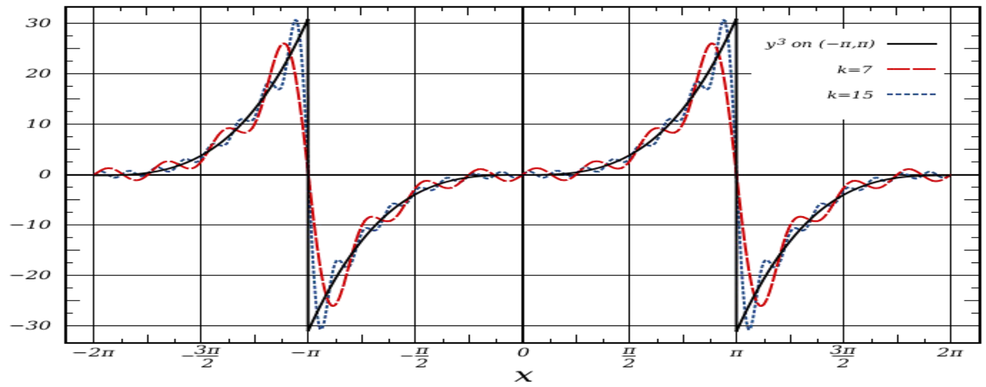
MAT 350
ENGINEERING MATHEMATICS

Fourier Series
and Fourier Transformation



Lecture: 10

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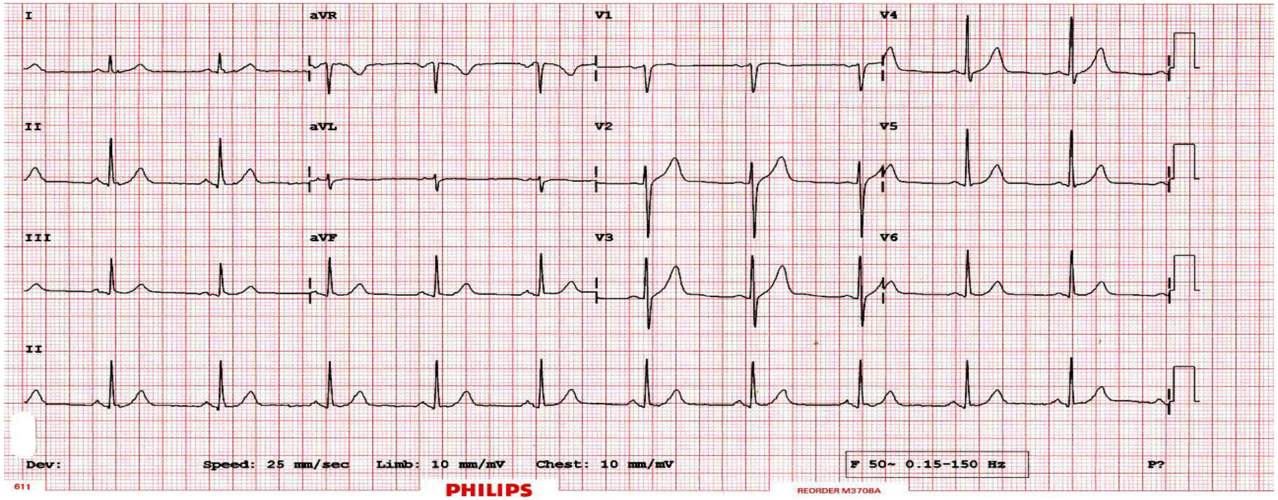
Introduction:

The central starting point of Fourier analysis is **Fourier series. They are infinite series** designed to represent general periodic functions in terms of simple ones, namely, cosines and sines.

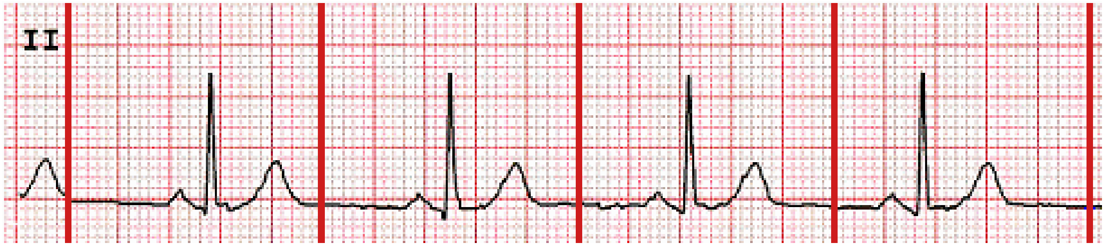
In a digital age, the *discrete Fourier transform plays an important role. Signals, such* as voice or music, are sampled and analyzed for frequencies. An important algorithm, in this context, is the *fast Fourier transform.*



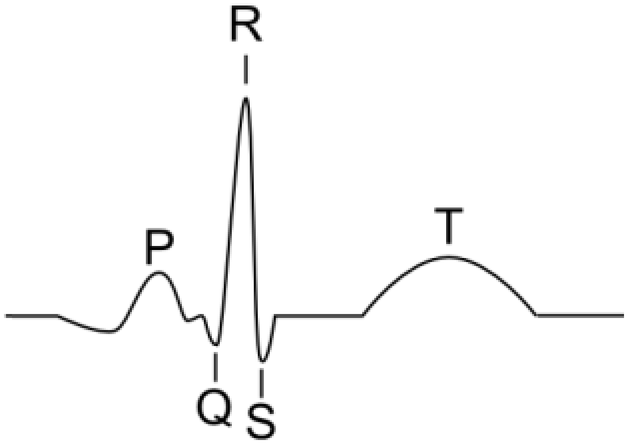
https://www.intmath.com/blog/mathematics/math-of-ecgs-fourier-series-4281

Modeling the Heartbeat Using Fourier Series

A heartbeat is roughly regular (if it isn't, it indicates something is wrong). Mathematically, we say something that repeats regularly is **periodic**. Such waves can be represented using a Fourier Series.



Apparently (according to the doctor), this ECG indicates my heart is quite healthy

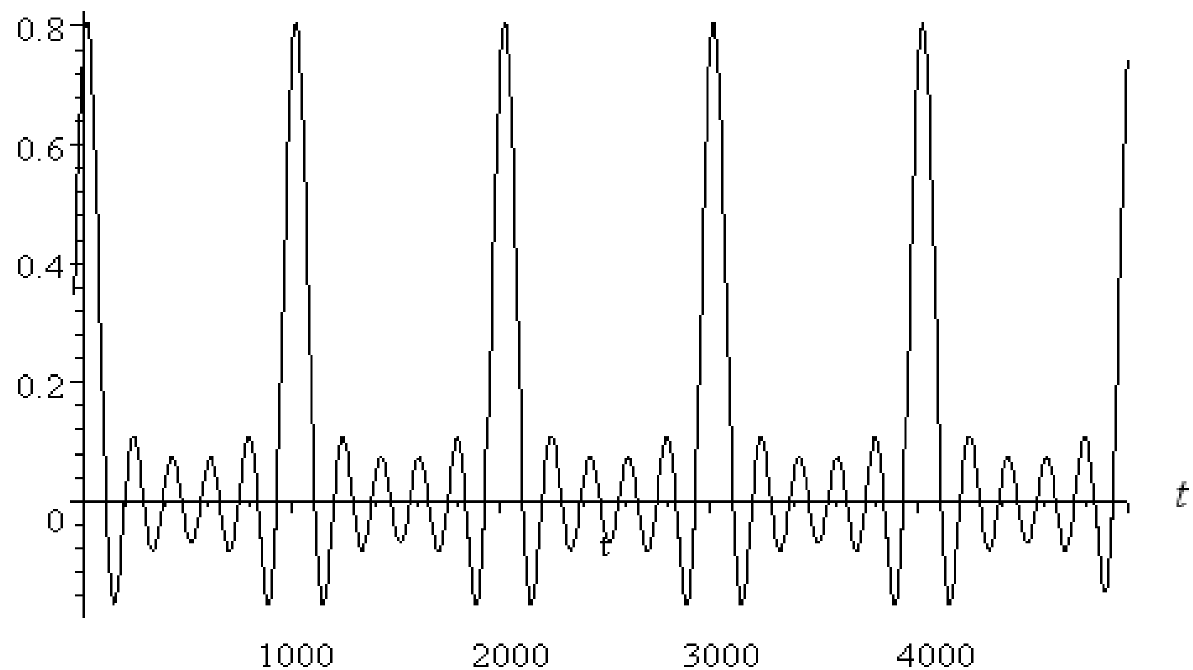


Modeling the Heartbeat Using Fourier Series

Assumptions

In my case, my heart rate was about 70 beats per minute. For the sake of simplicity, I'll assume 60 beats per minute or 1 per second. So the period = 1 second = 1000 milliseconds. A sample mathematical model ca be as follows:

$$f(t) = -\,0.0000156(t-20)^4 + 2.5$$
$$f(t) = f(t+1000)$$



Modeling the behavior of real **commodity prices** over a finite period **using Fourier Series**

Approximation of Primary Commodity Prices

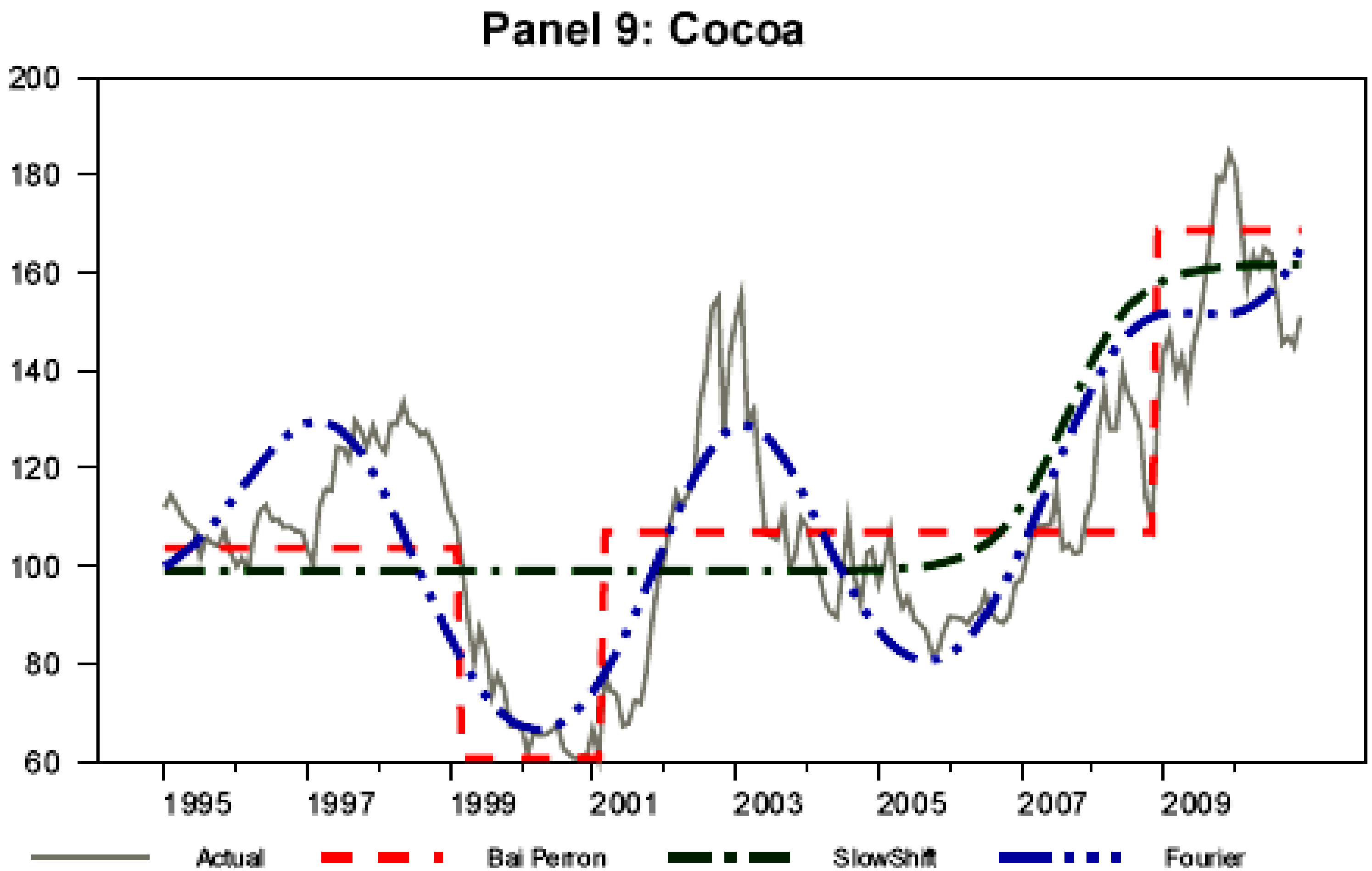


Figure: Commodity Prices, Actual Values and Shifting Means Obtained by Various Methods, 1995–2010

Fourier Series

Preliminaries:

A function f (x) is called a **periodic** function if f (x) is defined for all real x, except possibly at some points, and if there is some positive number p, called a period of f(x), such that

$$f(x + p) = f(x) \qquad (1)$$

for all x.

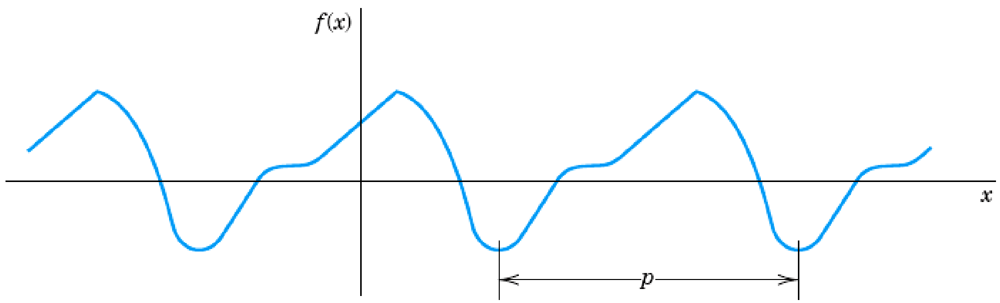


Fig. 258. Periodic function of period p

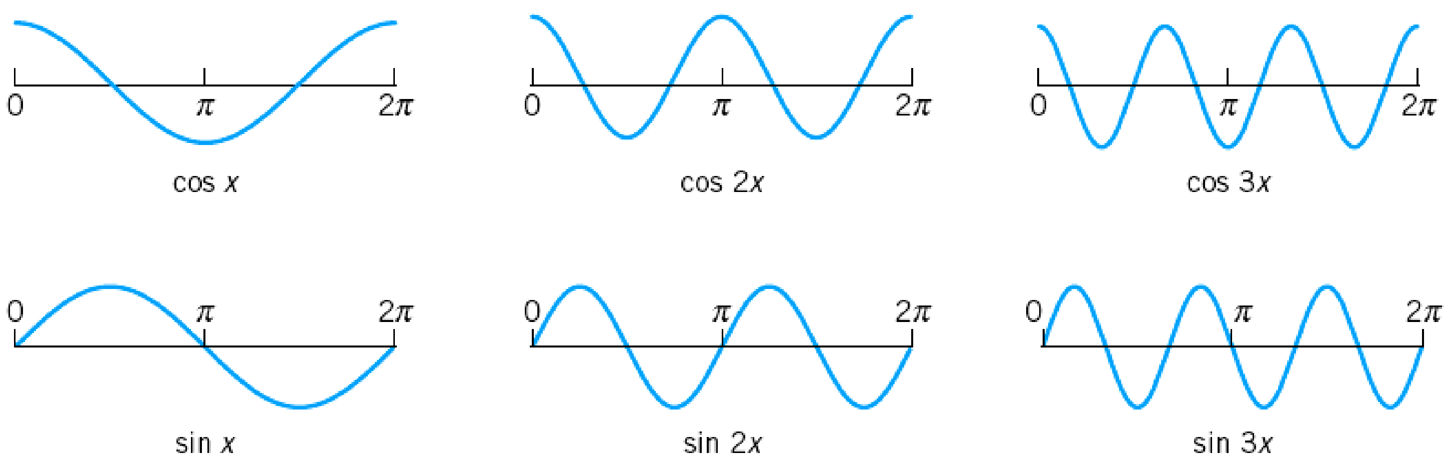
If $f(x)$ has period p, it also has the period $2p$ because (1) implies $f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$, etc.; thus for any integer $n = 1, 2, 3, \cdots$,

$$f(x + np) = f(x) \qquad \text{for all } x. \qquad (2)$$

Periodic Function:

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots . \tag{2b}$$

All these functions have the period . They form the so-called **trigonometric system.**



A trigonometric series is a combination of constant, sine, and cosine, as follows

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{3}$$

$a_0, a_1, b_1, a_2, b_2, \cdots$ are constants, called the **coefficients** of the series.

We see that each term has the period 2π. Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π.

Now suppose that f(x) is a given function of period and is such that it can be **represented by a series (3)**

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

Equation (4) is called the **Fourier series of f(x).**

The coefficients of (4) are the so-called **Fourier coefficients** of f(x), given by the **Euler formulas**

$$
\begin{aligned}
&\textbf{(0)} \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \\
&\textbf{(a)} \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, \cdots \\
&\textbf{(b)} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, \cdots .
\end{aligned}
\qquad (5)
$$

A Basic Example

Periodic Wave Equation:

Find the Fourier coefficients of the periodic function

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Class work

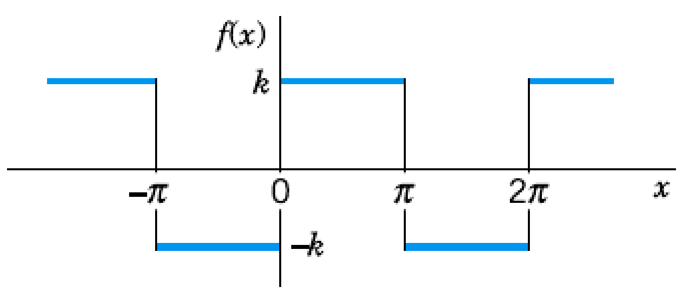


Fig. 260. Given function $f(x)$ (Periodic reactangular wave)

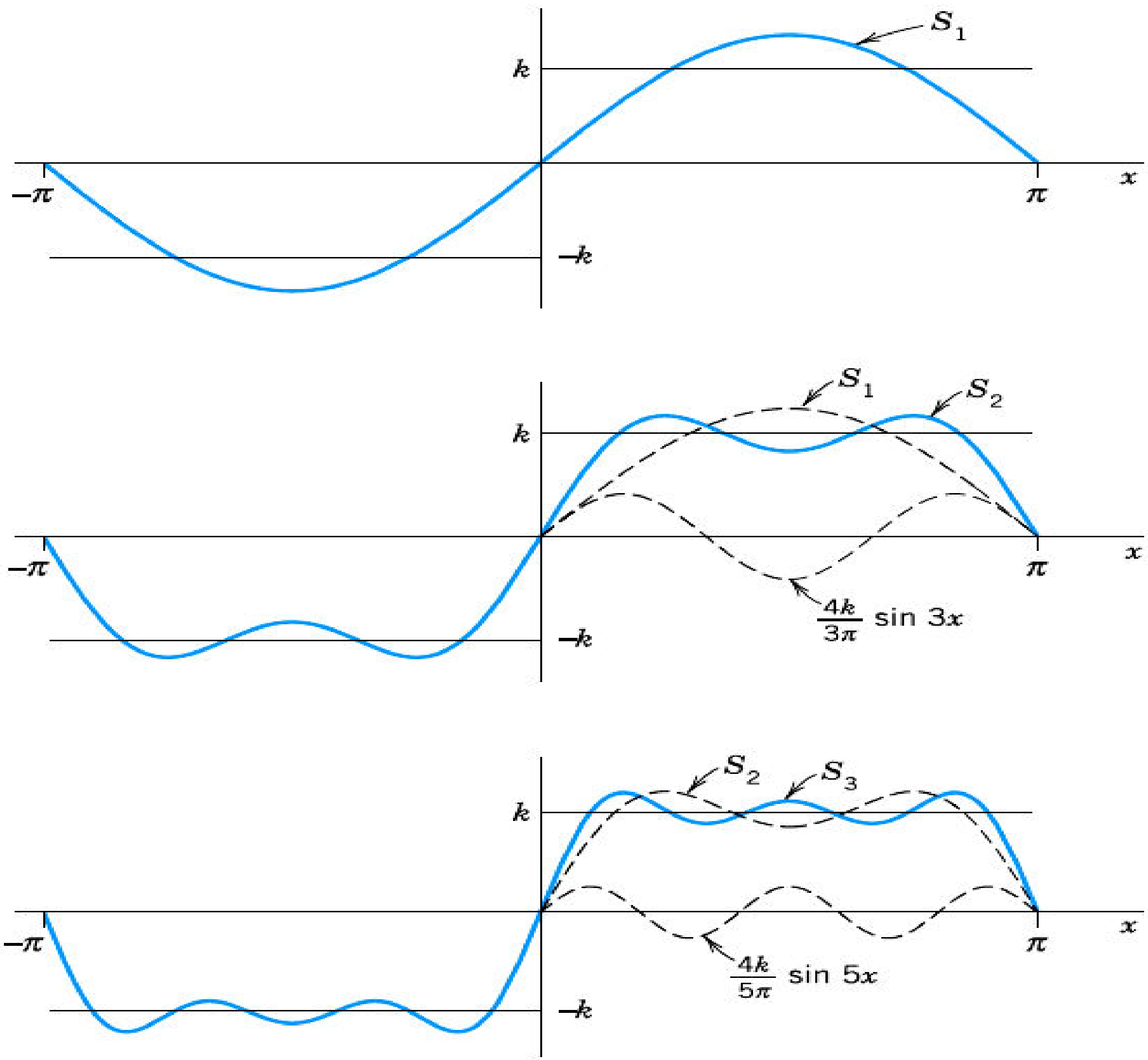


Fig. 261. First three partial sums of the corresponding Fourier series

Derivation of Fourier coefficients by the Euler formulas

Theorem 1 :

The trigonometric system (2b) is orthogonal on the interval $-\pi \leqq x \leqq \pi$ (also on $0 \leqq x \leqq 2\pi$ or any other interval of length 2π because of periodicity) that is, the integral of the product of any two functions in (2b) over that interval is 0, so that for any integers n and m,

$$\text{(a)} \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \neq m)$$

$$\text{(b)} \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \neq m) \qquad (6)$$

$$\text{(c)} \quad \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \neq m \text{ or } n = m).$$

Application of Theorem 1 to the Fourier Series:

We prove Eqn. 5(0).

Integrating on both sides of (4) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx.$$

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right).$$

The first term on the right equals $2\pi a_0$.

Integration shows that all the other integrals are 0.

Hence division by 2π gives 5(0)

Chapter 11.2

Arbitrary Period. Even and Odd Functions. Half-Range Expansions

Orientation. This section concerns three topics:

1. Transition from period 2π to any period $2L$, for the function f, simply by a transformation of scale on the x-axis.
2. Simplifications. Only cosine terms if f is even ("Fourier cosine series"). Only sine terms if f is odd ("Fourier sine series").
3. Expansion of f given for $0 \leqq x \leqq L$ in two Fourier series, one having only cosine terms and the other only sine terms ("half-range expansions").

Fourier series for a function f(x), where the period is not 2π but 2L

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

$$
\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots .
\end{aligned} \tag{7}
$$

EXAMPLE 1 Periodic Rectangular Wave

Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

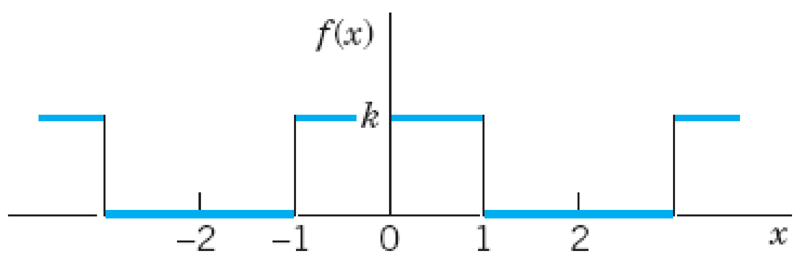


Fig. 263. Example 1

Solution.

Using relation (7), we obtain

$$a_0 = k/2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots,$$

$$a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

Using relation (7), we also obtain

$$b_n = 0 \text{ for } n = 1, 2, \cdots.$$

Hence, the Fourier Series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

Fourier Even and Odd functions:

If $f(x)$ is an **even function**, that is, $f(-x) = f(x)$

Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even})$$

with coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots.$$

If $f(x)$ is an **odd function**, that is, $f(-x) = -f(x)$

Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$

with coefficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Note

(a) $\displaystyle\int_{-L}^{L} g(x)\, dx = 2 \int_0^L g(x)\, dx$ for even g

(b) $\displaystyle\int_{-L}^{L} h(x)\, dx = 0$ for odd h

Summary

Even Function of Period 2π. If f is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2, \cdots$$

Odd Function of Period 2π. If f is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \cdots.$$

FOURIER INTEGRAL

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only.

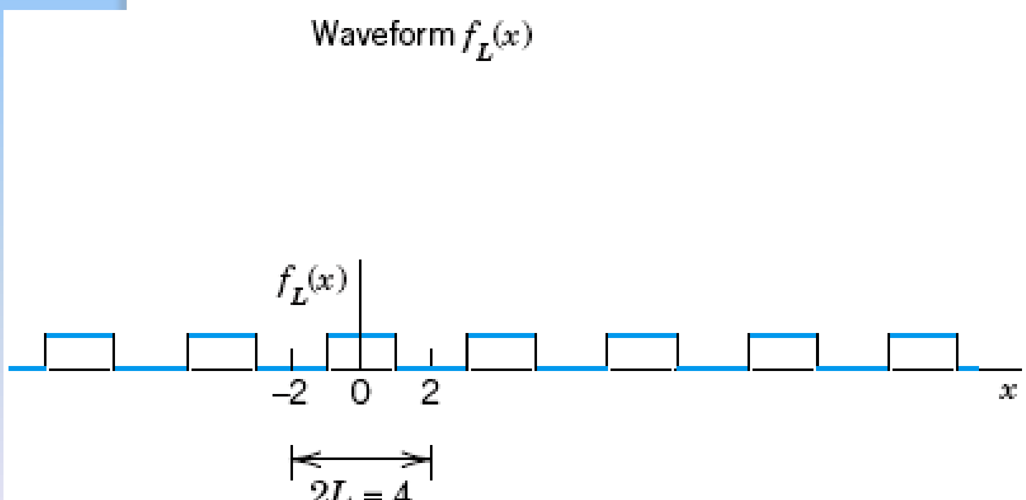
Since, of course, many problems involve functions that are *non-periodic and are of interest on the whole x-axis, we ask what can be done to extend the*
method of Fourier series to such functions. This idea will lead to "**Fourier integrals**."

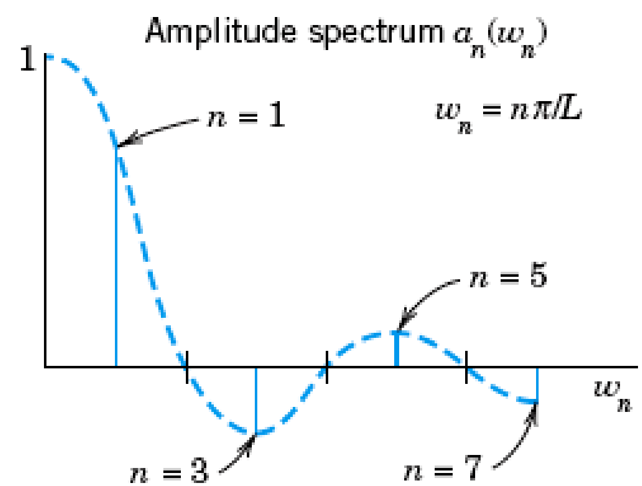
Rectangular Wave

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$





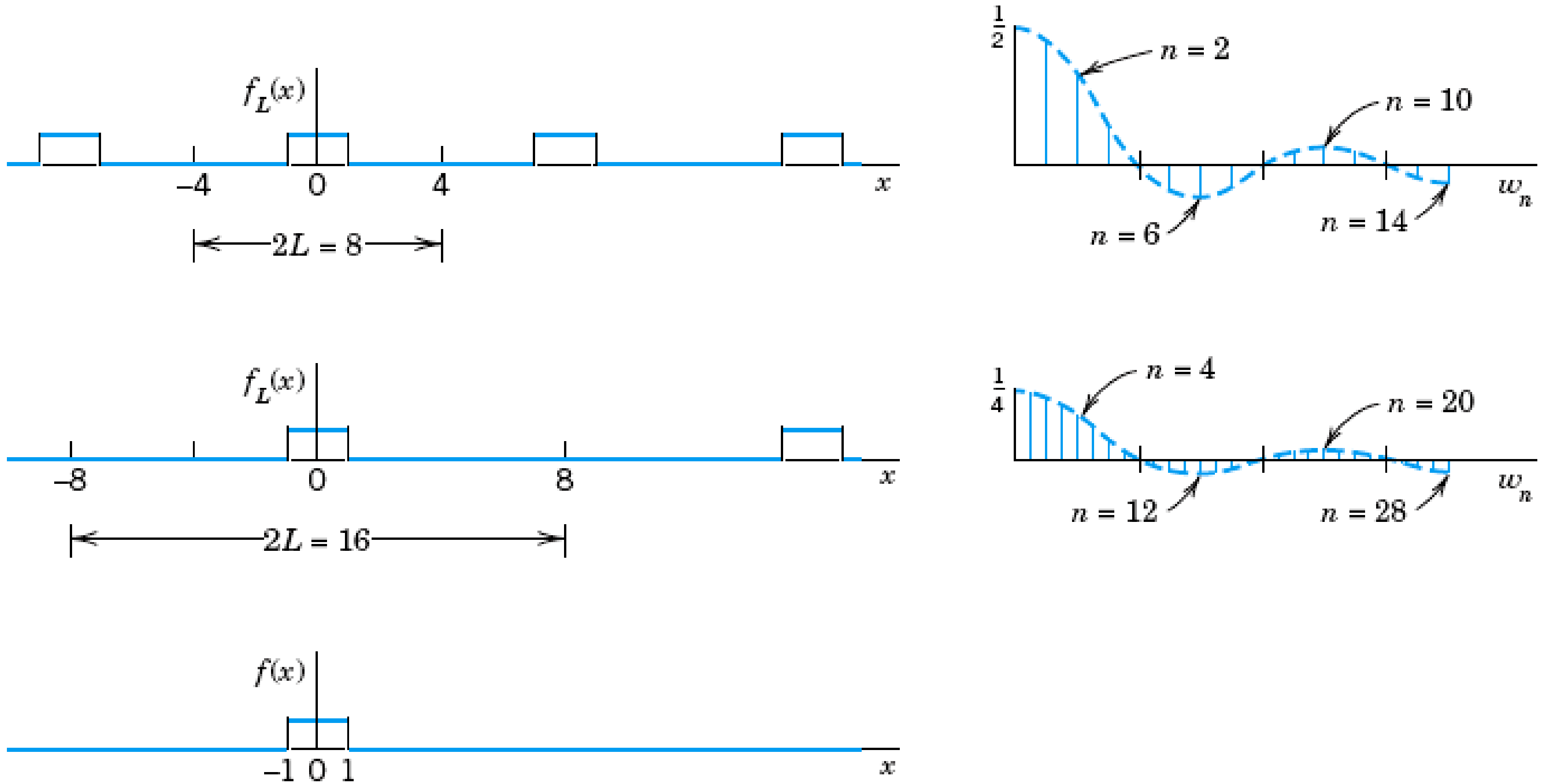


Fig. 280. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

Fourier Series:

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

Fourier Integral:

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx]\, dw.$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv,$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$$

Applications of Fourier Integrals

Single Pulse, Sine Integral. Dirichlet's Discontinuous Factor. Gibbs Phenomenon

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

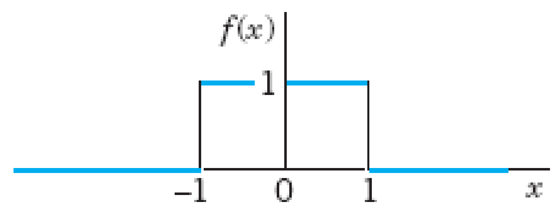


Fig. 281. Example 2

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

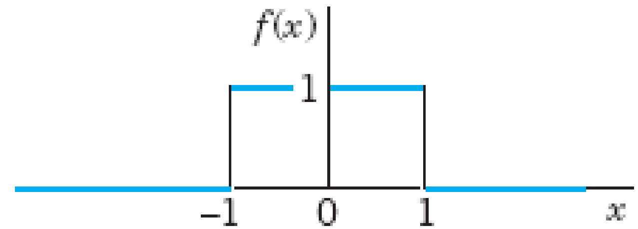


Fig. 281. Example 2

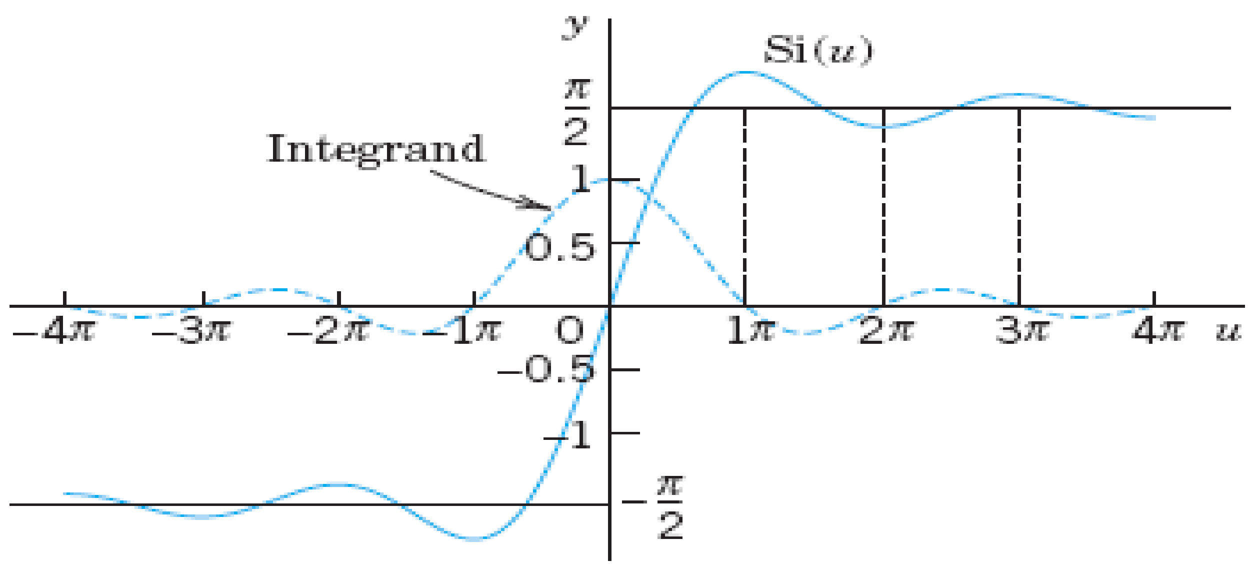


Fig. 282. Sine integral Si(u) and integrand

Fourier Cosine Integral and Fourier Sine Integral

Fourier cosine integral

$$f(x) = \int_0^\infty A(w) \cos wx \, dw$$

where

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv.$$

Fourier sine integral

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

where

$$B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv.$$

11.8 Fourier Cosine and Sine Transforms

An integral transform is a transformation in the form of an integral that produces from given functions new functions depending on a different variable.

Fourier transforms. can be obtained from the Fourier integral.

Fourier Cosine Transform

The Fourier cosine transform concerns ***even functions*** ***f(x). We obtain it from the Fourier*** cosine integral