# Chapter # 03
# (Topics in Differentiation)

**3.4 Related Rates:** In this section we will study related rates problems. In such problems one tries to find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change are known.

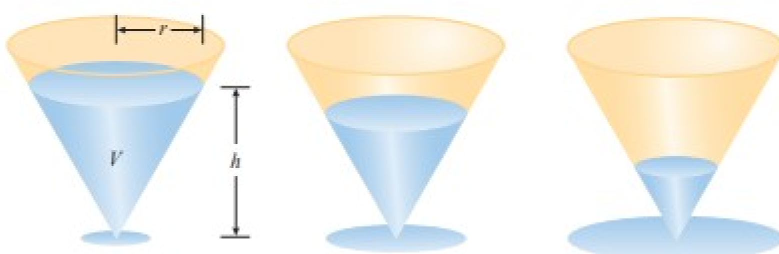
**Differentiating Equations to Relate Rates:** Following figure show a liquid draining through a conical filter. As the liquid drains, its volume ***V***, height ***h***, and radius ***r*** are functions of the elapsed time ***t***, and at each instant these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h$$

If we were interested in finding the rate of change of the volume ***V*** with respect to the time ***t***, we could begin by differentiating both sides of this equation with respect to t to obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Thus, to find $\boldsymbol{\frac{dV}{dt}}$ at a specific time ***t*** from this equation we would need to have values for ***r***, ***h***, $\boldsymbol{\frac{dh}{dt}}$, and $\frac{dr}{dt}$ at that time. This is called a related rates problem because the goal is to find an unknown rate of change by relating it to other variables whose values and whose rates of change at time ***t*** are known or can be found in some way.




**Example 1:** Suppose that ***x*** and ***y*** are differentiable functions of ***t*** and are related by the equation $\boldsymbol{y = x^3}$. Find $\boldsymbol{\frac{dy}{dt}}$ at time ***t*** = 1 if ***x*** = 2 and $\boldsymbol{\frac{dx}{dt} = 4}$ at time ***t*** = 1.

**Solution:** Using the chain rule to differentiate both sides of the equation $\boldsymbol{y = x^3}$ with respect to ***t*** yields

$$\frac{dy}{dt} = \frac{d}{dt}[x^3] = 3x^2\frac{dx}{dt}$$

Thus, the value of $\boldsymbol{\frac{dy}{dt}}$ at time ***t*** = 1 is

$$\left.\frac{dy}{dt}\right|_{t=1} = 3(2)^2 \left.\frac{dx}{dt}\right|_{t=1} = 12 \cdot 4 = 48$$

**Example 2:** Assume that oil spilled from a ruptured tanker spreads in a circular pattern Oil spill from a ruptured tanker whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

**Solution:** Let

$t$ = number of seconds elapsed from the time of the spill
$r$ = radius of the spill in feet after $t$ seconds
$A$ = area of the spill in square feet after $t$ seconds

We know the rate at which the radius is increasing, and we want to find the rate at which the area is increasing at the instant when $\boldsymbol{r = 60}$; that is, we want to find

$$\left.\frac{dA}{dt}\right|_{r=60} \quad \text{given that} \quad \frac{dr}{dt} = 2 \text{ ft/s}$$

This suggests that we look for an equation relating ***A*** and ***r*** that we can differentiate with respect to ***t*** to produce a relationship between $\boldsymbol{\frac{dA}{dt}}$ $\boldsymbol{and}$ $\boldsymbol{\frac{dr}{dt}}$. But ***A*** is the area of a circle of radius ***r***, so

$$A = \pi r^2$$

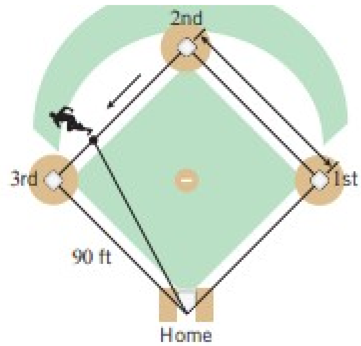
Differentiating both sides with respect to ***t*** yield

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Thus, when $\boldsymbol{r = 60}$ the area of the spill is increasing at the rate of

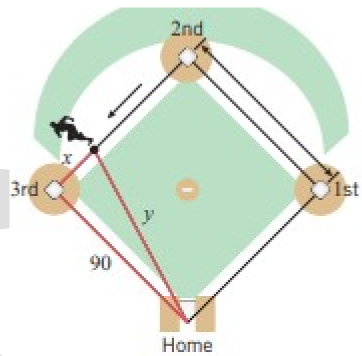
$$\left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2/\text{s} \approx 754 \text{ ft}^2/\text{s}$$

**Example 3:** A baseball diamond is a square whose sides are **90** ft long (following figure). Suppose that a player running from second base to third base has a speed of **30** ft/s at the instant when he is **20** ft from third base. At what rate is the player's distance from home plate changing at that instant?




**Solution:** We are given a constant speed with which the player is approaching third base, and we want to find the rate of change of the distance between the player and home plate at a particular instant. Thus, let

$t$ = number of seconds since the player left second base
$x$ = distance in feet from the player to third base
$y$ = distance in feet from the player to home plate




Thus, we want to find

$$\left.\frac{dy}{dt}\right|_{x=20} \quad \text{given that} \quad \left.\frac{dx}{dt}\right|_{x=20} = -30 \text{ ft/s}$$

As suggested by above figure, an equation relating the variables ***x*** and ***y*** can be obtained using the Theorem of Pythagoras

$$x^2 + 90^2 = y^2 \qquad (i)$$

Differentiating both sides of this equation with respect to ***t*** yields

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

from which we obtain

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \qquad (\textbf{\textit{ii}})$$

When ***x* = 20**, it follows from (***i***) that

$$y = \sqrt{20^2 + 90^2} = \sqrt{8500} = 10\sqrt{85}$$

so that (***ii***) yields

$$\left.\frac{dy}{dt}\right|_{x=20} = \frac{20}{10\sqrt{85}}(-30) = -\frac{60}{\sqrt{85}} \approx -6.51 \text{ ft/s}$$

The negative sign in the answer tells us that ***y*** is decreasing.

**Home Work: Exercise 3.4: Problem No. 7, 8, 9, 10, 13, 15, 17**