3.) Let $x_1, \dots, x_n$ be a random sample from an exponential distribution with parameter θ.

a.) Find MLE for θ.

b.) Show that this estimator is an unbiased and consistent estimator.

c.) Find a sufficient statistic for parameter θ.

Solution:

a.)
$$f(x) = \frac{1}{\theta} e^{-\frac{1}{\theta}x}, \quad x>0$$

The Likelihood function is:

$$L = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\Sigma x_i}$$

$$\ln L = -n \ln \theta - \frac{1}{\theta} \Sigma x_i$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0$$

$$\Rightarrow -n\theta + \Sigma x_i = 0$$

$$\therefore \hat{\theta} = \bar{x}$$

b.) Unbiasedness:

$$E(\hat{\theta}) = E(\bar{x}) = E\left(\frac{\Sigma x_i}{n}\right) = \frac{1}{n} \cdot n\theta = \theta$$

Consistency:

$$V(\hat{\theta}) = V\left(\frac{\Sigma x_i}{n}\right) = \frac{1}{n^2} \cdot n\theta^2 = \frac{\theta^2}{n}$$

$$\therefore \lim_{n\to\infty} V(\hat{\theta}) = 0$$

$\therefore \hat{\theta} = \bar{x}$ is an unbiased and consistent estimator of θ.

c.) Sufficiency:

$$\prod_{i=1}^{n} f(x_i ; \theta) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i}$$

$$= g\left(\sum_{i=1}^{n} x_i ; \theta\right) h(x_1, \dots, x_n)$$

where, $g\left(\sum_{i=1}^{n} x_i ; \theta\right) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i}$

$h(x_1, \dots, x_n) = 1$

Hence, $\sum x_i$ is sufficient estimator for θ.

4.) Let $x_1, \ldots, x_n$ be a random sample from a poisson distribution with parameter λ.

a.) Find MLE for the parameter λ.

b.) Show that this estimator is an unbiased and consistent estimator.

c.) Find a sufficient statistic for parameter λ.

Solution:

a.)

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0, 1, 2, \ldots$$

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i}$$

$$\ln L = -n\lambda + \Sigma x_i \ln\lambda + \ln\left(\frac{1}{\prod_{i=1}^{n} x_i}\right)$$

$$\frac{\partial \ln L}{\partial \lambda} = -n + \frac{\Sigma x_i}{\lambda} = 0$$

$$\Rightarrow \frac{-n\lambda + \Sigma x_i}{\lambda} = 0$$

$$\Rightarrow -n\lambda + \Sigma x_i = 0$$

$$\therefore \hat{\lambda} = \bar{x}$$

b.) Unbiasedness:

$$E(\hat{\lambda}) = E(\bar{x}) = E\left(\frac{\Sigma x_i}{n}\right) = \frac{1}{n} \cdot n\lambda = \lambda$$

Consistency:

$$V(\hat{\lambda}) = V\left(\frac{\Sigma x_i}{n}\right) = \frac{1}{n^2} \cdot n\lambda = \frac{\lambda}{n}$$

$$\lim_{n \to \infty} V(\hat{\lambda}) = 0$$

c.) Sufficiency:

$$\prod_{i=1}^{n} f(x_i ; \theta) = \frac{e^{-n\lambda} \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i !}$$

$$= e^{-n\lambda} \lambda^{\Sigma x_i} \cdot \frac{1}{\prod_{i=1}^{n} x_i !}$$

$$= g(\Sigma x_i ; \theta) \; h(x_i, \ldots, x_n)$$

where, $g\left(\sum_{i=1}^{n} x_i , \theta\right) = e^{-n\lambda} \lambda^{\Sigma x_i}$

$$h(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i !$$

Hence, $\sum_{i=1}^{n} x_i$ is a sufficient estimator for λ.

1.) Let $x_1, \ldots, x_m$ be a random sample from a binomial distribution with parameter θ. Find MLE for θ.

Solution:

$$f(x) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$$

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{m} f(x_i;\theta)$$

$$= \prod_{i=1}^{m} \binom{n}{x_i} \theta^{x_i} (1-\theta)^{n-x_i}$$

$$= \theta^{\sum_{i=1}^{m} x_i} (1-\theta)^{nm - \sum_{i=1}^{m} x_i} \prod_{i=1}^{m} \binom{n}{x_i}$$

$$\ln L(\theta) = \sum_{i=1}^{m} x_i \ln\theta + \left(nm - \sum_{i=1}^{m} x_i\right) \ln(1-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{\sum_{i=1}^{m} x_i}{\theta} - \frac{nm - \sum_{i=1}^{m} x_i}{1-\theta} = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{m} x_i - \cancel{\theta \sum_{i=1}^{m} x_i} - nm\theta + \cancel{\theta \sum_{i=1}^{m} x_i}}{\theta(1-\theta)} = 0$$

$$\Rightarrow nm\theta = \sum_{i=1}^{m} x_i$$

$$\therefore \hat{\theta} = \frac{\sum_{i=1}^{m} x_i}{mn} = \frac{\overline{x}}{n}$$

Unbiasedness:

$$E(\hat{\theta}) = E\left(\frac{\bar{x}}{n}\right) = E\left(\frac{\sum_{i=1}^{m} x_i}{nm}\right)$$

$$= \frac{mn\theta}{mn} = \theta$$

Consistency:

$$V(\hat{\theta}) = V\left(\frac{\sum_{i=1}^{m} x_i}{nm}\right) = \frac{1}{n^2 m^2} \cdot nm\theta(1-\theta)$$

$$\lim_{n\to\infty} V(\hat{\theta}) = \lim_{n\to\infty} \frac{1}{nm}\theta(1-\theta) = 0$$

$\therefore \hat{\theta}$ is an unbiased and consistent estimator of θ.

Sufficiency:

$$\prod_{i=1}^{m} f(x_i;\theta) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i} \prod_{i=1}^{m}\binom{n}{x_i}$$

$$= g(\Sigma x_i;\theta)\, h(x_i,\ldots,x_n)$$

where, $g(\Sigma x_i;\theta) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$

$h(x_1,\ldots,x_n) = \prod_{i=1}^{m}\binom{n}{x_i}$

Hence, $\sum_{i=1}^{n} x_i$ is sufficient estimator for θ.

2.) $X_1, \dots, X_n$ be a random sample from a bernoulli distribution with parameter θ.

a.) Find MLE for θ.

b.) Show that this estimator is an unbiased and consistent estimator.

c.) Find a sufficient statistic for parameter θ.

Solution:

a.) $$f(x) = \theta^x (1-\theta)^{1-x}$$

The Likelihood function is:

$$L = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

$$\ln L = \sum_{i=1}^{n} x_i \ln \theta + \left(n - \sum_{i=1}^{n} x_i\right) \ln(1-\theta)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1-\theta} = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i - \theta \cancel{\sum_{i=1}^{n} x_i} - n\theta + \theta \cancel{\sum_{i=1}^{n} x_i}}{\theta(1-\theta)} = 0$$

$$\therefore \hat{\theta} = \bar{x}$$

b.) Unbiasedness:

$$E(\hat{\theta}) = E(\bar{x}) = E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{n\theta}{n} = \theta$$

Consistency:

$$V(\hat{\theta}) = V\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{1}{n^2} n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

$$\therefore \lim_{n \to \infty} V(\hat{\theta}) = 0$$

c) Sufficiency:

$$\prod_{i=1}^{n} f(x_i ; \theta) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

$$= g\left(\sum_{i=1}^{n} x_i ; \theta\right) h=(x_1, \ldots, x_n)$$

where, $g\left(\sum_{i=1}^{n} x_i ; \theta\right) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}$

$h(x_1, \ldots, x_n) = 1$

Hence, $\sum_{i=1}^{n} x_i$ is sufficient estimator for θ.