Normal Distribution

Normal Distribution:

The probability density function of normal distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \begin{array}{c} -\infty < x < \infty, \\ -\infty < \mu < \infty, \\ 0 \leq \sigma^2 < \infty \end{array}$$

Expectation: $E(x) = \mu$

Variance: $V(x) = \sigma^2$

Standard normal distribution: When $\mu = 0$ and $\sigma^2 = 1$ then the normal distribution is called **standard normal distribution.**

The probability density function of standard normal distribution is:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

Property of a normal distribution:

1) It is symmetric.

2) Mean=Mode=Median.

3) It is unimodal.

4) The total area under the curve is equal to one.

5) The normal curve approaches, but never touches, the x-axis

Transformation:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}z^2} \sigma\,dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$$

Let, $z = \frac{x-\mu}{\sigma}$

$=> z = \frac{x}{\sigma} - \frac{\mu}{\sigma}$

$=> \frac{dz}{dx} = \frac{1}{\sigma}$

$=> dx = \sigma\,dz$

x	$-\infty$	∞
z	$-\infty$	∞

That is if $X \sim N(\mu, \sigma^2)$ and if you want to transform the normal distribution to standard distribution then the transform random variable is $Z = \frac{x-\mu}{\sigma}$ (Z score)

Probability Calculations for Normal Distributions:

$$P(a < x < b)$$

$$= P\left(\frac{a-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$

$$= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

$P(x < b)$

$= P(-\infty < x < b)$

$= P(\frac{-\infty - \mu}{\sigma} < \frac{x - \mu}{\sigma} < \frac{b - \mu}{\sigma})$

$= P(-\infty < Z < \frac{b-\mu}{\sigma})$

$= F(\frac{b - \mu}{\sigma})$

$P(x > a)$

$= P(a < x < \infty)$

$= P(\frac{a - \mu}{\sigma} < \frac{x - \mu}{\sigma} < \frac{\infty - \mu}{\sigma})$

$= P(\frac{a - \mu}{\sigma} < Z < \infty)$

$= F(\infty) - F(\frac{a - \mu}{\sigma})$

$= 1 - F(\frac{a - \mu}{\sigma})$

Example: A company manufactures concrete blocks that are used for construction purposes. Suppose that the weights of the individual concrete blocks **are normally distributed** with a mean value of $\mu = 11.0$ kg and a standard deviation of $\sigma = 0.3$ kg.

i) Calculate the probability that a concrete block weight is less than 10.5kg.

ii) Calculate the probability that a concrete block weight is within 10kg to 12kg.

iii) Calculate the probability that a concrete block weight is greater than 10.5kg.

Solution:

i)

$P(x < 10.5)$

$= P(-\infty < x < 10.5)$

$= P(\frac{-\infty - 11}{0.3} < \frac{x - 11}{0.3} < \frac{10.5 - 11}{0.3})$

$= P(-\infty < Z < -1.67)$

$= F(-1.67)$

$= 0.0475$

ii)

$P(10 < x < 12)$

$= P(\frac{10 - 11}{0.3} < \frac{x - 11}{0.3} < \frac{12 - 11}{0.3})$

$= P(-3.33 < Z < 3.33)$

$= F(3.33) - F(-3.33)$

$= 0.99957 - 0.00043$

$= 0.99914$

iii)

$P(x > 10.5)$

$= P(10.5 < x)$

$= P(10.5 < x < \infty)$

$$= P\left(\frac{10.5 - 11}{0.3} < \frac{x - 11}{0.3} < \frac{\infty - 11}{0.3}\right)$$

$$= P(-1.67 < Z < \infty)$$

$$= F(\infty) - F(-1.67)$$

$$= 1 - .04746$$

$$= 0.95254$$