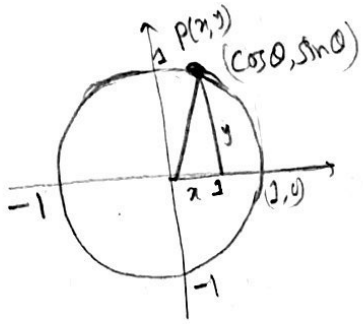
Trignometric Functions

Unit circle approach

Unit circle is a circle whose radius is 1 and whose center is at the origin of a rectangular coordinate system.

Circumference of circle with radius r is $2\pi r$.



Let t be a real number and let $P=(x,y)$ be the point on the unit circle that corresponds to t.

The sine function associates with t the y-coordinate of P and is denoted by $\sin t = y$

The cosine function associates with t the y-coordinate P and is denoted by $\cos t = x$

$\tan t = \frac{y}{x}$, $\operatorname{cosec} t = \frac{1}{y}$, $\sec t = \frac{1}{x}$, $\cot x = \frac{x}{y}$

<u>Example:</u> If $P = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ be the point on the unit circle.

Find values of sint, cost, tant, cosect.

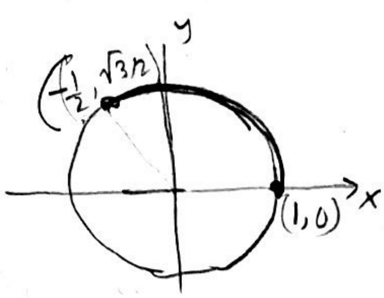
<u>Soln:</u> Here $x = -\frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$

$\therefore \sin t = y = \frac{\sqrt{3}}{2}$

$\cos t = x = -\frac{1}{2}$

$\tan t = \frac{\sin t}{\cos t} = \frac{y}{x} = -\sqrt{3}$

$\operatorname{cosec} t = \frac{1}{y} = \frac{2}{\sqrt{3}}$



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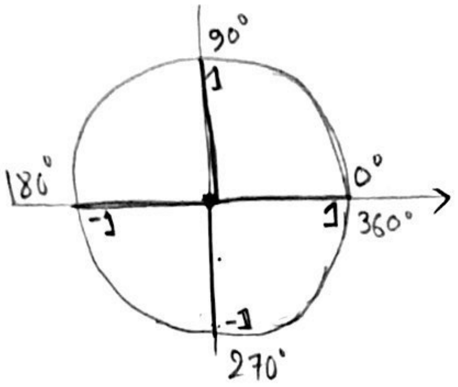
Multiples of 90°



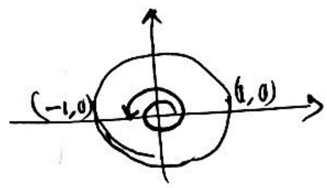
When we draw a unit circle and label angles that are multiples of 90°. These angles are known as quadrantal angles, have their terminal side on either the x-axis or the y-axis.

Quadrantal angles

θ (radians)	θ (degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
0	0°	0	1	0	undefined	1	undefined
$\frac{\pi}{2}$	90°	1	0	undefined	1	undefined	0
π	180°	0	−1	0	undefined	−1	undefined
$\frac{3\pi}{2}$	270°	−1	0	undefined	−1	undefined	0
2π	360°	0	1	0	undefined	1	undefined

The 0° angle is said to be coterminal angle with the 360° angle. Coterminal angles are angles drawn in standard position that share a terminal side.

Ex: Find the exact value of $\sin 3\pi$



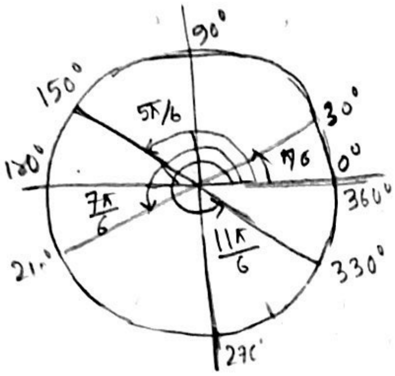
Soln: Here $\theta = 3\pi$

$\therefore \sin 3\pi = \sin(2\pi + \pi)$

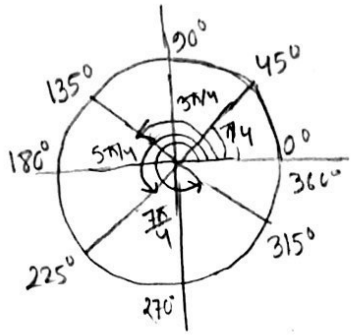
$= y = 0$

$\cos(-270°) = 0$

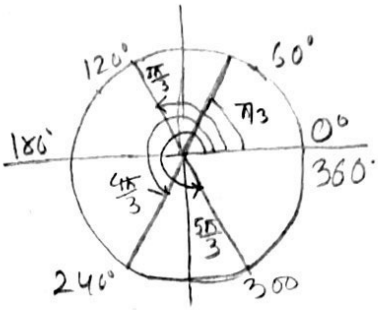
Multiples of 30°



Multiples of 45°



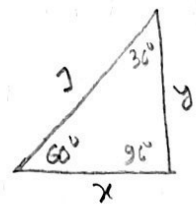
Multiples of 60°



For $\frac{\pi}{4} = 45°$, the point $P(x,y)$ lies on circle $x^2+y^2=1$ and bisect quadrant I. i.e P lies on the line $y=x$.

$\therefore x^2+y^2=1 \Rightarrow x^2+x^2=1 = 2x^2=1 \therefore x=\frac{1}{\sqrt{2}}$ and $y=\frac{1}{\sqrt{2}}$

For $\frac{\pi}{3} = 60°$, we have



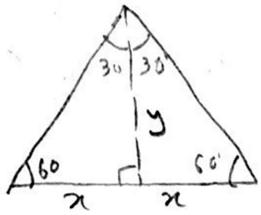
$x = \frac{1}{2}$, therefore by pythagorean theorem

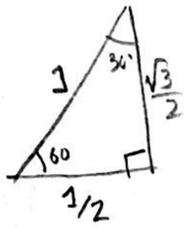
$x^2 + y^2 = 1$

$\Rightarrow \frac{1}{4} + y^2 = 1$

$\Rightarrow y^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore y = \frac{\sqrt{3}}{2}$





For $\frac{\pi}{6} = 30°$ we have $x = \frac{\sqrt{3}}{2}$, $y = 1/2$

θ(radians)	θ(degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

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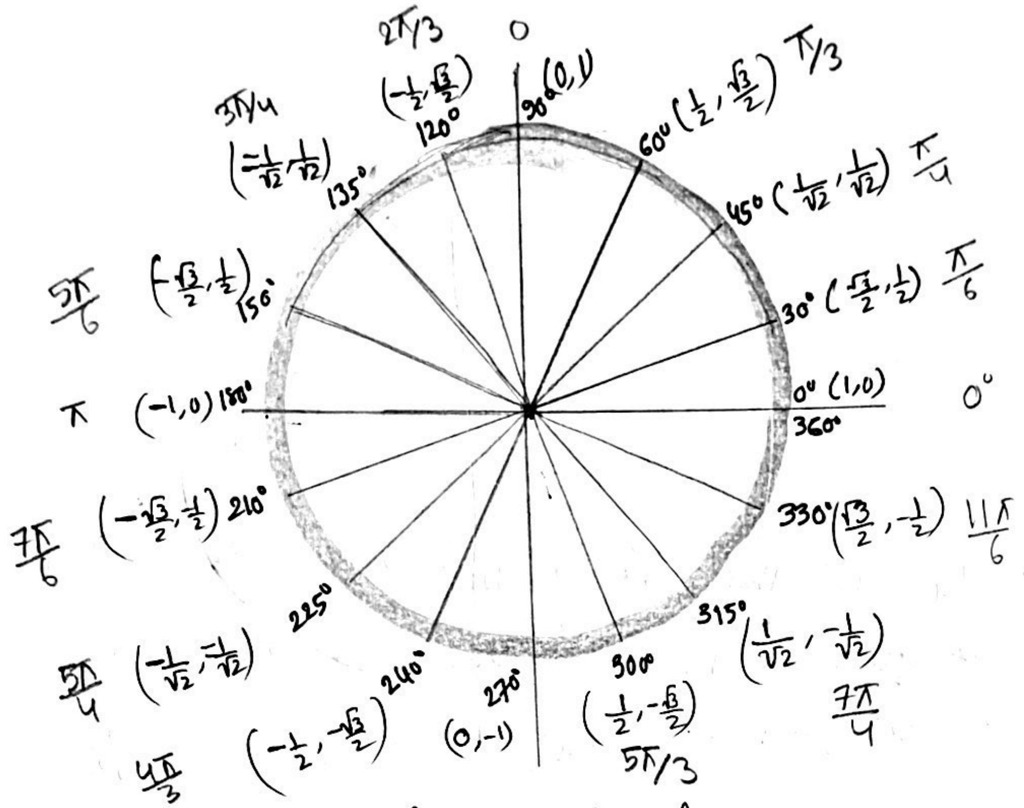

Ex: Find the exact value of each expression

a. $\sin 45^\circ \cos 180^\circ = \frac{1}{\sqrt{2}}(-1) = -\frac{1}{\sqrt{2}}$

b. $\tan\frac{\pi}{4} - \sin\frac{3\pi}{2} = 1 - (-1) = 2$

c. $\left(\sec\frac{\pi}{4}\right)^2 + \csc\frac{\pi}{2} = (\sqrt{2})^2 + 1 = 2+1 = 3$

Ex:

$\cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$ $\quad \sin 135^\circ = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan 315^\circ = \frac{1/\sqrt{2}}{-\frac{1}{\sqrt{2}}} = -1$

$\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ $\quad \sin(-60^\circ) = \sin\left(-\frac{\pi}{3}\right)$

$= -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$\cos 210^\circ = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$\cos\frac{8\pi}{3} = \cos\left(2\pi + \frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$

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Domain and Range of trignometric function:

Function	Domain	Range
$\sin\theta$	$(-\infty,\infty)$	$-1 \le \sin\theta \le 1$
$\cos\theta$	$(-\infty,\infty)$	$-1 \le \cos\theta \le 1$
$\tan\theta$	All real numbers except odd integer multiples of $\pi/2$	$(-\infty,\infty)$
$\operatorname{cosec}\theta$	All real numbers except integer multiples of π	$\operatorname{cosec}\theta \le -1$ or $\operatorname{cosec}\theta \ge 1$
$\sec\theta$	All real numbers except odd integer multiples of $\frac{\pi}{2}$	$\sec\theta \le -1$ or $\sec\theta \ge 1$
$\cot\theta$	All real numbers except integer multiples of π	$(-\infty,\infty)$

periodic function:

A function f is called periodic if there is a positive number P such that whenever θ in the domain of f so is $\theta+P$ and

$$f(\theta+P) = f(\theta)$$

If there is a smallest such number P, this smallest value is called the period of f.

$\sin(\theta+2\pi) = \sin\theta$ $\quad \tan(\theta+2\pi) = \tan\theta$ $\quad \sec(\theta+2\pi) = \sec\theta$

$\cos(\theta+2\pi) = \cos\theta$ $\quad \cot(\theta+2\pi) = \cot\theta$ $\quad \operatorname{cosec}\theta(\theta+2\pi) = \operatorname{cosec}\theta$

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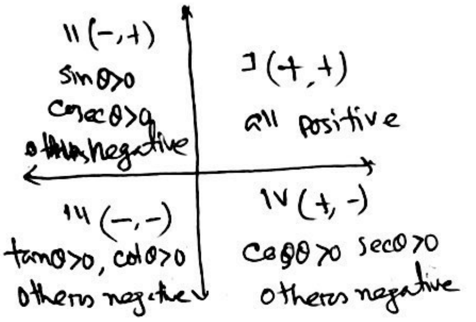
Find the exact value of

1. $\sin\frac{17\pi}{4} = \sin\left(4\pi + \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

2. $\cos 5\pi = \cos(4\pi + \pi) = \cos\pi = -1$

Determine the sign of trignometric functions:



Fundamental identities:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$

$\sin^2\theta + \cos^2\theta = 1$ $\quad \tan^2\theta + 1 = \sec^2\theta$ $\quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$

Ex: If $\sin\theta = \frac{1}{3}$ and $\cos\theta < 0$, find the exact value of each of the remaining five trignometric functions.

Soln: we know,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \sin^2\theta$$

$$\Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

Since, $\cos\theta < 0$, chose the minus sign and use $\sin\theta = \frac{1}{3}$

$$\therefore \cos\theta = -\sqrt{1-\sin^2\theta}$$

$$= -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{9-1}{9}} = -\sqrt{\frac{8}{9}} = \frac{-2\sqrt{2}}{3}$$

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$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/3}{\frac{-2\sqrt{2}}{3}} = -\frac{1}{2\sqrt{2}}$ $\cot\theta = \frac{1}{\tan\theta} = -2\sqrt{2}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{-2\sqrt{2}}{3}} = -\frac{3}{2\sqrt{2}}$ $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{1/3} = 3$

Even-ODD properties:

A function is even if $f(-\theta) = f(\theta)$

A function is odd if $f(-\theta) = -f(\theta)$

Cosine and secant functions are even functions, the others are odd functions.

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$ $\sec(-\theta) = \sec\theta$

$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

$\cot(-\theta) = -\cot\theta$

Find the exact value of

I. $\sin(-45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$ [odd function]

II. $\cos(-\pi) = \cos\pi = -1$ [even function]

III. $\cot\left(-\frac{3\pi}{2}\right) = -\cot\frac{3\pi}{2} = 0$ [odd function]

IV. $\tan\left(-\frac{37\pi}{4}\right) = -\tan\frac{37\pi}{4}$ [odd function]

$= -\tan\left(\frac{\pi}{4} + 9\pi\right)$ (period of π)

$= -\tan\pi/4 = -1$

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