\frac{1}{3}$.

Any line perpendicular to this line will have slope

$$m_1 m_2 = -1$$

$$\Rightarrow m_2 = -\frac{1}{m_1}$$

$$\Rightarrow m_2 = \frac{-1}{\left(-\frac{1}{3}\right)} = (-1)\cdot\left(-\frac{3}{1}\right) = 3$$

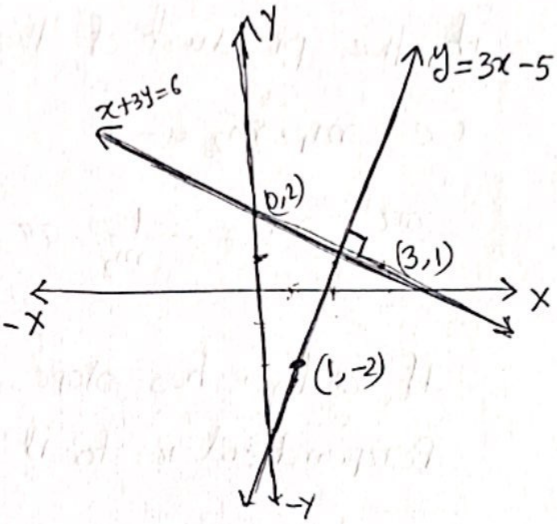
we require the point (1,-2) to be on the line with slope 3, ~~we~~ use the point-slope form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - (-2) = 3(x - 1)$$

$$\Rightarrow y + 2 = 3x - 3$$

$$\Rightarrow 3x - y = 5$$



Exercise:

Find an equation of line that is perpendicular to the line $y = \frac{1}{2}x + 4$ and contain Point $(1, -2)$.

Summary of the Chapter

1. Slope, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$; $x_1 \neq x_2$

2. Equation of a vertical line : $x = a$

   where a is the x-intercept.
   slope undefined.

3. Equation of a Horizontal line: $y = b$

   where b is the y-intercept, slope is zero.

4. Point-slope form of an equation of a line :

   $$y - y_1 = m(x - x_1)$$

   where m is the slope of the line and (x_1, y_1) point ~~contains~~ lies on the line.

5. Slope-intercept form of an equation of a line:

   $$y = mx + b$$

   where m is the slope and b is the y-intercept.

6. General form of equation:

   $$Ax + By = C$$

   If $B = 0$ then $Ax = C$ vertical line. If $A = 0$ then $By = C$ horizontal line.

7. Two lines are parallel if their slopes are equal.

   i.e $m_1 = m_2$

8. Two lines are perpendicular if their slopes are perpendicular i.e $m_1 m_2 = -1$

   or, $m_1 = -\frac{1}{m_2}$ or $m_2 = -\frac{1}{m_1}$

Exercise:

1. ~~Graph~~ Plot each pair of points and determine the slope containing them. Graph the line
   i. $(-2,3)$; $(2,1)$ ii) $(-1,2)$; $(-1,-2)$ iii) $(4,2)$; $(-5,2)$

2. Graph the line containing the point P and having slope m
   i. $P=(1,2)$; $m=3$ ii) $P=(0,3)$; slope undefined
   iii) $P=(2,-4)$; $m=0$ iv) $P=(1,3)$; $m=-\frac{2}{5}$

3. i) Find an equation of the line containing the points $(-1,2)$ and $(-1,-2)$. Graph the line.
   ii) Find an equation of the line containing the points $(2,0)$ and $(2,2)$. Graph the line.

4. Find the slope and y-intercept of each line. Graph the line.
   i) $x+y=1$ ii) $x=2$ iii) $y=\frac{1}{2}x+2$ iv) $y=5$

5. Find the intercepts of the graph of each equation.
   i) $x-\frac{2}{3}y=4$ ii) $7x+2y=21$ iii) $\frac{1}{2}x+\frac{1}{3}y=1$

6. The equations of two lines are given, determine if the lines are parallel, perpendicular or neither.
   i) $y=2x-3$, $y=2x+4$ ii) $y=4x+5$, $y=-4x+2$ iii) $y=-2x+3$, $y=-\frac{1}{2}x+2$

7. Find an eq^n perpendicular to the line $y=2x-3$ containing the point $(1,-2)$.

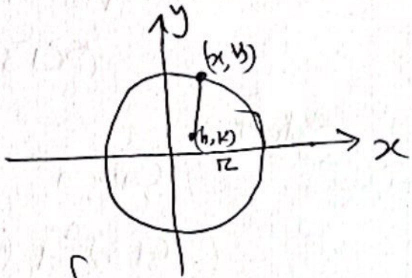
Circles

A circle is a set of points in the xy-plane that are a fixed distance r from a fixed point (h,k). The fixed distance r is called the radius and the fixed point (h,k) is called the center of the circle.

Let (x,y) represents the coordinates of any points on a circle with radius r and center (h,k). Then the distance between the points (x,y) and (h,k) must always equal to r. Thus by distance formula:

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

$$\Rightarrow (x-h)^2+(y-k)^2=r^2$$

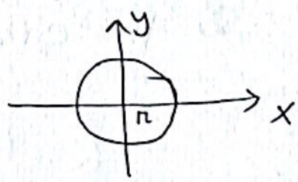


The standard form of an equation of circle with radius r and center (h,k) is

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

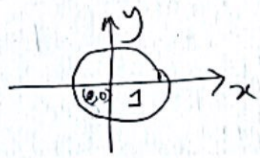
The standard form of an equation of a circle with radius r and center at the origin $(0,0)$ is

$$x^2+y^2=r^2$$



If the radius $r=1$, then the circle whose center is at origin is called unit circle and has the equation

$$x^2+y^2=1$$



Example:

Write the standard form of an equation of the circle with radius 5 and center $(-3,6)$.

Solution: Here radius $r=5$, $h=-3$, $k=6$.

we have, $$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow [x-(-3)]^2+(y-6)^2=5^2$$

$$\Rightarrow (x+3)^2+(y-6)^2=25$$

Example:

Graph the equation of circle.

$$(x+3)^2+(y-2)^2=16$$

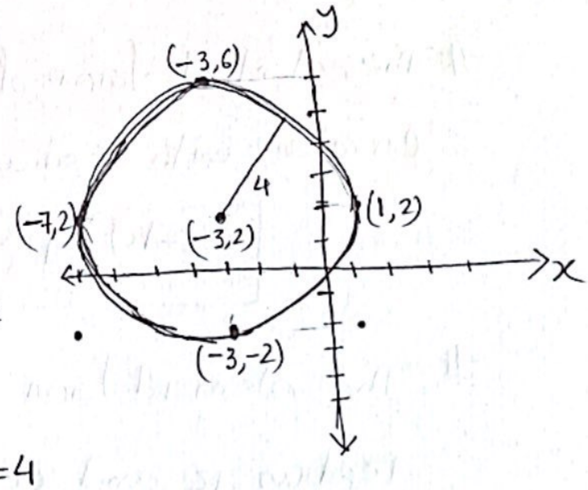
Solution:

Given

$$(x+3)^2+(y-2)^2=16$$

$$\Rightarrow [x-(-3)]^2+(y-2)^2=4^2$$

Compare with the standard form of the eq^n of circle $(x-h)^2+(y-k)^2=r^2$ we get $h=-3$, $k=2$ and $r=4$



So the circle has center $(-3,2)$ and radius 4.

Exercise:

1. Write the standard-form of an equation of the circle with radius 8 and center $(4,2)$.
2. Graph the equation of the circle
$$(x+2)^2+(y+1)^2=4$$

Example:

For the circle $(x+3)^2+(y-2)^2=16$, Find the intercepts.

Solution:

To find x-intercepts let $y=0$, then

$$(x+3)^2+(y-2)^2=16$$
$$\Rightarrow (x+3)^2+(0-2)^2=16$$
$$\Rightarrow (x+3)^2+4=16$$
$$\Rightarrow (x+3)^2=12$$
$$\Rightarrow x+3=\pm\sqrt{12}$$
$$\Rightarrow x=-3\pm\sqrt{12}$$

To find y-intercepts. let $x=0$, then

$$(0+3)^2+(y-2)^2=16$$
$$\Rightarrow 9+(y-2)^2=16$$
$$\Rightarrow (y-2)^2=7$$
$$\Rightarrow y-2=\pm\sqrt{7}$$
$$\Rightarrow y=2\pm\sqrt{7}$$

The x-intercepts are $-3-\sqrt{12}$ and $-3+\sqrt{12}$

The y-intercepts are $2-\sqrt{7}$ and $2+\sqrt{7}$

General form of the equation of a circle:

When its graph is a circle, the equation

$$x^2+y^2+ax+by+c=0$$

is refferred to as the general form of the equation of a circle.

**This eqn has a graph of a circle or a point or has no graph at all.

For example $x^2+y^2=0$ is the single point $(0,0)$.

The equation $x^2+y^2+5=0$ has no graph, because sum of squares of real number are never negative.

Problem:

Graph the equation

$$x^2+y^2+4x-6y+12=0$$

Soln: Group the terms involving x, group the term involving y and put the constant on the right hand side of the equation. The result is

$$(x^2+4x)+(y^2-6y)=-12$$

$$\Rightarrow (x^2+4x+4)+(y^2-6y+9)=-12+4+9$$

$$\Rightarrow (x+2)^2+(y-3)^2=1$$

$$\Rightarrow (x-(-2))^2+(y-3)^2=1$$

This eqn is the standard form of the equation of a circle with radius 1 and center $(-2,3)$.

1. Find the standard form of the equation and graph the circle

i) $x^2+y^2+4x-4y-1=0$ ii) $2x^2+2y^2-12x+8y-24=0$

2. Find the intercepts; center, radius

i. $2(x-3)^2+2y^2=8$ ii) $x^2+(y-1)^2=1$

Exercise / Homework:

1. Write the standard form and the general form of the circle of radius r and origin (h, k). Graph each circle.

i) $r=2; (h,k)=(0,2)$ ii) $r=\frac{1}{2}; (h,k)=(\frac{1}{2},0)$

iii) $r=4; (h,k)=(-2,1)$

2. Find the center and radius of each circle, also find the intercepts.

i) $2x^2+2y^2+8x+7=0$ ii) $x^2+y^2+4x+2y-20=0$

3. Find the standard form of the equation of each circle;

i) center at the origin and containing the point $(-2,3)$

ii) center $(2,3)$ and tangent to the x-axis

iii) with endpoints of a diameter at $(1,4)$ and $(-3,2)$