Fourier Analysis

Ref. Advanced Engineering Mathematics, Erwin Kreyszig, 10th Edn

Fourier series

-infinite series that represent general periodic functions in terms of sine and cosine, i.e.,

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots\cdots\cdots$$

$$= a_0 + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\}$$

where $a_0, a_1, b_1, b_2, \ldots\ldots$ are called Fourier coefficients of the series.

Each term in the above series has the period $\boldsymbol{2\pi}$*. Hence if the coefficients are such that the series converges, its sum will be a function of period* $\boldsymbol{2\pi}$*.*

Here the **Fourier coefficients** for functions of period $\boldsymbol{p = 2\pi}$:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \qquad n = 1,2, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \qquad n = 1,2, \ldots$$

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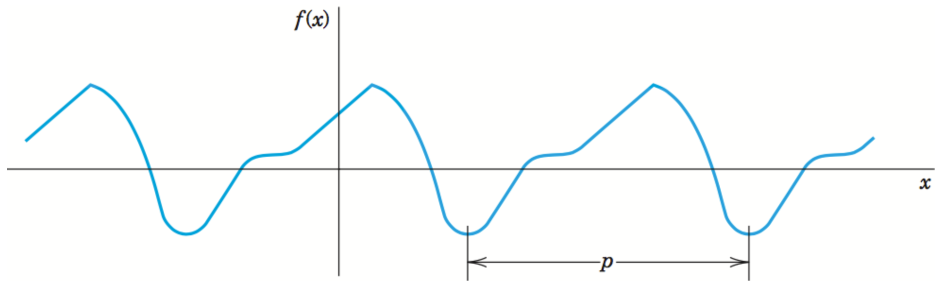
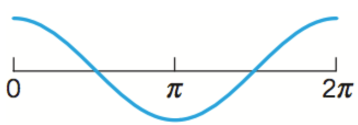
Periodic Function:

A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x, except possibly at some points, and if there some positive number $\boldsymbol{p}$, called a ***period*** of $f(x)$, such that

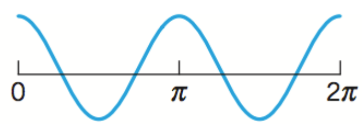
$$f(x+\boldsymbol{p}) = f(x) \Rightarrow f(x+n\boldsymbol{p}) = f(x) \quad for \;\; n = 1,2,3,\ldots$$



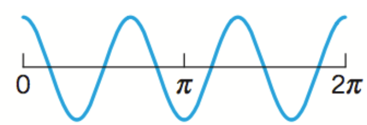
Examples. $\cos x$, $\cos 2x$, $\cos 3x$, $\sin x$, $\sin 2x$, $\sin 3x$



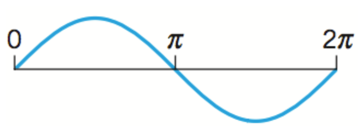
$\cos x$



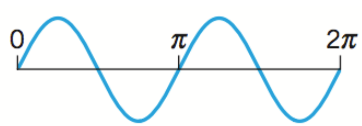
$\cos 2x$



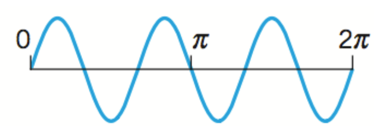
$\cos 3x$



$\sin x$



$\sin 2x$



$\sin 3x$

More examples of periodic function: $\tan x$, $\cot x$, $\tan 2x$, $\cot 2x$ etc.

Examples of non-periodic function: $x, x^2, x^3, e^x, \ln x$ etc.

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Example. Find the Fourier coefficients and then the Fourier series of the periodic function

$$f(x) = \begin{cases} -k, & if \ -\pi < x < 0 \\ k, & if \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

Solution. Here the **Fourier coefficients** are:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right] = \frac{1}{2\pi}\left[-\int_{-\pi}^{0} k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= \frac{k}{2\pi}\left[-\int_{-\pi}^{0} dx + \int_{0}^{\pi} dx\right] = \frac{k}{2\pi}\left[-x\Big|_{-\pi}^{0} + x\Big|_{0}^{\pi}\right] = \frac{k}{2\pi}[-(0+\pi) + (\pi - 0)] = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left[-\int_{-\pi}^{0} k\cos(nx)\,dx + \int_{0}^{\pi} k\cos(nx)\,dx\right]$$

$$= \frac{k}{n\pi}\left[-\sin nx\Big|_{-\pi}^{0} + \sin nx\Big|_{0}^{\pi}\right] = 0$$

Fourier Analysis

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Example. Find the Fourier coefficients and then the Fourier series of the periodic function

$$f(x) = \begin{cases} -k, & if \ -\pi < x < 0 \\ k, & if \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

Solution. Now,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi}\left[- \int_{-\pi}^{0} k \sin(nx)\, dx + \int_{0}^{\pi} k \sin(nx)\, dx \right]$$

$$= \frac{k}{n\pi}\left[\cos nx \Big|_{-\pi}^{0} - \cos nx \Big|_{0}^{\pi} \right] = \frac{k}{n\pi}[1 - \cos n\pi - \cos n\pi + 1] = \frac{2k}{n\pi}[1 - \cos n\pi]$$

Thus the Fourier series of the given function is written by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\} = \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi) \sin(nx)}{n}$$

$$= \frac{4k}{\pi}\left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)$$

$$\boxed{\begin{aligned} 1 - \cos n\pi &= 1 - \begin{cases} -1, & if\ n \text{ is odd} \\ 1, & if\ n \text{ is even} \end{cases} \\ &= \begin{cases} 2, & if\ n \text{ is odd} \\ 0, & if\ n \text{ is even} \end{cases} \end{aligned}}$$

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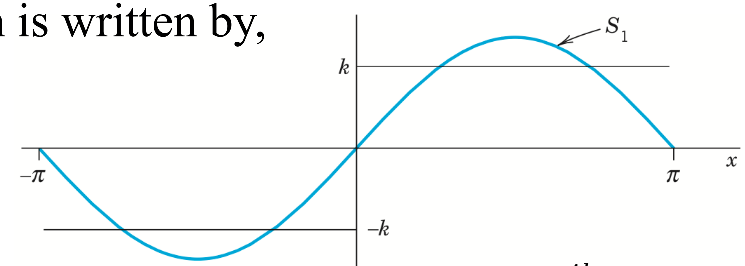
Example. Find the Fourier coefficients and then the Fourier series of the periodic function

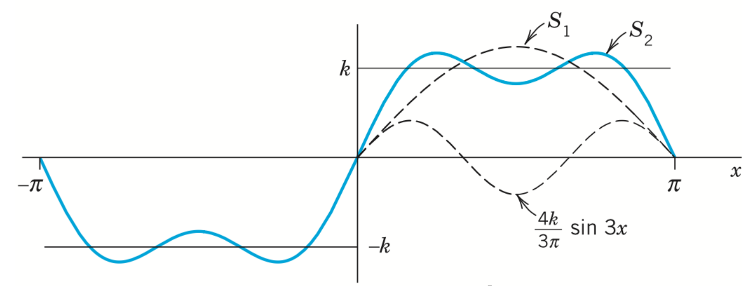
$$f(x) = \begin{cases} -k, & if \quad -\pi < x < 0 \\ k, & if \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

Solution. Thus the Fourier series of the given function is written by,

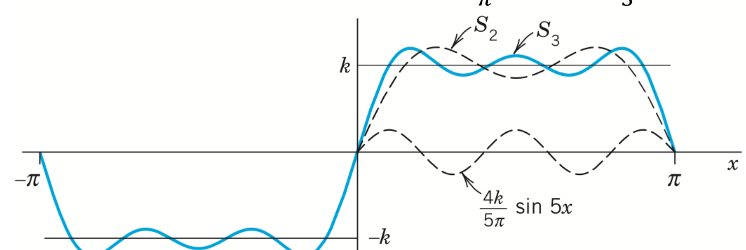
$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$



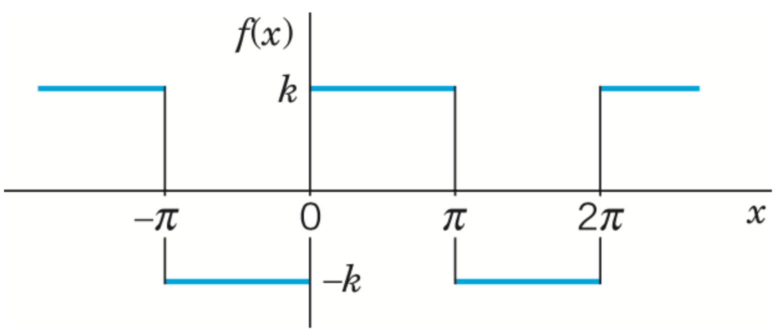
Fourier Approximation of $f(x)$ by $S_1 = \frac{4k}{\pi}\sin x$



Fourier Approximation of $f(x)$ by $S_2 = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x)$



Fourier Approximation of $f(x)$ by $S_3 = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x)$



Graph of $f(x)$

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Exercise problems: 11.1

Find the *fundamental period,* the smallest positive period, for

1. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$

2. $\cos nx$, $\sin nx$, $\cos \dfrac{2\pi x}{k}$, $\sin \dfrac{2\pi x}{k}$, $\cos \dfrac{2\pi n x}{k}$, $\sin \dfrac{2\pi n x}{k}$

Sketch $f(x)$ for $-\pi < x < \pi$ as given below

6. $f(x) = |x|$

7. $f(x) = |\sin x|$, $f(x) = \sin |x|$

8. $f(x) = e^{-|x|}$, $f(x) = |e^{-x}|$

9. $f(x) = \begin{cases} x & \text{if} \quad -\pi < x < 0 \\ \pi - x & \text{if} \quad 0 < x < \pi \end{cases}$

10. $f(x) = \begin{cases} -\cos^2 x & \text{if} \quad -\pi < x < 0 \\ \cos^2 x & \text{if} \quad 0 < x < \pi \end{cases}$

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π.

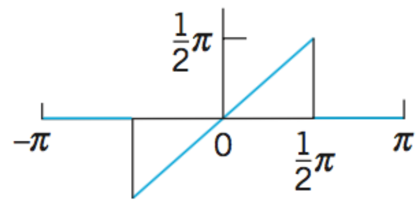
12. $f(x)$ in Prob. 6

13. $f(x)$ in Prob. 9

14. $f(x) = x^2 \quad (-\pi < x < \pi)$

15. $f(x) = x^2 \quad (0 < x < 2\pi)$

16.



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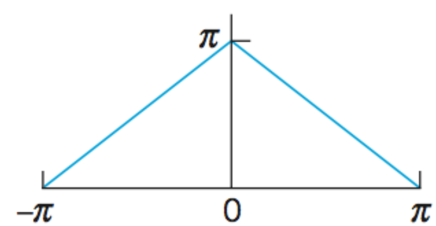

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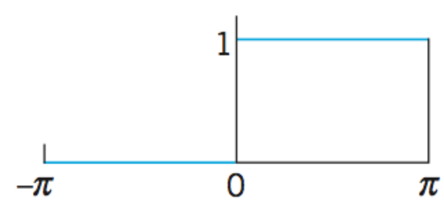

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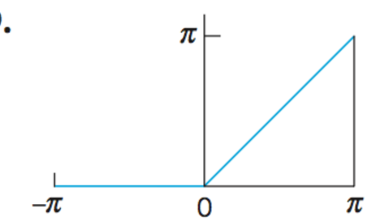

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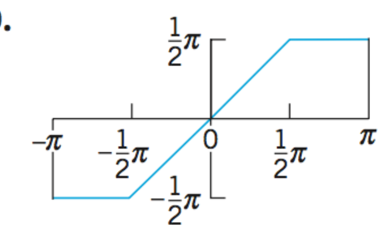

Fourier Analysis

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Fourier series for any period $p = 2L$

Using a suitable change of scale, we introduce a new variable v such that $x = \frac{p}{2\pi} v$ which implies $x = \frac{L}{\pi} v \Rightarrow v = \frac{\pi}{L} x$. Then, $v = \pm \pi$ corresponds to $x = \pm L$.

Then the Fourier series can be written in terms of new variable v as,

$$f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(nv) + b_n \sin(nv)\}$$

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} \left\{a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)\right\}$$

Here the **Fourier coefficients** for functions of period $\boldsymbol{p = 2L}$:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) dv, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \cos(nv)\, dv, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \sin(nv)\, dv$$

Now, since, $v = \frac{\pi}{L} x \Rightarrow dv = \frac{\pi}{L} dx$, hence we can write,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$n = 1, 2, 3, \ldots$$

Fourier Analysis

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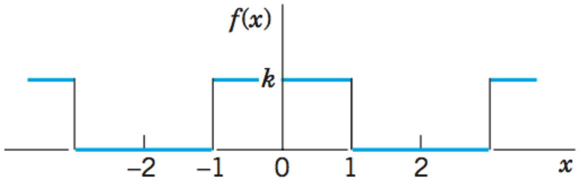
Example. Find the Fourier series of the periodic rectangular wave.

$$f(x) = \begin{cases} 0, & if \; -2 < x < -1 \\ k, & if \; -1 < x < 1 \\ 0, & if \quad 1 < x < 2 \end{cases}$$



Solution. Here $p = 2L = 4, L = 2$ and the **Fourier coefficients** are:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \frac{1}{4} \int_{-1}^{1} k\, dx = \frac{k}{4} \int_{-1}^{1} dx = \frac{k}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx = \frac{k}{2} \int_{-1}^{1} \cos\left(\frac{n\pi}{2} x\right) dx = \frac{2k}{2} \int_{0}^{1} \cos\left(\frac{n\pi}{2} x\right) dx$$

$$= \frac{2k}{n\pi} \left[\sin\left(\frac{n\pi}{2} x\right) \Big|_0^1 \right] = \frac{2k}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx = \frac{k}{2} \int_{-1}^{1} \sin\left(\frac{n\pi}{2} x\right) dx = \frac{2k}{2} \cdot 0 = 0$$

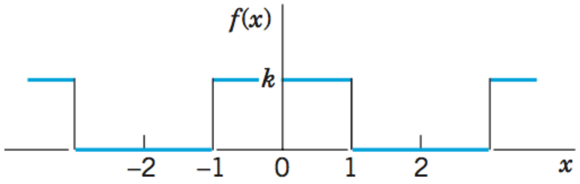
$$\left[\int_{-a}^{a} (even\ function)\, dx = 2 \int_{0}^{a} (even\ function)\, dx, \qquad \int_{-a}^{a} (odd\ function)\, dx = 0 \right]$$

Fourier Analysis

Ref. Advanced Engineering Mathematics, Erwin Kreyszig, 10th Edn

Example. Find the Fourier series of the periodic rectangular wave

$$f(x)=\begin{cases}0, & if\ -2<x<-1\\ k, & if\ -1<x<1\\ 0, & if\ \ 1<x<2\end{cases}$$



Solution. Thus the Fourier series of the given functions becomes,

$$f(x)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}x\right)$$

$$=\frac{k}{2}+\frac{2k}{\pi}\left[\cos\left(\frac{\pi}{2}x\right)-\frac{1}{3}\cos\left(\frac{3\pi}{2}x\right)+\frac{1}{5}\cos\left(\frac{5\pi}{2}x\right)-\cdots+\cdots-\cdots\right]$$

$$\left[\int_{-a}^{a}(even\ function)\,dx=2\int_{0}^{a}(even\ function)\,dx,\qquad \int_{-a}^{a}(odd\ function)\,dx=0\right]$$

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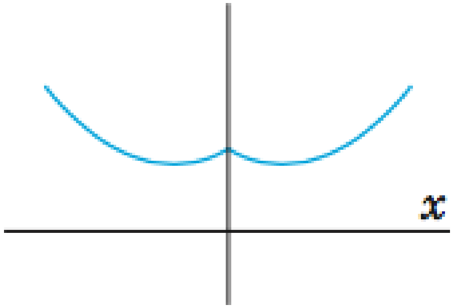
Fourier series for Even and Odd functions

If $f(x)$ is an ***even*** functions, i.e., $f(-x) = f(x)$ then the **Fourier coefficients** for functions of period $\boldsymbol{p = 2L}$ is found by,



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{2}{2L}\int_{0}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{1}{L}\cdot 0 = 0, \qquad n = 1, 2, 3, \ldots$$

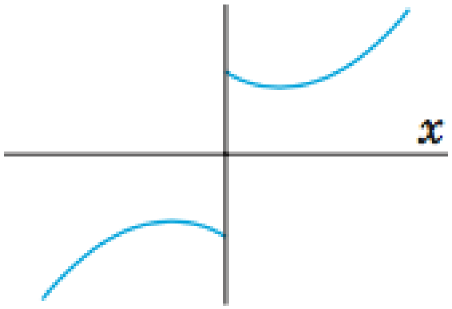
Thus, the Fourier series is written as **Fourier cosine series**,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right\} = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right)$$

$$\boxed{\left[\int_{-a}^{a}(even\ function)\,dx = 2\int_{0}^{a}(even\ function)\,dx, \qquad \int_{-a}^{a}(odd\ function)\,dx = 0\right]}$$

Fourier series for Even and Odd functions

If $f(x)$ is an ***odd*** functions, i.e., $f(-x) = -f(x)$ then the **Fourier coefficients** for functions of period $\boldsymbol{p = 2L}$ is found by,



$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \frac{1}{2L} \cdot 0 = 0,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx = \frac{1}{L} \cdot 0 = 0, \qquad n = 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx \qquad n = 1, 2, 3, \dots$$

Thus, the Fourier series is written as **Fourier sine series**,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right) \right\} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right)$$

$$\boxed{\left[\int_{-a}^{a} (even\ function)\, dx = 2 \int_{0}^{a} (even\ function)\, dx, \qquad \int_{-a}^{a} (odd\ function)\, dx = 0 \right]}$$

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Half range Fourier expansion: cosine series

Let $f(x)$ is a function defined for $0 < x < L$.

If $f(x)$ is extended as an ***even*** function, which is called *even periodic function* $f_1(x)$ of $f(x)$, then the **half-range Fourier cosine series** for $f(x)$of period $\boldsymbol{T = 2L}$ is written by

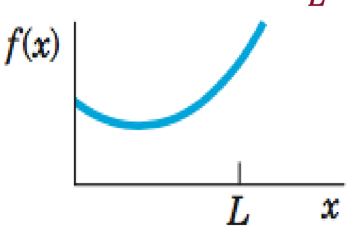
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

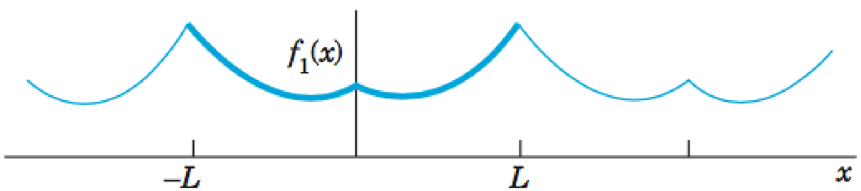
with the Fourier coefficients,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{2}{2L}\int_{0}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{1}{L}\cdot 0 = 0, \qquad n = 1, 2, 3, \ldots$$





Half range Fourier expansion: sine series

Let $f(x)$ is a function defined for $0 < x < L$.

If $f(x)$ is extended as an ***odd*** function, which is called *odd periodic function* $f_2(x)$ of $f(x)$,

then the **half-range Fourier sine series** for $f(x)$of period $\boldsymbol{T = 2L}$ is written by

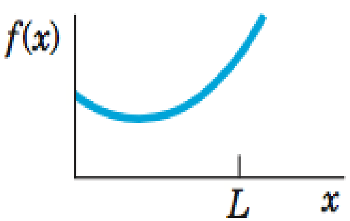
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right)$$

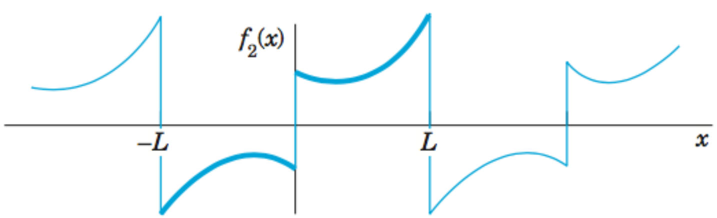
with the Fourier coefficients,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \frac{1}{2L} \cdot 0 = 0,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx = \frac{1}{L} \cdot 0 = 0, \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx \qquad n = 1, 2, 3, \ldots$$





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Exercise problems:11.2

1-7. Are the following functions even or odd or neither even nor odd?

1. e^x, $e^{-|x|}$, $x^3 \cos nx$, $x^2 \tan \pi x$, $\sinh x - \cosh x$

2. $\sin^2 x$, $\sin(x^2)$, $\ln x$, $x/(x^2+1)$, $x \cot x$

3. Sums and products of even functions

4. Sums and products of odd functions

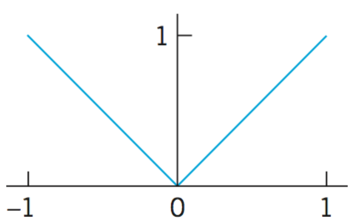
5. Absolute values of odd functions

6. Product of an odd times an even function

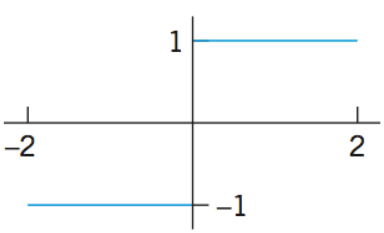
7. Find all functions that are both even and odd.

8-12. Is the given function even or odd or neither even nor odd? Find its Fourier series.

8.



9.



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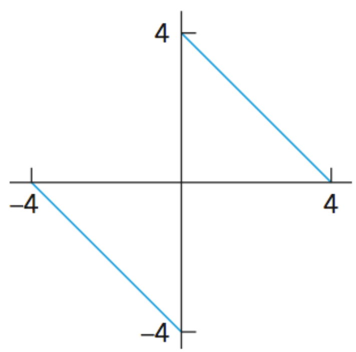
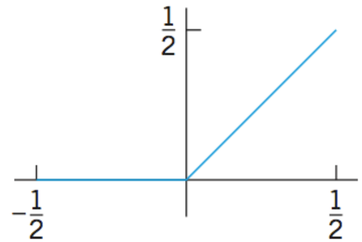


11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$

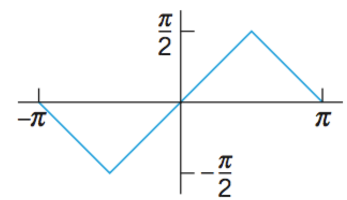
13-18. Is the given function even or odd or neither even nor odd? Find its Fourier series.

13.



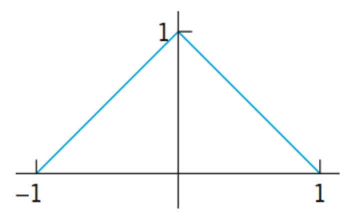
14. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$

15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$

17.



18. Rectifier. Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.

Fourier Analysis

Ref. Advanced Engineering Mathematics, Erwin Kreyszig, 10th Edn

Exercise problems:11.2

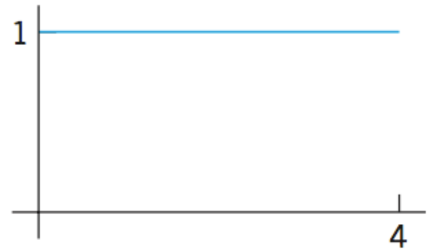
23-29. Find

(a) the Fourier cosine series,

(b) the Fourier sine series.

(c) Sketch $f(x)$ and its two periodic extensions.

23.



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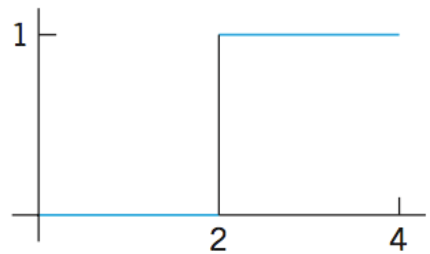

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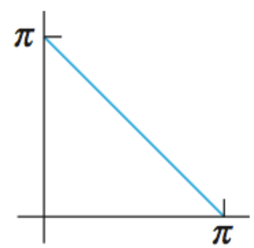

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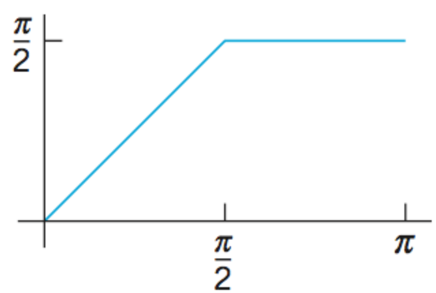

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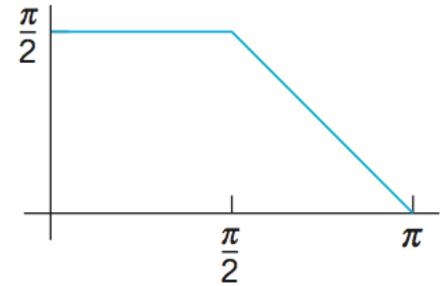
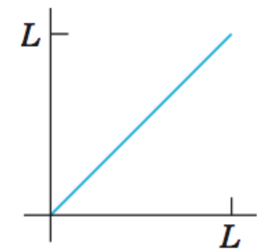


28.



29. $f(x) = \sin x \ (0 < x < \pi)$

Fourier Analysis

Ref. Advanced Engineering Mathematics, Erwin Kreyszig, 10th Edn

Fourier integrals

If $f_L(x)$ is any periodic function of period $\mathbf{2L}$ that can be written by Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(w_n x) + b_n \sin(w_n x)\} \qquad where \quad w_n = \frac{n\pi}{L}$$

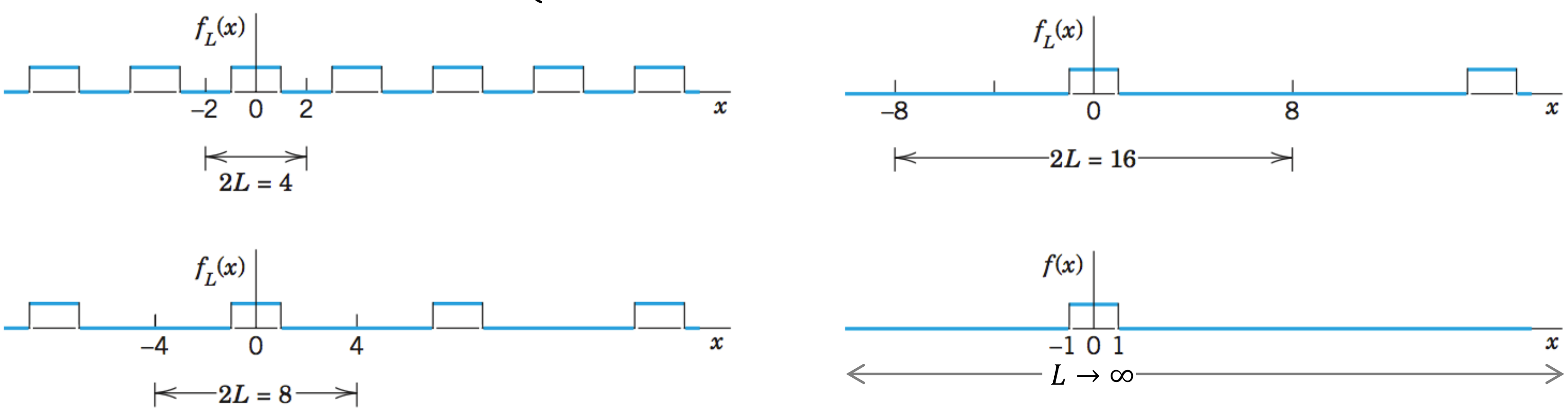
with the Fourier coefficients,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f_L(x)\, dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f_L(x) \cos(w_n x)\, dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f_L(x) \sin(w_n x)\, dx$$

$$n = 1, 2, 3, \ldots$$

Example. Consider the periodic rectangular wave

$$f_L(x) = \begin{cases} 0 \text{ if } -L < x < 1 \\ 1 \text{ if } -1 < x < 1 \\ 0 \text{ if } \quad 1 < x < L \end{cases}$$



Fourier Analysis

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Fourier integrals:

Now, if the period $L \to \infty$ and assume that the resulting non-periodic function becomes $f(x) = \lim_{L\to\infty} f_L(x)$ is absolutely integrable on the x-axis. Let $f(x)$ is piecewise continuous in every finite interval and has a right and left-hand derivative at every point on the interval, then the Fourier series becomes ***Fourier integral*** and can be written as,

$$f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]\,dw$$

Where, $A(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\cos(wx)\,dx,\qquad B(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\sin(wx)\,dx$

Fourier Cosine integral:

If $f(x)$ has a Fourier integral representation and is even, then the Fourier cosine integral is

$$f(x) = \int_0^\infty A(w)\cos(wx)\,dw \qquad \text{where} \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos(wx)\,dx \qquad [B(w) = 0]$$

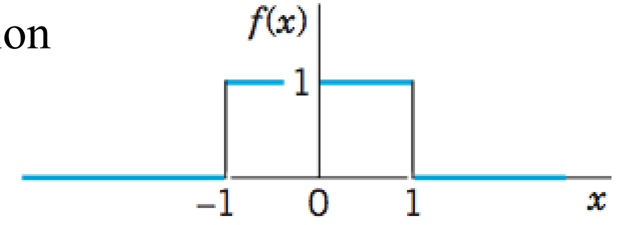
Fourier Sine integral:

If $f(x)$ has a Fourier integral representation and is odd, then the Fourier sine integral is

$$f(x) = \int_0^\infty B(w)\sin(wx)\,dw \qquad \text{where} \qquad B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin(wx)\,dx \qquad [A(w) = 0]$$

Example. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$



Solution. Here,

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(wx)\, dx = \frac{1}{\pi} \int_{-1}^{1} \cos(wx)\, dx = \frac{2}{\pi w} \sin wx \Big|_{x=0}^{x=1} = \frac{2 \sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(wx)\, dx = \frac{1}{\pi} \int_{-1}^{1} \sin(wx)\, dx = \frac{1}{\pi} \cdot 0 = 0$$

Therefore, the Fourier integral representation of the given function can be written as,

$$f(x) = \int_{0}^{\infty} [A(w) \cos(wx) + B(w) \sin(wx)]\, dw = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin w \cos(wx)}{w} dw$$

Here, the average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $\frac{1+0}{2} = \mathbf{\frac{1}{2}}$.

$$\int_{0}^{\infty} \frac{\sin w \cos(wx)}{w} dw = \frac{\pi}{2} f(x) = \frac{\pi}{2} \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases} = \frac{\pi}{2} \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ \mathbf{1/2} & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x > 1 \end{cases} = \begin{cases} \pi/2 & \text{if} \quad 0 \le x < 1 \\ \boldsymbol{\pi/4} & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$$

Fourier Analysis

Ref. Advanced Engineering Mathematics, Erwin Kreyszig, 10th Edn

Example. Find the Fourier cosine and sine integral representation of the function

$$f(x) = e^{-kx}, \qquad \text{where} \quad x > 0 \ \text{ and } \ k > 0$$

Solution. For the Fourier cosine integral,

$$A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos(wx)\,dx = \frac{2}{\pi}\int_0^\infty e^{-kx}\cos(wx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-kx}}{w^2+k^2}(w\sin wx - k\cos wx\,)\right]_{x=0}^{\infty} = \frac{2k}{\pi(w^2+k^2)}$$

Therefore, the Fourier cosine integral representation of the given function can be written as,

$$f(x) = e^{-kx} = \int_0^\infty A(w)\cos(wx)\,dw = \frac{2k}{\pi}\int_0^\infty \frac{\cos(wx)}{(w^2+k^2)}dw$$

$$\therefore \int_0^\infty \frac{\cos(wx)}{(w^2+k^2)}dw = \frac{\pi}{2k}e^{-kx} \qquad (x > 0 \ \text{ and } \ k > 0)$$

Example. Find the Fourier cosine and sine integral representation of the function

$$f(x) = e^{-kx}, \qquad \text{where} \quad x > 0 \text{ and } k > 0$$

Solution. For the Fourier sine integral,

$$B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin(wx)\,dx = \frac{2}{\pi}\int_0^\infty e^{-kx}\sin(wx)\,dx$$

$$= \frac{2}{\pi}\left[-\frac{e^{-kx}}{w^2+k^2}(w\cos wx + k\sin wx)\right]_{x=0}^{\infty} = \frac{2w}{\pi(w^2+k^2)}$$

Therefore, the Fourier sine integral representation of the given function can be written as,

$$f(x) = e^{-kx} = \int_0^\infty B(w)\sin(wx)\,dw = \frac{2}{\pi}\int_0^\infty \frac{w\sin(wx)}{(w^2+k^2)}\,dw$$

$$\therefore \int_0^\infty \frac{w\sin(wx)}{(w^2+k^2)}\,dw = \frac{\pi}{2}e^{-kx} \qquad (x > 0 \text{ and } k > 0)$$

Fourier Analysis

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Exercise problems: 11.7

Show that the integral represents the indicated function. The integral tells you which one, and its value tells you what function to consider.

1. $$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\,dx = \begin{cases} 0 & \text{if} \quad x < 0 \\ \pi/2 & \text{if} \quad x = 0 \\ \pi e^{-x} & \text{if} \quad x > 0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2}\,dw = \begin{cases} \frac{\pi}{2} \sin x & \text{if} \quad 0 \leqq x \leqq \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$$

3. $$\int_0^\infty \frac{1 - \cos \pi w}{w} \sin xw\,dw = \begin{cases} \frac{1}{2}\pi & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$$

4. $$\int_0^\infty \frac{\cos \frac{1}{2}\pi w}{1 - w^2} \cos xw\,dw = \begin{cases} \frac{1}{2}\pi \cos x & \text{if} \quad 0 < |x| < \frac{1}{2}\pi \\ 0 & \text{if} \quad |x| \geqq \frac{1}{2}\pi \end{cases}$$

5. $$\int_0^\infty \frac{\sin w - w \cos w}{w^2} \sin xw\,dw = \begin{cases} \frac{1}{2}\pi x & \text{if} \quad 0 < x < 1 \\ \frac{1}{4}\pi & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$$

6. $$\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4}\,dw = \frac{1}{2}\pi e^{-x} \cos x \quad \text{if} \quad x > 0$$

Fourier Analysis

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Exercise problems: 11.7

Represent $f(x)$ as Fourier Cosine integral.

7. $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

9. $f(x) = 1/(1 + x^2)$ $[x > 0.$ *Hir*

10. $f(x) = \begin{cases} a^2 - x^2 & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

Represent $f(x)$ as Fourier sine integral.

16. $f(x) = \begin{cases} x & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

17. $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

19. $f(x) = \begin{cases} e^{x} & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

20. $f(x) = \begin{cases} e^{-x} & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

Fourier Analysis

Fourier Transform: Fourier Cosine Transform

The Fourier cosine transform concerns ***even functions*** $f(x)$. From the Fourier cosine integral,

$$f(x) = \int_0^\infty A(w)\cos(wx)\,dw \qquad \text{where} \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos(wx)\,dx = \sqrt{\frac{2}{\pi}}F_c(w)$$

where, $F_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(wx)\,dx$ is called the Fourier cosine transform of $f(x)$ and

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(w)\cos(wx)\,dw$$ is called the inverse Fourier cosine transform of $F_c(w)$

Fourier Transform: Fourier sine Transform

The Fourier sine transform concerns ***odd functions*** $f(x)$. From the Fourier sine integral,

$$f(x) = \int_0^\infty B(w)\sin(wx)\,dw \qquad \text{where} \qquad B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin(wx)\,dx = \sqrt{\frac{2}{\pi}}F_s(w)$$

where, $F_s(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(wx)\,dx$ is called the Fourier sine transform of $f(x)$ and

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(w)\sin(wx)\,dw$$ is called the inverse Fourier sine transform of $F_c(w)$

Fourier Analysis

Example. Find the Fourier cosine and sine transform of the function

$$f(x) = \begin{cases} k & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad\quad x > a \end{cases}$$

Solution. Here the Fourier cosine transform of $f(x)$ is,

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(wx)\,dx = \sqrt{\frac{2}{\pi}} \int_0^a k\cos(wx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{k\sin(wx)}{w}\right]_{x=0}^a = \sqrt{\frac{2}{\pi}} \frac{k\sin(wa)}{w}$$

Furthermore, the Fourier sine transform of $f(x)$ is,

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(wx)\,dx = \sqrt{\frac{2}{\pi}} \int_0^a k\sin(wx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{k\cos(wx)}{w}\right]_{x=0}^a = \sqrt{\frac{2}{\pi}} \frac{k}{w}(1 - \cos wa)$$