DAY-3:

7.3: Integration of Trigonometric Functions

→ There are 7 groups, where each group has 3 sub-groups.

Group -1: All six trigonometric functions with power 1

1) $\int \sin x \; dx = -\cos x + C$
2) $\int \cos x \; dx = \sin x + C$

3) $\int \tan x \; dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{\cos x} \, \boldsymbol{\sin x \, dx}$

[Set $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$, that is, $du = -\sin x \; dx$. Hence, $-du = \boldsymbol{\sin x \; dx}$]

$= \int \frac{1}{u} (-1) du = -\int \frac{1}{u} \, du$; [Note: When $\frac{1}{u}$ is given, we only know that $u \neq 0$]

$= -\ln |u| + C$

$= -\ln |\cos x| + C$

$= \ln |\cos x|^{-1} + C$; $[n \log_b x = \log_b x^n]$

$= \ln |(\cos x)^{-1}| + C$

$= \ln |\frac{1}{\cos x}| + C$

$\int \tan x \; dx = \ln |\sec x| + C$

4) $\int \cot x \; dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{1}{\sin x} \cos x \, dx$

$= \int \frac{1}{u} \, du$; $set \;\; u = \sin x \rightarrow du = \cos x \; dx$

$= \ln|\sin x| + C$

Example: $\int 3x^2 \, dx = x^3 + C$

5) $\int \sec x \; dx = \int \sec x \cdot 1 \, dx$; $1 = \frac{\sec x + \tan x}{\sec x + \tan x}$

$$= \int \frac{\sec x \ (\tan x + \sec x\ )}{\tan x \ + \ \sec x} \quad dx$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} \ dx = = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \ dx \ ;$$

[Set $u = \tan x + \sec x$, then $\frac{du}{dx} = \sec^2 x \ + \sec x \ \tan x$. Hence, $du = (\sec^2 x \ + \sec x \ \tan x\ )dx$]

$$\int \sec x \ dx = \int \frac{1}{u} \ du \quad = \quad \ln|u| + C$$

$$\int \sec x \ dx \quad = \quad \ln|\tan x + \sec x| + C$$

Definition: Logarithmic Derivative

Since $\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)} = \frac{\frac{d}{dx}(D)}{D}$, $\frac{f'(x)}{f(x)}$ fraction is called the logarithmic derivative. And hence,

$$\int \frac{f'(x)}{f(x)} \ dx \ = \ln|f(x)| + C.$$

Note: (1) $\int \sec x \tan x \, dx \ = \sec x + C$ and $\int \sec^2 x \, dx \ = \tan x + C$

(2) $\int \csc x \cot x \, dx \ = -\csc x + C$ and $\int \csc^2 x \, dx \ = -\cot x + C$

Also, $\cot^2 x + 1 = \csc^2 x \ \Rightarrow \ \mathbf{1 = \csc^2 x - \cot^2 x}$

6) $\int \csc x \ dx$ Homework

$$\int \csc x \ dx = \int \csc x \ \cdot 1 \ dx = \int \csc x \ \cdot \frac{\csc x + \cot x}{\csc x + \cot x} \ dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \ dx$$

$$= \int \frac{\csc x \cot x + \csc^2 x}{\csc x + \cot x} \ dx \ ;$$

set $u = \csc x + \cot x$, then $\dfrac{du}{dx} = -\csc x \cot x - \csc^2 x \rightarrow -du = (\csc x \cot x + \csc^2 x)dx$

$$= \int \frac{1}{u}(-1)\, du = -\ln|u| + C$$

$$= -\ln|\csc x + \cot x| + C$$

$= \ln|\csc x + \cot x|^{-1} + C$; Note: $x^{-1} = \frac{1}{x}$.

$= \ln\left|\dfrac{1}{\csc x + \cot x}\right| + C$; Formula: $1 + \cot^2 x = \csc^2 x$

$$= \ln\left|\frac{\csc^2 x - \cot^2 x}{\csc x + \cot x}\right| + C$$

$$= \ln|\csc x - \cot x| + C,$$

Alternative Method:

$\int \csc x \; dx = \int \frac{1}{\sin x}\, dx =$

$$= \int \frac{1}{2\,\sin(\frac{x}{2})\,\cos\left(\frac{x}{2}\right)}\, dx$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right)}{2\,\sin(\frac{x}{2})\,\cos\left(\frac{x}{2}\right)\,\sec^2\left(\frac{x}{2}\right)}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2\left(\frac{x}{2}\right)}{\sin(\frac{x}{2})\,\cos\left(\frac{x}{2}\right)\dfrac{1}{\cos^2\left(\frac{x}{2}\right)}}\, dx$$

$= \displaystyle\int \frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)}{\tan(\frac{x}{2})}\, dx$; $Set\ u = \tan(\frac{x}{2})$, then $du = \dfrac{1}{2}\sec^2\left(\frac{x}{2}\right) dx.$

$$= \int \frac{1}{u}\, du\, dx = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

Group -2: All six trigonometric functions with power 2

Formulas:

(i) $\sin^2 x = \frac{1}{2}[1-\cos(2x)]$

(ii) $\cos^2 x = \frac{1}{2}[1+\cos(2x)]$

(iii) $\tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

(iv) $cot^2 x + 1 = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1$

1) $(a) \int \sin^2 x \; dx = \int \frac{1}{2}[1-\cos(2x)] \; dx$

$$= \frac{1}{2} \int [1-\cos(2x)] \; dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C \quad ;$$

$$\left[\text{Formula:} \quad \int \cos(kx) \; dx = \frac{\sin(kx)}{k} + C \right]$$

$$\int \sin^2 x \; dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$(b) \int \sin^2(3x) \; dx$$

$$= \int \frac{1}{2}[1-\cos(6x)] \; dx$$

$$= \frac{1}{2} \int [1-\cos(6x)] \; dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(6x)}{6}\right] + C$$

2) $(a)\ \int \cos^2 x\ \ dx$

$$= \int \frac{1}{2}[1+\cos(2x)]\ \ dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin(2x)\ + C$$

$(b)\ \displaystyle\int \cos^2(5x)\ \ dx$

$$= \int \frac{1}{2}[1+\cos(10x)]\ \ dx$$

$$= \frac{1}{2}\int [1+\cos(10x)]\ \ dx$$

$$= \frac{1}{2}\left[x + \frac{\sin(10x)}{10}\right] + C$$

$$= \frac{1}{2}x + \frac{1}{20}\sin(10x) + C$$

3) $\int \tan^2 x\ \ dx\ =\ \int [\ \sec^2 x\ -1]\,dx\ =\ \tan x\ - x\ + C$

4) $\int \cot^2 x\ \ dx\ =\ \int [\ \csc^2 x\ -1]\,dx\ =\ -\cot x\ - x\ + C$

5) $\int \sec^2 x\ \ dx\ = \tan x + C$

6) $\int \csc^2 x\ \ dx\ =\ -\cot x + C$

Group -3: All six trigonometric functions with power n, any integer $n \geq 2$.

Reduction Formulas for Integration:

(1) * $\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

(2) $\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$

(3) * $\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx$

(4) $\int \cot^n x\, dx = -\frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\, dx$

(5) * $\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$

(6) $\int \csc^n x\, dx = -\frac{1}{n-1}\csc^{n-2} x \cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\, dx$

HINT: To derive the formula [Homework]

(1) $\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

Start with $I = \int \sin^n x\, dx$

$= \int \sin^{n-1} x \; \sin x \; dx$; **Set** $u = \sin^{n-1} x$, $dv = \sin x\, dx$

***Also,* for** $\int \sec^n x\, dx$, **set**

$I = \int \sec^n x\, dx = \int \sec^{n-2} x \cdot \sec^2 x\, dx$;

$u = \sec^{n-2} x$ **and** $dv = \sec^2 x\, dx$

Definition: Co-functions

(i) Sine and Cosine are co-functions
(ii) Tangent and Cotangent are co-functions
(iii) Secant and Cosecant are co-functions

1) Evaluate $\int \sec^5 x \, dx$

Solution: We know that for $n \geq 2$,

$$\int \sec^{n} x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \ldots\ldots\ldots \quad (1)$$

Here $\int \sec^5 x \, dx$; Given $n = 5,\ n-1 = 4,\ \ n-2 = 3$

$$\int \sec^5 x \, dx = \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx \ ;$$

$; [\text{here } n = 3, \quad n-1 = 2, \ \ n-2 = 1]$

$$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4}\left[\frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx\right]$$

$$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \int \sec x \, dx$$

$$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln|\sec x + \tan x| + C$$

Homework:

2) $\int \sec^7 x \, dx \ + \int \sec^5 x \, dx$

$$= \frac{1}{6} \sec^5 x \tan x + \frac{5}{6} \int \sec^5 x \, dx \ + \int \sec^5 x \, dx$$
$$= \frac{1}{6} \sec^5 x \tan x + \frac{11}{6} \int \sec^5 x \, dx$$

Please complete!

3) $\int \sin^6 x \, dx \ + \int \sin^4 x \, dx$

4) $\int \tan^6 x \, dx \ + \int \tan^5 x \, dx$

Group-4: $\int \boldsymbol{sinA\ cosB\ dx}$; $\int \boldsymbol{sinA\ sinB\ dx}$; $\int \boldsymbol{cosA\ cosB\ dx}$ **here** $\mathbf{A \neq B}$.

Formulas:

1) $\sin A \cos B = \frac{1}{2}\,[\sin(A-B) + \sin(A+B)]$
2) $\sin A \sin B = \frac{1}{2}\,[\cos(A-B) - \cos(A+B)]$
3) $\cos A \cos B = \frac{1}{2}\,[\cos(A-B) + \cos(A+B)]$

Definitions

1) $y = \sin x$ is an odd function, that is, $\sin(-x) = -\sin x$
$$[f(-x) = -f(x)]$$
2) $y = \cos x$ is an even function, that is, $\cos(-x) = \cos x$
$$[f(-x) = f(x)]$$

Here,

$$\cos(5x)\sin(2x) = \sin(2x)\cos(5x)$$

$$= \frac{1}{2}\,[\sin(-3x) + \sin(7x)] = \frac{1}{2}\,[-\sin(3x) + \sin(7x)]$$

Example:1 Evaluate

$$\int_0^{\frac{\pi}{2}} \boldsymbol{sin\,(3x)\,sin\,(6x)\ dx}$$

Solution:

$$\int_0^{\frac{\pi}{2}} sin\,(3x)\,sin\,(6x)\ dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos(3x-6x) - \cos(3x+6x)]\ dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[\cos(-3x) - \cos(9x)]\ dx\ ; \quad \cos(-3x) = \cos(3x)$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} [\cos(3x) - \cos(9x)]\ dx$$

$$= \frac{1}{2}\left[\frac{\sin(3x)}{3} - \frac{\sin(9x)}{9}\right]_{0}^{\frac{\pi}{2}}$$

$$= \left[\frac{1}{6}\ \sin(3x) - \frac{1}{18}\ \sin(9x)\right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{6}\sin\left(3\frac{\pi}{2}\right) - 0 - \frac{1}{18}\sin\left(9\ \frac{\pi}{2}\right) - 0 \quad ;$$

$$\left[\sin 0 = 0, \quad \sin\left(9\ \frac{\pi}{2}\right) = \sin\left(2\pi\ \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin\left(3\frac{\pi}{2}\right) = -1\right]$$

$$= \frac{1}{6}(-1) - \frac{1}{18}(1) = -\frac{1}{6} - \frac{1}{18}$$

$$= -\frac{4}{18} = -\frac{2}{9}$$

Example: 2 Evaluate

$$\int_{\mathbf{0}}^{\frac{\boldsymbol{\pi}}{\mathbf{2}}} cos\ (3x)cos(6x)\ \boldsymbol{dx}$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[\cos(3x - 6x) + \cos(3x + 6x)]\ dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[\cos(-3x) + \cos(9x)]\ dx$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} [\cos(3x) + \cos(9x)]\ dx$$

$$= \frac{1}{2}\left[\frac{\sin(3x)}{3} + \frac{\sin(9x)}{9}\right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{6}\sin\left(3\ \frac{\pi}{2}\right) - 0 + \frac{1}{18}\sin\left(9\ \frac{\pi}{2}\right) - 0$$

$$= \frac{1}{6}(-1) + \frac{1}{18}\ (1) = -\frac{1}{6} + \frac{1}{18}$$

$$= -\frac{2}{18} = -\frac{1}{9}$$

DAY-4, 24th February, 2021

Group: 5 $\int \boldsymbol{sin^n x \ cos^m x \ \ dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

→Formula: $\sin^2 x + \cos^2 x = 1.$

→There are 3 $-$cases.

Case-1: When $\boldsymbol{m}$ is odd in $\int \boldsymbol{sin^n x \ cos^m x \ \ dx}$

Steps:

1) Split off a factor $\cos x \, dx$
2) Write $\cos^2 x \ = \ 1 - \sin^2 x$
3) Set $u = \sin x$. Then $\frac{du}{dx} = \cos x$, that is, $\cos x \ dx = du$

Example: 3

$(a) \displaystyle\int sin^{20} x \ cos^7 x \ \ dx$

Solution:

$$\int sin^{20} x \ cos^7 x \ \ dx$$

$$= \int sin^{20} x \ cos^6 x \ \cos x \, dx$$

$$= \int sin^{20} x \ (cos^2 x)^3 \ \cos x \, dx$$

$$= \int sin^{20} x \ (1 - \sin^2 x)^3 \ \cos x \, dx \quad ;$$

[Set $\mathrm{u} = \sin x$, $\frac{du}{dx} = \cos x$, that is, $\cos x \, dx = du$]

$$= \int u^{20}(1 - u^2)^3 du$$

$$= \int u^{20} \, (1 - 3u^2 + 3u^4 - u^6) du$$

$$= \int [u^{20} - 3u^{22} + 3u^{24} - u^{26}]du$$

$$= \left[\frac{1}{21}u^{21} - \frac{3}{23}\ u^{23} + \frac{3}{25}u^{25} - \frac{1}{27}u^{27}\right] + C$$

$$= \frac{1}{21}sin^{21}x - \frac{3}{23}\ sin^{23}x + \frac{3}{25}sin^{25}x - \frac{1}{27}sin^{27}x + C$$

Example: 3 $(b) \int sin^4\, x\ \ cos^5x\ \ \ dx$

Solution: Given, $\int sin^4\, x\ \ cos^5x\ \ \ dx$

$$= \int sin^4\, x\ \ cos^4x\ \ \cos x\, dx$$

$$= \int sin^4\, x\ \ (cos^2x)^2\ \ \cos x\, dx$$

$$= \int sin^4\, x\ \ (1 - \sin^2 x)^2\ \cdot\ \cos x\, dx$$

[Set $u = \sin x$. Then $\frac{du}{dx} = \cos x$, that is, $\cos x\ \ dx = \ du$]

$$= \int u^4\ \ (1 - u^2)^2\ \ du$$

$$= \int u^4\ \ [1 - 2u^2 + u^4]\ \ du$$

$$= \int\ [u^4 - 2u^6 + u^8]\ \ du$$

$$= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$= \frac{1}{5}sin^5x - \frac{2}{7}sin^7x + \frac{1}{9}sin^9x + C$$

Example: 3 $(C) \int sin^9\, x\ \ cos^7x\, dx$

Solution:

$$\int sin^9 x\ \ cos^7 x\, dx$$

$$= \int sin^9 x\ \ cos^6 x\ \ \cos x\, dx$$

$$= \int sin^9 x\ \ (cos^2 x)^3 \cos x\, dx$$

$$= \int sin^9 x\ \ (1-\sin^2 x)^3 \cos x\, dx \quad ; \text{ Set } u = \sin x\,. \text{Then } du = \cos x\ \ dx$$

$$= \int u^9\ \ (1-u^2)^3\, du$$

$$= \int u^9\ \ [1-3u^2+3u^4-u^6]\, du$$

$$= \int [u^9 - 3u^{11} + 3u^{13} - u^{15}]\, du$$

$$= \frac{1}{10}u^{10} - \frac{3}{12}u^{12} + \frac{3}{14}u^{14} - \frac{1}{16}u^{16} + C$$

$$= \frac{1}{10}sin^{10}x - \frac{3}{12}sin^{12}x + \frac{3}{14}sin^{14}x - \frac{1}{16}sin^{16}x + C$$

Case-2: When n is odd in $\int sin^n x\ \ cos^m x\ \ \ dx$

Steps:

1) Split off a factor $\sin x\, dx$
2) Write $\sin^2 x\ = 1-\cos^2 x$
3) Set $u = \cos x\,.$ Then $\frac{du}{dx} = -sin\ x \Rightarrow \sin x\, dx = -du$

Example:4 (a) Evaluate $\int sin^7 x\ \ cos^8 x\ \ \ dx$

Solution:

$$\int sin^7 x\ \ cos^8 x\ \ \ dx$$

$$= \int sin^6 x \;\; cos^8 x \;\; \sin x \;\; dx$$

$$= \int (sin^2 x)^3 \;\; cos^8 x \;\; \sin x \;\; dx$$

$$= \int (1 - \cos^2 x\,)^3 \;\; cos^8 x \;\; \sin x \;\; dx$$

[Set $u = \cos x$. Then $\frac{du}{dx} = -sin\ x \Rightarrow \sin x\, dx = -du$]

$$\int sin^7 x \;\; cos^8 x \;\; dx = \int (1 - u^2\,)^3 \;\; u^8 \;\; (-1)\, du$$

$$= -\int (1 - u^2\,)^3 \;\; u^8 \quad du$$

$$= -\int [1 - 3u^2 + 3u^4 - u^6]\, u^8 \quad du$$

$$= -\int [u^8 - 3u^{10} + 3u^{12} - u^{14}]\;\; du$$

$$= -\left[\frac{1}{9}u^9 - \frac{3}{11}u^{11} + \frac{3}{13}u^{13} - \frac{1}{15}u^{15}\right] + C$$

$$= -\left[\frac{1}{9}cos^9 x - \frac{3}{11}cos^{11} x + \frac{3}{13}cos^{13} x - \frac{1}{15}cos^{15} x\right] + C$$

Example:4(b) Evaluate $\int sin^5 x \;\; cos^6 x \;\; dx$ → Please submit!

Evaluate 4 (c) $\int sin^{11} x \;\; cos^5 x \;\; dx$

Case-3: When n and m both are even integers

Step: Write $\sin^2 x = \frac{1}{2}[1-\cos(2x)]$ and $\cos^2 x = \frac{1}{2}[1+\cos(2x)]$; and whenever we get $\cos^2\theta$, we must apply the formula $\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)]$.

Example: Evaluate $\int sin^4 x\ cos^4 x\ \ dx$

Solution:

$$\int sin^4 x\ cos^4 x\ \ dx$$

$$= \int (sin^2 x\ cos^2 x)^2\ \ dx$$

$$= \int \left(\frac{1}{2}[1-cos\,(2x)]\frac{1}{2}[1+cos\,(2x)]\right)^2\ \ dx$$

$$= \frac{1}{4}\cdot\frac{1}{4}\int (\,[1-cos\,(2x)][1+cos\,(2x)]\,)^2\ \ dx$$

$$= \frac{1}{16}\int \left(1-cos^2\,(2x)\right)^2\ \ dx$$

$$= \frac{1}{16}\int \left(1-\frac{1}{2}[1+cos\,(4x)]\right)^2\ \ dx$$

$$= \frac{1}{16}\int \left(1-\frac{1}{2}-\frac{1}{2}cos\,(4x)\right)^2 dx$$

$$= \frac{1}{16}\int \left(\frac{1}{2}-\frac{1}{2}cos\,(4x)\right)^2 dx$$

$$= \frac{1}{16}\cdot\ \frac{1}{4}\int \left(1-cos\,(4x)\right)^2 dx$$

$$= \frac{1}{64}\int [1-2\,cos(4x)+cos^2(4x)]\,dx$$

$$= \frac{1}{64}\int [1-2\,cos(4x)]dx\ \ +\frac{1}{64}\int cos^2(4x)\,dx$$

$$= \frac{1}{64}\ [x - 2\ \frac{\sin(4x)}{4}] + \frac{1}{64}\ \int \frac{1}{2}[1 + \cos(8x)]\, dx$$

$$= \frac{1}{64}\ \left[x - 2\ \frac{\sin(4x)}{4}\right] + \frac{1}{64}\,\frac{1}{2}\ \int [1 + \cos(8x)]\, dx$$

$$= \frac{1}{64}\ x - \frac{1}{128}\sin(4x) + \frac{1}{128}\ \left[x + \frac{\sin(8x)}{8}\right] + C$$

$$= \frac{3}{128}\ x - \frac{1}{128}\sin(4x) + \frac{1}{1024}\ \sin(8x) + C$$

<u>Alternative Method 1:</u>

Evaluate

$$\int sin^4\, x\ \ cos^4 x\ \ \ dx$$

$$= \int sin^4\, x\ \ (1 - \sin^2 x)^2\ \ \ dx$$

$$= \ \int [sin^4\, x - 2\sin^6 x + \sin^8 x\ ]\, dx$$

$= \int \sin^4 x\ \ dx\ \ - 2\ \ \int \sin^6 x\ \ \ dx\ \ + \int \sin^8 x\ \ dx \rightarrow$ Then apply reduction formula for integration.

<u>Alternative Method 2:</u>

Evaluate

$$\int sin^4\, x\ \ cos^4 x\ \ \ dx\ \ = \frac{1}{16}\int \left(1 - cos^2\,(2x)\right)^2\ \ \ dx\ \ = \frac{1}{16}\int \left(sin^2\,(2x)\right)^2\ \ \ dx$$

$$= \frac{1}{16}\int sin^4\,(2x)\ \ \ dx\ ;set\ \ \ u = 2x, then\ \frac{1}{2}du = dx$$

$= \frac{1}{32}\ \int sin^4\,(u)\ \ du \rightarrow$ Apply reduction formula.

Warning: $\int sin^4 x\ cos^4 x\ \ dx\ \ \neq (\int sin^4 x\ \ dx\ )(\int cos^4 x\ \ dx)$

Homework

*****Evaluate** $\boldsymbol{\int sin^6 x\ cos^4 x\ \ dx\ = \int (sin^4 x\ cos^4 x) sin^2 x\ \ dx}$ **Homework**

Evaluate $\int sin^6 x\ cos^5 x\ \ dx$

Evaluate $\int sin^{99} x\ cos^7 x\ \ dx \rightarrow$ case-1

Group: 6 $\boldsymbol{\int tan^n x\ sec^m x\, dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

Group: 7 $\boldsymbol{\int cot^n x\ csc^m x\, dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

Group: 6 $\boldsymbol{\int tan^n x\ sec^m x\ \ dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

There are 3-cases in Group 6

Case-1: If m is even in $\boldsymbol{\int tan^n x\ sec^m x\ \ dx}$

Steps:

1) Split off the factor $\boldsymbol{\sec^2 x\, dx}$; [Do not change this factor. Save it for du]
2) Write $\sec^2 x = 1 + \tan^2 x$
3) Set $u = \tan x$. Then we get $du = \sec^2 x\ dx$

Example: 5 $\int tan^{100} x\ sec^2 x\, dx = \int u^{100}\, du = \frac{1}{101} \tan^{101} x + C$

$$Set\ \ u = \tan x\ .\ \ Then\ du = \sec^2 x\ dx$$

→Note that step (2) is not needed for Example 5.

Example: 6 $\int tan^9 x \ sec^6 x \ \ dx$

$$= \int tan^9 x \ sec^4 x \ \mathbf{sec^2 \, x \, dx}$$

$$= \int tan^9 x \ (sec^2 x)^2 \ \mathbf{sec^2 \, x \, dx}$$

$$= \int tan^9 x \ (1 + \tan^2 x)^2 \ \mathbf{sec^2 \, x \, dx} \ ; \quad \text{Set} \ \ u = \tan x \, , \text{then} \ du = \sec^2 x \ dx$$

$$= \int u^9 \ (1 + u^2)^2 \ du$$

$$= \int u^9 \ [1 + 2u^2 + u^4] \ du$$

$$= \int [u^9 + 2u^{11} + u^{13}] \ du$$

$$= \frac{1}{10} u^{10} + \frac{2}{12} u^{12} + \frac{1}{14} u^{14} + C$$

$$= \frac{1}{10} tan^{10} x + \frac{2}{12} tan^{12} x + \frac{1}{14} tan^{14} x + C$$

Group: 6 $\int \boldsymbol{tan^n \, x \ \ sec^m x \ \ dx}$ **; here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

Case-2: If n is odd in $\int \boldsymbol{tan^n \, x \ \ sec^m x \, dx}$

<u>Steps:</u>

1) Split off the factor $sec \, x \ \tan x \, dx$
2) Write $\tan^2 x = \sec^2 x - 1$
3) Set $u = \sec x$. Then we get $du = \sec \, x \tan x \ dx$

Example: 7 $(a) \int tan^9 x \ sec^6 x \ \ dx$

$$= \int tan^8 x \ sec^5 x \ \sec x \tan x \ \ dx$$

$$= \int (tan^2 x)^4 \ sec^5 x \ \sec x \tan x \ \ dx$$

$$= \int (\sec^2 x - 1\,)^4 \;\; sec^5 x \;\sec x \tan x \quad dx$$

$= \int (u^2 - 1)^4 \, u^5 \, du \;\rightarrow$ Please complete!

Example: 7(b) $\int tan^7\, x \;\; sec^{10} x \quad dx$

Solution:

$$\int tan^7\, x \;\; sec^{10} x \quad dx$$

$$= \int tan^6\, x \;\; sec^9 x \;\; \sec x \tan x \, dx$$

$$= \int (tan^2\, x)^3 \;\; sec^9 x \;\; \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^3 \;\; sec^9 x \;\; \sec x \tan x \, dx \;\; ;$$

$[\text{Set } u = sec\, x\,, \text{then we get } du = \sec x \tan x \;\; dx]$

$$= \int (u^2 - 1)^3 \;\; u^9 \, du$$

$$= \int [u^6 - 3u^4 + 3u^2 - 1] \;\; u^9 \, du$$

$$= \int [u^{15} - 3u^{13} + 3u^{11} - u^9] \, du$$

$$= \frac{1}{16} u^{16} - \frac{3}{14} u^{14} + \frac{3}{12} u^{12} - \frac{1}{10} u^{10} + C$$

$$= \frac{1}{16} \sec^{16} x - \frac{3}{14} \; sec^{14} x + \frac{1}{4} \; sec^{12} x - \frac{1}{10} sec^{10} x + C$$

Case-3: If n is even **and** m is odd in $\int \boldsymbol{tan^n x \;\; sec^m x \;\; dx}$

Step: Write $\tan^2 x = \sec^2 x - 1$, and then we will get sum of integrals of the form $\int \sec^{\mathrm{k}} x \;\; dx$ for $k \geq 2$. So, apply the reduction formula

$$\int \mathbf{sec^k}\, \boldsymbol{x} \;\; \boldsymbol{dx} = \frac{\mathbf{1}}{\boldsymbol{k-1}}\mathbf{sec}^{\boldsymbol{k-2}}\, \boldsymbol{x} \;\; \mathbf{tan}\, \boldsymbol{x} + \frac{\boldsymbol{k-2}}{\boldsymbol{k-1}} \;\; \int \mathbf{sec^{k-2}}\, \boldsymbol{x} \;\; \boldsymbol{dx} \ldots \ldots \ldots \quad (\mathbf{1})$$

Example: 8 $\int tan^4 x \;\; sec^3 x \;\; dx = \int (tan^2 x)^2 \;\; sec^3 x \;\; dx$

$= \displaystyle\int (Sec^2 x - 1)^2 \;\; sec^3 x \;\; dx$

$= \displaystyle\int [\sec^4 x - 2\sec^2 x + 1]\, sec^3 x \;\; dx$

$= \displaystyle\int [\sec^7 x - 2\sec^5 x + sec^3 x] \;\; dx$

$= \int \sec^7 x \;\; dx \;\; - 2 \;\; \int \sec^5 x \;\; dx + \;\; \int \sec^3 x \;\; dx \rightarrow$ Apply reduction formula.

Complete!

Group: 7 $\int \boldsymbol{cot^n x \;\; csc^m x\, dx}$ **; here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers .**

Assignment-1

Please submit on 1st March, 2021

Homework: Write all the steps for all 3-cases for Group-7 and then solve following exercises

Example: 9 $\int cot^9 x \;\; csc^6 x \;\; dx$

Example: 10 $\int cot^7 x \;\; csc^{16} x \;\; dx$

Example: 11 $\int \cot^4 x \;\; csc^3 x \;\; dx$

Group: 7 $\int \cot^n x \ \csc^m x \, dx$ **; here m and n are positive integers .**

Case-3: If n is even **and** m is odd

Step: Write $\cot^2 x = \csc^2 x - 1$, and then we will get sum of integrals of the form $\int \csc^k x \ dx$ for $k \geq 2$. So, apply the reduction formula

(1) … … … $\int \csc^k x \ dx = -\frac{1}{k-1}\csc^{k-2} x \ \cot x + \frac{k-2}{k-1} \ \int \csc^{k-2} x \ dx$

Example: 11 $\int \cot^4 x \ \ csc^3 x \ \ dx = \int (\cot^2 x)^2 \ \ csc^3 x \ \ dx$

$$= \int (\csc^2 x - 1)^2 \ \ csc^3 x \ \ dx$$

$$= \int [\csc^4 x - 2\csc^2 x + 1] \ \ csc^3 x \ \ dx$$

$$= \int [\csc^7 x - 2\csc^5 x + csc^3 x \,] \, dx$$

$$\left[\int \csc^k x \ dx = -\frac{1}{k-1}\csc^{k-2} x \ \cot x + \frac{k-2}{k-1} \ \int \csc^{k-2} x \ dx\right]$$

$$= \int \csc^7 x \ dx - 2\int \csc^5 x \ dx + \int \csc^3 x \ dx \, ; k = 7, \qquad k-1 = 6, k-2 = 5$$

$$= -\frac{1}{6}\csc^5 x \cot x + \frac{5}{6}\int \csc^5 x \ dx - 2\int \csc^5 x \ dx + \int \csc^3 x \ dx$$

$$= -\frac{1}{6}\csc^5 x \cot x - \frac{7}{6}\int \csc^5 x \ dx + \int \csc^3 x \ dx$$

$$= -\frac{1}{6}\csc^5 x \cot x - \frac{7}{6}\left[-\frac{1}{4}\csc^3 x \cot x + \frac{3}{4}\int \csc^3 x \ dx\right] + \int \csc^3 x \ dx$$

… … Continue !!