# Chapter # 05
# (Integration)

**5.3 Integration by Substitution:** In this section we will study a technique, called substitution, that can often be used to transform complicated integration problems into simpler ones.

***u*-Substitution:** The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that $\boldsymbol{F}$ is an antiderivative of $\boldsymbol{f}$ and that $\boldsymbol{g}$ is a differentiable function. The chain rule implies that the derivative of $\boldsymbol{F(g(x))}$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \qquad (1)$$

or since $\boldsymbol{F}$ is an antiderivative of $\boldsymbol{f}$,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \qquad (2)$$

For our purposes it will be useful to let $\boldsymbol{u = g(x)}$ and to write $\boldsymbol{\frac{du}{dx} = g'(x)}$ in the differential form $\boldsymbol{du = g'(x)dx}$. With this notation (2) can be expressed as

$$\int f(u)\,du = F(u) + C \qquad (3)$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

is called the ***method of u-substitution***.

**Example 1:** Evaluate

$$\int (x^2+1)^{50} \cdot 2x\,dx.$$

**Solution:** If we let $\boldsymbol{u = x^2 + 1}$, then $\boldsymbol{\frac{du}{dx} = 2x}$, which implies that $\boldsymbol{du = 2xdx}$. Thus, the given integral can be written as

$$\int (x^2+1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C$$

**Example 2:**

$$\int \sin(x+9)\,dx = \int \sin u\,du = -\cos u + C = -\cos(x+9) + C$$

$$\boxed{\begin{array}{c} u = x+9 \\ du = 1\cdot dx = dx \end{array}}$$

$$\int (x-8)^{23}\,dx = \int u^{23}\,du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

$$\boxed{\begin{array}{c} u = x-8 \\ du = 1\cdot dx = dx \end{array}}$$

**Example 3:** Evaluate

$$\int \cos 5x\,dx.$$

**Solution:**

$$\int \cos 5x\,dx = \int (\cos u)\cdot \frac{1}{5}\,du = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$

$$\boxed{\begin{array}{c} u = 5x \\ du = 5\,dx \text{ or } dx = \frac{1}{5}\,du \end{array}}$$

**Example 4:**

$$\int \frac{dx}{\left(\frac{1}{3}x-8\right)^5} = \int \frac{3\,du}{u^5} = 3\int u^{-5}\,du = -\frac{3}{4}u^{-4} + C = -\frac{3}{4}\left(\frac{1}{3}x-8\right)^{-4} + C$$

$$\boxed{\begin{array}{c} u = \frac{1}{3}x-8 \\ du = \frac{1}{3}\,dx \text{ or } dx = 3\,du \end{array}}$$

**Example 5:** Evaluate

$$\int \frac{dx}{1+3x^2}.$$

**Solution:** Substituting

$$u = \sqrt{3}x, \quad du = \sqrt{3}\,dx$$

yields

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}} \int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}} \tan^{-1} u + C = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}x) + C$$

**Example 6:**

$$\begin{aligned}
\int \left(\frac{1}{x} + \sec^2 \pi x\right) dx &= \int \frac{dx}{x} + \int \sec^2 \pi x \, dx \\
&= \ln|x| + \int \sec^2 \pi x \, dx \\
&= \ln|x| + \frac{1}{\pi} \int \sec^2 u \, du \qquad \boxed{\begin{array}{c} u = \pi x \\ du = \pi \, dx \text{ or } dx = \frac{1}{\pi} du \end{array}} \\
&= \ln|x| + \frac{1}{\pi} \tan u + C = \ln|x| + \frac{1}{\pi} \tan \pi x + C
\end{aligned}$$

**Example 7:** Evaluate

$$\int \sin^2 x \cos x \, dx.$$

**Solution:** If we let $\boldsymbol{u = \sin x}$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x \, dx$$

Thus,

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

**Example 10:** Evaluate

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx.$$

**Solution:** Substituting

$$u = e^x, \quad du = e^x \, dx$$

yields

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x) + C$$

**Example 11:** Evaluate

$$\int x^2\sqrt{x-1}\,dx.$$

**Solution:** The composition $\sqrt{x-1}$ suggests the substitution

$$u = x-1 \quad \text{so that} \quad du = dx$$

Therefore

$$x^2 = (u+1)^2 = u^2 + 2u + 1$$

so that

$$\int x^2\sqrt{x-1}\,dx = \int (u^2+2u+1)\sqrt{u}\,du = \int (u^{5/2} + 2u^{3/2} + u^{1/2})\,du$$
$$= \tfrac{2}{7}u^{7/2} + \tfrac{4}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$
$$= \tfrac{2}{7}(x-1)^{7/2} + \tfrac{4}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C$$

**Formula:**

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

**Example 14:** Evaluate

$$\int \frac{dx}{\sqrt{2-x^2}}.$$

**Solution:** Let, $u = x \;\; and \;\; a = \sqrt{2}$ .

Therefore,

$$\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + C$$

$\boxed{\textbf{Home Work: Exercise 5.3: Problem No. 15-56, 61 and 62}}$