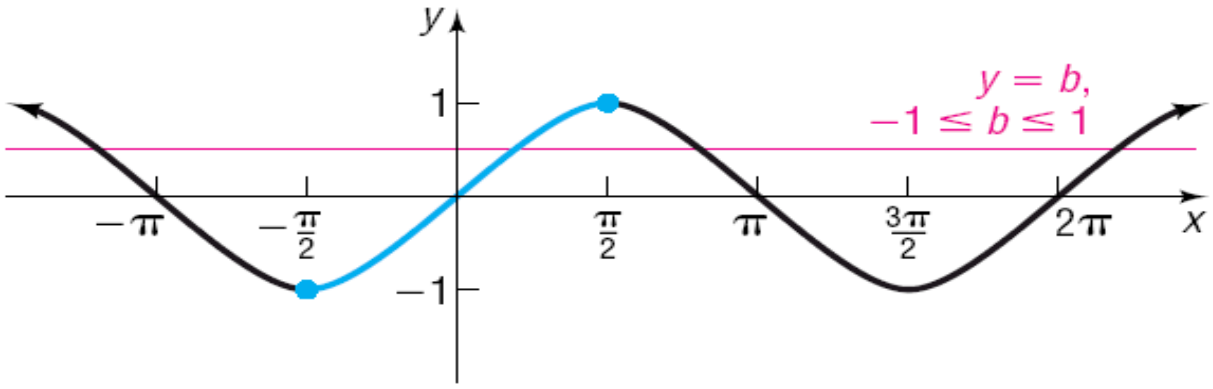
Section 8.1: The Inverse Sine, Cosine, and Tangent Functions

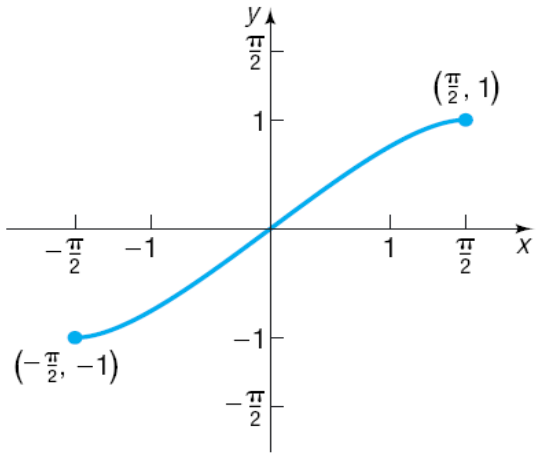
Review of Properties of Functions and Their Inverses

1. $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
2. Domain of f = range of f^{-1}, and range of f = domain of f^{-1}.
3. The graph of f and the graph of f^{-1} are symmetric with respect to the line $y = x$.
4. If a function $y = f(x)$ has an inverse function, the equation of the inverse function is $x = f(y)$. The solution of this equation is $y = f^{-1}(x)$.

The Inverse Sine Function



$$y = \sin x, -\infty < x < \infty, -1 \le y \le 1$$

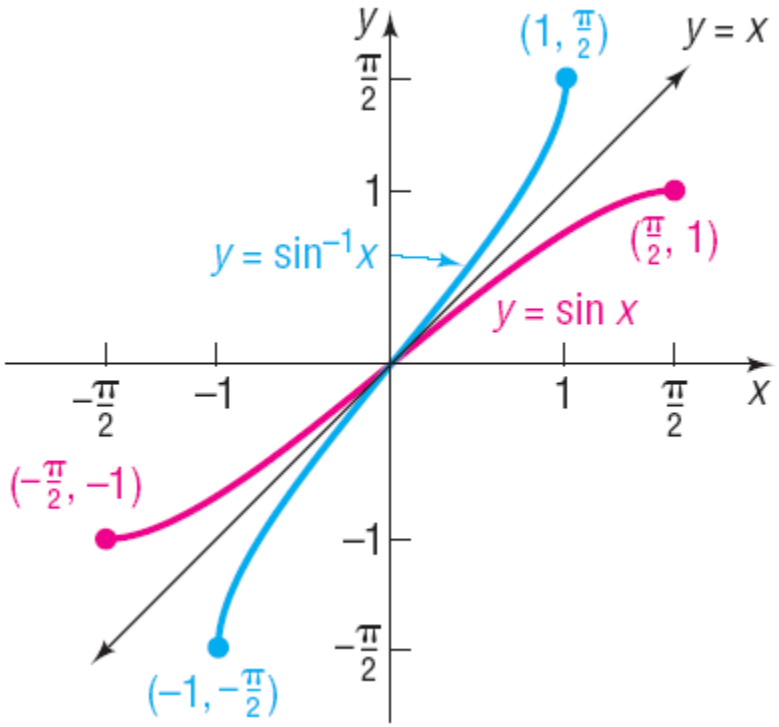


$$y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}, -1 \le y \le 1$$

Definition:

$$y = \sin^{-1} x \quad \text{means} \quad x = \sin y$$

$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$



$$y = \sin^{-1} x, -1 \le x \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

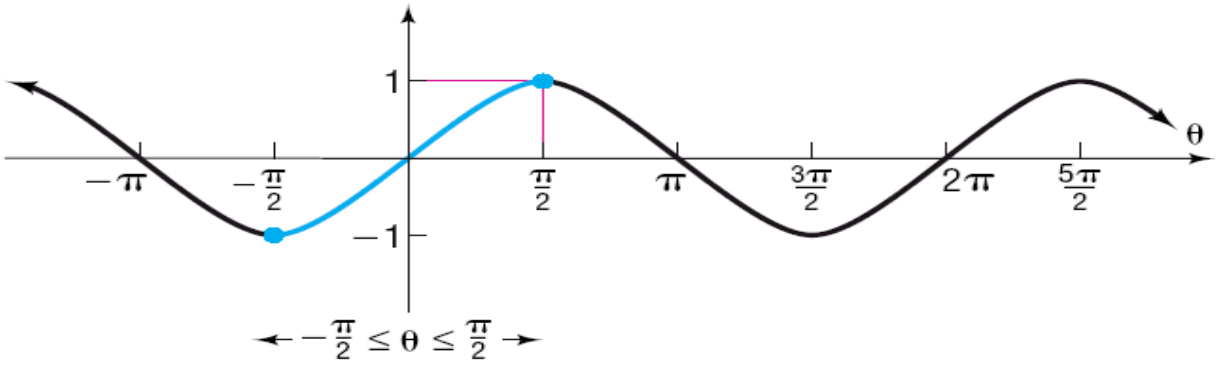
1 Find the Exact Value of an Inverse Sine Function

Example: 1

Find the exact value of: $\sin^{-1} 1$

We seek the angle $\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 1.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1



Example: 2 Find the exact value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Example: 3 Find the exact value of $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

2 Find an Approximate Value of an Inverse Sine Function

Use your calculator: Example: 4

Find an approximate value of:

(a) $\sin^{-1}\frac{1}{3}$ (b) $\sin^{-1}\left(-\frac{1}{4}\right)$

3 Use Properties of Inverse Functions to Find Exact Values of Certain Composite Functions

Properties of Inverse Functions

$$f^{-1}(f(x)) = \sin^{-1}(\sin x) = x, \quad \text{where } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

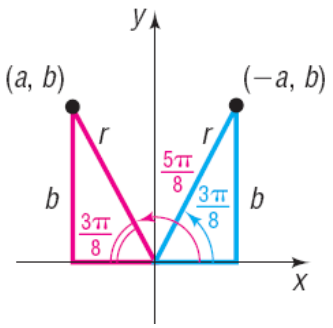
$$f(f^{-1}(x)) = \sin(\sin^{-1} x) = x, \quad \text{where } -1 \le x \le 1$$

Example:5:

Find the exact value of each of the following composite functions:

(a) $\sin^{-1}\left(\sin\frac{\pi}{8}\right)$

(b) $\sin^{-1}\left(\sin\frac{5\pi}{8}\right)$

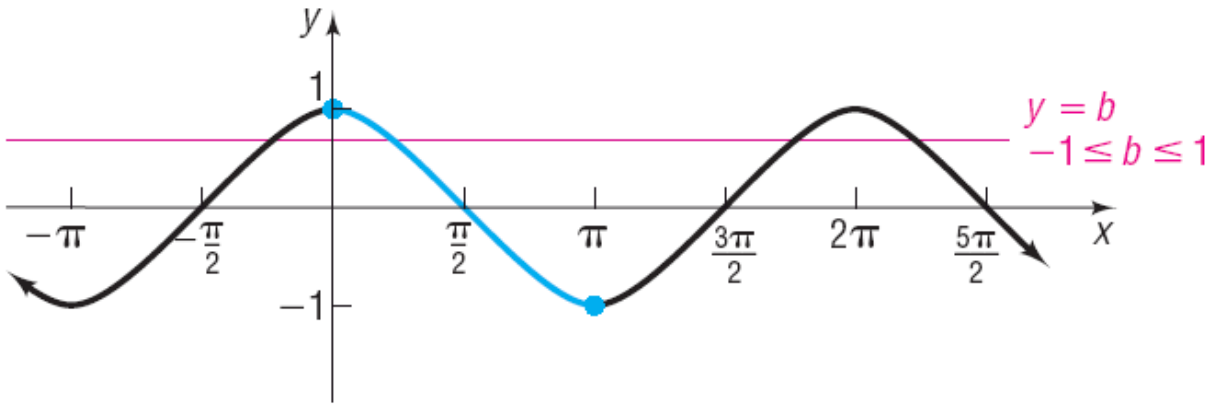


Example:6

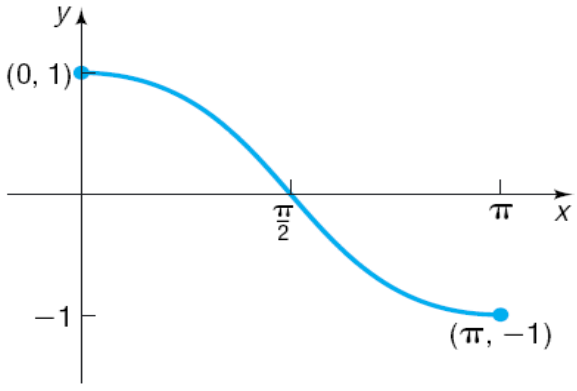
Find the exact value, if any, of each composite function.

(a) $\sin(\sin^{-1} 0.5)$ (b) $\sin(\sin^{-1} 1.8)$

The Inverse Cosine Function

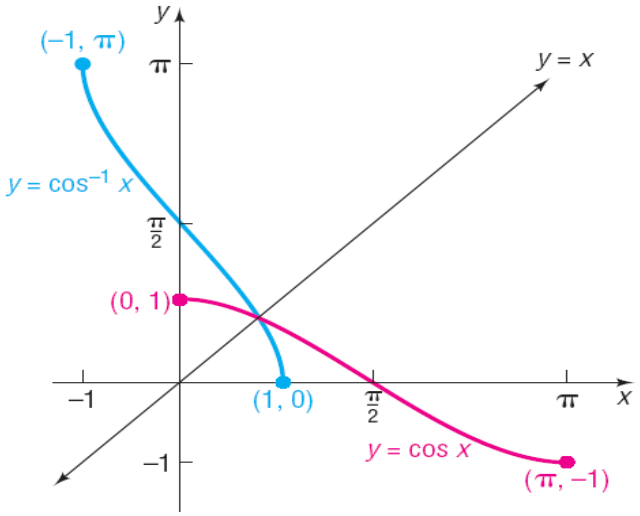


$y = \cos x, \quad -\infty < x < \infty, \quad -1 \le y \le 1$



$y = \cos x, 0 \le x \le \pi, -1 \le y \le 1$

$$y = \cos^{-1} x \quad \text{means} \quad x = \cos y$$
$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi$$



$$y = \cos^{-1} x, \; -1 \le x \le 1, \; 0 \le y \le \pi$$

Example:7 (a) Find the exact value of $\cos^{-1}\left(\frac{1}{2}\right)$

θ	$\cos\theta$
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$
π	-1

Example:7(b) Find the exact value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Properties of Inverse Functions

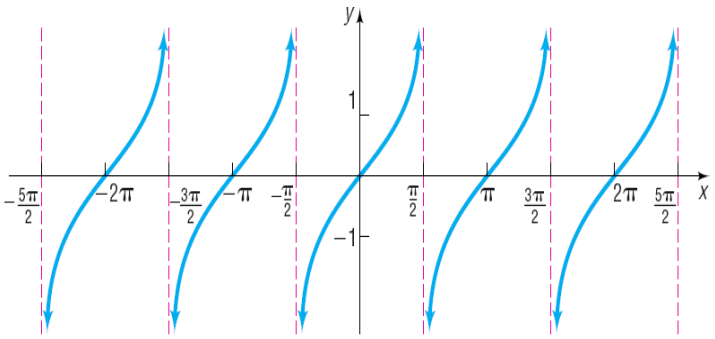
$$f^{-1}(f(x)) = \cos^{-1}(\cos x) = x, \quad \text{where } 0 \le x \le \pi$$
$$f(f^{-1}(x)) = \cos(\cos^{-1} x) = x, \quad \text{where } -1 \le x \le 1$$

Example:8 Find the exact value of $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

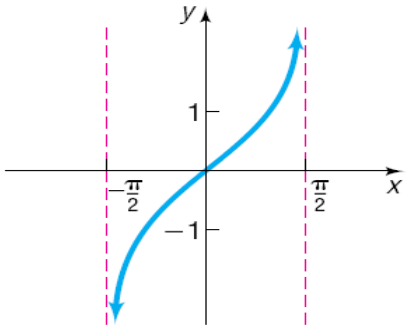
Example:9 Find the exact value of $\cos\left(\cos^{-1}\left(\frac{5}{2}\right)\right)$

Example:10 Find the exact value of $\cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$

The Inverse Tangent Function



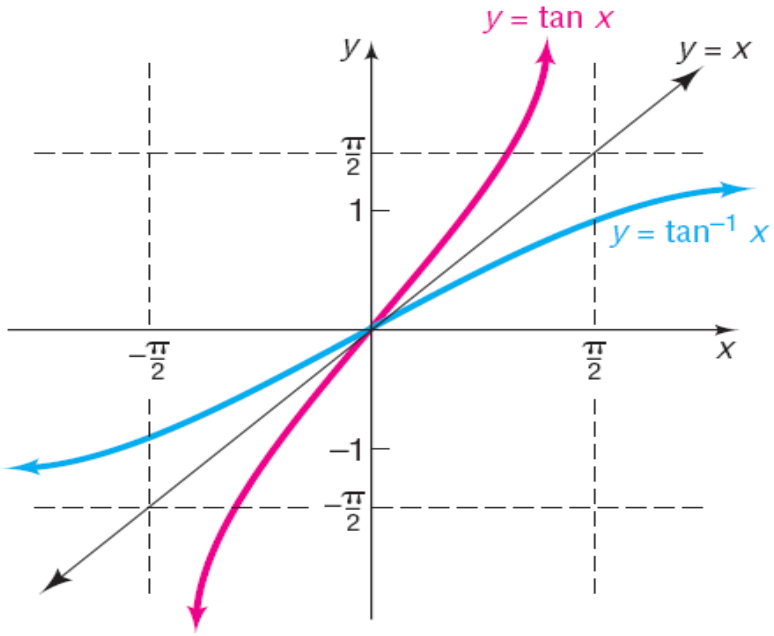
$y = \tan x, -\infty < x < \infty, x$ not equal to odd multiples of $\frac{\pi}{2}, -\infty < y < \infty$



$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}, -\infty < y < \infty$

$y = \tan^{-1} x$ means $x = \tan y$

where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$



$y = \tan^{-1} x, -\infty < x < \infty, -\frac{\pi}{2} < y < \frac{\pi}{2}$

Example: 11 Find the exact value of $\tan^{-1}\sqrt{3}$

θ	$\tan\theta$
$-\frac{\pi}{2}$	Undefined
$-\frac{\pi}{3}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3}$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	Undefined

Example: 12 Find the exact value of $\tan^{-1} 0$

Properties of Inverse Functions

$$f^{-1}(f(x)) = \tan^{-1}(\tan x) = x \quad \text{where } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(f^{-1}(x)) = \tan(\tan^{-1} x) = x \quad \text{where } -\infty < x < \infty$$

Section 8.2 : The Inverse Trigonometric Functions (Continued)

1 Find the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Example: 1 Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}\right)$

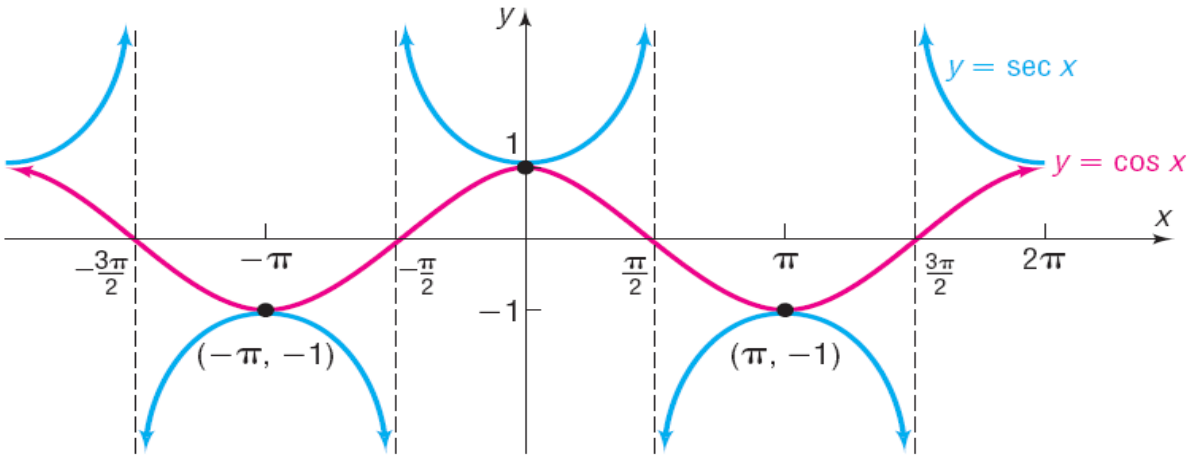
Example: 2 Find the exact value of $\tan\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Example: 3 Find the exact value of $\sin\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

2 Define the Inverse Secant, Cosecant, and Cotangent Functions

$$y = \sec^{-1} x \quad \text{means} \quad x = \sec y$$

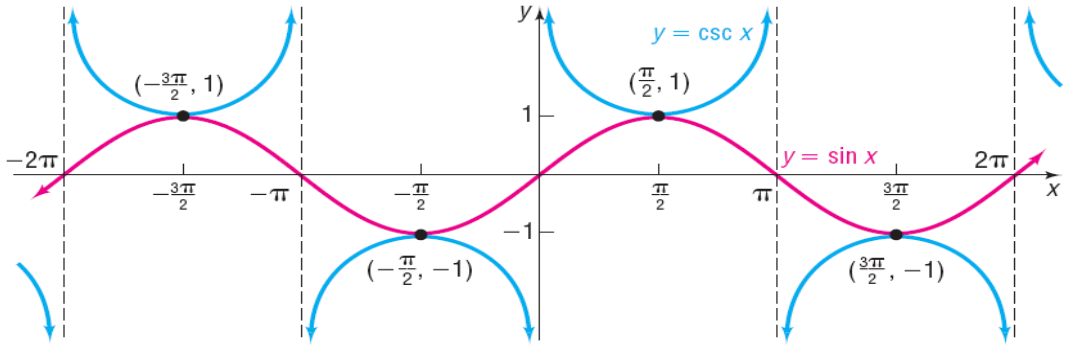
$$\text{where} \quad |x| \geq 1 \quad \text{and} \quad 0 \leq y \leq \pi, \quad y \neq \frac{\pi}{2}^{*}$$



$y = \sec x, -\infty < x < \infty, x$ not equal to odd multiples of $\frac{\pi}{2}, |y| \geq 1$

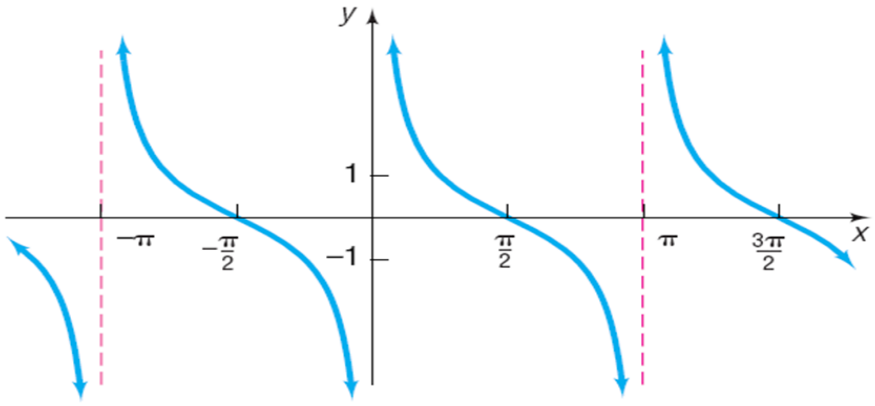
$$y = \csc^{-1} x \quad \text{means} \quad x = \csc y$$

$$\text{where} \quad |x| \geq 1 \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq 0^{\dagger}$$



$y = \csc x, -\infty < x < \infty, x$ not equal to integer multiples of $\pi, |y| \geq 1$

$$y = \cot^{-1} x \quad \text{means} \quad x = \cot y$$
$$\text{where} \quad -\infty < x < \infty \quad \text{and} \quad 0 < y < \pi$$



$y = \cot x, -\infty < x < \infty, x$ not equal to integer multiples of $\pi, -\infty < y < \infty$

Example: 4 Find the exact value of $\sec^{-1}\sqrt{2}$

Example: 5 Find the exact value of $\csc^{-1} 2$

Example: 6 Find the exact value of $\cot^{-1}\sqrt{3}$

Section 8.3 : Trigonometric Equations

Example: 1 Determine whether $\theta = \frac{\pi}{4}$ is a solution of the equation

$$2\sin\theta + \sqrt{2} = 0.$$

Is $\theta = \frac{5\pi}{4}$ a solution?

Example: 2 Solve the equation $2\cos\theta - \sqrt{3} = 0$ on the interval $0 \le \theta \le 2\pi$.

Example: 3 Solve the equation $2\sin^2\theta - 3\sin\theta + 1 = 0$ on the interval $0 \le \theta \le 2\pi$.

Example: 4 Use your graphing calculator to find all solutions in $[0, 2\pi]$ for $x^2 - 2\sin(2x) = 3x$. Round to two decimal places.

Example: 5 Use your graphing calculator to find all solutions in $[0, 4\pi]$ for $3\cos x + x = 4$. Round to two decimal places.