Exercise 10.1

45. $x = \sqrt{t}$, $y = 2t+4$, $t = 1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{\frac{1}{2\sqrt{t}}} = 4\sqrt{t} \qquad \left.\frac{dy}{dx}\right|_{t=1} = 4$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{2}{\sqrt{t}}}{\frac{1}{2\sqrt{t}}} = 4 \qquad \left.\frac{d^2y}{dx^2}\right|_{t=1} = 4$$

46. $x = \frac{1}{2}t^2 + 1$, $y = \frac{1}{3}t^3 - t$, $t = 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^2-1}{t} = t - \frac{1}{t} \qquad \left.\frac{dy}{dx}\right|_{t=2} = \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{1+\frac{1}{t^2}}{t} = \frac{t^2+1}{t^3} \qquad \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{5}{8}$$

47. $x = \sec t$, $y = \tan t$, $t = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\sec t . \tan t} = \operatorname{Cosec} t \qquad \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{2}{\sqrt{3}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\operatorname{Cosec} t . \operatorname{Cot} t}{\sec t . \tan t} = -\operatorname{Cot}^3 t \qquad \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = -\frac{1}{3\sqrt{3}}$$

48. $x = \sinh t,\ y = \cosh t,\ t = 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sinh t}{\cosh t} = \tanh t \qquad \left.\frac{dy}{dx}\right|_{t=0} = 0$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\operatorname{sech}^2 t}{\cosh t} = \operatorname{sech}^3 t \qquad \left.\frac{d^2y}{dx^2}\right|_{t=0} = 1$$

49. $x = \theta + \cos\theta,\ y = 1 + \sin\theta,\ \theta = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{\frac{dx}{d\theta}} = \frac{\cos\theta}{1-\sin\theta} \qquad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{\sqrt{3}/2}{1-\frac{1}{2}} = \sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{(1-\sin\theta)\times -\sin\theta + \cos^2\theta}{(1-\sin\theta)^2} \times \frac{1}{(1-\sin\theta)}$$

$$= \frac{1}{(1-\sin\theta)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{6}} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$$

50. $x = \cos\phi,\ y = 3\sin\phi,\ \phi = \frac{5\pi}{6}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\phi}}{\frac{dx}{d\phi}} = \frac{3\cos\phi}{-\sin\phi} = -3\cot\phi \qquad \left.\frac{dy}{dx}\right|_{\phi=\frac{5\pi}{6}} = 3\sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\phi}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\phi}} = \frac{+3\,Cosec^2\phi}{-Sin\phi} = -3\,Cosec^3\phi$$

$$\left.\frac{d^2y}{dx^2}\right|_{\phi=\frac{5\pi}{6}} = -24$$

65. $x = t^2,\ y = \frac{1}{3}t^3 \quad (0 \le t \le 1)$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^1 \sqrt{(2t)^2 + (t^2)^2}\ dt.$$

$$= \int_0^1 t\sqrt{4+t^2}\ dt$$

let, $u = 4+t^2 \Rightarrow du = 2t\,dt$

$t=0,\ u=4$

$t=1,\ u=5$

$$= \int_0^5 \frac{1}{2}u^{1/2}\,du$$

$$= \left[\frac{1}{3}u^{3/2}\right]_4^5$$

$$= \frac{1}{3}\left(5\sqrt{5} - 8\right)$$

66. $x = \sqrt{t} - 2$, $y = 2t^{3/4}$ $(1 \le t \le 16)$

Let, $t = u^2$ so, $x = u - 2$, $y = 2u^{3/2}$ $(1 \le u \le 4)$

$$L = \int_1^4 \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

$$= \int_1^4 \sqrt{1 + 9u}\, du$$

$$= \left[\frac{2}{27}(1+9u)^{3/2}\right]_1^4 = \frac{2}{27}\left(37\sqrt{37} - 10\sqrt{10}\right)$$

67. $x = \cos 3t$, $y = \sin 3t$ $(0 \le t \le \pi)$

$$L = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^\pi \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^\pi \sqrt{9(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^\pi 3\, dt$$

$$= 3\pi$$

69. $x = \sin t + \cos t$, $y = \sin t - \cos t$ $(0 \le t \le \pi)$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2(\cos^2 t + \sin^2 t)}\, dt = \sqrt{2}\,\pi$$