Chapter 4.4

Polynomial and Rational Inequalities

4.4.1 Polynomial Inequalities

Example 1 Solution of a Polynomial Inequality Using Its Graph

Solve $(x+3)(x-1)^2 > 0$ by graphing $f(x) = (x+3)(x-1)^2$

Solution:

Step 1: End behavior: The graph of f resembles that of the power function $y = x^3$ for large values of $|x|$, i.e.

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty$$

Step 2: The *y*-intercept is $f(0) = 3$. To find the *x*-intercepts, solve the equation $f(x) = 0$.

Therefore, we get $f(x) = (x+3)(x-1)^2 = 0$ giving $x = -3$ or $x = 1$.

Thus, the *y*-intercept is 3 and the *x*-intercepts are -3 and 1.

Step 3: The intercept -3 is a zero of multiplicity 1, so the graph of f crosses the *x*-axis at -3. The other intercept 1 is a zero of multiplicity 2, so the graph of f touches the *x*-axis at 1.

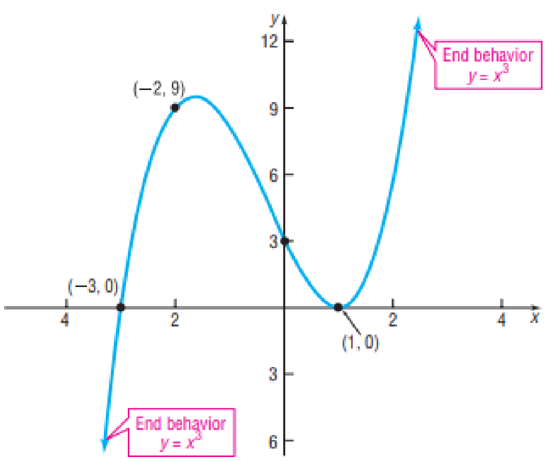
Step 4: Since f is a polynomial function of degree 3, the graph of f will have at most two turning points.

Step 5: Behavior near zeros:

Near -3: $f(x) = (x+3)(x-1)^2 \approx (x+3)(-3-1)^2 = 16(x+3)$ which is a line with slope 16

Near 1: $f(x) = (x+3)(x-1)^2 \approx (1+3)(x-1)^2 = 4(x-1)^2$ which is a parabola that opens up

Figure 45



Use the real zeros -3 and 1 to divide the real number line into three intervals:

$(-\infty, -3)$, $(-3, 1)$, $(1, +\infty)$

To draw the conclusion, prepare the following table:

Interval	Number picked	Value of *f*	Point on graph	Location of graph
$(-\infty, -3)$	-4	$f(-4) = -25$	$(-4, -25)$	Below the *x*-axis
$(-3, 1)$	-2	$f(-2) = 9$	$(-2, 9)$	Above the *x*-axis
$(1, +\infty)$	2	$f(2) = 5$	$(2, 5)$	Above the *x*-axis

We evaluated f at -4, -2 and 2 to help establish the scale on the *y*-axis.

Step 6: **Figure 45** illustrates the information obtained from Step 1 to Step 5.

From the graph, we see that $f(x) > 0$ for $-3 < x < 1$ or $x > 1$. Because the original inequality is strict, the solution set is

$$S = \{x : -3 < x < 1 \ \ or \ \ x > 1\}$$

or, in interval notation $(-3, 1) \cup (1, \infty)$.

Example (Extra) Solution of a Polynomial Inequality Algebraically

Solve $(x+3)(x-1)^2 > 0$ algebraically and graph the solution set.

Solution:

Step I: The *y*-intercept is $f(0) = 3$. To find the *x*-intercepts, solve the equation $f(x) = 0$.

Therefore, we get $f(x) = (x+3)(x-1)^2 = 0$ giving $x = -3$ or $x = 1$.

Thus, *y*-intercept is 3 and the *x*-intercepts are $x = -3$ and 1.

Step II: Use the real zeros -3 and 1 to divide the real number line into three intervals:

$$(-\infty, -3), \quad (-3, 1), \quad (1, +\infty)$$

Step III: To draw the conclusion, prepare the following table:

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty, -3)$	-4	$f(-4) = -25$	$(-4, -25)$	Below the *x*-axis
$(-3, 1)$	-2	$f(-2) = 9$	$(-2, 9)$	Above the *x*-axis
$(1, +\infty)$	2	$f(2) = 5$	$(2, 5)$	Above the *x*-axis

From the Table, we can conclude that $f(x) > 0$ for all numbers *x* for which $-3 < x < 1$ or $x > 1$. Because the original inequality is strict, the solution set of the given inequality is

$$S = \{x : -3 < x < 1 \ \ or \ \ x > 1\}$$

or, in interval notation $(-3, 1) \cup (1, \infty)$.

Example 2 Solution of a Polynomial Inequality Algebraically

Solve the inequality $x^4 > x$ algebraically and graph the solution set.

Solution: See the Textbook.

Table 16

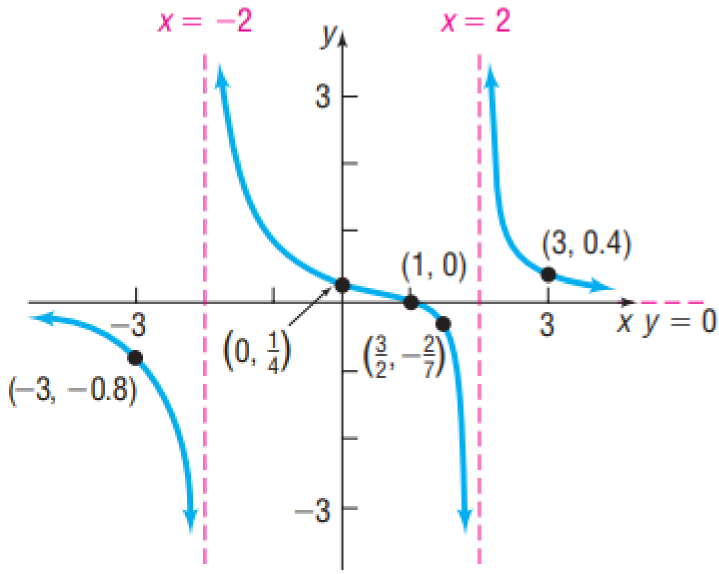
4.4.2 Rational Inequalities

Example 3 Solution of a Rational Inequality Using Its Graph

Solve $\dfrac{x-1}{x^2-4} \geq 0$ by graphing $R(x) = \dfrac{x-1}{x^2-4}$.

Solution: See the Textbook (Example 1, Section 4.3, **Figure 47**)

Figure 47



Step I: We have $R(x)=\dfrac{x-1}{x^2-4}=\dfrac{x-1}{(x+2)(x-2)}$

Therefore, $Dom(R)=\{x: x\neq -2, x\neq 2\}$.

Step II: Since there are no common factors between the numerator and denominator, R is in lowest terms.

Step III: Since 0 is in the domain of R, the y-intercept is $f(0)=\frac{1}{4}$. To find x-intercepts, solve $x-1=0$ or $x=1$. Therefore, the only real zero of the numerator is 1, i.e. the only x-intercept of the graph of R is 1.

Near 1: $R(x)=\dfrac{x-1}{(x+2)(x-2)}\approx\dfrac{x-1}{(-1+2)(-1-2)}=-\dfrac{1}{3}(x+1)$ which is a line with slope $-\dfrac{1}{3}$

Plot the point $(1,0)$ and draw a line through $(1,0)$ with a negative slope.

Step IV: Since R is in lowest terms, the vertical asymptotes are $x=-2$ and $x=2$.

Step V: Since R is proper, the horizontal asymptote is $y=0$. The graph of R intersected the horizontal asymptote at $(1,0)$ because

$$R(x)=\frac{x-1}{(x+2)(x-2)}=0\Rightarrow x-1=0\Rightarrow x=1$$

Step VI: Since the zero of the numerator is 1 and the zeros of the denominator are -2 and 2, divide the x-axis into four intervals:

$$(-\infty,-2),\quad (-2,1),\quad (1,2),\quad (2,+\infty)$$

Step VII: To draw the conclusion, prepare the following table:

Interval	Number picked	Value of R	Point on graph	Location of graph
$(-\infty,-2)$	-3	$R(-3)=-\frac{4}{5}$	$\left(-3,-\frac{4}{5}\right)$	Below the x-axis
$(-2,1)$	0	$R(0)=\frac{1}{4}$	$\left(0,\frac{1}{4}\right)$	Above the x-axis

$(1,2)$	$\frac{3}{2}$	$R\left(\frac{3}{2}\right)=-\frac{2}{7}$	$\left(\frac{3}{2},-\frac{2}{7}\right)$	Below the *x*-axis
$(2,+\infty)$	3	$R(3)=\frac{2}{5}$	$\left(3,\frac{2}{5}\right)$	Above the *x*-axis

From the graph, we see that $R(x)\geq 0$ for $-2<x\leq 1$ or $x>2$. Therefore, the solution set of the given inequality is

$$S=\{x:-2<x\leq 1 \ or \ x>2\}$$

or, in interval notation $(-2,\,1]\cup(2,\infty)$.

Example (Extra) Solution of a Rational Inequality Algebraically

Solve $\dfrac{x-1}{x^2-4}\geq 0$ algebraically and graph the solution set.

Solution: See the Textbook (Example 1, Section 4.3, **Figure 47**)

Step I: To find the real zeros (x-intercepts of the graph), solve $x-1=0$ or $x=1$. Therefore, the only real zero of the numerator is 1.

Step II: Since the zero of the numerator is 1 and the zeros of the denominator are -2 and 2, divide the real number line into four intervals:

$$(-\infty,-2),\quad (-2,1),\quad (1,2),\quad (2,+\infty)$$

Step III: To draw the conclusion, prepare the following table:

Interval	Number picked	Value of *R*	Point on graph	Location of graph
$(-\infty,-2)$	-3	$R(-3)=-\frac{4}{5}$	$\left(-3,-\frac{4}{5}\right)$	Below the *x*-axis
$(-2,1)$	0	$R(0)=\frac{1}{4}$	$\left(0,\frac{1}{4}\right)$	Above the *x*-axis
$(1,2)$	$\frac{3}{2}$	$R\left(\frac{3}{2}\right)=-\frac{2}{7}$	$\left(\frac{3}{2},-\frac{2}{7}\right)$	Below the *x*-axis
$(2,+\infty)$	3	$R(3)=\frac{2}{5}$	$\left(3,\frac{2}{5}\right)$	Above the *x*-axis

From the above table, we can conclude that $R(x)\geq 0$ for all real numbers x for which $-2<x\leq 1$ or $x>2$. We did not include -2 and 2 in the solution set because they are not in the domain of R. Therefore, the solution set of the given inequality is

$$S=\{x:-2<x\leq 1 \ or \ x>2\}$$

or, in interval notation $(-2,\,1]\cup(2,\infty)$.

Example 4 Solution of a Rational Inequality Algebraically

Solve the inequality $\dfrac{4x+5}{x+2} \geq 3$ algebraically and graph the solution set.

Solution: See the Textbook.

Table 17

4.4 Assess your understanding

Skill Building

Solve the inequality by using the graph of the function.
[**Hint:** The graphs were drawn in Problem 69-74 of Section 4.1]

9. Solve $f(x) < 0,$ where $f(x) = x^2(x-3)$

Solution:

Step 1: End behavior: The graph of f resembles that of the power function $y = x^3$ for large values of $|x|$, i.e.

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty$$

Step 2: The y-intercept is $f(0) = 0.$ To find the x-intercepts, solve the equation $f(x) = 0.$

Therefore, we get $f(x) = x^2(x-3) = 0$ giving $x = 0$ or $x = 3.$

Thus, the y-intercept is 0 and the x-intercepts are 0 and 3.

Step 3: The intercept 0 is a zero of multiplicity 2, so the graph of f touches the x-axis at 0.

The other intercept 3 is a zero of multiplicity 1, so the graph of f crosses the x-axis at 3.

Step 4: Since f is a polynomial function of degree 3, the graph of f will have at most two turning points.

Step 5: Behavior near zeros:

Near 0: $f(x) = x^2(x-3) \approx x^2(0-3) = -3x^2$ which is a parabola that opens down

Near 3: $f(x) = x^2(x-3) \approx 3^2(x-3) = 9(x-3)$ which is a line with slope 9

Step 6: Use the real zeros -3 and 1 to divide the real number line into three intervals:

$$(-\infty, 0), \quad (0, 3), \quad (3, +\infty)$$

To draw the conclusion, prepare the following table:

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty, 0)$	-1	$f(-1) = -4$	$(-1, -4)$	Below the x-axis
$(0, 3)$	2	$f(2) = -4$	$(2, -4)$	Below the x-axis
$(3, +\infty)$	4	$f(4) = 16$	(4, 16)	Above the x-axis

We evaluated f at $-1,$ 2 and 4 to help establish the scale on the y-axis.

69. Step 1: $y = x^3$

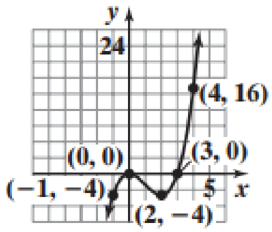
Step 2: x-intercepts: 0, 3; y-intercept: 0

Step 3: 0: multiplicity 2, touches; 3: multiplicity 1, crosses

Step 4: At most 2 turning points

Step 5: Near 0: $f(x) \approx -3x^2$; Near 3: $f(x) \approx 9(x-3)$

Step 6:



From the graph, we see that $f(x) < 0$ for $x < 0$ or $0 < x < 3$. Because the original inequality is strict, the solution set of the given inequality is

$$S = \{x : x < 0 \text{ or } 0 < x < 3\}$$

or, in interval notation $(-\infty, 0) \cup (0, 3)$.

11. Solve $f(x) \geq 0$, where $f(x) = (x+4)(x-2)^2$

Solution:

Step 1: End behavior: The graph of f resembles that of the power function $y = x^3$ for large values of $|x|$, i.e.

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty$$

Step 2: The y-intercept is $f(0) = 16$. To find the x-intercepts, solve the equation $f(x) = 0$.

Therefore, we get $f(x) = (x+4)(x-2)^2 = 0$ giving $x = -4$ or $x = 2$.

Thus, the y-intercept is 16 and the x-intercepts are -4 and 2.

Step 3: The intercept -4 is a zero of multiplicity 1, so the graph of f crosses the x-axis at -4.

The other intercept 2 is a zero of multiplicity 2, so the graph of f touches the x-axis at 2.

Step 4: Since f is a polynomial function of degree 3, the graph of f will have at most two turning points.

Step 5: Behavior near zeros:

Near -4: $f(x) = (x+4)(x-2)^2 \approx (x+4)(-4-2)^2 = 36(x+4)$ which is a line with slope 36

Near 2: $f(x) = (x+4)(x-2)^2 \approx (2+4)(x-2)^2 = 6(x-2)^2$ which is a parabola that opens up

Step 6: Use the real zeros -4 and 2 to divide the real number line into three intervals:

$$(-\infty, -4), \quad (-4, 2), \quad (2, +\infty)$$

To draw the conclusion, prepare the following table:

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty, -4)$	-5	$f(-5) = -49$	$(-5, -49)$	Below the x-axis
$(-4, 2)$	-2	$f(-2) = 32$	$(-2, 32)$	Above the x-axis
$(2, +\infty)$	4	$f(4) = 32$	(4, 32)	Above the x-axis

We evaluated f at -5, -2 and 4 to help establish the scale on the y-axis.

71. Step 1: $y = x^3$

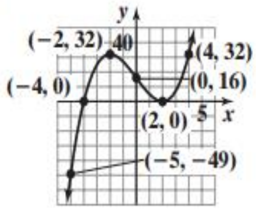
Step 2: x-intercepts: $-4, 2$; y-intercept: 16

Step 3: -4: multiplicity 1, crosses; 2: multiplicity 2, touches

Step 4: At most 2 turning points

Step 5: Near -4: $f(x) \approx 36(x+4)$; Near 2: $f(x) \approx 6(x-2)^2$

Step 6:



From the graph, we see that $f(x) \geq 0$ for $x \geq -4$. Because the original inequality is not strict, the solution set of the given inequality is

$$S = \{x : x \geq -4\}$$

or, in interval notation $[-4, \infty)$.

13. Solve $f(x) \leq 0$, where $f(x) = -2(x+2)(x-2)^3$

Solution set $S = \{x : x \leq -2 \ or \ x \geq 2\}; (-\infty, -2] \cup [2, \infty)$

Solve the inequality by using the graph of the function.

[**Hint:** The graphs were drawn in Problems 7-10 of Section 4.3]

15. Solve $R(x) > 0$, where $R(x) = \dfrac{x+1}{x(x+4)}$

Solution: (a) $Dom(R) = \{x : x \neq 0, x \neq -4\}$.

(b) Since there are no common factors between the numerator and denominator, R is in lowest terms.

(c) Since 0 is not in the domain of R, there is no y-intercept. To find x-intercepts, solve $x+1=0$ or $x=-1$. Therefore, the only real zero of the numerator is $x=-1$, i.e. the only x-intercept of the graph of R is -1.

Near -1: $R(x) = \dfrac{x+1}{x(x+4)} \approx \dfrac{x+1}{(-1)(-1+4)} = -\dfrac{1}{3}(x+1)$

Plot the point $(-1,0)$ and draw a line through $(-1,0)$ with a negative slope.

(d) Since R is in lowest terms, the vertical asymptotes are $x=0$ and $x=-4$.

(e) Since R is proper, the horizontal asymptote is $y=0$ intersected at $(-1,0)$ because

$$R(x) = \frac{x+1}{x(x+4)} = 0 \Rightarrow x+1=0 \Rightarrow x=-1$$

(f) Now construct a table.

Interval	Number picked	Value of *R*	Point on graph	Location of graph
$(-\infty,-4)$	-5	$R(-5)=-\frac{4}{5}$	$\left(-5,-\frac{4}{5}\right)$	Below the *x*-axis
$(-4,-1)$	-2	$R(-2)=\frac{1}{4}$	$\left(-2,\frac{1}{4}\right)$	Above the *x*-axis
$(-1,0)$	$-\frac{1}{2}$	$R\left(-\frac{1}{2}\right)=-\frac{2}{7}$	$\left(-\frac{1}{2},-\frac{2}{7}\right)$	Below the *x*-axis
$(0,+\infty)$	1	$R(1)=\frac{2}{5}$	$\left(1,\frac{2}{5}\right)$	Above the *x*-axis

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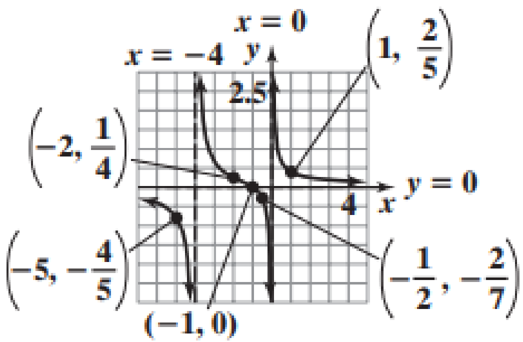

From the graph, we see that $R(x)>0$ for $-4<x<-1$ or $x>0$. Because the original inequality is strict, the solution set of the given inequality is

$$S=\{x:-4<x<-1 \ or \ x>0\}$$

or, in interval notation $(-4,-1)\cup(0,\infty)$.

17. Solve $R(x)\le 0$, where $R(x)=\frac{3x+3}{2x+4}$

Solution: (a) $Dom(R)=\{x:x\neq -2\}$.

(b) Since there are no common factors between the numerator and denominator, *R* is in lowest terms.

(c) Since 0 is in the domain of *R*, the *y*-intercept is $R(0)=\frac{3}{4}$.To find *x*-intercepts, solve $3(x+1)=0$ or $x=-1$. Therefore, the only real zero of the numerator is -1, i.e. the only *x*-intercept of the graph of *R* is -1.

Near -1: $R(x)=\frac{3(x+1)}{2(x+2)}\approx\frac{3(x+1)}{2(-1+2)}=\frac{3}{2}(x+1)$

Plot the point $(-1,0)$ and draw a line through $(-1,0)$ with a positive slope.

(d) Since *R* is in lowest terms, the only vertical asymptote is $x=-2$.

(e) Since *R* is proper, the horizontal asymptote is $y=\frac{3}{2}$ which does not intersect at $(-1,0)$ because

$$R(x)=\frac{3x+3}{2x+4}=\frac{3}{2}\Rightarrow 6x+12=6x+6\Rightarrow 12=6 \text{ is absurd}$$

(f) Now construct a table.

Interval	Number picked	Value of *R*	Point on graph	Location of graph
$(-\infty,-2)$	-3	$R(-3)=3$	$(-3,3)$	Above the *x*-axis
$(-2,-1)$	$-\dfrac{3}{2}$	$R\left(-\dfrac{3}{2}\right)=-\dfrac{3}{2}$	$\left(-\dfrac{3}{2},-\dfrac{3}{2}\right)$	Below the *x*-axis
$(-1,+\infty)$	0	$R(0)=\dfrac{3}{4}$	$\left(0,\dfrac{3}{4}\right)$	Above the *x*-axis

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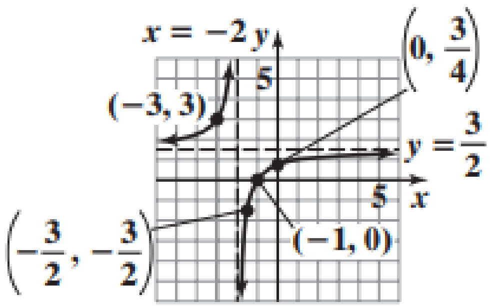

From the graph, we see that $R(x)\le 0$ for $-2<x\le -1$. Because the original inequality is not strict, the solution set of the given inequality is

$$S=\{x:-2<x\le -1\}$$

or, in interval notation $(-2,-1]$.

Solve the following inequalities algebraically.

19. $(x-5)^2(x+2)<0$

Solution set $S=\{x:x<-2\};(-\infty,-2)$

21. $x^3-4x^2>0$

Solution set $S=\{x:x>4\};(4,\infty)$

23. $2x^3>-8x^2$

Solution set $S=\{x:-4<x<0 \text{ or } x>0\};(-4,0)\cup(0,\infty)$

25. $(x-1)(x-2)(x-3)\le 0$

Solution set $S=\{x:x\le 1 \text{ or } 2\le x\le 3\};(-\infty,1]\cup[2,3]$

27. $x^3-2x^2-3x>0$

Solution set $S=\{x:-1<x<0 \text{ or } x>3\};(-1,0)\cup(3,\infty)$

29. $x^4>x^2$

Solution set $S=\{x:x<-1 \text{ or } x>1\};(-\infty,-1)\cup(1,\infty)$

31. $x^4 > 1$

Solution set $S = \{x : x < -1 \ or \ x > 1\}; (-\infty, -1) \cup (1, \infty)$

33. $\dfrac{x+1}{x-1} > 0$

Solution set $S = \{x : x < -1 \ or \ x > 1\}; (-\infty, -1) \cup (1, \infty)$

35. $\dfrac{(x-1)(x+1)}{x} \le 0$

Solution set $S = \{x : x \le -1 \ or \ 0 < x \le 1\}; (-\infty, -1] \cup (0, 1]$

37. $\dfrac{(x-2)^2}{x^2 - 1} \ge 0$

39. $\dfrac{x+4}{x-2} \le 1$

41. $\dfrac{3x-5}{x+2} \le 2$

43. $\dfrac{1}{x-2} < \dfrac{2}{3x-9}$