ASSIGNMENT

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Course : MAT120

Section : 11

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Ans to the QNO: 1(a)

(i)

$$f(x) = \frac{1}{x^2-1} \;;\; x>1$$

$$\Rightarrow y = \frac{1}{x^2-1}$$

$$\Rightarrow x = \frac{1}{y^2-1}$$

$$\Rightarrow y^2-1 = \frac{1}{x}.$$

$$\Rightarrow y^2 = 1+\frac{1}{x}.$$

$$\Rightarrow y = \pm\sqrt{\frac{1+x}{x}}.$$

$$\therefore f^{-1}(x) = \pm\sqrt{\frac{1+x}{x}}.$$

$\therefore$ Domain of f^{-1} : $(0,\infty)$

$\therefore$ Range of f^{-1} : $(1,\infty)$

(Ans).

(ii).

if $x \geq 2$,

$f(x) = \log(x-1)$

$\Rightarrow y = \log(x-1)$

$\Rightarrow x = \log(y-1)$

$\Rightarrow 10^x = y-1$

$\Rightarrow y = 10^x+1$

$\therefore f^{-1}(x) = 10^x+1$

if $x<2$,

$f(x) = x-2$

$\Rightarrow y = x-2$

$\Rightarrow x = y-2$

$\Rightarrow y = x+2$

$\therefore f^{-1}(x) = x+2.$

$x=2;\ \log(2-1) = \log(1) = 0.$

$x=2;\ 2-2=0.$

$$\therefore f^{-1}(x) = \begin{cases} 10^x+1\ ; & x \geq 0 \\ x+2\ ; & x<0 \end{cases}$$

$\therefore$ Domain of f^{-1} : $(-\infty, \infty)$

$\therefore$ Range of f^{-1} : $(-\infty, \infty)$.

(Ans).

Ans to the QNO : 01 (b)

(i)

$$f(x) = \frac{3-x}{1-x}.$$

$$\Rightarrow y = \frac{3-x}{1-x}.$$

$$\Rightarrow x = \frac{3-y}{1-y}.$$

$$\Rightarrow x(1-y) = 3-y$$

$$\Rightarrow x - xy = 3-y.$$

$$\Rightarrow y - xy = 3-x.$$

$$\Rightarrow y(1-x) = 3-x.$$

$$\Rightarrow y = \frac{3-x}{1-x}.$$

$$\therefore f^{-1}(x) = \frac{3-x}{1-x} \text{ (Ans)}.$$

(ii)

symmetric reflected over $y=x$.

Ans to the QNO: 01(C)

(i)

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Let,

$A = \sin^{-1}x \Rightarrow x = \sin A \ldots\ldots (i)$

and $B = \cos^{-1}x \Rightarrow x = \cos B \ldots\ldots (ii)$

From (i) and (ii),

$x = \sin A = \cos B.$

$\Rightarrow \sin A = \sin\left(\frac{\pi}{2} - B\right).$

$\Rightarrow A = \frac{\pi}{2} - B$

$\Rightarrow A + B = \frac{\pi}{2}.$

$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(proved).

(ii)

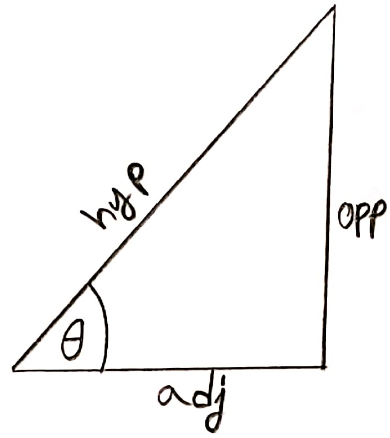
$\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Let,

$\sin^{-1}x = \theta$ . . . . (i)

$\Rightarrow \sin\theta = x$

$\Rightarrow \sin\theta = \frac{x}{1} = \frac{opp}{hyp}$



Therefore, $opp = x$ and $hyp = 1$

Using the Pythagorean theorem,

$(hyp)^2 = (adj)^2 + (opp)^2$

$\Rightarrow (adj)^2 = (hyp)^2 - (opp)^2$

$\Rightarrow (adj) = \sqrt{(hyp)^2 - (opp)^2}$

$\Rightarrow (adj) = \sqrt{(1)^2 - (x)^2}$

$\therefore adj = \sqrt{1-x^2}$

Now, $\cos\theta = \frac{adj}{hyp}$

$\Rightarrow \cos\theta = \frac{\sqrt{1-x^2}}{1}$

$\Rightarrow \cos(\sin^{-1}x) = \sqrt{1-x^2}$ [From (i)]

$\therefore \cos(\sin^{-1}x) = \sqrt{1-x^2}$

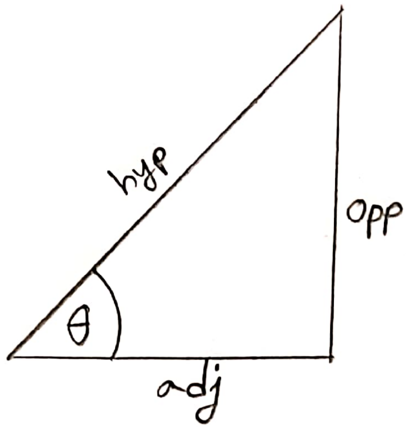
(Proved).

(iii)

$\sin(\cos^{-1}x) = \sqrt{1-x^2}$

Let,

$\cos^{-1}x = \theta \quad \ldots\ldots \text{(i)}$



$\Rightarrow \cos\theta = x$

$\Rightarrow \cos\theta = \dfrac{x}{1} = \dfrac{adj}{hyp}$

Therefore, $adj = x$ and $hyp = 1$

Using the Pythagorean theorem,

$(hyp)^2 = (adj)^2 + (opp)^2$

$\Rightarrow (opp) = \sqrt{(hyp)^2 - (adj)^2}$

$\Rightarrow (opp) = \sqrt{(1)^2 - (x)^2}$

$\therefore opp = \sqrt{1-x^2}$

Now,

$\sin\theta = \dfrac{opp}{hyp}$

$\Rightarrow \sin\theta = \dfrac{\sqrt{1-x^2}}{1}$

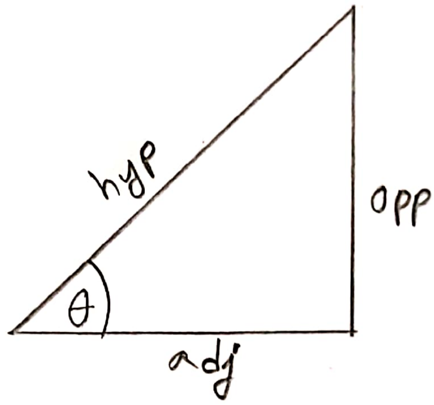
$\Rightarrow \sin(\cos^{-1}x) = \sqrt{1-x}$ [From (i)]

$\therefore \sin(\cos^{-1}x) = \sqrt{1-x}$

(proved)

(iv)

$$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$$



Let,

$\sin^{-1}x = \theta \; \ldots\ldots \; (i)$

$\Rightarrow \sin\theta = x = \frac{x}{1} = \frac{opp}{hyp}$

Therefore, opp $= x$ and hyp $= 1$

Using the Pythagorean theorem,

$(hyp)^2 = (adj)^2 + (opp)^2$

$\Rightarrow (adj) = \sqrt{(hyp)^2 - (opp)^2}$

$\Rightarrow (adj) = \sqrt{(1)^2 - (x)^2}$

$\therefore adj = \sqrt{1-x^2}$

Now,

$\tan\theta = \frac{opp}{adj}$

$\Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$ [From (i)]

$\therefore \tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

(Proved)

Ans to the Q NO : 2

(a)

(i) $\log_{10}(2+x) = 4$

$\Rightarrow 10^4 = 2+x$

$\Rightarrow x = 10000-2$

$\therefore x = 9998$

(Ans).

(ii) $\ln\left(\frac{1}{x}\right) + \ln(2x^3) = \ln 3$

$\Rightarrow \ln\left(\frac{2x^3}{x}\right) = \ln 3$

$\Rightarrow \ln(2x^2) = \ln 3$

$\Rightarrow 2x^2 = 3$

$\Rightarrow x^2 = \frac{3}{2}$

$\therefore x = \sqrt{\frac{3}{2}}$

(Ans).

(iii) $\ln 4x - 3\ln(x^2) = \ln 2$

$\Rightarrow \ln 4x - \ln(x^2)^3 = \ln 2$

$\Rightarrow \ln 4x - \ln x^6 = \ln 2$

$\Rightarrow \ln \frac{4x}{x^6} = \ln 2$

$\Rightarrow \ln \frac{4}{x^5} = \ln 2$

$\Rightarrow \frac{4}{x^5} = 2$

$\Rightarrow x^5 = \frac{4}{2}$

$\Rightarrow x^5 = 2$

$\therefore x = \sqrt[5]{2}$

(Ans).

(iv) $e^{-2x} - 3e^{-x} = -2$

$\Rightarrow (e^{-x})^2 - 3e^{-x} + 2 = 0.$

$\Rightarrow (e^{-x})^2 - 2.e^{-x} - e^{-x} + 2 = 0.$

$\Rightarrow e^{-x}(e^{-x} - 2) - 1(e^{-x} - 2) = 0$

$$\Rightarrow (e^{-x}-2)(e^{-x}-1)=0$$

Now,

$e^{-x}-2=0$

$\Rightarrow e^{-x}=2$

$\Rightarrow -x=\ln 2$

$\therefore x=-\ln 2$

Or,

$e^{-x}-1=0$

$\Rightarrow e^{-x}=1$

$\Rightarrow -x=\ln 1$

$\therefore x=0.$

$\therefore x=(-\ln 2, 0)$

(Ans).

Ans to the Question : 02(b)

Given,

$x^k=e^x$

$\Rightarrow \ln(x^k)=\ln(e^x)$

$\Rightarrow \ln x^k=x.$

$\Rightarrow k\ln x=x$

$\Rightarrow \frac{\ln x}{x}=\frac{1}{k}$ (showed).

Again,

$$\frac{\ln x}{x} = \frac{1}{K}$$

$$\Rightarrow k \ln x = x.$$

$$\Rightarrow \ln x^K = x.$$

$$\Rightarrow x^K = e^x$$

(showed).

Ans to the QNO : 02 (c)

Let $f(x) = \frac{ax+b}{cx+d}$, what conditions on a, b, c and d guarantee that f^{-1} exists? Find $f^{-1}(x)$.

Now, let $f(x) = ax+b$ and $g(x) = cx+d$, determine the conditions on a, b, c and d guarantee that f^{-1} exists if $f(g(x)) = g(f(x))$.

$\therefore f(g(x)) = a(cx+d)+b = acx + ad + b$

$\therefore g(f(x)) = c(ax+b)+d = acx + bc + d.$

Then,

$$acx + ad + b = acx + bc + d.$$

$$\Rightarrow ad + b = bc + d.$$

$$\Rightarrow f(d) = g(b).$$

Therefore, the conditions on a, b, c, d guarantee that f^{-1} if $f(g(x)) = g(f(x))$.

Again,

$$f(f^{-1}(x)) = x.$$

$$\Rightarrow f(f^{-1}(x)) = \frac{a(f^{-1}(x)) + b}{c(f^{-1}(x)) + d} = x.$$

$$\Rightarrow \frac{a(f^{-1}(x)) + b}{c(f^{-1}(x)) + d} = x.$$

$$\Rightarrow a(f^{-1}(x)) + b = x[c(f^{-1}(x)) + d]$$

$$\Rightarrow a(f^{-1}(x)) + b = cx(f^{-1}(x)) + dx$$

$$\Rightarrow a(f^{-1}(x)) - cx(f^{-1}(x)) = dx - b$$

$$\Rightarrow (a - cx) \cdot f^{-1}(x) = dx - b.$$

$$\Rightarrow f^{-1}(x) = \frac{dx-b}{a-cx}.$$

$$\Rightarrow f^{-1}(x) = \frac{-(b-dx)}{-(cx-a)}$$

$$\therefore f^{-1}(x) = \frac{b-dx}{cx-a}.$$

(Ans).

Ans to the QNO: 03(a)

$$\lim_{x\to 2} \frac{1-x}{(x+2)(x-2)}$$

$$\therefore \lim_{x\to 2^+} \left(\frac{1-x}{(x+2)(x-2)}\right)$$

$$= \lim_{x\to 2^+} \left((1-x).\frac{1}{(x+2)(x-2)}\right).$$

$$= \lim_{x\to 2^+} (1-x). \lim_{x\to 2^+} \left(\frac{1}{(x+2)(x-2)}\right).$$

$= (1-2). \infty$ [for x approaching 2 from the right $x>2 \Rightarrow (x+2)(x-2)>0$. The denominator is a positive quantity approaching 0 from the right]

$= (-1).\infty$

$= -\infty$

$$\therefore \lim_{x\to 2^-}\left(\frac{1-x}{(x+2)(x-2)}\right)$$

$$= \lim_{x\to 2^-}\left((1-x)\frac{1}{(x+2)(x-2)}\right).$$

$$= \lim_{x\to 2^-}(1-x).\ \lim_{x\to 2^-}\left(\frac{1}{(x+2)(x-2)}\right)$$

$$= (1-2).(-\infty)$$

[For x approaching 2 from the left $x<2 \Rightarrow (x+2)(x-2)<0$
The denominator is a negative quantity approaching 0 from the left]

$$= (-1).(-\infty).$$

$$= \infty$$

Here,

$$\lim_{x\to 2^-}\frac{1-x}{(x+2)(x-2)} \neq \lim_{x\to 2^+}\frac{1-x}{(x+2)(x-2)}.$$

Therefore, the limit does not exist.

$$\therefore \lim_{x\to 2}\frac{1-x}{(x+2)(x-2)} = \text{diverges}$$

(Ans).

Ans to the QNO: 03(b)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x(x+2)}\right).$$

$$=\lim_{x\to 0}\left(\frac{x+2}{x(x+2)}-\frac{2}{x(x+2)}\right).$$

$$=\lim_{x\to 0}\left(\frac{x+2-2}{x(x+2)}\right).$$

$$=\lim_{x\to 0}\left(\frac{x}{x(x+2)}\right).$$

$$=\lim_{x\to 0}\left(\frac{1}{x+2}\right).$$

$$=\frac{1}{0+2}.$$

$$=\frac{1}{2}$$ (Ans).

Ans to the QNO: 03 (c)

(i)

Here, $x \to -3$

So, $f(x) = x+1 \; ; \; x \leq -2$

$$\therefore \lim_{x \to -3} f(x) = \lim_{x \to -3} (x+1).$$

$$= -3+1 = -2$$

(Ans).

(ii)

Here, $x \to -2$

So, $f(x) = x+1 \; ; \; x \leq -2$

$$\therefore \lim_{x \to -2} f(x) = \lim_{x \to -2} (x+1). = -2+1 = -1$$

(Ans).

(iii)

Here, $x \to 0$

So, $f(x) = e^x + 1 \; ; \; x \geq 0.$

$$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} (e^x+1) = e^0+1$$
$$= 1+1 = 2$$

(Ans).

(iv)

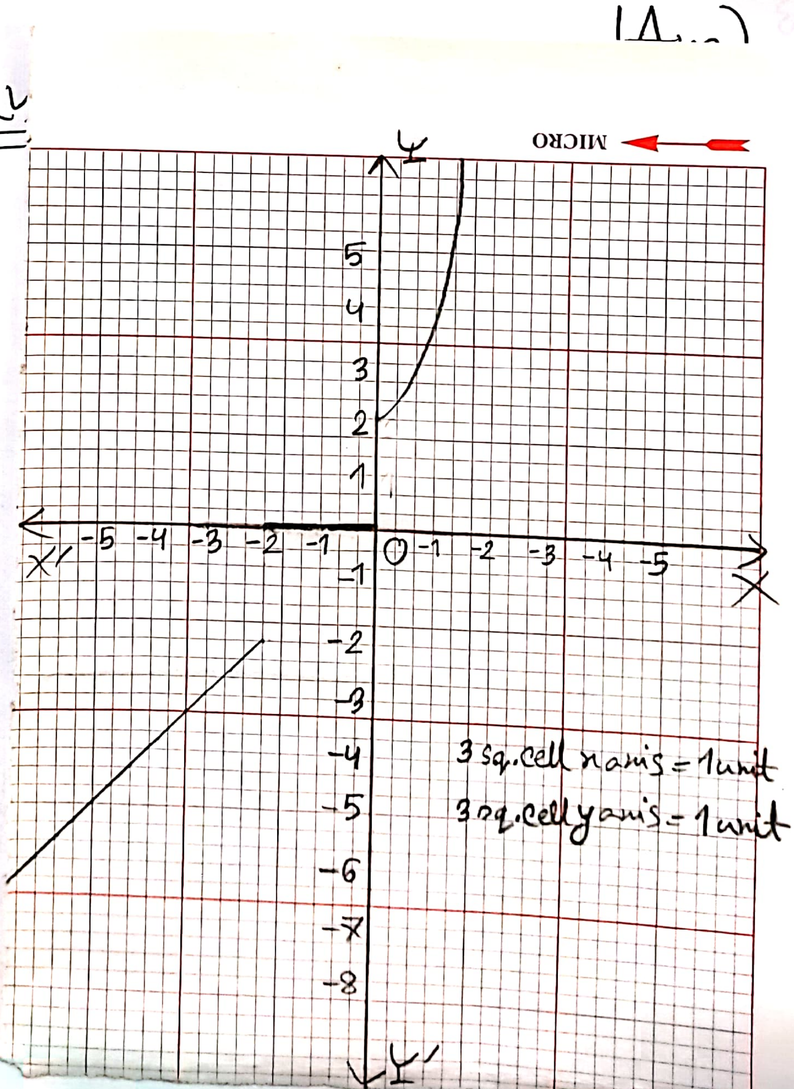
Here, $x \to 1$

So, $f(x) = e^x+1$; $x \geq 0$.

$$\therefore \lim_{x\to 1} f(x) = \lim_{x\to 1} (e^x+1) = e^1+1$$
$$= e+1$$

(Ans)

(v



Ans to the QNO : 04

(a)

$$f(x) = \begin{cases} \dfrac{3}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$$

when $x \neq 1$ then the function $\frac{3}{x-1}$ is continuous. The only point at which the function $f(x)$ can be dis continuous is $x=1$.

$\therefore$ The left hand side limit of $f(x)$ at $x=1$ is:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{3}{x-1} = -\infty$$

$\therefore$ The right hand side limit of $f(x)$ at $x=1$ is:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{3}{x-1} = \infty$$

Therefore, the limit at $x=1$ doesn't exist and the function is not continuous at $x=1$.

(Ans).

(b)

$$f(x) = \begin{cases} x^2+5, & x>2 \\ m(x+1)+k, & -1<x\leq 2 \\ 2x^3+x+7, & x\leq -1 \end{cases}$$

Here, polynomials are continuous everywhere except $x=-1$ and $x=2$. So, $x=-1$ and $x=2$ are the only possible discontinuities for f.

So, we need to find constants m and k such that

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 2^-} f(-1) = 4$ and

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} = f(2)$ are satisfied.

$$\therefore \lim_{n \to 1^+} f(n) = \lim_{n \to 1^+} (2n^3 + n + 7)$$

$$= 2.(-1)^3 - 1 + 7 = 4 \ldots\ldots (i)$$

$$\therefore \lim_{n \to 1^-} f(n) = \lim_{n \to 1^-} (m(n+1) + k)$$

$$= m.0 + k = k \ldots\ldots\ldots (ii)$$

From (i) and (ii),

$$\lim_{n \to 1^+} f(n) = \lim_{n \to 2^-} = f(-1) = 4$$

$$\therefore k = 4$$

Again,

$$\lim_{n \to 2^+} f(n) = \lim_{n \to 2^+} (n^2 + 5)$$

$$= (2)^2 + 5 = 9 \ldots\ldots\ldots (iii)$$

$$\lim_{n \to 2^-} f(n) = \lim_{n \to 2^-} (m(n+1) + k)$$

$$= \lim_{n \to 2^-} (m(n+1) + 4)$$

$$= m.3 + 4 \ldots\ldots\ldots\ldots (iv)$$

From (iii) and (iv),

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} = f(2)$$

$$\therefore m = \frac{5}{3}$$

Therefore,

$$f(x) = \begin{cases} x^2+5 & , x>2 \\ \frac{5}{3}x+\frac{17}{3} & , -1<x\leq 2 \\ 2x^3+x+7 & , x\leq -1 \end{cases}$$