SEEK KNOWLEDGE

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7

$$X' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix} X + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{4t}.$$

where $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}$

Finding the characteristic equation of the coefficient matrix:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} 2-\lambda & 3 \\ 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(5-\lambda) = 0.$$

$\therefore \lambda = 1, 2, 5.$

For $\lambda = 1$,

$$\begin{pmatrix} 1-1 & 1 & 1 \\ 0 & 2-1 & 3 \\ 0 & 0 & 5-1 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [\text{Since } e^t \neq 0]$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $K_2 + K_3 = 0$

$K_2 + 3K_3 = 0$

$4K_3 = 0$

$\therefore K_3 = 0$

$\therefore K_2 = 0$

$\therefore K_1 = 1$

$K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

So, $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t$

For $\lambda = 2$,

$$\begin{pmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 3 \\ 0 & 0 & 5-2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [\text{Since } e^{2t} \neq 0]$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$-K_1 + K_2 + K_3 = 0$

$3K_3 = 0$

$3K_3 = 0.$

$\therefore K_3 = 0$

$\therefore K_2 = 1$

$\therefore K_1 = 1$

$K = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

So, $X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$

For $\lambda = 5$,

$$\begin{pmatrix} 1-5 & 1 & 1 \\ 0 & 2-5 & 3 \\ 0 & 0 & 5-5 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [\text{Since } e^{5t} \neq 0]$$

$$\Rightarrow \begin{pmatrix} -4 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$-4K_1 + K_2 + K_3 = 0$

$-3K_2 + 3K_3 = 0$

$\therefore K_3 = 2$

$\therefore K_2 = 2$

$\therefore K_1 = 1$

$$K = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

So, $X_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{5t}$.

Therefore, $X_c = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{5t}$

Since, $F(t) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{4t}$

$$\phi(t) = \begin{pmatrix} e^{t} & e^{2t} & e^{5t} \\ 0 & e^{2t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}$$

$$\det(\phi(t)) = \det\begin{pmatrix} e^{t} & e^{2t} & e^{5t} \\ 0 & e^{2t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}$$

$$= e^{t}\begin{vmatrix} e^{2t} & 2e^{5t} \\ 0 & 2e^{5t} \end{vmatrix} - e^{2t}\begin{vmatrix} 0 & 2e^{5t} \\ 0 & 2e^{5t} \end{vmatrix} + e^{5t}\begin{vmatrix} 0 & e^{2t} \\ 0 & 0 \end{vmatrix}$$

$$= e^{t}.\, e^{2t}.\, 2e^{5t} - e^{2t}.0 + e^{5t}.0$$

$$= 2e^{8t} - 0 + 0 = 2e^{8t} \neq 0$$

Let, $A = \begin{pmatrix} e^{t} & e^{2t} & e^{5t} \\ 0 & e^{2t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}$

$$= \begin{pmatrix} \begin{vmatrix} e^{2t} & 2e^{5t} \\ 0 & 2e^{5t} \end{vmatrix} & \begin{vmatrix} 0 & 2e^{5t} \\ 0 & 2e^{5t} \end{vmatrix} & \begin{vmatrix} 0 & e^{2t} \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} e^{2t} & e^{5t} \\ 0 & 2e^{5t} \end{vmatrix} & \begin{vmatrix} e^{2t} & e^{5t} \\ 0 & 2e^{5t} \end{vmatrix} & \begin{vmatrix} e^{t} & e^{2t} \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} e^{2t} & e^{5t} \\ e^{2t} & 2e^{5t} \end{vmatrix} & \begin{vmatrix} e^{t} & e^{5t} \\ 0 & 2e^{5t} \end{vmatrix} & \begin{vmatrix} e^{t} & e^{2t} \\ 0 & e^{2t} \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{7t} & 0 & 0 \\ 2e^{7t} & 2e^{7t} & 0 \\ 2e^{7t} & e^{7t}-2e^{7t} & e^{3t} \end{pmatrix}.$$

$$= \begin{pmatrix} 2e^{7t} & 0 & 0 \\ -2e^{7t} & 2e^{7t} & 0 \\ e^{7t} & -2e^{7t} & e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^{7t} & -2e^{7t} & e^{7t} \\ 0 & 2e^{7t} & -2e^{7t} \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$\phi^{-1}(t) = \frac{1}{2e^{8t}} \begin{pmatrix} 2e^{7t} & -2e^{7t} & e^{7t} \\ 0 & 2e^{7t} & -2e^{7t} \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & -e^{-t} & \frac{1}{2}e^{-t} \\ 0 & e^{-t} & -e^{-t} \\ 0 & 0 & \frac{1}{2}e^{-5t} \end{pmatrix}$$

$$\therefore X_p = \phi(t) \int \phi^{-1}(t) \cdot F(t)\, dt$$

$$= \begin{pmatrix} e^{t} & e^{2t} & e^{5t} \\ 0 & e^{2t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix} \int \begin{pmatrix} e^{-t} & -e^{-t} & \frac{1}{2}e^{-t} \\ 0 & e^{-t} & -e^{-t} \\ 0 & 0 & \frac{1}{2}e^{-5t} \end{pmatrix} \begin{pmatrix} e^{4t} \\ -e^{4t} \\ 2e^{4t} \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{t} & e^{2t} & e^{5t} \\ 0 & e^{2t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix} \int \begin{pmatrix} e^{3t} + e^{3t} + e^{3t} \\ 0 - e^{3t} - 2e^{3t} \\ 0 - 0 - e^{-t} \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{t} & e^{2t} & e^{5t} \\ 0 & e^{2t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix} \int \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \\ -e^{-t} \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{t} & e^{2t} & e^{5t} \\ 0 & e^{2t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t} - e^{5t} + e^{4t} \\ 0 - e^{5t} + 2e^{4t} \\ 0 - 0 + 2e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{4t} - e^{5t} \\ 2e^{4t} - e^{5t} \\ 2e^{4t} \end{pmatrix}$$

$$X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{5t}$$

$$+ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} e^{4t} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{5t}.$$

(Ans).

8

$$X' = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{pmatrix} X + \begin{pmatrix} 5 \\ -10 \\ 40 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{pmatrix}$

Finding the characteristic equation of the coefficient matrix:

$\det(A - \lambda I) = 0.$

$$\Rightarrow \begin{vmatrix} 0-\lambda & 0 & 5 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 5 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & -\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (-\lambda) \begin{vmatrix} 5-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 5 \begin{vmatrix} 0 & 5-\lambda \\ 5 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)(-\lambda)(5-\lambda) + 5.(-5.(5-\lambda)) = 0.$$

$$\Rightarrow \lambda^2(5-\lambda) + (-25)(5-\lambda) = 0.$$

$$\Rightarrow (5-\lambda)(\lambda-5)(\lambda+5) = 0.$$

$\therefore \lambda = 5, 5, -5$

For $\lambda = 5$,

$$\begin{pmatrix} 0-5 & 0 & 5 \\ 0 & 5-5 & 0 \\ 5 & 0 & 0-5 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \left[\text{Since } e^{5t} \neq 0\right]$$

$$\Rightarrow \begin{pmatrix} -5 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & -5 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$-5K_1 + 5K_3 = 0$

$5K_1 - 5K_3 = 0$

For, K_2 we choose an arbitrary value.

Choosing $K_1 = 0$ yields $K_3 = 0$ and let $K_2 = 1$.

$\therefore K_1 = 0$

$\therefore K_2 = 1$

$\therefore K_3 = 0$

So, $K = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\therefore X_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{5t}$

Alternatively if $K_1 = 1$ and $K_3 = 1$, then $K_2 = 0$

So,

$K = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{5t}.$

For $\lambda = -5$,

$$\begin{pmatrix} 0+5 & 0 & 5 \\ 0 & 5+5 & 0 \\ 5 & 0 & 0+5 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [\text{Since } e^{-5t} \neq 0]$$

$$\Rightarrow \begin{pmatrix} 5 & 0 & 5 \\ 0 & 10 & 0 \\ 5 & 0 & 5 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$5K_1 + 5K_3 = 0$

$10K_2 = 0$

$5K_1 + 5K_3 = 0$

$\therefore K_2 = 0$

$\therefore K_3 = 1$

$\therefore K_1 = -1$

$$K = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So,

$$X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-5t}.$$

Therefore,

$$X_c = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{5t} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{5t} + c_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-5t}$$

Since, $F(t) = \begin{pmatrix} 5 \\ -10 \\ 40 \end{pmatrix}$

$$\phi(t) = \begin{pmatrix} 0 & e^{5t} & -e^{5t} \\ e^{5t} & 0 & 0 \\ 0 & e^{5t} & e^{-5t} \end{pmatrix}$$

$$\det(\phi(t)) = \det\begin{pmatrix} 0 & e^{5t} & -e^{-5t} \\ e^{5t} & 0 & 0 \\ 0 & e^{5t} & e^{-5t} \end{pmatrix}$$

$$= 0.\begin{vmatrix} 0 & 0 \\ e^{5t} & e^{-5t} \end{vmatrix} - e^{5t}\begin{vmatrix} e^{5t} & 0 \\ 0 & e^{-5t} \end{vmatrix} - e^{-5t}\begin{vmatrix} e^{5t} & 0 \\ 0 & e^{5t} \end{vmatrix}$$

$$= 0 - e^{5t}.e^{0} - e^{-5t}.e^{10t}$$

$$= -e^{5t} - e^{5t} = -2e^{5t}.$$

Let, $$A = \begin{pmatrix} 0 & e^{5t} & -e^{-5t} \\ e^{5t} & 0 & 0 \\ 0 & e^{5t} & e^{-5t} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} 0 & 0 \\ e^{5t} & e^{-5t} \end{vmatrix} & \begin{vmatrix} e^{5t} & 0 \\ 0 & e^{-5t} \end{vmatrix} & \begin{vmatrix} e^{5t} & 0 \\ 0 & e^{5t} \end{vmatrix} \\ \begin{vmatrix} e^{5t} & -e^{-5t} \\ e^{5t} & e^{-5t} \end{vmatrix} & \begin{vmatrix} 0 & -e^{-5t} \\ 0 & e^{-5t} \end{vmatrix} & \begin{vmatrix} 0 & e^{5t} \\ 0 & e^{5t} \end{vmatrix} \\ \begin{vmatrix} e^{5t} & -e^{-5t} \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & -e^{-5t} \\ e^{5t} & 0 \end{vmatrix} & \begin{vmatrix} 0 & e^{5t} \\ e^{5t} & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{0} & e^{10t} \\ e^{0}+e^{0} & 0 & 0 \\ 0 & e^{0} & -e^{10t} \end{pmatrix} = \begin{pmatrix} 0 & 1 & e^{10t} \\ 2 & 0 & 0 \\ 0 & 1 & -e^{10t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & e^{10t} \\ -2 & 0 & 0 \\ 0 & -1 & -e^{10t} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & -1 \\ e^{10t} & 0 & -e^{10t} \end{pmatrix}$$

$$\therefore \phi^{-1}(t) = \frac{1}{-2e^{5t}} \begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & -1 \\ e^{10t} & 0 & -e^{10t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{-5t} & 0 \\ \frac{1}{2e^{5t}} & 0 & \frac{1}{2e^{5t}} \\ -\frac{1}{2}e^{5t} & 0 & \frac{1}{2}e^{5t} \end{pmatrix} = \begin{pmatrix} 0 & e^{-5t} & 0 \\ \frac{1}{2}e^{-5t} & 0 & \frac{1}{2}e^{-5t} \\ -\frac{1}{2}e^{5t} & 0 & \frac{1}{2}e^{5t} \end{pmatrix}$$

$$X_p = \phi t \int \phi^{-1}(t) F(t)\, dt.$$

$$= \begin{pmatrix} 0 & e^{5t} & -e^{-5t} \\ e^{5t} & 0 & 0 \\ 0 & e^{5t} & e^{-5t} \end{pmatrix} \int \begin{pmatrix} 0 & e^{-5t} & 0 \\ \frac{1}{2}e^{-5t} & 0 & \frac{1}{2}e^{-5t} \\ -\frac{1}{2}e^{5t} & 0 & \frac{1}{2}e^{5t} \end{pmatrix} \begin{pmatrix} 5 \\ -10 \\ 40 \end{pmatrix} dt$$

$$= \begin{pmatrix} 0 & e^{5t} & -e^{-5t} \\ e^{5t} & 0 & 0 \\ 0 & e^{5t} & e^{-5t} \end{pmatrix} \int \begin{pmatrix} 0 - 10e^{-5t} + 0 \\ \frac{5}{2}e^{-5t} - 0 + 20e^{-5t} \\ -\frac{5}{2}e^{5t} - 0 + 20e^{5t} \end{pmatrix} dt.$$

$$= \begin{pmatrix} 0 & e^{5t} & -e^{-5t} \\ e^{5t} & 0 & 0 \\ 0 & e^{5t} & e^{-5t} \end{pmatrix} \int \begin{pmatrix} -10e^{-5t} \\ \frac{45}{2}e^{-5t} \\ \frac{35}{2}e^{5t} \end{pmatrix} dt.$$

$$= \begin{pmatrix} 0 & e^{5t} & -e^{-5t} \\ e^{5t} & 0 & 0 \\ 0 & e^{5t} & e^{-5t} \end{pmatrix} \begin{pmatrix} 2e^{-5t} \\ -\frac{9}{2}e^{-5t} \\ \frac{7}{2}e^{5t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{9}{2}e^{0} & -\frac{7}{2}e^{0} \\ 2e^{0} & 0 & 0 \\ 0 & -\frac{9}{2}e^{0} & \frac{7}{2}e^{0} \end{pmatrix} = \begin{pmatrix} -\frac{9}{2}-\frac{7}{2} \\ 2 \\ -\frac{9}{2} \quad \frac{7}{2} \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \\ -1 \end{pmatrix}$$

$$X = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{5t} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{5t} + c_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}e^{-5t} + \begin{pmatrix} -8 \\ 2 \\ -1 \end{pmatrix}$$

(Ans).