First Order Linear ODEs (Exact/Non-exact)

- ***Linear differential equation of order ONE :***

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Example.

$$2(y - 4x^2)dx + 2xdy = 0 \Rightarrow \frac{dy}{dx} + \frac{1}{x}y = 4x \qquad \text{[Exact ODE]}$$

$$2(y - 4x^2)dx + xdy = 0 \Rightarrow \frac{dy}{dx} + \frac{2}{x}y = 8x \qquad \text{[Non−exact ODE]}$$

$$2xy\,dx + dy = 0 \Rightarrow \frac{dy}{dx} + 2xy = 0 \qquad \text{[Non−exact, separable ODE]}$$

$$(y - x)dx + dy = 0 \Rightarrow \frac{dy}{dx} + y = x \qquad \text{[Non−exact ODE]}$$

❖ In some instances a first order linear ODEs (non-exact) can be solved by separation of variables.

First Order Linear ODEs (Exact/Non-exact)

Steps to solve a First order linear ODES (non-exact and non-separable):

(i) Put the equation into the standard form: $\frac{dy}{dx} + P(x)y = Q(x)$.

(ii) Obtain the integrating factor with no integral constant adding: $f(x) = e^{\int P(x)dx}$.

(iii) Multiply the both sides of the standard form equation by the integrating factor.

The resultant equation will be then exact ODEs.

(iv) Solve the resultant exact ODEs.

Linear First Order Non-Homogeneous ODE:	$\frac{dy}{dx} + P(x)y = Q(x)$.
Linear First Order Homogeneous ODE:	$\frac{dy}{dx} + P(x)y = 0$.

First Order Linear ODEs (Exact/Non-exact)

Example. Solve the differential equations,

$$2(y-4x^2)dx + xdy = 0$$

[The ODE is not exact]

Solution. The given equation is not exact since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2y-8x^2) = 2, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x) = 1$$

Now, rewrite the given equation into the standard form as

$$\frac{dy}{dx} + \frac{2(y-4x^2)}{x} = 0 \;\Rightarrow\; \frac{dy}{dx} + \frac{2}{x}y = 8x \quad \text{when } x \neq 0$$

Then the integrating factor yields,

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

Now multiplying the integrating factor in the both side of the given ODE yields,

$$x^2\frac{dy}{dx} + 2xy = 8x^3 \Rightarrow x^2dy + 2xydx = 8x^3dx \Rightarrow (x^2)dy + y\,d(x^2) = 8x^3dx$$

$$\Rightarrow d(x^2y) = 8x^3dx \Rightarrow \int d(x^2y) = \int 8x^3\,dx$$

$$\boxed{d(f\cdot g) = f\cdot dg + g\cdot df}$$

$$\therefore \boldsymbol{x^2y = 2x^4 + c}$$

First Order Linear ODEs (Exact/Non-exact)

Example. Solve the differential equations,

$$\frac{dy}{dx} - 3y = 6$$

[The ODE is not separable]

Solution. Here, the integrating factor becomes,

$$e^{\int(-3)dx} = e^{-3x}$$

Now multiplying the integrating factor in the both side of the given ODE yields,

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x} \Rightarrow e^{-3x}dy - 3e^{-3x}ydx = 6e^{-3x}dx$$

$$\Rightarrow (e^{-3x})dy + y\, d(e^{-3x}) = 6e^{-3x}dx$$

$$\Rightarrow d(ye^{-3x}) = 6e^{-3x}dx$$

$$\boxed{d(f \cdot g) = f \cdot dg + g \cdot df}$$

$$\Rightarrow \int d(ye^{-3x}) = 6\int e^{-3x}\, dx$$

Therefore, the desired solution becomes,

$$\therefore \boldsymbol{ye^{-3x} = -2e^{-3x} + c \Rightarrow y = ce^{3x} - 2}$$

First Order Linear ODEs (Exact/Non-exact)

Example. Solve the initial value problem,

$$\frac{dy}{dx} + y = x, \qquad y(0) = 4$$ [The ODE is not separable]

Solution. Here, the integrating factor becomes,

$$e^{\int (1)dx} = e^x$$

Now multiplying the integrating factor in the both side of the given ODE yields,

$$e^x\frac{dy}{dx} + e^x y = xe^x \Rightarrow e^x dy + e^x y dx = xe^x dx \Rightarrow (e^x)dy + y\, d(e^x) = xe^x dx$$

$$\Rightarrow d(ye^x) = xe^x dx \Rightarrow \int d(ye^x) = \int xe^x\, dx$$

$$\boxed{d(f \cdot g) = f \cdot dg + g \cdot df}$$

$$\therefore \boldsymbol{ye^x = xe^x - e^x + c \Rightarrow y = x - 1 + ce^{-x}}$$

From the initial condition, we have, when $x = 0$, $y = 4$. Therefore, $4 = 0 - 1 + c \Rightarrow c = 5$.

Therefore the particular solution of the problem is, $\boldsymbol{y = x - 1 + 5e^{-x}}$

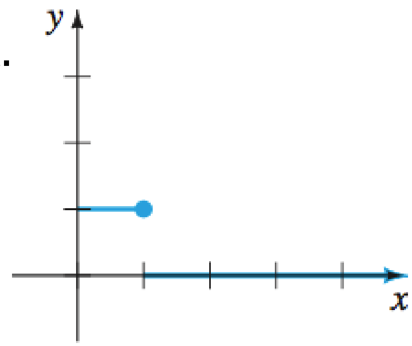
Here, $\boldsymbol{ce^{-x}}$ in the general solution is called a transient term since $\boldsymbol{e^{-x} \to 0}$ as $\boldsymbol{x \to \infty}$.

First Order Linear ODEs (Exact/Non-exact)

Example. Solve the initial value problem,

$$\frac{dy}{dx} + y = f(x), \qquad y(0) = 0 \quad \text{where} \quad f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

Solution. Here, $f(x)$ discontinuous at $x = 1$ as can be seen in the graph.



Here, the integrating factor becomes,

$$e^{\int (1)dx} = e^x$$

Now multiplying the integrating factor in the both side of the given ODE yields,

$$e^x \frac{dy}{dx} + e^x y = f(x)e^x \Rightarrow e^x dy + e^x y dx = f(x)e^x dx$$

$$\Rightarrow (e^x)dy + y\, d(e^x) = f(x)e^x dx$$

$$\Rightarrow d(ye^x) = f(x)e^x dx$$

$$\Rightarrow \int d(ye^x) = \int f(x)e^x\, dx$$

First Order Linear ODEs (Exact/Non-exact)

Example. Solve the initial value problem,

$$\frac{dy}{dx} + y = f(x), \qquad y(0) = 0 \quad \text{where } f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Solution. Continued...

$$ye^x = \begin{cases} \int e^x\,dx\,, 0 \le x \le 1 \\ 0, \qquad x > 1 \end{cases} = \begin{cases} e^x + c_1, 0 \le x \le 1 \\ c_2, \qquad x > 1 \end{cases} \Rightarrow y = \begin{cases} 1 + c_1 e^{-x}, & 0 \le x \le 1 \\ c_2 e^{-x}, & x > 1 \end{cases}$$

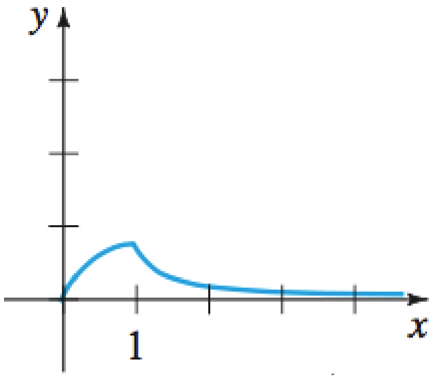
From the initial condition, we have, when $x = 0$, $y = 0$. Therefore, we must have $c_1 = -1$.

For the continuity of $y(x)$ at any point, it is required that $y(x)$ must be continuous at $x = 1$ and then we must have

$$\lim_{x \to 1^+} y(x) = y(1) \Rightarrow \lim_{x \to 1^+} c_2 e^{-x} = 1 + c_1 e^{-1} \Rightarrow c_2 e^{-1} = 1 - e^{-1} \Rightarrow c_2 = e - 1.$$

Thus the desired solution yields,

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1 \\ (e-1)e^{-x}, & x > 1 \end{cases}$$



First Order Linear ODEs (Exact/Non-exact)

Example. Solve the differential equations,

$$(y+1)dx + (4x - y)dy = 0$$

[The ODE is not linear in y, but in x]

Solution. The given equation is not linear in y, but it is linear in x. Rewriting the equation,

$$\frac{dx}{dy} + \frac{4x - y}{y+1} = 0 \Rightarrow \frac{dx}{dy} + \frac{4}{y+1}x = \frac{y}{y+1} = 1 - \frac{1}{y+1}$$

Here, the integrating factor becomes,

$$e^{\int\left(\frac{4}{y+1}\right)dy} = e^{4\ln(y+1)} = e^{\ln(y+1)^4} = (y+1)^4$$

Now multiplying the integrating factor in the both side of the ODE yields,

$$(y+1)^4 dx + 4(y+1)^3 x dy = [(y+1)^4 - (y+1)^3]dy$$

$$\Rightarrow (y+1)^4 dx + x d\{(y+1)^4\} = [(y+1)^4 - (y+1)^3]dy$$

$$\boxed{d(f\cdot g) = f\cdot dg + g\cdot df}$$

$$\Rightarrow d[x(y+1)^4] = [(y+1)^4 - (y+1)^3]dy$$

$$\Rightarrow \int d[x(y+1)^4] = \int [(y+1)^4 - (y+1)^3]\,dy \Rightarrow x(y+1)^4 = \frac{(y+1)^5}{5} - \frac{(y+1)^4}{4} + \frac{c}{20}$$

$$\Rightarrow x = \frac{y+1}{5} - \frac{1}{4} + \frac{c}{20}(y+1)^{-4} \Rightarrow \mathbf{20x = 4y - 1 + c(y+1)^{-4}}$$

Reduction to First Order Linear ODEs

Bernoulli's Equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad where\ n \in \mathbb{R}$$

For $\boldsymbol{n} = \mathbf{0}, \mathbf{1}$, the above equation becomes linear.

For $\boldsymbol{n} \neq \mathbf{0}, \mathbf{1}$, the above equation becomes non-linear and can be reduced into linear by substituting $u = y^{1-n}$

Example. Solve the following Bernoulli differential equation,

$$x\frac{dy}{dx} + y = x^2y^2$$

Solution. The given equation is not linear in y. Rewriting the equation in the form,

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2 \Rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x$$

Now substitute, $u = y^{-1} \Rightarrow \frac{du}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow y^{-2}\frac{dy}{dx} = -\frac{du}{dx}$. Thus the ODE becomes,

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x \Rightarrow -\frac{du}{dx} + \frac{1}{x}u = x \Rightarrow \frac{du}{dx} - \frac{1}{x}u = -x$$ [The ODE is linear in u]

Reduction to First Order Linear ODEs

Example. Solve the following Bernoulli differential equation,

$$x\frac{dy}{dx} + y = x^2y^2$$

Solution. **Continued…**

Here, the integrating factor becomes, $e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$

Now multiplying the integrating factor in the both side of the ODE yields,

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -\frac{x}{x} \Rightarrow \frac{du}{dx} - \frac{u}{x} = -x \Rightarrow xdu - udx = -x^2dx$$

$$\Rightarrow \frac{xdu - udx}{x^2} = -\frac{x^2dx}{x^2}$$

$$\boxed{d\left(\frac{f}{g}\right) = \frac{g\cdot df - f\cdot dg}{g^2}}$$

$$\Rightarrow d\left(\frac{u}{x}\right) = -dx \Rightarrow \int d\left(\frac{u}{x}\right) = -\int dx$$

$$\Rightarrow \frac{u}{x} = -x + c \Rightarrow u = -x^2 + cx \Rightarrow \frac{1}{y} = -x^2 + cx$$

The desired solution becomes, $\boldsymbol{y = \frac{1}{cx - x^2}}$

First Order Linear ODEs (Exact/Non-exact)

Exercises 2.3

H.W. from the text book

Find the general solution of the given differential equation. Determine whether there are any transient terms in the general solution.

1. $\dfrac{dy}{dx} = 5y$

2. $\dfrac{dy}{dx} + 2y = 0$

3. $\dfrac{dy}{dx} + y = e^{3x}$

4. $3\dfrac{dy}{dx} + 12y = 4$

5. $y' + 3x^2y = x^2$

6. $y' + 2xy = x^3$

7. $x^2y' + xy = 1$

8. $y' = 2y + x^2 + 5$

9. $x\dfrac{dy}{dx} - y = x^2 \sin x$

10. $x\dfrac{dy}{dx} + 2y = 3$

11. $x\dfrac{dy}{dx} + 4y = x^3 - x$

12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$

13. $x^2y' + x(x + 2)y = e^x$

14. $xy' + (1 + x)y = e^{-x} \sin 2x$

15. $y\,dx - 4(x + y^6)\,dy = 0$

16. $y\,dx = (ye^y - 2x)\,dy$

17. $\cos x\dfrac{dy}{dx} + (\sin x)y = 1$

Solve the given initial-value problem. Give the largest interval over which the solution is defined

25. $\dfrac{dy}{dx} = x + 5y, \quad y(0) = 3$

26. $\dfrac{dy}{dx} = 2x - 3y, \quad y(0) = \frac{1}{3}$

27. $xy' + y = e^x, \quad y(1) = 2$

28. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

31. $x\dfrac{dy}{dx} + y = 4x + 1, \quad y(1) = 8$

32. $y' + 4xy = x^3e^{x^2}, \quad y(0) = -1$

33. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

34. $x(x + 1)\dfrac{dy}{dx} + xy = 1, \quad y(e) = 1$

35. $y' - (\sin x)y = 2 \sin x, \quad y(\ \ /2) = 1$

36. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

First Order ODEs - Exact equations

Exercises 2.3

H.W. from the text book

Solve the given initial-value problem.

37. $\dfrac{dy}{dx} + 2y = f(x), y(0) = 0,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

38. $\dfrac{dy}{dx} + y = f(x), y(0) = 1,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

39. $\dfrac{dy}{dx} + 2xy = f(x), y(0) = 2,$ where

$$f(x) = \begin{cases} x, & 0 \le x \quad 1 \\ 0, & x \ge 1 \end{cases}$$

40. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x), y(0) = 0,$ where

$$f(x) = \begin{cases} x, & 0 \le x \quad 1 \\ -x, & x \ge 1 \end{cases}$$

53. A heart pacemaker consists of a switch, a battery of constant voltage E_0, a capacitor with constant capacitance C_0, and the heart as a resistor with constant resistance R_0. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{R_0 C_0} E, \quad \text{where} \quad E(4) = E_0.$$

First Order ODEs – Bernoulli's equations

Exercises 2.5

H.W. from the text book

Solve the following Bernoulli's differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$

16. $\dfrac{dy}{dx} - y = e^x y^2$

17. $\dfrac{dy}{dx} = y(xy^3 - 1)$

18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

38. In the study of population dynamics one of the most famous models for a growing but bounded population is the **logistic equation**

$$\frac{dP}{dt} = P(a - bP)$$

where a and b are positive constants. Solve the DE using the fact that it is a Bernoulli equation.