Textbook: Precalculus by Howard anton, Irl Bivens and Stephen Davis

Chapter 4: Polynomial and Rational Functions

We have the following functions:

(i) Polynomial functions: $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
where $a_n, a_{n-1}, \ldots, a_1, a_0$ are all real numbers and the number n is a nonnegative integer.

(ii) Rational functions: $R(x) = \dfrac{p(x)}{q(x)}$ if $q(x) \neq 0$, i.e. every rational function is a ratio of polynomial functions.

In this chapter, we will consider two general classes of functions, namely, polynomial functions and rational functions and examine their properties.

4.1 Polynomial Functions and Models

4.1.1 Identify Polynomial Functions and Their Degree

Consider a *linear function*

$$p_1(x) = f(x) = a_1 x + a_0 \tag{1}$$

since we know the slope-intercept form of a line is $y = mx + b$ where the number b is the y-intercept and m is the slope of the line defined by $m = \tan\theta$.
Then we consider a *quadratic function*

$$f(x) = a_2 x^2 + a_1 x + a_0 \tag{2}$$

Equations (1) and (2) are examples of *polynomial functions*.

Definition

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are all real numbers and the number n is a nonnegative integer.
The domain of a polynomial function is the set of all real numbers.

A polynomial function is a function whose rule is given by a polynomial in one variable.

The **degree** of a polynomial function is the largest power of x that appears.
The zero polynomial function

$$f(x) = 0x^n + 0x^{n-1} + \cdots + 0x + 0$$

is not assigned a degree.

Example 1

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

(a) $f(x) = 2 - 3x^4$ (b) $g(x) = \sqrt{x}$ (c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

(d) $F(x)=0$ (e) $G(x)=8$ (f) $h(x)=-2x^3(x-1)^2$

Solution: (a) The given function f is a polynomial function of degree 4.

(b) The given function g is not a polynomial function because in $g(x)$, power of x is a rational number not a nonnegative integer.

(c) The given function h is not a polynomial function. It is the ratio of two distinct polynomials, and the polynomial in the denominator is of positive degree.

(d) The given function F is the zero polynomial function; it is not assigned a degree.

(e) The given function G is a nonzero constant function. It is a polynomial function of degree 0 since $G(x)=8=8x^0$.

(f) The given function $h(x)=-2x^3(x-1)^2$ can be simplified in the form as

$$h(x)=-2x^3(x-1)^2=-2x^3(x^2-2x+1)=-2x^5+4x^4-2x^3$$

So h is a polynomial function of degree 5.

We have already discussed in detail polynomial functions of different degrees. Now let us give the summary of the properties of the graphs of these polynomial functions.

(i) The *zero function* $f(x)=0$ is a polynomial function with no degree whose graph is the x-axis.

(ii) The *constant function* $f(x)=a_0$ with $a_0 \neq 0$, is a polynomial function of degree zero whose graph is the horizontal line with y-intercept a_0.

(iii) The *linear function* $f(x)=a_1x+a_0$ with $a_1 \neq 0$, is a polynomial function of degree one whose graph is a nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0.

(iv) The *quadratic function* $f(x)=a_2x^2+a_1x+a_0$ with $a_2 \neq 0$, is a polynomial function of degree two whose graph is a parabola which opens up if $a_2>0$ and the graph opens down if $a_2<0$.

(v) The *power function* $f(x)=ax^n$ is a *monomial function* of degree n where a is a nonzero real number and n is a positive integer.

Examples of power functions are

$$f(x)=3x,\ g(x)=-5x^2,\ h(x)=8x^3,\ p(x)=-5x^4$$

The graph of a power function of degree 1, $f(x)=ax$, is a straight line, with slope a, that passes through the origin.

The graph of a power function of degree 2, $f(x)=ax^2$, is a parabola, with vertex at the origin, that opens up if $a>0$ and opens down if $a<0$.

To graph a power function of the form $f(x)=x^n$, we need to use a compression or a stretching and a reflection about the x-axis which will enable us to obtain the graph of $g(x)=ax^n$.

Now we begin with power functions of even degree of the form $f(x)=x^n$, $n\geq 2$ and n even. The domain of f is the set of all real numbers, and the range is the set of nonnegative real numbers. Such a power function is an even function and so its graph is symmetric with respect to the y-axis. Its graph always contains the origin and the points $(-1,1)$ and $(1,1)$.

If $n = 2$, the graph is the parabola that opens up, with vertex at the origin. If $n \geq 4$, the graph of $f(x) = x^n$, for n even, will be closer to the x-axis than the parabola $y = x^2$, if $-1 < x < 1$, $x \neq 0$, and farther from the x-axis than the parabola $y = x^2$, if $x < -1$ or if $x > 1$. Figure 2(a) illustrates this conclusion.
Figure 2(b) shows the graphs of $y = x^4$ and $y = x^8$ for the purpose of comparison.

Properties of Power Functions $f(x) = x^n$, when n is an Even Positive Integer:

1. f is an even function, so its graph is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(-1,1)$, $(0,0)$ and $(1,1)$.
4. As the exponent n increases in magnitude, the function increases more rapidly when $x < -1$ or if $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Now we consider power functions of odd degree of the form $f(x) = x^n$, $n \geq 3$ and and n odd. The domain and the range of f are the set of real numbers. Such a power function is an odd function, so its graph is symmetric with respect to the origin. Its graph always contains the origin and the points $(-1,-1)$ and $(1,1)$.
The graph of $f(x) = x^n$ when $n = 3$ has been shown several times and is repeated in Figure 4. If $n \geq 5$, the graph of $f(x) = x^n$, for n odd, will be closer to the x-axis than that of $y = x^3$, if $-1 < x < 1$ and farther from the x-axis than that of $y = x^3$, if $x < -1$ or if $x > 1$. Figure 4 illustrates this conclusion.
Figure 5 shows the graph of $y = x^3$ and the graph of $y = x^9$ for further comparison.

Properties of Power Functions $f(x) = x^n$, when n is an Odd Positive Integer:

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(-1,-1)$, $(0,0)$ and $(1,1)$.
4. As the exponent n increases in magnitude, the function increases more rapidly when $x < -1$ or if $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

4.1.2 Graph Polynomial Functions Using Transformations

The methods of shifting, compression, stretching, and reflection studied in Section 2.5, when used with the facts just presented, will enable us to graph polynomial functions that are transformations of power functions.

Example 2
Graph the function $y = 1 - x^5$ using transformations.
Solution: See the **Textbook.**
Sketch of the solution:

Step I: Draw the original graph of $y = f(x) = x^5$. Since power of x is an odd integer, the graph contains the points $(-1,-1)$, $(0,0)$ and $(1,1)$.

Step II: To graph $y = f(x) = -x^5$, use the transformation of reflection about the x-axis, and then the graph contains the points $(-1,1)$, $(0,0)$ and $(1,-1)$.

Step III: To graph $y = f(x) = 1 - x^5$, use the transformation vertical shift up by 1 unit and then the graph contains the points $(-1,2)$, $(0,1)$ and $(1,0)$.

See Figure 7

Example 3

Graph the function $y = \frac{1}{2}(x-1)^4$ using transformations.

Solution: See the **Textbook.**

Sketch of the solution:

Step I: Draw the original graph of $y = f(x) = x^4$. Since power of x is an even integer, the graph contains the points $(-1,1)$, $(0,0)$ and $(1,1)$.

Step II: To graph $y = f(x) = (x-1)^4$, use the transformation of horizontal shift right by 1 unit and then the graph contains the points $(0,1)$, $(1,0)$ and $(2,1)$.

Step III: To graph $y = \frac{1}{2}(x-1)^4$, use the transformation compression by a factor of $\frac{1}{2}$ unit and then the graph contains the points $\left(0, \frac{1}{2}\right)$, $(1,0)$ and $\left(2, \frac{1}{2}\right)$.

See Figure 8

4.1.3 Identify the Real Zeros of a Polynomial Function and Their Multiplicity

Figure 9 shows the graph of a polynomial function with four x-intercepts. Notice that at the x-intercepts the graph must either cross the x-axis or touch the x-axis. Consequently, between consecutive x-intercepts the graph is either above the x-axis or the graph is below the x-axis.

If a polynomial function f is factored completely, it is easy to locate the x-intercepts of the graph by solving the equation $f(x) = 0$ using the Zero-Product Property.

For example, if $y = f(x) = (x-1)^2(x+3)$, then the solutions of the equation

$$f(x) = (x-1)^2(x+3) = 0$$

are 1 and -3. That is, $f(1) = 0$ and $f(-3) = 0$.

Definition

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f.

As a consequence of this definition, the following statements are equivalent.

Finding a Polynomial Function from Its Zeros

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x-r$ is a factor of f.
4. r is a solution to the equation $f(x)=0$.

So the real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x)=0$.

Example 4

Find a polynomial function of degree 3 whose zeros are -3, 2 and 5.

Solution: If r is a real zero of a polynomial function f, then $x-r$ is a factor of f. This means that $x-(-3)=x+3, x-2$, and $x-5$ are factors of f. As a result, any polynomial function of the form

$$f(x)=a(x+3)(x-2)(x-5)$$

where a is a nonzero real number, qualifies. The value of a causes a stretch, compression, or reflection, but does not affect the x-intercepts of the graph.

See Figure 10

If the same factor $x-r$ occurs more than once, r is called a **repeated,** or **multiple, zero of *f*.**

More precisely, we have the following definition.

Definition

If $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity *m* of *f*.**

Example 5

For the polynomial

$$f(x)=5(x-2)(x+3)^2\left(x-\frac{1}{2}\right)^4$$

In this problem, we can say that 2 is a zero of multiplicity 1 because the exponent on the factor $x-2$ is 1; -3 is a zero of multiplicity 2 because the exponent on the factor $x+3$ is 2; and the number $\frac{1}{2}$ is a zero of multiplicity 4 because the exponent on the factor $x-\frac{1}{2}$ is 4.

4.1 Assess Your Understanding

Determine which functions are polynomial functions. For those that are, state the degree. For those that are not, tell why not.

15. $f(x)=4x+x^3$ **16.** $f(x)=5x^2+4x^4$ **17.** $g(x)=\dfrac{1-x^2}{2}$

18. $h(x)=3-\dfrac{1}{2}x$ **19.** $f(x)=1-\dfrac{1}{x}$ **20.** $f(x)=x(x-1)$

21. $g(x) = x^{3/2} - x^2 + 2$ **22.** $h(x) = \sqrt{x}(\sqrt{x} - 1)$ **23.** $F(x) = 5x^4 - \pi x^3 + \frac{1}{2}$

24. $F(x) = \frac{x^2 - 5}{x^3}$ **25.** $G(x) = 2(x-1)^2(x^2+1)$ **26.** $G(x) = -3x^2(x+2)^3$

Solutions to the above problems:

15. The given function f is a polynomial function of degree 3.
16. The given function f is a polynomial function of degree 4.
17. The given function g is a polynomial function of degree 2.
18. The given function h is a polynomial function of degree 1.
19. The given function f is not a polynomial function because in $f(x)$, power of x is a negative integer.
20. The given function f is a polynomial function of degree 2.
21. The given function g is not a polynomial function because first term in $g(x)$ has a rational power of x.
22. The given function $h(x) = \sqrt{x}(\sqrt{x} - 1)$ can be simplified in the form as

$$h(x) = \sqrt{x}(\sqrt{x} - 1) = x - \sqrt{x}$$

The given function h is not a polynomial function because in $h(x)$, the second term has a rational power of x.

23. The given function F is a polynomial function of degree 4.
24. The given function F is not a polynomial function because it is the ratio of a polynomial function and a monomial function; and the monomial function in the denominator is of positive degree.
25. The given function $G(x) = 2(x-1)^2(x^2+1)$ can be simplified in the form as

$$G(x) = 2(x-1)^2(x^2+1) = 2(x^2 - 2x + 1)(x^2+1)$$
$$= 2(x^4 + x^2 - 2x^3 - 2x + x^2 + 1) = 2x^4 - 4x^3 + 4x^2 - 4x + 2$$

So the function G is a polynomial function of degree 4.
26. Do yourself.

Use transformations of the graph of $y = x^4$ or $y = x^5$ to graph each of the following function:

27. $f(x) = (x+1)^4$ **28.** $f(x) = (x-2)^5$ **29.** $f(x) = x^5 - 3$ **30.** $f(x) = x^4 + 2$

31. $f(x) = \frac{1}{2}x^4$ **32.** $f(x) = 3x^5$ **33.** $f(x) = -x^5$ **34.** $f(x) = -x^4$

35. $f(x) = (x-1)^5 + 2$ **36.** $f(x) = (x+2)^4 - 3$ **37.** $f(x) = 2(x+1)^4 + 1$

38. $f(x) = \frac{1}{2}(x-1)^5 - 2$ **39.** $f(x) = 4 - (x-2)^5$ **40.** $f(x) = 3 - (x+2)^4$

Solution: Collect the Figures from the part **Answers** given at the last part of the **Textbook.**

27. Sketch of the solution:

Step I: Draw the original graph of $y = f(x) = x^4$. Since power of x is an even integer, the graph contains the points $(-1,1)$, $(0,0)$ and $(1,1)$.

Step II: To graph $y = f(x) = (x+1)^4$, use the transformation of horizontal shift 1 unit left and then the graph contains the points $(-2,1)$, $(-1,0)$ and $(0,1)$.

See Textbook page AN 16

28. Sketch of the solution:

Step I: Draw the original graph of $y = f(x) = x^5$. Since power of x is an odd integer, the graph contains the points $(-1,-1)$, $(0,0)$ and $(1,1)$.

Step II: To graph $f(x) = (x-2)^5$, use the transformation horizontal shift 2 unit right and then the graph contains the points $(1,-1)$, $(2,0)$ and $(3,1)$.

See Textbook page AN 16

29. Sketch of the solution:

Step I: Draw the original graph of $y = f(x) = x^5$. Since power of x is an odd integer, the graph contains the points $(-1,-1)$, $(0,0)$ and $(1,1)$.

Step II: To graph $f(x) = x^5 - 3$, use the transformation of vertical shift 3 unit down and then the graph contains the points $(-1,-4)$, $(0,-3)$ and $(1,-2)$.

See Textbook page AN 16

37. Sketch of the solution:

Step I: Draw the original graph of $y = f(x) = x^4$. Since power of x is an even integer, the graph contains the points $(-1,1)$, $(0,0)$ and $(1,1)$.

Step II: To graph $y = f(x) = (x+1)^4$, use the transformation of horizontal shift left by 1 unit and then the graph contains the points $(-2,1)$, $(-1,0)$ and $(0,1)$.

Step III: To graph $f(x) = 2(x+1)^4$, use the transformation of stretching by a factor of 2 unit and then the graph contains the points $(-2,2)$, $(-1,0)$ and $(0,2)$.

Step IV: To graph $f(x) = 2(x+1)^4 + 1$, use the transformation of vertical shift up by 1 unit and then the graph contains the points $(-2,3)$, $(-1,1)$ and $(0,3)$.

See Textbook page AN 16

39. Sketch of the solution:

Step I: Draw the original graph of $y = f(x) = x^5$. Since power of x is an odd integer, the graph contains the points $(-1,-1)$, $(0,0)$ and $(1,1)$.

Step II: To graph $y = f(x) = -x^5$, use the transformation of reflection about the x-axis, and then the graph contains the points $(-1,1)$, $(0,0)$ and $(1,-1)$.

Step III: To graph $y = f(x) = -(x-2)^5$, use the transformation of horizontal shift right by 2 unit and then the graph contains the points $(1,1)$, $(2,0)$ and $(3,-1)$.

Step IV: To graph $f(x)=4-(x-2)^5$, use the transformation of vertical shift up by 4 unit and then the graph contains the points $(1,5)$, $(2,4)$ and $(3,3)$.

See Textbook page AN 16

In Problems 41–48, form a polynomial function whose real zeros and degree are given. Answers will vary depending on the choice of a leading coefficient.

41. Zeros: $-1,1,3$; degree 3

42. Zeros: $-2,2,3$; degree 3

43. Zeros: $-3,0,4$; degree 3

44. Zeros: $-4,0,2$; degree 3

45. Zeros: $-4,-1,\ 2,\ 3$; degree 4

46. Zeros: $-3,-1,\ 2,\ 5$; degree 4

47. Zeros: -1; multiplicity 1; 3, multiplicity 2, degree 3

48. Zeros: -2; multiplicity 2; 4, multiplicity 1, degree 3

Solutions to the Problems

41. Zeros: $-1,1,3$; degree 3

Solution: Since zeros of f are $-1,1$ and 3, the polynomial is

$$f(x)=(x+1)(x-1)(x-3)=(x^2-1)(x-3)=x^3-3x^2-x+3 \text{ where } a=1$$

43. Zeros: $-3,0,4$; degree 3

Solution: Since zeros of f are $-3,0$ and 4, the polynomial is

$$f(x)=(x+3)(x-0)(x-4)=(x^2+3x)(x-4)=x^3+3x^2-4x^2-12x$$
$$=x^3-x^2-12x \quad \text{where } a=1$$

45. Zeros: $-4,-1,\ 2,\ 3$; degree 4

Solution: Since zeros of f are $-4,-1,\ 2$ and 3, the polynomial is

$$f(x)=(x+4)(x+1)(x-2)(x-3)=(x^2+5x+4)(x^2-5x+6)$$
$$=x^4-15x^2+10x+24 \quad \text{where } a=1$$

47. Zeros: -1; multiplicity 1; 3, multiplicity 2, degree 3

Solution: Using the given information, the polynomial is of the form

$$f(x)=(x+1)(x-3)^2=(x+1)(x^2-6x+9)$$
$$=x^3-5x^2+3x+9 \quad \text{where } a=1$$