27/1/19

Attendance - 5%

Quiz (2/3) - 15%

Mid (1) - 25%

Assesment (1) - 15%

Final - 35%

Assignment - 5%

90 - A

20 attendences is must

Differentiation → rate of change at a time interval

$\frac{dE}{dI}$ dep. var / Ind. var. = How expenditure changes according to income

Order $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 1$

Degree $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$ degree = ~~3~~ 1 → degree of the order

Order = ~~2~~ 2

Non linear

* Zigzag term - ~~Sin~~ sin, cos, $\frac{d^2y}{dx^2} + \sin y = 0$

* Power of y - y^2

* Co-efficient of derivative not y func^n - (1-y) not be x

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$a_n(x)$ = Constant / function of x

2nd order $n=2$, $\quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

1st order $n=1$, $\quad a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

If $g(x)=0$, homogenous diff. eq^n

and $g(x)\neq 0$, non-homogenous diff. eq^n

29/11/19

Mathematical models

→ Relation between the variables and identification

Population Dynamics

If $P(t)$ denotes the total population at time t,

$$\frac{dP}{dt} \propto P$$

Newton's law of coolling

$$\frac{dT}{dt} \propto (T - T_m)$$

How temp. is changing

Kirchhoff's 2nd law

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q =$$

Lecture 2

Variable Seperable

1st order diff eqⁿ is said to be variable seperable if it can be written as

$$\boxed{f(x)dx = h(y)dy}$$

Then the ODE is called seperable ODE

Ex:
$$\frac{dy}{dx} = xy \quad \Rightarrow \frac{1}{y}dy = x dx$$

$$\frac{dy}{dx} = e^{x+y} \quad \Rightarrow e^{-y}dy = e^x dx$$

$$\frac{dy}{dx} = \sin(x+y) \rightarrow \text{Not seperable ODE}$$

How to solve seperable ODE?

Once an ODE is seperable, we find the solution by integration.

$$\int e^{-y}dy = \int e^x dx$$

$$\Rightarrow -e^{-y} = e^x + c$$

$$\Rightarrow e^{-y} = -e^x - c$$

$$\Rightarrow \ln(e^{-y}) = \ln(-e^x - c) \quad \Rightarrow -y = \ln(-e^{+x} - c)$$

As there are many values of C, So the solution is called "General Solution".

Chapter 2.2 (Danish)

25. Solve $x^2 \frac{dy}{dx} = y - xy$, $y(-1) = -1$

$\Rightarrow \cancel{x^2 \frac{dy}{dx}} = \cancel{\frac{y-xy}{x^2}} = \cancel{\frac{y}{x^2}} - \cancel{\frac{1}{x}}$

$\Rightarrow \cancel{x^2 dy} = \cancel{y\,dx} - \cancel{xy\,dx}$

$\Rightarrow \cancel{x^2}$

$\Rightarrow x^2 \frac{dy}{dx} = y(1-x)$

$\Rightarrow \frac{x^2}{1-x} \times \frac{dy}{dx} = y$

$\Rightarrow \int \frac{1-x}{x^2} dx = \int \frac{1}{y} dy$ → Seperable ODE

$\Rightarrow \int \frac{1}{y} dy = \int \frac{1-x}{x^2} dx$

$x^2 = z$

$2x\,dx = dz$

$\Rightarrow \ln|y| = \frac{x^{-1}}{-1} - \ln|x| + c$

$\frac{1}{x^2} - \frac{1}{x}$

$= -\frac{1}{x} - \ln|x| + c$

$\Rightarrow y = e^{-\frac{1}{x} - \ln|x| + c}$

$= e^{-\frac{1}{x}} \cdot e^{-\ln|x|} \cdot e^{c}$

$= e^{-1/x} \cdot \frac{1}{x} \cdot e^{c} = \frac{e^{-1/x}}{x} \cdot K$

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$$\Rightarrow y = k \cdot \frac{e^{-\frac{1}{x}}}{x} \longrightarrow \text{general soln}$$

So, $y(-1) = k \times \dfrac{e^{-\frac{1}{-1}}}{-1} = -1$

$$\Rightarrow k \times e = 1$$

$$\Rightarrow k = \frac{1}{e}$$

So, $y = \dfrac{1}{e} \times \dfrac{1}{x} \times e^{-\frac{1}{x}} \longrightarrow$ particular solution

(যখন constant এর fixed value পাচ্ছি)

$$\frac{dP}{dt} = kt$$

$$\Rightarrow dP = kt\,dt$$

$$\Rightarrow \int dP = k \int t\,dt$$

$$\Rightarrow P = k\frac{t^2}{2} + C$$

$$\frac{dP}{dt} = kP$$

$$\Rightarrow dP \times \frac{1}{P} = k\,dt$$

$$\Rightarrow \ln|P| = kt + C$$

$$\Rightarrow P = e^{kt+C} = e^{kt} \cdot e^{C} = e^{kt} \times P_0$$

$$\Rightarrow \boxed{P = P_0 e^{kt}}$$

population growth constant

initial population

Population at certain time (t)

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(13) $N = N_0 e^{kt}$

$t = 4$ hrs , $\to N = 2N_0$

$2N_0 = N_0 e^{k \times 4} \Rightarrow k = 0.173$ (increasing growth rate)

$t = ?$ when $N = 8N_0$

$8N_0 = N_0 e^{0.173 \times t} \Rightarrow t = \frac{\ln 8}{0.173} \approx 12$ H

(15) $\frac{dT}{dt} \propto T - T_m$

T is the inside temp

T_m is the outside temp.

Lets T_m be the average of $50°F - 40°F$

So $T_m = 45$

$\Rightarrow \frac{dT}{dt} = k(T - T_m)$

$\Rightarrow \frac{dT}{dt} = k(T - 45)$

$\Rightarrow \frac{dT}{T-45} = k\,dt$

$\Rightarrow \ln|T - 45| = kt + c$

$\Rightarrow \cancel{\ln}\ T - 45 = e^{kt+c}$

$\Rightarrow T = e^{kt} \cdot e^{c} + 45$

$\Rightarrow \boxed{T(t) = T_0 e^{kt} + 45}$ $[T_0 = e^{c}]$

$\Rightarrow \cancel{T(t) = 65 \times e^{k \times 4} + 45}$

10 pm

$T(0) = 70°F$ $T(9) = 65$

10pm - 2am

$T(8) = ?$

10pm - 6Am

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$\boxed{3/2/19}$

Solve: ~~Example~~ $2xyy' = y^2 - x^2$

$\Rightarrow y' = \frac{y^2 - x^2}{2xy}$

$\Rightarrow \frac{dy}{dx} = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}$ ——— (1)

Let, $\boxed{\frac{y}{x} = u}$ $\Rightarrow y = ux \Rightarrow y' = u + x\frac{du}{dx}$

So, (1) $\Rightarrow u + x\frac{du}{dx} = \frac{u}{2} - \frac{1}{2u}$

$\Rightarrow x\frac{du}{dx} = \frac{u}{2} - \frac{1}{2u} - u = \frac{u^2 - 1 - 2u^2}{2u} = \frac{-u^2 - 1}{2u}$

$\Rightarrow \frac{2u}{u^2+1}du = -\frac{dx}{x}$

$\Rightarrow \int \frac{2u}{u^2+1}du = -\int \frac{dx}{x}$

Let, $u^2 + 1 = z$

$\Rightarrow 2u\,du = dz$

$\Rightarrow \ln|u^2+1| = -\ln|x| + \ln c$

$\Rightarrow e^{\ln|u^2+1|} = e^{-\ln|x| + \ln c} = e^{-\ln x} \cdot e^{\ln c}$

$\Rightarrow u^2 + 1 = \frac{1}{x} \times c$

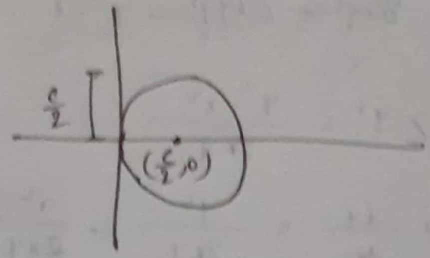
$\Rightarrow \left(\frac{y}{x}\right)^2 + 1 = \frac{c}{x}$

$\Rightarrow y^2 + x^2 = \frac{c}{x} \times x^2 = cx$ ——→ General solution

$\Rightarrow x^2 - cx + \frac{c^2}{4} + y^2 = \frac{c^2}{4}$

$\Rightarrow \left(x-\frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$

Center of circle $\left(\frac{c}{2}, 0\right)$ radius: $\frac{c}{2}$



First order differential Eqn

Chapter 2.4

Exact eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Show the ODE $2xy\,dx + (x^2-1)\,dy = 0$ exact

$$\frac{\partial}{\partial y}(2xy) = 2x \qquad \frac{\partial}{\partial x}(x^2) = 2x$$

It is exact ODE

How to solve an exact ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{—— (1)}$$

(1) is exact, So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ —— (2)

Find a solution of $f(x,y) = 0$ of (1)

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$\frac{\partial f}{\partial x} = M - 3(a)$, $\frac{\partial f}{\partial y} = N - 3(b)$

$\{3(a) \Rightarrow \frac{\partial f}{\partial x} = M(x,y)$

$\Rightarrow \int \frac{\partial f}{\partial x} dx = \int M(x,y) dx + c$

$\Rightarrow \boxed{f(x,y) = \int M(x,y) dx + g(y)}$ —— (4)

Diff. w.r.t y, $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x,y) dx + \frac{\partial}{\partial y} g(y)$

$\Rightarrow \boxed{N = \frac{\partial}{\partial y} \int M(x,y) dx + g'(y)}$ — (5)

We integrate $g'(y)$ to get $g(y)$ as,

$g(y) = \int g'(y) dy$

(4) ~~⇒~~ Put $g(y)$ in (4) to get general soln.

(#) Solve, the exact eqn $2xy\, dx + (x^2 - 1) dy = 0$

$f(x,y) = \int 2xy\, dx + g(y)$ — (1)

$(x^2 - 1) = \frac{\partial}{\partial y} \int 2xy\, dx + g'(y)$ —— (2)

~~(1) = f(x.~~ (2) $\Rightarrow x^2 - 1 = \frac{\partial}{\partial y} (\cancel{2}y) \times \frac{x^2}{\cancel{2}} + g'(y)$

$\Rightarrow x^2 - 1 = x^2 + g'(y)$

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$\Rightarrow g'(y) = -1$ ~~fee~~

$\Rightarrow \int g'(y)\,dy = -\int dy$

$\Rightarrow g(y) = -y + c$

$(1) \Rightarrow f(x,y) = \int 2xy\,dx + \cancel{y} - y + c$

$= 2y \times \frac{x^2}{2} + \cancel{-}y + c = x^2 y - \cancel{x}y + c = 0$

$\boxed{5/2/19}$

Quiz (Sunday) ~~in~~ including today [Chap 2.1, 2.2, 2.3, 2.4]

22) (A) Test whether the following ODE is exact :

$(x+y)^2\,dx + (2xy + x^2 - 1)\,dy = 0 \qquad y(1) = 1$

If exact, find the ~~given &~~ general solution and particular solution

We know that, $M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x+y)^2 = \frac{\partial}{\partial y}(x^2+y^2+2xy)$

$= 0 + 2y + 2x$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy + x^2 - 1)$$

$$= 2y + 2x$$

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, So the ODE is exact

So, $f(x,y) = \int M(x,y)\,dx + g(y)$ —— (1)

$N = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)$ —— (2)

$(1) \Rightarrow f(x,y) = \int (x^2+y^2+2xy)\,dx + g(y)$

$(2) \Rightarrow$ ~~N=~~ $(2xy + x^2 - 1) = \frac{\partial}{\partial y}\int (x^2+y^2+2xy)\,dx + g'(y)$

$\Rightarrow 2xy + x^2 - 1 = \frac{\partial}{\partial y}\left(\frac{x^3}{3} + y^2x + \cancel{2}yx\frac{x^2}{\cancel{2}}\right) + g'(y)$

$\Rightarrow \cancel{2xy} + \cancel{x^2} - 1 = 0 + \cancel{2x y} + \cancel{x^2} + g'(y)$

$\Rightarrow g'(y) = -1 \Rightarrow \int g'(y)\,dy = -\int dy \Rightarrow g(y) = -y + C$

$(1) \Rightarrow f(x,y) = \int (x^2 + y^2 + 2xy)\,dx - y + C$

$\Rightarrow f(x,y) = \frac{x^3}{3} + y^2x + x^2y - y + C = 0$ → general sol^n

$\Rightarrow f(1,1) = \frac{1}{3} + 1 + \cancel{1} - \cancel{1} + C = 0$ $[\because y(1) = 1]$

$\Rightarrow C = -\frac{4}{3}$

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Particular solⁿ :> $f(x,y) = \frac{x^3}{3} + y^2 x + x^2 y - y + -\frac{4}{3}$ → Particular solⁿ

Lecture 3

Chapter 2.3

General form of a first order ODE:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{--- (1) (General form)}$$

↳ derivative এর সামনে কিছু থাকবে

or, $\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$

$\Rightarrow \frac{dy}{dx} + P(x)y = f(x)$ — (2) where, $P(x) = \frac{a_0(x)}{a_1(x)}$

(Standard form) and $f(x) = \frac{g(x)}{a_1(x)}$

We want to make (2) and exact differential,

To make it exact differential, we multiply (2) with a function known as integrating factor,

$$\boxed{\mu(x) = e^{\int P(x)dx}} \quad \text{--- (3)}$$

$y = y(x)$

$\frac{d}{dx}(x^2 y) = x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2)$

$\Rightarrow \frac{d}{dx}(x^2 y) = x^2\frac{dy}{dx} + 2xy$ (exact differential)

Then (2) becomes,

$$\mu(x)\cdot\left[\frac{dy}{dx}+P(x)y\right] = f(x)\times\mu(x)$$

$$\Rightarrow \mu(x)\frac{dy}{dx}+\mu(x)P(x)y = f(x)\mu(x) \quad \text{———} (4) \quad (\text{Exact differential})$$

(4) is exact differential.

$$\frac{d}{dx}(\mu(x)y) = f(x)\mu(x)$$

$$\Rightarrow \int\frac{d}{dx}(\mu(x)y)\,dx = \int f(x)\mu(x)\,dx$$

$$\Rightarrow \mu(x)y = \int f(x)\mu(x)\,dx + C$$

$$\Rightarrow \boxed{y = \frac{1}{\mu(x)}\left[\int f(x)\mu(x)\,dx + C\right]} \quad (\text{General solution})$$

④ Solve, $x^2\frac{dy}{dx}+2xy=0$

It is general form. Make it standard form.

$$\frac{dy}{dx}+\frac{2xy}{x^2} = \frac{0}{x^2}$$

$$\Rightarrow \frac{dy}{dx}+\frac{2}{x}y = 0$$

So, $P(x) = \frac{2}{x}$

$$\mu(x) = e^{\int\frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$\therefore$ Multiply $u(x)$ to standard form;

$$x^2\left[\frac{dy}{dx}+\frac{2}{x}y\right]=0\times x^2$$

$$\Rightarrow x^2\frac{dy}{dx}+2xy=0$$

$$\Rightarrow \frac{d}{dx}\left(x^2y\right)=0$$

$$\Rightarrow \int\frac{d}{dx}(x^2y)\,dx=\int 0\,dx$$

$$\Rightarrow x^2y=\text{Const}\quad [C=\text{Constant}]$$

$$\Rightarrow y=\frac{C}{x^2}\ \rightarrow \text{General solution}$$

(11) $x\frac{dy}{dx}+4y=x^3-x$

It is general form. Make it standard form

$$\frac{dy}{dx}+\frac{4y}{x}=\frac{x^3-x}{x}$$

$$\Rightarrow \frac{dy}{dx}+\frac{4}{x}y=x^2-1$$

So, $P(x)=\frac{4}{x}$

$u(x) = e^{\int \frac{4}{x} dx} = e^{4 \ln x} = e^{\ln x^4} = x^4$

Multiply $u(x)$ to standard form

$$x^4 \left[\frac{dy}{dx} + \frac{4}{x} y \right] = x^4 (x^2 - 1)$$

$$\Rightarrow x^4 \frac{dy}{dx} + 4x^3 y = x^6 - x^4$$

$$\Rightarrow \frac{d}{dx}(x^4 y) = x^6 - x^4$$

$$\Rightarrow \int \frac{d}{dx}(x^4 y)\, dx = \int (x^6 - x^4)\, dx$$

$$\Rightarrow x^4 y = \frac{x^7}{7} - \frac{x^5}{5} + C$$

$$\Rightarrow y = \frac{x^3}{7} - \frac{x}{5} + C$$

23) $x \frac{dy}{dx} + (3x + 1) y = e^{-3x}$

Make it standard form as it is general form.

$$\frac{dy}{dx} + \frac{1}{x}(3xy + y) = \frac{e^{-3x}}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{3x+1}{x} y = \frac{e^{-3x}}{x}$$

$P(x) = \frac{3x+1}{x} = 3 + \frac{1}{x}$

So, $\mu(x) = e^{\int P(x)\,dx} = e^{\int \left(3+\frac{1}{x}\right)dx} = e^{3x + \ln x}$

$= e^{3x} \cdot e^{\ln x}$

$= xe^{3x}$

Multiply $\mu(x)$ to standard form,

$$xe^{3x}\left[\frac{dy}{dx} + \left(\frac{3x+1}{x}\right)y\right] = \frac{e^{-3x}}{x} \times xe^{3x}$$

$$\Rightarrow \frac{d}{dx}\left(xe^{3x}y\right) = 1$$

$$\Rightarrow \int \frac{d}{dx}\left(xe^{3x}y\right)dx = \int 1\,dx$$

$$\Rightarrow xe^{3x}y = x + c$$

$$\Rightarrow y = \frac{1}{xe^{3x}}(x+c)$$

$\cancel{\int x^3 e^{-3x}\,dx}$

$\cancel{= x^3 \int e^{-3x}dx - \int\left[\frac{d}{dx}(x^3)\int e^{-3x}dx\right]dx}$

$\cancel{= x^3 \times \frac{e^{-3x}}{-3} + \int 3x^2 \times \frac{e^{-3x}}{-3}\,dx}$

$=$

10/2/19

$L\frac{di}{dt} + iR = E(t)$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \quad \text{---} (2)$$

$i = \frac{dq}{dt}$

Standard linear ~~eq~~ ODE

$$P(t) = \frac{R}{L}$$

$$\mu(t) = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

Multiply $\mu(t)$ with (2) $\Rightarrow e^{\frac{Rt}{L}}\left(\frac{di}{dt} + \frac{R}{L} i\right) = e^{\frac{Rt}{L}} \times \frac{E}{L}$

$$\Rightarrow \frac{d}{dt}\left(e^{\frac{Rt}{L}} i\right) = e^{\frac{Rt}{L}} \times \frac{E}{L}$$

$$\Rightarrow \int \frac{d}{dt}\left(e^{\frac{Rt}{L}} i\right) dt = \int e^{\frac{Rt}{L}} \times \frac{E}{L}\, dt$$

$$\Rightarrow e^{\frac{Rt}{L}} i = \frac{E}{\cancel{L}} \times \frac{\cancel{L} e^{\frac{Rt}{L}}}{R} + C$$

$$= \frac{E}{R} e^{\frac{Rt}{L}} + C$$

$$\Rightarrow i(t) = \frac{E}{R} + Ce^{-\frac{Rt}{L}} \quad \text{---} (3) \text{ general solution}$$

By default, $t = 0$, $i = 0$ So, $0 = \frac{E}{R} + C \times e^0 \Rightarrow C = -\frac{E}{R}$

$(3) \Rightarrow i(t) = \frac{E}{R} - \frac{E}{R} e^{-\frac{Rt}{L}} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ Particular Solution

Ex-7

$$\frac{1}{2}\frac{di}{dt} + 10i = 12$$

$\boxed{\text{Bernoulli's equation}}$

$\frac{dy}{dx} + P(x)y = f(x)y^n$ for $n \neq 0, n \neq 1$ ← Non linear First order

$\Rightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x)$

$\Rightarrow (\quad) + P(x)u = f(x)$ $\boxed{u = y^{1-n}}$

(A) Solve, $x\frac{dy}{dx} + y = x^2y^2$

$y=2, n=2$, so it is non-linear.

$\frac{dy}{dx} + \frac{y}{x} = xy^2$ —(2) (Standard form)

$u = y^{1-2} = \frac{1}{y}$ $\Rightarrow y = \frac{1}{u}$ $\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= -\frac{1}{u^2}\cdot\frac{du}{dx}$

So,

(2) ⇒ So, $-\frac{1}{u^2}\frac{du}{dx} + \frac{1}{ux} = \frac{x}{u^2}$

$\Rightarrow \frac{du}{dx} - \frac{1}{x}u = -x$ —(3) (Standard form)

$P(x) = -\frac{1}{x}, \quad f(x) = -x$

$\mu(x) = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = -\frac{1}{x}$

$(3) \Rightarrow -\frac{1}{x}\left(\frac{du}{dx} - \frac{1}{x}u\right) = -x \times \frac{1}{x}$

$\Rightarrow \int \frac{d}{dx}\left(-\frac{1}{x} \times u\right) dx = -\int dx$

$\Rightarrow -\frac{1}{x}u = -x + C$

$\Rightarrow u = -x^2 + Cx$

$\Rightarrow \frac{1}{y} = -x^2 + Cx$

$\Rightarrow y = \frac{1}{Cx - x^2}$

(21) $x^2 \frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

$\Rightarrow \frac{dy}{dx} - \frac{2x}{x^2} y = \frac{3}{x^2} y^4$ — (1)

$\Rightarrow n = 4$

$u = y^{1-4} \Rightarrow u = \frac{1}{y^3} \Rightarrow y^3 = \frac{1}{u}$

$\cancel{\Rightarrow 3y^2 \frac{dy}{dx} = -\frac{1}{u^2} \frac{du}{dx}}$

$\Rightarrow 3y^2 \frac{dy}{dx} = -\frac{1}{u^2} \frac{du}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{u^2 \times 3y^2} \frac{du}{dx} = \frac{-1}{3 \times \left(\frac{1}{u}\right)^{2/3}} \times \frac{1}{u^2} \times \frac{du}{dx}$

$= \frac{-1}{3} \times \left(\frac{1}{u}\right)^{4/3} \frac{du}{dx}$

$(5) \Rightarrow -\frac{1}{3}\left(\frac{1}{u}\right)^{\frac{4}{3}} \frac{du}{dx} - \frac{2}{x} \times \left(\frac{1}{u}\right)^{1/3} = 3\left(\frac{1}{u}\right)^{4/3} \times \frac{1}{x^2}$

$\boxed{12/2/19}$

$\boxed{\text{Lecture 4}}$

n-th order differential eq^n

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

~~Subject (to) to~~ If right hand side is zero, homogenous and nonzero is non-homogenous.

Solving 2nd order ODE

General form is

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \quad \text{---} (1)$$

here, $a_2(x)$, $a_1(x)$, $a_0(x)$ is func^n of x or Constants.

If $g(x) = 0$, it is called homogenous

If $g(x) \neq 0$, it is called non-homogenous.

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$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \quad \text{——} (2) \rightarrow \text{Homogenous } n^{th} \text{ order}$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{a_1(x)}{a_2(x)}\frac{dy}{dx} + \frac{a_0(x)}{a_2(x)}y = 0$$

$$\Rightarrow y'' + a(x)\frac{dy}{dx} + b(x)y = 0 - (3) \text{ where, } a(x) = \frac{a_1(x)}{a_2(x)} \text{ and } b(x) = \frac{a_0(x)}{a_2(x)}$$

$\rightarrow$ Standard form

$$(3) \Rightarrow y'' + a(x)y' + b(x)y = 0 \quad \text{——} (4)$$

Let, $y = e^{\lambda x}$ be a trial solution of (4), λ is any scalar.

$$\Rightarrow y' = \lambda e^{\lambda x}$$

$$\Rightarrow y'' = \lambda^2 e^{\lambda x}$$

Substitute all these in (4),

$$\lambda^2 e^{\lambda x} + a(x)\lambda e^{\lambda x} + b(x)e^{\lambda x} = 0$$

$$\Rightarrow e^{\lambda x}\left[\lambda^2 + a(x)\lambda + b(x)\right] = 0$$

$e^{\lambda x} \neq 0$, $\lambda^2 + a\lambda + b = 0$ —(v) $\rightarrow$ Characteristic equation

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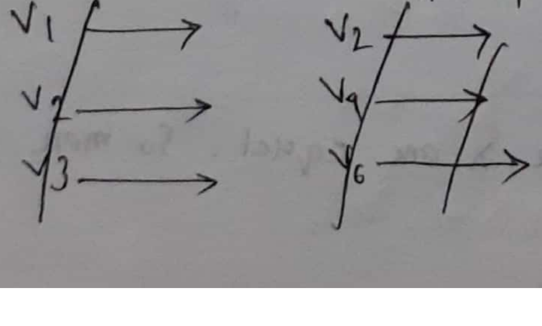
Ref →



linearly dependent $\vec{v}_1 = (1,2,3)$ $\vec{v}_2 = (2,4,6)$ $\vec{v}_2 = 2\vec{v}_1$

$\vec{v}$ $(1,2,3)$ $\vec{v} \Rightarrow (1,-1,0)$

We want to solve (5) [root is for algebraic equation]

we will have two roots : λ_1 and λ_2

They have 3 different types

Type 1 : Real and distinct $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$

$\lambda^2 + a\lambda + b = 0$

$\Rightarrow \lambda_1, \lambda_2 = \dfrac{-a \pm \sqrt{a^2 - 4 \times 1 \times b}}{2 \times 1} = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

$y_1 = e^{\lambda_1 x}$ $\quad y_2 = e^{\lambda_2 x}$ $\quad$ y_1 and y_2 are fundamental Solution

They are linearly independent because $\lambda_1 \neq \lambda_2$

So, $y = c_1 y_1 + c_2 y_2$

$= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Type 2: Real and equal, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 = \lambda_2 = \lambda$ (let)

$y_1 = e^{\lambda x}$ $\quad y_2 = e^{\lambda x}$

They are linearly dependent because λ are equal. So make it linearly independent, $y_2 = x e^{\lambda x}$

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General solution, $y = c_1 y_1 + c_2 y_2$

$$= c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

Type 3: In Complex roots, $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

$y_1 = e^{(\alpha + i\beta)x}$ $y_2 = e^{(\alpha - i\beta)x}$

They are linearly independent because powers are different

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

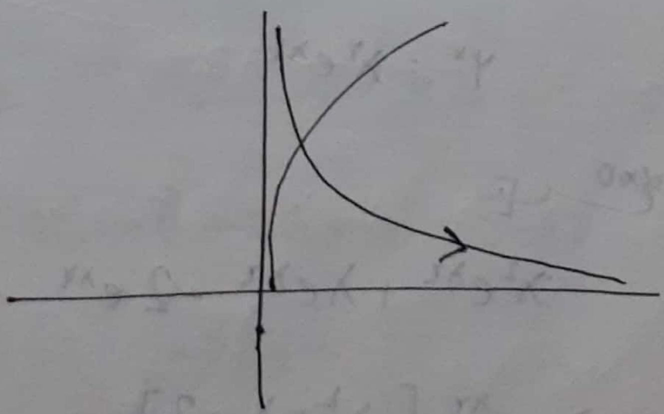
$$= c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x}$$

$$= e^{\alpha x}\left(c_1 e^{i\beta x} - c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$, A and B are contant

Type 1:

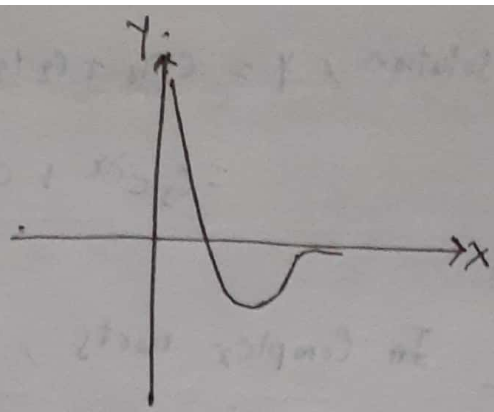
$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$



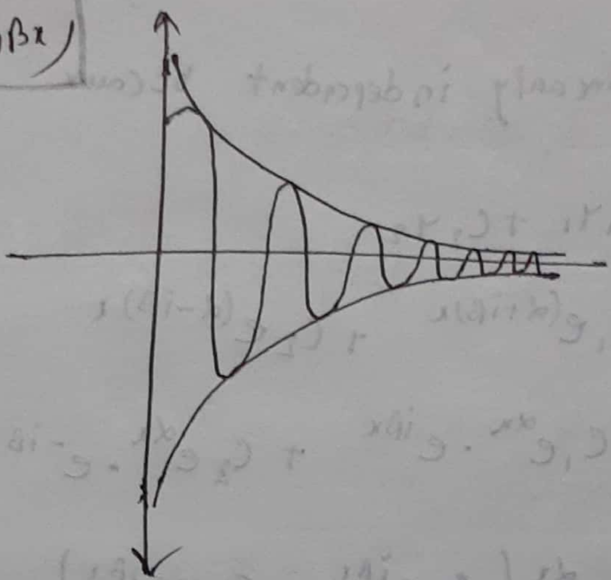
Type 2: $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$

$(c_1 = c_2) \Rightarrow y = c_1 (e^{\lambda x} + x e^{\lambda x})$

$\Rightarrow \boxed{y = c_1 (1+x) e^{\lambda x}}$



Type 3: $\boxed{y = e^{\alpha x} (A \cos\beta x + B \sin\beta x)}$



(11) $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$

Let, the trial solution be $y = e^{\lambda x}$ — (1)

$y' = \lambda e^{\lambda x}$ $\quad$ $y'' = \lambda^2 e^{\lambda x}$

~~(1) ⇒ 4 = e^{λ×0}~~

Substituting it, $\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0$

$\Rightarrow e^{\lambda x} [\lambda^2 + \lambda - 2] = 0$

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$e^{\lambda x} \neq 0$ so, $\lambda^2 + \lambda - 2 = 0$

$\Rightarrow \lambda^2 + 2\lambda - \lambda - 2 = 0$

$\Rightarrow \lambda(\lambda+2) - 1(\lambda+2) = 0$

$\Rightarrow (\lambda-1)(\lambda+2) = 0$

$\Rightarrow \lambda_1 = -2 \quad \lambda_2 = 1$

So, the roots are type 1.

$y = c_1 e^{-2x} + c_2 e^{x}$ —— (2) (General solution)

Putting $y(0) = 4$ so, (2) $\Rightarrow 4 = c_1 e^{-2\times 0} + c_2 e^{0}$

$\Rightarrow c_1 + c_2 = 4$ — (3)

(2) $\Rightarrow y' = c_1 \times -2 e^{-2x} + c_2 e^{x}$

$\Rightarrow -5 = c_1 \times -2 e^{-2\times 0} + c_2 e^{0}$ [$y'(0) = -5$]

$\Rightarrow \cancel{c_1} - 2c_1 + c_2 = -5$ —— (4)

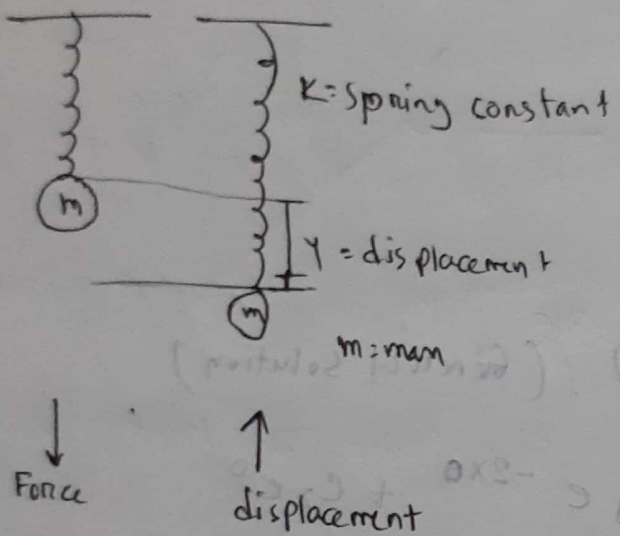
(3) $\times 2$ + (4) $\Rightarrow \cancel{2c_1} + 2c_2 \cancel{\;} \cancel{-2c_1} + c_2 = 8 - 5$

$\Rightarrow 3c_2 = 3 \quad \Rightarrow c_2 = 1$

(3) $\Rightarrow c_1 + 1 = 4 \quad \Rightarrow c_1 = 3$

So, (2) => $y = 3e^{-2x} + e^{x}$ (Particular solution)

$\boxed{17/2/19}$ $\boxed{\text{Lecture 5}}$ → Assessment



Hook's law: $F \propto -y$

$$\Rightarrow F = -ky \quad \text{—— (1)}$$

Newton's law: $F = ma$ —— (2)

(1) and (2) =>

$$ma = -ky$$

$$\Rightarrow \boxed{m \times \frac{d^2y}{dt^2} = -ky} \text{—— (3)}$$

From 3, we make it into standard form

(3) => $my'' + ky = 0$

$$\Rightarrow y'' + \frac{k}{m}y = 0 \quad \text{—— (4)}$$

Let the trial solution be $y = e^{\lambda t}$

$$\Rightarrow y' = \lambda e^{\lambda t}$$

$$\Rightarrow y'' = \lambda^2 e^{\lambda t}$$

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Mid 1 Sunday
March 3

So, (4) $\Rightarrow \lambda^2 e^{\lambda t} + \frac{k}{m} e^{\lambda t} = 0$

$\Rightarrow e^{\lambda t} \left(\lambda^2 + \frac{k}{m}\right) = 0$

As $e^{\lambda x} \neq 0$, so, $\lambda^2 + \frac{k}{m} = 0$

$\Rightarrow \lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$ $\left[\sqrt{\frac{k}{m}} = \omega_0\right.$, angular frequency

So, It is type 3, complex root.

So, general solution, $y = e^{0xt}(A\cos\omega_0 t + B\sin\omega_0 t)$

$= A\cos\omega_0 t + B\sin\omega_0 t$ ——— (5)

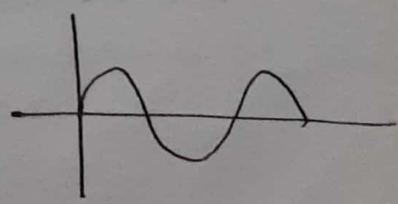
Eq^n (5) is called the simple harmonic oscillation of the spring.

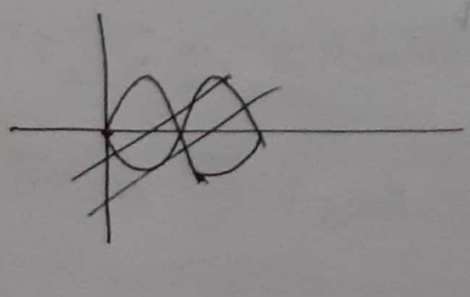
The frequency is $f = \frac{\omega_0}{2\pi}$ Hz

This f is known as natural frequency of the system.

An alternative representation of (5) can be $y(t) = \cos(\omega t - \delta)$ ——— (6)

where, $C = \sqrt{A^2+B^2}$ (Amplitude)





$\delta = \tan^{-1}\frac{B}{A}$ (phase)

$\downarrow$

the ratio of two different ~~phase~~ angle

Eqn (6) is called undamped simple harmonic ~~motion~~ oscillation.

Example 1

$W = 98N \qquad Y = 1.09\,m$

$$m\frac{d^2y}{dt^2} + ky = 0 \quad \text{——} \quad (1)$$

$98N = m \times 9.8 \implies m = \frac{98 \times 10}{98} = 10\ kg$

~~Ka~~ $98\,N = k \times 1.09 \implies k = \frac{98}{1.09} = 89.91 = 90$

$$(1) \implies 10\frac{d^2y}{dt^2} + 90y = 0$$

$$\implies \frac{d^2y}{dt^2} + 9y = 0$$

$$\implies y'' + 9y = 0 \quad \text{——} \quad (2)$$

Let the trial solution be $y = e^{\lambda t}$

$$\implies y' = \lambda e^{\lambda t} \implies y'' = \lambda^2 e^{\lambda t}$$

So, $(2) \implies \lambda^2 e^{\lambda t} + 9e^{\lambda t} = 0$

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$\Rightarrow e^{\lambda t}(\lambda^2+9)=0$

As $e^{\lambda t} \neq 0$ so, $\lambda^2 = -9$

$\Rightarrow \lambda = \pm 3i$

So, the general solution,

$y = e^{0 \times t}(A\cos 3t + B\sin 3t)$

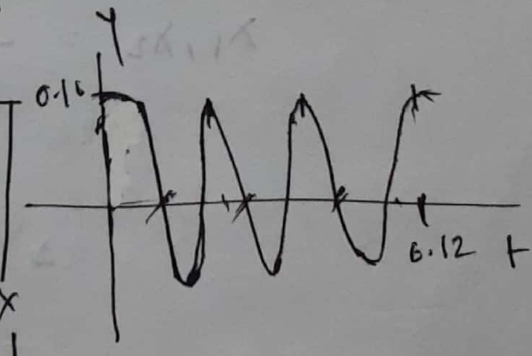
$\Rightarrow y(t) = A\cos 3t + B\sin 3t$

Here, $y(0) = 16$ cm velocity, $y'(0) = 0\ ms^{-1}$

If we put these values, we get $A = 0.16$, $B = 0$

So, $y(t) = 0.16\cos 3t + 0$

$\Rightarrow y(t) = 0.16\cos 3t$



F_1 = damper force, $> F$

$F_1 \propto -y'$ $\Rightarrow F_1 = -cy'$ c: damper constant

F_1 is the disturbing force used to reduced velocity.

So, $m\frac{d^2y}{dt^2} = -ky + (-cy')$

$\Rightarrow m\frac{d^2y}{dt^2} = -ky - cy'$

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$\Rightarrow m\frac{d^2y}{dt^2} + ky + c\frac{dy}{dt} = 0$

$\Rightarrow \frac{d^2y}{dt^2} + \frac{c}{m}\cdot\frac{dy}{dt} + \frac{k}{m}y = 0 \quad — (7)$

Let the $y = e^{\lambda t}$ $\quad \Rightarrow y' = \lambda e^{\lambda t}$ $\quad \Rightarrow y'' = \lambda^2 e^{\lambda t}$

So, (7) $\Rightarrow \lambda^2 e^{\lambda t} + \frac{c}{m}\lambda e^{\lambda t} + \frac{k}{m}e^{\lambda t} = 0$

$\Rightarrow e^{\lambda t}\left(\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}\right) = 0$

As, $e^{\lambda t} \neq 0$ so, $\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

$\cancel{\Rightarrow m\lambda^2 + c\lambda + k = 0}$

$\cancel{\lambda_1, \lambda_2 = }$

$$\lambda_1, \lambda_2 = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\times1\times\frac{k}{m}}}{2\times1}$$

$$= -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$$

Say, $\alpha = \frac{c}{2m}$, $\quad \beta = \frac{1}{2m}\sqrt{c^2 - 4km}$

$\lambda_1, \lambda_2 = -\alpha \pm \beta$

λ_1 and λ_2 are of three types.

Type 1: $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$

It happens when $\sqrt{c^2-4km} > 0$

So, $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{(-\alpha+\beta)t} + c_2 e^{(-\alpha-\beta)t}$ → Over damping

Type 2: $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 = \lambda_2$

It happens when $\sqrt{c^2-4km} = 0$ so, $\lambda_1 = \lambda_2 = -\frac{c}{2m}$

So, $y = c_1(1+t)e^{\lambda t} = c_1(1+t)e^{-\frac{c}{2m}t}$ → Critically damping (Most comfortable)

Type 3: λ_1, λ_2 is complex

$$\left.\begin{aligned} \lambda_1 &= -\alpha + i\beta \\ \lambda_2 &= -\alpha - i\beta \end{aligned}\right\} \text{ it happens when } \sqrt{(c^2-4km)} < 0$$

So, $y = e^{-\alpha t}(A\cos\beta t + B\sin\beta t)$ → Under damping

19/2/19

From start to 28/2/19 → Mid sylabus → March 3

4 question - 1 hr

- Modeling question (populatn, temp, damping) = 2
- Show exact / = 1

Ex-2

$$m\frac{d^2y}{dt^2} + ky = 0 \quad \text{—— (1)}$$

$W = mg = 2\,lb \Rightarrow m = \frac{2}{32}$ slug $= \frac{1}{16}$ slug $\qquad y = 6'' = \frac{6}{12}'$

$F = -ky \Rightarrow 2 = -k \times (6/12)' \Rightarrow k = \left|-\frac{24}{6}\right| = 4$

$$(1) \Rightarrow \frac{1}{16}\frac{d^2y}{dt^2} + 4y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + 64y = 0 \quad \text{—— (1)}$$

Let the trial solution be $y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}$

$$(1) \Rightarrow \lambda^2 e^{\lambda x} + 64 e^{\lambda x} = 0$$

$$\Rightarrow e^{\lambda x}(\lambda^2 + 64) = 0$$

$$\Rightarrow \lambda^2 + 64 = 0 \quad [\because e^{\lambda x} \neq 0]$$

$$\Rightarrow \lambda = \pm 8i$$

So the general solution is, $y = e^{0 \wedge t}(A\cos 8t + B\sin 8t)$

$$= A\cos 8t + B\sin 8t$$

So the conditions are $Y(0) = 8'' = \left(\frac{8}{12}\right)'$

$$Y'(0) = \frac{4}{3} \text{ ft s}^{-1}$$

$$Y(0) = \frac{8}{12} = A\cos 0 + B\sin 0$$

$$\Rightarrow A = \frac{2}{3}$$

$$Y'(t) = -8A\sin 8t + 8B\cos 8t$$

$$\Rightarrow Y'(0) = -8A\sin 0 + 8B\cos 0 = -\frac{4}{3}$$ [bcz it is (-)ve upward velocity]

$$\Rightarrow 8B = -\frac{4}{3}$$

$$\Rightarrow B = -\frac{1}{6}$$

So, the particular solution is, $Y = \frac{2}{3}\cos 8t + \frac{1}{6}\sin 8t$

Amplitude, $C = \sqrt{A^2 + B^2}$ ft

$$= \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2}$$

$$= \frac{\sqrt{17}}{6} \text{ ft}$$

phase, $\delta = \tan^{-1}\left(\frac{-\frac{1}{6}}{\frac{2}{3}}\right)$

$$= 0.02$$

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Ex-9

$W = mg \Rightarrow \therefore m = \frac{8}{32} = \frac{1}{4}$ slug

$F = -ky \Rightarrow 8 = -k \times 2 \Rightarrow k = \left|-\frac{8}{2}\right| = 4$

As the damping force is wanted, So, $F_d = -2y'$

$\frac{1}{4} y'' + 4y = 0$ Without dampened

$\Rightarrow y'' + 16y = 0$

Let, the trial solⁿ be $y = e^{\lambda t}$

$\lambda^2 e^{\lambda t} + 16 e^{\lambda t} = 0$

$\Rightarrow e^{\lambda t}(\lambda^2 + 16) = 0$

$\Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$

So, the general solⁿ will be,

$y(t) = A\cos 4t + B\sin 4t$

$\frac{1}{4} y'' + 4y = -2y'$ With dampened

$\Rightarrow y'' + 8y' + 16y = 0$

Let the trial solⁿ be $y = e^{\lambda t}$

$\lambda^2 e^{\lambda t} + 8\lambda e^{\lambda t} + 16 e^{\lambda t} = 0$

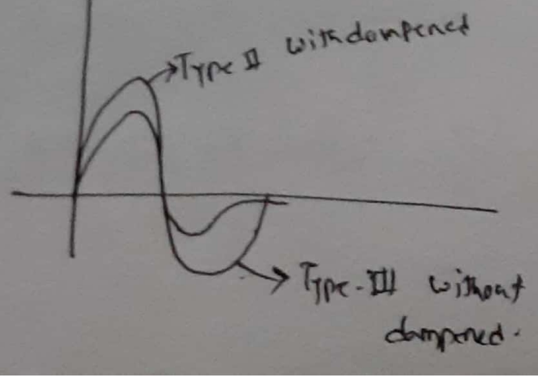
$\Rightarrow e^{\lambda t}(\lambda^2 + 8\lambda + 16) = 0$

$\Rightarrow \lambda^2 + 8\lambda + 16 = 0$

$\Rightarrow \lambda = -4, -4$

So, the general soln is

$y(t) = C_1 e^{-4t} + C_2 t e^{-4t}$



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The conditions are, $y(0) = 0$ $\quad y'(0) = -3 \text{ ft/s}$

$y(0) = 0 \Rightarrow C_1 e^0 + C_2 \times 0 \times e^0$

$\Rightarrow C_1 = 0$

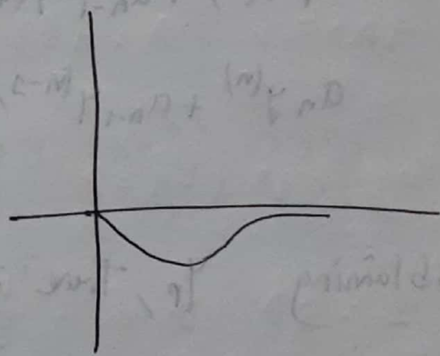
$y'(t) = -4 C_1 e^{-4t} + C_2 (e^{-4t} - 4te^{-4t})$

$\Rightarrow y'(0) = -3 = -4 \times 0 \times e^0 + C_2 (e^0 - 4 \times 0 \times e^0)$

$\Rightarrow C_2 \times (1 - 40) = -3$

$\Rightarrow C_2 = -3$

So, the particular solⁿ is $y(t) = -3te^{-4t}$



Ex-1 (I)

$C = 100 \text{ kg/s}$

So, damped force, $F_d = -100 \frac{dy}{dt}$

So, the eqⁿ is $10 \frac{d^2y}{dt^2} = -90y - 100 \frac{dy}{dt}$

$\downarrow$ Spring $\quad \downarrow$ damper

Lecture 6

Non homogenous eqn

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots\dots + a_1 y' + a_0 y = g(x)$ —— (1)

The general soln of (1) is, $y = y_c + y_p$

y_c = Complementary funcn (inherited)

y_p = any particular solution

y_c is obtained from homogenous part.

~~$a_n y(n) + a_{n-1} y(n-1)$~~ putting right hand side 0 in (1) =>

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

For obtaining y_p, there is several approaches;

(1) Superposition principle

(2) Inverse Operator method

(3) Method of substitution

(4) Variation of parameters

29/2/19

Superposition ~~method~~ Principle

The output will directly follow the input. The input is always the right hand side ~~func~~ in differential equation.

So, if $g(x)$ is sinosidual funcn, then y_p will be sinosidual funcn.

(Q) Solve the following ODE, using Superposition principle,

$$y'' + 4y' + 4y = 2x^2 - 3x + 6$$

Solution: The general solution y: $y = y_c + y_p$

For y_c, we solve, $y'' + 4y' + 4y = 0$ — (1)

let, the trial solution of (1) is $y = e^{\lambda t}$ $\Rightarrow y' = \lambda e^{\lambda t}$ $\Rightarrow y'' = \lambda^2 e^{\lambda t}$

So, substituting in (1) $\Rightarrow \lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 4e^{\lambda t} = 0$

$$\Rightarrow e^{\lambda t}(\lambda^2 + 4\lambda + 4) = 0$$

$$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda + 2)^2 = 0$$

$$\Rightarrow \lambda = -2, -2$$

So, the ~~particular~~ complimentary solution is, $y_c = C_1 e^{-2x} + C_2 x e^{-2x}$

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For y_p: We let, $y_p = Ax^2 + Bx + C$ ← [We write it because our input is second order quadratic equation]

Then we can write our main ODE as, $y_p'' + 4y_p' + 4y = 2x^2 \cancel{+8x} - 3x + 6$

$y_p = Ax^2 + Bx + C$

$\Rightarrow y_p' = 2Ax + B \cancel{+C}$

$\Rightarrow y_p'' = 2A$

$\Rightarrow 2A + 4 \times (2Ax + B \cancel{+C}) + 4(Ax^2 + Bx + C)$

$= 2x^2 - 3x + 6$

$\Rightarrow 2A + 8Ax + 4B \cancel{+4C} + 4Ax^2 + 4Bx + 4C$

$= 2x^2 - 3x + 6$

$\Rightarrow 4Ax^2 + (8A + 4B)x + (2A + 4B + 4C) = 2x^2 - 3x + 6$ —(2)

Equating eqn (2) $\Rightarrow$

$2 = 4A \Rightarrow A = \frac{1}{2}$

$8A + 4B = -3 \quad \Rightarrow 8 \times \frac{1}{2} + 4B = -3 \quad \Rightarrow B = -\frac{7}{4}$

$2A + 4B + 4C = 6$

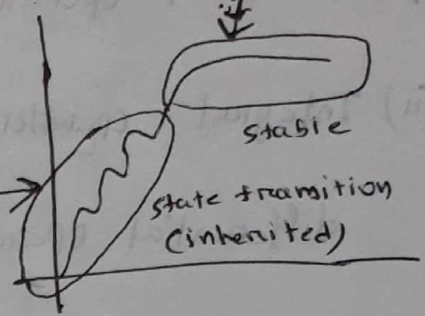
$\Rightarrow \cancel{2} \times \frac{1}{\cancel{2}} + \cancel{4} \times -\frac{7}{\cancel{4}} + 4C = 6$

$\Rightarrow C = \frac{\cancel{12}}{\cancel{8}}\ 3$

So, $y_p = \frac{1}{2}x^2 - \frac{7}{4}x + 3$

So, the general solution is, $y = y_p + y_c$

$$= c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2}x^2 - \frac{7}{4}x + 3$$



The particular solution form

$g(x)$	Form of y_p
1	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
x^3-x	Ax^3+Bx^2+Cx+D
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos 4x$	$A\cos 4x + B\sin 4x$
e^{5x}	Ae^{5x}
x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$

$\longrightarrow C_0\cos(\omega t - \delta)$

$= A\cos\beta t + B\sin\beta t$

where, $C = \sqrt{A^2+B^2}$

$\delta = \tan^{-1}\frac{B}{A}$

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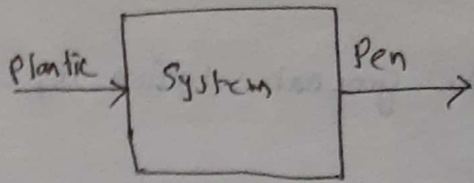
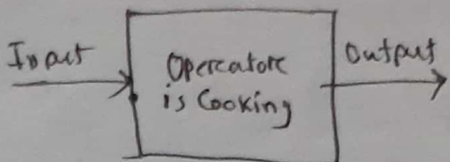
Inverse Operator Method

There are 2 types of operator in this method,



(i) Differential operator : D



(ii) Integral operator [Inverse differential operator] : $\frac{1}{D} = D^{-1}$

we can't solve e^x, $\sin x$ ... in the right side of ODE by this method

Example : $D(2x) = 2 \quad [\because \frac{d}{dx}(2x) = 2]$

$$D^{-1}(2x) = \int 2x\,dx = x^2$$

$$(1-x)^{-1} = 1 + x + x^2 + \cdots\cdots + x^n + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 + \cdots\cdots \pm x^n + \cdots$$

Hence:
$$(1-D)^{-1} = 1 + D + D^2 + \cdots\cdots + D^n + \cdots,$$

$$(1+D)^{-1} = 1 - D + D^2 + \cdots\cdots \pm D^n + \cdots$$

$$(1-D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots\cdots$$

$$(1+D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots\cdots$$

(#) Solve the following ODE using inverse operator method

$$y'' - 3y' + 2y = 4x$$

Solution: The general solution is, $y = y_c + y_p$

Let, the trial solution for y_c is $y_c = e^{\lambda x}$ $\Rightarrow y_c' = \lambda e^{\lambda x}$ $\Rightarrow y_c'' = \lambda^2 e^{\lambda x}$

Substituting eq^n (1) $\Rightarrow y'' - 3y' + 2y = 0$

$$\Rightarrow \lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2 e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \quad [e^{\lambda x} \neq 0]$$

$$\Rightarrow (\lambda - 2)(\lambda - 1) = 0$$

$$\Rightarrow \lambda_1 = 2, \ \lambda_2 = 1$$

The complimentary sol^n will be, $y_c = c_1 e^{2x} + c_2 e^{x}$

For y_p: We write our ODE as

$$y_p'' - 3y_p' + 2y_p = 4x$$

$$\Rightarrow D^2 y_p - 3Dy_p + 2y_p = 4x$$

$$\Rightarrow y_p(D^2 - 3D + 2) = 4x$$

$$\Rightarrow y_p = \frac{1}{D^2 - 3D + 2} 4x$$

$$= (D^2 - 3D + 2)^{-1} 4x$$

$$= \left[2\left(\frac{D^2}{2} - \frac{3D}{2} + 1\right)\right]^{-1} 4x$$

$D y_p = \frac{d}{dx}(y_p)$

$D^2 y_p = \frac{d^2}{dx^2}(y_p)$

$$= \frac{1}{2}\left[1-\left(\frac{3D}{2}-\frac{D^2}{2}\right)\right]^{-1} 4x$$

$$= \frac{1}{2}\left[1+\left(\frac{3D}{2}-\frac{D^2}{2}\right)+\underbrace{\left(\frac{3D}{2}-\frac{D^2}{2}\right)^2}_{\frac{9}{4}D^2-\frac{3}{4}D^3+\frac{D^4}{4}}+\cdots\cdots\right] 4x$$

$$= \frac{1}{2}\left[4x+\frac{3}{2}D(4x)+0+0+\cdots+0\right]$$

4x এর double derivative নিলে সব zero

$$= \frac{1}{2}\left[4x+\frac{3}{2}\times 4\right]$$

$$= 2x+3$$

So, the general solution is $y = y_c + y_p = c_1 e^x + c_2 e^{2x} + 2x + 3$

If R.H.S is $4x^2$

$$\frac{1}{2}\left[4x^2+\frac{3}{2}D(4x^2)-\frac{1}{2}D^2(4x^2)+\frac{9}{4}D^2(4x^2)-0\cdots\right]$$

$$= \frac{1}{2}\left[4x^2+\frac{3}{2}\times 8x-\frac{1}{2}\times 8+\frac{9}{4}\times 8+0\right]$$

$$= \frac{1}{2}\left[4x^2+12x-4+18\right]$$

$$= 2x^2+6x+7$$

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Lecture 6

Method of substitution

Shift of exponents

We are solving problem of type: $y'' + y' + y = e^x$

How to solve:

We know for y_p, we will have: $f(D)\, y_p = e^{ax}$

$$(D^2 + D + 1) y_p = e^1$$
$$f(D)\, y_p = e^{ax},\quad a = 1$$

$$y_p = \frac{e^{ax}}{f(D)}, \quad f(D) \neq 0$$

Type I: Substitute 'D' by 'a'

If $f(a) \neq 0$, then $y_p = \frac{1}{f(a)} e^{ax}$

Type II: Substitute 'D' by 'a' and $f(a) = 0$. So we will fail to type I

Substitute technique

Then use the shifting as: $y_p = e^{ax} \frac{1}{f(D+a)}$

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(4)

Solve : $y'' - y' - 2y = e^x$

Let the solution be $y = y_c + y_p$

For y_c : Let the trial solution be $y = e^{\lambda x}$ $\Rightarrow y' = \lambda e^{\lambda x}$ $\Rightarrow y'' = \lambda^2 e^{\lambda x}$

Substituting, we get $\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2e^{\lambda x} = 0$

$\Rightarrow e^{\lambda x}(\lambda^2 - \lambda - 2) = 0$

$\Rightarrow \lambda^2 - \lambda - 2 = 0 \quad [e^{\lambda x} \neq 0]$

$\Rightarrow \lambda^2 - 2\lambda + \lambda - 2 = 0$

$\Rightarrow (\lambda - 2)(\lambda + 1) = 0$

$\Rightarrow \lambda = 2, -1$

So, $y_c = c_1 e^{2x} + c_2 e^{-x}$

For y_p : $D^2 y_p - D y_p - 2y_p = e^x$

$\Rightarrow (D^2 - D - 2) y_p = e^x$

So, here $f(D) = D^2 - D - 2 = (D-2)(D+1)$

$a = 1$

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Substituting 'D' by 'a', So $f(a) = (a-2)(a+1)$

$= (1-2)(1+1) = -2 \neq 0$

So, it is type I bcz $f(a) \neq 0$

So, $y_p = \frac{1}{f(a)} e^{ax} = -\frac{1}{2} e^{x}$

So, $y = c_1 e^{2x} + c_2 e^{-x} - \frac{1}{2} e^{x}$

(#) If R.H.S is e^{-x}, so, $f(a) = (a-2)(a+1) = 0$

So, it is type II bcz $f(a) = 0$

Now, $(D^2 - D - 2) y_p = e^{-x}$

$\Rightarrow y_p = \frac{1}{D^2 - D - 2} e^{-x}$

$= \frac{1}{\underbrace{(D-2)}_{\substack{\text{good part} \\ \text{bcz } (-1-2) \\ = 3}} \underbrace{(D+1)}_{\substack{\text{bad part} \\ \text{bcz } (-1+1) \\ = 0}}} e^{-x}$

$\Rightarrow y_p = e^{-x} \frac{1}{(-1-2)\underbrace{(D+1+a)}_{\text{D is shifted by D+a}}}$ $\left[\because y_p = e^{ax} \frac{1}{f(D+a)} \right]$

$$= e^{-x} \frac{1}{-3(D-1+1)}$$

$$= e^{-x} \frac{1}{-3D}$$

$$= -\frac{1}{3} e^{-x} \frac{1}{D}$$

$$= -\frac{1}{3} e^{-x} \int 1 dx$$

$$= -\frac{1}{3} e^{-x} x$$

(4) Solve $y'' - 4y' + 4y = e^{2x}$

Let, the solution be, $y = y_c + y_p$

For y_c: Let the trial solution of y_c is $y = \lambda e^{\lambda x}$ $\Rightarrow$ ~~λ~~ $y' = \lambda e^{\lambda x}$

$\Rightarrow y'' = \lambda^2 e^{\lambda x}$

Substituting, $\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 4e^{\lambda x} = 0$

$\Rightarrow e^{\lambda x}(\lambda^2 - 4\lambda + 4) = 0$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow (\lambda - 2)^2 = 0$

$\Rightarrow \lambda = 2, 2$

So, $y_c = C_1 e^{2x} + C_2 x e^{2x}$

For y_p: $D^2 y_p - 4Dy_p + 4y_p = e^{2x}$

$\Rightarrow (D^2 - 4D + 4) y_p = e^{2x}$

Here, $f(D) = D^2 - 4D + 4 = (D-2)(D-2)$

$a = 2$

$f(a) = (a-2)(a-2) = 0$

So, it is type II bcz $f(a) = 0$

Now, $(D^2 - 4D + 4) y_p = e^{2x}$

$$\Rightarrow y_p = \frac{1}{D^2 - 4D + 4} e^{2x}$$

$$= \frac{1}{(D-2)(D-2)} e^{2x}$$

$$= e^{2x} \frac{1}{(D-2+2)(D-2+2)}$$

$$= e^{2x} \frac{1}{D \times D}$$

$$= e^{2x} \frac{1}{D^2}$$

$$= e^{2x} \iint \left[1 \, dx \right] dx$$

$$= e^{2x} \int x \, dx$$

$$= e^{2x} \times \frac{x^2}{2}$$

So, $y = C_1 e^{2x} + C_2 x e^{2x} + \frac{x^2 e^{2x}}{2}$

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Lecture 7

Variation of parameters

y_p will follow $y_c = c_1y_1 + c_2y_2$ — (3), where y_1 and y_2 are the fundamental solutions of $\underline{y'' + P(x)y' + Q(x)y = 0}$ —— (2)

Standard form → no term with y''

So, $y_p = u_1y_1 + u_2y_2$ — (4), u_1 and u_2 are func^n of x. It is a solution of main ODE, $y'' + P(x)y' + Q(x)y = f(x)$ — (1)

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

y_p'

y_p''

Substituting y_p, y_p', y_p'', we get equality,

$y_1u_1' + y_2u_2' = 0$

$y_1'u_1' + y_2'u_2' = f(x)$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

$\Rightarrow \quad AX = B$

Using Crammer's rule, Solution vector $\left\{ X = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} \right.$ can be found as,

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$$u_1' = \frac{|A_1|}{|A|} \qquad u_2' = \frac{|A_2|}{|A|}$$

$$A_1 = \begin{pmatrix} 0 & y_2 \\ f(x) & y_2' \end{pmatrix} \qquad A_2 = \begin{pmatrix} y_1 & 0 \\ y_1' & f(x) \end{pmatrix}$$

$$u_1 = \int u_1' \, dx \qquad u_2 = \int u_2' \, dx$$

$$\text{Wronskian}, \; W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

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Lecture 8

Cauchy - Euler equatn (Higher order ODEs with variable Co-eff)

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$a_n(x), a_{n-1}(x), \ldots\ldots a_1(x)$ are constant / func^n of x.

Above is n^{th} order ODE of Constant Co-efficient

~~$a_n(x)$~~

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

Here, $a_n, a_{n-1}, \ldots, a_1$ are constants. Also called

Cauchy-euler eqn. It is n^{th} order ODE of variable coefficient.

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + y = e^x \rightarrow \text{variable co-eff.}$$

$$2x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = e^x \longrightarrow \text{Constant co-eff.}$$

2nd order Cauchy-euler eqn

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = g(x) \quad \text{---} \quad (1)$$

Let, $y = y_c + y_p$ as $g(x) \neq 0$

For y_c: $ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0 \quad \text{---} \quad (2)$

Let, the trial solution of (2) be $y = x^\lambda$

$$\Rightarrow y' = \lambda x^{\lambda-1}$$

$$\Rightarrow y'' = \lambda(\lambda-1) x^{\lambda-2}$$

Substituting all these in (2),

$$ax^2 \times \lambda(\lambda-1) x^{\lambda-2} + bx \cdot \lambda x^{\lambda-1} + cx^\lambda = 0$$

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$\Rightarrow a\lambda(\lambda-1)x^\lambda + b\lambda x^\lambda + Cx^\lambda = 0$

$\Rightarrow x^\lambda [a\lambda^2 - a\lambda + b\lambda + C] = 0$

$\Rightarrow a\lambda^2 + (b-a)\lambda + C = 0 \quad [\because x^\lambda \neq 0] \rightarrow$ auxiliary eqn / characteristic eqn

$$\Rightarrow \lambda = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4aC}}{2a}$$

Type 1: λ_1, λ_2 are real and $\lambda_1 \neq \lambda_2$

$Y_1 = x^{\lambda_1}$ and $Y_2 = x^{\lambda_2}$ are fundamental solution of auxiliary eqn

$Y = C_1 Y_1 + C_2 Y_2 = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$ → general soln

Type 2: λ_1, λ_2 are real and $\lambda_1 = \lambda_2$

$Y_1 = x^\lambda$ and $Y_2 = (\ln x)x^\lambda$

$Y = C_1 Y_1 + C_2 Y_2 = C_1 x^\lambda + C_2 (\ln x) x^\lambda$

Type 3: λ_1, λ_2 are complex

$\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - i\beta$

$Y = x^\alpha [C_1 \cos\beta(\ln x) + C_2 \sin\beta(\ln x)]$

Particular solution is finding and replacing the constant's value.

(A) $x^2y'' - 3xy' + 3y = 2x^4e^x$

$y = y_c + y_p$

For y_c: Let the trial solution $y = x^\lambda \Rightarrow y' = \lambda x^{\lambda-1}$

$\Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2}$

$x^2\lambda(\lambda-1)x^{\lambda-2} - 3x \times \lambda x^{\lambda-1} + 3x^\lambda = 0$

$\Rightarrow \lambda(\lambda-1)x^\lambda - 3\lambda x^\lambda + 3x^\lambda = 0$

$\Rightarrow \lambda^2 - \lambda - 3\lambda + 3 = 0 \quad [\because x^\lambda \neq 0]$

$\Rightarrow \lambda(\lambda-1) - 3(\lambda-1) = 0$

$\Rightarrow (\lambda-3)(\lambda-1) = 0$

$\Rightarrow \lambda = 1, 3$ Fundamental solution is $y_1 = x$ and $y_2 = x^3$

So, $y_c = C_1x + C_2x^3$ is general solution

For y_p: $y_p = u_1y_1 + u_2y_2$, here $y_1 = x$, $y_2 = x^3$

$\Rightarrow y_1' = 1$, $y_2' = 3x^2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3$$

$x^2y'' - 3xy' = 2x^4e^x \Rightarrow y'' - \frac{3}{x}y' = 2x^2e^x$

$$u_1' = \frac{\begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix}}{2x^3} = -\cancel{2}x^{5-3}e^x = -\cancel{2}x^2e^x$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix}}{2x^3} = \cancel{2}x^{3-3}e^x = e^x$$

$$u_1 = \int u_1'\,dx = -\int x^2e^x\,dx$$

$$= -x^2\int e^x dx + \int\left[\frac{d}{dx}(x^2)\int e^x dx\right]dx$$

$$= -x^2e^x + \int 2x\,e^x dx$$

$$= -x^2e^x + 2x\,e^x - 2\int e^x dx$$

$$= -x^2e^x + 2xe^x - 2e^x \quad \cancel{= x^2e^x - 3e^x}$$

$$u_2 = \int u_2'\,dx = \int e^x dx = e^x$$

So, $y_p = (2xe^x - x^2e^x - 2e^x)x + e^x x^3$

$$= 2x^2e^x - \cancel{x^3e^x} - 2xe^x + \cancel{x^3e^x}$$

$$= 2x^2e^x - 2xe^x$$

$$y = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$$

Changing to constant co-efficient

(1) $x^2y'' - xy' + y = \ln x$ $\longrightarrow$ variable co-efficient

Sol^n: We want to rewrite it as an ODE of constant co-efficient.

For this, we substitute, $\underline{x = e^t \Rightarrow t = \ln x}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \cdot \frac{dy}{dt}$$

$$\Rightarrow \underbrace{x\frac{dy}{dx}}_{\text{variable co-eff.}} = \underbrace{\frac{dy}{dt}}_{\text{constant co-eff.}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt} \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x} \cdot \underline{\frac{d}{dt}\left(\frac{dy}{dx}\right)} + \frac{dy}{dt} \times \left(-\frac{1}{x^2}\right)$$

Continuous as e^t

so, $f_{xy} = f_{yx}$

$$= \frac{1}{x} \times \frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right) - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x^2} \times \frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

So, substituting

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} + y = t$$

$$\Rightarrow \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t \longrightarrow \text{Constant co-efficient}$$

CH $y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t} \Rightarrow y'' = \lambda^2 e^{\lambda t}$

$\lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow (\lambda - 1)^2 = 0 \quad \Rightarrow \lambda = 1$

$y_c = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

$y_p = At + B \Rightarrow y_p' = A \quad \Rightarrow y_p'' = 0$

$0 - 2A + At + B = t$

$\Rightarrow \cancel{-2} \; A = 1 \qquad B = 2A = 2$

$y_p = \cancel{2}\, t + 2$

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Lecture

Laplace transform

Laplace transform transfers *

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

t: time, real number

s: frequency complex number

Laplace transformation ($\mathcal{L}$)

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)}$$ → Improper integral

provided that the integral converges.

Here, e^{-st} is called the kernel of the laplace transformation.

Convergent integral

divergent integral

$\int_a^b f(x)\,dx = \infty$ (divergent)

$= P$ (any constant) Convergent

$\int_0^\infty f(x)\,dx = \lim_{A\to\infty} \int_0^A f(x)\,dx$

Example:

$f(t) = 1$. Evaluate laplace transformation

$\mathcal{L}\{f(t)\}$

Soln: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\infty e^{-st}\,dt$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\, dt$$

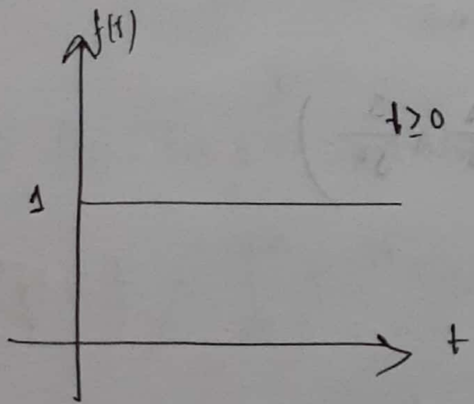
$$= \lim_{A\to\infty} \left[\frac{e^{-st}}{-s} \right]_0^A$$

$$= -\frac{1}{s} \lim_{A\to\infty} \left(e^{-sA} - e^0\right)$$

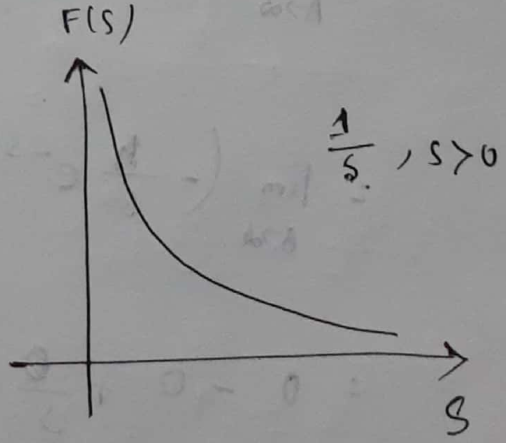
$$= -\frac{1}{s} \lim_{A\to\infty} \left(e^{-sA} - 1\right) \qquad \frac{1}{e^{\infty}} = 0$$

$$= -\frac{1}{s}(0-1) = \frac{1}{s}, \quad s > 0$$

If we take $s<0$, we can get ~~dive~~ ∞, which is divergent. As we want integral converges, we need $s>0$ to form laplace tranform.



You can't differentiate it, you can't analyze it.



You can differentiate it, You can analyze it.

If $f(t) = t$, Evaluate $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\, t\, dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st} t\, dt$$

$$= \lim_{A\to\infty} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^A$$

$$= \lim_{A\to\infty} \left(-\frac{A}{s} e^{-sA} - \frac{1}{s^2} e^{-sA} + \frac{A}{s} e^0 + \frac{1}{s^2} e^0 \right)$$

$$= \lim_{A\to\infty} \left(-\frac{A}{s} e^{-sA} - \frac{1}{s^2} e^{-sA} + \frac{A}{s} + \frac{1}{s^2} \right)$$

$$= 0 - 0 + \frac{0}{s} + \frac{1}{s^2}$$

$$= \frac{1}{s^2}, \quad s > 0$$

$$\int e^{-st} t\, dt$$

$$= t \int e^{-st} dt - \int \left[\frac{d}{dt}(t) \int e^{-st} dt \right] dt$$

$$= -\frac{t}{s} e^{-st} + \frac{1}{s} \int e^{-st} dt$$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}$$

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(#) $f(t) = e^{-3t}$, Evaluate $\mathcal{L}\{f(t)\}$

Sol^n: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{-3t} \, dt$

$$= \int_0^\infty e^{-(s+3)t} \, dt$$

$$= \lim_{A \to \infty} \int_0^A e^{-(s+3)t} \, dt$$

$$= \lim_{A \to \infty} \left[- \frac{e^{-(s+3)t}}{(s+3)} \right]_0^A$$

$$= \lim_{A \to \infty} - \frac{e^{-(s+3)A}}{s+3} + \frac{e^0}{s+3}$$

$$= \frac{1}{s+3} \quad , \ s+3 > 0 \implies s > -3$$

(#) $f(t) = \sin 2t$, Evaluate $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \sin 2t \, dt$$

$$= \lim_{A \to \infty} \int_0^A e^{-st} \sin 2t \, dt$$

$$\cancel{= \lim_{A \to \infty} \sin 2t \int_0^A e^{-st} dt - \int_0^A \left[\frac{d}{dt} (\sin 2t) \int e^{-st} \right] dt}$$

$$= \lim_{A\to\infty}\left[\sin 2t \times \frac{e^{-st}}{-s}\right]_0^A - \lim_{A\to\infty}\int_0^A\left[\frac{d}{dt}(\sin 2t)\int e^{-st}\,dt\right]dt$$

$$= \lim_{A\to\infty}\left(\sin 2A\,\frac{e^{-sA}}{-s} - 0\right) - \lim_{A\to\infty}\int_0^A 2\cos 2t\,\frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{2}{s}\lim_{A\to\infty}\int_0^A \cos 2t\, e^{-st}\,dt$$

$$= \frac{2}{s}\lim_{A\to\infty}\left[\cos 2t \times \frac{e^{-st}}{-s}\right]_0^A - \frac{2}{s}\lim_{A\to\infty}\int_0^A\left[\frac{d}{dt}(\cos 2t)\int e^{-st}dt\right]dt$$

$$= \frac{2}{s}\lim_{A\to\infty}\left(\cos 2A\,\frac{e^{-sA}}{-s} - \cos 0 \times \frac{e^{0}}{-s}\right) + \frac{2}{s^2}\int_0^A -2\sin 2t\, e^{-st}\,dt$$

$$= -\frac{2}{s^2}(0-1) + \lim_{A\to\infty}\frac{2}{s^2}\int_0^A -2\, e^{-st}\sin 2t\,dt$$

$$= \frac{2}{s^2} + -\frac{4}{s^2}\lim_{A\to\infty}\int_0^A e^{-st}\sin 2t\,dt$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\int_0^\infty e^{-st}\sin 2t\,dt$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} + \frac{4}{s^2}\mathcal{L}\{f(t)\} = \frac{2}{s^2}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2}{\cancel{s^2}} \times \frac{\cancel{s^2}}{s^2+4} = \frac{2}{s^2+4}, \quad s>0$$

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(A) $f(t) = \cos 2t$ Evaluate $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cos 2t \, dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st} \cos 2t \, dt$$

$$= \lim_{A\to\infty} \left[\cos 2t \times \frac{e^{-st}}{-s} \right]_0^A - \lim_{A\to\infty} \int_0^A \left[\frac{d}{dt}(\cos 2t) \int e^{-st} dt \right] dt$$

$$= \lim_{A\to\infty} \left(\cancel{\cos 2A \frac{e^{-sA}}{-s}}^{\to 0} - 1 \times \frac{e^{-0}}{-s} \right) - \lim_{A\to\infty} \int_0^A \left(+2\sin 2t \times \frac{e^{-st}}{+s} \right) dt$$

$$= 0 + \frac{1}{s} - \frac{2}{s} \lim_{A\to\infty} \int_0^A e^{-st} \sin 2t \, dt$$

$$= \frac{1}{s} - \frac{2}{s} \lim_{A\to\infty} \left[\sin 2t \times \frac{e^{-st}}{-s} \right]_0^A + \frac{2}{s} \lim_{A\to\infty} \int_0^A \left[\frac{d}{dt}(\sin 2t) \int e^{-st} dt \right] dt$$

$$= \frac{1}{s} - \frac{2}{s} \lim_{A\to\infty} \left(\sin 2A \frac{e^{-sA}}{-s} - 0 \right) + \frac{2}{s} \lim_{A\to\infty} \int_0^A 2\cos 2t \frac{e^{-st}}{-s} dt$$

$$= \frac{1}{s} - \frac{2}{s} \times 0 + \frac{2}{s} \times \frac{2}{-s} \lim_{A\to\infty} \int_0^A e^{-st} \cos 2t \, dt$$

$$= \frac{1}{s} - \frac{4}{s^2} \mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} + \frac{4}{s^2} \mathcal{L}\{f(t)\} = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s} \times \frac{s^2}{s^2+4} = \frac{s}{s^2+4}, \quad s > 0$$

$\boxed{12/3/19}$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $n>0$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$ $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$ $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$

Inverse Laplace transform

$\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}^{-1}$ (inverse laplace)

$\mathcal{L}\{1\} = \frac{1}{s}$ $\mathcal{L}\{t\} = \frac{1}{s^2}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ $\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}$

Properties:

$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

$= F(s) + G(s)$

$\mathcal{L}^{-1}\{F(s)+G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}$

$s^2Y(s) - sY(0) - Y'(0)$

$sY(s) - Y(0)$

Laplace tram formation for derivatives:

$f(t)$

$\downarrow$

$f'(t)$ – First derivative

$\downarrow$

$f''(t)$ – Second derivative

We know, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

Now, $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

Let, $f(t)$ is bounded func

$$= \lim_{A\to\infty} \int_0^A e^{-st} f'(t)\,dt$$

$$= \lim_{A\to\infty} \left[e^{-st} \int_0^A f'(t)\,dt - \int_0^A \left[\frac{d}{dt}(e^{-st}) \cdot \int f'(t)\,dt \right] dt \right]$$

$$= \lim_{A\to\infty} \left[e^{-st} f(t) \right]_0^A - \int_0^A -se^{-st} f(t)\,dt$$

$$= \lim_{A\to\infty} \left(e^{-sA} f(A) - e^0 f(0) \right) + s \lim_{A\to\infty} \int_0^A e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Also, $\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$

$$= \lim_{A\to\infty} \int_0^A e^{-st} f''(t)\,dt$$

$$= \lim_{A\to\infty} e^{-st} \int_0^A f''(t)\,dt - \int_0^A \left[\frac{d}{dt}(e^{-st}) \int f''(t)\,dt\right] dt$$

$$= \lim_{A\to\infty} \left[e^{-st} f'(t)\right]_0^A - \int_0^A -se^{-st} f'(t)\,dt$$

$$= \lim_{A\to\infty} \left[e^{-sA} f'(A) - e^0 . f'(0)\right] + s \lim_{A\to\infty} \int_0^A e^{-st} f'(t)\,dt$$

$$= 0 - f'(0) + s\,\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s\,(sF(s) - f(0))$$

$$= s^2 F(s) - sf(0) - f'(0)$$

Theorem:

Suppose n is the order of the derivative

$$\mathcal{L}\{f^n\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0)$$

$$- \ldots\ldots\ldots - f^{n-1}(0)$$

So, $\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$

(4) Solve, $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$ → if $y'(1) = 5$ we cannot use Laplace

$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$

$\Rightarrow s^2 Y(s) - s y(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{1}{s+4}$

$\Rightarrow s^2 Y(s) - s \times 1 - 5 - 3sY(s) + 3 \times 1 + 2Y(s) = \frac{1}{s+4}$

$\Rightarrow [s^2 - 3s + 2] Y(s) = \frac{1}{s+4} + 2 + 5$

$\Rightarrow (s-1)(s-2) Y(s) = \frac{s^2+6s+9}{(s+4)}$

$\Rightarrow Y(s) = \frac{s^2+6s+9}{(s+4)(s-1)(s-2)}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\right\}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{-\frac{16}{15(s-1)} + \frac{25}{6(s-2)} + \frac{1}{30(s+4)}\right\}$

$$= \underbrace{-\frac{16}{15}e^{t} + \frac{25}{6}e^{2t}}_{y_c} + \underbrace{\frac{1}{30}e^{-4t}}_{y_p}$$

$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2}$

$\Rightarrow A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1) = s^2 + 6s + 9$

~~$\Rightarrow A \times -4 \times$~~

$s = -4$, $A \times -5 \times -6 = 16 - 24 + 9$

$\Rightarrow A = \frac{1}{30}$

$s = 1$, $B = -\frac{16}{15}$

$s = 2$, $C = \frac{25}{6}$

Quiz 2 → Laplace

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$\boxed{17/3/19}$

Assessment → April 2 ⟶ Lesson 8, 9, 12, 13, 14, 17

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Properties: Translation on the s-axis

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, a is real number

$\mathcal{L}\{t\} = \frac{1}{s^2}$ $\quad \mathcal{L}\{e^{2t} f(t)\} = \frac{1}{(s-2)^2}$

$\mathcal{L}\{e^{3t} f(t)\} = \frac{1}{s^2}\Big|_{s \to s-3}$

$= \frac{1}{(s-3)^2}$

$\mathcal{L}\{e^t \sin 2t\} = \frac{2}{s^2+4}\Big|_{s \to s-1}$

$= \frac{2}{(s-1)^2+4}$

$\mathcal{L}\{e^{-4t} t^3\} = \frac{3!}{s^4}\Big|_{s \to s+4}$

$= \frac{3!}{(s+4)^4}$

$\mathcal{L}\{e^{-t} t^2\} = \frac{2!}{s^3}\Big|_{s \to s+1}$

$= \frac{4}{(s+1)^3}$

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$$\mathcal{L}^{-1}\left\{\frac{2!}{(s+1)^3}\right\} = e^{-t}t^2$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$= \frac{1}{2}e^{-t}\sin 2t$$

$$\cancel{\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}}$$

$$\cancel{= 2\mathcal{L}^{-1}\left\{\frac{s}{s-3}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}}$$

$$\cancel{= \mathcal{L}^{-1}\left\{\frac{2s}{(s-3)^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}}$$

$$\cancel{+5\,\mathcal{L}\{e^{3t}t^2\}}$$

$$\cancel{\frac{2s+5}{(s-3)(s-3)} = \frac{A}{s-3} + \frac{B}{s-3}}$$

$$\Rightarrow \; = 2\frac{s}{s-3} + \frac{5}{s-3}$$

$$\cancel{2s+5 = A(s-3)}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-3} + \frac{11}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11 \cdot e^{3t}t$$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$\Rightarrow 2s+5 = A(s-3) + B$$

$s=3$, $B=11$

$s=0$, $A=2$

Ex-3

$$y'' - 6y' + 9y = t^2 e^{3t} \qquad y(0) = 2 \;, \; y'(0) = 17$$

$$\Rightarrow \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) - [6\, sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s^2 - 6s + 9)\, Y(s) - s \times 2 - 17 + 6 \times 2 = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s-3)(s-3)Y(s) = \frac{2}{(s-3)^3} + 2s + 5$$

$$\Rightarrow Y(s) = \frac{2}{(s-3)^5} + \frac{2s+5}{(s-3)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\} + \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{(s-3)} + \frac{B}{(s-3)^2}$$

$$= \frac{2}{s-3} + \frac{11}{(s-3)^2}$$

$$\Rightarrow Y(t) = \frac{2}{4!} \times e^{3t}t^4 + 2e^{3t} + 11e^{3t}t$$

$$= \frac{1}{12} e^{3t}t^4 + 2e^{3t} + 11e^{3t}t$$

$\boxed{\text{Lecture 11}}$

System of First order ODE

System of Linear equation has boundary / mean common area.

$\frac{dP}{dD} = 3P - 2D \qquad \frac{dP}{ds} = S + 3P$ How P is changing w.r.t D and S ?

In this case, we need system of differential eqn.

(**) { Eigen value
Eigen vector

Matrix Form of a linear System

$$\frac{dD}{dP} = D + S - 2t \qquad \frac{dS}{dP} = D - S + 3P$$

$$A = \begin{array}{c} \begin{array}{cc} D & S \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array} \qquad F = \begin{array}{c} P \\ \begin{pmatrix} -2P \\ 3P \end{pmatrix} \end{array} \qquad X = \begin{pmatrix} D \\ S \end{pmatrix} \rightarrow \text{Solution vector}$$

Co-efficient matrix (independent variables) — Input

$$\dot{X} = \begin{pmatrix} \frac{dD}{dP} \\ \frac{dS}{dP} \end{pmatrix}$$

If input $= 0$, system is homogenous

input $\neq 0$, system is non-homogenous.

$$\therefore \dot{X} = AX + F$$

(a) $\frac{dx}{dt} = 2x + 4y - t$ $A = \begin{pmatrix} 2 & 4 \\ 5 & -7 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$\frac{dy}{dt} = 5x - 7y$ $\mathring{X} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ $F(t) = \begin{pmatrix} -t \\ 0 \end{pmatrix}$

$$\mathring{X} = AX + F$$

Two

- Vectors are linearly ~~in~~ dependent -

(a) Both of them are same (b) One is scalarly multiple of other.

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 5 \\ 6 & 10 \end{pmatrix}$$

$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ eigen value : λ_1, λ_2

Char. eq^n : $|a - \lambda I| = 0$

λ_1 λ_2

eign vector (K_1) eign vector (K_2)

$(A-\lambda I)K_1 = 0$ $(A-\lambda I)K_2 = 0$

$X_1 = K_1 e^{\lambda_1 t}$ $X_2 = K_2 e^{\lambda_2 t}$ (Fundamental sol^n)

General sol^n : $X = C_1 X_1 + C_2 X_2 = C_1 K_1 e^{\lambda_1 t} + C_2 K_2 e^{\lambda_2 t}$

① Eigen ~~value~~ values vectors represents the energy function i.e how much energy is passing through your system.

Solve: $\frac{dx}{dt} = 2x + 3y$

$\frac{dy}{dt} = 2x + y$

Solution: Co-efficient vector, $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

~~Solution vector~~, $\dot{X} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ Solution Vector, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$\dot{X} = AX$ it is homogenous as $f(t) = 0$

For the eigen values of A,

$|A - \lambda I| = 0$

$A - \lambda I = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$= \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}$

$|A - \lambda I| = (2-\lambda)(1-\lambda) - 6$

~~$\Rightarrow 2 (2-\lambda$~~

$\Rightarrow (2-\lambda)(1-\lambda) - 6 = 0$

$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow \lambda^2 - 4\lambda + \lambda - 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda + 1) = 0 \quad \Rightarrow \lambda = 4, -1$

Say, $\lambda_1 = -1$ and $\lambda_2 = 4$

For $\lambda_1 = -1$, we seek the eigen vector K_1 as

$$(A - \lambda_1 I) K_1 = 0$$

$$\Rightarrow \begin{pmatrix} 2-\lambda_1 & 3 \\ 2 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-(-1) & 3 \\ 2 & 1-(-1) \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3P_1 + 3P_2 \\ 2P_1 + 2P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \Rightarrow 3P_1 + 3P_2 = 0 \quad \Rightarrow P_1 + P_2 = 0 \\ 2P_1 + 2P_2 = 0 \quad \Rightarrow P_1 + P_2 = 0 \end{array}\right\} \quad P_1 + P_2 = 0$$

Therefore, free variables exist (as eqn < vars), let, $P_2 = 1$

$$\therefore P_1 + 1 = 0 \quad \Rightarrow P_1 = -1$$

$$\therefore K_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Fundamental solution, $X_1 = K_1 e^{\lambda_1 t}$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-1 \cdot t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 4$, we seek the eigen vector K_2

$$(A - \lambda_2 I) K_2 = 0$$

$$\Rightarrow \begin{pmatrix} 2-4 & 3 \\ 2 & 1-4 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2P_1 + 3P_2 \\ 2P_1 - 3P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -2P_1 + 3P_2 = 0 \quad \text{---} (i) \\ 2P_1 - 3P_2 = 0 \quad \cancel{\Rightarrow P_1 = P} \text{---} (ii) \end{array}\right\} 2P_1 - 3P_2 = 0$$

~~(i)+(ii) => $P_1 = 0$ (ii) => $0 - 3P_2 = 0$ => $P_2 = 0$~~

Therefore, free variables exist. Let $P_2 = 1$

$$\therefore 2P_1 - 3 = 0 \Rightarrow P_1 = \frac{3}{2}$$

$$\therefore K_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

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Fundamental solution, $X_2 = K_2 e^{\lambda_2 t}$

$$= \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} e^{4t}$$

General solution, $X = C_1 X_1 + C_2 X_2$

$$= C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} e^{4t}$$ [with 1 as free variable]

If $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So, $t = 0$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-0} + C_2 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} e^{0}$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + \frac{3}{2} C_2 \\ -C_1 + C_2 \end{pmatrix}$$

$\Rightarrow C_1 + \frac{3}{2} C_2 = 1$

$-C_1 + C_2 = +1$

$\Rightarrow C_2 = \frac{4}{5}$ 1

$C_1 = -\frac{1}{5}$ $\cancel{2}$

Example 2. Solve, $\frac{dx}{dt} = -4x + y + z$, $\frac{dy}{dt} = x + 5y - z$, $\frac{dz}{dt} = y - 3z$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-4-\lambda)\left[(5-\lambda)(-3-\lambda)+1\right] - 1(-3-\lambda) + 1(1-0) = 0$$

$$\Rightarrow (-4-\lambda)(5-\lambda)(3-\lambda) \cancel{-4} \cancel{-\lambda} \cancel{+3} \cancel{+\lambda} \cancel{+1} = 0$$

$$\Rightarrow (-4-\lambda)(5-\lambda)(3-\lambda) = 0$$

$\Rightarrow$ ~~For~~ So, $\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = 5$

$-4-(-3)$

For, $\lambda_1 = -3$, we seek eigen vector K_1

$$(A - \lambda_1 I) K_1 = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 1 & 8 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -p_1 + p_2 + p_3 \\ p_1 + 8p_2 - p_3 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$\Rightarrow -P_1 + P_2 + P_3 = 0 \quad — (i)$

$P_1 + 8P_2 - P_3 = 0 \quad — (ii)$

$P_2 = 0 \quad — (iii)$

$\left.\begin{array}{l} (i) \Rightarrow -P_1 + P_3 = 0 \\ (ii) \Rightarrow P_1 - P_3 = 0 \end{array}\right\} \begin{array}{l} P_1 - P_3 = 0 \\ P_2 = 0 \end{array}$

So, let, $P_3 = 1$ as free variable exists.

so, $P_1 - 1 = 0 \Rightarrow P_1 = 1$

$$K_1 = \begin{pmatrix} 1 \\ \cancel{1}0 \\ \cancel{0}1 \end{pmatrix}$$

fundamental solution, $X_1 = \begin{pmatrix} 1 \\ \cancel{1}0 \\ \cancel{0}1 \end{pmatrix} e^{-3t}$

Similarly for $\lambda_2 = -4$, we get eigen vector,

$K_2 = \begin{pmatrix} -10 \\ 1 \\ -1 \end{pmatrix}$ fundamental solⁿ, $X_2 = \begin{pmatrix} -10 \\ 1 \\ -1 \end{pmatrix} e^{-4t}$

Similarly, for $\lambda_3 = 5$, we get eigen vector,

$$K_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$$ So, Fundamental Solⁿ; $X_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}$

General solution, $X = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} -10 \\ 0 \\ 1 \end{pmatrix} e^{-4t}$

$$+ C_3 \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}$$

31/3/19

Assignment + Assessment = 7/4/19

5 ~~[illegible]~~ short question +2 Solution

TA - 01714872286 - Kapsain

SA - 10:00 - 2:00, 5:00 - 6:00

S - 8:00 - 11:00, 5:00 - 6:00

M - 8:00 - 2:00 T - 8:00 - 11:00

Repeated Eigen value

Solve: $X' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} X$

$$A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-9-\lambda) + 36 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda+3)^2 = 0$$

$$\Rightarrow \lambda_1 = -3, \ \lambda_2 = -3$$

For, $\lambda_1 = -3$, we get eigen vector K_1

$$(A - \lambda_1 I) K_1 = 0$$

$$\Rightarrow \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 6P_1 - 18P_2 &= 0 \\ 2P_1 - 6P_2 &= 0 \end{aligned}$$

$$\Rightarrow P_1 - 3P_2 = 0$$

Let $P_2 = 1$, so, $K_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Fundamental solution, $X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$

For $\lambda_2 = -3$, we get eigen vector K_2

$$(A - \lambda_2 I) K_2 = K_1$$

$$\Rightarrow \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 6l_1 - 18l_2 &= 3 \\ 2l_1 - 6l_2 &= 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} l_1 - 3l_2 &= \frac{1}{2} \\ l_1 - 3l_2 &= \frac{1}{2} \end{aligned}$$

$$\Rightarrow l_1 - 3l_2 = \frac{1}{2}$$

Let, $P_2 = 1$, So, $K_2 = \begin{pmatrix} \frac{7}{2} \\ 1 \end{pmatrix}$

Fundamental solution,

$$X_2 = \begin{pmatrix} \frac{7}{2} \\ 1 \end{pmatrix} e^{-3t} + t \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$$

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So, the General solution is

$$X = C_1 X_1 + C_2 X_2$$

$$= C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + C_2 \left[\begin{pmatrix} 7 \\ 2 \end{pmatrix} e^{-3t} + t \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} \right]$$

$\boxed{\text{Lecture 12}}$

Non homogenous linear system

$F(t) \neq 0$

So, $\mathring{X} = AX + F(t)$

Solve: $X' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X + \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}$

$$X_c = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$$

For X_p: $F(t) = \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix} = \begin{pmatrix} 6t \\ -10t \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$$= \begin{pmatrix} 6 \\ -10 \end{pmatrix} t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\downarrow \qquad \downarrow$$

$$A t + B$$

Let, $x_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\Rightarrow x_p' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + 0 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Substitute we get, $x_p' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} x_p + F(t)$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right] + \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix} + \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 6a_1 t + 6b_1 + a_2 t + b_2 + 6t \\ 4a_2 t + 4b_2 + 3a_2 t + 3b_2 - 10t + 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} t(6a_1 + a_2 + 6) + 6b_1 + b_2 \\ t(4a_1 + 3a_2 - 10) + 4b_1 + 3b_2 + 4 \end{pmatrix}$$

$\Rightarrow$

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Equating both sides,

$$6a_1 + a_2 + 6 = 0$$

$$4a_1 + 3a_2 - 10 = 0$$

$$\Rightarrow a_1 = -2$$

$$a_2 = 6$$

$$6a_1 + a_2 = -6 \quad \text{--- (i)}$$

$$4a_1 + 3a_2 = 10 \quad \text{--- (ii)}$$

$$\text{(i)} \times 4 \Rightarrow 24a_1 + 4a_2 = -24$$

$$\text{(ii)} \times 6 \Rightarrow 24a_1 + 18a_2 = 60$$

$$(-) \qquad -14a_2 = -84$$

$$\Rightarrow a_2 = 6$$

$$\text{(i)} \Rightarrow 6a_1 + 6 = -6$$

$$\Rightarrow a_1 = -2$$

$$6b_1 + b_2 = a_1 = -2$$

$$4b_1 + 3b_2 + 4 = a_2 = 6$$

$$\Rightarrow b_1 = -\frac{4}{7}$$

$$b_2 = \frac{10}{7}$$

$$6b_1 + b_2 = -2 \quad \text{--- (iii)}$$

$$4b_1 + 3b_2 = 6 - 4 = 2 \quad \text{--- (iv)}$$

$$\text{(iii)} \times 3 \Rightarrow 18b_1 + 3b_2 = -6$$

$$\text{(iv)} \Rightarrow 4b_1 + 3b_2 = 2$$

$$(-) \qquad 14b_1 = -8$$

$$\Rightarrow b_1 = -\frac{8}{14} = -\frac{4}{7}$$

$$\text{(iii)} \Rightarrow \cancel{6}^{3} \times -\frac{8}{\cancel{14}_{7}} + b_2 = -2$$

$$\Rightarrow b_2 = \frac{24}{7} - 2 = \frac{10}{7}$$

$$\text{So, } x_p = \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}$$

General solution, $x = x_c + x_p$

10.(a)

$$I' = AI + F(t)$$

$$\Rightarrow \begin{pmatrix} \frac{di_2}{dt} \\ \frac{di_3}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L_1} & -\frac{R_1}{L_1} \\ -\frac{R_1}{L_2} & -\frac{R_1+R_2}{L_2} \end{pmatrix} \begin{pmatrix} i_2 \\ i_3 \end{pmatrix} + \begin{pmatrix} \frac{E}{L_1} \\ \frac{E}{L_2} \end{pmatrix}$$

$$\Rightarrow I' = \begin{pmatrix} -2 & -2 \\ -2 & -5 \end{pmatrix} I + \begin{pmatrix} 60 \\ 60 \end{pmatrix}$$

Complimentary Solution $I_c = C_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t}$ → Fundamental solution

Let, $I_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \Rightarrow I_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So, Particular Integral, $I_p = \begin{pmatrix} 30 \\ 0 \end{pmatrix}$

General Solution $I = C_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t} + \begin{pmatrix} 30 \\ 0 \end{pmatrix}$

$$\left.\begin{matrix} i_2(0) = 0 \\ i_3(0) = 0 \end{matrix}\right\} I(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$C_1 = -12$

$C_2 = -6$

Particular solution, $J = -12\begin{pmatrix}2\\-1\end{pmatrix}e^{-t} + -6\begin{pmatrix}1\\2\end{pmatrix}e^{-6t} + \begin{pmatrix}30\\0\end{pmatrix}$

$i_1 = i_2 + i_3$

$= -24e^{-t} - 6e^{-6t} + 30 + 12e^{-t} + -12e^{-6t} + 0$

$= -12e^{-t} - 18e^{-6t} + 30$

$\boxed{2/4/19}$

$\boxed{\text{Lecture 13}}$

Power series Solution of ODE

Series — 1, 2, 3, ...

Power series — x^0, x^1, x^2 . . . . (power of x)

~~Taylor~~ Maclaurine series:

$$\sum_{n=0}^{\infty} C_n x^n = C_0 x^0 + C_1 x^1 + C_2 x^2 + C_3 x^3 + \dots + C_n x^n + \dots$$

power of variable

indexing

$$= C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

(power series of x)

Taylor Series:

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 (x-a)^0 + C_1 (x-a)^1 + C_2 (x-a)^2 + \dots + C_n (x-a)^n + \dots$$

$$= C_0 + C_1 (x-a) + C_2 (x-a)^2 + \dots + C_n (x-a)^n + \dots$$

[power series of (x-a)]

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Every differential equation has either maclaurine type solution or taylor type solution.

Analytical function - Continuous..

An analytical differential equation ⇒ Maclaurine type

Not " " " ⇒ Taylor type

$y' + x = 0 \longrightarrow$ Analytical $\Longrightarrow$ Maclaurine

$y' + \frac{1}{x} = 0 \longrightarrow$ not analytical $\Longrightarrow$ Taylor

$y' + \frac{1}{x-1} = 0 \longrightarrow$ not analytical $x=1 \Longrightarrow$ Taylor

power is different

$$\sum_{n=0}^{\infty} C_n x^{n+1} + \sum_{n=1}^{\infty} C_n n x^{n-1}$$

Indexing is different

If we add/sub two power series, we need
(i) Unique power
(ii) Unique indexing

Let, $n+1 = k$
$n = k-1$
when $n=0$, then $k = 0+1$
$= 1$

Let, $n-1 = k$
$n = k+1$
When $n=1$, then $k = 1-1 = 0$

$$\sum_{k=1}^{\infty} C_{k-1} x^k$$

$$\sum_{k=0}^{\infty} C_{k+1} (k+1) x^k$$

power is same

$$\sum_{k=1}^{\infty} C_{k-1} x^k + C_{0+1}(0+1)x^0 + \sum_{k=1}^{\infty} C_{k+1}(k+1)x^k$$

$$= C_1 + \sum_{k=1}^{\infty} \left[C_{k-1} + C_{k+1}(k+1)\right]x^k$$

<u>Solve:</u> $\displaystyle\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1}$

Let, $n-2=k$

$\Rightarrow n = k+2$

When $n=2$, then $k=0$

Let, $n+1=k$

$\Rightarrow n = k-1$

When $n=0$, then $k=1$

$$\sum_{k=0}^{\infty} (k+2)(k+1)C_{k+2} x^k + \sum_{k=1}^{\infty} C_{k-1} x^k$$

$$= \sum_{k=1}^{\infty} (k+2)(k+1)C_{k+2} x^k + \sum_{k=1}^{\infty} C_{k-1} x^k + (0+2)(0+1)C_{0+2} x^0$$

$$= \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} + C_{k-1}\right]x^k + 2C_2$$

Solve: $y = \sum_{n=0}^{\infty} C_n x^n$ $\qquad y' + y = 0$ — (1)

Let, $y = \sum_{n=0}^{\infty} C_n x^n$ is a solution of (1). We take maclaurine type as it is analytical

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$

$$(1) \Rightarrow \sum_{n=1}^{\infty} C_n n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$$

Let, $n-1 = k$ $\qquad$ Let, $n = k$

$\Rightarrow n = k+1$ $\qquad$ When $n=0$, then $k=0$

When $n=1$, then $k=0$

$$\Rightarrow \sum_{k=0}^{\infty} C_{k+1}(k+1) x^k + \sum_{k=0}^{\infty} C_k x^k = 0.$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[C_{k+1}(k+1) + C_k\right] x^k = 0$$

As, $x^k \neq 0$, so, $C_{k+1}(k+1) + C_k = 0$ for $k = 0, 1, 2 \ldots$

$$\Rightarrow C_{k+1} = -\frac{C_k}{k+1}$$

$k=0, \quad C_1 = -C_0$

$k=1, \quad C_2 = -\frac{1}{2} C_1 = \frac{1}{2} C_0 = \frac{1}{2!} C_0$

$k=2, \quad C_3 = -\frac{1}{3} C_2 = -\frac{1}{6} C_0 = \frac{-1}{3!} C_0$

$C_4 = \frac{1}{4!} C_0$

$C_5 = -\frac{1}{5!} C_0$

So, $y = \sum_{C_n}^{\infty} C_n x^n$

$= C_0 x^0 + C_1 x^1 + C_2 x^2 + \dots + C_n x^n + \dots$

$= C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n + \dots$

$= C_0 - C_0 x + \frac{1}{2!} C_0 x^2 - \frac{1}{3!} C_0 x^3 + \dots$

$= C_0 \left(1 - x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \dots \right)$

$= C_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

9/4/19

Zill
- 3.1 – Linear eqn and modeling – 1 qu
- 6.1, 6.2 – Series soln – 1 qu
- 7.1, 7.2, 7.3 – Laplace, inverse laplace, shifting – 1 qs.
- 8.3 – System of ODE (non homogeneous) – 1 qs.

11 – Fourier (Kreyszig') – 1 qsts.

May 2 Marks = 35

Answer ~~4/5~~ 5/6

Series solution ⇒ Quiz (Next Class) – 16/4/19

Solving Second order differentiation

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0$$

$$\Rightarrow \quad y'' + P(x)\, y' + Q(x)\, y = 0$$

A point is ordinary if it is valid for $P(x)$ and $Q(x)$

A point is singular if it is unvalid either for $P(x)$ OR ~~and~~ $Q(x)$ or both of them

For ordinary points, we take Maclaurine series

For singular points, we take Taylor series

$y'' + \frac{1}{x} y' + y = 0$

$x = 0$, $P(x)$ = undefined. So, $x=0$ is singular point and all other points are ordinary point.

$y'' + \frac{1}{x} y' + (1-x)y = 0$

$x = 1$ and 0 are two singular points.

Ex:

Solve, $y'' + xy = 0$ — (1)

Soln: The diff. eqn has no singular points, so we use Maclaurine series,

Let, $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution of (1)

$\Rightarrow y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$

$\Rightarrow y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$

$y'' + xy = 0$

$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} x c_n x^{n+1} = 0$$

Let $n+1 = k$ $\Rightarrow n = k-1$

When $n=0$, then $k = 1$

Let, $n-2 = k$

$\Rightarrow n = k+2$

When, $n=2$, then $k=0$

$$\Rightarrow \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k-1} x^{k} = 0$$

$$\Rightarrow \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k-1} x^k + 2c_2 = 0$$

$$\Rightarrow 2c_2 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) + c_{k-1}\right] x^k = 0$$

$\Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0$

as initiates from $k=1$

$c_{k+2}(k+2)(k+1) + c_{k-1} = 0$ for $k = 1, 2, 3, \ldots$

$$\Rightarrow c_{k+2} = -\frac{1}{(k+2)(k+1)} c_{k-1}$$

$k=1$, $c_3 = -\frac{1}{6} c_0$

C_2

$$k=2,\ \cancel{C_4 = -\frac{1}{12} C_1 = -\frac{1}{12\times 6} C_0 = \frac{1}{72} C_0}$$

$$k=3,\ \cancel{C_5 = -\frac{1}{20} C_4 = \frac{1}{72\times 20} C_0}$$

$$k=4,\ \cancel{C_6 = -\frac{1}{30} \times \frac{1}{72\times 20}}$$

$$k=2,\ C_4 = -\frac{C_1}{3\times 4}$$

$$k=3,\ C_5 = -\frac{1}{20} C_2 = -\frac{1}{20} \times 0 = 0$$

$$k=4,\ C_6 = -\frac{1}{30} \times C_3 = -\frac{1}{30} \times -\frac{1}{6} C_0 = \frac{1}{180} C_0$$

$$k=5,\ C_7 = -\frac{1}{42} C_4 = -\frac{1}{42} \times -\frac{1}{12} C_1 = \frac{1}{42\times 12} C_1$$

$$k=6,\ C_8 = -\frac{1}{10\times 9} C_5 = 0$$

$$k=7,\ C_9 = -\frac{1}{9\times 8} C_6 = -\frac{1}{72\times 180} C_0$$

$$k=8,\ C_{10} = -\frac{1}{10\times 9} C_7 = -\frac{1}{10\times 9 \times 42\times 12} C_1$$

Let, $y = \sum_{n=0}^{\infty} c_n x^n$

$$= c_0 + c_1 x + \cancel{c_2 x^2} + c_3 x^3 + c_4 x^4 + \cancel{c_5 x^5} + c_6 x^6 + c_7 x^7 + \cancel{c_8 x^8} +$$

$$c_9 x^9 + c_{10} x_{10} + \cdots\cdots\cdots$$

$$= c_0 + c_1 x - \frac{1}{2\times 3} c_0 x^3 - \frac{1}{3\times 4} c_1 x^4$$

$$+ \frac{1}{2\times 3\times 5\times 6} c_0 x^6 + \frac{1}{3\times 4\times 6\times 7} c_1 x^7$$

$$- \frac{1}{2\times 3\times 5\times 6\times 8\times 9} c_0 x^9 - \frac{1}{3\times 4\times 6\times 7\times 9\times 10} c_1 x^{10} + \cdots$$

$$= c_0 \left(1 - \frac{1}{2\times 3} x^3 + \frac{1}{2\times 3\times 5\times 6} x^6 - \frac{1}{2\times 3\times 5\times 6\times 8\times 9} x^9 + \cdots\right)$$

$$+ c_1 \left(x - \frac{1}{3\times 4} x^4 + \frac{1}{3\times 4\times 6\times 7} x^7 - \frac{1}{3\times 4\times 6\times 7\times 9\times 10} x^{10} + \cdots\right)$$

$$= c_0 y_1 + c_1 y_2$$

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Quiz 9 → Sunday

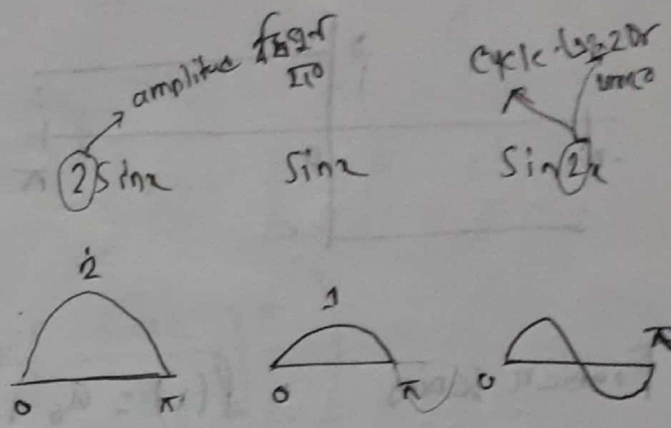
→ Fourier series

Lecture 10

Fourier series

Fourier series works with periodic function.

$$f(x+np)=f(x) \quad , n = \text{integers}.$$



$$f(x) \approx \sum_{n=1}^{\infty} a_0 + a_n \cos nx + b_n \sin nx$$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)} \rightarrow \text{period}, P = 2\pi$$

Where, a_0, a_n and b_n are called fourier co-efficients.

They are evaluated using Euler method:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\, dx + \int_{0}^{\pi} k\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left(\frac{k}{n}\left[\cos nx\right]_{-\pi}^{0} - \frac{k}{n}\left[\cancel{\sin nx}\cos nx\right]_{0}^{\pi}\right)$$

$$= \frac{k}{n\pi}\left(1-\cos n\pi\right) - \frac{k}{n\pi}\left(\cos n\pi - 1\right)$$

$$= \frac{2k}{n\pi}\left(1-\cos n\pi\right)$$

n is even, $\cos n\pi = 1$ So, $b_n = 0$

n is odd, $\cos n\pi = -1$ So, $b_n = \frac{4k}{n\pi}$

So, $f(x) = 0 + 0 + \sum_{n=1}^{\infty} b_n \sin nx$

$$= b_1\sin x + \cancel{b_2}^{0}\sin 2x + b_3\sin 3x + \cancel{b_4}^{0}\sin 4x + b_5\sin 5x + \cancel{b_6}^{0}\sin 6x + \cdots$$

$$= b_1\sin x + b_3\sin 3x + b_5\sin 5x + \cdots$$

We find the partial sums first to find the infinite sum.

$S_1 = b_1 \sin x = \frac{4k}{\pi} \sin x$

$S_2 = b_3 \sin 3x + S_1 = \frac{4k}{3\pi} \sin 3x + \frac{4k}{\pi} \sin x =$

$S_3 = b_5 \sin 5x + S_2 = \frac{4k}{5\pi} \sin 5x + \frac{4k}{3\pi} \sin 3x + \frac{4k}{\pi} \sin x$

$\boxed{20/4/19}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ ——— } (i)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

To get a_0, we integrate (i) w.r.t x over the limit $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \underbrace{\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos nx\,dx}_{\text{even functn}} + \underbrace{\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin nx\,dx}_{\text{odd functn}}$$

$$= a_0 [x]_{-\pi}^{\pi} + 0 + 0$$

$$= 2\pi a_0$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

To find a_m, we multiply (1) with $\cos mx$ and integrate over $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \underbrace{\int_{-\pi}^{\pi} a_0 \cos mx\,dx}_{\text{evenfunctn}} + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos nx \cos mx\,dx$$

$$+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin nx \cos mx\,dx$$

$\int_{-\pi}^{\pi} \sin nx \cos mx\,dx = 0$ [$n \neq m$ or $n = m$]

$\Rightarrow$

$$= \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx$$

$$= \sum_{n=1}^{\infty} a_n \cdot \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

$$= \sum_{n=1}^{\infty} a_n \pi, \text{ if } n = m$$

$$= a_m \pi$$

$$\Rightarrow a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$ [we can substitute m by n]

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To find b_n, we multiply (1) with $\sin mx$ and integrate over $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi} a_0 \sin mx\,dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos nx \sin mx\,dx$$

$$+\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin nx \sin mx\,dx$$

$$= 0 + \sum 0 + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx\,dx$$

$$= \sum_{n=1}^{\infty} b_n \cdot \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

$$= \sum_{n=1}^{\infty} b_n \pi \quad , \text{if } n = m$$

$$= b_m \pi$$

$$\Rightarrow b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \quad [\text{we can substitute } m \text{ by } n]$$

Fourier series for an arbitrary period:

$P = 2L \quad , \quad L = 1, 2, 3, \ldots\ldots, n$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}\right)x + b_n \sin\left(\frac{n\pi}{L}\right)x\right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}\right)x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}\right)x$$

Y axis reflection
$f(x) = f(-x)$
and Even functn

Solve: $f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$ $\quad P = 2L = 4 \quad , \quad L = 2$

$$a_0 = \frac{1}{2 \times 2} \int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{4}\left[\int_{-2}^{-1} 0 \times dx + \int_{-1}^{1} k\,dx + \int_{1}^{2} 0 \times dx\right]$$

$$= \frac{1}{4} \cdot \int_{-1}^{1} k\,dx$$

$$= \frac{k}{4} \left[x \right]_{-1}^{1}$$

$$= \frac{k}{4} \times [1 + 1]$$

$$= \frac{k}{2}$$

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{n\pi}{2}\right) x \, dx$$

$$= \frac{1}{2} \left[\int_{2}^{-1} 0 \times \cos \frac{n\pi}{2} x \, dx + \int_{-1}^{1} k \cos \frac{n\pi}{2} x \, dx + \int_{1}^{2} 0 \times \cos \frac{n\pi}{2} x \, dx \right]$$

$$= \frac{1}{2} k \int_{-1}^{1} \cos \frac{n\pi}{2} x \, dx$$

$$= \frac{k}{2} \times \frac{2}{n\pi} \left[\sin \frac{n\pi x}{2} \right]_{-1}^{1}$$

$$= \frac{k}{n\pi} \left[\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right] = \frac{2k}{n\pi} \sin \frac{n\pi}{2}$$

~~n = even, $a_n = 0$~~

~~n = odd, $a_n =$~~

$$\sin \frac{n\pi}{2} = \begin{cases} 1, & \text{if } n = 1, 5, 9, \ldots \\ -1, & \text{if } n = 3, 7, 11, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$a_n = \begin{cases} \frac{2k}{n\pi} & \text{if } n = 1, 5, 9, \ldots \\ -\frac{2k}{n\pi} & \text{if } n = 3, 7, 11, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin nx\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} k\sin\left(\frac{n\pi}{2}\right)x\,dx$$

$$= \frac{k}{2}\times\frac{2}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right)x\right]_{-1}^{1}$$

$$= \frac{k}{n\pi}\times 0 = 0$$

So, $$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{2}x + 0$$

$$= \frac{k}{2} + a_1\cos\frac{\pi}{2}x + \cancelto{0}{a_2\cos\pi x} + a_3\cos\frac{3\pi}{2}x + \cancelto{0}{a_4\cos\frac{4\pi}{2}x}$$

$$+ a_5\cos\frac{5\pi}{2}x + \cancelto{0}{a_6\cos\frac{6\pi}{2}x} + a_7\cos\frac{7\pi}{2}x + \cancelto{0}{a_8\cos\frac{8\pi}{2}x} +$$

$$a_9\cos\frac{9\pi}{2}x + \cdots$$

$$= \frac{k}{2} + \frac{2k}{\pi}\cos\frac{\pi}{2}x - \frac{2k}{3\pi}\cos\frac{3\pi}{2}x + \frac{2k}{5\pi}\cos\frac{5\pi}{2}x$$

$$- \frac{2k}{7\pi}\cos\frac{7\pi}{2}x + \cdots$$

Fourier even and odd funcⁿ

If the functn is even, there will exists even co-efficients

" " " odd, " " odd "

a_0 = even co-efficient

a_n = even [as Cosnx = even functⁿ]

b_n = odd [as Sinnx = odd functⁿ]

যদি domain change করে আর range change হয়, তাহলে odd functⁿ.

$\cos(-\theta) = \cos\theta$ তাই এইটা even functn।

যেমন $[-\pi, \pi]$ এইখানে $\cos(-\pi) = \cos\pi$

$\cos(\pi) = \cos\pi$

২য় ক্ষেত্রে $f(-\pi) = -K \neq f(\pi) = K$, So এইটা odd function.

Fourier Integral

$$f(x) = \int_0^\infty [A(\omega)\cos \omega x + B(\omega)\sin \omega x]\, d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos \omega v\, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin \omega v\, dv$$