Mat-116

logarithmic function

The logarithmic function to the base a, where $a>0$ and $a \neq 1$ is denoted by $y = \log_a x$ and is defined by

$$y = \log_a x \text{ if and only if } x = a^y$$

Example: If $y = \log_3 x$ then $x = 3^y$

i.e $4 = \log_3 81$ then $81 = 3^4$

Similarly if $a^4 = 24$ then $4 = \log_a 24$

Example:

Find the exact value of

i. $\log_2 16$

so $y = \log_2 16$

$\Rightarrow 2^y = 16$

$\Rightarrow 2^y = 2^4$

$\Rightarrow y = 4$

Therefore $\log_2 16 = 4$

ii. $\log_3 \frac{1}{27}$

$y = \log_3 \frac{1}{27}$

$\Rightarrow 3^y = \frac{1}{27}$

$\Rightarrow 3^y = 3^{-3}$

$\Rightarrow y = -3$

Therefore $\log_3 \frac{1}{27} = -3$

Domain & Range:

Domain of logarithmic funⁿ = Range of exponential function $= (0, \infty)$

Range of logarithmic funⁿ = Domain of exponential function $= (-\infty, \infty)$

Ex: Find the domain of $f(x) = \log_2(x+3)$

Sol^n: The domain of f consists of all x for which $x+3>0$

i.e $x>-3$. $\therefore$ Domain $= (-3, \infty)$

Ex: Find the domain of $g(x) = \log_5\left(\frac{1+x}{1-x}\right)$

Sol^n: The domain of g consists of all x for which

$$\frac{1+x}{1-x} > 0$$

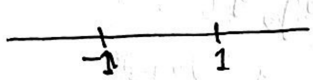
Solving this inequality, the domain of $g(x)$ consists of all x between -1 and 1. i.e $-1<x<1$.

In interval notation $(-1, 1)$.

Do you know, how?

$1+x=0 \Rightarrow x=-1$

$1-x=0 \Rightarrow x=1$



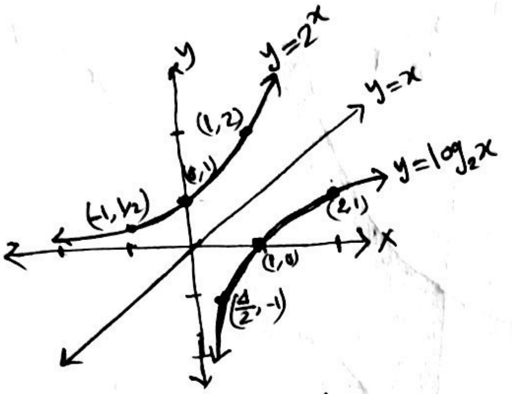
Interval	$(1+x)/(1-x)$	Sign
$x<-1$	$(-)/(+)$	$-$
$-1<x<1$	$(+)/(+)$	$+$
$x>1$	$(+)/(-)$	$-$

Thus $\frac{1+x}{1-x} > 0$ only when x lies between -1 and 1.

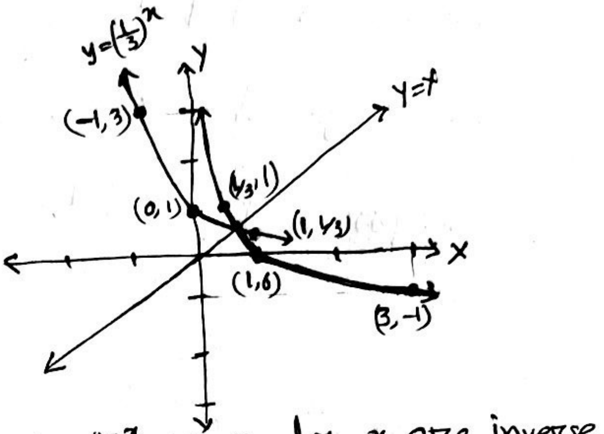
Properties of $\log_a x$:

1. The domain is the set of positive real numbers or $(0, \infty)$.

   The range is set of all real numbers or $(-\infty, \infty)$
2. The x-intercept is 1. There is no y-intercept.
3. The y-axis $(x=0)$ is a vertical asymptote.
4. A logarithmic function is decreasing if $0<a<1$ and increasing if $a>1$.
5. The graph of f contains points $(1,0)$, $(a,1)$ and $\left(\frac{1}{a}, -1\right)$



$y = 2^x$ and $y = \log_2 x$ are inverse functions to each other.

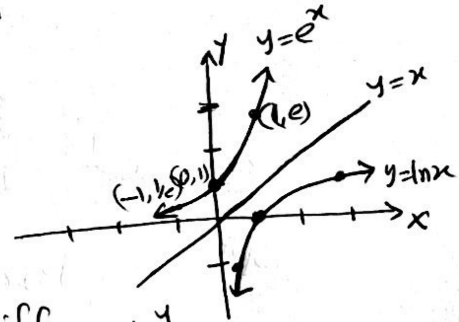


$y = (\frac{1}{3})^x$ and $y = \log_{1/3} x$ are inverse functions to each other.

Natural logarithm:

If the base of a logarithmic function is the number e, then we have the natural logarithm

$y = \ln x$ if and only if $x = e^y$



Note: If the base a of the logarithmic function is not indicated, it is understood to be 10. i.e $y = \log x$ iff $x = 10^y$.

Ex: $f(x) = 3\log(x-1)$

Soln: Domain $= \{x \mid x > 1\}$

Vertical asymptote $x = 1$.

Range $= (-\infty, \infty)$

To find $f^{-1}(x)$ begin with $y = 3\log(x-1)$. Then interchange x and y.

$\therefore\ x = 3\log(y-1)$

$\Rightarrow \frac{x}{3} = \log(y-1)$

$\Rightarrow y - 1 = 10^{x/3}$

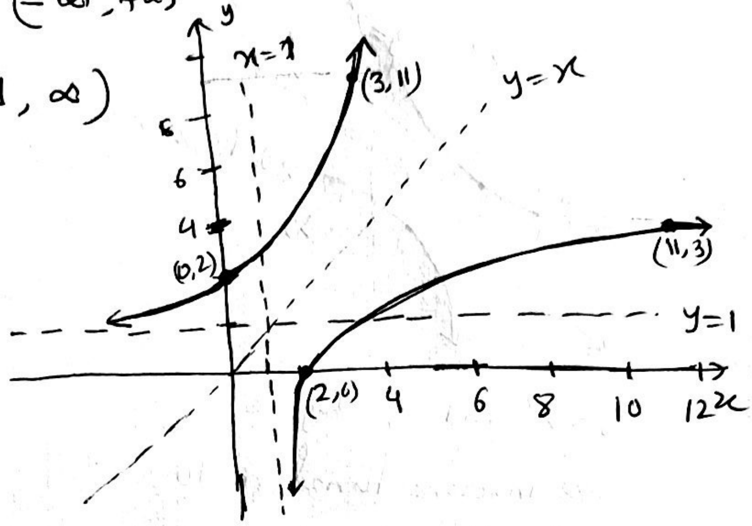
$\Rightarrow y = 10^{x/3} + 1$

$\therefore f^{-1}(x) = 10^{x/3} + 1$

The domain of $f^{-1}(x) = (-\infty, +\infty)$

The range of $f^{-1}(x) = (1, \infty)$

Graph of f and f^{-1}.



Ex: Solve $\log_3(4x-7)=2$

Soln:
$$\log_3(4x-7) = 2$$
$$\Rightarrow 4x-7 = 3^2$$
$$\Rightarrow 4x = 9+7 = 16$$
$$\Rightarrow x = 4.$$

Ex: Solve $\log_x 64 = 2$

Soln: $\log_x 64 = 2 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$

Ex. Solve $e^{2x} = 5$

Soln:
$$e^{2x} = 5$$

we know $y = \ln x$ if $x = e^y$

$$\therefore \ln 5 = 2x$$
$$\Rightarrow 2x = \ln 5 \Rightarrow x = \frac{\ln 5}{2}$$

$\therefore$ The solution set is $\left\{\frac{\ln 5}{2}\right\}$

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Ex: Show that $\log_a 1 = 0$

Soln: $y = \log_a 1$

$\Rightarrow a^y = 1$

$\Rightarrow a^y = a^0$

$\Rightarrow y = 0$

$\therefore \log_a 1 = 0$

Ex: Show that $\log_a a = 1$

Soln: $y = \log_a a$

$\Rightarrow a^y = a$

$\Rightarrow a^y = a^1$

$\Rightarrow y = 1$

$\therefore \log_a a = 1$

Properties of logarithms:

1. $\log_a a^r = r$
2. $a^{\log_a M} = M$
3. $\log_a(MN) = \log_a M + \log_a N$
4. $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
5. $\log_a M^r = r\log_a M$
6. $a^x = e^{x\ln a}$

Ex: $\log_a\left(x\sqrt{x^2+1}\right) = \log_a x + \log_a \sqrt{x^2+1}$

$= \log_a x + \log_a (x^2+1)^{1/2}$

$= \log_a x + \frac{1}{2}\log_a(x^2+1)$; $x>0$

Ex: $\ln \frac{x^2}{(x-1)^3} = \ln x^2 - \ln(x-1)^3$

$= 2\ln x - 3\ln(x-1)$; $x>1$

Ex: $\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} = \log_a\sqrt{x^2+1} - \log_a\left[x^3(x+1)^4\right]$

$= \log_a(x^2+1)^{\frac{1}{2}} - \log_a x^3 - \log_a(x+1)^4$

$= \frac{1}{2}\log_a(x^2+1) - 3\log_a x - 4\log_a(x+1)$; $x>0$

Ex: $\log_a 7 + 4\log_a 3 = \log_a 7 + \log_a 3^4$ $\quad r\log_a M = \log_a M^r$

$= \log_a 7 + \log_a 81$

$= \log_a (7 \cdot 81)$ $\quad \log_a M + \log_a N = \log_a (MN)$

$= \log_a 567$

Ex: $\frac{2}{3}\ln 8 - \ln(5^2 - 1) = \ln 8^{2/3} - \ln(25 - 1)$

$8^{2/3} = (\sqrt[3]{8})^2$
$= 2^2$
$= 4$

$= \ln 4 - \ln 24$

$= \ln\left(\frac{4}{24}\right) = \ln\left(\frac{1}{6}\right)$

$= \ln 1 - \ln 6 = -\ln 6.$

If $M = N$, then $\log_a M = \log_a N$

If $\log_a M = \log_a N$ then $M = N$

Ex: Approximate $\log_2 7$.

Soln: $y = \log_2 7$

$\Rightarrow 2^y = 7$

$\Rightarrow \ln 2^y = \ln 7$

$\Rightarrow y \ln 2 = \ln 7 \Rightarrow y = \frac{\ln 7}{\ln 2}$

$\therefore y \approx 2.8074.$

If $a \neq 1, b \neq 1$ then $\log_a M = \frac{\log_b M}{\log_a a}$ $\quad$ Either $b = 10$ or $b = e$.

Ex: $\log_5 89 = \frac{\log 89}{\log 5} \approx 2.7889$

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Ex: Solve $2\log_5 x = \log_5 9$

Soln:
$$2\log_5 x = \log_5 9$$
$$\Rightarrow \log_5 x^2 = \log_5 9$$
$$\Rightarrow x^2 = 9$$
$$\therefore x = +3 \text{ or } x = -3$$

Since the domain is $x > 0$, so we discard $x = -3$.

Ex: $\log_5(x+6) + \log_5(x+2) = 1$

Soln. The domain of the variable requires that $x+6 > 0$ and $x+2 > 0$, so $x > -6$ and $x > -2$. This means any solution must satisfy $x > -2$.

$$\therefore \log_5(x+6) + \log_5(x+2) = 1$$
$$\Rightarrow \log_5[(x+6)(x+2)] = 1$$
$$\Rightarrow (x+6)(x+2) = 5^1$$
$$\Rightarrow x^2 + 6x + 2x + 12 = 5$$
$$\Rightarrow x^2 + 8x + 7 = 0$$
$$\Rightarrow x^2 + 7x + x + 7 = 0 \Rightarrow x(x+7) + 1(x+7) = 0$$
$$\Rightarrow (x+7)(x+1) = 0$$
$$\therefore x = -1 \text{ or } x = -7$$

only $x = -1$ satisfies the restriction $x > -2$.

$\therefore$ The solution set is $\{-1\}$.

Ex: Solve $5^{x-2} = 3^{3x+2}$

Soln:

$$5^{x-2} = 3^{3x+2}$$

$$\Rightarrow \ln 5^{x-2} = \ln 3^{3x+2}$$

$$\Rightarrow (x-2)\ln 5 = (3x+2)\ln 3$$

$$\Rightarrow x\ln 5 - 2\ln 5 = 3x\ln 3 + 2\ln 3$$

$$\Rightarrow (\ln 5 - 3\ln 3)\, x = 2\ln 5 + 2\ln 3$$

$$\Rightarrow x = \frac{2(\ln 5 + \ln 3)}{\ln 5 - 3\ln 3}$$

$\therefore$ The solution set is $\left\{ \frac{2(\ln 5 + \ln 3)}{\ln 5 - \ln 3} \right\}$

Ex: Solve $4^x - 2^x - 12 = 0$

Soln:

$$4^x - 2^x - 12 = 0$$

$$\Rightarrow (2^2)^x - 2^x - 12 = 0$$

$$\Rightarrow 2^{2x} - 2^x - 12 = 0$$

Let $2^x = u$

$$\therefore u^2 - u - 12 = 0$$

$$\Rightarrow u^2 - 4u + 3u - 12 = 0$$

$$\Rightarrow (u-4)(u+3) = 0$$

$\therefore u - 4 = 0$ or $u + 3 = 0$

$\Rightarrow u = +4$ $\qquad \Rightarrow u = -3$

$\Rightarrow 2^x = 4$ $\qquad \Rightarrow 2^x = -3$ But $2^x > 0$

$\Rightarrow 2^x = 2^2$ $\qquad \therefore$ The only solution set is $\{2\}$

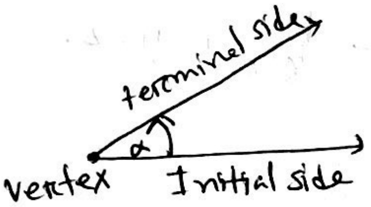
$\therefore x = 2$

Angles and measure

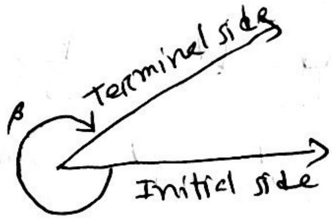
If two rays are drawn with a common vertex, they form an angle.

Rotation counterclockwise → angle positive

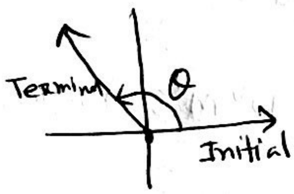
" clockwise → angle negative

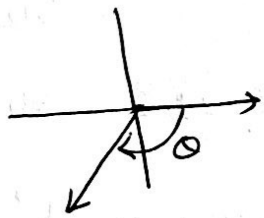


counterclockwise rotation positive angle.



An angle θ is said to be in standard position if its vertex is at origin of a rectangular coordinate system and its initial side coincides with positive x-axis.

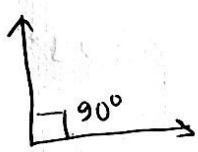




Degrees:

The angle formed by rotating the initial side exactly once in the counterclockwise direction until it coincides with itself (1 revolution) is said to measure 360 degrees.

1 revolution



$\frac{1}{2}$ revolution

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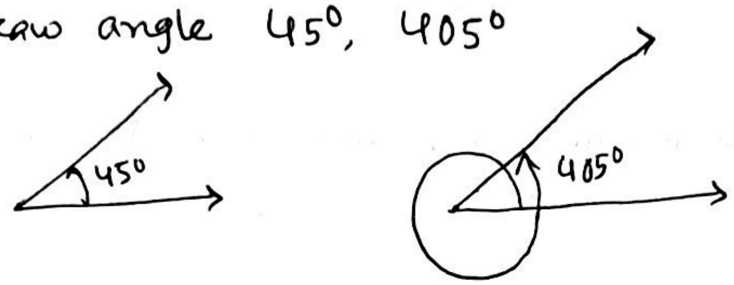
Draw angle 45^0, 405^0



convert bet^n decimals and degrees:

one minute, $1' = \frac{1}{60}$ degree

one second, $1'' = \frac{1}{60}$ minute $= \frac{1}{3600}$ degree.

i.e $1^0 = 60'$, $1' = 60''$

<u>Ex:</u> Convert $50^0 6' 21''$ to a decimal in degrees.

<u>Sol^n:</u> $50^0 6' 21'' = 50^0 + 6' + 21''$

$= 50^0 + 6 \cdot \left(\frac{1}{60}\right)^0 + 21 \left(\frac{1}{3600}\right)^0$

$= 50^0 + 0.1^0 + 0.0058^0$

$= 50.1058^0$

<u>Ex:</u> Convert 21.256^0 to the $D^0 M' S''$ form.

<u>Sol^n:</u> $21.256^0 = 21^0 + 0.256^0$

$= 21^0 + (0.256)(60')$

$= 21^0 + 15.36'$

$= 21^0 + 15' + 0.36'$

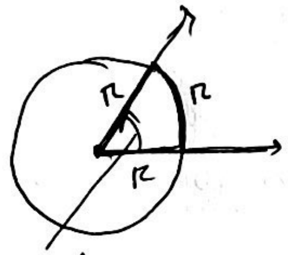
$= 21^0 + 15' + (0.36)(60'')$

$= 21^0 + 15'' + 21.6''$

$\approx 21^0 15' 22''$

Radians:

If the radius of the circle is r and the length of the arc subtended by the central angle is also r, then the measure of angle is 1 radian.



1 Radian

Arc length:

For a circle of radius r, a central angle of θ radians subtends an arc whose length s is

$$s = r\theta$$

<u>Ex:</u> Find the length of the arc of a circle of radius 2 meters subtended by a central angle of 0.25 radian.

<u>Soln:</u> Here $r = 2$, $\theta = 0.25$

$\therefore$ $s = r\theta = 2(0.25) = 0.5$ meter.

\# $1 \text{ degree} = \frac{\pi}{180}$ radians

$1 \text{ radians} = \frac{180}{\pi}$ degrees

Ex: Convert degree to radians

$60^\circ = 60 \cdot \frac{\pi}{180}$ radian $= \frac{\pi}{3}$ radians

$150^\circ = 150 \cdot \frac{\pi}{180}$ radian $= \frac{5\pi}{6}$ radians

-45°, 90°

Ex: convert each angle in radians to degrees.

$\frac{\pi}{6}$ radians $= \frac{\pi}{6} \cdot 1$ radian $= \frac{\pi}{6} \cdot \frac{180}{\pi} = 30^\circ$

$\frac{3\pi}{2}$ radians $= \frac{3\pi}{2} \cdot \frac{180}{\pi}$ degrees $= 270^\circ$

$\frac{7\pi}{3}$, 3 radians.

Area of a sector

$$A = \frac{1}{2} r^2 \theta \quad ; \theta \text{ must be in radians}$$

Ex: Find the area of the sector of a circle of radius 2 feet formed by an angle of 30°.

Soln: We know $A = \frac{1}{2} r^2 \theta$

Here $r = 2$ $\quad \theta = 30^\circ = 30 \cdot \frac{\pi}{180} = \frac{\pi}{6}$ radians

$\therefore A = \frac{1}{2} \cdot 2^2 \cdot \frac{\pi}{6} = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$ ~~radians~~

≈ 1.05 square feet.