Mat - 116 Lec - 11

Polynomial Functions

A Polynomial function is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1} \ldots a_1, a_0$ are real numbers and n is non negative integers.

* The domain of a Polynomial function is the set of all real numbers.
* The degree of a Polynomial function is the largest power of x that appears.
* The zero polynomial function $f(x) = 0 + 0\cdot x + 0\cdot x^2 + \cdots + 0x^n$ is not assigned a degree.

Example: Identify polynomial functions:

1. $f(x) = 2 - 3x^4$ 2. $g(x) = \sqrt{x}$ 3. $\frac{x^2-2}{x^3-1}$ 4. $F(x) = 0$
5. $G(x) = 8$ 6. $H(x) = -2x^3(x-1)^2$

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0 \quad a_0 \neq 0$	constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1 x + a_0$	Linear function	non vertical, non-horizontal line with slope a_1 and y-intercept a_0.
2	$f(x) = a_2 x^2 + a_1 x + a_0$, $a_2 \neq 0$	Quadratic Function	Parabola. graph opens up $a_2 > 0$, opens down $a_2 < 0$

Power-functions:

A power function of degree n is a monomial function of the form $f(x) = ax^n$

where a is a real number, $a \neq 0$ and $n > 0$ is an integer.

Example:

$f(x) = 5x$ degree 1

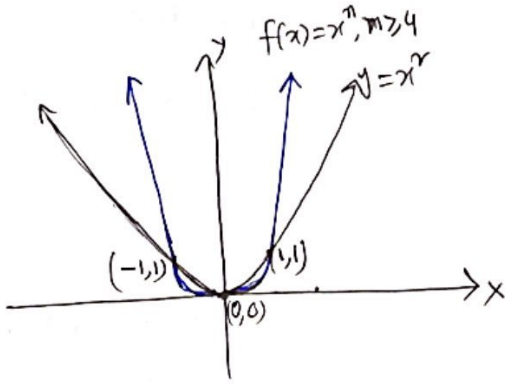
$f(x) = -5x^2$ degree 2

$f(x) = 8x^3$ degree 3

$f(x) = -5x^4$ degree 4

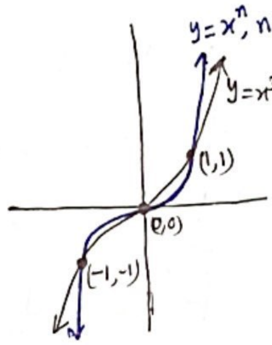
#1 Properties of power function $f(x) = x^n$, n is a positive even integer:

1. If n is even integer, then f is an even function, so the graph is symmetric about y-axis.
2. The domain is the set of all real numbers. The range is the set of non-negative real numbers.
3. The graph always contains the points $(-1,1)$, $(0,0)$ and $(1,1)$.
4. As the exponent n increases, the function increases more rapidly when $x < -1$ or $x > 1$, but for x near origin $(-1 < x < 1)$, the graph tends to flatten out and lie closer to the x-axis.



Properties of power function, $f(x)=x^n$, n is a positive odd integer.

1. Since n is odd, so the function f is odd and the graph is symmetric with respect to the origin.
2. The domain and range are the set of all real numbers.
3. The graph always contains the points $(-1,-1)$, $(0,0)$ and $(1,1)$.
4. As the exponent n increases, the function increases more rapidly when $x<-1$ or $x>1$, but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.



Example: i. Graph $f(x)=1-x^5$

ii) Graph $f(x)=(x-1)^4$

Real zero of polynomial function and their multiplicity:

If f is a function and r is a real number for which $f(r)=0$, then r is called a real zero of f.

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x-r$ is a factor of f.
4. r is a solution of $f(x)=0$.

If $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is not a factor of f, then r is called a zero of multiplicity m of f.

Example:

$$f(x) = 5(x-2)(x+3)^2(x-\tfrac{1}{2})^4$$

Here,

2 is a zero of multiplicity 1, because the exponent on the factor $x-2$ is 1.

-3 is a zero of muliliplicity 2, because the exponent on the factor $x+3$ is 2.

$\frac{1}{2}$ is a zero of multiplicity 4, because the exponent on the factor $x-\frac{1}{2}$ is 4.

If r is a zero of even multiplicity:

i. The sign of $f(x)$ doesnot change from one side to the other side of r.

ii) The graph of f touches the x-axis at r.

If r is a zero of odd multiplicity:

i) The sign of $f(x)$ changes from one side to the other side of r.

ii) The graph of f crosses the x-axis at r.

Example:

$$f(x) = x^2(x-2)$$

y-intercept, let $x=0$ gives $f(x)=0$

so y-intercept is zero.

x-intercept, let $f(x)=0$

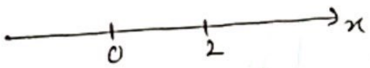
$$\Rightarrow x^2(x-2)=0$$

$$\therefore x=0 \text{ and } x=2$$

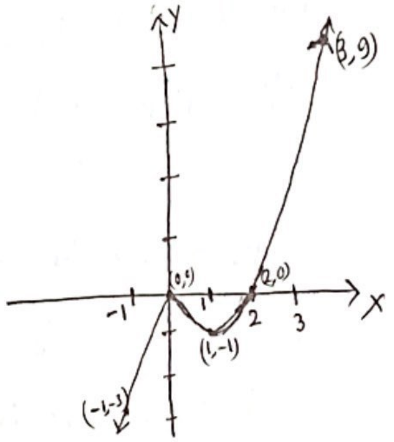
So x-intercept 0 and 2.

0 is a zero of multiplicity 2, so the graph of $f(x)$ touches at zero.

2 is a zero of multiplicity 1, so the graph of $f(x)$ crosses at 2.



interval	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
Number	-1	1	3
Value of f	$f(-1)=3$	$f(1)=-1$	$f(3)=9$



Behavior near zero; $f(x) = x^2(x-2)$

$$\approx x^2(0-2)$$

$$= -2x^2 \text{ (Parabola, opens down)}$$

Behavior near $x=2$; $f(x) = x^2(x-2)$

$$= 2^2(x-2)$$

$$= 4(x-2) \text{ (linear function)}$$

~~with~~

Turning Points:

If f is a polynomial function of degree n, then the graph of f has at most $n-1$ turning points.

If the graph of a polynomial function f has $n-1$ turning points, the degree of f is at least n.

For example, $f(x) = x^2(x-2)$ is the graph of a polynomial function of degree 3 and has $(3-1) = 2$ turning points.

End behavior:

For large values of x, either positive or negative, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

resembles the graph of the power function

$$y = a_n x^n$$

For example: if $f(x) = -2x^3 + 5x^2 + x - 4$, then the graph of f will behave like the graph of $y = -2x^3$ for very large values of x, either positive or negative.

Similarly $f(x) = -2x^4 + x^3 + 4x^2 - 7x + 1$, the graph of f will resemble the graph of the power function $y = -2x^4$ for large $|x|$.

Analyzing the graph of a polynomial function:

Step 1: Determine the end behavior of the graph of the function.

Step 2: Find the x and y-intercepts.

Step 3: Determine the zeros of the functions and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

Step 4: Determine the maximum number of turning points on the graph of the function.

Step 5: Determine the behavior of the graph of f near each x-intercept.

Step 6: Use the information in steps 1 to 5 to draw a complete graph of the function.

Example: $f(x) = x^2(x-4)(x+1) = (x^3 - 4x^2)(x+1) = x^4 - 4x^3 + x^3 - 4x^2$

$= x^4 - 3x^3 - 4x^2$

Step 1: End behavior: the graph of f resembles that of the power function $y = x^4$ for large values of $|x|$.

Step 2: The y-intercept is 0.

For x-intercept $f(x) = 0$

$$x^2(x-4)(x+1) = 0$$

So $x^2 = 0 \qquad x-4=0 \qquad x+1=0$

$\Rightarrow x = 0$ or $x = 4$ or $x = -1$

The x-intercepts are $-1, 0$ and 4.

Step 3: The intercept 0 is a zero of multiplicity 2, so the graph will touch x-axis.

4 and -1 are zeros of multiplicity 1, so the graph of f will cross the x-axis at 4 and -1.

Step 4: The graph of f will contain at most 3 turning points.

Step 5: Behavior near -1, 0 and 4.

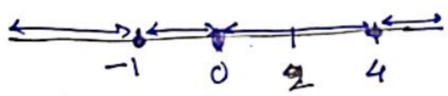
near -1: $f(x) = x^2(x-4)(x+1) \approx (-1)^2(-1-4)(x+1) = -5(x+1)$ A line with slope -5.

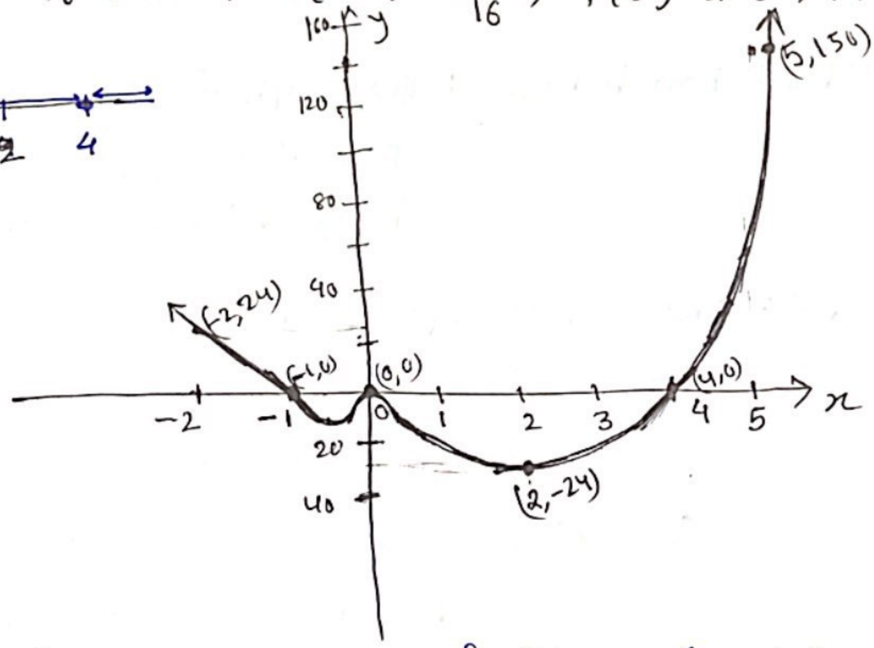
Near 0: $f(x) = x^2(x-4)(x+1) \approx x^2(0-4)(0+1) = -4x^2$ A parabola opening down.

Near 4: $f(x) = x^2(x-4)(x+1) \approx 4^2(x-4)(4+1) = 80(x-4)$ A line with slope 80.

Step 6: Gather all info from 1 to 5. Evaluate $f(x)$ at $x = -2, -\frac{1}{2}$ and 2, 5.

$f(-2) = 24$, $f(-1/2) = -\frac{9}{16}$, $f(5) = 150$, $f(2) = -24$.





Exercise: Analyze the graph of $f(x) = (2x+1)(x-3)^2$.

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Solution: $f(x) = (2x+1)(x-3)^2$

$= (2x+1)(x^2-6x+9)$

$= 2x^3-12x^2+18x+x^2-6x+9$

$f(x) = 2x^3-11x^2+12x+9$

Step 1: End behavior: The graph of f behaves like $y=2x^3$ for large values of $|x|$.

Step 2: y-intercept, let $x=0$, $f(0)=9$

$\therefore y = 9$

x-intercept $f(x)=0$

$\therefore 2x^3-11x^2+12x+9=0$

$\Rightarrow (2x+1)(x-3)^2=0$

$\therefore 2x+1=0$ or $(x-3)^2=0$

$\therefore x=-1/2$ or $x=3$

The x-intercepts are $x=-1/2$ and 3.

Step 3: The x-intercept $-1/2$ is a zero of multiplicity 1, so the graph of f crosses the x-axis at $x=-1/2$

The x-intercept 3 is a zero of multiplicity 2, so the graph of f touches the x axis at $x=3$.

Step 4: The graph of the function has $3-1=2$ turning points.

Step 5: Behavior near $x=-1/2$ and $x=3$.

Near $-\frac{1}{2}$: $f(x)=(2x+1)(x-3)^2$

$\approx (2x+1)\left(-\frac{1}{2}+3\right)^2$

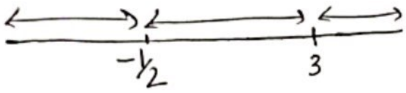
$= (2x+1)\frac{25}{4}$

$= \frac{25}{2}x + \frac{25}{4}$ A line with slope $\frac{25}{2}$

Near 3: $f(x) = (2x+1)(x-3)^2$

$\approx (2.3+1)(x-3)^2 = 7(x-3)^2$; a parrabola opens up.

step 6: collect all information from 1 to 5. Evaluate f at $x = -1, 1$ and 4.



$f(-1) = -16$, $f(1) = 12$, $f(4) = 9$.