4 Higher-Order Differential Equations

Initial-Value Problem In Section 1.2 we defined an initial-value problem for a general nth-order differential equation. For a linear differential equation an **nth-order initial-value problem** is

$$\textit{Solve:}\quad a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

$$\textit{Subject to:}\quad y(x_0) = y_0,\quad y'(x_0) = y_1,\ \ldots,\quad y^{(n-1)}(x_0) = y_{n-1}.$$

THEOREM 4.1.1 Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

Boundary-Value Problem Another type of problem consists of solving a linear differential equation of order two or greater in which the dependent variable y or its derivatives are specified at *different points*. A problem such as

$$\textit{Solve:}\quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\textit{Subject to:}\quad y(a) = y_0,\quad y(b) = y_1$$

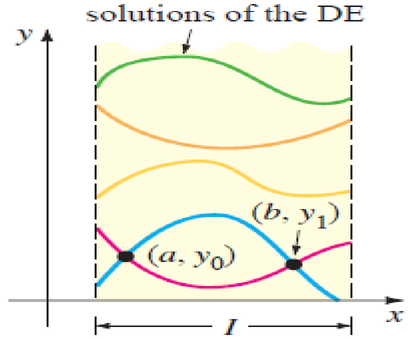


FIGURE 4.1.1 Solution curves of a BVP that pass through two points

4.1.2 HOMOGENEOUS EQUATIONS

A linear nth-order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{6}$$

is said to be **homogeneous,** whereas an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{7}$$

with $g(x)$ not identically zero, is said to be **nonhomogeneous.** For example,

Differential Operators In calculus differentiation is often denoted by the capital letter D—that is, $dy/dx = Dy$. The symbol D is called a **differential operator** because it transforms a differentiable function into another function. For example, $D(\cos 4x) = -4 \sin 4x$ and $D(5x^3 - 6x^2) = 15x^2 - 12x$. Higher-order derivatives can be expressed in terms of D in a natural manner:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y \qquad \text{and, in general,} \qquad \frac{d^ny}{dx^n} = D^ny,$$

where y represents a sufficiently differentiable function. Polynomial expressions involving D, such as $D + 3$, $D^2 + 3D - 4$, and $5x^3D^3 - 6x^2D^2 + 4xD + 9$, are also differential operators. In general, we define an ***n*th-order differential operator** or **polynomial operator** to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x). \tag{8}$$

THEOREM 4.1.2 Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation (6) on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x),$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

DEFINITION 4.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n - 1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

THEOREM 4.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

THEOREM 4.1.6 General Solution—Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (7) on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (6) on I. Then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p,$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

$$y = \textit{complementary function} + \textit{any particular solution}$$
$$= y_c + y_p.$$

How to Solve

4.2 REDUCTION OF ORDER

F=ma. resisting force Spring force External force = Q.

General form → $\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0}$ 2nd Order Linear O.D.E. (1)

y_1 of the DE. The basic idea described in this section is that *equation* (1) *can be reduced to a linear first-order DE by means of a substitution* involving the known solution y_1. A second solution y_2 of

m

resisting Constant/Coeff.

Spring Const./Coeff.

Unknown fnc.

$y = u(x)y_1(x)$

G.S.

$\boxed{4.2}$ $y = c_1y_1 + c_2y_2$

EXAMPLE 1 A Second Solution by Reduction of Order

y'' → u'

$y = u y_1$ linked

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

Let $y = u(x)y_1$ General.

SOLUTION If $y = u(x)y_1(x) = u(x)e^x$, then the Product Rule gives

$$y' = ue^x + e^xu', \quad y'' = ue^x + 2e^xu' + e^xu'',$$

and so

$$y'' - y = e^x(u'' + 2u') = 0.$$

$u'' + 2u' = 0$

$w' + 2w = 0$

$u' = w$

$u'' = w'$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Using the integrating factor e^{2x}, we can write $\frac{d}{dx}[e^{2x}w] = 0$. After integrating, we get $w = c_1e^{-2x}$ or $u' = c_1e^{-2x}$. Integrating again then yields $u = -\frac{1}{2}c_1e^{-2x} + c_2$. Thus

2.3. I.F.

$\frac{du}{dx} = c_1 \int e^{-2x} dx$

G.S. $$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2e^x. \tag{2}$$

$y_1 = e^x$

$y_2 = e^{-x}$

4.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

$y = e^{-x} + e^x$

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

Solving $ay' + by = 0$ for y' yields $y' = ky$, where k is a constant. This observation reveals the nature of the unknown solution y; the only nontrivial elementary function whose derivative is a constant multiple of itself is an exponential function e^{mx}. Now the new solution method: If we substitute $y = e^{mx}$ and $y' = me^{mx}$ into $ay' + by = 0$, we get

$$ame^{mx} + be^{mx} = 0 \qquad \text{or} \qquad e^{mx}(am + b) = 0.$$

$\cosh x = \frac{e^{x} + e^{-x}}{2}$

$y(x) = c.$

$y(x) = x.$

$y(x) = x^2$

Auxiliary Equation

We begin by considering the special case of the second-order equation

$a_2(x) = 2$

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$

$$ay'' + by' + cy = 0, \tag{2}$$

m, basis, spe. const.

$$am^2 + bm + c = 0. \tag{3}$$

- m_1 and m_2 real and distinct ($b^2 - 4ac > 0$),
- m_1 and m_2 real and equal ($b^2 - 4ac = 0$), and
- m_1 and m_2 conjugate complex numbers ($b^2 - 4ac < 0$).

Case I: Distinct Real Roots

$$y = c_1e^{m_1x} + c_2e^{m_2x}. \tag{4}$$

Case II: Repeated Real Roots

one exponential solution, $y_1 = e^{m_1 x}$.

$$y_2 = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}}\, dx = e^{m_1 x} \int dx = x e^{m_1 x}. \tag{5}$$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}. \tag{6}$$

Case III: Conjugate Complex Roots

$y = \sin mx$, $y = \ln mx$

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x). \tag{8}$$

$y = e^{mx}$, $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x/2} & e^{3x} \\ -\frac{1}{2}e^{-x/2} & 3e^{3x} \end{vmatrix}$

EXAMPLE 1 Second-Order DEs

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

$ax^2 + bx + c = 0$

1) IR, Unequal
2) IR, equal
3) Complex

(a) $2m^2 - 5m - 3 = (2m + 1)(m - 3) = 0$, $m_1 = -\frac{1}{2}$, $m_2 = 3$

From (4), $y = c_1 e^{-x/2} + c_2 e^{3x}$.

(b) $m^2 - 10m + 25 = (m - 5)^2 = 0$, $m_1 = m_2 = 5$

From (6), $y = c_1 e^{5x} + c_2 x e^{5x}$.

(c) $m^2 + 4m + 7 = 0$, $m_1 = -2 + \sqrt{3}i$, $m_2 = -2 - \sqrt{3}i$

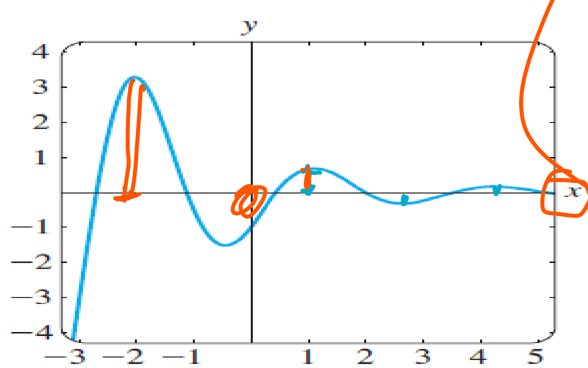
From (8) with $\alpha = -2$, $\beta = \sqrt{3}$, $y = e^{-2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$

$y = c_1 y_1 + c_2 y_2$
$= c_1 e^{-2x} \cdot e^{\sqrt{3}ix} + c_2 e^{-2x} \cdot e^{-\sqrt{3}ix}$
$= e^{-2x}(c_1 e^{\sqrt{3}ix} + c_2 e^{-\sqrt{3}ix})$

EXAMPLE 2 An Initial-Value Problem

Solve $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

As $x \to \infty$
$y(x) \to 0$



As $x \to \infty$
$y(x) \to 0$

FIGURE 4.3.1 Solution curve of IVP in Example 2

Hence the solution of the IVP is $y = e^{-x/2}(-\cos 2x + \frac{3}{4} \sin 2x)$. In Figure 4.3.1 we see that the solution is oscillatory, but $y \to 0$ as $x \to \infty$

EXAMPLE 3 Third-Order DE

Solve $y''' + 3y'' - 4y = 0$.

$$m^3 + 3m^2 - 4 = (m - 1)(m^2 + 4m + 4) = (m - 1)(m + 2)^2,$$

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}.$$

EXAMPLE 4 Fourth-Order DE

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$.

The auxiliary equation $m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$

$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}.$$

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$$

EXERCISES 4.3

In Problems 1–14 find the general solution of the given second-order differential equation.

9. $y'' + 9y = 0$

10. $3y'' + y = 0$

11. $y'' - 4y' + 5y = 0$

12. $2y'' + 2y' + y = 0$

13. $3y'' + 2y' + y = 0$

14. $2y'' - 3y' + 4y = 0$

In Problems 15–28 find the general solution of the given higher-order differential equation.

22. $y''' - 6y'' + 12y' - 8y = 0$

23. $y^{(4)} + y''' + y'' = 0$

24. $y^{(4)} - 2y'' + y = 0$

33. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$

34. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$

35. $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7$

36. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, \ y''(0) = 1$

39. $y'' + y = 0, \quad y'(0) = 0, y'(\pi/2) = 0$

40. $y'' - 2y' + 2y = 0, \quad y(0) = 1, y(\pi) = 1$

Discussion Problems

59. Two roots of a cubic auxiliary equation with real coeffi cients are $m_1 = -\frac{1}{2}$ and $m_2 = 3 + i$. What is the corresponding homogeneous linear differential equation? Discuss: Is your answer unique?

60. Find the general solution of $2y''' + 7y'' + 4y' - 4y = 0$ if $m_1 = \frac{1}{2}$ is one root of its auxiliary equation.

61. Find the general solution of $y''' + 6y'' + y' - 34y = 0$ if it is known that $y_1 = e^{-4x} \cos x$ is one solution.

62. To solve $y^{(4)} + y = 0$, we must find the roots of $m^4 + 1 = 0$. This is a trivial problem using a CAS but can also be done by hand working with complex numbers. Observe that $m^4 + 1 = (m^2 + 1)^2 - 2m^2$. How does this help? Solve the differential equation.

63. Verify that $y = \sinh x - 2 \cos (x + \pi/6)$ is a particular solution of $y^{(4)} - y = 0$. Reconcile this particular solution with the general solution of the DE.

64. Consider the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi/2) = 0$. Discuss: Is it possible to determine values of λ so that the problem possesses **(a)** trivial solutions? **(b)** nontrivial solutions?

4.4 UNDETERMINED COEFFICIENTS—SUPERPOSITION APPROACH*

Method of Undetermined Coefficient The first of two ways we shall consider for obtaining a particular solution y_p for a nonhomogeneous linear DE is called the **method of undetermined coefficients** The underlying idea behind this method is a conjecture about the form of y_p, an educated guess really, that is motivated by the kinds of functions that make up the input function $g(x)$. The general method is limited to linear DEs such as (1) where

- the coefficients a_i, $i = 0, 1, \ldots, n$ are constants and
- $g(x)$ is a constant k, a polynomial function, an exponential function $e^{\alpha x}$, a sine or cosine function $\sin \beta x$ or $\cos \beta x$, or finite sums and product of these functions.

EXAMPLE 1 **General Solution Using Undetermined Coefficient**

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$. (2)

SOLUTION **Step 1.** We first solve the associated homogeneous equation $y'' + 4y' - 2y = 0$. From the quadratic formula we find that the roots of the auxiliary equation $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Step 2. Now, because the function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3. The general solution of the given equation is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

EXAMPLE 2 Particular Solution Using Undetermined Coefficient

Find a particular solution of $y'' - y' + y = 2 \sin 3x$.

SOLUTION A natural first guess for a particular solution would be $A \sin 3x$. But because successive differentiations of $\sin 3x$ produce $\sin 3x$ *and* $\cos 3x$, we are prompted instead to assume a particular solution that includes both of these terms:

$$y_p = A \cos 3x + B \sin 3x.$$

EXAMPLE 3 Forming y_p by Superposition

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$. (3)

SOLUTION **Step 1.** First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is found to be $y_c = c_1e^{-x} + c_2e^{3x}$.

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

EXAMPLE 4 A Glitch in the Method

Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

The difficulty here is apparent on examining the complementary function $y_c = c_1e^x + c_2e^{4x}$. Observe that our assumption Ae^x is already present in y_c. This means that e^x is a solution of the associated homogeneous differential equation, and a constant multiple Ae^x when substituted into the differential equation necessarily produces zero.

$$y_p = Axe^x.$$

TABLE 4.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

EXAMPLE 8 An Initial-Value Problem

Solve $y'' + y = 4x + 10 \sin x$, $y(\pi) = 0$, $y'(\pi) = 2$.

$$y_p = Ax + B + Cx \cos x + Ex \sin x.$$

EXAMPLE 9 Using the Multiplication Rule

Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

SOLUTION The complementary function is $y_c = c_1e^{3x} + c_2xe^{3x}$. And so, based on

$$y_p = Ax^2 + Bx + C + Ex^2e^{3x}.$$

$$y = c_1e^{3x} + c_2xe^{3x} + \tfrac{2}{3}x^2 + \tfrac{8}{9}x + \tfrac{2}{3} - 6x^2e^{3x}.$$

EXERCISES 4.4

In Problems 1–26 solve the given differential equation by undetermined coefficients

5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$
11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$
12. $y'' - 16y = 2e^{4x}$
13. $y'' + 4y = 3 \sin 2x$
14. $y'' - 4y = (x^2 - 3) \sin 2x$
15. $y'' + y = 2x \sin x$

27. $y'' + 4y = -2, \quad y(\pi/8) = \frac{1}{2}, y'(\pi/8) = 2$

28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11, \quad y(0) = 0, y'(0) = 0$

29. $5y'' + y' = -6x, \quad y(0) = 0, y'(0) = -10$

30. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$

31. $y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, y'(0) = 1$

39. $y'' + 3y = 6x, \quad y(0) = 0, y(1) + y'(1) = 0$

40. $y'' + 3y = 6x, \quad y(0) + y'(0) = 0, y(1) = 0$

4.5 UNDETERMINED COEFFICIENTS—ANNIHILATOR APPROACH

EXAMPLE 3 General Solution Using Undetermined Coefficient

Solve $y'' + 3y' + 2y = 4x^2$. (9)

$$y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

Step 2. Now, since $4x^2$ is annihilated by the differential operator D^3, we see that $D^3(D^2 + 3D + 2)y = 4D^3x^2$ is the same as

$$D^3(D^2 + 3D + 2)y = 0. \tag{10}$$

$$m^3(m + 1)(m + 2) = 0,$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + c_5 e^{-2x}.$$

$$y_p = A + Bx + Cx^2,$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2.$$

4.6 VARIATION OF PARAMETERS

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{5}$$

$$y'' + P(x)y' + Q(x)y = f(x) \tag{6}$$

$$y_c = c_1y_1(x) + c_2y_2(x)$$

$$y = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{7}$$

$$y_1u_1' + y_2u_2' = 0$$

$$y_1'u_1' + y_2'u_2' = f(x)$$

$$u_1' = \frac{W_1}{W} = -\frac{y_2f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W}, \tag{9}$$

where $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}. \tag{10}$$

EXAMPLE 1 General Solution Using Variation of Parameters

Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.

$$y_c = c_1e^{2x} + c_2xe^{2x}.$$

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x + 1)xe^{4x},$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x + 1)e^{4x},$$

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x, \quad u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1.$$

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.$$

EXAMPLE 2 General Solution Using Variation of Parameters

Solve $4y'' + 36y = \csc 3x$.

EXAMPLE 3 General Solution Using Variation of Parameters

Solve $y'' - y = \dfrac{1}{x}$.

EXERCISES 4.6

In Problems 1–18 solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$
2. $y'' + y = \tan x$
3. $y'' + y = \sin x$
4. $y'' + y = \sec\theta \tan\theta$

13. $y'' + 3y' + 2y = \sin e^x$
14. $y'' - 2y' + y = e^t \arctan t$
15. $y'' + 2y' + y = e^{-t} \ln t$
16. $2y'' + 2y' + y = 4\sqrt{x}$
17. $3y'' - 6y' + 6y = e^x \sec x$

In Problems 19–22 solve each differential equation by variation of parameters, subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

19. $4y'' - y = xe^{x/2}$
20. $2y'' + y' - y = x + 1$
21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$
22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

4.7 CAUCHY-EULER EQUATION

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

EXAMPLE 1 Distinct Roots

Solve $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = 0.$

$$y = x^m \quad m^2 - 3m - 4 = 0. \quad y = c_1 x^{-1} + c_2 x^4.$$

EXAMPLE 2 Repeated Roots

Solve $4x^2 \dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + y = 0.$

SOLUTION The substitution $y = x^m$ yields

the general solution is $y = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x$.

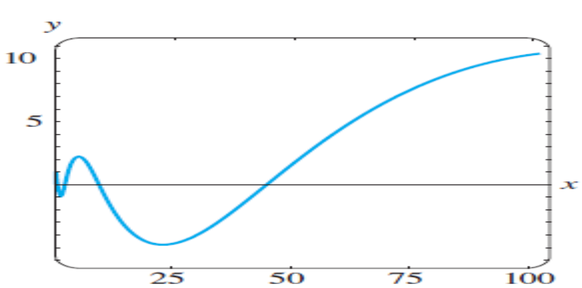
For higher-order equations, if m_1 is a root of multiplicity k, then it can be shown that

$$x^{m_1}, \quad x^{m_1} \ln x, \quad x^{m_1}(\ln x)^2, \ldots, \quad x^{m_1}(\ln x)^{k-1}$$

EXAMPLE 3 An Initial-Value Problem

Solve $4x^2 y'' + 17y = 0,\ y(1) = -1,\ y'(1) = -\frac{1}{2}$.

$$y = x^{1/2}[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)].$$



(b) solution for $0 < x \leq 100$

FIGURE 4.7.1 Solution curve of IVP in Example 3

$y = -x^{1/2} \cos(2 \ln x)$.

EXAMPLE 5 Variation of Parameters

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

we make the identification $f(x) = 2x^2e^x$. Now with $y_1 = x$, $y_2 = x^3$, and

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x,$$

we fin $\quad u_1' = -\frac{2x^5e^x}{2x^3} = -x^2e^x \quad$ and $\quad u_2' = \frac{2x^3e^x}{2x^3} = e^x.$

$$y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x.$$

Reduction to Constant Coefficient

EXAMPLE 6 Changing to Constant Coefficient

Solve $x^2y'' - xy' + y = \ln x$.

SOLUTION With the substitution $x = e^t$ or $t = \ln x$, it follows that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \qquad \leftarrow \text{Chain Rule}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) \qquad \leftarrow \text{Product Rule and Chain Rule}$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Substituting in the given differential equation and simplifying yields

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t.$$

$$y = c_1e^t + c_2te^t + 2 + t.$$

$$y = c_1x + c_2x\ln x + 2 + \ln x.$$

EXERCISES 4.7

In Problems 1–18 solve the given differential equation.

1. $x^2y'' - 2y = 0$

2. $4x^2y'' + y = 0$

29. $xy'' + y' = x, \quad y(1) = 1, y'(1) = -\frac{1}{2}$

30. $x^2y'' - 5xy' + 8y = 8x^6, \quad y\left(\frac{1}{2}\right) = 0, y'\left(\frac{1}{2}\right) = 0$

33. $x^2y'' + 10xy' + 8y = x^2$

34. $x^2y'' - 4xy' + 6y = \ln x^2$

35. $x^2y'' - 3xy' + 13y = 4 + 3x$