**Makeup Class**

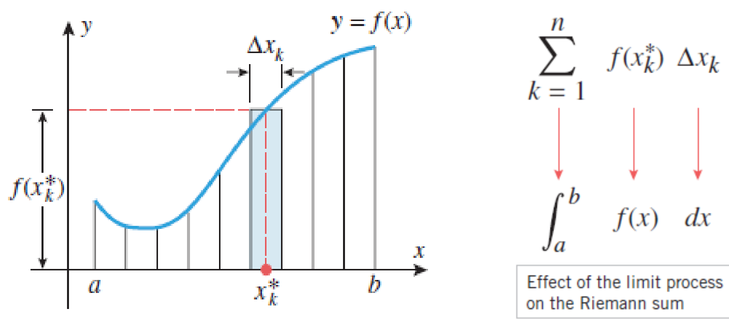
**23rd March, Tuesday at 8pm**

**MAT-130, Lecture-9**

**Chapter 6: Applications of Integration**

**Section 6.1 Area Between Two Curves**

**MAT-120: A REVIEW OF RIEMANN SUMS (AREA UNDER THE CURVE $y = f(x)$ on $[a, b]$ )**



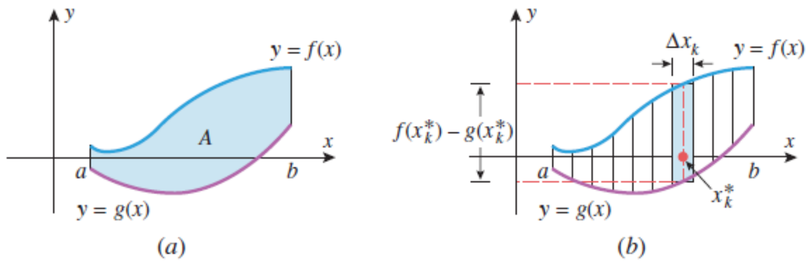

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

Here $n \to \infty$.

Note that $g(x) = 0$ is the lower boundary. Hence, $A = \int_a^b [f(x) - 0]dx = \int_a^b f(x)\,dx$

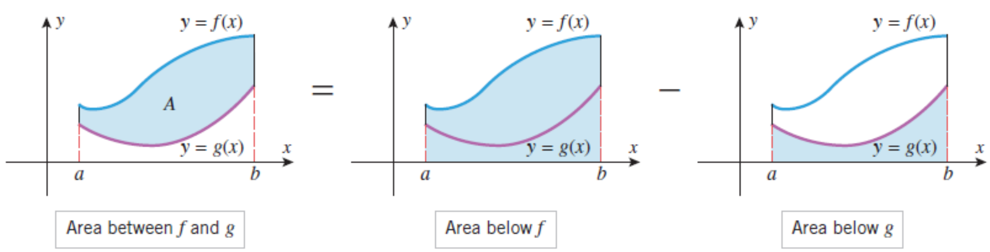
**AREA BETWEEN $y = f(x)$ AND $y = g(x)$**

> **6.1.1 FIRST AREA PROBLEM** Suppose that $f$ and $g$ are continuous functions on an interval $[a, b]$ and
>
> $$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$
>
> [This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area $A$ of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$

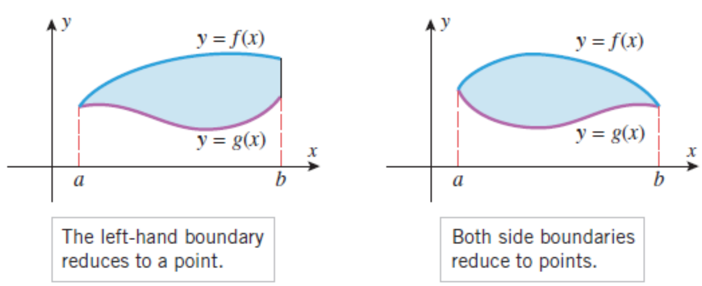



> **6.1.2 AREA FORMULA** If $f$ and $g$ are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all $x$ in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is
>
> $$A = \int_a^b [f(x) - g(x)]\, dx \tag{1}$$

### GEOMERTICAL INTERPRETATION

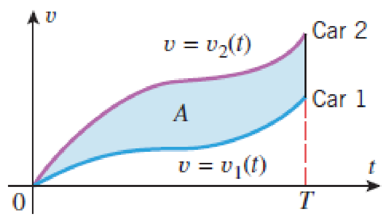



### POINT AS A BOUNDARY




### Example 1 [Worked out example from book]

The figure below shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area $A$ between the curves over the interval $0 \leq t \leq T$.




**Note: Graphing will help you a lot to know the boundaries.**

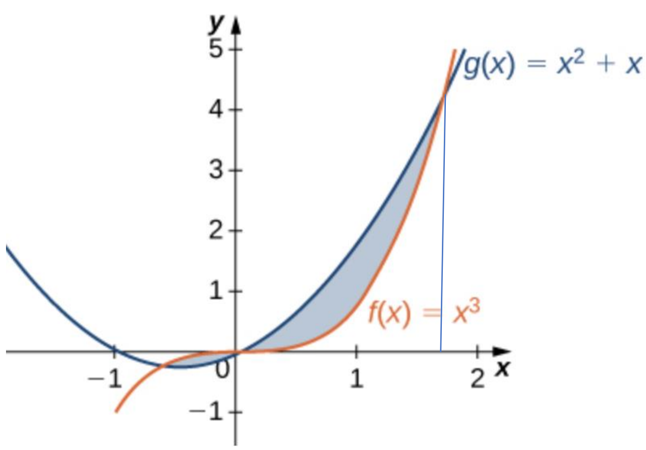
**Formula: Area** $=\int_a^b[\text{Upper boundary} - \text{Lower boundary}]dx$

**Example 2**

Find the area of the region bounded by the curves $f(x) = x^3$ and $g(x) = x^2 + x$ in the **first quadrant.**

Solution: Given $f(x) = x^3$ and $g(x) = x^2 + x$.

Note that $g(x) = x^2 + x$ is a parabola opening upward, with $x -$ intercepts $x = 0, -1$ since $g(x) = x(x + 1)$.




To find the points of intersection, set $f(x) = g(x) \Rightarrow x^3 - x^2 - x = 0$

$$\Rightarrow x(x^2 - x - 1) = 0$$

$$\Rightarrow x = 0, \quad x^2 - x - 1 = 0$$

$$x^2 - x - 1 = 0 \;\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

That is, $x = 0, \ \frac{1+\sqrt{5}}{2}, \ \frac{1-\sqrt{5}}{2}$

Interval: $I = [a, b] = \left[0, \ \frac{1+\sqrt{5}}{2}\right]$

The area of the region is

$$A = \int_0^{\frac{1+\sqrt{5}}{2}} [g(x) - f(x)] \ dx = \int_0^{\frac{1+\sqrt{5}}{2}} [x^2 + x - x^3]dx, \qquad \text{please complete}$$

**Example 3**

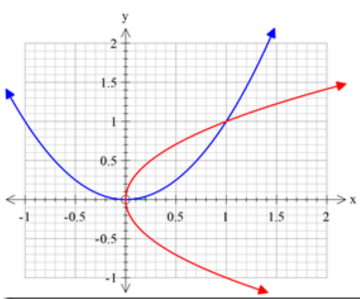
Find the area of the region **bounded** by the curves $y = x^2$ and $y^2 = x$.

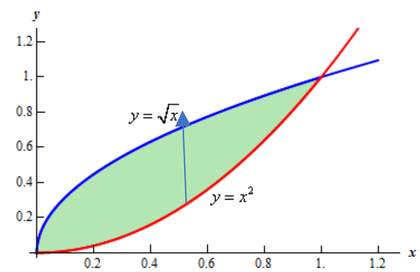
Solution: Given $y = x^2$ and $y^2 = x$

From $y^2 = x$, we get $y = \pm\sqrt{x}$.

Here we get $y = \sqrt{x}$, the half-portion of the red parabola which is above the $x$ −axis [in first quadrant]

And $y = -\sqrt{x}$, the half-portion of the red parabola which is below the $x$ −axis [in forth quadrant]




To find the interval which is given by the points of intersection,

set $x^2 = \sqrt{x} \quad \Rightarrow x^4 = x$

$\Rightarrow x^4 - x = 0 \quad \Rightarrow x(x^3 - 1) = 0$

We get $x = 0, 1$. Interval $= [0, 1]$

The area of the region is

$$A = \int_0^1 [\sqrt{x} - x^2] dx$$

$$= \int_0^1 \left[x^{\frac{1}{2}} - x^2\right] dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} - 0 = \frac{1}{3}\ unit^2$$

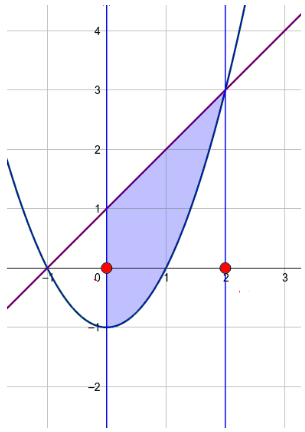
**Example 4**

Find the area of the region **bounded** by the curves $y = x^2 - 1$ and $y = x + 1$ on the interval $[0,2]$.

Solution: Given $y = x^2 - 1$ and $y = x + 1$ on the interval $[0,2]$.

To find the points of intersection:

Set $x^2 - 1 = x + 1 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow \quad x = -1, 2.$



The area of the region is

$$A = \int_0^2 [(x+1) - (x^2 - 1)]dx$$

$$= \int_0^2 [-x^2 + x + 2]dx = -\frac{8}{3} + 2 + 4 = -\frac{8}{3} + 6 = \frac{10}{3}\ unit^2$$

**Example 5**

Find the area of the region **bounded** by the curves $y = \sin x$ and $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

**<u>Solution:</u>**

Given $y = \sin x$ and $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

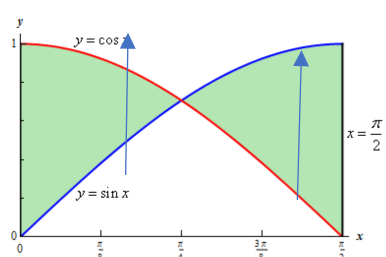
To find the point of intersection, set $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$ ,

which belongs to the given interval.

Intervals are $I_1 = \left[0, \frac{\pi}{4}\right]$ and $I_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

On $I_1 = \left[0, \frac{\pi}{4}\right]$, $y = \cos x$ is the upper boundary

On $I_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $y = \sin x$ is the upper boundary.



$$\text{Area } A = \int_{0}^{\frac{\pi}{4}} [\cos x - \sin x]\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x]\, dx = ??$$

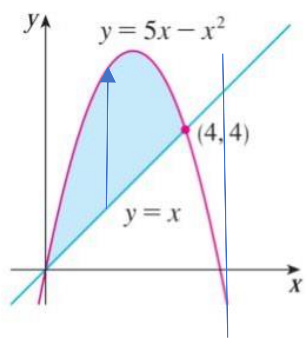
**Finding the Limits of Integration for the Area Between Two Curves**

**Step 1.** Sketch the region and then draw a vertical line segment through the region at an arbitrary point $x$ on the $x$ −axis, connecting the top and bottom boundaries.

**Step 2.** The $y$ −coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

**Step 3.** To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$.

**Example 6 (a)** Find the area of the region bounded by the functions $y = 5x - x^2$ and $y = x$ on the interval $[0,5]$.




Points of intersection $x = 0, 4$

Note that $[0,5] = [0,4] \cup [4,5]$

On the interval $[0,4]$: Upper boundary $y = 5x - x^2$ and lower bounbdary $y = x$

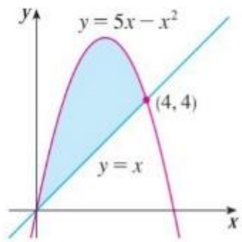
On the interval $[4,5]$: Upper boundary $y = x$ and lower bounbdary $y = 5x - x^2$

$$\text{Area } A = \int_0^4 [5x - x^2 - x]dx + \int_4^5 [x - (5x - x^2)]\, dx$$

Complete!

Note: $\int_4^4 F(x)\, dx = 0$ for any function $F(x)$.

**Example 6** **(b)** Find the area of the shaded region given below.



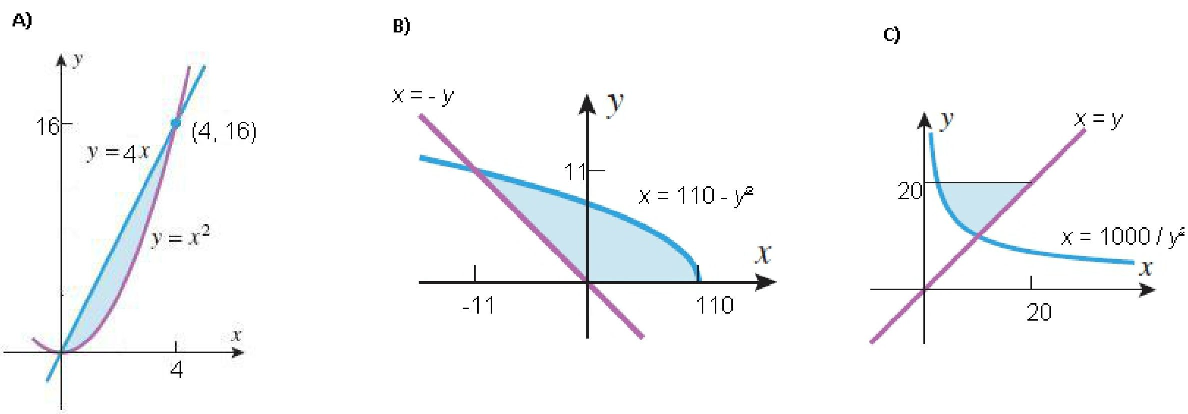

**Solution:**

$$A = \int_0^4 [(5x - x^2) - x] \; dx.$$

Please submit Example 7 by tomorrow, 23rd March, 2021.

**Example 7**

**Find the area of the given graphs in the shaded regions.**

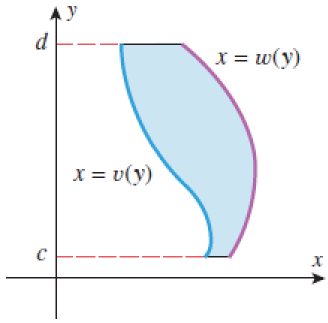


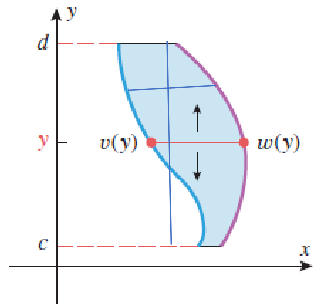

## Lecture-10

## REVERSING THE ROLES OF $x$ AND $y$ : $x = w(y)$, $x = v(y)$

**6.1.4 AREA FORMULA** If $w$ and $v$ are continuous functions and if $w(y) \geq v(y)$ for all $y$ in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \qquad (4)$$




**Formula:** **If the boundaries are given by functions of $y$, then Area =** $\int_c^d [\textbf{Right boundary} - \textbf{Left boundary}]\,dy$

***Finding the Limits of Integration for the Area Between Two Curves***

**Step 1.** Sketch the region and then draw a horizontal line segment through the region at an arbitrary point $y$ on the $y$ –axis, connecting the left and right boundaries.

**Step 2.** The $x$ –coordinate of the right endpoint of the line segment sketched in Step 1 will be $w(y)$, the left one $v(y)$, and the length of the line segment will be $w(y) - v(y)$. This is the integrand in (1).

**Step 3.** To determine the limits of integration, imagine moving the line segment top and then bottom. The bottommost position at which the line segment intersects the region is $y = c$ and the topmost is $y = d$.

**Example 8**

Find the area of the region **bounded** by the curves $y^2 = x$ and $y = x - 2$.

[NOTE: If we consider these curves as functions of $x$, then we get three curves given by

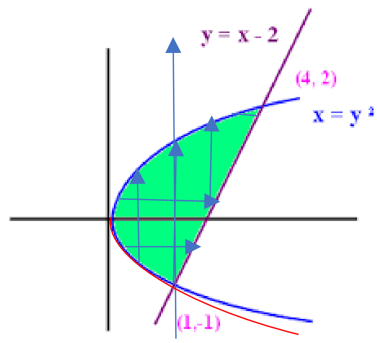
From the equation $y^2 = x \quad \Rightarrow \quad y = \pm\sqrt{x}$

$y = x - 2,\ \ y = \sqrt{x}\,,\ y = -\sqrt{x},\ \ INTERVAL = [0, 4]\ .$

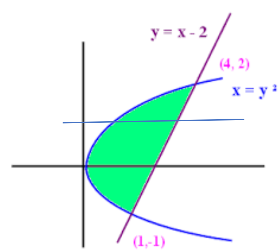
$y = \sqrt{x} \rightarrow$ Upper part of the parabola which is above the x-axis

$y = -\sqrt{x} \rightarrow$ Lower part of the parabola which is below the x-axis

$$A = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_1^4 [\sqrt{x} - (x - 2)]dx\ = ???$$




Alternative method:
Find the area of the region **bounded** by the curves $y^2 = x$ and $y = x - 2$.




Note that $y^2 = x$ is not a function of $x$, but $x = y^2$ and $x = y + 2$ are functions of $y$.
The area of the region is

$$A = \int_{-1}^{2} [(y + 2) - y^2] \, dy$$

**Example 9**

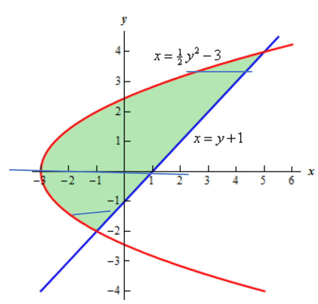
Find the area of the region **bounded** by the curves $x = \frac{1}{2}y^2 - 3$ and $x = y + 1$.

Solution: The region is bounded on the left by $x = v(y) = \frac{1}{2}y^2 - 3$ and on the right by $x = w(y) = y + 1$.

To find the interval, left find the points of intersection.

Set $\frac{1}{2}y^2 - 3 \ = \mathrm{y} + 1 \ \Rightarrow \mathrm{y}^2 - 2y - 8 = 0 \ \Rightarrow (y - 4)(y + 2) = 0, \ i.e., \ y = -2, 4.$

Interval $= [c, d] = [-2, 4]$.

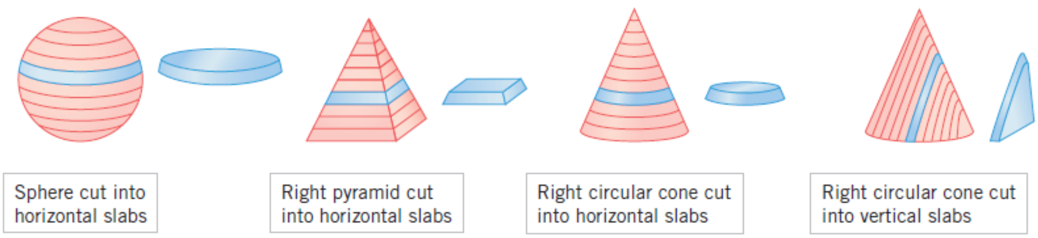


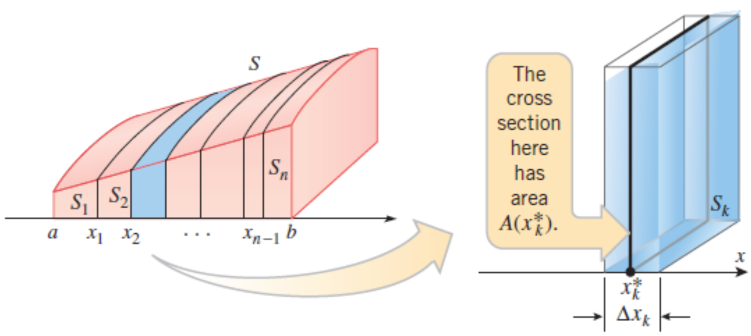
**Area** $=\int_c^d$**[Right boundary − Left boundary]**$dy$

$$= \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy = \int_{-2}^{4} \left[-\frac{1}{2}y^2 + y + 4\right] dy$$

$$= \left[-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right]_{-2}^{4} = \frac{1}{6}\left[-y^3 + 3y^2 + 24y\right]_{-2}^{4} = \frac{1}{6}[-72 + 36 + 144] = -12 + 6 + 24 = 18\, unit^2.$$

## SECTION 6.2: VOLUME BY SLICING: DISC AND WASHER METHOD




Adding these approximations yields the following Riemann sum that approximates the volume $V$:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\, dx$$

In summary, we have the following result.

**6.2.2 VOLUME FORMULA** Let $S$ be a solid bounded by two parallel planes perpendicular to the $x$-axis at $x = a$ and $x = b$. If, for each $x$ in $[a, b]$, the cross-sectional area of $S$ perpendicular to the $x$-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

**6.2.3 VOLUME FORMULA** Let $S$ be a solid bounded by two parallel planes perpendicular to the $y$-axis at $y = c$ and $y = d$. If, for each $y$ in $[c, d]$, the cross-sectional area of $S$ perpendicular to the $y$-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

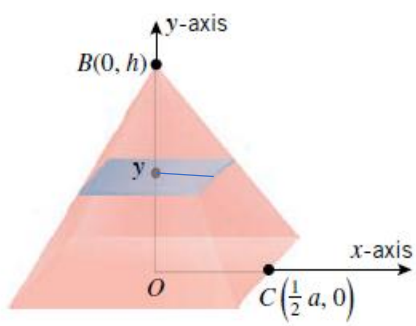
Summary:

The volume of the solid $S$ is given by

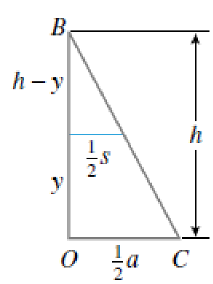
$V = \int_a^b (\text{Area of the cross} - \text{section})\ dx$ Or $V = \int_c^d (\text{Area of the cross} - \text{section})\ dy$

**Example 1**

**Derive the formula** for the volume of a right pyramid whose altitude is $h$ and whose base is a square with sides of length $\boldsymbol{a}$.



(a)



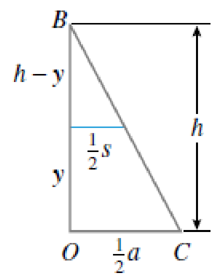
(b)

Project the Pyramid along the y-axis, placing the height of the pyramid along the axis with the center of the base at the origin.

Now, take any cross-section of the pyramid at any $y,\ 0 \le y \le h.$ The cross-section is a square that is perpendicular to the $y$ −axis.

Volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy \ldots \ldots \ldots (1)$

Let the length of a side of the cross-section be $s$. Then the area of the cross-section is
$A(y) = s^2 \ldots \ldots \ldots (2)$
By the similar triangle property on the triangles



(b)

$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \Rightarrow \frac{s}{a} = \frac{h-y}{h}.$ Hence $s = \frac{a}{h}(h-y)$

From equation (2): Area of the cross-section is $A(y) = \left[\frac{a}{h}(h-y)\right]^2 = \frac{a^2}{h^2}(h-y)^2$
Then volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$

$$= \int_0^h \frac{a^2}{h^2}(h-y)^2\ dy$$

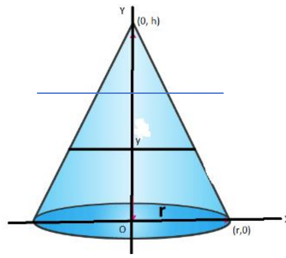
$$= \frac{a^2}{h^2} \int_0^h \left[h^2 - 2hy + y^2\right]\ dy$$

$$= \frac{a^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$

$$V = \frac{1}{3}a^2 h \quad unit^3.$$

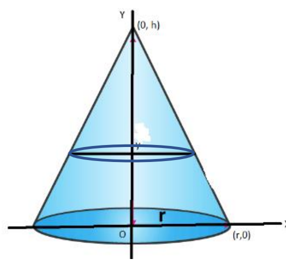
**Example 2**

Derive the formula for the volume of a **right circular cone** whose altitude is $h$ and whose base is a circle of radius $r$.




Solution: Project the right circular cone placing the height of the cone along the $y$ −axis with the center of the base at the origin.

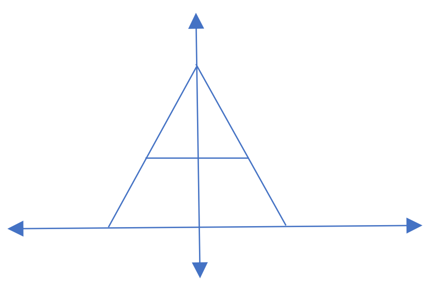
Now, take any cross-section of the cone at any $y,\ 0 \le y \le h.$ The cross-section is a disk that is perpendicular to the $y$ −axis.




$$\text{Volume } V = \int_0^h (\text{Area of the crosssection})\ dy \ \ldots\ldots\ \ldots\ (1)$$

Let the radius of the cross-section be $r_1$. Then the area of the cross-section is

$$A(y) = \pi r_1^2 \ \ldots\ldots\ldots (2)$$



By the similar triangle property on the triangles

$$\frac{2r_1}{2r} = \frac{h-y}{h} \ \Rightarrow \frac{r_1}{r} = \frac{h-y}{h}. \quad \text{Hence} \quad r_1 = \frac{r}{h}\,(h-y)$$

From equation (2): Area of the cross-section is $A(y) = \pi\left[\frac{r}{h}\,(h-y)\right]^2 = \pi\frac{r^2}{h^2}(h-y)^2$

Then volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$

$$= \pi \int_0^h \frac{r^2}{h^2}(h-y)^2\ dy$$

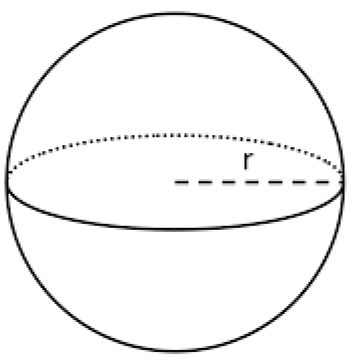
$$= \frac{\pi r^2}{h^2} \int_0^h [h^2 - 2hy + y^2]\ dy$$

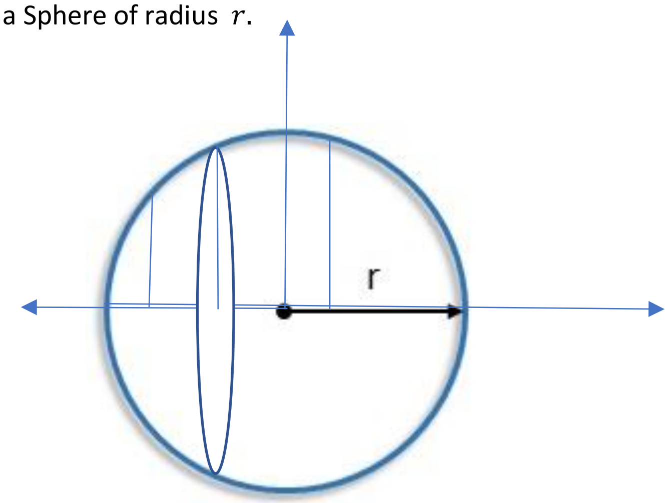
$$= \frac{\pi r^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$

$$V = \frac{\pi}{3} r^2 h \quad unit^3.$$

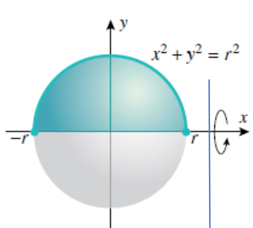
**Example 3**

Derive the formula for the volume of a Sphere of radius $r$.




The projection of the sphere on the $xy$ −plane is a disk of radius r, bounded by the circle $x^2 + y^2 = r^2$.
But $x^2 + y^2 = r^2$ is not a function. The upper-half circle represents a function of $x$ given by
$y = \sqrt{r^2 - x^2}$.



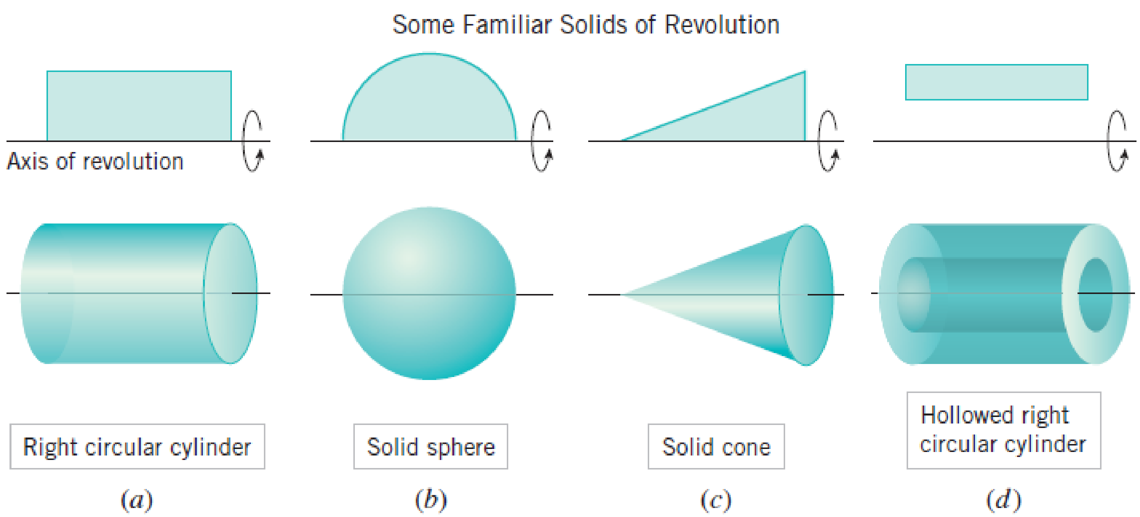

Interval = $[-r, r]$ , The cross-section at any $x$, $-r \le x \le r$, is a disk of radius, **say $r_1$** .
Here $r_1 = \sqrt{r^2 - x^2} \ - 0 = \ \sqrt{r^2 - x^2}$.
Area of the cross-section $A(x) = \pi r_1^2 = \pi(r^2 - x^2)$.

volume $V = \int_a^b (\text{Area of the cross} - \text{section})\ dx$
$V = \int_{-r}^{r} \pi\left(r^2 - x^2\right)\ dx = \frac{4}{3}\pi r^3$. [complete !!]

[Note: $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$
Lower-half circle$y = -\sqrt{r^2 - x^2}$,
Upper-half circle $y = \sqrt{r^2 - x^2}$ ]

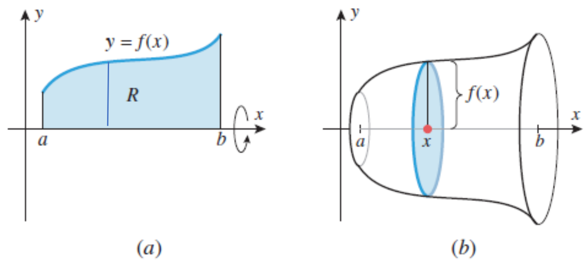
## SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type




## VOLUMES BY DISKS PERPENDICULAR TO THE $x$ –AXIS

Let $f$ be continuous and nonnegative on $[a, b]$, and let $R$ be the region that is bounded above by $y = f(x)$, below by the $x$ –axis, and on the sides by the vertical lines *x* = *a* and *x* = *b*. Then the volume of the solid of revolution that is generated by revolving the region *R* about the $x$ –axis is given by

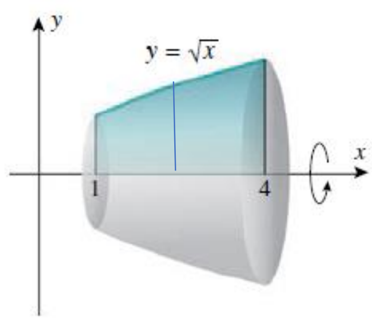



The cross-section is a disk of radius $r = f(x)$ that is perpendicular to the $x$ —axis. Hence, the volume is

$$V = \int_a^b \pi[f(x)]^2 dx$$

**Example 4** Find the volume of the solid that is obtained when the region **under the curve** $y = \sqrt{x}$ **over** the interval $[\mathbf{1}, \mathbf{4}]$ is revolved about the $x$ —axis.

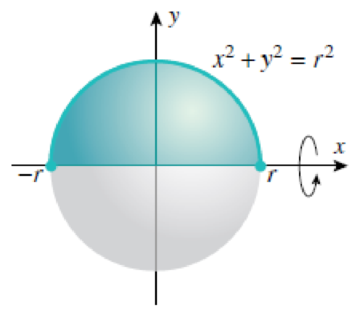
**Solution:**




The Volume $V = \int_a^b \pi[f(x)]^2 dx = \int_1^4 \pi[\sqrt{x}]^2 dx$ complete!!!

Homework

**Example 5** Find the volume of the solid generated by revolving the circle $x^2 + y^2 = r^2$ about the $x$ —axis.

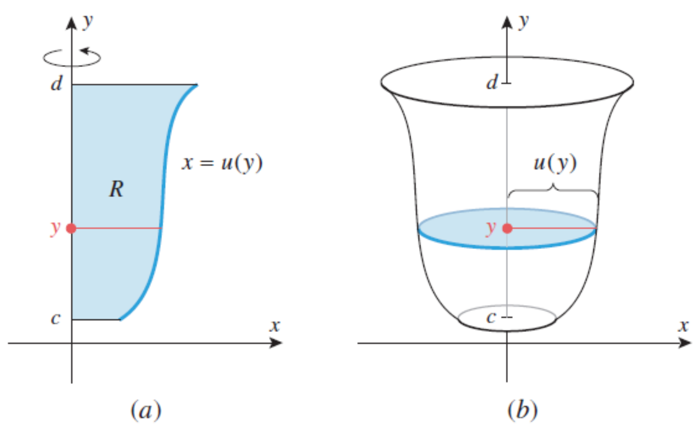



### VOLUMES BY DISKS PERPENDICULAR TO THE $y$ −AXIS

Let $x = u(y)$ be continuous and nonnegative on $[c, d]$, and let $R$ be the region that is bounded on the right by $x = u(y)$, on the left by the $y$ −axis, and at the bottom and top by the horizontal lines $y = c$ and $y = d$. Then the volume of the solid of revolution that is generated by revolving the region $R$ about the $y$ −axis is given by

$$V = \int_c^d \pi[u(y)]^2 dy$$

Note that the cross-section is a disk of radius $r = u(y)$ that is perpendicular to the $y$ −axis.



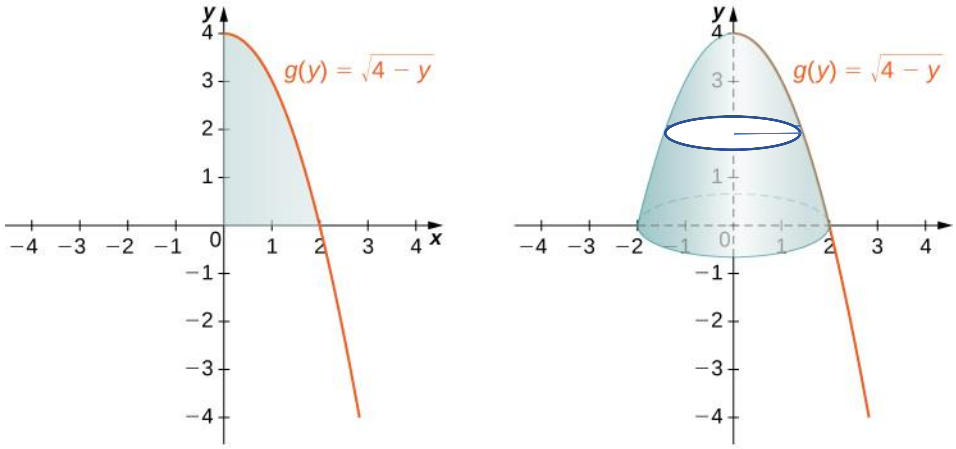

Disks

**Information comes from the origin $R$ . Interval, radius, height come from the region.**

**Example 6 [Complete !!!]**

Find the volume of the solid that is obtained when the region $R$ is revolved about the $\boldsymbol{y}$ **−axis,** where $R$ is bounded by the curve $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{y}) = \sqrt{4 - y}$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]



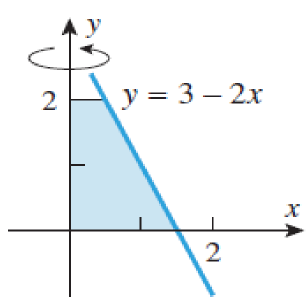

(a) (b)

$V = \int_c^d \pi[u(y)]^2 dy = \int_0^4 \pi\left[\sqrt{4-y}\right]^2 dy$ complete!

**Example 7**

Find the volume of the solid that is obtained when the region $R$ revolved about the $y$ −axis, where $R$ is bounded by the curve $y = 3 - 2x$, $y = 2$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]




**Interval, Shape of the cross-section, Area of the cross-section, Variable of the function that gives you the area of the cross-section.**

$$\boldsymbol{y = 3 - 2x}\ \textbf{. That is,} \qquad \boldsymbol{x = \frac{1}{2}(3 - y)}$$

**Radius of the cross-section is** $\boldsymbol{= \frac{1}{2}(3 - y)}$

$$I = [0, 2], \text{Cross} - \text{section is a disk}, \qquad A(y) = \pi\left(\frac{3}{2} - \frac{y}{2}\right)^2$$