Calculus and Analytical Geometry-I
(Introduction, Functions, Families of Functions)

Introduction: One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a "function," which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus, including polynomials, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.

0.1 Functions: In this section we will define and develop the concept of a "function," which is the basic mathematical object that scientists and mathematicians use to describe relationships between variable quantities. Functions play a central role in calculus and its applications.

Definition: If a variable ***y*** depends on a variable ***x*** in such a way that each value of ***x*** determines exactly one value of ***y***, then we say that ***y*** is a function of ***x***.

Four common methods for representing functions are:

- **(i)** Numerically by tables
- **(ii)** Geometrically by graphs
- **(iii)** Algebraically by formulas
- **(iv)** Verbally

Definition: A function ***f*** is a rule that associates a unique output with each input. If the input is denoted by ***x***, then the output is denoted by ***f* (*x*)** (read "***f*** of ***x***").

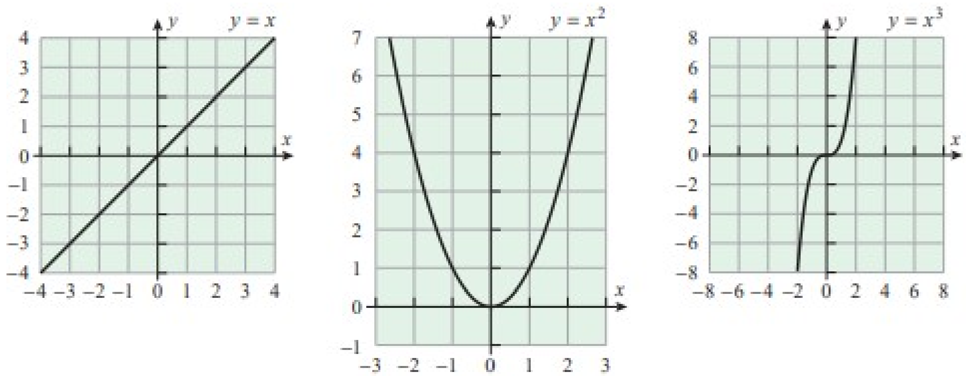
Example 2: The equation

$$y = 3x^2 - 4x + 2$$

has the form $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ in which the function ***f*** is given by the formula

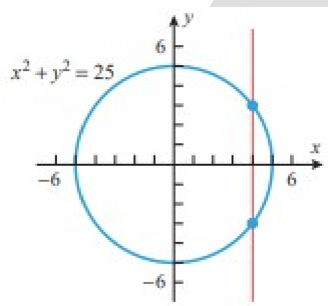
$$f(x) = 3x^2 - 4x + 2$$

Graphs of Function: If f is a real-valued function of a real variable, then the graph of f in the xy-plane is defined to be the graph of the equation $y = f(x)$.



The Vertical Line Test: A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.

Example 3: The graph of the equation $x^2 + y^2 = 25$ is a circle of radius 5 centered at the origin and hence there are vertical lines that cut the graph more than once. Thus this equation does not define y as a function of x.



The Absolute Value Function: Recall that the absolute value or magnitude of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Properties of absolute value: If a and b are real numbers, then

(a)	$|-a| = |a|$	A number and its negative have the same absolute value.
(b)	$|ab| = |a|\,|b|$	The absolute value of a product is the product of the absolute values.
(c)	$|a/b| = |a|/|b|, b \neq 0$	The absolute value of a ratio is the ratio of the absolute values.
(d)	$|a + b| \leq |a| + |b|$	The ***triangle inequality***

Note: A statement that is correct for all real values of x is

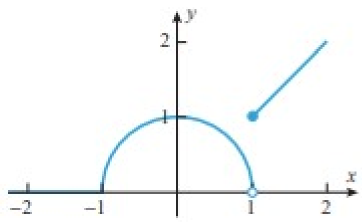
$$\sqrt{x^2} = |x|$$

Piecewise-Defined Functions: The absolute value function $\boldsymbol{f(x) = |x|}$ is an example of a function that is defined piecewise in the sense that the formula for f changes, depending on the value of ***x***.

Example 4: Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0, & x \le -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \ge 1 \end{cases}$$

Solution:



Domain and Range: If ***x*** and ***y*** are related by the equation $\boldsymbol{y = f(x)}$, then the set of all allowable inputs (***x***-values) is called the domain of ***f***, and the set of outputs (***y***-values) that result when ***x*** varies over the domain is called the range of ***f***.

Definition: If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the natural domain of the function.

Example 6: Find the natural domain of

(a) $f(x) = x^3$ (b) $f(x) = 1/[(x-1)(x-3)]$

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

Solution:

(*a*) The function f has real values for all real x, so its natural domain is the interval $(-\infty, \infty)$.

(*b*) The function f has real values for all real x, except x = 1 and x = 3, where divisions by zero occur. Thus, the natural domain is $\{x: x \neq 1 \ and \ x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, \infty)$

(*c*) Since $f(x) = tanx = \frac{sinx}{cosx}$, the function f has real values except where cos x = 0, and this occurs when x is an odd integer multiple of π/2. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

(*d*) The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x-3)(x-2) \geq 0$$

This inequality is satisfied if $x \leq 2 \ or \ x \geq 3$, so the natural domain of f is $(-\infty, 2] \cup [3, \infty)$

Home Work: Exercise 0.1: Problem No. 7 - 10; 15-18, 23-24 and 27(a)

0.2 New Functions From Old: Just as numbers can be added, subtracted, multiplied, and divided to produce other numbers, so functions can be added, subtracted, multiplied, and divided to produce other functions. In this section we will discuss these operations and some others that have no analogs in ordinary arithmetic.

Arithmetic Operations on Functions: Given functions ***f*** and ***g***, we define

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\\ (f/g)(x) &= f(x)/g(x)\end{aligned}$$

For the functions $\boldsymbol{f+g},\ \boldsymbol{f-g}\ \boldsymbol{and}\ \boldsymbol{fg}$ we define the domain to be the intersection of the domains of ***f*** and ***g***, and for the function $\frac{\boldsymbol{f}}{\boldsymbol{g}}$ we define the domain to be the intersection of the domains of ***f*** and ***g*** but with the points where $\boldsymbol{g(x) = 0}$ excluded (to avoid division by zero).

Example 2: Show that if $\boldsymbol{f(x) = \sqrt{x},\ \ g(x) = \sqrt{x}\ \ and\ h(x) = x}$, then the domain of $\boldsymbol{fg}$ is not the same as the natural domain of $\boldsymbol{h}$.

Solution: The natural domain of $h(x) = x \ is\ (-\infty,\ \infty)$. Note that

$$(fg)(x) = \sqrt{x}\sqrt{x} = x = h(x)$$

on the domain of $\boldsymbol{fg}$. The domains of both ***f*** and ***g*** are $[0, \infty)$, so the domain of $\boldsymbol{fg}$ is

$$[0, +\infty) \cap [0, +\infty) = [0, +\infty)$$

Since the domains of $\boldsymbol{fg}$ and $\boldsymbol{h}$ are different, it would be misleading to write $\boldsymbol{(fg)(x) = x}$ without including the restriction that this formula holds only for $\boldsymbol{x \geq 0}$.

Composition of Functions: Given functions ***f*** and ***g***, the composition of ***f*** with ***g***, denoted by $\boldsymbol{f \circ g}$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $\boldsymbol{f \circ g}$ is defined to consist of all ***x*** in the domain of ***g*** for which $\boldsymbol{g(x)}$ is in the domain of ***f***.

Example 3: Let $f(x) = x^2 + 3 \ and \ g(x) = \sqrt{x}$, Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution: (*a*) $(f \circ g)(x) = f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x})^2 + 3 = x + 3$

Since the domain of $\boldsymbol{g}$ is $[\mathbf{0}, \infty)$ and the domain of $\boldsymbol{f}$ is $(-\infty, \infty)$, the domain of $\boldsymbol{f \circ g}$ consists of all $\boldsymbol{x}$ in $[0, \infty)$ such that $\boldsymbol{g}(\boldsymbol{x}) = \sqrt{\boldsymbol{x}}$ lies in $(-\infty, \infty)$; thus, the domain of $\boldsymbol{f \circ g}$ is $[0, \infty)$. Therefore

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

(*b*) $(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$

Since the domain of $\boldsymbol{f}$ is $(-\infty, \infty)$ and the domain of $\boldsymbol{g}$ is $[0, \infty)$, the domain of $\boldsymbol{g \circ f}$ consists of all $\boldsymbol{x}$ in $(-\infty, \infty)$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^2 + \mathbf{3}$ lies in $[0, \infty)$. Thus, the domain of $\boldsymbol{g \circ f}$ is $(-\infty, \infty)$. Therefore,

$$(g \circ f)(x) = \sqrt{x^2 + 3}$$

There is no need to indicate that the domain is $(-\infty, \infty)$, since this is the natural domain of $\sqrt{x^2 + 3}$.

Expressing Function as a Composition: Many problems in mathematics are solved by "decomposing" functions into compositions of simpler functions. For example, consider the function $\boldsymbol{h}$ given by

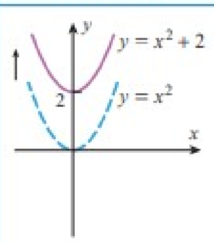
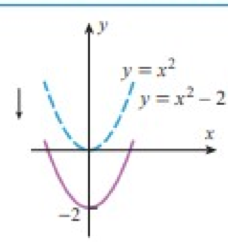
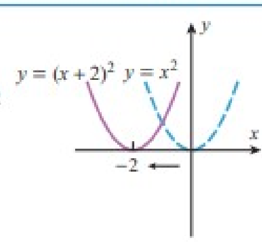
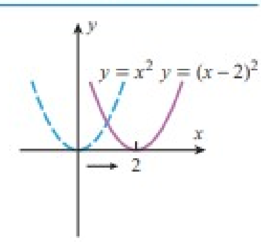
$$h(x) = (x + 1)^2$$

Now let $g(x) = x + 1 \ and \ f(x) = x^2$

$$\therefore \ h(x) = (x + 1)^2 = [g(x)]^2 = f(g(x)).$$

Translations:

TRANSLATION PRINCIPLES

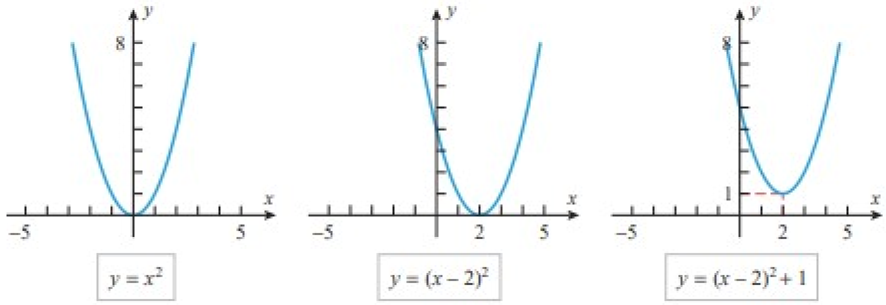
OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				

Example 8: Sketch the graph of $y = x^2 - 4x + 5$.

Solution: Completing the square on the first two terms yields

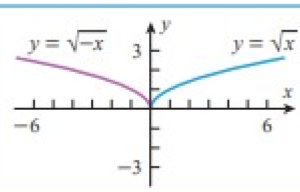
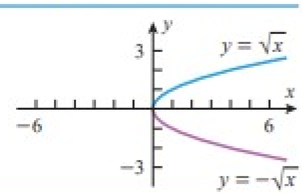
$$y = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$$

In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $\boldsymbol{x-2}$, and up 1 unit because of the +1.



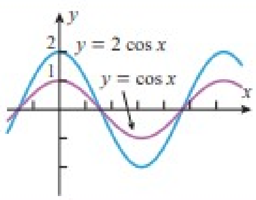
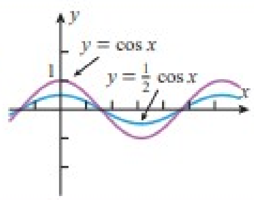
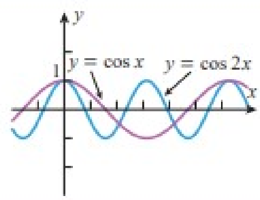
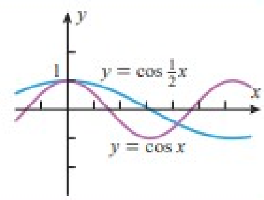
Reflections: The graph of $\boldsymbol{y = f(-x)}$ is the reflection of the graph of $\boldsymbol{y = f(x)}$ about the $\boldsymbol{y}$-axis because the point $(\boldsymbol{x}, \boldsymbol{y})$ on the graph of $\boldsymbol{f(x)}$ is replaced by $(\boldsymbol{-x}, \boldsymbol{y})$. Similarly, the graph of $\boldsymbol{y = -f(x)}$ is the reflection of the graph of $\boldsymbol{y = f(x)}$ about the $\boldsymbol{x}$-axis because the point $(\boldsymbol{x}, \boldsymbol{y})$ on the graph of $\boldsymbol{f(x)}$ is replaced by $(\boldsymbol{x}, \boldsymbol{-y})$.

REFLECTION PRINCIPLES

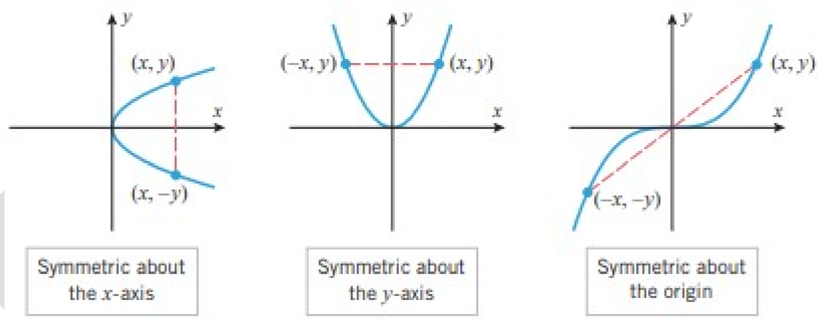
OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
EXAMPLE		

Stretches and Compressions:

STRETCHING AND COMPRESSING PRINCIPLES

OPERATION ON $y = f(x)$	Multiply $f(x)$ by c $(c > 1)$	Multiply $f(x)$ by c $(0 < c < 1)$	Multiply x by c $(c > 1)$	Multiply x by c $(0 < c < 1)$
NEW EQUATION	$y = cf(x)$	$y = cf(x)$	$y = f(cx)$	$y = f(cx)$
GEOMETRIC EFFECT	Stretches the graph of $y = f(x)$ vertically by a factor of c	Compresses the graph of $y = f(x)$ vertically by a factor of $1/c$	Compresses the graph of $y = f(x)$ horizontally by a factor of c	Stretches the graph of $y = f(x)$ horizontally by a factor of $1/c$
EXAMPLE				

Symmetry: The following figure illustrates three types of symmetries: symmetry about the **x**-axis, symmetry about the **y**-axis, and symmetry about the origin.



Theorem (Symmetry Tests):

(a) A plane curve is symmetric about the ***y***-axis if and only if replacing ***x*** by ***−x*** in its equation produces an equivalent equation.

(b) A plane curve is symmetric about the ***x***-axis if and only if replacing ***y*** by ***−y*** in its equation produces an equivalent equation.

(c) A plane curve is symmetric about the origin if and only if replacing both ***x*** by ***−x*** and ***y*** by ***−y*** in its equation produces an equivalent equation.

Even and Odd Functions: A function ***f*** is said to be an even function if

$$f(-x) = f(x)$$

and is said to be an odd function if

$$f(-x) = -f(x)$$

Home Work: Exercise 0.2: Problem No. 5 - 15; 27-32, 51-54 and 59