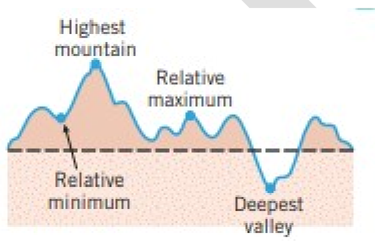
Chapter # 04
(The Derivative in Graphing and Applications)

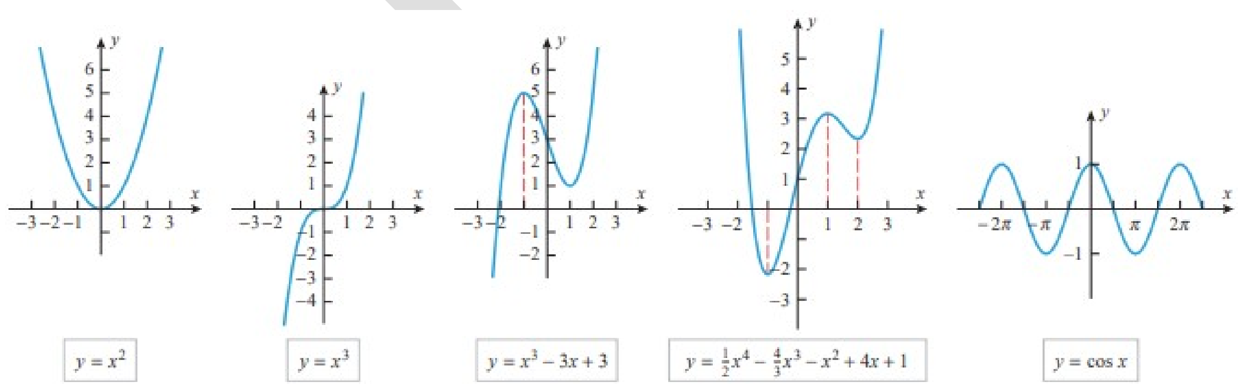
4.2 Analysis of Function II (Relative Extrema; Graphing Polynomials): In this section we will develop methods for finding the high and low points on the graph of a function and we will discuss procedures for analyzing the graphs of polynomials.

Relative Maxima and Minima: A function $\boldsymbol{f}$ is said to have a relative maximum at $\boldsymbol{x_0}$ if there is an open interval containing $\boldsymbol{x_0}$ on which $\boldsymbol{f(x_0)}$ is the largest value, that is, $\boldsymbol{f(x_0) \geq f(x)}$ for all $\boldsymbol{x}$ in the interval. Similarly, $\boldsymbol{f}$ is said to have a relative minimum at $\boldsymbol{x_0}$ if there is an open interval containing $\boldsymbol{x_0}$ on which $\boldsymbol{f(x_0)}$ is the smallest value, that is, $\boldsymbol{f(x_0) \leq f(x)}$ for all $\boldsymbol{x}$ in the interval. If $\boldsymbol{f}$ has either a relative maximum or a relative minimum at $\boldsymbol{x_0}$, then $\boldsymbol{f}$ is said to have a relative extremum at $\boldsymbol{x_0}$.

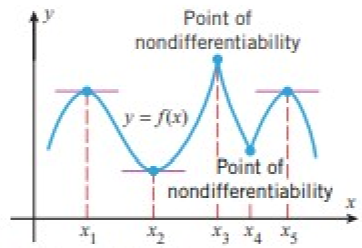


Example 1: Using graph

- $\boldsymbol{f(x) = x^2}$ has a relative minimum at ***x* = 0** but no relative maxima.
- $\boldsymbol{f(x) = x^3}$ has no relative extrema.
- $\boldsymbol{f(x) = x^3 - 3x + 3}$ has a relative maximum at ***x* = −1** and a relative minimum at ***x* = 1.**
- $\boldsymbol{f(x) = 12x^4 - 43x^3 - x^2 + 4x + 1}$ has relative minima at ***x* = −1** and ***x = 2*** and a relative maximum at ***x* = 1.**
- $\boldsymbol{f(x) = \cos x}$ has relative maxima at all even multiples of $\boldsymbol{\pi}$ and relative minima at all odd multiples of $\boldsymbol{\pi}$.



Theorem: Suppose that $\boldsymbol{f}$ is a function defined on an open interval containing the point $\boldsymbol{x_0}$. If $\boldsymbol{f}$ has a relative extremum at $\boldsymbol{x = x_0}$, then $\boldsymbol{x = x_0}$ is a critical point of $\boldsymbol{f}$; that is, either $\boldsymbol{f'(x_0) = 0}$ or $\boldsymbol{f}$ is not differentiable at $\boldsymbol{x_0}$.



▲ **Figure 4.2.3** The points x_1, x_2, x_3, x_4, and x_5 are critical points. Of these, x_1, x_2, and x_5 are stationary points.

Example 3: Find all critical points of $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$.

Solution: The function $\boldsymbol{f}$ is continuous everywhere and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

We see from this that $\boldsymbol{f'(x) = 0}$ **if** $\boldsymbol{x = 2}$ $\boldsymbol{and}$ $\boldsymbol{f'(x)}$ is undefined if ***x* = 0**. Thus ***x* = 0** and ***x* = 2** are critical points and **x = 2** is a stationary point.

Theorem (First Derivative Test): Suppose that $\boldsymbol{f}$ is continuous at a critical point $\boldsymbol{x_0}$.

(a) If $\boldsymbol{f'(x)} > 0$ on an open interval extending left from $\boldsymbol{x_0}$ and $\boldsymbol{f'(x)} < 0$ on an open interval extending right from $\boldsymbol{x_0}$, then $\boldsymbol{f}$ has a relative maximum at $\boldsymbol{x_0}$.

(b) If $\boldsymbol{f'(x)} < 0$ on an open interval extending left from $\boldsymbol{x_0}$ and $\boldsymbol{f'(x)} > 0$ on an open interval extending right from $\boldsymbol{x_0}$, then $\boldsymbol{f}$ has a relative minimum at $\boldsymbol{x_0}$.

(c) If $\boldsymbol{f'(x)}$ has the same sign on an open interval extending left from $\boldsymbol{x_0}$ as it does on an open interval extending right from $\boldsymbol{x_0}$, then $\boldsymbol{f}$ does not have a relative extremum at $\boldsymbol{x_0}$.

Theorem (Second Derivative Test): Suppose that $\boldsymbol{f}$ is twice differentiable at the point $\boldsymbol{x_0}$.

(*a*) If $f'(\boldsymbol{x_0}) = 0$ and $f''(\boldsymbol{x_0}) > 0$, then $\boldsymbol{f}$ has a relative minimum at $\boldsymbol{x_0}$.

(*b*) If $f'(\boldsymbol{x_0}) = 0$ and $f''(\boldsymbol{x_0}) < 0$, then $\boldsymbol{f}$ has a relative maximum at $\boldsymbol{x_0}$.

(*c*) If $f'(\boldsymbol{x_0}) = 0$ and $f''(\boldsymbol{x_0}) = 0$, then the test is inconclusive; that is, $\boldsymbol{f}$ may have a relative maximum, a relative minimum or neither at $\boldsymbol{x_0}$.

Example 5: Find the relative extrema of $\boldsymbol{f(x) = 3x^5 - 5x^3}$.

Solution: We have,

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x + 1)(x - 1)$$
$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

Solving, $\boldsymbol{f'(x) = 0}$ yields the stationary points ***x* = 0, *x* = −1, and *x* = 1**.

At $\boldsymbol{x = 1}$: $\boldsymbol{f''(1) = 30(1)(2 - 1) = 30} > 0 \Rightarrow f$ has relative minimum at $\boldsymbol{x = 1}$.

∴ The relative minimum value is $f(1) = 3 - 5 = -2$.

At $\boldsymbol{x = -1}$: $\boldsymbol{f''(-1) = 30(-1)(2 - 1) = -30 < 0 \Rightarrow f}$ has relative maximum at $\boldsymbol{x = -1}$.

∴ The relative maximum value is $f(-1) = -3 + 5 = 2$.

At $\boldsymbol{x = 0}$: $\boldsymbol{f''(0) = 0}$ **inconclusive** at $\boldsymbol{x = 0}$, so we will try the first derivative test at that point. A sign analysis of $\boldsymbol{f'}$ is given in the following table:

INTERVAL	$15x^2(x+1)(x-1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$

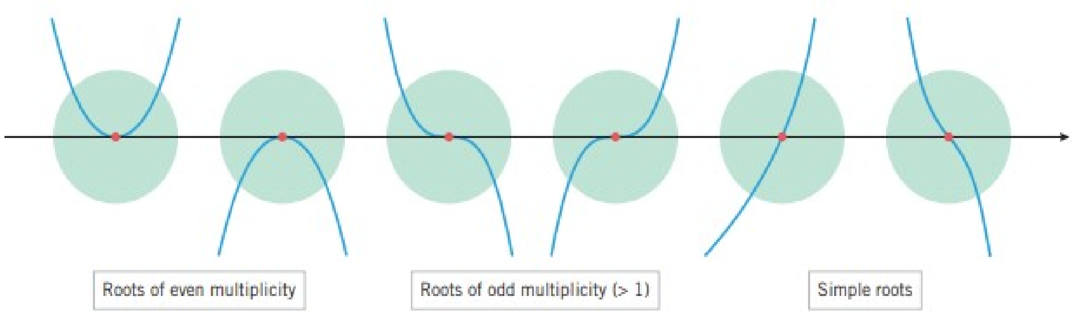
Since there is no sign change in $\boldsymbol{f'}$ at ***x* = 0**, there is neither a relative maximum nor a relative minimum at that point.

The geometric implications of multiplicity: Suppose that $\boldsymbol{p(x)}$ is a polynomial with a root of multiplicity $\boldsymbol{m\ at\ x = r}$.

(*a*) *If m is even, then the graph of $y = p(x)$ is tangent to the x-axis at $x = r$, does not cross the x-axis there, and does not have an inflection point there.*

(*b*) *If m is odd and greater than 1, then the graph is tangent to the x-axis at $x = r$, crosses the x-axis there, and also has an inflection point there.*

(*c*) *If $m = 1$ (so that the root is simple), then the graph is not tangent to the x-axis at $x = r$, crosses the x-axis there, and may or may not have an inflection point there.*

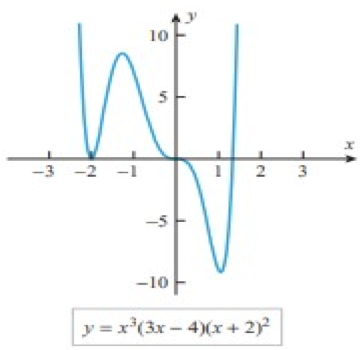


Example 6: Make a conjecture about the behavior of the graph of

$$y = x^3(3x-4)(x+2)^2$$

in the vicinity of its x-intercepts, and test your conjecture by generating the graph.

Solution: The x-intercepts occur at $x = 0, x = \frac{4}{3},\ and\ x = -2$. The root $x = 0$ has multiplicity **3**, which is odd, so at that point the graph should be tangent to the x-axis, cross the x-axis, and have an inflection point there. The root $x = -2$ has multiplicity **2**, which is even, so the graph should be tangent to but not cross the x-axis there. The root $x = \frac{4}{3}$ is simple, so at that point the curve should cross the x-axis without being tangent to it.



Analysis of Polynomials: Common properties of all polynomials-

- The natural domain of a polynomial is $(-\infty, \infty)$.
- Polynomials are continuous everywhere.
- Polynomials are differentiable everywhere, so their graphs have no corners or vertical tangent lines.
- The graph of a nonconstant polynomial eventually increases or decreases without bound as $x \to \infty\ or\ as\ x \to -\infty$ is $x \to \pm\infty$, depending on the sign of the term of highest degree and whether the polynomial has even or odd degree [see Formulas.
- The graph of a polynomial of degree n **(> 2)** has at most n, x-intercepts, at most $(n-1)$ relative extrema, and at most $(n-2)$ inflection points. This is because the ***x***intercepts, relative extrema, and inflection points of a polynomial $p(x)$ are among the real solutions of the equations $p(x) = 0$, $p'(x) = 0\ and\ p''(x) = 0$, and the polynomials in these equations have degree n, $(n-1)\ and\ (n-2)$, respectively. Thus, for example, the graph of a quadratic polynomial has at most two x-intercepts, one relative extremum, and no inflection points; and the graph of a cubic polynomial has at most three x-intercepts, two relative extrema, and one inflection point.

Example 8: Sketch the graph of the equation $y = x^3 - 3x + 2$ and identify the locations of the intercepts, relative extrema, and inflection points.

Solution: Given,

$$y = x^3 - 3x + 2$$

For $\boldsymbol{x}$-intercepts, $\quad y = 0 \;\Rightarrow\; x^3 - 3x + 2 = 0 \;\Rightarrow (x+2)(x-1)^2 = 0 \;\Rightarrow\; x = -2 \,\&\, x = 1$

For $\boldsymbol{y}$-intercepts, $\quad x = 0 \;\Rightarrow\; y = 2$

End behavior: We have

$$\lim_{x\to+\infty}(x^3 - 3x + 2) = \lim_{x\to+\infty} x^3 = +\infty$$
$$\lim_{x\to-\infty}(x^3 - 3x + 2) = \lim_{x\to-\infty} x^3 = -\infty$$

so the graph increases without bound as $\boldsymbol{x} \to +\infty$ and decreases without bound as $\boldsymbol{x} \to -\infty$.

Derivatives:

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$$
$$\frac{d^2y}{dx^2} = 6x$$

Stationary points for relative extrema: $\frac{dy}{dx} = 0 \Rightarrow 3(x-1)(x+1) = 0 \quad \therefore\; x = 1 \;\&\; x = -1$

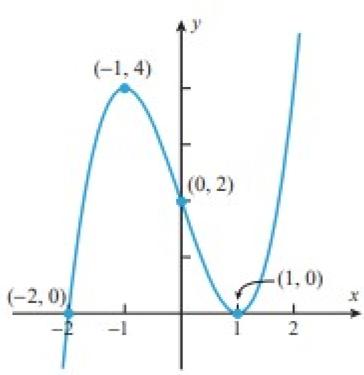
At $x = 1$: $\frac{d^2y}{dx^2} = 6 > 0 \;\Rightarrow\; \textit{Relative minimum at } x = 1$

At $x = -1$: $\frac{d^2y}{dx^2} = -6 < 0 \;\Rightarrow\; \textit{Relative maximum at } x = -1$

Stationary points for inflection points: $\frac{d^2y}{dx^2} = 0 \;\Rightarrow x = 0$

For $(-\infty, 0)$: $\frac{d^2y}{dx^2} < 0 \;\Rightarrow y \;\textit{ is concave down on } (-\infty, 0)$

For $(0, \infty)$: $\frac{d^2y}{dx^2} > 0 \;\Rightarrow y \;\textit{ is concave up on } (0, \infty)$



$y = x^3 - 3x + 2$

Home Work: Exercise 4.2: Problem No. 3-8, 33-40, 47 and 51-54