Chapter # 01
(Limits and Continuity)

1.5 Continuity: A thrown baseball cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of the ball as an unbroken curve. In this section, we translate "***unbroken curve***" into a precise mathematical formulation called ***continuity***, and develop some fundamental properties of continuous curves.

Definition: A function ***f*** is said to be continuous at $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

If one or more of the conditions of this definition fails to hold, then we will say that ***f*** has a discontinuity at $x = c$.

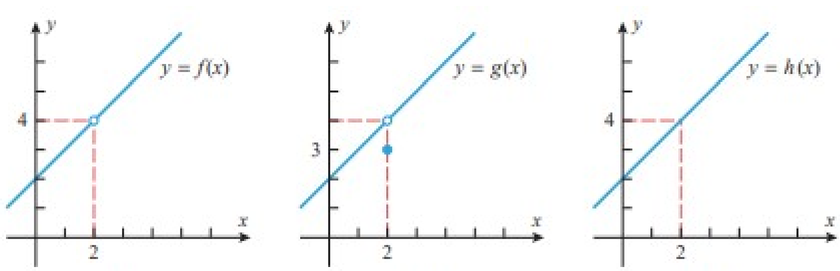
Example 1: Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution: In each case we must determine whether the limit of the function as **x→2** is the same as the value of the function at **x = 2**. In all three cases the functions are identical, except at **x =** 2, and hence all three have the same limit at **x = 2**, namely

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The function ***f*** is undefined at **x = 2**, and hence is not continuous at **x = 2** (Figure ***a***). The function ***g*** is defined at **x = 2**, but its value there is ***g*(2) = 3**, which is not the same as the limit as ***x*** approaches ***2***; hence, ***g*** is also not continuous at **x = 2** (Figure ***b***). The value of the function ***h*** at **x = 2** is ***h*(2) = 4**, which is the same as the limit as **x** approaches **2**; hence, ***h*** is continuous at ***x* = 2** (Figure ***c***).



(*a*) (*b*) (*c*)

Continuous on an Interval: If a function ***f*** is continuous at each number in an open interval $(\boldsymbol{a}, \boldsymbol{b})$, then we say that ***f*** is continuous on $(\boldsymbol{a}, \boldsymbol{b})$. This definition applies to infinite open intervals of the form $(\boldsymbol{a}, +\infty)$, $(-\infty, \boldsymbol{b})$, and $(-\infty, +\infty)$. In the case where ***f*** is continuous on $(-\infty, +\infty)$, we will say that ***f*** is continuous everywhere.

Definition: A function ***f*** is said to be continuous on a closed interval [a, b] if the following conditions are satisfied:

1. ***f*** is continuous on (***a***, ***b***).

2. ***f*** is continuous from the right at ***a***.

3. ***f*** is continuous from the left at ***b***.

Example 2: What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution: Because the natural domain of this function is the closed interval **[−3, 3]**, we will need to investigate the continuity of ***f*** on the open interval **(−3, 3)** and at the two endpoints. If ***c*** is any point in the interval **(−3, 3)**,

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

which proves ***f*** is continuous at each point in the interval **(−3, 3)**. The function ***f*** is also continuous at the endpoints since

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$
$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$

Thus, ***f*** is continuous on the closed interval **[−3, 3]**.

Theorem: If the functions ***f*** and ***g*** are continuous at ***c***, then

(*a*) ***f*** + ***g*** is continuous at ***c***.

(*b*) ***f*** − ***g*** is continuous at ***c***.

(*c*) ***fg*** is continuous at ***c***.

(*d*) ***f /g*** is continuous at ***c*** if $\boldsymbol{g(c) \neq 0}$ and has a discontinuity at **c** if ***g*(*c*)** = 0.

Continuity of Polynomials and Rational Functions:

Theorem:

(*a*) A polynomial is continuous everywhere.

(*b*) A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.

Example 4: Show that $|x|$ is continuous everywhere.

Solution: We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

so $|x|$ is the same as the polynomial x on the interval $(\mathbf{0}, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, \mathbf{0})$. But polynomials are continuous everywhere, so $x = 0$ is the only possible discontinuity for $|x|$. Since $|0| = 0$, to prove the continuity at $x = 0$ we must show that

$$\lim_{x \to 0} |x| = 0$$

Because the piecewise formula for $|x|$ changes at 0, it will be helpful to consider the onesided limits at ***0*** rather than the two-sided limit. We obtain

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Thus, $|x|$ is continuous at **x = 0**.

Continuity of Compositions:

Theorem: theorem If $\lim_{x\to c} g(x) = L$ and if the function f is continuous at L, then

$\lim_{x\to c} f(g(x)) = f(L)$. *That is,*

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

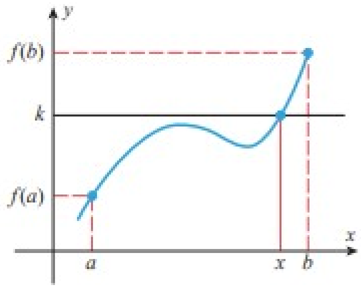
This equality remains valid if $\lim_{x\to c}$ *is replaced everywhere by one of* $\lim_{x\to c^+}$, $\lim_{x\to c^-}$, $\lim_{x\to +\infty}$, *or* $\lim_{x\to -\infty}$.

Theorem:

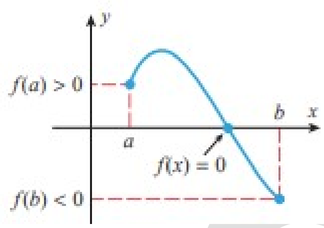
(a) If the function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

(b) If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

Intermediate-Value Theorem: If ***f*** is continuous on a closed interval **[*a*, *b*]** and ***k*** is any number between ***f*** **(*a*)** and **f(*b*)**, inclusive, then there is at least one number ***x*** in the interval **[*a*, *b*]** such that ***f*(*x*) = *k***.



Theorem: If ***f*** is continuous on **[*a*, *b*]**, and if ***f*(*a*)** and ***f*(*b*)** are nonzero and have opposite signs, then there is at least one solution of the equation $\boldsymbol{f(x) = 0}$ in the interval **(*a*, *b*)**.



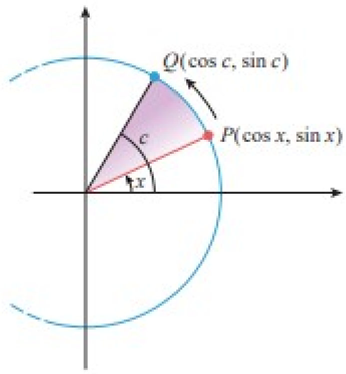
Home Work: Exercise 1.5: Problem No. 11-22, 29-32

1.6 Continuity of Trigonometric, Exponential and Inverse Functions: In this section we will discuss the continuity properties of trigonometric functions, exponential functions, and inverses of various continuous functions. We will also discuss some important limits involving such functions.

Continuity of Trigonometric Functions: The graphs of $\boldsymbol{\sin x}$ and $\boldsymbol{\cos x}$ are drawn as continuous curves. We will not formally prove that these functions are continuous, but we can motivate this fact by letting ***c*** be a fixed angle in radian measure and ***x*** a variable angle in radian measure. If the angle ***x*** approaches the angle ***c***, then the point $\mathbf{P}(\mathbf{cos\,x}, \boldsymbol{\sin x})$ moves along the unit circle toward $\mathbf{Q}(\mathbf{cos\,c}, \boldsymbol{\sin c})$, and the coordinates of ***P*** approach the corresponding coordinates of ***Q***. This implies that

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c$$

Thus, $\boldsymbol{\sin x}$ and $\boldsymbol{\cos x}$ are continuous at the arbitrary point ***c***; that is, these functions are continuous everywhere.



As x approaches c the point P approaches the point Q.

Theorem: If ***c*** is any number in the natural domain of the stated trigonometric function, then

$$\lim_{x \to c} \sin x = \sin c \qquad \lim_{x \to c} \cos x = \cos c \qquad \lim_{x \to c} \tan x = \tan c$$
$$\lim_{x \to c} \csc x = \csc c \qquad \lim_{x \to c} \sec x = \sec c \qquad \lim_{x \to c} \cot x = \cot c$$

Example 1: Find the limit

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right)$$

Solution: Since the cosine function is continuous everywhere, it follows

$$\lim_{x \to 1} \cos(g(x)) = \cos\left(\lim_{x \to 1} g(x)\right)$$

provided $\boldsymbol{\lim_{x \to 1} g(x)}$ exists. Thus,

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right) = \lim_{x \to 1} \cos(x + 1) = \cos\left(\lim_{x \to 1} (x + 1)\right) = \cos 2$$

Theorem: If $\boldsymbol{f}$ is a one-to-one function that is continuous at each point of its domain, then $\boldsymbol{f^{-1}}$ is continuous at each point of its domain; that is, $\boldsymbol{f^{-1}}$ is continuous at each point in the range of $\boldsymbol{f}$.

Example 2: $\boldsymbol{sin^{-1}x}$ is the inverse of the restricted ***sine*** function whose domain is the interval **[−π/2, π/2]** and whose range is the interval **[−1, 1]**. Since $\boldsymbol{\sin x}$ is continuous on the interval **[−π/2, π/2]**, the above theorem implies $\boldsymbol{sin^{-1}x}$ is continuous on the interval **[−1, 1]**.

Theorem: Let $\boldsymbol{b} > 0,\ b \neq 0$.

(*a*) *The function* b^x *is continuous on* $(-\infty, +\infty)$.

(*b*) *The function* $\log_b x$ *is continuous on* $(0, +\infty)$.

Example 3: Where is the following function continuous?

$$f(x) = \frac{\tan^{-1} x + \ln x}{x^2 - 4}$$

Solution: The fraction will be continuous at all points where the numerator and denominator are both continuous and the denominator is nonzero. Since $\boldsymbol{tan^{-1}x}$ is continuous everywhere and $\mathbf{ln}\,\boldsymbol{x}$ is continuous if ***x*** > 0, the numerator is continuous if ***x*** > 0. The denominator, being a polynomial, is continuous everywhere, so the fraction will be continuous at all points where ***x*** > 0 and the denominator is nonzero. Thus, ***f*** is continuous on the intervals $(\mathbf{0}, \mathbf{2})$ $\boldsymbol{and}$ $(\mathbf{2}, \infty)$.

Theorem (The Squeezing Theorem): Let ***f***, ***g***, and ***h*** be functions satisfying

$$g(x) \leq f(x) \leq h(x)$$

for all ***x*** in some open interval containing the number ***c***, with the possible exception that the inequalities need not hold at ***c***. If ***g*** and ***h*** have the same limit as ***x*** approaches ***c***, say

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then ***f*** also has this limit as ***x*** approaches ***c***, that is

$$\lim_{x \to c} f(x) = L$$

Theorem:

$$(a)\quad \lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad (b)\quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Example 4: Find

$$\text{(a)}\ \lim_{x \to 0} \frac{\tan x}{x} \qquad \text{(b)}\ \lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} \qquad \text{(c)}\ \lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$$

Solution: (*a*)

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) = \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \left(\lim_{x \to 0} \frac{1}{\cos x} \right) = (1)(1) = 1$$

(*b*)

$$\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} = \lim_{\theta \to 0} 2 \cdot \frac{\sin 2\theta}{2\theta} = 2 \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta}$$

Now make the substitution $\boldsymbol{x = 2\theta}$, and use the fact that $x \to 0\ as\ \theta \to 0$. This yields

$$\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} = 2 \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} = 2 \lim_{x \to 0} \frac{\sin x}{x} = 2(1) = 2$$

(*c*)

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 5x}{x}} = \lim_{x \to 0} \frac{3 \cdot \dfrac{\sin 3x}{3x}}{5 \cdot \dfrac{\sin 5x}{5x}} = \frac{3 \cdot 1}{5 \cdot 1} = \frac{3}{5}$$

Home Work: Exercise 1.6: Problem No. 17-40, 49-55