Linear System of ODE

A system of n-linear ODE in n-unknowns:

$$
\begin{aligned}
&P_{11}(D)x_1 + P_{12}(D)x_2 + \cdots + \cdots + P_{1n}(D)x_n = b_1(t)\\
&P_{21}(D)x_1 + P_{22}(D)x_2 + \cdots + \cdots + P_{2n}(D)x_n = b_2(t)\\
&\qquad\quad \dots \qquad\qquad \dots \qquad\qquad\qquad \dots \qquad\qquad \dots\\
&P_{n1}(D)x_1 + P_{n2}(D)x_2 + \cdots + \cdots + P_{mn}(D)x_n = b_n(t)
\end{aligned}
$$

where the P_{ij} are polynomials of various degrees in the differential operator D.
The system of first-order ODEs in normal form (a special case of the above system) is,

$$
\begin{aligned}
\frac{d}{dt}(x_1) &= g_1(t, x_1, x_2, \dots, x_n) = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\
\frac{d}{dt}(x_2) &= g_2(t, x_1, x_2, \dots, x_n) = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\
&\vdots\\
\frac{d}{dt}(x_n) &= g_n(t, x_1, x_2, \dots, x_n) = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}
$$

If all $f_i(t) = 0$, $i = 1,2, \dots, n$ then the above system is said to be **homogeneous**; otherwise the system is called **non-homogeneous.**

Linear System of ODE

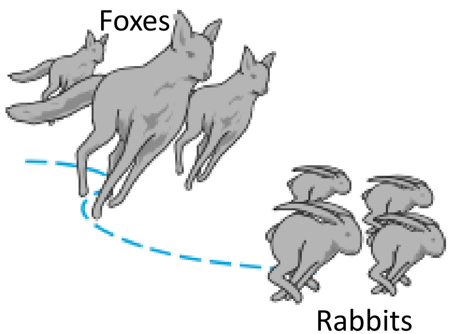
Example 01: Homogeneous Form

$$\begin{aligned}\frac{d}{dt}(x) &= 3x + 4y \\ \frac{d}{dt}(y) &= 5x - 7y\end{aligned} \Rightarrow \begin{aligned}D(x) &= 3x + 4y \\ D(y) &= 5x - 7y\end{aligned} \Rightarrow \begin{aligned}(D-3)x - 4y &= 0 \\ 5x - (D+7)y &= 0\end{aligned}$$

Example 02: Non-homogeneous Form

$$\begin{aligned}\frac{d}{dt}(x) &= 6x + y + z + t \\ \frac{d}{dt}(y) &= 8x + 7y - z + 10t \\ \frac{d}{dt}(z) &= 2x + 9y - z + 6t\end{aligned} \Rightarrow \begin{aligned}D(x) &= 6x + y + z + t \\ D(y) &= 8x + 7y - z + 10t \\ D(z) &= 2x + 9y - z + 6t\end{aligned} \Rightarrow \begin{aligned}(D-6)x - y - z &= t \\ -8x + (D-7)y + z &= 10t \\ -2x - 9y + (D+1)z &= 6t\end{aligned}$$

- **Predator-Prey Population Model:** $$\begin{aligned}\frac{dP_1}{dt} &= aP_1 - bP_1P_2 \\ \frac{dP_2}{dt} &= -mP_2 + nP_1P_2\end{aligned}$$



where P_1: population of rabbits, P_2: population of foxes

a: growth rate of rabbits, b: killing rate of rabbits,

m: death rate of foxes, and n: growth rate of foxes.

Linear System of ODE: Homogeneous Form

Solution of Homogeneous system of ODE by Elimination

Example 01:

$$\begin{aligned}\frac{d}{dt}(x) &= 3x + 4y \\ \frac{d}{dt}(y) &= 5x - 7y\end{aligned} \Rightarrow \begin{aligned}D(x) &= 3x + 4y \\ D(y) &= 5x - 7y\end{aligned} \Rightarrow \begin{aligned}(D-3)x - 4y &= 0 \qquad (1) \\ 5x - (D+7)y &= 0 \qquad (2)\end{aligned}$$

Solution. Multiplying Eq. (1) by 5 and operating Eq. (2) by $(D-3)$ and then subtracting eliminates x from the given system, i.e.,

$$\begin{aligned}5(D-3)x - 20y &= 0 \qquad (1) \\ 5(D-3)x - (D-3)(D+7)y &= 0 \qquad (2)\end{aligned}$$

Now, Eq. (1) $-$Eq. (2) $\Rightarrow$

$(D-3)(D+7)y - 20y = 0 \Rightarrow \{(D+7)(D-3) - 20\}y = 0 \Rightarrow (D^2 + 4D - 41)y = 0$

From which, the auxiliary roots are, $m = -2 \pm 3\sqrt{5} \Rightarrow m_1 = -2 + 3\sqrt{5}, m_2 = -2 - 3\sqrt{5}$

Thus, $y(t) = c_1 e^{(-2+3\sqrt{5})t} + c_2 e^{(-2-3\sqrt{5})t} = e^{-2t}\left(c_1 e^{3\sqrt{5}t} + c_2 e^{-3\sqrt{5}\,t}\right)$

Linear System of ODE: Homogeneous Form

Solution of Homogeneous system of ODE by Elimination

Example 01:

$$\begin{aligned}\frac{d}{dt}(x) &= 3x + 4y \\ \frac{d}{dt}(y) &= 5x - 7y\end{aligned} \Rightarrow \begin{aligned}D(x) &= 3x + 4y \\ D(y) &= 5x - 7y\end{aligned} \Rightarrow \begin{aligned}(D-3)x - 4y &= 0 \qquad (1) \\ 5x - (D+7)y &= 0 \qquad (2)\end{aligned}$$

Solution. Now substituting the solution of $y(t)$ into Eq. (1) we obtain,

$$x(t) = \frac{4}{D-3}y = \frac{4}{D-3}\left[c_1 e^{(-2+3\sqrt{5})t} + c_2 e^{(-2-3\sqrt{5})t}\right]$$

$$= 4\left[\frac{c_1}{\left(-2+3\sqrt{5}\right)-3}e^{(-2+3\sqrt{5})t} + \frac{c_2}{\left(-2-3\sqrt{5}\right)-3}e^{(-2-3\sqrt{5})t}\right]$$

$$= 4\left[\frac{c_1}{-5+3\sqrt{5}}e^{(-2+3\sqrt{5})t} + \frac{c_2}{-5-3\sqrt{5}}e^{(-2-3\sqrt{5})t}\right].$$

Linear System of ODE: Homogeneous Form

Solution of Homogeneous system of ODE by Elimination

Example 02:

$$Dx + (D + 2)y = 0 \qquad (1)$$
$$(D - 3)x - 2y = 0 \qquad (2)$$

Solution. Operating Eq. (1) by $(D - 3)$ and Eq. (2) by D and then subtracting eliminates x from the given system, i.e.,

$$D(D - 3)x + (D + 2)(D - 3)y = 0 \qquad (1)$$
$$D(D - 3)x - 2D\,y = 0 \qquad (2)$$

Now, Eq. (1) −Eq. (2) ⇒

$(D - 3)(D + 2)y + 2Dy = 0 \Rightarrow \{(D + 2)(D - 3) + 2D\}y = 0 \Rightarrow (D^2 + D - 6)y = 0$

From which, the auxiliary equation is $m^2 + m - 6 = 0 \Rightarrow (m + 3)(m - 2) = 0$

and the auxiliary roots are, $m = -3, 2$

Thus, $y(t) = c_1 e^{-3t} + c_2 e^{2t}$

Linear System of ODE: Homogeneous Form

Solution of Homogeneous system of ODE by Elimination

Example 02:

$$Dx + (D+2)y = 0 \qquad (1)$$
$$(D-3)x - 2y = 0 \qquad (2)$$

Solution. Now substituting the solution of $y(t)$ into Eq. (2) we obtain,

$$x(t) = \frac{2}{D-3}y = \frac{2}{D-3}[c_1e^{-3t} + c_2e^{2t}]$$

$$= 2\left[\frac{c_1}{(-3)-3}e^{-3t} + \frac{c_2}{(2)-3}e^{2t}\right]$$

$$= -\frac{1}{3}c_1e^{-3t} - 2c_2e^{2t}.$$

Linear System of ODE: Non-Homogeneous Form

Solution of Non-homogeneous system of ODE by Elimination

Example 03:

$$\begin{aligned}\frac{dx}{dt} &= -y + t \\ \frac{dy}{dt} &= x - t\end{aligned} \Rightarrow \begin{aligned} Dx + y &= t \qquad (1) \\ x - Dy &= t \qquad (2)\end{aligned}$$

Solution. Operating Eq. (2) by D and Eq. (1) by 1 and then subtracting eliminates x from the given system, i.e.,

$$Dx + y = t \qquad (1)$$

$$Dx - D^2y = 1 \qquad (2)$$

Now, Eq. (1) − Eq. (2) ⇒

$$D^2y + y = t - 1 \Rightarrow (D^2 + 1)y = t - 1$$

From which, the auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Thus, the complementary solution for $y(\mathrm{t})$ is, $y_c(t) = c_1 \cos t + c_2 \sin t$.

The particular solution for $y(t)$, is

$$y_p(t) = \frac{1}{1 + D^2}(t - 1) = (1 + D^2)^{-1}(t - 1) = (1 - \mathrm{D}^2)(\mathrm{t} - 1) = (\mathrm{t} - 1)$$

Linear System of ODE: Non-Homogeneous Form

Solution of Non-homogeneous system of ODE by Elimination

Example 03:

$$\begin{aligned} \frac{dx}{dt} &= -y + t \\ \frac{dy}{dt} &= x - t \end{aligned} \quad \Rightarrow \quad \begin{aligned} Dx + y &= t \qquad (1) \\ x - Dy &= t \qquad (2) \end{aligned}$$

Solution. Therefore, the solution of $y(t)$ becomes,

$$y(t) = y_c(t) + y_p(t) = c_1 \cos t + c_2 \sin t + \mathrm{t} - 1$$

Now substituting the solution of $y(t)$ into Eq. (2) we obtain,

$$x(t) = t + Dy = -c_1 \sin t + c_2 \cos t + t + 1.$$

Linear System of ODE: Non-Homogeneous Form

Solution of Non-homogeneous system of ODE by Elimination

Example 04:

$$D^2x - 4y = e^t \qquad (1)$$
$$4x - D^2y = e^t \qquad (2)$$

Solution. Operating Eq. (1) by D^2 and multiplying Eq. (2) by 4 and then subtracting eliminates y from the given system, i.e.,

$$D^4x - 4D^2y = e^t \qquad (1)$$
$$16x - 4D^2y = 4e^t \qquad (2)$$

Now, Eq. (1) − Eq. (2) ⇒

$$D^4x - 16x = -3e^t \Rightarrow (D^4 - 16)x = -3e^t$$

From which, the auxiliary equation is $m^4 - 16 = 0 \Rightarrow (m^2 + 4)(m^2 - 4) = 0$

And the auxiliary roots are, $m = \pm 2, \pm 2i$.

Thus, the complementary solution for $x(\mathrm{t})$ is,

$$x_c(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos 2t + c_4 \sin 2t.$$

The particular solution for $x(t)$, is

$$x_p(t) = -3\frac{1}{D^4 - 16}(e^t) = -\frac{3}{1^4 - 16}(e^t) = \frac{1}{5}e^t$$

Linear System of ODE: Non-Homogeneous Form

Solution of Non-homogeneous system of ODE by Elimination

Example 04:

$$D^2x - 4y = e^t \qquad (1)$$

$$4x - D^2y = e^t \qquad (2)$$

Solution. Therefore, the solution of $x(t)$ becomes,

$$x(t) = x_c(t) + x_p(t) = c_1e^{2t} + c_2e^{-2t} + c_3\cos 2t + c_4\sin 2t + \frac{1}{5}e^t$$

ow substituting the solution of $y(t)$ into Eq. (1) we obtain,

$$y(t) = \frac{1}{4}(D^2x - e^t) = \frac{1}{4}\left(4c_1e^{2t} + 4c_2e^{-2t} - 4c_3\cos 2t - 4c_4\sin 2t + \frac{1}{5}e^t - e^t\right)$$

$$= c_1e^{2t} + c_2e^{-2t} - c_3\cos 2t - c_4\sin 2t - \frac{1}{5}e^t.$$

Linear System of ODE

Exercise 4.9

Solve the following system of differential equations by systematic elimination.

1. $\dfrac{dx}{dt} = 2x - y$

$\dfrac{dy}{dt} = x$

2. $\dfrac{dx}{dt} = 4x + 7y$

$\dfrac{dy}{dt} = x - 2y$

3. $\dfrac{dx}{dt} = -y + t$

$\dfrac{dy}{dt} = x - t$

4. $\dfrac{dx}{dt} - 4y = 1$

$\dfrac{dy}{dt} + x = 2$

5. $(D^2 + 5)x - 2y = 0$

$-2x + (D^2 + 2)y = 0$

6. $(D + 1)x + (D - 1)y = 2$

$3x + (D + 2)y = -1$

7. $\dfrac{d^2x}{dt^2} = 4y + e^t$

$\dfrac{d^2y}{dt^2} = 4x - e^t$

8. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$

9. $Dx + D^2y = e^{3t}$

$(D + 1)x + (D - 1)y = 4e^{3t}$

10. $D^2x - Dy = t$

$(D + 3)x + (D + 3)y = 2$

11. $(D^2 - 1)x - y = 0$

$(D - 1)x + Dy = 0$

12. $(2D^2 - D - 1)x - (2D + 1)y = 1$

$(D - 1)x + Dy = -1$

13. $2\dfrac{dx}{dt} - 5x + \dfrac{dy}{dt} = e^t$

$\dfrac{dx}{dt} - x + \dfrac{dy}{dt} = 5e^t$

14. $\dfrac{dx}{dt} + \dfrac{dy}{dt} = e^t$

$-\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + y = 0$

15. $(D - 1)x + (D^2 + 1)y = 1$

$(D^2 - 1)x + (D + 1)y = 2$

16. $D^2x - 2(D^2 + D)y = \sin t$

$x + Dy = 0$