Chapter 3.1

Q.4. An archer hits a bull's-eye with a probability of 0.09, and the result of different attempts can be taken to be independent of each other. If the archer shoots nine arrows, calculate the probability that:

(a) Exactly two arrows score bull's-eyes.

(b) At least two arrows score bull's-eyes.

What is Expected number of bull's-eyes scored?

Ans.

Given that,

Probability of a bull's-eyes hit $P = 0.09$

Archer shoots arrows $n = 9$

(a) Probability that Exactly two arrows score bull's-eyes is -

$$P(X=2) = {}^{9}C_{2} \times (0.09)^{2} \times (1-0.09)^{9-2}$$

$$= \frac{9!}{2!\,7!} \times (0.09)^2 \times (0.91)^7$$

$$= \frac{9\times 8\times 7!}{2\times 1\times 7!} \times (0.09)^2 \times (0.91)^7$$

$$= 36 \times (0.09)^2 \times (0.91)^7$$

$$= 0.1506 \text{ Ans.}$$

(b) At least two arrows score bull's-eyes. Therefore, Probability is -

$$P(X\geq 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9)$$

$$P(X\geq 2) = 1 - P(X=1)$$

$$= 1 - \left[{}^{9}C_{1} \times (0.09)^1 \times (1-0.09)^{9-1} \right]$$

Q.5. A fair die is rolled eight times. Calculate the probability that there are:

(a) Exactly five even numbers

(b) Exactly one 6

(c) No 4s

Ans. When a fair die is rolled, function of all the situations, $S = \{1, 2, 3, 4, 5, 6\}$

Probability of a single number $= \frac{1}{6}$

Probability of even number is come at one time rolling $= \{2, 4, 6\} = \frac{3}{6} = \frac{1}{2}$

given that, fair die rolled eight times.

So $n = 8$

(a) Exactly five even numbers,

the probability mass function of $B(8, \frac{1}{2})$

$$P(X=5) = C_5^8 \times (0.5)^5 \times (0.5)^{8-5}$$

$$= \frac{8!}{5!\,3!} \times (0.5)^5 \times (0.5)^3$$

$$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} \times (0.5)^8$$

$$= 56 \times (0.5)^8$$

$$= 0.21875 \text{ Ans.}$$

(b) Exactly one 6.

Probability mass function of $B(8, \frac{1}{6})$

$$P(X=1) = C_1^8 \times \left(\frac{1}{6}\right)^1 \times \left(1 - \frac{1}{6}\right)^{8-1}$$

$$= \frac{8!}{1!\,7!} \times \frac{1}{6} \times \left(\frac{5}{6}\right)^7$$

$$= \frac{8 \times 7!}{1 \times 7!} \times \left(\frac{1}{6}\right)^8 \times (5)^7$$

$= 8 \times \left(\frac{1}{6}\right)^8 \times (5)^7$

$= 0.3721$ Ans

(c) No 4s.

Probablity mass function of $B(8, \frac{1}{6})$.

$P(X=0) = C_0^8 \times \left(\frac{1}{6}\right)^0 \times \left(1-\frac{1}{6}\right)^{8-0}$

$= \frac{8!}{0!\,8!} \times 1 \times \left(\frac{5}{6}\right)^8$

$= \left(\frac{5}{6}\right)^8$

$= 0.2326$ Ans.

Q.6. A multiple-Choice quiz consists of ten questions, each with five possible answer of which only one is correct. A student passes the quiz if seven or more correct answers are obtained. What is the probability that a student who guesses blindly at all of the questions will pass the quiz? What is the probability of passing the quiz if, on each question, a student can eliminate three incorrect answers and then guesses between the remaining two?

Ans. quiz consists of ten questions, so $n = 10$

each question have five choice and only one is correct, so probability of question is right $= \frac{1}{5} = 0.2$

if seven or more answer is correct, student will pass the quiz.

(a) Probability mass function of $B(10, 0.2)$

Probability that student guesses blindly and will pass the exam is,

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= C_7^{10} \times (0.2)^7 \times (1-0.2)^{10-7} + C_8^{10} \times (0.2)^8 \times (1-0.2)^{10-8} + C_9^{10} \times (0.2)^9 \times (1-0.2)^{10-9} + C_{10}^{10} \times (0.2)^{10} \times (1-0.2)^0$$

$$= \frac{10!}{7!\,3!} \times (0.2)^7 \times (0.8)^3 + \frac{10!}{8!\,2!} \times (0.2)^8 \times (0.8)^2 + \frac{10!}{9!\,1!} \times (0.2)^9 \times (0.8)^1 + \frac{10!}{10!\,0!} \times (0.2)^{10} \times 1$$

$$= \frac{10\times9\times8\times7!}{7!\times3\times2\times1}\times(0.2)^7\times(0.8)^3 + \frac{10\times9\times8!}{8!\times2\times1}\times(0.2)^8\times(0.8)^2 +$$

$$\frac{10\times9!}{9!\times1}\times(0.2)^9\times(0.8) + \frac{10!}{10!\times1}\times(0.2)^{10}$$

$$= (0.2)^7\left[120\times(0.8)^3 + 45\times(0.2)\times(0.8)^2 + 10\times(0.2)^2\times(0.8) + (0.2)^3\right]$$

$$= (0.2)^7\left[61.44 + 5.76 + 0.32 + 0.008\right]$$

$$= 67.528\times(0.2)^7 = 0.000864 \text{ Ans.}$$

(b) If student can eliminate three options, so probability that a question is correct $= \frac{1}{2} = 0.5$

Probability mass function of $B(10, 0.5)$

student will pass the quiz, therefore probability

$$P(X\geq7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= C_7^{10}\times(0.5)^7\times(0.5)^{10-7} + C_8^{10}\times(0.5)^8\times(0.5)^{10-8} + C_9^{10}\times(0.5)^9\times(0.5)^{10-9} + C_{10}^{10}\times(0.5)^{10}\times(0.5)^0$$

$$= \frac{10!}{7!\,3!}\times(0.5)^7\times(0.5)^3 + \frac{10!}{8!\,2!}\times(0.5)^8\times(0.5)^2 + \frac{10!}{9!\,1!}\times(0.5)^9\times(0.5)^1 + \frac{10!}{10!\,0!}\times(0.5)^{10}\times1$$

$$= (0.5)^{10}\left[\frac{10\times9\times8\times7!}{7!\times3\times2\times1} + \frac{10\times9\times8!}{8!\times2\times1} + \frac{10\times9!}{9!\times1} + \frac{10!}{10!\times1}\right]$$

$$= (0.5)^{10}\times\left[120 + 45 + 10 + 1\right]$$

$$= (0.5)^{10}\times176 = 0.171875 \text{ Ans.}$$

Q.9. A Company receives 60% of its order over the internet. Within a Collection of 18 independently Placed orders, what is the probability that

(a) between eight and ten of the orders are received over the internet?

(b) no more than four of the orders are recieved over the internet?

Ans.

Probability of receiving one order over the internet

is $P = \frac{60}{100} = 0.6$

Collection of independently placed order $n = 18$

(a) $X \sim B(18, 0.6)$

Probability that between eight and ten of the order are received over the internet is $= P(X=8) + P(X=9) + P(X=10)$

$$P(X=9) = C_9^{18} \times (0.6)^9 \times (1-0.6)^{18-9}$$

$$= \frac{18!}{9!\,9!} \times (0.6)^9 \times (0.4)^9$$

$$= \frac{18\times17\times16\times15\times14\times13\times12\times11\times10\times9!}{9\times8\times7\times6\times5\times4\times3\times2\times1\times 9!} \times (0.6)^9 \times (0.4)^9$$

$$= 48620 \times (0.6)^9 \times (0.4)^9$$

$$= 0.12844$$

$$P(X=8) = C_8^{18} \times (0.6)^8 \times (1-0.6)^{18-8}$$

$$= \frac{18!}{8!\,10!} \times (0.6)^8 \times (0.4)^{10}$$

$$= \frac{18\times17\times16\times15\times14\times13\times12\times11\times10!}{8\times7\times6\times5\times4\times3\times2\times1\times10!} \times (0.6)^8 \times (0.4)^{10}$$

$= 43758 \times (0.6)^8 \times (0.4)^{10}$

$= 0.07707$

$P(X=10) = C^{18}_{10} \times (0.6)^{10} \times (1-0.6)^{18-10}$

$= \frac{18!}{10!\,8!} \times (0.6)^{10} \times (0.4)^8$

$= 43758 \times (0.6)^{10} \times (0.4)^8$

$= 0.1734$

So, $P = P(X=8) + P(X=9) + P(X=10)$

$= 0.12844 + 0.07707 + 0.1734$

$= 0.37891$ Ans.

(b) Probability of no more than four of the orders are recieved is,

$P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= C^{18}_0 \times (0.6)^0 \times (0.4)^{18-0} + C^{18}_1 (0.6)^1 \times (0.4)^{18-1} + C^{18}_2 \times (0.6)^2 \times (0.4)^{18-2} + C^{18}_3 (0.6)^3 \times (0.4)^{18-3} + C^{18}_4 (0.6)^4 \times (0.4)^{18-4}$

$= \frac{18!}{0!\,18!} \times (0.4)^{18} + \frac{18!}{1!\,17!} \times (0.6) \times (0.4)^{17} + \frac{18!}{2!\,16!} \times (0.6)^2 \times (0.4)^{16} + \frac{18!}{3!\,15!} \times (0.6)^3 \times (0.4)^{15} + \frac{18!}{4!\,14!} \times (0.6)^4 \times (0.4)^{14}$

$= (0.4)^{14} \left[(0.4)^4 + 18 \times (0.6) \times (0.4)^3 + 9 \times 17 \times (0.6)^2 \times (0.4)^2 + 3 \times 17 \times 16 \times (0.6)^3 \times (0.4) + 3 \times 17 \times 4 \times 15 \times (0.6)^4 \right]$

$= (0.4)^{14} \left[0.0256 + 0.6912 + 8.8128 + 70.5024 + 396.576 \right]$

$= 476.608 \times (0.4)^{14}$

$= 0.001279$ Ans.

Q.11. Investments are made in ten companies, and for each company there is a probability of 0.65 that the investment will deliver a profit. What is the probability that at least half of the investments will deliver a profit?

Ans.

Investments are made in ten companies, so $n = 10$

Probability of investment will deliver a profit in each company is $p = 0.65$

At least half of the investment deliver profit means at least 5 companies get the profit.

So probability of mass function $B(10, 0.65)$ is

$$P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= {}^{10}C_5 \times (0.65)^5 \times (1-0.65)^5 + {}^{10}C_6 \times (0.65)^6 \times (1-0.65)^4 + {}^{10}C_7 \times (0.65)^7 \times (1-0.65)^3 + {}^{10}C_8 \times (0.65)^8 \times (1-0.65)^2 + {}^{10}C_9 \times (0.65)^9 \times (1-0.65)^1 + {}^{10}C_{10} \times (0.65)^{10} \times (1-0.65)^0$$

$$= \frac{10!}{5!\,5!} \times (0.65)^5 \times (0.35)^5 + \frac{10!}{6!\,4!} \times (0.65)^6 \times (0.35)^4 + \frac{10!}{7!\,3!} \times (0.65)^7 \times (0.35)^3 + \frac{10!}{8!\,2!} \times (0.65)^8 \times (0.35)^2 + \frac{10!}{9!\,1!} \times (0.65)^9 \times (0.35)^1 + \frac{10!}{10!\,0!} \times (0.65)^{10} \times (0.35)^0$$

$$= 252 \times (0.65)^5 \times (0.35)^5 + 210 \times (0.65)^6 \times (0.35)^4 + 120 \times (0.65)^7 \times (0.35)^3 + 45 \times (0.65)^8 \times (0.35)^2 + 10 \times (0.65)^9 \times (0.35) + (0.65)^{10}$$

Chapter 3.2

Q.2. if X has a negative binomial distribution with Parameters $p = 0.6$ and $r = 3$, calculate :

A. $P(X=5)$ B. $P(X=8)$

C. $P(X \leq 7)$ D. $P(X \geq 7)$

Ans. negative binomial distribution with Parameter p and r. The probability mass function is

$$P(X=x) = {}^{x-1}C_{r-1} \, (1-p)^{x-r} \, p^r , \quad \text{Here } p = 0.6,\ r = 3$$

(A) $P(X=5) = {}^{5-1}C_{3-1} \, (1-0.6)^{5-3} \times (0.6)^3$

$= {}^{4}C_{2} \times (0.4)^2 \times (0.6)^3$

$= \frac{4!}{2!\,2!} \times (0.4)^2 \times (0.6)^3$

$= 6 \times (0.4)^2 \times (0.6)^3$

$= 0.20736$

(B) $P(X=8) = {}^{8-1}C_{3-1} \, (1-0.6)^{8-3} \times (0.6)^3$

$= {}^{7}C_{2} \times (0.4)^5 \times (0.6)^3$

$= \frac{7 \times 6 \times 5!}{5! \times 2 \times 1} \times (0.4)^5 \times (0.6)^3$

$= 21 \times (0.4)^5 \times (0.6)^3$

$= 0.04644$

(C) $P(X \leq 7) = P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7)$

$= {}^{3-1}C_{3-1} \times (1-0.6)^{3-3} \times (0.6)^3 + {}^{4-1}C_{3-1} \, (1-0.6)^{4-3} \times (0.6)^3 +$
${}^{5-1}C_{3-1} \, (1-0.6)^{5-3} \times (0.6)^3 + {}^{6-1}C_{3-1} \, (1-0.6)^{6-3} \times (0.6)^3 +$
${}^{7-1}C_{3-1} \, (1-0.6)^{7-3} (0.6)^3$

$= (0.6)^3 [\ ^2C_2 \times (0.4)^0 \times 1 + \ ^3C_2 \times (0.4) + \ ^4C_2 \times (0.4)^2 \times 1 +$
$\ ^5C_2 \times (0.4)^3 + \ ^6C_2\ (0.4)^4]$

$= (0.6)^3 [1 + 3 \times 0.4 + 6 \times (0.4)^2 + 10 \times (0.4)^3 + 15 \times (0.4)^4]$

$= (0.6)^3 [1 + 1.2 + 0.96 + 0.64 + 0.384]$

$= 4.184 \times (0.6)^3 = 0.903744$ Ans

(ii) $P(X \geq 7) = 1 - P(X \leq 6)$

$= 1 - [P(X=3) + P(X=4) + P(X=5) + P(X=6)]$

$= 1 - [\ ^2C_2 \times (0.4)^0 \times (0.6)^3 + \ ^3C_2 \times (0.4) \times (0.6)^3 + \ ^4C_2 \times$
$(0.4)^2 \times (0.6)^3 + \ ^5C_2 \times (0.4)^3 \times (0.6)^3]$

$= 1 - [(0.6)^3 + 1.2 \times (0.6)^3 + 0.96 \times (0.6)^3 + 0.64 \times$
$(0.6)^3]$

$= 1 - [0.216 + 0.2592 + 0.20736 + 0.13824]$

$= 1 - 0.8208 = 0.1792$ Ans.

Q.4. Suppose that $X_1, \cdots X_r$ are independent random variables, each with a geometric distribution with parameter p. Explain why

$$Y = X_1 + \cdots + X_r$$

has a negative binomial distribution with parameters p and r. Use this relationship to establish the mean and variance of a negative binomial distribution.

Ans. Notice that a negative binomial distribution with parameters p and r can be thought of as the number of trials ~~at~~ up to and including the r^{th} success in a sequence of independent Bernoulli trials with a constant success probability p, which can be considered to be the number of trials up to and including the first success, pulse the number of trials after the first success and up to and including the ~~first~~ second success, plus the number of trials after the second success and up to including the third success, so on. Each of these r components has a geometric distribution with parameter p.

$$P(X = x) = C_{r-1}^{x-1} \; (1-p)^{x-r} p^r$$

and for $y = X_1 + X_2 + \cdots + X_r$

Expected value is $E(X) = \dfrac{r}{p}$

So that $Var(X) = E(X^2) - (E(X))^2$

$$= \frac{r}{p^2} - \frac{r}{p} = \frac{r - rp}{p^2} = \frac{r(1-p)}{p^2}$$

Q.5. An archer hit's the bull's eye with a probability of 0.09, and the result of different attempts can be taken to be independent of each other.

(a) If the archer shoots a series of arrows, what is the probability that the first bull's eye is scored with the fourth arrow.

(b) What is the probability that the third bull's eye is scored with the tenth arrow.

(c) What is the expected number of arrow shot before the ~~third~~ first bull's eye is scored?

(d) What is the expected number of arrows shot before the third eye is scored.

Ans. Given that, $p = 0.09$

(a) first bull's eye scored with fourth arrow,

So Probability is

$$P(X=4) = p(1-p)^{x-1}$$

$$= 0.09 \times (1-0.09)^{4-1}$$

$$= 0.09 \times (0.91)^3$$

$$= 0.0678$$

(b) $r = 3$, $x = 10$

$$P(X=10) = {}^{x-1}C_{r-1} (1-p)^{x-r} p^r$$

$$= {}^{10-1}C_{3-1} (1-0.09)^{10-3} (0.09)^3$$

$$= {}^{9}C_{2} (0.91)^7 (0.09)^3$$

$$= 36 \times (0.91)^7 \times (0.09)^3$$

$$= 0.01356$$

(C) $r = 1, \quad p = 0.09$

Expected numbers of arrow shot before first bull's eye scored is

$$\Rightarrow \quad E(X) = \frac{r}{p}$$

$$= \frac{1}{0.09} = 11.11$$

(D) Expected numbers of arrow shot before third bull's eye scored,

here $r = 3, \quad p = 0.09$

$$\Rightarrow \quad E(X) = \frac{r}{p} = \frac{3}{0.09} = 33.33 \text{ Ans.}$$

Q. (9) Recall problem 3.1.9. in which a company receive 60% of its orders over the internet. within a certain period of time:-

(a) What is the probability that the fifth order receives is the first Internet order?

(b) What is the probability that the eight order recieves is the fourth Internet order?

Ans.

Probability of recieving orders over the internet.

is $p = \frac{60}{100} = 0.6$

(a) The fifth order received is the first internet order, so that here $r=1$, $x=5$

therefore, probability is

$$P(X=5) = {}^{5-1}C_{1-1} \times (1-0.6)^{5-1} \times (0.6)^1$$

$$= {}^{4}C_{0} \times (0.4)^4 \times (0.6)^1 \qquad \because {}^{4}C_{0} = 1$$

$$= 0.01536 \text{ Ans}$$

(b) The eigth order received is the fourth internet order, so that here $r=4$, $x=8$

$$P(X=8) = {}^{8-1}C_{4-1} \times (0.6)^4 \times (1-0.6)^{8-4}$$

$$= {}^{7}C_{3} \times (0.4)^4 \times (0.6)^4$$

$$= \frac{7\times6\times5\times4!}{3\times2\times4!} \times (0.4)^4 \times (0.6)^4$$

$$= 35 \times (0.4)^4 \times (0.6)^4$$

$$= 0.1161 \text{ Ans}$$

Chapter 3.4

Q.2. A Committee consist of eight right - wing members and seven left - wing members. A subcommittee is formed by randomly choosing five of the committee members. Draw a line graph of the probability mass function of the number of right-wing members serving on the subcommittee.

Ans. A Committe consists wing-members is $N = 8+7 = 15$

right-wing members serving on the subcommittee,
So $r = 8$

A subcommittee is formed by randomly choosing five of the committee members, So $n = 5$

therefore probability is $\Rightarrow P(X=x) = \dfrac{C_x^r \times C_{n-x}^{N-r}}{C_n^N}$

$$P(X=0) = \frac{C_0^8 \times C_{5-0}^{15-8}}{C_5^{15}}$$

$$= \frac{1 \times C_5^7}{C_5^{15}}$$

$$= \frac{\frac{7 \times 6 \times 5!}{2 \times 5!}}{\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10!}{5 \times 4 \times 3 \times 2 \times 1 \times 10!}}$$

$$= \frac{7 \times 3}{3 \times 13 \times 11 \times 7} = \frac{1}{143} = 0.006993$$

$$P(X=1) = \frac{C_1^8 \times C_{5-1}^{15-8}}{C_5^{15}}$$

$$= \frac{8 \times C_4^7}{3 \times 13 \times 11 \times 7} = \frac{8 \times 35}{3 \times 13 \times 11 \times 7}$$

$$= \frac{40}{429} = 0.09324$$

$$P(X=2) = \frac{{}^{8}C_{2} \times {}^{15-8}C_{5-2}}{{}^{15}C_{5}}$$

$$= \frac{\frac{8\times7}{2} \times {}^{7}C_{3}}{3\times13\times11\times7}$$

$$= \frac{4\times7\times7\times5}{3\times13\times11\times7}$$

$$= \frac{140}{429} = 0.3263$$

$$P(X=3) = \frac{{}^{8}C_{3} \times {}^{15-8}C_{5-3}}{{}^{15}C_{5}}$$

$$= \frac{\frac{8\times7\times6}{3\times2} \times {}^{7}C_{2}}{3\times13\times11\times7}$$

$$= \frac{56\times21}{3\times13\times11\times7}$$

$$= \frac{168}{429} = 0.3916$$

$$P(X=4) = \frac{{}^{8}C_{4} \times {}^{15-8}C_{5-4}}{{}^{15}C_{5}}$$

$$= \frac{\frac{8\times7\times6\times5}{4\times3\times2\times1} \times {}^{7}C_{1}}{3\times13\times11\times7}$$

$$= \frac{70\times7}{3\times13\times11\times7} = \frac{70}{429} = 0.1631$$

$$P(X=5) = \frac{{}^{8}C_{5} \times {}^{15-8}C_{5-5}}{{}^{15}C_{5}} = \frac{\frac{8\times7\times6}{3\times2\times1} \times {}^{7}C_{0}}{3\times13\times11\times7}$$

$$= \frac{56}{3\times13\times11\times7} = \frac{8}{429} = 0.01864$$

Q.3. A box contains 17 balls of which 10 are red and 7 are blue. A sample of 5 balls is chosen at random and placed in a jar. Calculate the probability that:

(a) The jar contains exactly 3 red balls.

(b) The jar contains exactly 1 red ball.

(c) The jar contains more blue balls than red balls.

Ans.

A box contains total balls $N = 17$

red balls are $r = 10$

A sample of balls chosen random, so $n = 5$

(a) The jar contains exactly 3 red balls.

therefore, probability is $P(X=x) = \dfrac{C_x^r \times C_{n-x}^{N-r}}{C_n^N}$

$$P(X=3) = \frac{C_3^{10} \times C_{5-3}^{17-10}}{C_5^{17}}$$

$$= \frac{\frac{10\times9\times8\times7!}{3\times2\times7!} \times C_2^7}{\frac{17\times16\times15\times14\times13\times12!}{5\times4\times3\times2\times1\times12!}}$$

$$= \frac{120\times21}{17\times4\times7\times13}$$

$$= \frac{90}{221} \text{ Ans}$$

(b) The jar contains exactly 1 red balls

$$P(X=1) = \frac{C_1^{10} \times C_{5-1}^{17-10}}{C_5^{17}}$$

$$= \frac{10 \times {}^{7}C_4}{17 \times 4 \times 7 \times 13}$$

$$= \frac{10 \times 35}{17 \times 4 \times 7 \times 13}$$

$$= \frac{25}{442} \text{ Ans}$$

(C) The jar contains more blue balls than red balls means maximum 2 red ball in jar. therefor, probability is

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{{}^{10}C_0 \times {}^{17-10}C_{5-0}}{{}^{17}C_5} + \frac{{}^{10}C_1 \times {}^{17-10}C_{5-1}}{{}^{17}C_5} + \frac{{}^{10}C_2 \times {}^{17-10}C_{5-2}}{{}^{17}C_5}$$

$$= \frac{1}{{}^{17}C_5} \left[1 \times {}^{7}C_5 + 10 \times {}^{7}C_4 + 45 \times {}^{7}C_3 \right]$$

$$= \frac{1}{{}^{17}C_5} \left[21 + 10 \times 35 + 45 \times 35 \right]$$

$$= \frac{1946}{17 \times 4 \times 7 \times 13}$$

$$= \frac{139}{442} \text{ Ans}$$

Q.7. There are 11 items of a product on a shelf in a retail outlet, and unknown to the customers, 4 of the items are outdated. Suppose that a customer take 3 items at random.

(a) What is the probability that none of the outdated products are selected by the customer?

(b) What is the probability that ~~(none of the outdated)~~ exactly 2 of the items taken by the customer are outdated?

Ans.

Total items of a product is $N = 11$

Customer take three items at random so $n = 3$

outdated itmes in a product is $r = 4$

So that probability of none of the outdated products are selected by the customer is

(a)
$$P(x=0) = \frac{{}^{4}C_{0} \times {}^{11-4}C_{3-0}}{{}^{11}C_{3}}$$

$$= \frac{{}^{4}C_{0} \times {}^{7}C_{3}}{{}^{11}C_{3}}$$

$$= \frac{1 \times \frac{7\times6\times5\times4!}{3\times2\times1\times4!}}{\frac{11\times10\times9\times8!}{3\times2\times1\times8!}}$$

$$= \frac{35}{165}$$

$$= \frac{7}{33} = 0.2121 \text{ Ans}$$

(b) Probability that exactly 2 of the items taken by the customer are outdated is

$$P(X=2) = \frac{{}^{4}C_{2} \times {}^{11-4}C_{3-2}}{{}^{11}C_{3}}$$

$$= \frac{\frac{4\times3}{2} \times {}^{7}C_{1}}{165}$$

$$= \frac{2\times3\times7}{165}$$

$$= \frac{14}{55} = 0.2545 \text{ Ans}$$

Q. 8. A plate has 15 cupcakes on it, of which 9 are chocolate and 6 are strawberry. A child randomly selects 5 of the cupcakes and eats them. What is the probability that the number of chocolate cupcakes remaining on the plate is between 5 and 7 inclusive?

Ans.

A plate has total number of cup cakes $N = 15$

Chocolate in cupcakes is $r = 9$

A child randomly select 5 cupcakes, so $n = 5$

Probability that the number of chocolate cupcakes remaining on the plate is between 5 and 7 is

$$\Rightarrow P(5 \le X \le 7) = P(X=5) + P(X=6) + P(X=7)$$

$$= \frac{C_4^9 \times C_{5-4}^{15-9}}{C_5^{15}} + \frac{C_3^9 \times C_{5-3}^{15-9}}{C_5^{15}} + \frac{C_2^9 \times C_{5-2}^{15-9}}{C_5^{15}}$$

∵ if remaining 5 than he select 4. Thus, remaining 6 & 7, so selected 3 & 2.

$$= \frac{\frac{9\times8\times7\times6}{4\times3\times2} \times C_1^6}{\frac{15\times14\times13\times12\times11}{5\times4\times3\times2\times1}} + \frac{\frac{9\times8\times7}{3\times2} \times C_2^6}{C_5^{15}} + \frac{\frac{9\times8}{2} \times C_3^6}{C_5^{15}}$$

$$= \frac{1}{C_5^{15}} \left[126\times6 + 84\times15 + 36\times20 \right]$$

$$= \frac{1}{3\times7\times13\times11} \left[756 + 1260 + 720 \right]$$

$$= \frac{2736}{3003} = 0.9111 \text{ Ans}$$

Chapter 3.4

Q.6. On average there are four traffic accidents in a city during one hour of rush-hour traffic. Use the poisson distribution to calculate the probability that in one such hour there are

(a) no accidents

(b) at least six accidents.

Ans.

4 traffic accidents occurs in during one hour in a city,

~~So, the probability of 1 traffic accident in one such hour is~~ [illegible]

So the probability in poisson distribution can be distributed as $\lambda = 4$

(a) Probability that no accidents occurs in such one hour is,

$$P(X = x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$P(X = 0) = \frac{e^{-4} \cdot (4)^{0}}{0!}$$

$$= e^{-4} = 0.01832$$

(b) at least six accident occurs in one hours than the probability is,

$$P(X \geq 6) = 1 - P(X \leq 5)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)]$$

$$= 1 - \left[\frac{e^{-4} \cdot (4)^{0}}{0!} + \frac{e^{-4} \cdot (4)^{1}}{1!} + \frac{e^{-4} \cdot (4)^{2}}{2!} + \frac{e^{-4} \cdot (4)^{3}}{3!} + \frac{e^{-4} \cdot (4)^{4}}{4!} + \frac{e^{-5} \cdot (4)^{5}}{5!} \right]$$

$$= 1 - e^{-4}\left[1 + 4 + \frac{(4)^2}{2!} + \frac{(4)^3}{3!} + \frac{(4)^4}{4!} + \frac{(4)^5}{5!}\right]$$

$$= 1 - e^{-4}\left[1 + 4 + 8 + \frac{64}{6} + \frac{(4)^4}{24} + \frac{(4)^5}{120}\right]$$

$$= 1 - e^{-4}\left[1 + 4 + 8 + 10.6667 + 10.6667 + 8.5333\right]$$

$$= 1 - 42.8667 \times e^{-4}$$

$$= 1 - 0.78513 = 0.21487 \text{ Ans}$$

Q. 7. Recall that the Poisson distribution with a parameter value of $\lambda = np$ can be used to approximate the $B(n, p)$ distribution when n is very large and the success probability p is very small.

A box contains 500 electrical switches, each one of which has a probability of 0.005 of being defective. Use the Poisson distribution to make an approximate calculation of the probability that the box contains no more than 3 defective switches.

Ans.

A box contains total electrical switches $n = 500$

Probability of each switch is defective $p = 0.005$

So that the probability mass function is $B(500, 0.005)$

This function can be approximated by a Poisson distribution with $\lambda = 500 \times 0.005$

$$\Rightarrow \lambda = 2.5$$

now, the probability that the box contains no more than 3 defective switches is given by,

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{e^{-2.5} \times (2.5)^0}{0!} + \frac{e^{-2.5} \times (2.5)^1}{1!} + \frac{e^{-2.5} \times (2.5)^2}{2!} + \frac{e^{-2.5} \times (2.5)^3}{3!}$$

$$= e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2} + \frac{(2.5)^3}{6}\right]$$

$$= e^{-2.5}\left[1 + 2.5 + 3.125 + 2.6042\right]$$

$$= e^{-2.5} \times 9.2292 = 0.7576 \text{ Ans}$$

Q.8. In a scanning process, the number of misrecorded pieces of information has a Poisson distribution with parameter $\lambda = 9.2$.

(a) What is the probability that there are between six and ten misrecorded pieces of information?

(b) What is the probability that there are no more than four misrecorded pieces of information?

Ans.

The number of misrecorded pieces of information has given a Poisson distribution with $\lambda = 9.2$

(a) therefore, probability that there are between six and 10 misrecorded pieces of information is.

$$P(6 \le X \le 10) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= \frac{e^{-9.2}(9.2)^6}{6!} + \frac{e^{-9.2}(9.2)^7}{7!} + \frac{e^{-9.2}(9.2)^8}{8!} + \frac{e^{-9.2}(9.2)^9}{9!} + \frac{e^{-9.2}(9.2)^{10}}{10!}$$

$$= (9.2)^6 e^{-9.2}\left[\frac{(9.2)^0}{6!} + \frac{9.2}{7!} + \frac{(9.2)^2}{8!} + \frac{(9.2)^3}{9!} + \frac{(9.2)^4}{10!}\right]$$

$$= (9.2)^6 \times e^{-9.2}\left[\frac{1}{720} + \frac{9.2}{720\times7} + \frac{(9.2)^2}{720\times7\times8} + \frac{(9.2)^3}{720\times7\times8\times9} + \frac{(9.2)^4}{720\times7\times8\times9\times10}\right]$$

$$= \frac{(9.2)^6 \times e^{-9.2}}{720}\left[1 + 1.3143 + 1.51143 + 1.5450 + 1.4214\right]$$

$$= \frac{(9.2)^6 \times e^{-9.2}}{720} \times 6.7921 = 0.57795 \text{ Ans}$$

(b) Probability that there are no more than four misrecorded pieces of information is,

$$P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{e^{-9.2}\cdot(9.2)^0}{0!} + \frac{e^{-9.2}\cdot(9.2)^1}{1!} + \frac{e^{-9.2}\cdot(9.2)^2}{2!} + \frac{e^{-9.2}\cdot(9.2)^3}{3!} + \frac{e^{-9.2}\cdot(9.2)^4}{4!}$$

$$= e^{-9.2}\left[1 + 9.2 + \frac{(9.2)^2}{2} + \frac{(9.2)^3}{6} + \frac{(9.2)^4}{24}\right]$$

$$= e^{-9.2}\left[1 + 9.2 + 42.32 + 129.7813 + 298.4971\right]$$

$$= 480.7984 \times e^{-9.2}$$

$$= 0.04858$$ Ans

Q. 9. Suppose that the number of errors in a company's accounts has a poisson distribution with a mean on 4.7. what is the probability that there will be exactly three errors?

A. 0.127 B. 0.157 C. 0.187 D. 0.217

Ans. The number of error in a company's accounts has a poisson distribution with a mean $\lambda = 4.7$

Probability that there will be exactly three errors in company's accounts is given by,

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$\Rightarrow P(X=3) = \frac{e^{-4.7} \cdot (4.7)^3}{3!}$$

$$= \frac{e^{-4.7} \cdot (4.7)^3}{6}$$

$$= 17.3038 \times e^{-4.7}$$

$$= 0.15738 \text{ Ans}$$

So, option (B) is correct Answer.

Chapter 4.1

Q.1. Suppose that $X \sim U(-3, 8)$. find :

(a) $E(X)$
(b) The standard
(c) The upper quartile of the distribution.
(d) $P(0 \le X \le 4)$

Ans.

$X \sim U(-3, 8)$ is said to have a uniform distribution. So the Expected value is

(a) $$E(X) = \frac{a+b}{2}$$
$$= \frac{-3+8}{2} = \frac{5}{2}$$
$$= 2.5$$

and Variance is $\text{Var}(X) = \frac{(b-a)^2}{12}$

$$= \frac{(8+3)^2}{12} = \frac{121}{12}$$

(b) The standard deviation is $\sigma = \sqrt{\text{var}(X)}$

$$= \sqrt{\frac{(8+3)^2}{12}}$$
$$= \frac{11}{2\sqrt{3}}$$
$$= \frac{11}{3.4641} = 3.175 \text{ Ans}$$

(c) The upper quantile is $P = \frac{3}{4}$

So the upper quantile of distribution is,

$$\Rightarrow A = (1-P)a + Pb$$
$$= \left(1 - \frac{3}{4}\right) \times (-3) + \frac{3}{4} \times 8$$
$$= -\frac{3}{4} + 6 = \frac{21}{4}$$
$$= 5.25 \text{ Ans}$$

(d) $P(0 \le X \le 4) = $?

The probability in uniform distribution function for U(a, b) is given by,

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} \frac{1}{b-a}\, dx$$

Here $x_1 = 0$, $x_2 = 4$ and $\frac{1}{b-a} = \frac{1}{8-(-3)} = \frac{1}{11}$

$$= \int_0^4 \frac{1}{11}\, dx$$

$$= \frac{1}{11} \times |x|_0^4$$

$$= \frac{1}{11}(4-0)$$

$$= \frac{4}{11} \text{ Ans}$$

Q.2. A new battery supposedly with a charge of 1.5 volts actually has a voltage with a uniform distribution between 1.43 and 1.60 volts.

(a) What is the expection of the voltage?

(b) What is the standard deviation of the voltage.

(c) What is the cumulative distribution function of the voltge.

(d) What is the probability that a battery has a voltage less than 1.48 volts.

(e) If a box contains 50 batteries what are the expectation and variance of the number of batteries in the box with a voltage less than 1.5 volts?

Ans. A new battery supposedly with a charge of = 1.5 volts

A uniform distribution is $x \sim U(1.43, 1.60)$

(a) the expectation of the voltage is,

$$E(x) = \frac{a+b}{2}$$

$$= \frac{1.43+1.60}{2} = \frac{3.03}{2}$$

$$= 1.515$$

(b) The variance in the voltage is

$$Var(X) = \frac{(b-a)^2}{12}$$

$$= \frac{(1.60 - 1.43)^2}{12}$$

$$= \frac{(0.17)^2}{12}$$

so, the standard deviation is $\sigma = \sqrt{var(X)}$

$$\therefore \sigma = \sqrt{\frac{(0.17)^2}{12}}$$

$$= \frac{0.17}{2\sqrt{3}}$$

$$= \frac{0.17}{3.4641} = 0.0491 \text{ Ans}$$

(C) The cumulative distribution function for a uniform distribution is,

$$F(x) = \frac{x-a}{b-a} \quad \text{for } a \le x \le b$$

$$= \frac{x-1.43}{1.60-1.43}$$

$$F(x) = \frac{x-1.43}{0.17} \quad \text{for } 1.43 \le x \le 1.60$$

(d) Probability that a battery has a voltage less than 1.48 volts is,

$$P(X < 1.48) = F(1.48)$$

$$= \frac{x-1.43}{0.17} \quad \text{where } x = 1.48.$$

$$= \frac{1.48-1.43}{0.17}$$

$$= \frac{0.05}{0.17} = 0.2941 \text{ Ans}$$

(e) box contains total batteries $n = 50$

probability of batteries in the box with a voltage less than 1.5 volts is,

$$P(X < 1.5) = F(1.5)$$

$$= \frac{1.5 - 1.43}{0.17}$$

$$P = \frac{0.07}{0.17}$$

$$P = 0.4118$$

The number of batteries with a voltage less than 1.5 volts has a binomial distribution with $n = 50$ and $P = 0.4118$, so

the expected voltage is,

$$E(X) = np$$

$$= 50 \times 0.4118 = 20.59$$

and the variance in voltage is,

$$Var(X) = np(1-p)$$

$$= 50 \times 0.4118 (1 - 0.4118)$$

$$= 50 \times 0.4118 \times 0.5882$$

$$= 12.1110 \text{ Ans}$$

Q.3. A computer random-number generator produces numbers that have a uniform distribution between 0 and 1.

(a) If 20 random numbers are generated, what are the expectation and variance of the number of them that lie in each of the four intervals $[0.00, 0.30)$, $[0.30, 0.50)$, $[0.50, 0.75)$ and $[0.75, 1.00)$?

(b) What is the probability that exactly five numbers lie in each of the four intervals?

Ans. A computer produces numbers that have a uniform distribution as $X \sim U(0,1)$

(a) 20 random numbers are generated, so $n=20$

Probability of interval $[0.00, 0.30]$ is $P_1 = \int_0^{0.30} \frac{1}{1-0}\, dx$

$$= |x|_0^{0.30}$$

$$= (0.30-0) = 0.30$$

Probability of interval $[0.50, 0.75)$ is $P_3 = \int_{0.50}^{0.75} \frac{1}{1-0}\, dx$

$$= |x|_{0.50}^{0.75}$$

$$= (0.75-0.50) = 0.25$$

Probability of interval $[0.30, 0.50)$ is $P_2 = \int_{0.30}^{0.50} \frac{1}{1-0}\, dx$

$$\Rightarrow P_2 = |x|_{0.30}^{0.50}$$

$$\Rightarrow P_2 = (0.50-0.30) = 0.20$$

Probability of interval $[0.75, 1.00)$ is $P_4 = \int_{0.75}^{1.00} \frac{1}{1-0} dx$

$$= |x|_{0.75}^{1.00}$$

$$= 1.00 - 0.75 = 0.25$$

now, the expectation of interval $[0.00, 0.30)$ is

$$E(X_1) = nP_1 = 20 \times 0.30 = 6.$$

The expectation of interval $[0.30, 0.50)$ is

$$E(X_2) = nP_2 = 20 \times 0.20 = 4$$

The expectation of interval $[0.50, 0.75)$ is

$$E(X_3) = nP_3 = 20 \times 0.25 = 5$$

The expectation of interval $[0.75, 1.00)$ is

$$E(X_4) = nP_4 = 20 \times 0.25 = 5$$

Thus, variance of first interval is

$$Var(X_1) = nP_1(1-P_1)$$
$$= 20 \times 0.30 \times (1-0.30)$$
$$= 20 \times 0.30 \times 0.70 = 4.2$$

variance of second interval is

$$Var(X_2) = nP_2(1-P_2)$$
$$= 20 \times 0.20 \times 0.80$$
$$= 3.2$$

variance of third interval is

$$Var(X_3) = nP_3(1-P_3)$$
$$= 20 \times 0.25 \times 0.75$$
$$= 3.75$$

variance of fourth interval is-

$$Var(X_4) = nP_4(1-P_4)$$
$$= 20 \times 0.25 \times 0.75$$
$$= 3.75$$

(b) exactly five numbers lie in each of the four intervals is the multinomial distribution with $x_1 = 5$, $x_2 = 5$, $x_3 = 5$, $x_4 = 5$ and $n=20$

So the required probability is,

$$P(X_1=5, X_2=5, X_3=5, X_4=5) = \frac{20!}{5!\,5!\,5!\,5!} \times (0.30)^5 \times (0.20)^5 \times (0.25)^5 \times (0.25)^5$$

$$= \frac{2.4329 \times 10^{18}}{0.20736 \times 10^{9}} \times (0.30)^5 \times (0.20)^5 \times (0.25)^{10}$$

$$= 1.17327 \times 10^{10} \times (0.30)^5 \times (0.20)^5 \times (0.25)^{10}$$

$$= 0.0087 \text{ Ans}$$

Q.5. Suppose that a metal pin has a diameter that has a Uniform distribution between 4.182 mm and 4.185 mm.

(a) What is the probability that a pin will fit into a hole that has a diameter of 4.184 mm?

(b) If a pin does fit into the hole, what is the probability that the difference between the diameter of the hole and the diameter of the pin is less than 0.0005 mm?

Ans.

metal pin has a diameter that has a uniform distribution between $X \sim U(4.182, 4.185)$

(a) The cumulative distribution function is given by

$$F(x) = \frac{x-a}{b-a} \quad \text{for } a \le x \le b$$

$$f(x) = \frac{x - 4.182}{4.185 - 4.182} \quad \text{for } 4.182 \le x \le 4.185$$

$$f(x) = \frac{x - 4.182}{0.003} \quad \text{for } 4.182 \le x \le 4.185$$

the probability that a pin will fit into hole that has a diameter of 4.184 mm.

$$P(X = 4.184) = F(4.184) = \frac{4.184 - 4.182}{4.185 - 4.182}$$

$$= \frac{0.002}{0.003} = \frac{2}{3} \text{ Ans.}$$

(b) given that, holes diameter is = 4.184 mm.

difference between the diameter of the hole and the diameter of pin is less than 0.0005 mm

So the range of pin's diameter is

$$= \quad 4.1835 \leq x \leq 4.1840$$

and the probability of in this diameter range of pin is

$$\Rightarrow \quad P(4.1835 \leq x \leq 4.1840) = \int_{4.1835}^{4.1840} \frac{1}{4.185 - 4.182}\, dx$$

$$= \frac{1}{0.003} \left| x \right|_{4.1835}^{4.1840}$$

$$= \frac{1}{0.003} \left[4.1840 - 4.1835 \right]$$

$$= \frac{0.0005}{0.003} = \frac{1}{6}$$

and the pin fits in hole when the diameter of holes is less than or equal to 4.1840. so the probability of pin fits in hole $\Rightarrow$

$$P(x \leq 4.1840) = F(4.1840)$$

$$= \frac{4.1840 - 4.1820}{4.185 - 4.182}$$

$$= \frac{0.0020}{0.003} = \frac{2}{3}$$

So the probability as required is,

$$P(4.1835 \leq x \leq 4.1840 \mid x \leq 4.1840) = \frac{P(4.1835 \leq x \leq 4.1840)}{P(x \leq 4.1840)}$$

$$= \frac{1/6}{2/3}$$

$$= \frac{1}{4} \quad \underline{\text{Ans.}}$$

Chapter 4.2

Q.2. Suppose that you are waiting for a friend to call you and that the time you wait in minutes has an exponential, distribution with parameter $\lambda = 0.1$

(a) What is the expectation of your waiting time?

(b) What is the probability that your will wait longer than 10 minutes?

(c) What is the probability that you will wait less than 5 minutes?

(d) Suppose that after 5 minutes you are still waiting for the call. What is the distribution of your additional waiting time? in this case, what is the probability that your total waiting time is longer than 15 minutes?

(e) Suppose now that the time you wait in minutes for the call has a $U(0, 20)$ distribution. What is the expectation of your watting time? If after 5 minutes you are still waiting for the call, what is the distribution of your additional waiting time.

Ans. the time you wait in minutes has an exponential, distribution with parameter $\lambda = 0.1$

(a) So the expection of wating time is,

$$\Rightarrow \quad E(x) = \frac{1}{\lambda} = \frac{1}{0.1} = 10$$

(b) The comulative distribution function in expontial distribution is given by,

$$\Rightarrow \quad F(x) = 1 - e^{-\lambda x} \qquad \text{for } x \geq 0$$

Probability that I'll wait longer than 10 minutes is,

$$P(X \geq 10) = 1 - P(X \leq 10)$$
$$= 1 - F(10)$$
$$= 1 - (1 - e^{-0.1 \times 10})$$
$$= 1 - [1 - e^{-1}]$$
$$= e^{-1} = 0.36788 \text{ Ans}$$

(c) Probability that i can wait less than 5 minutes is,

$$P(X < 5) = F(5)$$
$$= 1 - e^{-0.1 \times 5}$$
$$= 1 - e^{-0.5}$$
$$= 1 - 0.6065$$
$$= 0.3935 \text{ Ans}$$

(d) The additional waiting time is = 5 minute.

and the parameter $\lambda = 0.1$

So this is also an exponential distribution.

total waiting time is longer than 15 minutes

So the additional waiting time is = 15-5 = 10 minutes longer than

So the required Probability,

$$P(X \geq 10) = 1 - P(X \leq 10)$$
$$= 1 - F(10)$$
$$= 1 - (1 - e^{-0.1 \times 10})$$
$$= e^{-1} = 0.36788$$

0.98774

(e) Now the waiting time is the Uniform distribution with $U(0,20)$.

The Expectation of waiting time is

$$E(X) = \frac{a+b}{2} = \frac{0+20}{2}$$

$$= 10$$

If after 5 minutes still wating for call, So the remaining wating time is $= 20 - 5 = 15$

So that this the uniform distribution with $U(0,15)$. Ans

Q.3. The time in days between breakdowns of a machine is exponentially distributed with $\lambda = 0.2$.

Ans.

The time in days between breakdowns of a machine is exponentially distributed with $\lambda = 0.2$.

(a) expected time between machine breakdowns is,

$$E(X) = \frac{1}{\lambda}$$

$$= \frac{1}{0.2} = 5$$

(b) and variance of machine breakdowns is.

$$Var(X) = \frac{1}{\lambda^2}$$

$$= \left(\frac{1}{0.2}\right)^2 = 5^2 = 25$$

so that standard deviation is $\sigma = \sqrt{var(X)}$

$$= \sqrt{25} = 5 \text{ Ans}$$

(C) we know that the median of the exponential distribution is $M = \frac{0.693}{\lambda}$

$$= \frac{0.693}{0.2}$$

$$= 3.465$$

(d) the probability that after the machine is repaired it lasts at least a week before failing again so,

$$P(X \geq 7) = 1 - P(X \leq 7)$$

$$= 1 - F(7)$$

$$= 1 - (1 - e^{-0.2 \times 7})$$

$$= e^{-1.4}$$

$$= 0.2466 \text{ Ans}$$

(e) machine has performed satisfactorily for six days,

Probability that it lasts at least two more days before breaking down is memoryless property of the exponential distribution and given by,

$$P(X \geq 2) = 1 - F(2)$$

$$= 1 - (1 - e^{-0.2 \times 2})$$

$$= e^{-0.4}$$

$$= 0.6703 \text{ Ans}$$

Ans. 5. A double exponential distribution, has a probability density function is given as

$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x-\theta|} \quad \text{for } -\infty \le x \le \infty$$

So the cumulative function of this distribution is,

$$F(x) = \int_{-\infty}^{x} \frac{1}{2}\lambda e^{-\lambda|x-\theta|}\,dx \quad \text{for } -\infty \le x \le \theta$$

$$F(x) = \frac{1}{2}\times\lambda \int_{-\infty}^{x} e^{-\lambda|x-\theta|}\,dx = \frac{1}{2}e^{-\lambda(\theta-x)} \quad \text{for } -\infty \le x \le \theta$$

$$\Rightarrow F(x) = \frac{\lambda}{2}\int_{-\infty}^{\theta} e^{-\lambda|x-\theta|}\,dx + \frac{\lambda}{2}\int_{\theta}^{x} e^{-\lambda|x-\theta|}\,dx$$

$$= \frac{\lambda}{2}\times\frac{1}{\lambda}\left|-e^{-\lambda|x-\theta|}\right|_{-\infty}^{\theta} + \frac{\lambda}{2}\times\frac{1}{\lambda}\left|-e^{-\lambda|\theta-x|}\right|_{\theta}^{x}$$

$$= \frac{1}{2}\left[-e^{-\infty}+e^{0}\right] + \left(-\frac{1}{2}\right)\left[-e^{0}+e^{-\lambda(x-\theta)}\right]_{\theta}$$

$$= \frac{1}{2} + \frac{\lambda}{2}\int_{\theta}^{x} e^{-\lambda(x-\theta)}\,dx$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-\lambda(x-\theta)}$$

$$\Rightarrow F(x) = 1 - \frac{1}{2}e^{-\lambda(x-\theta)} \quad \text{for } \theta \le x \le \infty$$

(a) $P(X \le 0) = F(0)$, given that $\lambda = 3, \theta = 2$

$$= \cancel{1 - \frac{1}{2}e^{-3(2-0)}}$$

$$= \cancel{1 - \frac{1}{2}e^{-6}}$$

$$= \cancel{1}$$

$$= \frac{1}{2}e^{-\lambda(\theta-x)}$$

$$= \frac{1}{2}e^{-3(2-0)}$$

$$= \frac{1}{2} e^{-6}$$

$$= 0.00123 \text{ Ans}$$

(b) $$P(X \geq 1) = 1 - P(X \leq 1)$$

$$= 1 - F(1)$$

$$= \cancel{1} - [\tfrac{1}{2} e^{-3 \times (2-1)})$$

$$= \cancel{e^{-2}}$$

$$= 1 - \left[\frac{1}{2} e^{-3(2-1)}\right]$$

$$= 1 - \frac{1}{2} e^{-3}$$

$$= 1 - 0.02489$$

$$= 0.97511 \text{ Ans.}$$

Ans.7. The arrival times of workers at a factory at a factory first-aid room satisfy a Poisson process with an average of 1·8 per hour.

(a) the value of the parameter λ of the Poisson process is $\lambda = 1{\cdot}8$

(b) expectation of the time between two arrivals at the first aid-room is given by

$$E(X) = \frac{1}{\lambda}$$
$$= \frac{1}{1{\cdot}8} = 0.5555 \text{ Ans}$$

(c) The cumulative function in exponential distribution is $F(x) = 1 - e^{-\lambda x}$

So, at least 1 hour between two arrivals at the first-aid room is exponential distribution. So required probability is,

$$P(X \geq 1) = 1 - P(X \leq 1)$$
$$= 1 - F(1)$$
$$= 1 - (1 - e^{-1.8 \times 1})$$
$$= e^{-1.8} = 0.1653 \text{ Ans}$$

(d) the distribution of the number of workers visiting the first aid room during a 4-hours is poisson distribution with $\lambda = 4\lambda$,

$$= 4 \times 1.8$$
$$= 7.2$$

(e) So, the probability that at least four workers visit the first aid room during a 4-hour period is Poisson distribution with $\lambda = 4 \times 1.8 = 7.2$

So,

$$P(X \geq 4) = 1 - P(X \leq 3)$$

$$= 1 - \left[P(X=0) + P(X=1) + P(X=2) + P(X=3)\right]$$

$$= 1 - \left[\frac{e^{-7.2}(7.2)^0}{0!} + \frac{e^{-7.2}(7.2)^1}{1!} + \frac{e^{-7.2}(7.2)^2}{2!} + \frac{e^{-7.2}(7.2)^3}{3!}\right]$$

$$= 1 - e^{-7.2}\left[1 + 7.2 + \frac{(7.2)^2}{2} + \frac{(7.2)^3}{6}\right]$$

$$= 1 - e^{-7.2}\left[8.2 + 25.92 + 62.208\right]$$

$$= 1 - 96.328 \times e^{-7.2}$$

$$= 1 - 0.07192$$

$$= 0.92808 \text{ Ans}$$

Ans. 9. Poisson process with $\lambda = 0.8$

(a) Probability that the time between two adjacent events is longer than 1.5 is exponential distribution with Parameter $\lambda = 0.8$

Cumulative function of this distribution is

$$F(x) = 1 - e^{-\lambda x}$$

So the required Probability is,

$$P(X \geq 1.5) = 1 - P(X \leq 1.5)$$
$$= 1 - F(1.5)$$
$$= 1 - (1 - e^{-0.8 \times 1.5})$$
$$= 1 - (1 - e^{-1.2})$$
$$= 1 - 1 + e^{-1.2}$$
$$= e^{-1.2} = 0.3012 \text{ Ans}$$

(b) ~~in~~ numbers of events in a period of length 2 is Poisson distribution with Parameter

$$\lambda = 2 \times \lambda_1$$
$$= 2 \times 0.8$$
$$= 1.6$$

So the required Probability is,

$$P(X \geq 3) = 1 - P(X \leq 2)$$
$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$
$$= 1 - \left[\frac{e^{-1.6} \times (1.6)^0}{0!} + \frac{e^{-1.6} \times (1.6)^1}{1!} + \frac{e^{-1.6} \times (1.6)^2}{2!} \right]$$

$$= 1 - e^{-1.6}\left[1 + 1.6 + \frac{(1.6)^2}{2}\right]$$

$$= 1 - e^{-1.6}[2.6 + 1.28]$$

$$= 1 - 3.88 \times e^{-1.6}$$

$$= 1 - 0.7834$$

$$= 0.2166 \text{ Ans}$$

Ans. 11. Customer arrive at a service window according to a Poisson process with parameter $\lambda = 0.2$ per minute.

(a) the time between two successive arrivals is less than 6 minutes is -

$$P(X \le 6) = F(6)$$
$$= 1 - e^{-0.2 \times 6}$$
$$= 1 - e^{-1.2}$$
$$= 1 - 0.3012$$
$$= 0.699$$

(b) the probability that there will be exactly three arrivals during a given 10-minute period is -

number of arrivals has a Poisson distribution with parameter $\lambda = 10\lambda,$
$$= 10 \times 0.2$$
$$= 2$$

so that, the required probability is,

$$P(X=3) = \frac{e^{-2} \times (2)^3}{3!}$$
$$= \frac{e^{-2} \times 8}{6}$$
$$= 0.180 \text{ Ans}$$

Chapter 5.1

Ans. 1. given that $Z \sim N(0,1)$, Here $\mu = 0$, $\sigma^2 = 1$
$\sigma = 1$

(a) $P(Z \leq 1.34) = P(-\infty \leq Z \leq 1.34)$

$= \phi($ ~~1.34-0~~

we know that,

$$P(a \leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(-\infty \leq Z \leq 1.34) = \Phi\left(\frac{1.34-0}{1}\right) - \Phi\left(\frac{-\infty-0}{1}\right)$$

$$= \Phi(1.34)$$

$$= 0.9099 \text{ Ans}$$

(b) $P(Z \geq -0.22) = 1 - P(Z \leq -0.22)$

$$= 1 - \left[\Phi\left(\frac{-0.22-0}{1}\right) - \Phi\left(\frac{-\infty-0}{1}\right)\right]$$

$$= 1 - [\Phi(-0.22)]$$

$$= 1 - 0.4129$$

$$= 0.5871 \text{ Ans}$$

(c) $P(-2.19 \leq Z \leq 0.43) = \Phi\left(\frac{0.43-0}{1}\right) - \Phi\left(\frac{-2.19-0}{1}\right)$

$$= \Phi(0.43) - \Phi(-2.19)$$

$$= 0.6664 - 0.0143$$

$$= 0.6521 \text{ Ans}$$

(d) $P(0.09 \leq Z \leq 1.76) = \Phi\left(\frac{1.76-0}{1}\right) - \Phi\left(\frac{0.09-0}{1}\right)$

$$= \Phi(1.76) - \Phi(0.09)$$

$= 0.9608 - 0.5359$

$= 0.4249$ Ans

(e) $P(|z| \leq 0.38) = P(-0.38 \leq z \leq 0.38)$

$= \phi\left(\frac{+0.38-0}{1}\right) - \phi\left(\frac{-0.38-0}{1}\right)$

$= \phi(+0.38) - \phi(-0.38)$

$= 0.6480 - 0.3520$

$= 0.2960$ Ans

(f) The value of x for which $P(z \leq x) = 0.55$

we know that the cumulative fn of a Normal distribution is,

$$\Phi(x) = \int_{-\infty}^{x} \phi(x)\, dx$$

$$\Rightarrow 0.55 = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$\Rightarrow 0.55 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx$$

Solving this

$\Rightarrow x = 0.1257$

(g) The value of x for which $P(z \geq x) = 0.72$

we know that the cumulative fn of a normal distribution is,

$$\Rightarrow \phi(x) = \int_{-\infty}^{x} \phi(x)\, dx$$

$$\Rightarrow 1 - 0.72 = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

solving this

$x = -0.5828$

(h) The value of x for which $P(|z| \leq x) = 0.31$

we know that,

$$P(|z| \leq x) = P(-x \leq z \leq x)$$

$$\Rightarrow \quad 0.31 = \phi(x) - \phi(-x)$$

$$\Rightarrow \quad 0.31 = (2 \times \phi(x)) - 1$$

$$\Rightarrow \quad 1.31 = 2 \times \phi(x)$$

$$\Rightarrow \quad \phi(x) = 0.655$$

$$\Rightarrow \quad 0.655 = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

Solving this

$x = 0.3989$ Ans

Ans. 3. Given that $X \sim N(10, 2)$

So, $\mu = 10$, $\sigma^2 = 2$

$\Rightarrow \sigma = \sqrt{2} = 1.4142$

(a) $\Rightarrow P(X \le 10.34) = P(-\infty \le x \le 10.34)$

$= \phi\left(\frac{10.34-10}{1.4142}\right) - \phi\left(\frac{-\infty-10}{1.4142}\right)$

$= \phi\left(\frac{0.34}{1.4142}\right)$

$= \phi(0.2404)$

$= 0.5950$ Ans

(b) $P(X \ge 11.98) = 1 - P(X \le 11.98)$

$= 1 - P(-\infty \le x \le 11.98)$

$= 1 - \left[\phi\left(\frac{11.98-10}{1.4142}\right) - \phi\left(\frac{-\infty-10}{1.4142}\right)\right]$

$= 1 - \left[\phi\left(\frac{1.98}{1.4142}\right) - 0\right]$

$= 1 - \phi(1.4001)$

$= 1 - 0.9193$

$= 0.0807$ Ans

(c) $P(7.67 \le X \le 9.90) = \phi\left(\frac{9.90-10}{1.4142}\right) - \phi\left(\frac{7.67-10}{1.4142}\right)$

$= \phi\left(\frac{-0.1}{1.4142}\right) - \phi\left(\frac{-2.33}{1.4142}\right)$

$= \phi(-0.0707) - \phi(-1.6476)$

$= 0.4718 - 0.0497$

$= 0.4221$ Ans

(d) $P(10.88 \le X \le 13.22) = \phi\left(\frac{13.22-10}{1.4142}\right) - \phi\left(\frac{10.88-10}{1.4142}\right)$

$= \phi\left(\frac{3.22}{1.4142}\right) - \phi\left(\frac{0.88}{1.4142}\right)$

$= \phi(2.2769) - \phi(0.6223)$

$= 0.9886 - 0.7331$

$= 0.2555$ Ans

(e) $P(|Z| \ge 0.91) = 1 - P(|Z| \le 0.91)$

$= 1 - [P(-0.91 \le Z \le 0.91)]$

$= 1 - \left[\phi\left(\frac{0.91-10}{1.4142}\right) - \phi\left(\frac{-0.91-10}{1.4142}\right)\right]$

$= 1 - \left[\phi\left(\frac{-9.09}{1.4142}\right) - \phi\left(\frac{-10.91}{1.4142}\right)\right]$

$= 1 - [\phi(-6.4277) - \phi(-7.7146)]$

$= 1 - [$

(e) $P(|X-10| \le 3) = P(7 \le X \le 13)$

$= \left[\phi\left(\frac{13-10}{1.4142}\right) - \phi\left(\frac{7-10}{1.4142}\right)\right]$

$= \phi\left(\frac{3}{1.4142}\right) - \phi\left(\frac{-3}{1.4142}\right)$

$= \phi(2.1213) - \phi(-2.1213)$

$= [2\phi(2.1213)] - 1$

$= [2 \times 0.9831] - 1$

$= 1.9662 - 1$

$= 0.9662$ Ans

(f) The value of x which $P(X \leq x) = 0.81$

$$\Rightarrow \quad \phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$\Rightarrow \quad 0.81 = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

by solving this

$$\Rightarrow \quad x = 11.2415$$

(g) The value of x which $P(X \geq x) = 0.04$

$$\Rightarrow \quad P(X \geq x) = 1 - P(X \leq x)$$

$$\Rightarrow \quad 0.04 = 1 - \phi(x)$$

$$\Rightarrow \quad \phi(x) = 0.96$$

$$\Rightarrow \quad \phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$\Rightarrow \quad 0.96 = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

By solving this,

$$x = 12.4758$$

(h) The value of x for which $P(|X-10| \geq x) = 0.63$

$$\Rightarrow \quad P(|X-10| \geq x) = 1 - P(|X-10| \leq x)$$

$$= 1 - P(10 - x \leq X \leq 10 + x)$$

$$= 1 - [\phi(10+x) - \phi(10-x)]$$

$$= 1 - [2\phi(10+x) - 1]$$

$$\Rightarrow \quad 0.63 = 2 - 2\phi(10+x)$$

$$\Rightarrow \quad 2\phi(10+x) = 1.37$$

$$\Rightarrow \quad \phi(10+x) = 0.685$$

By solving this, $x = 0.6812$ Ans

<u>Ans</u>. 4. given that $X \sim N(-7, 14)$, Find:-

Here, $\mu = -7$, $\sigma^2 = 14 \Rightarrow \sigma = \sqrt{14} = 3.7417$

(a) $P(X \le 0) = P(-\infty \le X \le 0)$

$$= \phi\left(\frac{0-(-7)}{3.7417}\right) - \phi\left(\frac{-\infty-(-7)}{3.7417}\right)$$

$$= \phi\left(\frac{7}{3.7417}\right)$$

$$= \phi(1.8708)$$

$$= 0.9693$$

(b) $P(X \ge -10) = 1 - P(-\infty\, X \le -10)$

$$= 1 - P(-\infty \le X \le -10)$$

$$= 1 - \left[\phi\left(\frac{-10-(-7)}{3.7417}\right) - \phi\left(\frac{-\infty-(-7)}{3.7417}\right)\right]$$

$$= 1 - \left[\phi\left(\frac{-3}{3.7417}\right) - 0\right]$$

$$= 1 - \phi(-0.8018)$$

$\boxed{= 1 - 0.2[illegible]3}$

$$= 1 - 0.2113$$

$$= 0.7887$$

(c) $P(-15 \le X \le -1) = \phi\left(\frac{-1-(-7)}{3.7417}\right) - \phi\left(\frac{-15-(-7)}{3.7417}\right)$

$$= \phi\left(\frac{-1+7}{3.7417}\right) - \phi\left(\frac{-15+7}{3.7417}\right)$$

$$= \phi\left(\frac{6}{3.7417}\right) - \phi\left(\frac{-8}{3.7417}\right)$$

$$= \phi(1.6035) - \phi(-2.1381)$$

$$= 0.9456 - 0.01625$$

$$= 0.92935 \text{ Ans}$$

(d) $P(-5 \le X \le 2) = \phi\left(\frac{2-(-7)}{3.7417}\right) - \phi\left(\frac{-5-(-7)}{3.7417}\right)$

$= \phi\left(\frac{9}{3.7417}\right) - \phi\left(\frac{2}{3.7417}\right)$

$= \phi(2.4053) - \phi(0.5345)$

$= 0.9919 - 0.7035$

$= 0.2884$

(e) $P(|X+7| \ge 8) = 1 - P(|X+7| \le 8)$

$= 1 - P(-15 \le X \le 1)$

$= 1 - \left[\phi\left(\frac{+1-(-7)}{3.7417}\right) - \phi\left(\frac{-15-(-7)}{3.7417}\right)\right]$

$= 1 - \left[\phi\left(\frac{8}{3.7417}\right) - \phi\left(\frac{-8}{3.7417}\right)\right]$

$= 1 - \left[2\phi\left(\frac{8}{3.7417}\right) - 1\right]$

$\because \phi(x) - \phi(-x) = 2\phi(x) - 1$

$= 1 - [2\phi(2.1381) - 1]$

$= 1 - [2 \times 0.9837 - 1]$

$= 2 - 1.9674$

$= 0.0326$ Ans

(f) The value of x for which $P(X \le x) = 0.75$

we know that $\Rightarrow \phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

$\Rightarrow 0.75 = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

by solving this,

$x = -4.4763$

(g) The value of x for which $P(X \geq x) = 0.27$

$$\Rightarrow \quad P(X \geq x) = 1 - P(X \leq x)$$

$$\Rightarrow \quad P(X \leq x) = 1 - 0.27$$

$$\Rightarrow \quad P(X \leq x) = 0.73$$

we know that $\Rightarrow \quad \phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

$$\Rightarrow \quad 0.73 = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

by solving this we get,

$\Rightarrow x = -4.7071$ Ans

(h) The value of x for which $P(|X+7| \leq x) = 0.44$

$$\Rightarrow \quad P(|X+7| \leq x) = P(-(x+7) \leq X \leq x-7)$$

$$\Rightarrow \quad P(-(x+7) \leq X \leq x-7) = \phi(x-7) - \phi[-(x+7)]$$

$$\Rightarrow \quad 0.44 = \phi(x-7) - \phi(-(x+7))$$

by solving this we get,

$x = 2.18064$ Ans

Ans. 7. given that $X \sim N(\mu, \sigma^2)$

we know that, $P(X \leq \mu + \sigma Z_\alpha) = P(-\infty \leq X \leq \mu + \sigma Z_\alpha)$

$$\Rightarrow \quad = \phi\left(\frac{\mu + \sigma Z_\alpha - \mu}{\sigma}\right) - \phi\left(\frac{-\infty - \mu}{\sigma}\right)$$

$$= \phi\left(\frac{\sigma Z_\alpha}{\sigma}\right) - \phi(-\infty)$$

$$= \phi(Z_\alpha) - 0$$

$$= \phi(Z_\alpha) = 1 - \alpha$$

Thus, $P(\mu - \sigma Z_{\alpha/2} \leq X \leq \mu + \sigma Z_{\alpha/2}) = \phi\left(\frac{\mu + \sigma Z_{\alpha/2} - \mu}{\sigma}\right) - \phi\left(\frac{\mu - \sigma Z_{\alpha/2} - \mu}{\sigma}\right)$

$$= \phi\left(\frac{\sigma Z_{\alpha/2}}{\sigma}\right) - \phi\left(\frac{-\sigma Z_{\alpha/2}}{\sigma}\right)$$

$$= \phi(Z_{\alpha/2}) - \phi(-Z_{\alpha/2})$$

$$= 2\phi(Z_{\alpha/2}) - 1$$

$\because [\phi(x) - \phi(-x) = 2\phi(x) - 1]$

$$= 2\left(1 - \frac{\alpha}{2}\right) - 1$$

$\because \phi(Z_\alpha) = 1 - \alpha$

$\Rightarrow \phi(Z_{\alpha/2}) = 1 - \frac{\alpha}{2}$

$$= 2 - \alpha - 1$$

$$= 1 - \alpha$$

Hence proved

Ans. 9. The thickness of glass sheets produced by a certain process are normally distributed with a mean of $\mu = 3.00$ mm

and standard deviation $\sigma = 0.12$ mm

(a) we know that the cumulative function in normal distribution is,

$$\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

So the probability that a glass sheet is thicker than 3.2 mm is,

$$\Rightarrow P(X \geq 3.2) = P(3.2 \leq X \leq \infty)$$

$$= \phi\left(\frac{\infty - 3.00}{0.12}\right) - \phi\left(\frac{3.2 - 3.00}{0.12}\right)$$

$$= \phi(\infty) - \phi\left(\frac{0.2}{0.12}\right)$$

$$= 1 - \phi(1.6667)$$

$$= 1 - 0.9522$$

$$= 0.0478 \text{ Ans}$$

(b) the probability that a glass sheet is thinner than 2.7 mm is,

$$\Rightarrow P(X \leq 2.7) = P(-\infty \leq X \leq 2.7)$$

$$= \phi\left(\frac{2.7 - 3.00}{0.12}\right) - \phi\left(\frac{-\infty - 3.00}{0.12}\right)$$

$$= \phi\left(\frac{-0.3}{0.12}\right) - \phi(-\infty)$$

$$= \phi(-2.5) - 0$$

$$= \phi(-2.5) = 0.00621 \text{ Ans}$$

(C) Probability that a glass sheet has a thickness within the interval $[3.00-C, 3.00+C]$ is,

$$\Rightarrow P(3.00-C \le X \le 3.00+C) = \phi\left(\frac{3.00+C-3.00}{0.12}\right) - \phi\left(\frac{3.00-C-3.00}{0.12}\right)$$

$$= \phi\left(\frac{C}{0.12}\right) - \phi\left(\frac{-C}{0.12}\right)$$

$$= 2\phi\left(\frac{C}{0.12}\right) - 1$$

given that $P(3.00-C \le X \le 3.00+C) = 0.99$

$$\Rightarrow 2\phi\left(\frac{C}{0.12}\right) - 1 = 0.99$$

$$\Rightarrow 2\phi\left(\frac{C}{0.12}\right) = 0.99+1$$

$$\Rightarrow \phi\left(\frac{C}{0.12}\right) = \frac{1.99}{2}$$

$$\Rightarrow \frac{C}{0.12} = \phi^{-1}(0.995)$$

$$\Rightarrow C = 0.12 \times \phi^{-1}(0.995)$$

from computer software package

$$\Rightarrow C = 0.12 \times 2.5758$$

$$= 0.3091 \text{ Ans}$$

Ans. 11. The thickness of metal plates made by a particular machine are normally distributed with a mean of $\mu = 4.3$ mm and a standard deviation of $\sigma = 0.12$ mm

(a) for the upper quartile we have the cumulative function of Normal distribution,

$$\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

Here given that, $\phi(x) = 0.75$

$$\Rightarrow x = \phi^{-1}(0.75)$$

from computer software package,

we get, $x = 4.3809$

So the upper quartile is 4.3809.

Thus, for the lower quartile,

$$\phi(x) = 0.25$$

$$\Rightarrow x = \phi^{-1}(0.25)$$

from computer software package

$$\Rightarrow x = 4.2191$$

So the lower quartile is 4.2191.

(b) Probability that a metal plate has a thickness within the interval $[4.3-C, 4.3+C]$ is,

$$P(4.3-C \le X \le 4.3+C) = \phi\left(\frac{4.3+C-4.3}{0.12}\right) - \phi\left(\frac{4.3-C-4.3}{0.12}\right)$$

$$= \phi\left(\frac{C}{0.12}\right) - \phi\left(\frac{-C}{0.12}\right)$$

$$= 2\phi\left(\frac{c}{0.12}\right) - 1 \qquad \because \phi(x) - \phi(-x) = 2\phi(x) - 1$$

given that, $P(4.3 - c \le X \le 4.3 + c) = \frac{80}{100}$

$$= 0.8$$

$$\Rightarrow \quad 2\phi\left(\frac{c}{0.12}\right) - 1 = 0.8$$

$$\Rightarrow \quad 2\phi\left(\frac{c}{0.12}\right) = 0.8 + 1$$

$$\Rightarrow \quad \phi\left(\frac{c}{0.12}\right) = \frac{1.8}{2}$$

$$\Rightarrow \quad \frac{c}{0.12} = \phi^{-1}(0.9)$$

$$\Rightarrow \quad c = 0.12 \times 1.2816$$

$$\Rightarrow c = 0.1538 \text{ Ans}$$

Ans. 13. The resistance in milliohms of 1 copper cable at a certain temperature is normally distributed with mean $\mu = 23.8$ and variance $\sigma^2 = 1.28$

$$\sigma = \sqrt{1.28}$$
$$= 1.1314$$

(a) Probability that a 1-meter segment of copper cable has a resistance less than 23.0 is,

$$P(X \le 23.0) = P(-\infty \le X \le 23.0)$$
$$= \phi\left(\frac{23.0 - 23.8}{1.1314}\right) - \phi\left(\frac{-\infty - 23.8}{1.1314}\right)$$
$$= \phi\left(\frac{-0.8}{1.1314}\right) - \phi(0)$$
$$= \phi(-0.7071) - 0$$
$$= \phi(-0.7071) = 0.2398$$

(b) Probability that a 1-meter segment of copper cable has a resistance greater than 24.0 is.

$$P(X \ge 24.0) = 1 - P(-\infty \le X \le 24.0)$$
$$= 1 - \left[\phi\left(\frac{240 - 23.8}{1.1314}\right) - \phi\left(\frac{-\infty - 23.8}{1.1314}\right)\right]$$
$$= 1 - \left[\phi\left(\frac{0.2}{1.1314}\right) - \phi(-\infty)\right]$$
$$= 1 - [\phi(0.1768) - 0]$$
$$= 1 - 0.57017$$
$$= 0.42983 \text{ Ans}$$

(C) ~~What is~~ the Probability that a 1 - meter segment of copper cable has a resistance between 24.2 and 24.5 is,

$$P(24.2 \le X \le 24.5) = \Phi\left(\frac{24.2-23.8}{1.1314}\right) - \Phi\left(\frac{24.5-23.8}{1.1314}\right)$$

$$= \Phi\left(\frac{0.4}{1.1314}\right) - \Phi\left(\frac{0.7000}{1.1314}\right)$$

$$= \Phi(0.35355) - \Phi(0.6187)$$

$$= 0.6381 - .$$

$$= \Phi\left(\frac{24.5-23.8}{1.1314}\right) - \Phi\left(\frac{24.2-23.8}{1.1314}\right)$$

$$= \Phi\left(\frac{0.7}{1.1314}\right) - \Phi\left(\frac{0.4}{1.1314}\right)$$

$$= \Phi(0.6187) - \Phi(0.3535)$$

$$= 0.7319 - 0.6381$$

$$= 0.0938 \text{ Ans}$$

(d) the upper quartile of the resistance level is,

$$P(X \le x) = \Phi(x) = 0.75$$

$$\Rightarrow \frac{x-23.8}{1.1314} = \Phi^{-1}(0.75)$$

$$\Rightarrow x - 23.8 = 0.674 \times 1.1314$$

$$\Rightarrow x = 0.7626 + 23.8$$

$$\Rightarrow x = 24.5626 \text{ Ans}$$

(e) 95th Percentile of the resistance level is $= \frac{95}{100}$

$= 0.95$ th

quantile of resistance level is,

$$\Rightarrow \phi\left(\frac{x-23.8}{1.1314}\right) = 0.95$$

$$\Rightarrow \frac{x-23.8}{1.1314} = \phi^{-1}(0.95)$$

$$\Rightarrow \frac{x-23.8}{1.1314} = 1.645$$

$$\Rightarrow x-23.8 = 1.645 \times 1.1314$$

$$\Rightarrow x = 1.8612 + 23.8$$

$$= 25.66 \text{ Ans.}$$

Ans 18. A has an expected return of

dollars μ = \$30,000

Standard deviation σ = \$4000

Probability that the return will be at least 25,000 \$ is,

$$P(X \geq 25{,}000) = P(25000 \leq X \leq \infty)$$

$$= \phi\left(\frac{\infty - 30{,}000}{4000}\right) - \phi\left(\frac{25000 - 30{,}000}{4000}\right)$$

$$= \phi(\infty) - \phi\left(\frac{-5000}{4000}\right)$$

$$= \phi(\infty) - \phi(-1.25)$$

$$= 1 - \phi(-1.25)$$

$$= 1 - 0.1056$$

$$= 0.8944$$ Ans

Chapter 5.2

Ans 1. given that, $X \sim N(3.2, 6.5)$, $Y \sim (-2.1, 3.5)$
$Z \sim N(12.0, 7.5)$

(a) for $X+Y$, there is

$$\mu = \mu_1 + \mu_2 = 3.2 + (-2.1)$$
$$= 1.1$$
$$\sigma^2 = \sigma_1^2 + \sigma_2^2 = 6.5 + 3.5$$
$$= \sqrt{10}$$

So, the probability is,

$$P(X+Y \geq 0) = P(0 \leq X+Y \leq \infty)$$
$$= \phi\left(\frac{\infty - 1.1}{\sqrt{10}}\right) - \phi\left(\frac{0-1.1}{\sqrt{10}}\right)$$
$$= \phi(\infty) - \phi\left(\frac{-0.1}{\sqrt{10}}\right)$$
$$= \phi(\infty) - \phi\left(\frac{-0.100}{3.1623}\right)$$
$$\cancel{1 - 0.4960}$$
$$= 1 - \phi\left(\frac{-0.1}{3.1623}\right)$$
$$= 1 - \phi(-0.0316)$$
$$= 1 - 0.4874$$
$$=$$

$$= 1 - \phi\left(\frac{-1.1}{3.1623}\right)$$
$$= 1 - \phi(-0.3478)$$
$$= 1 - 0.3639$$
$$= 0.6361 \text{ Ans}$$

(b) ~~for $2X + 3Y + 4Z \leq 10$, there is~~,
~~$\mu = \mu_1$~~.

(b) for $x+y-2z$, there is,

$$\mu = \mu_1 + \mu_2 - 2\mu_3$$
$$= 3.2 + (-2.1) - 2(12.0)$$
$$= 1.1 - 24 = -22.9$$

and
$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2^2\sigma_3^2$$
$$= 6.5 + 3.5 + 4 \times 7.5$$
$$= 10 + 30 = 40$$
$$\sigma = \sqrt{40} = 6.3245$$

So, the probability is,

$$P(x+y-2z \le -20) = P(-\infty \le x+y-2z \le -20)$$
$$= \phi\left(\frac{-20+22.9}{6.3245}\right) - \phi\left(\frac{-\infty+22.9}{6.3245}\right)$$
$$= \phi\left(\frac{2.9}{6.3245}\right) - \phi(-\infty)$$
$$= \phi(0.4585) - 0$$
$$= 0.6767 \text{ Ans}$$

(c) for $3x+5y$, there is

$$\mu = 3\mu_1 + 5\mu_2 = 3\times 3.2 + 5\times(-2.1)$$
$$= 9.6 - 10.5 = -0.9$$
$$\sigma^2 = 3^2\sigma_1^2 + 5^2\sigma_2^2 = 9\times 6.5 + 25\times 3.5$$
$$= 58.5 + 87.5 = 146$$
$$\Rightarrow \sigma = \sqrt{146} = 12.083$$

So, the probability is,

$$P(3x+5y \ge 1) = P(1 \le 3x+5y \le \infty)$$
$$= \phi\left(\frac{\infty+0.9}{12.083}\right) - \phi\left(\frac{1+0.9}{12.083}\right)$$
$$= \phi(\infty) - \phi\left(\frac{1.9}{12.083}\right)$$

$$\cancel{= 1 - \phi(0.0083)}$$
$$= 1 - \phi(0.1572)$$
$$= 1 - 0.5625$$
$$= 0.4375 \text{ Ans}$$

(d) for $4X - 4Y + 2Z$, there is,

$$u = 4u_1 - 4u_2 + 2u_3$$
$$= 4 \times 3.2 - 4 \times (-2.1) + 2 \times (12.0)$$
$$= 12.8 + 8.4 + 24$$
$$= 45.2$$

$$\sigma^2 = 4^2\sigma_1^2 + 4^2\sigma_2^2 + 2^2\sigma_3^2$$
$$= 16 \times 6.5 + 16 \times 3.5 + 4 \times 7.5$$
$$= 160 + 30 = 190$$
$$\sigma = \sqrt{190} = 13.784$$

$$P(4X - 4Y + 2Z \leq 25) = P(-\infty \leq 4X - 4Y + 2Z \leq 25)$$
$$= \phi\left(\frac{25 - 45.2}{13.784}\right) - \phi\left(\frac{-\infty - 45.2}{13.784}\right)$$
$$= \phi\left(\frac{-20.2}{13.784}\right) - \phi(-\infty)$$
$$= \phi(-1.4655) - 0$$
$$= 0.0714 \text{ Ans}$$

(e) for $X + 6Y + Z$, there is,

$$u = u_1 + 6u_2 + u_3$$
$$= 3.2 + 6 \times (-2.1) + 12$$
$$= 3.2 - 12.6 + 12$$
$$= 2.6$$

$$\sigma^2 = \sigma_1^2 + 36\sigma_2^2 + \sigma_3^2$$
$$= 6.5 + 36 \times 3.5 + 7.5$$
$$\sigma^2 = 140 \Rightarrow \sigma = \sqrt{140} = 11.8322$$

So, the probability is,

$$P(|x+6y+z| \geq 2) = 1-\left[P(-2 \leq x+6y+z \leq 2)\right]$$
$$= 1-\left[\phi\left(\frac{2-2.6}{11.8322}\right) - \phi\left(\frac{-2-2.6}{11.8322}\right)\right]$$
$$= 1-\left[\phi\left(\frac{-0.6}{11.8322}\right) - \phi\left(\frac{-4.6}{11.8322}\right)\right]$$
$$= 1-\left[\phi(-0.0507) - \phi(-0.3888)\right]$$
$$= 1-\left[0.4797 - 0.3487\right]$$
$$\cancel{= 1 - 0.1715}$$
$$= 1-\left[0.4797 - 0.3487\right]$$
$$= 1 - 0.1310 = 0.8690 \text{ Ans}$$

(F) for $2x-y$ there is,

$$\mu = 2\mu_1 - \mu_2$$
$$= 2\times 3.2 - (-2.1)$$
$$= 6.4 + 2.1 = 8.5$$
$$\sigma^2 = 4\sigma_1^2 + \sigma_2^2 = 4\times 6.5 + 3.5$$
$$\sigma^2 = 29.5$$
$$\Rightarrow \sigma = 5.4314$$

So, $P(|2x-y-6| \leq 1) = P(5 \leq 2x-y \leq 7)$

$$= \phi\left(\frac{7-8.5}{5.4314}\right) - \phi\left(\frac{5-8.5}{5.4314}\right)$$
$$= \phi\left(\frac{-1.5}{5.4314}\right) - \phi\left(\frac{-3.5}{5.4314}\right)$$
$$= \phi(-0.2762) - \phi(-0.6444)$$
$$= 0.3912 - 0.2597$$
$$= 0.1315 \text{ Ans}$$

Ans. 3.

given that $X_i \sim (\mu_i, \sigma_i^2)$

independent variable so $\mu_i = 0$, $\sigma^2 = 0$

(a) So the

$$P(|X_i| \leq 0.5) = P(-0.5 \leq X_i \leq 0.5)$$
$$= \phi(0.5) - \phi(-0.5)$$
$$= 2\phi(0.5) - 1$$
$$= 2 \times 0.6915 - 1$$
$$= 0.3830$$

(b) given that this is the normal distribution with

$\mu = 0$, $\sigma^2 = \frac{1}{8} \Rightarrow \sigma = \frac{1}{\sqrt{8}} = 0.3535$

So,

$$P(|X| \leq 0.5) = P(-0.5 \leq X \leq 0.5)$$
$$= \phi\left(\frac{0.5-0}{0.3535}\right) - \phi\left(\frac{-0.5-0}{0.3535}\right)$$
$$= 2\phi\left(\frac{0.5}{0.3535}\right) - 1$$
$$= 2\phi(1.4144) - 1$$
$$= 2 \times 0.9214 - 1$$
$$= 1.8428 - 1 = 0.8428 \text{ Ans}$$

(C) given that this is the normal distribution with

$\mu = 0$, $\sigma^2 = \frac{1}{n} \Rightarrow \sigma = \frac{1}{\sqrt{n}}$

So, $P(|X| \leq 0.5) = P(-0.5 \leq X \leq 0.5)$

$$= \phi\left(\frac{0.5-0}{1/\sqrt{n}}\right) - \phi\left(-\frac{0.5-0}{1/\sqrt{n}}\right)$$
$$= 2\phi\left(\frac{0.5}{1/\sqrt{n}}\right) - 1$$

given that $P(|X| \leq 0.5) \geq 0.99$

$\Rightarrow 2\phi(0.5\sqrt{n}) - 1 = 0.99$

$\Rightarrow \phi(0.5\sqrt{n}) = \frac{1.99}{2}$

$\Rightarrow 0.5\sqrt{n} = \phi^{-1}(.995) \Rightarrow$

$$0.5\sqrt{n} = 2.5758$$

$$\Rightarrow \sqrt{n} = \frac{2.5758}{0.5}$$

$$\Rightarrow \sqrt{n} = 5.1516$$

$$\Rightarrow n = (5.1516)^2 = 26.54$$

which is satisfied for $n \geq 27$ Ans

Ans. 9. Sugar packets have weights with $N(1.03, 0.014^2)$ distribution. A box contains 22 sugar packets.

(a) This is the normal distribution of the total weight of sugar in a box with parameter,

$$\mu = 22 \times \mu_1 = 22 \times 1.03 = 22.66$$

and $$\sigma^2 = 22 \times \sigma_1^2 = 22 \times 0.014^2$$

$$= 22 \times 1.96 \times 10^{-4} = 0.0043$$

$$\sigma = \sqrt{0.0043} = 0.0657$$

(b) for the upper quartile of the total weight of sugar in a box is given by

$$P(X \leq x) = 0.75$$

$$\Rightarrow \quad P(X \leq x) = \phi\left(\frac{x - 22.66}{0.0657}\right) - \phi\left(\frac{-\infty - 22.66}{0.0657}\right)$$

$$= \phi\left(\frac{x - 22.66}{0.0657}\right) - \phi(-\infty)$$

$$\Rightarrow \quad \phi\left(\frac{x - 22.66}{0.0657}\right) = 0.75$$

$$\Rightarrow \quad \frac{x - 22.66}{0.0657} = \phi^{-}(0.75)$$

$$\Rightarrow \quad x - 22.66 = 0.674 \times 0.0657$$

$$\Rightarrow \quad x = 0.0443 + 22.66$$

$$= 22.7043$$

Thus, for lower quartile,

$$P(X \leq x) = 0.25$$

$$\Rightarrow \quad P(X \le x) = \phi\left(\frac{x - 22.66}{0.0657}\right) - \phi\left(\frac{-\infty - 22.66}{0.0657}\right)$$

$$0.25 = \phi\left(\frac{x - 22.66}{0.0657}\right) - \phi(-\infty)$$

$$\Rightarrow \quad \phi\left(\frac{x - 22.66}{0.0657}\right) = 0.25$$

$$\Rightarrow \quad \frac{x - 22.66}{0.0657} = \phi^{-1}(0.25)$$

$$\Rightarrow \quad x - 22.66 = 0.0657 \times (-0.674)$$

$$\Rightarrow \quad x = -0.0443 + 22.66$$

$$x = 22.616 \text{ Ans}$$

Ans. 11. $X_1 \ldots X_{15}$ are independent identically distributed $N(4.5, 0.88)$ random variables, with an average x.

for an average x,

$$\mu = 4.5 \quad \text{and} \quad \sigma^2 = \frac{0.88}{15}$$

$$= 0.0587$$

$$\sigma = 0.2422$$

(a) $$P(4.2 \le X \le 4.9) = \phi\left(\frac{4.9-4.5}{0.2422}\right) - \phi\left(\frac{4.2-4.5}{0.2422}\right)$$

$$= \phi\left(\frac{0.4}{0.2422}\right) - \phi\left(\frac{-0.3}{0.2422}\right)$$

$$= \phi(1.6515) - \phi(-1.2386)$$

$$= 0.9507 - 0.1077$$

$$= 0.8430 \text{ Ans}$$

(b) given that, $P(4.5 - c \le X \le 4.5 + c) = 0.99$

and we know that,

$$P(4.5-c \le X \le 4.5+c) = \phi\left(\frac{4.5+c-4.5}{0.2422}\right) - \phi\left(\frac{4.5-c-4.5}{0.2422}\right)$$

$$= \phi\left(\frac{c}{0.2422}\right) - \phi\left(\frac{-c}{0.2422}\right)$$

$$= 2\phi\left(\frac{c}{0.2422}\right) - 1$$

$$\Rightarrow 2\phi\left(\frac{c}{0.2422}\right) = 0.99 + 1$$

$$\Rightarrow \phi\left(\frac{c}{0.2422}\right) = \frac{1.99}{2}$$

$$\Rightarrow \frac{c}{0.2422} = \phi^{-1}(0.995)$$

$$\Rightarrow c = 0.2422 \times 2.576 = 0.6239 \text{ Ans}$$

Ans. 19. Company A has an expected return of

$E(X) = \mu_1 = \$\,30,000$

and $\sigma_1 = \$\,4000$

Company B has an expected return of

$E(X) = \mu_2 = \$\,45,000$

and $\sigma_2 = \$\,3000$

total return from both investment is,

$X = X_1 + X_2$

So, $\mu = \mu_1 + \mu_2 = 30,000 + 4500$

$= 75000$

and $\sigma^2 = \sigma_1^2 + \sigma_2^2 = (4000)^2 + (3000)^2$

$= (16+9) \times 10^6$

$= 25 \times 10^6$

$\sigma = 5000$

So, the required probability is,

$P(X \geq 85,000) = P(85000 \leq X \leq \infty)$

$= \phi\left(\frac{\infty - 75,000}{5000}\right) - \phi\left(\frac{85000 - 75000}{5000}\right)$

$= \phi(\infty) - \phi\left(\frac{10,000}{5000}\right)$

$= \phi(\infty) - \phi(2)$

$= 1 - 0.9772$

$= 0.0228$ Ans

Chapter 5.3

Ans 5. The number of cracks in a ceramic tile has a poisson distribution with parameter $\lambda = 2.4$

(a) if there is 500 ceramic tiles, then the apporoximate distribution is normal distribution

with $\mu = 500 \times 2.4 = 1200$

and $\sigma^2 = 500 \times 2.4 = 1200$

$\sigma = \sqrt{1200} = 34.641$

(b) the probability that there are more than 1250 cracks in 500 ceramic tiles is,

$$P(X \geq 1250) = 1 - \phi\left(\frac{1250-1200}{34.641}\right)$$

$$= 1 - \phi\left(\frac{50}{34.641}\right)$$

$$= 1 - \phi(1.4434)$$

$$= 1 - 0.9255$$

$$= 0.0745 \text{ Ans}$$

Ans. 7. Probability that a purchaser unpacks a newly television set it doesn't work properly is $p = 0.0007$

Company sells 2,50,000 t.v. sets a year, so

$$n = 2,50,000$$

So, the probability that there will be no more than 200 unhappy purchaser in a year is,

$$P(X \leq 200) = \cancel{\phi\left(200 - (2,5\right.} \; \phi\left(\frac{200 + 0.5 - (2,50,000 \times 0.0007)}{\sqrt{2,50,000 \times 0.0007 \times 0.9993}}\right)$$

$$= \phi\left(\frac{200.5 - 175}{\sqrt{174.8775}}\right)$$

$$= \phi\left(\frac{25.5}{13.2241}\right)$$

$$= \phi(1.9283)$$

$$= 0.9731 \text{ Ans.}$$

Ans. 9. A day's sales in \$1000 units at a gas station have a gamma distribution with $k = 5$ and $\lambda = 0.9$

The Expected value of a day's sales in 1000\$ is

$$E(X) = \frac{k}{\lambda} = \frac{5}{0.9} = 5.5556$$

and $Var(X) = \frac{k}{\lambda^2} = \frac{5}{0.9^2} = \frac{5}{0.81} = 6.1728$

The yearly income can be approximated by a normal distribution with

$$\mu = 365 \times E(X)$$
$$= 365 \times 5.5556 = 2027.794$$

and $\sigma^2 = 365 \times 6.1728 = 2253.072$

$$\sigma = 47.4665$$

So the probability that in one year the gas station takes in more than \$2 million is,

$$P(X \geq 2000) = 1 - \phi\left(\frac{2000 - 2027.794}{\sqrt{2253.072}}\right)$$
$$= 1 - \phi\left(\frac{-27.794}{47.4665}\right)$$
$$= 1 - \phi(-0.5855)$$
$$= 1 - 0.2791$$
$$= 0.7209 \text{ Ans}$$

Ans. 13. The lifetime of batteries are independent with an exponential distribution with a mean of $E(X) = 48$ days. $\because E(x) = \frac{1}{\lambda} = 84$

$\Rightarrow \lambda = \frac{1}{84} = \frac{1}{84}$

There is a random selection of 350 batteries

So $n = 350$

Probability that batteries have lifetime between 60 and 100 is,

$$P(60 \le X \le 100) = F(100) - F(60)$$

$$= \left(1 - e^{-100/84}\right) - \left(1 - e^{-60/84}\right)$$

$$= e^{-60/84} - e^{-100/84}$$

$$\cancel{= e^{-1.25} - e^{-2.0833}}$$

$$\cancel{= 0.2865 - 0.1245}$$

$$\cancel{= 0.1620}$$

$$= e^{-60/84} - e^{-100/84}$$

$$= e^{-0.7143} - e^{-1.1905}$$

$$= 0.4895 - 0.3041$$

$$= 0.1854$$

So, now using normal distribution,

Probability that at least 55 of the batteries have lifetimes between 60 and 100 days is,

$$P(Y \ge 55) = 1 - \Phi\left(\frac{55 - 0.5 - (350 \times 0.1854)}{\sqrt{350 \times 0.1854 \times 0.8146}}\right)$$

$$= 1 - \phi\left(\frac{54.5 - 64.89}{\sqrt{52.8594}}\right)$$

$$= 1 - \phi\left(\frac{-10.39}{7.2704}\right)$$

$$= 1 - \phi(-1.4291)$$

$$= 1 - 0.07649$$

$$= 0.92351 \text{ Ans}$$

Ans.15. the components have weight that are independent and uniformaly distributed between 890 and 892.

(a) The probability that the components weighs more than 891.2 is,

$$P(X \geq 891.2) = \frac{892-891.2}{892-890}$$

$$= \frac{0.8}{2} = 0.4$$

Using the negative binomial distribution the required probability is given by as $P = 0.4$ and $r = 3$

$$\Rightarrow P(X=6) = C^{6-1}_{3-1} (1-0.4)^{6-3} (0.4)^3$$

$$= C^{5}_{2} (0.6)^3 (0.4)^3$$

$$= 10 \times (0.6)^3 \times (0.4)^3$$

$$= 0.13824 \text{ Ans}$$

(b) If a box contains 200 components, so $n = 200$

Probability that components have weigh more than 890.7 is,

$$P(X \geq 890.7) = \frac{892-890.7}{892-890} = \frac{1.3}{2} = 0.65$$

Using normal distribution probability that at least 100 component weigh more than 890.7 is,

$$P(Y \geq 100) = 1 - \phi\left(\frac{100 - 0.5 - (200 \times 0.65)}{\sqrt{200 \times 0.65 \times 0.35}}\right)$$

$$= 1 - \phi\left(\frac{99.5 - 130}{\sqrt{45.5}}\right)$$

$$= 1 - \phi\left(\frac{-30.5}{6.7454}\right)$$

$$= 1 - \phi(-4.5216)$$

$$= 1 - 0.00000307$$

$$= 0.99999693 \text{ Ans}$$

Chapter 7.2

Ans. 1. given that $E(X_1) = 21$, $Var(X_1) = 10$

and $E(X_2) = 21$, $Var(X_2) = 15$

and given that,

$$\hat{21}_1 = \frac{X_1}{2} + \frac{X_2}{2}$$

$$\hat{21}_2 = \frac{X_1}{4} + \frac{3X_2}{4}$$

$$\hat{21}_3 = \frac{X_1}{6} + \frac{X_2}{3} + 9$$

(a) We know that, a point estimate is biased if

$$bias(\hat{\theta}) = E(\hat{\theta}) - \theta$$

So, Here

$$bias(\hat{u}_1) = E(\hat{u}_1) - 21$$

$$\Rightarrow E(\hat{21}_1) = E\left[\frac{X_1}{2}\right] + E\left(\frac{X_2}{2}\right)$$

$$= \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2)$$

$$E(\hat{u}_1) = \frac{1}{2} \times 21 + \frac{1}{2} \times 21 = 21$$

and $u_1 = 21$

So, $bias(\hat{u}_1) = 21 - 21 = 0$

So $\hat{21}_1$ is unbiased.

$$bias(\hat{21}_2) = E(\hat{21}_2) - 21_2$$

$$E(\hat{u}_2) = E\left(\frac{X_1}{4}\right) + E\left(\frac{3X_2}{4}\right)$$

$$= \frac{1}{4}E(X_1) + \frac{3}{4}E(X_2)$$

$$= \frac{1}{4} \times 21 + \frac{3}{4} \times 21 = 21$$

$$\text{bias}(\hat{u}_2) = 21 - 21 = 0$$

So, The point Estimate $\hat{u}_2$ is unbiased.

$$\text{bias}(\hat{u}_3) = E(\hat{u}_3) - u_3$$

$$\Rightarrow \quad E(\hat{u}_3) = E\left(\frac{X_1}{6}\right) + E\left(\frac{X_2}{3}\right) + 9$$

$$= \frac{1}{6} E(X_1) + \frac{1}{3} E(X_2) + 9$$

$$= \frac{21}{6} + \frac{21}{3} + 9$$

$$= \frac{21}{2} + 9$$

$$\Rightarrow \quad \text{bias}(\hat{u}_3) = \frac{21}{2} + 9 - 21$$

$$= 9 - \frac{21}{2}$$

(b) We know that,

$$\text{Var}(\hat{u}_1) = \frac{\text{Var}(X_1)}{2^2} + \frac{\text{Var}(X_2)}{2^2}$$

$$= \frac{10}{4} + \frac{15}{4} = \frac{25}{4} = 6.25$$

and $\text{Var}(\hat{u}_2) = \dfrac{\text{Var}(X_1)}{4^2} + \left(\dfrac{3}{4}\right)^2 \text{Var}(X_2)$

$$= \frac{10}{16} + \frac{9}{16} \times 15$$

$$= \frac{10+135}{16} = \frac{145}{16} = 9.0625$$

Thus, $\text{Var}(\hat{u}_3) = \dfrac{\text{Var}(X_1)}{6^2} + \dfrac{\text{Var}(X_2)}{3^2}$

$$= \frac{10}{36} + \frac{15}{9}$$

$$= \frac{10+60}{36} = \frac{70}{36} = 1.9444 \text{ Ans}$$

The Point estimate $\hat{x}_3$ has the smallest variance.

(C) we know that $MSE(\hat{\theta}) = var(\hat{\theta}) + (bias)^2$

Here $MSE(\hat{x}_1) = var(\hat{x}_1) + [bias(\hat{x}_1)]^2$

$= 6.25 + 0^2 = 6.25$

$MSE(\hat{x}_2) = var(\hat{x}_2) + [bias(\hat{x}_2)]^2$

$= 9.0625 + 0^2$

$= 9.0625$

$MSE(\hat{x}_3) = var(\hat{x}_3) + [bias(\hat{x}_3)]^2$

$= \left(9 - \frac{x}{2}\right)^2 + 1.9444$

given that $x = 8$

$\Rightarrow \quad MSE(\hat{x}_3) = (9-4)^2 + 1.9444$

$= 25 + 1.9444$

$= 26.9444$ Ans

Ans. 2. given that $E(X_1) = \mu$, $var(X_1) = 7$

and $E(X_2) = \mu$, $var(X_2) = 13$

$E(X_3) = \mu$ $var(X_3) = 20$

Point Estimates are.

$$\hat{\mu}_1 = \frac{X_1}{3} + \frac{X_2}{3} + \frac{X_3}{3}$$

$$\hat{\mu}_2 = \frac{X_1}{4} + \frac{X_2}{3} + \frac{X_3}{5}$$

$$\hat{\mu}_3 = \frac{X_1}{6} + \frac{X_2}{3} + \frac{X_3}{4} + 2$$

(a) $bais(\hat{\mu}_1) = E(\hat{\mu}_1) - \mu_1$

$$\Rightarrow E(\hat{\mu}_1) = \frac{1}{3}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{3}E(X_3)$$

$$= \frac{\mu}{3} + \frac{\mu}{3} + \frac{\mu}{3} = \mu$$

$bias(\hat{\mu}_1) = \mu - \mu = 0$

So, Point Estimate $\hat{\mu}_1$ is unbiased.

$bias(\hat{\mu}_2) = E(\hat{\mu}_2) - \mu_2$

$$\Rightarrow E(\hat{\mu}_2) = \frac{1}{4}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{5}E(X_3)$$

$$= \frac{\mu}{4} + \frac{\mu}{3} + \frac{\mu}{5}$$

$$= \frac{15\mu + 20\mu + 12\mu}{60} = \frac{47}{60}\mu$$

$bias(\hat{\mu}_2) = \frac{47}{60}\mu - \mu$

$$= \frac{-13}{60}\mu = -0.2167\mu$$

$bias(\hat{\mu}_3) = E(\hat{\mu}_3) - \mu_3$

$$\Rightarrow E(\hat{\mu}_3) = \frac{1}{6}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{4}E(X_3) + 2$$

$$= \frac{u}{6} + \frac{u}{3} + \frac{u}{4} + 2$$

$$= \frac{2u+4u+3u}{12} + 2$$

$$= \frac{9}{12}u + 2 = \frac{3}{4}u + 2$$

$$bais(\hat{u}_3) = \frac{3}{4}u + 2 - u$$

$$= 2 - \frac{u}{4}$$

(b)

$$var(\hat{u}_1) = \frac{var(x_1)}{3^2} + \frac{var(x_2)}{3^2} + \frac{var(x_3)}{3^2}$$

$$= \frac{1}{9}\left[var(x_1) + var(x_2) + var(x_3)\right]$$

$$= \frac{1}{9}\left[7 + 13 + 20\right]$$

$$= \frac{40}{9} = 4.4444 \text{ Ans}$$

$$var(\hat{u}_2) = \frac{var(x_1)}{16} + \frac{var(x_2)}{9} + \frac{var(x_3)}{25}$$

$$= \frac{1}{3600}\left[225\, var(x_1) + 400\, var(x_2) + 144\, var(x_3)\right]$$

$$= \frac{1}{3600}\left[225 \times 7 + 400 \times 13 + 20 \times 144\right]$$

$$= \frac{1}{3600}\left[1575 + 5200 + 2880\right]$$

$$= \frac{9655}{3600} = 2.6819 \text{ Ans}$$

$$var(\hat{u}_3) = \frac{var(x_1)}{6^2} + \frac{var(x_2)}{3^2} + \frac{var(x_3)}{4^2}$$

$$= \frac{var(x_1)}{36} + \frac{var(x_2)}{9} + \frac{var(x_3)}{16}$$

$$= \frac{4 \times 7 + 16 \times 13 + 20 \times 9}{144}$$

$$= \frac{28 + 208 + 180}{144}$$

$$= \frac{416}{144} = 2.8889 \text{ Ans}$$

The point estimate $\hat{u}_2$ has the smallest variance.

(C) $MSE(\hat{u}_1) = var(\hat{u}_1) + (\text{bias } \hat{u}_1)^2$

$$= 4.4444 + 0^2 = 4.444 \text{ Ans}$$

$MSE(\hat{u}_2) = var(\hat{u}_2) + (\text{bias } \hat{u}_2)^2$

$$= 2.6819 + (-0.21672u)^2$$

$$= 2.6819 + 0.0469u^2$$

given that $u = 3$

$$\Rightarrow MSE(\hat{u}_2) = 2.6819 + 0.0469 \times 3^2$$

$$= 2.6819 + 0.4226$$

$$= 3.1045$$

$MSE(\hat{u}_3) = var(\hat{u}_3) + (\text{bias } \hat{u}_3)^2$

$$= 2.8889 + \left(2 - \frac{u}{4}\right)^2$$

$$= 2.8889 + \left(2 - \frac{3}{4}\right)^2$$

$$= 2.8889 + \left(\frac{5}{4}\right)^2$$

$$= 2.8889 + 1.5625$$

$$= 4.4514 \text{ Ans}$$

Ans. 3. given that $E(x_1) = \mu$ $var(x_1) = 4$, $E(x_2) = \mu$ and $var(x_2) = 6$

(a) $\hat{\mu}_1 = \frac{x_1}{2} + \frac{x_2}{2}$

So, $$Var(\hat{\mu}_1) = \frac{Var(x_1)}{2^2} + \frac{Var(x_2)}{2^2}$$
$$= \frac{Var(x_1)}{4} + \frac{Var(x_2)}{4}$$
$$= \frac{1}{4}[4+6]$$
$$= \frac{10}{4} = 2.5$$

(b) $\hat{\mu} = p\,x_1 + (1-p)\,x_2$

So, $$var(\hat{\mu}) = p^2\,var(x_1) + (1-p)^2\,var(x_2)$$
$$= 4p^2 + 6(1-p)^2$$

for minimum varince $4p^2 = 6(1-p)^2$

So, it gives $p = 0.6$

The value $p = 0.6$ produces the smallest variance which is $var(\hat{\mu}) = 4(0.6)^2 + 6(1-0.6)^2$
$$= 4 \times (0.6)^2 + 6 \times (0.4)^2$$
$$= 1.44 + 0.96 = 2.4$$

(c) The relative efficiency is $= \frac{Var(\hat{\mu}_2)}{Var(\hat{\mu}_1)}$
$$= \frac{2.4}{2.5} = \frac{2.4}{2.5}$$
$$= 0.96 \text{ Ans}$$

Ans. 7. given that $X \sim N(\mu, \sigma^2)$ and

$$\hat{\mu} = \frac{X + \mu_0}{2}$$

$$\hat{\mu} = \frac{X}{2} + \frac{\mu_0}{2}$$

$$\text{bias}(\hat{\mu}) = E(\hat{\mu}) - \mu$$

$$\Rightarrow \quad E(\hat{\mu}) = \frac{E(X)}{2} + \frac{\mu_0}{2}$$

$$= \frac{\mu}{2} + \frac{\mu_0}{2}$$

$$\text{bias}(\hat{\mu}) = \frac{\mu+\mu_0}{2} - \mu = \frac{\mu_0 - \mu}{2}$$

$$\text{Var}(\hat{\mu}) = \frac{\text{Var}(X)}{2^2} = \frac{\text{Var}(X)}{4}$$

$$= \frac{\sigma^2}{4}$$

$$\text{MSE}(\hat{\mu}) = \text{Var}(\hat{\mu}) + (\text{bias}\,\hat{\mu})^2$$

$$= \frac{\sigma^2}{4} + \frac{(\mu_0 - \mu)^2}{4}$$

given that $|\mu - \mu_0| \leq \sqrt{3}\,\sigma$

so, $|\mu - \mu_0|_{max} = \sqrt{3}\sigma$

$$\Rightarrow \quad \text{MSE}(\hat{\mu})_{max} = \frac{\sigma^2}{4} + \frac{3\sigma^2}{4}$$

$$= \sigma^2$$

and we know that $X \sim N(\mu, \sigma^2)$

$\text{Var}(\hat{X}) = \sigma^2$

and bias $= \mu - \mu = 0$

$\mathrm{MSE}(\hat{X}) = \sigma^2 + 0^2 = \sigma^2$

$\Rightarrow \quad \mathrm{MSE}(\hat{X}) \geq \mathrm{MSE}(\hat{u})$

because max value of $\mathrm{MSE}(\hat{u})$ is can be equal to $\mathrm{MSE}(\hat{X})$.

Chapter 7.3

Ans. 3. a sample $X_1 \cdots X_n$ of normally distributed random variables with u_0 and variance $\sigma_0^2 = 7$

(a) if $n = 15$. then for $u_0 = \hat{x}$, mean is

$$X = X_1 + X_2 + \cdots + X_{15}$$

$$\hat{X} = \frac{X_1 + X_2 + \cdots + X_{15}}{15}$$

$$E(\hat{X}) = \frac{E(X_1) + E(X)_2 + \cdots + E(X_{15})}{15}$$

$$= \frac{u_0 + u_0 + \cdots + u_0}{15} = u_0$$

So, for $u_0 = \hat{x}$

$$u = u_0 - u_0 = 0$$

and $$\sigma^2 = \frac{\sigma_0^2 + \sigma_0^2 + \cdots + \sigma_0^2}{15^2}$$

$$= \frac{15\sigma_0^2}{15^2} = \frac{\sigma_0^2}{15} = \frac{7}{15} = 0.46667$$

$$\Rightarrow \sigma = \cancel{1.8074}\ 0.6831$$

So, required probability is,

$$P(|u - \hat{x}| \leq 0.4) = P(-0.4 \leq u - \hat{x} \leq 0.4)$$

$$= \phi\left(\frac{0.4 - 0}{0.6831}\right) - \phi\left(\frac{-0.4 - 0}{0.6831}\right)$$

$$= 2\phi\left(\frac{0.4}{1.8074}\right) - 1$$

$$= 2\phi(0.2213) - 1$$

$$= 2 \times 0.5876 - 1$$

$$= 1.1752 - 1 = 0.1752$$

$$= 2\phi\left(\frac{0.4}{0.6831}\right) - 1$$

$$= 2\phi(0.5856) - 1$$

$$= 2 \times 0.7209 - 1$$

$$= 1.4418 - 1 = 0.4418 \text{ Ans}$$

(b) when $n = 50$, then

$$X = X_1 + \cdots + X_{50}$$

$$\hat{X} = \frac{X_1 + \cdots + X_{50}}{50}$$

$$E(\hat{X}) = \frac{E(X_1) + \cdots + E(X_{50})}{50}$$

$$= \frac{u_0 + \cdots + u_0}{50} = u_0$$

$$= u_0$$

and $\sigma_{\cdot}^2 = \frac{\sigma_0^2}{50} = \frac{7}{50} = 0.14$

So, for $u - \hat{X}$, ~~0~~

$$u = u_0 - u_0 = 0$$

and $\sigma^2 = 0.14$

$$\Rightarrow \sigma = 0.3742$$

So, required probability is,

$$P(|u - \hat{X}| \leq 0.4) = P(-0.4 \leq u - \hat{X} \leq 0.4)$$

$$= \phi\left(\frac{0.4 - 0}{0.3742}\right) - \phi\left(\frac{-0.4 - 0}{0.3742}\right)$$

$$= 2\phi\left(\frac{0.4}{0.3742}\right) - 1$$

$$= 2 \times \phi(1.0689) - 1$$

$$= 2 \times 0.8574 - 1$$

$$= 1.7148 - 1 = 0.7148 \text{ Ans}$$

Ans. 7. A sample $x_1 \cdots x_n$ of normally distributed with mean u and $n = 21$.

(a) we know that if $x_1 \cdots x_n$ normally distributed with mean u then

$$\frac{\sqrt{n}(\hat{x}-u)}{s} \sim t_{n-1}$$

so, $$\frac{\sqrt{21}\ \hat{x}-u}{s} \sim t_{20}$$

$$\Rightarrow \frac{\hat{x}-u}{s} \sim \frac{1}{\sqrt{21}} t_{20}$$

given that $P(|\hat{x}-u/s| \leq c) = 0.95$

$$\Rightarrow P\left(\left|\frac{1}{\sqrt{21}} t_{20}\right| \leq c\right) = 0.95$$

$$\Rightarrow P\left(-c \leq \frac{1}{\sqrt{21}} t_{20} \leq c\right) = 0.95$$

$$\Rightarrow P\left(-\sqrt{21}c \leq t_{20} \leq \sqrt{21}c\right) = 0.95$$

$$\Rightarrow 2t_{\sqrt{21}c,20} - 1 = 0.95$$

$$\Rightarrow 2t_{\sqrt{21}c,20} = 1.95$$

$$\Rightarrow t_{\sqrt{21}c,20} = 0.975$$

$$\Rightarrow \sqrt{21}c = t_{20}^{-1}(0.975)$$

$$\Rightarrow c = \frac{2.086}{\sqrt{21}} = \frac{2.086}{4.5826}$$

$$= 0.4552 \text{ Ans}$$

(b) Thus, given that $P(|(\hat{x}-u)/S| \leq c) = 0.99$

$$\Rightarrow P\left(\left|\frac{1}{\sqrt{21}} t_{20}\right| \leq c\right) = 0.99$$

$$\Rightarrow P\left(-\cancel{C} \le \frac{1}{\sqrt{21}} t_{20} \le C\right) = 0.99$$

$$\Rightarrow P\left(-\sqrt{21}\,C \le t_{20} \le \sqrt{21}\,C\right) = 0.99$$

$$\Rightarrow \quad 2\,t_{\sqrt{21}C,\,20} - 1 = 0.99$$

$$\Rightarrow \quad 2\,t_{\sqrt{21}C,\,20} = 1.99$$

$$\Rightarrow \quad t_{\sqrt{21}C,\,20} = 0.995$$

$$\Rightarrow \quad \sqrt{21}\,C = t_{20}^{-1}(0.995)$$

$$\Rightarrow \quad C = \frac{2.845}{\sqrt{21}}$$

$$= \frac{2.845}{4.5826} = 0.6208 \text{ Ans}$$

Ans. 8. In a consumer survey total representative

Sample of people $n = 450$

Proportion of Sample of people that prefer product A to product B is
$p = 234$

so the point estimate of p is,

$$\hat{p} = \frac{p}{n} = \frac{234}{450}$$

$$= 0.52$$

Standard error of point estimate of p is

$$s.e.(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= \sqrt{\frac{0.52 \times (1-0.52)}{450}}$$

$$= \sqrt{\frac{0.52 \times 0.48}{450}}$$

$$= \sqrt{\frac{0.2496}{450}}$$

$$= \sqrt{5.5467 \times 10^{-4}}$$

$$= 0.0236 \text{ Ans}$$

Ans. 9. Total breaking strengths of 35 pieces of Cotton thread are measured, so $n = 35$

The sample mean $\bar{x} = 974.3$

and the sample variance is $s^2 = 452.1$

$\Rightarrow \quad s = \sqrt{452.1} = 21.2626$

Point estimate of the average breaking strength of this type of cotton thread is $\hat{x}$,

then $\hat{x} = \bar{x} = 974.3$

And, standard error of point estimate of $\bar{x}$ is,

$$s.e.(\hat{x}) = \sqrt{\frac{s}{n}}$$

$$= \frac{s}{\sqrt{n}} = \frac{21.2626}{\sqrt{35}}$$

$$= \frac{21.2626}{5.9161}$$

$$= 3.5940 \text{ Ans}$$

Ans. 22. The weight of bricks are normally distributed with $\mu = 110.0$ and $\sigma = 0.4$

Weight of 22 bricks are measured, so $n = 22$

Let resulting point estimate of μ is $\hat{\mu}$

So, probability is,

$$P(109.9 \le \hat{\mu} \le 110.1) = P(109.9 \le \bar{X} \le 110.1)$$

We know that, for $\bar{X}$,

$$\mu = 110.0$$

$$\text{and } \sigma^2 = \frac{\sigma_0^{\,2}}{22} = \frac{0.4^2}{22}$$

$$= \frac{0.16}{22} = 0.007272$$

$$\Rightarrow \sigma = 0.0853$$

So, required probability using normal distribution is,

$$P(109.9 \le \bar{X} \le 110.1) = \phi\left(\frac{110.1 - 110.0}{0.0853}\right) - \phi\left(\frac{109.9 - 110.0}{0.0853}\right)$$

$$= \phi\left(\frac{0.1}{0.0853}\right) - \phi\left(\frac{-0.1}{0.0853}\right)$$

$$= 2\phi(1.1723) - 1$$

$$= 2 \times 0.8795 - 1$$

$$= 1.759 - 1 = 0.759 \text{ Ans.}$$

Ans. 27. The failure time of a component has an exponential distribution with parameter $\lambda = 0.02$ per minute.

The experiments takes components $n = 110$

Probability that the component will last longer than one hour is–

$$P(X \geq 60) = 1 - \left(1 - e^{-0.02 \times 60}\right)$$

$$= e^{-0.02 \times 60}$$

$$= e^{-1.2} = 0.3012$$

Let Y be the number of components that last longer than one hour. So, required probability is,

$$P(0.3012 - 0.05 \leq \hat{Y} \leq 0.3012 + 0.05)$$

$$= P\left(0.2512 \leq \frac{Y}{110} \leq 0.3512\right)$$

$$= P(27.632 \leq Y \leq 38.632)$$

Using normal approximation,

$$P(27.632 \leq Y \leq 38.632) = \phi\left(\frac{38.632 + 0.5 - 110 \times 0.3012}{\sqrt{110 \times 0.3012 \times 0.6988}}\right) - \phi\left(\frac{27.632 - 0.5 - 110 \times 0.3012}{\sqrt{110 \times 0.3012 \times 0.6988}}\right)$$

$$= \phi\left(\frac{39.132 - 33.132}{\sqrt{23.1526}}\right) - \phi\left(\frac{27.132 - 33.132}{\sqrt{23.1526}}\right)$$

$$= \phi\left(\frac{6}{4.8117}\right) - \phi\left(\frac{-6}{4.8117}\right)$$

$$= 2\phi(1.2470) - 1$$

$$= 2 \times 0.8938 - 1 = 1.7876 - 1 = 0.7876 \text{ Ans}$$

Ans. 34.

the data set 7, 9, 14, 15, 22

Sample mean $\hat{\mu} = \bar{x} = \dfrac{7+9+14+15+22}{5}$

$= \dfrac{67}{5} = 13.4$

and $E(x^2) = 7^2 + 9^2 + 14^2 + 15^2 + 22^2$

$= 49 + 81 + 196 + 225 + 484$

$= 1035$

Variance of sample $\sigma^2 = E(x^2) - n\bar{x}^2 \,/ n-1$

$= \dfrac{1035 - (13.4)^2 \times 5}{4}$

$= \dfrac{1035 - 179.56 \times 5}{4}$

$= \dfrac{1035 - 897.8}{4}$

$= \dfrac{137.2}{4} = 34.3$

$\Rightarrow \sigma = \sqrt{34.3}$

$\Rightarrow \sigma = 5.8566$

and standard error of mean is

$\Rightarrow s.e.(\hat{\mu}) = \dfrac{\sigma}{\sqrt{n}}$

$= \dfrac{5.8566}{\sqrt{5}}$

$= \dfrac{5.8566}{2.2361} = 2.62$

So Option D is Correct.

Chapter 7.4

Ans. 1. 23 Observations are collected from a Poisson distribution and the sample average is $x = 5.63$, and $n = 23$

A point estimate of the parameter of the Poisson distribution is $\hat{\lambda}$,

So, $\hat{\lambda} = \bar{x} = 5.63$

Standard error of the Poisson distribution parameter $\hat{\lambda}$ is,

$$S.e.(\hat{\lambda}) = \sqrt{\frac{\hat{\lambda}}{n}}$$

$$= \sqrt{\frac{5.63}{23}}$$

$$= \sqrt{0.2448}$$

$$= 0.4948$$ Ans

Ans. 3. A set of independent data observation $x_1 \dots x_n$ that have an exponential distribution with Parameter λ.

Using the method of moments we know,

$$E(X) = \frac{1}{\lambda} = \bar{x}$$

$$\Rightarrow \quad \lambda = \frac{1}{\bar{x}}$$

The likelihood is,

$$L(x_1, \dots x_n, \lambda) = \lambda^n e^{-\lambda(x_1 + \dots + x_n)}$$

Which is maximized at $\hat{\lambda} = \frac{1}{\bar{x}}$

Ans. 5. A set of independent data observation $x_1 \cdots x_n$ that have a gamma distribution with $K = 5$ and Parameter λ.

using the method of moments, we know that the expectation for gamma distribution is,

$$E(X) = \frac{K}{\lambda} = \bar{x}$$

$$= \frac{5}{\lambda} = \bar{x} \quad \text{--}(1)$$

So, that the point estimate of λ is $\hat{\lambda}$ given as,

$$\Rightarrow \quad \hat{\lambda} = \cancel{\frac{\bar{x}}{5}} \; \frac{5}{\bar{x}} \qquad \text{from eq (1)}$$

now, the likelihood is,

$$L(x_1 \cdots x_n, \lambda) = \left\{\frac{1}{(K-1)!}\right\}^n \times \lambda^{Kn} \times x_1^{K-1} \times \cdots \times x_n^{K-1} \times e^{-\lambda(x_1 + \cdots + x_n)}$$

$$= \left(\frac{1}{4!}\right)^n \times \lambda^{5n} \times x_1^4 \times \cdots \times x_n^4 \times e^{-\lambda(x_1 + \cdots + x_n)}$$

$$= \left(\frac{1}{24}\right)^n \times \lambda^{5n} \times x_1^4 \times \cdots \times x_n^4 \times e^{-\lambda(x_1 + \cdots + x_n)}$$

which is maximized at $\hat{\lambda} = \frac{5}{\bar{x}}$

Chapter 8.1

Ans. 1. A sample of data observation is $n = 31$

mean $\bar{x} = 53.42$ and $s = 3.05$

given that 95% two sided t - interval for the population mean.

with this condn the confidence interval is,

$\Rightarrow x = t_{30}^{-1}(0.95)$

$\Rightarrow t_{0.025, 30} = 2.042$

$\Rightarrow \boxed{x = 2.042}$

$$\Rightarrow \left(53.42 - \frac{2.042 \times 3.05}{\sqrt{31}}, 53.42 + \frac{2.042 \times 3.05}{\sqrt{31}}\right)$$

$$\Rightarrow \left(53.42 - \frac{6.2281}{5.5678}, 53.42 + \frac{6.2281}{5.5678}\right)$$

$$\Rightarrow (53.42 - 1.1186, 53.42 + 1.1186)$$

$$\Rightarrow (52.3014, 54.5386)$$ Ans

Ans. 3. breaking strengths of a random sample of bundles of wool fibers $n = 20$

Sample mean $\bar{x} = 436.5$

and $s = 11.90$

At 90% Confidence, $1 - \alpha = 0.9$

$\Rightarrow \alpha = 0.1$

So, the critical point is $t_{\alpha/2, n-1} = t^{-1}(0.9)$

$\Rightarrow t_{0.005, 19} = 1.729$

The confidence interval is,

$$\left(\bar{x} - \frac{t_{0.005, 19} \times s}{\sqrt{n}}, \ \bar{x} + \frac{t_{0.005, 19} \times s}{\sqrt{n}} \right)$$

$$\left(436.5 - \frac{1.729 \times 11.90}{\sqrt{20}}, \ 436.5 + \frac{1.729 \times 11.90}{\sqrt{20}} \right)$$

$$\left(436.5 - \frac{20.5751}{4.4721}, \ 436.5 + \frac{20.5751}{4.4721} \right)$$

$$(436.5 - 4.6008, \ 436.5 + 4.6008)$$

$$(431.8992, \ 441.1008)$$

At 95% Confidence, $1 - \alpha = 0.95$

$\Rightarrow \alpha = 0.05$

So, the critical point is $t_{\alpha/2, n-1}$

$\Rightarrow t_{0.025, 19} = 2.093$

The confidence interval is,

$$\left(436.5 - \frac{2.093 \times 11.90}{\sqrt{20}}, \ 436.5 + \frac{2.093 \times 11.90}{\sqrt{20}} \right)$$

$$\left(436.5 - \frac{24.9067}{4.4721}, \ 436.5 + \frac{24.9067}{4.4721} \right)$$

$\Rightarrow (436.5 - 5.5693,\ 436.5 + 5.5693)$

$\Rightarrow (430.9307,\ 442.0693)$ Ans

At 99% confidence, $1-\alpha = 0.99$

$\Rightarrow \alpha = 0.01$

So, the critical point is $t_{\alpha/2, n-1}$

$\Rightarrow t_{0.005, 19} = 2.861$

the confidence interval is,

$$\left(436.5 - \frac{2.861 \times 11.90}{\sqrt{20}},\ 436.5 + \frac{2.861 \times 11.90}{\sqrt{20}}\right)$$

$$\left(436.5 - \frac{34.0459}{4.4721},\ 436.5 + \frac{34.0459}{4.4721}\right)$$

$$(436.5 - 7.6130,\ 436.5 + 7.6130)$$

$(428.887,\ 444.113)$ Ans

Even the 99% confidence level does not contain the value 450.0, and so 450.0 is not a plausible value for the average breaking strength.

Ans. 5. A sample of data observation $n = 28$

Sample mean $\bar{x} = 0.0328$

$\sigma = 0.015$

At 95% confidence the $1-\alpha = 0.9$

$\Rightarrow \alpha = 0.05$

the critical point is $t_{\alpha/2,\, n-1}$

$\Rightarrow t_{0.025,27} = 1.960$

the confidence interval is,

$$\left(0.0328 - \frac{1.960 \times 0.015}{\sqrt{28}},\ 0.0328 + \frac{1.960 \times 0.015}{\sqrt{28}}\right)$$

$$\left(0.0328 - \frac{0.0294}{5.2915},\ 0.0328 + \frac{0.0294}{5.2915}\right)$$

$$(0.0328 - 0.0056,\ 0.0328 + 0.0056)$$

$(0.0272,\ 0.0384)$ Ans

Ans. 7. a Population Standard deviation is no larger than 10

So, $s = 10$

given that 95% two-sided t-interval for the population mean that has a length at most 5.0.

with $1-\alpha = 0.95$

$\alpha = 0.05$

Critical point $t_{0.025,\,n-1} \simeq 2.0$ a sufficient sample size can be estimated as

$$n \geq 4 \times \left(\frac{t_{0.025,n-1}\, s}{L_0} \right)^2$$

$$= 4 \times \left(\frac{2 \times 10}{5} \right)^2$$

$$= 4 \times 4^2 = 64$$

A sample size of about $n = 64$ should be sufficient.

Ans. 11. Sample size of breaking strength $n = 20$

given that $L_0 = 10.0$ and $s = 11.90$

for 99%, $1-\alpha = 0.99$

$\alpha = 0.01$

Critical point $t_{\alpha/2, n-1}$

$$= t_{0.005, n-1} \simeq t_{0.005, 19}$$
$$\simeq 2.861$$

So, A total sample size of

$$n \geq 4 \times \left(\frac{t_{0.005, n-1} \times s}{L_0}\right)^2$$

$$= 4 \times \left(\frac{2.861 \times 11.90}{10.0}\right)^2$$

$$= 4 \times \left(\frac{34.0459}{10}\right)^2$$

$$= 4 \times (3.40459)^2$$

$$= 46.3649$$

$$\simeq 47$$

Therefore, an additional sample of at least $47-20 = 27$ observations should be sufficient.

Chapter 8.2

Ans. 1. A Sample of observations $n = 18$

Sample mean $\bar{x} = 57.74$

and $s = 11.20$

given that $H_0 : \mu = 55$. verses $H_A : \mu \neq 55.0$

(a) we know that, the test static is given by,

$$t = \frac{\sqrt{n}\,(\bar{x} - \mu_0)}{s}$$

$$= \frac{\sqrt{18}\,(57.74 - 55.0)}{11.20}$$

$$= \frac{4.2426 \times 2.74}{11.20}$$

$$= \frac{11.6247}{11.20} = 1.0379$$

$$\simeq 1.04$$

The p value is given by - $2 \times P(t_{17} \geq 1.04)$

$= 2 \times [1 - P(t_{17} \leq 1.04)]$

$= 2 \times [1 - t_{1.04, 17}]$

$= 2 \times [1 - 0.8435]$

$= 2 \times 0.1565 = 0.313$ Ans

(b) we know that, the tests static is given by -

$$t = \frac{\sqrt{n}\,(x - \mu_0)}{s} = \frac{\sqrt{18}\,(57.74 - 65.0)}{11.20}$$

$$= \frac{4.2426 \times (-7.26)}{11.20}$$

$$= -\frac{30.8013}{11.20} = -2.7501$$

The p value is $= P(t_{17} \leq -2.75)$

$= t_{-2.75, 17} = 0.0068$ Ans

Ans. 3. A sample of observations $n = 13$

Sample mean $\bar{x} = 2.879$

and $\sigma = 0.325$

(a) for $H_0: \mu = 3.0$ verses $H_A: \mu \neq 3.0$

we know that, the test statistics is

$$Z = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} = \frac{\sqrt{13}\,(2.879 - 3.0)}{0.325}$$

$$= \frac{3.6056 \times (-0.121)}{0.325}$$

$$= -\frac{0.4363}{0.325} = -1.3424 \simeq -1.34$$

the p value is $2 \times P(Z \leq -1.34)$

$$= 2 \times \Phi(-1.34)$$

$$= 2 \times 0.0901$$

$$= 0.1802 \text{ Ans}$$

(b) for $H_0: \mu = 3.1$ verses $H_A: \mu < 3.1$

the test statistic is

$$Z = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} = \frac{\sqrt{13}\,(2.879 - 3.1)}{0.325}$$

$$= \frac{3.6056 \times (-0.221)}{0.325}$$

$$= \frac{-0.7968}{0.325} = -2.4518 \simeq -2.45$$

The p value is $P(Z \leq -2.45)$

$$= \Phi(-2.45)$$

$$= 0.007 \text{ Ans}$$

Ans. 5. given that, for $H_0: \mu = 3.0$ mm verses $H_A: \mu \neq 3.0$

a sample of glass sheets $n = 41$

(a) given that $\alpha = 0.1$

The critical point $t_{\alpha/2, n-1}$

$= t_{0.05, 40} = 1.684$

and the null hypothesis accepted when $|t| \leq 1.684$

means $(-1.684 \leq t \leq 1.684)$

(b) given that $\alpha = 0.01$

The critical point $t_{\alpha/2, n-1}$

$= t_{0.005, 40}$

$= 2.704$

and the null hypothesis is rejected when $|t| > 2.704$

means $\{ 1 - [|t| < 2.704)$

(c) sample mean is $x = 3.04$

and $s = 0.124$

So, The test statistics is,

$$t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s}$$

$$= \frac{\sqrt{41}\,(3.04 - 3.00)}{0.124}$$

$$= \frac{6.4031 \times 0.04}{0.124} = \frac{0.2561}{0.124} = 2.0655$$

$$\simeq 2.066$$

The null - hypothesis is rejected at size $\alpha = 0.10$ and accepted at size $\alpha = 0.01$

(d) The p value is given by -

$$\Rightarrow \quad 2 \times P(t_{19} \geq 2.443)$$

$$\Rightarrow \quad 2 \times [1 - P(t_{19} \leq 2.443)]$$

$$\Rightarrow \quad 2 \times [1 - 0.9875]$$

$$\Rightarrow \quad 2 \times 0.0125$$

$$= \quad 0.025 \text{ Ans}$$

Ans 7. for $H_0: \mu = 1.025$ kg verses H_A $\mu \neq 1.025$ kg

A sample of $n=16$ sugar packets is obtained.

(a) Given that $\alpha = 0.10$

Critical point $t_{\alpha/2, n-1}$

$= t_{0.05, 15}$

$= 1.753$

and the null hypothesis is accepted when $|t| \leq 1.753$

(b) given that $\alpha = 0.01$

Critical point $= t_{\alpha/2, n-1}$

$= t_{0.005, 15}$

$= 2.947$

and the null hypothesis is rejected when $|t| > 2.947$

(c) The test statistic is,

where sample mean $\bar{x} = 1.053$ kg

and $s = 0.058$ kg

$$t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s} = \frac{\sqrt{16} \times (1.053 - 1.025)}{0.058}$$

$$= \frac{4 \times 0.028}{0.058} = \frac{0.112}{0.058}$$

$$= 1.931$$

The null hypothesis is rejected at size $\alpha = 0.10$ and accepted at size $\alpha = 0.01$.

(d) The p - value is given by-

$$\Rightarrow \quad 2 \times P(t_{15} \geq 1.931)$$

$$= 2 \times [1 - P(t_{15} \leq 1.931)]$$

$$= 2 \times [1 - 0.9635]$$

$$= 2 \times 0.0365$$

$$= 0.073 \text{ Ans}$$

Ans 9. for $H_0: \mu \le 0.065$ verses $H_A: \mu > 0.065$

A sample of $n = 61$ bottles of the chemical solution is obtained.

(a) given that $\alpha = 0.10$

Critical point for t-statistic is $t_{\alpha, n-1}$

$= t_{0.100, 60}$

$= 1.296$

and the null hypothesis is accepted when $t \le 1.296$

(b) given that $\alpha = 0.01$

Critical point for t-statistics is $t_{\alpha, n-1}$

$= t_{0.01, 60}$

$= 2.390$

and the null hypothesis is rejected when $t > 2.390$

(c) given that, Sample mean $\bar{x} = 0.0768$
and $s = 0.0231$

the test Statistic is,

$$t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s} = \frac{\sqrt{61} \times (0.0768 - 0.065)}{0.0231}$$

$$= \frac{7.8102 \times 0.0118}{0.0231} = \frac{0.0922}{0.0231}$$

$$= 3.99$$

The null hypothesis is rejected at size $\alpha = 0.01$ and consequently also at size $\alpha = 0.10$

(d) the p value is $P(t_{60} \geq 3.990)$

$= 1 - P(t_{60} \leq 3.990)$

$= 1 - t_{3.99, 60}$

$= 1 - 0.9999$

$= 0.0001$ Ans

Ans. 11. A machine is set to cut metal plates to a length of 44.350 mm.

a random sample of metal plates $n = 24$

sample mean of $\bar{x} = 44.364$

and $s = 0.019$

Consider the hypotheses $H_0 : \mu = 44.350$ versus $H_A : \mu \neq 44.350$

The test statistics is,

$$t = \frac{\sqrt{n}\,(\bar{x} - \mu_0)}{s} = \frac{\sqrt{24} \times (44.364 - 44.350)}{0.019}$$

$$= \frac{4.8990 \times 0.014}{0.019}$$

$$= \frac{0.06859}{0.019} = 3.61$$

The P-value is $\Rightarrow 2 \times P(t_{23} \geq 3.61)$

$= 2 \times 0.0007 = 0.0014$

There is sufficient evidence to conclude that the machine is miscalibrated.

Ans. 13. a sample of randomly measures sulfur level

$n = 15$

Sample mean $\bar{x} = 14.82$

and $s = 2.91$

Consider the hypotheses $H_0 : x \leq 12.50$ verses $H_A : x > 12.50$

The test statistic is,

$$t = \frac{\sqrt{n}(\bar{x} - x_0)}{s}$$

$$= \frac{\sqrt{15} \times (14.82 - 12.50)}{2.91}$$

$$= \frac{3.8730 \times 2.32}{2.91} = \frac{8.9854}{2.91}$$

$$= 3.09$$

The p- value is given by -

$$P(t_{14} \geq 3.09)$$

$$= 1 - P(t_{14} \leq 3.09)$$

$$= 1 - t_{3.09, 14}$$

$$= 1 - 0.996 = 0.004$$

There is sufficient evidence to conclude that the Chemical plant is in violation of the working code.