Course Name : Physics – I
Course # PHY 107

Notes-3 : Motion in One Dimensions

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Topics to be covered

1. Definitions of Physical Quantities
2. Geometrical Interpretation of instantaneous and average velocities
3. Geometrical Interpretation of Acceleration
4. Equations of Motions
5. Integral form of Displacement and Velocity
6. Free Fall: Definition and Properties
7. Examples

- One Dimension means that there is only one coordinate, and conventionally it is the x-axis (horizontally), and y-axis for vertical motion (like free-fall motion, to be covered later on).
- The direction of a vector quantity is given by the sign of the parameter whether it is positive or negative. By convention, to the right is positive and to the left is negative directions.
- Definition of physical quantities:
  1. Position: x(t) = Coordinate x as function of time t = (t,x(t))
  2. Displacement, $\Delta x \equiv x(t) - x(t_0) = x - x_0,$
     $\left(\text{where}\, t_0 \,\text{is the initial time and t is any time later}\right)$
  3. Distance, $d = \begin{cases} \text{Actual Path} \\ |\Delta x| \end{cases}.$

4. Instantaneous velocity (or simply velocity): It is defined as,

$$v \equiv \frac{dx(t)}{dt} = \frac{dx}{dt} = \text{First derivative of the position curve}.$$

5. Average velocity: It is defined as,

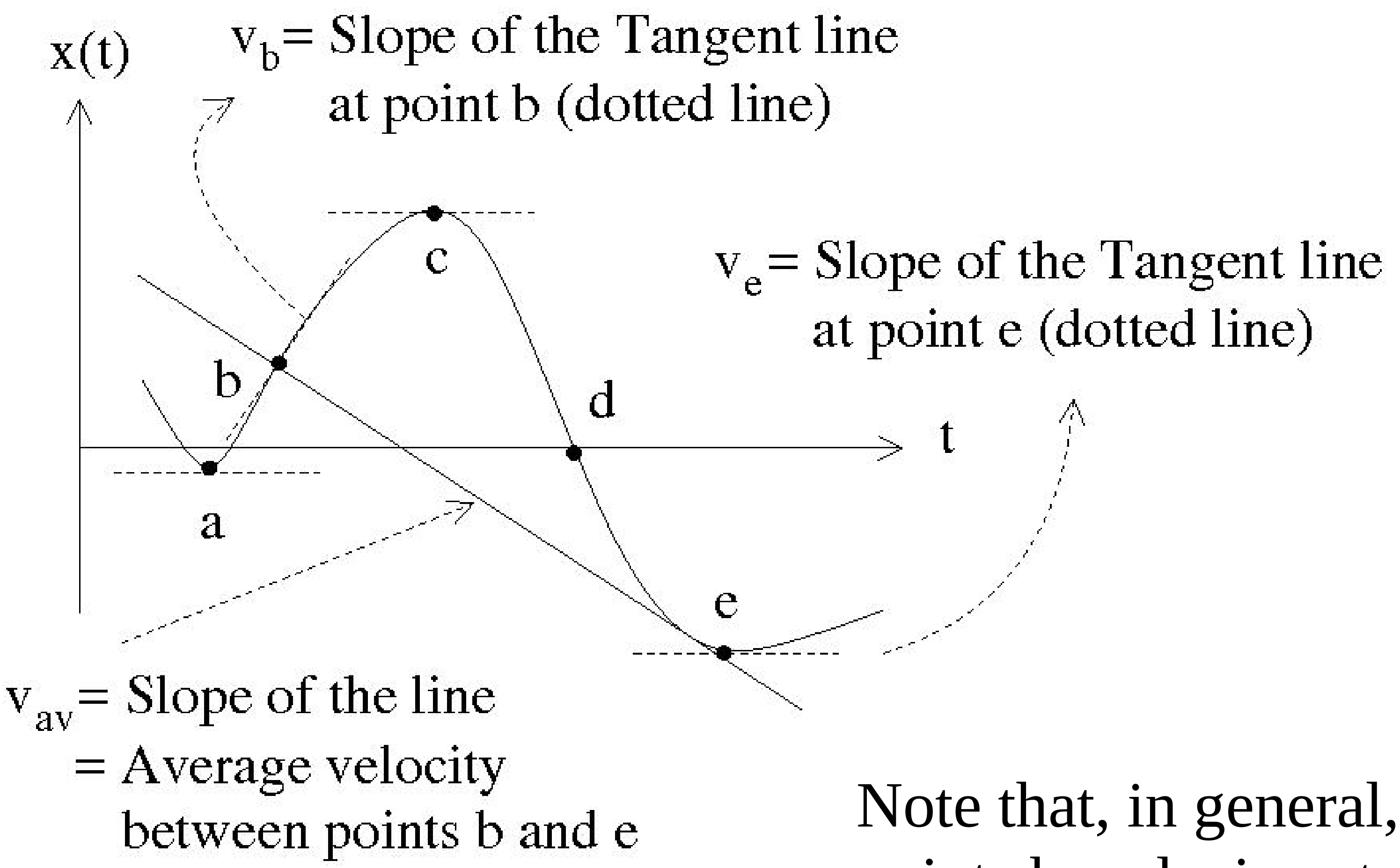
$$v_{\text{av}} \equiv \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0} \; (\text{Definition})$$

$$= \text{Slope of the straight line that passes through the points } (t, x) \text{ and } (t_{0,} x_0).$$

6. Speed: It has two meanings defined by

$$v = \begin{cases} |v_{\text{av}}| = \text{Magnitude of the average velocity} \\ \dfrac{d}{t} = \dfrac{\text{Actual Path}}{\text{Time}} \end{cases}.$$

The position vs. time graph:



Note the flowing:

$v_a = v_c = v_e = 0$

(because the slope of the dotted line is zero)

The curvature at a is positive

So, $a_a = \dfrac{d^2x}{dt^2} > 0.$

But the curvature at d is zero, because the line is straight through the point d.

Note that, in general, the average velocity between the points b and e is not equal to the average velocities at points b and e i.e. $v_{\mathrm{av}} = \dfrac{\Delta x}{\Delta t} = \dfrac{x_e - x_b}{t_e - t_b} \neq \dfrac{v_b + v_e}{2}$

7. The instantaneous acceleration is defined as

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$= \text{First derivetive of the velocity graph at time t}$$

$$= \text{Slope of the tangent line on th evelocity graph at time t}$$

$$= \text{Second derivative of the position graph at time t}$$

$$= \text{Curvature of the position graph at time t}$$

- Note that the acceleration is NOT constant in general, and is a function of time.
- It just gives how fast the velocity itself is changing. If the velocity does not change (means constant), the a = 0.
- DO NOT CONFUSE ACCELERATION WITH VELOCITY. THESE ARE COMPLETELY DIFFERENT !!!

Some properties of motion:

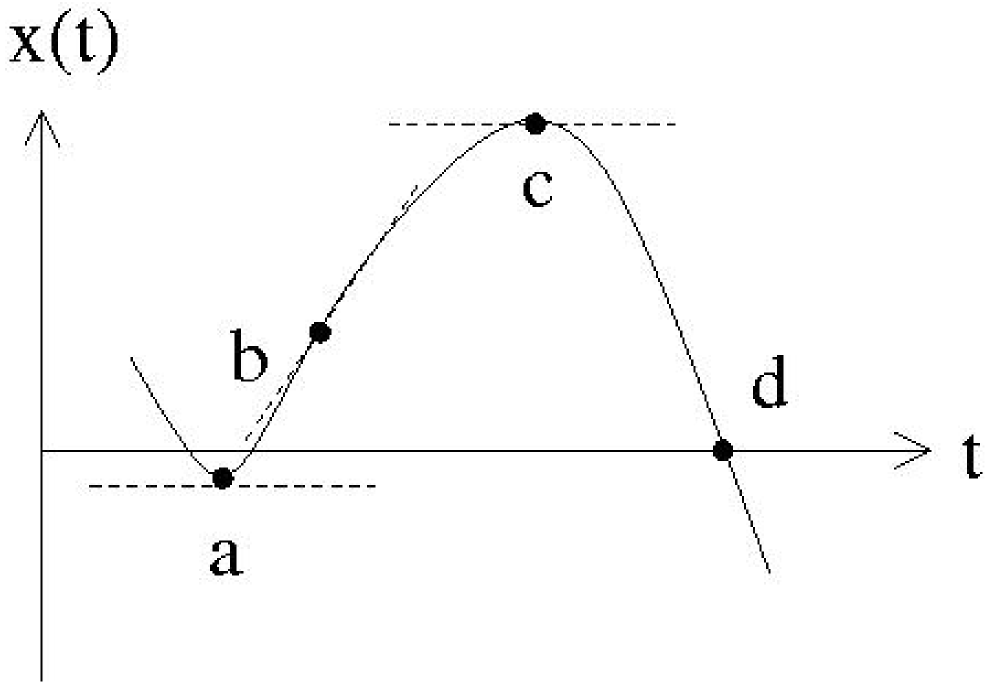
Based on the mathematical properties of the first and second derivative, it can be concluded that:

1) Speeding up implies that either $v > 0$ and $a > 0$ or $v < 0$ and $a < 0$. That is, both velocity and acceleration must have the same sign, either both positive or both negative.

2) Slowing down implies that either $v > 0$ and $a < 0$ or $v < 0$ and $a > 0$. That is, velocity and acceleration must have opposite signs.

3) Rest (actual rest) means that both $v = 0$ AND $a = 0$.

4) Momentarily at rest (also known as the `turning point'): This requires that ONLY $v = 0$, BUT $a \neq 0$.



- Note that the points a and c are turning point, because velocities are zero, but acceleration are not zero.
- The velocities changes directions at the turning points, from positive to negative or negative to positive directions.
- From a → b, it is speeding up, because velocity and acceleration are positive.
- From b → c, it is slowing down, because the velocity is positive, but the acceleration is negative.
- Again from c → d, it is speeding up, because both the velocity and the acceleration are negative.

Equations of Motion: Constant acceleration

- Since a=constant, the position x must be a quadratic function of time t, or linear function of time.
- The position x can not be a function of time with negative powers.
- The position can also not be a function of time with fractional powers.
- Otherwise the acceleration, being a double derivative of position with respect to time, will not be a constant.

- Let's choose the initial time equals zero, *i.e.*

$$t_0 = 0 .$$

- From the definition of average velocity, we find:

$$x = x_0 + v_{\text{av}} t = x_0 + \bar{v} t .$$

- From the definition of average acceleration, we also find,

$$v = v_0 + a_{\text{av}} t = v_0 + \bar{a} t = v_0 + at .$$

$$\left(\text{when a = constant, } a_{\text{av}} = \bar{a} = a\right)$$

- Using $\bar{v} = (v + v_0)/2$, we also get,

$$x = x_0 + v_0 t + (1/2) a t^2 .$$

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- Finally, removing time t, we also get,

$$v^2 = v_0^2 + 2a(x - x_0) .$$

Short Summary:

- Only for constant acceleration, the equations of motions are:

$$(1)\ x = x_0 + \bar{v}\, t$$
$$(2)\ v = v_0 + at$$
$$(3)\ x = x_0 + v_0 t + \frac{1}{2} at^2$$
$$(4)\ v^2 = v_0^2 + 2a(x - x_0)$$

- Only for constant acceleration, the average velocity can also be expressed as the arithmetic average of two velocities at the final and initial times, *i.e.*

$$v_{\text{av}} \equiv \bar{v} = \frac{\Delta x}{\Delta t} = \frac{v + v_0}{2} \ (\text{iff the acceleration is constant})$$

Integral Form of Displacement and Velocity:

- From the definition of velocity, we write:

$$v=\frac{dx}{dt} \Rightarrow dx = vdt \Rightarrow \text{Integratring gives: } \int_{x_0}^{x} dx = \int_{t_0}^{t} v\, dt$$

$$\Delta x \equiv x - x_0 = \int_{t_0}^{t} v\, dt = \text{Area under the curve in a v-t graph or A .}$$

$$= v\, \Delta t \text{ (iff v is constant).}$$

- Note in the v-t graph that

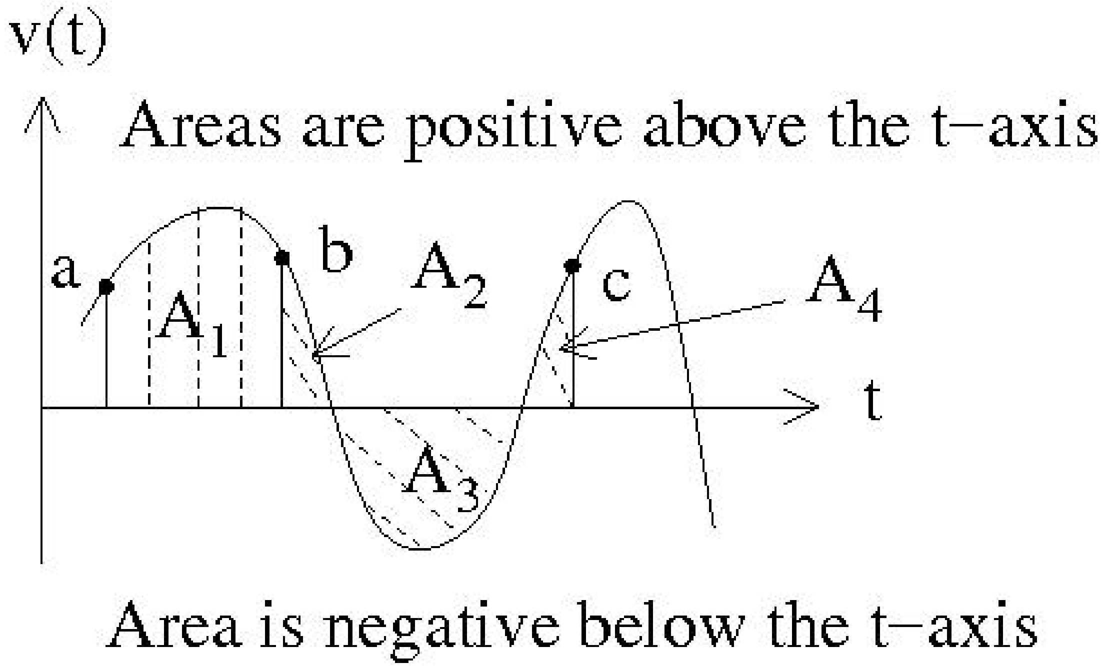
$A_1>0,\ A_2>0\ ,A_4>0,$ but $A_3<0$.

So, the displacements are:

$x_b - x_a = A_1$.

$x_c - x_b = A_2 + A_4 - A_3$.

$x_c - x_a = (A_1 + A_2 + A_4) - A_3$.



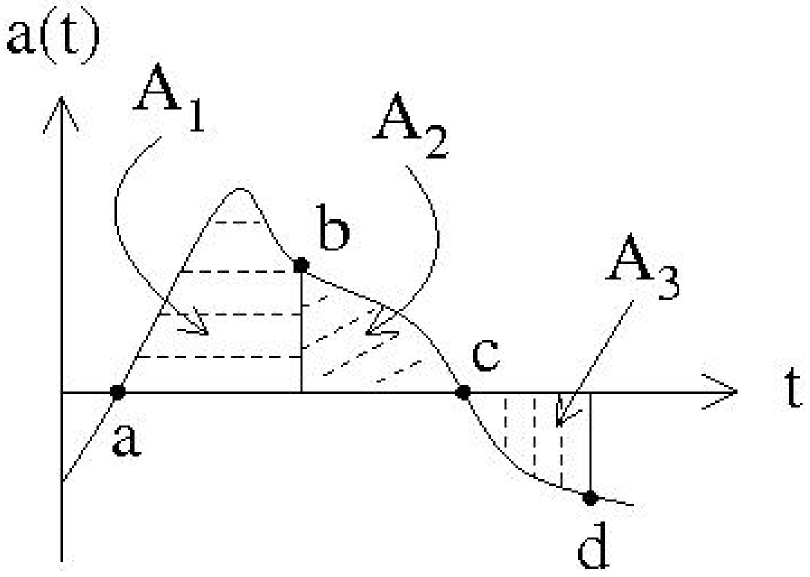
The Acceleration vs. Time Graph:

- Similarly, from the a-t graph, it is easy to see that the areas are:

  $A_1 > 0$, $A_2 > 0$, and $A_3 < 0$.

  Therefore, the velocity changes are:

  $$v_b - v_a = A_{1},$$
  $$v_c - v_a = A_1 + A_2,$$
  $$v_d - v_a = (A_1 + A_2) - A_3.$$



- Note that when the acceleration is constant, the graph is horizontal. So, the region under the curve is perfectly rectangular, and the area under the curve $(a)(\Delta t)$ where *a* is the height of the rectangle and Δt is the width of the rectangle.

Free Fall

- The free fall is an example of one-dimensional motion.
- The position is represented by the vertical axis, the y-axis. So the notation has been changed accordingly.
- The conditions to be satisfied for any Free-Fall motion:

  1) The acceleration is given by: $a=-g=-9.80\ \mathrm{m/s^2}$.

  2) The acceleration is fixed (both magnitude and direction) by the Gravitational force, NOT by any other means.

- Therefore, the equations of motion for Free-Fall become:

$$y=y_0+\bar{v}t\ ;\quad v=v_0-gt$$

$$y=y_0+v_0t-\frac{1}{2}gt^2\ ;\ v^2=v_0^2-2g\left(y-y_0\right).$$

Free Fall Properties:

- The time to go up equal the time to go down if the displacement is zero:

$$t_{up} = t_{down} = \frac{v_0}{g}\ (v_0 = \text{initial speed}).$$

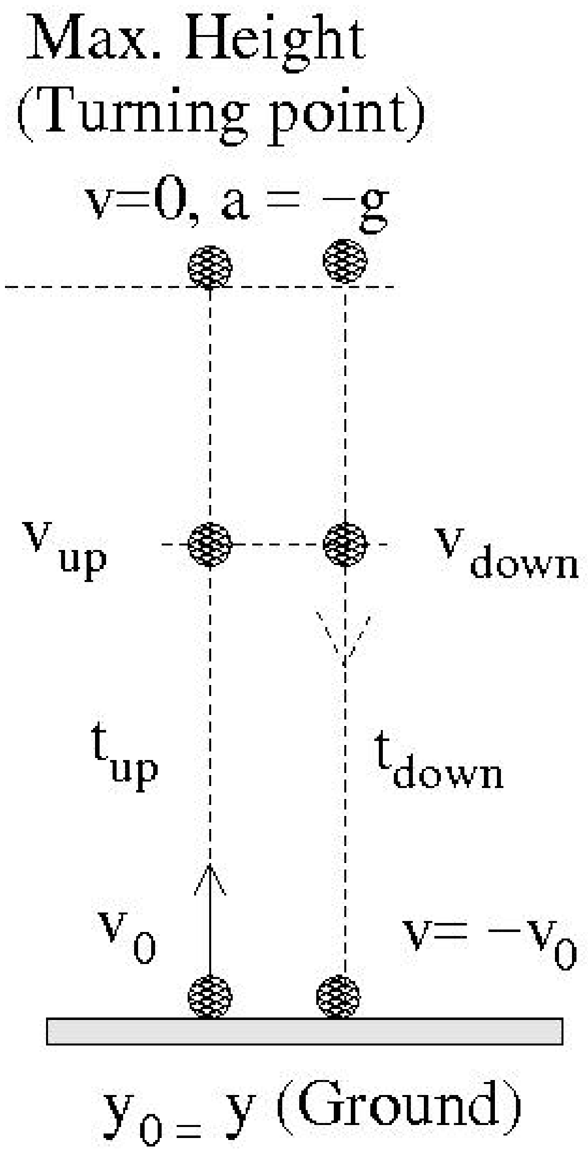
- At any level (*i.e.* same height from the ground):

$$v_{up} = -v_{down} \Rightarrow |v_{up}| = |v_{down}|.$$

- The total time of the flight (when displacement equals zero) is

$$T = 2t_{up} = 2t_{down} = \frac{2v_0}{g}.$$

- At the max. height, the velocity changes direction, but the acceleration is still non-zero constant. Hence it is a turning point.



Example:

The position of an object moving along an x axis is given by $x = 3t - 4t^2 + t^3$, where x is in meters and t in seconds. Find the following:

a) The object's displacement between t=0 to t= 4s.;

b) What is its average velocity for the time interval from t = 2s to t= 4s?

c) Find the turning point of the object.

d) When will the acceleration be zero?

Solution:

a) The displacement is: $\Delta x = x_4 - x_0 = [(3\times4 - 4\times4^2 + 4^3) - (0)]\,\text{m} = 12\,\text{m}.$

b) The average velocity is.

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{x_4 - x_2}{t_4 - t_2} = \frac{(3\times4 - 4\times4^2 + 4^3) - (3\times2 - 4\times2^2 + 2^3)}{4-2}\ \text{m/s} = 7\ \text{m/s}.$$

c) At the turning point, v=0. Therefore,

$$v = \frac{dx}{dt} = 3t^2 - 8t + 3 = 0 \Rightarrow t = \frac{-(-8) \pm \sqrt{(-8)^{-4} \times 3\times3}}{2\times3}\ \text{sec} = 0.45\,\text{sec or } 2.2\ \text{sec}.$$

Now, $x(t = 0.45\,\text{sec}) = [3(0.45) - 4(0.45)^2 + (0.45)^3]\,\text{m} = 0.63\,\text{m},$

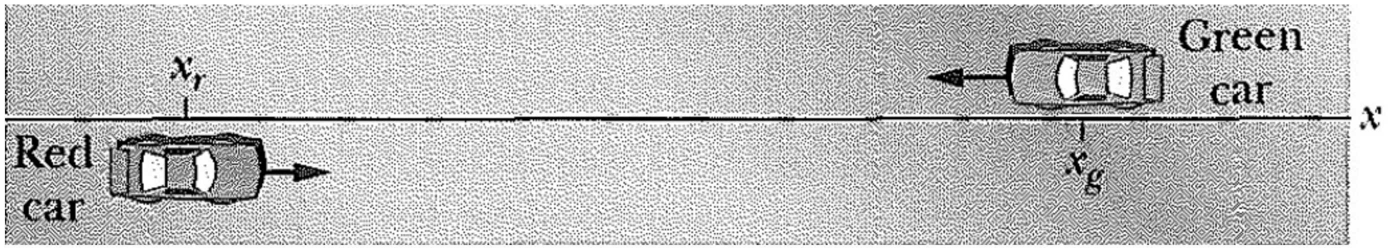
and $x(t = 2.2\,\text{sec}) = [3(2.2) - 4(2.2)^2 + (2.2)^3]\,\text{m} = 2.1\,\text{m}.$

d) The acceleration is :

$$a = \frac{dx^2}{dt^2} = -8 + 6t.\quad \text{So}\ , a = 0\ \text{gives}\ t = 1.33\,\text{sec}.$$

Problem # 2.34:

In the figure below, a red car and a green car, identical except for the color, move toward each other in adjacent lanes and parallel to an x axis. At time $t = 0$, the red car is at $x_r = 0$ and the green car is at $x_g = 220\,\text{m}$. If the red car has a constant velocity of 20 km/h, the cars pass each other at $x = 44.5$ m, and if it has a constant velocity of 40 km/h, they pass each other at $x = 76.6$ m. What are (a) the initial velocity and (b) the constant acceleration of the green car

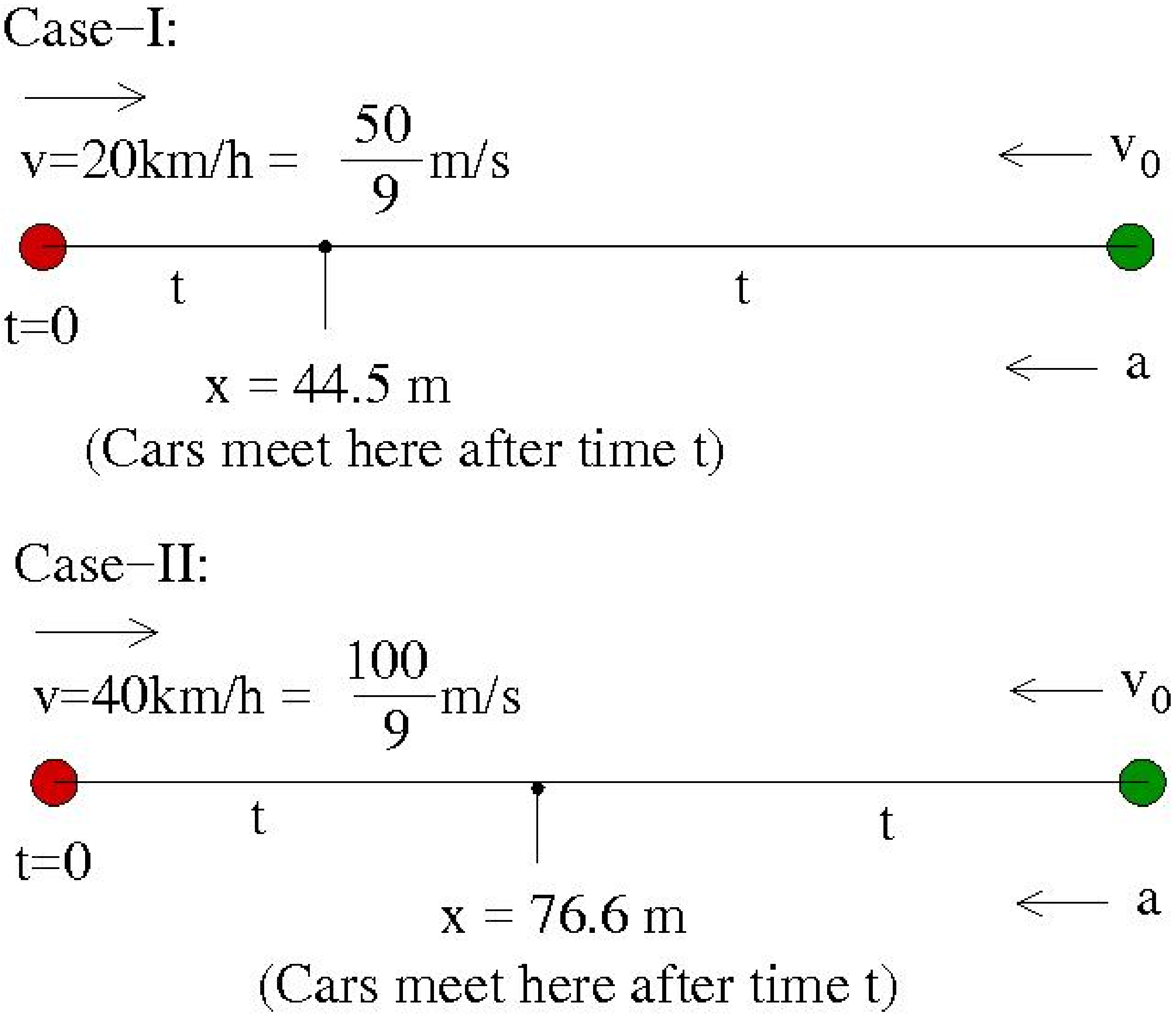


Solution:

The schematic diagram is:

Here, v_0 and a are unknown.

- In the Case-I, we find t from the red car, and write an equation for the green car.
- In the Case-II, we again find t From the red car, and write an equation for the green car.
- Thus, we get two equations with two unknown and solve.



- Case-I: Suppose after time t the cars meet at a position (or distance) 44.5 m

$$\text{Now },v=20\text{ km/h}=\frac{50}{9}\text{ m/sec}.\quad \text{So},t=\frac{x}{v}=\frac{44.5\times 9}{50}\text{sec}=8.0\ \text{sec}.$$

In time t, the green car's position changes as:

$$x=x_0+v_0t+\frac{1}{2}at^2\Rightarrow 44.5=220+8v_0+\frac{1}{2}a(8)^2\Rightarrow 32.0a+8.0v_0+175.5=0.$$

- Case-II: In this case, the cars meet at a position (or distance) 76.6 m.

$$\text{Now, }v=40\text{ km/h}=\frac{100}{9}\text{ m/sec}.\quad \text{So},t=\frac{x}{v}=\frac{76.6\times 9}{100}\text{sec}=6.9\ \text{sec}.$$

In time t, the green car's position changes as:

$$x=x_0+v_0t+\frac{1}{2}at^2\Rightarrow 76.6=220+6.9v_0+\frac{1}{2}a(6.9)^2\Rightarrow 23.8a+6.9v_0+143.4=0.$$

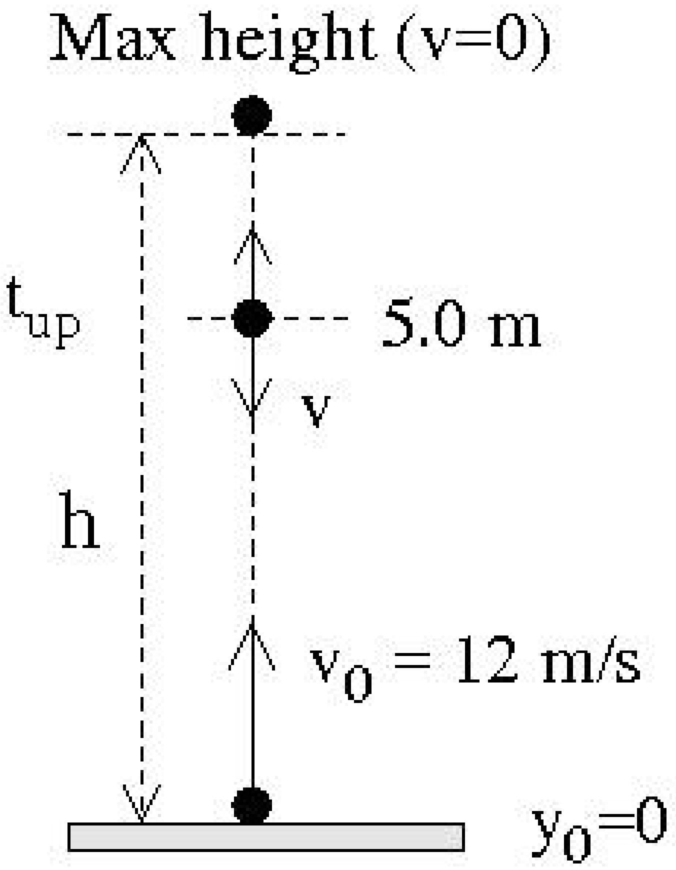
- Solving these equations (red colored equations) for initial velocity and acceleration using the method of elimination, we find that

$$\text{Initial velocity of the green car },v_0=-13.5\text{ m/s}.$$

$$\text{The acceleration of the green car },a=-2.1\text{ m/s}^2.$$

Problem:

- A baseball is tossed up with initial speed of 12 m/s. (a) How long does it take to reach the maximum height? (b) Find the maximum height. (c) How long dies it take to reach 5.0m above the release point? (d) What is the velocity at 5.0m above the release point.



Solution: The ground is the release point $(y_0=0)$.

a) Here, $v=v_0+at$ gives $0=12-9.8\,t_{up}$

$$\Rightarrow t_{up}=(12/9.8)\,\text{sec}=1.22\,\text{sec}.$$

b) In this case, we get,

$$v^2=0=v_0^2+2a(x-x_0)\Rightarrow h\equiv|x-x_0|=\frac{v_0^2}{2g}=\frac{(12)^2}{(2\times9.8)}\,\text{m}=7.35\,\text{m}.$$

c) Here, y = 5.0m. Therefore,

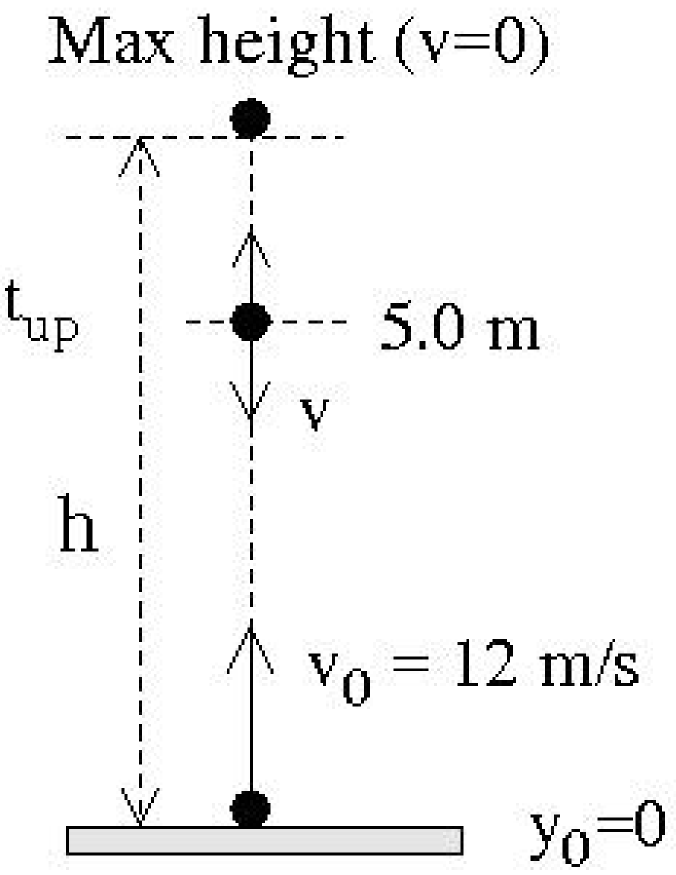
$$y = y_0 + v_0 t + \frac{1}{2} a t^2$$

$$5 = 0 + 12t - \frac{1}{2} 9.8 t^2 \Rightarrow 4.9t^2 - 12t + 5 = 0$$

$$t = \frac{-(-12)\, pm \sqrt{(-12)^2 - 4\times 4.9 \times 5}}{2\times 9.8}\,\text{sec}$$

$$= \color{red}{0.53\,\text{sec},\quad 1.92\,\text{sec}}.$$

Note that 0.53 sec is the time while going up and 1.92 sec is the time while going down.



(d) There are two velocities at y=5.0m above the release point: one is going up and one is for coming down.

Therefore, the velocities are:

$$v(t=0.53\,\text{sec}) = v_0 + at$$
$$= (12 - 9.8 \times 0.53)\,\text{m/s} = 6.81\,\text{m/s}.$$
$$v(t=1.92\,\text{sec}) = v_0 + at$$
$$= (12 - 9.8 \times 1.92)\,\text{m/s} = -6.81\,\text{m/s}.$$