

1) Consider the differential equations for extended unicycle model:

$$\begin{aligned}\dot{x}(t) &= V(t) \cos \theta(t) \\ \dot{y}(t) &= V(t) \sin \theta(t) \\ \dot{V}(t) &= a(t) \\ \dot{\theta}(t) &= \omega(t)\end{aligned}$$

Show that this system is differentially flat with flat output $z = [x, y]$.

2) For the system in problem 1, assume the initial conditions are:

$$\begin{aligned}x(0) &= 0, y(0) = 0, V(0) = 0.5, \theta(0) = -\pi/2 \\ x(T) &= 5, y(T) = 5, V(T) = 0.5, \theta(T) = -\pi/2 \text{ where } T = 10\end{aligned}$$

Parametrize the flat output trajectory using five basis functions, $u_1 = 1, u_2 = t, u_3 = t^2, u_4 = t^3, u_5 = t^4$.

a) Write the matrix-vector system, at $k = 5$. Clearly write all the elements for the matrix A and vector x, b .
b) Write the formula for the pseudo-inverse of A .

Solve Question 1 and 2 all problems briefly



Problem 1: Showing Differential Flatness for the Extended Unicycle Model

The extended unicycle model is given by:

$$\dot{x}(t) = V(t) \cos \theta(t)$$

$$\dot{y}(t) = V(t) \sin \theta(t)$$

$$\dot{V}(t) = a(t)$$

$$\dot{\theta}(t) = \omega(t)$$

We need to show that this system is differentially flat with flat output $z = (x, y)$.

Solution Approach:

1. **Definition of Differential Flatness:** A system is said to be differentially flat if there exists a flat output $z = (z_1, z_2)$ such that all states and inputs of the system can be expressed in terms of z and a finite number of its time derivatives.

2. **Relating State Variables to Flat Output:**

- Given $z_1 = x(t)$ and $z_2 = y(t)$, we can express $V(t)$, $\theta(t)$, $a(t)$, and $\omega(t)$ in terms of $x(t)$ and $y(t)$:

- From $\dot{x}(t) = V(t) \cos \theta(t)$ and $\dot{y}(t) = V(t) \sin \theta(t)$, we can solve for:

$$V(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

$$\theta(t) = \tan^{-1} \left(\frac{\dot{y}(t)}{\dot{x}(t)} \right)$$

- The acceleration $a(t)$ and angular velocity $\omega(t)$ can be obtained by differentiating $V(t)$ and $\theta(t)$ with respect to time:

$$a(t) = \frac{dV(t)}{dt}, \quad \omega(t) = \frac{d\theta(t)}{dt}$$

3. **Conclusion:** Since all the states (x, y, V, θ) and inputs (a, ω) can be expressed in terms of the flat outputs (x, y) and their derivatives, the system is differentially flat.

Problem 2: Parametrization and Matrix-Vector Formulation

Given:

- Initial conditions: $x(0) = 0, y(0) = 0, V(0) = 0.5, \theta(0) = -\pi/2$
- Final conditions: $x(T) = 5, y(T) = 5, V(T) = 0.5, \theta(T) = -\pi/2$
- Time horizon: $T = 10$
- Basis functions: $\psi_1 = 1, \psi_2 = t, \psi_3 = t^2, \psi_4 = t^3, \psi_5 = t^4$

a) Matrix-Vector System Formulation

1. Parametrizing Trajectory:

- The flat output $x(t)$ and $y(t)$ can be expressed as linear combinations of the basis functions:

$$x(t) = c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + c_4\psi_4 + c_5\psi_5$$

$$y(t) = d_1\psi_1 + d_2\psi_2 + d_3\psi_3 + d_4\psi_4 + d_5\psi_5$$

- This can be written in matrix form as $\mathbf{Ax} = \mathbf{b}$.

2. Constructing the Matrix \mathbf{A} and Vectors \mathbf{x} and \mathbf{b} :

- Matrix \mathbf{A} : It will consist of the values of the basis functions and their derivatives at $t = 0$ and $t = T$.
- Vector \mathbf{x} : It consists of the coefficients c_1, c_2, c_3, c_4, c_5 (for $x(t)$) and d_1, d_2, d_3, d_4, d_5 (for $y(t)$).
- Vector \mathbf{b} : It represents the initial and final conditions, such as $x(0), y(0), x(T), y(T), V(0), \theta(0), V(T), \theta(T)$.

3. Structure of \mathbf{A} and \mathbf{b} :

- \mathbf{A} is a 10×10 matrix formed using the basis functions evaluated at $t = 0$ and $t = T$.
- \mathbf{b} is a 10×1 vector containing the initial and final values.

b) Formula for the Pseudo-Inverse of \mathbf{A}

If \mathbf{A} is not square or is ill-conditioned, we use the pseudo-inverse:

$$\mathbf{A}^+ = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top$$

where \mathbf{A}^+ denotes the Moore-Penrose pseudo-inverse of \mathbf{A} .

This completes the solution for Problem 1 and Problem 2.

3) Assume there is no noise in the system of Problem 1. Show how you can implement an open-loop controller for the mission of Problem 2. Specifically, show the generated control inputs.

solve question 3. You need to analyze question 1 and question 2 here, then answer the question 3



Problem 3: Implementing an Open-Loop Controller for the System in Problem 1 and 2

Assumptions and System Setup

- Assumption: There is no noise in the system from Problem 1.
- We are tasked with implementing an open-loop controller to drive the system described in Problem 1 along the trajectory specified in Problem 2.

Open-Loop Controller Concept

An open-loop controller calculates the control inputs in advance, without any feedback, based on a desired trajectory. In this case, we will use the parameterized flat output trajectory from Problem 2 to generate the control inputs $V(t)$ and $\theta(t)$.

Step 1: Compute Desired States and Derivatives

From Problem 2, the desired trajectories for $x(t)$ and $y(t)$ are parameterized using the basis functions:

$$x(t) = c_1 + c_2t + c_3t^2 + c_4t^3 + c_5t^4$$

$$y(t) = d_1 + d_2t + d_3t^2 + d_4t^3 + d_5t^4$$

To generate the required control inputs $V(t)$ and $\omega(t)$, we need the first and second derivatives of $x(t)$ and $y(t)$:

1. First Derivatives:

$$\dot{x}(t) = c_2 + 2c_3t + 3c_4t^2 + 4c_5t^3$$

$$\dot{y}(t) = d_2 + 2d_3t + 3d_4t^2 + 4d_5t^3$$

2. Second Derivatives:

$$\ddot{x}(t) = 2c_3 + 6c_4t + 12c_5t^2$$

$$\ddot{y}(t) = 2d_3 + 6d_4t + 12d_5t^2$$

Step 2: Calculate the Control Inputs

1. Velocity $V(t)$:



Step 2: Calculate the Control Inputs

1. Velocity $V(t)$:

$$V(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

2. Orientation $\theta(t)$:

$$\theta(t) = \tan^{-1} \left(\frac{\dot{y}(t)}{\dot{x}(t)} \right)$$

3. Angular Velocity $\omega(t)$: To find $\omega(t)$, we differentiate $\theta(t)$ with respect to time:

$$\omega(t) = \frac{d\theta(t)}{dt} = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}^2(t) + \dot{y}^2(t)}$$

4. Acceleration $a(t)$:

$$a(t) = \frac{dV(t)}{dt} = \frac{\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)}{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}$$

Step 3: Implementation of the Open-Loop Controller

1. Control Inputs: The open-loop controller will output $V(t)$ and $\omega(t)$ as functions of time t using the equations derived above.
2. Procedure:
 - Precompute $V(t)$, $\theta(t)$, $a(t)$, and $\omega(t)$ using the coefficients c_i and d_i obtained from Problem 2.
 - Implement these control inputs in a simulation or real system to follow the desired trajectory $(x(t), y(t))$.

Generated Control Inputs

- Velocity $V(t)$: A continuous function that dictates how fast the unicycle should move along the path.
- Angular Velocity $\omega(t)$: A continuous function that adjusts the orientation of the unicycle to follow the desired trajectory.

Conclusion

This open-loop control scheme relies on the accurate calculation of $V(t)$ and $\omega(t)$ from the parameterized trajectory. Since there is no noise in the system, this approach will guide the unicycle along the desired path without deviation. However, in a real-world scenario, feedback might be needed to handle disturbances and ensure accurate tracking.

can you answer that briefly



Brief Solution to Problem 3

Open-Loop Controller for the Extended Unicycle Model

We need to implement an open-loop controller using the trajectory generated in Problem 2.

1. Compute Control Inputs:

- Use the parameterized trajectories for $x(t)$ and $y(t)$:

$$x(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4$$

$$y(t) = d_1 + d_2 t + d_3 t^2 + d_4 t^3 + d_5 t^4$$

- Calculate the first derivatives:

$$\dot{x}(t) = c_2 + 2c_3 t + 3c_4 t^2 + 4c_5 t^3, \quad \dot{y}(t) = d_2 + 2d_3 t + 3d_4 t^2 + 4d_5 t^3$$

- Compute velocity $V(t)$ and orientation $\theta(t)$:

$$V(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}, \quad \theta(t) = \tan^{-1} \left(\frac{\dot{y}(t)}{\dot{x}(t)} \right)$$

2. Calculate Angular Velocity and Acceleration:

- Angular velocity:

$$\omega(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}^2(t) + \dot{y}^2(t)}$$

- Acceleration:

$$a(t) = \frac{\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)}{V(t)}$$

3. Implementation:

- Use the computed $V(t)$ and $\omega(t)$ as open-loop control inputs to guide the unicycle along the desired trajectory without feedback.

This setup ensures the unicycle follows the desired path under the assumption of no noise or disturbances.

40. (i) Consider the differential equations for the (a) simple car and (b) differential drive car model. Identify the states, controls and constant parameters in each model.

$$\begin{aligned} \dot{x} &= v \cos \theta \\ \dot{y} &= v \sin \theta \\ \dot{\theta} &= \frac{v}{L} \tan \phi \end{aligned} \quad \text{(a)} \quad \begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} &= \begin{bmatrix} \frac{r}{2} \cos \theta & \frac{r}{2} \cos \theta \\ \frac{r}{2} \sin \theta & \frac{r}{2} \sin \theta \\ \frac{r}{L} & -\frac{r}{L} \end{bmatrix} \begin{bmatrix} \omega_r \\ \omega_l \end{bmatrix} \end{aligned} \quad \text{(b)}$$

(ii) Using a numerical integration scheme using the Euler method for the differential drive car model. Approximate the control inputs.

solve all problems in question no 4 briefly



Let's go through the problems step-by-step and solve them.

Problem 4 (I) - Identify the States, Controls, and Constant Parameters

(a) Simple Car Model

The simple car model is described by the differential equations:

$$\begin{aligned} \dot{x} &= v \cos \theta \\ \dot{y} &= v \sin \theta \\ \dot{\theta} &= \frac{v}{L} \tan \phi \end{aligned}$$

- **States:** The state variables for the system are x , y , and θ . These represent the car's position in the x - y plane and its orientation, respectively.
- **Controls:** The control inputs are v (linear velocity) and ϕ (steering angle).
- **Constant Parameters:** The constant parameter in this model is L , which represents the wheelbase (distance between the front and rear axles) of the car.

(b) Differential Drive Car Model

The differential drive model is given by:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} \cos \theta & \frac{r}{2} \cos \theta \\ \frac{r}{2} \sin \theta & \frac{r}{2} \sin \theta \\ \frac{r}{L} & -\frac{r}{L} \end{bmatrix} \begin{bmatrix} \omega_r \\ \omega_l \end{bmatrix}$$

- **States:** The state variables are x , y , and θ , which represent the position and orientation of the car.
- **Controls:** The control inputs are ω_r and ω_l , which represent the angular velocities of the right and left wheels, respectively.
- **Constant Parameters:** The constant parameters are r (the radius of each wheel) and L (the distance between the wheels).



Problem 4 (II) - Numerical Integration Using the Euler Method

Problem 4 (II) - Numerical Integration Using the Euler Method

To set up a numerical integration scheme using the Euler method for the differential drive car model, we can write the state update equations as follows.

Given the time step Δt , the Euler update for each state variable is:

$$x_{k+1} = x_k + \dot{x} \Delta t$$

$$y_{k+1} = y_k + \dot{y} \Delta t$$

$$\theta_{k+1} = \theta_k + \dot{\theta} \Delta t$$

Where \dot{x} , \dot{y} , and $\dot{\theta}$ are calculated from the differential drive model equations:

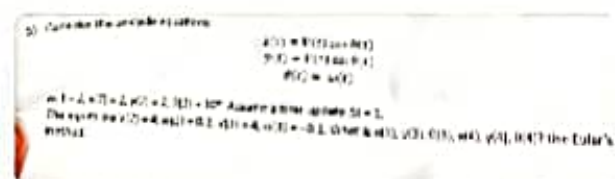
$$\dot{x} = \frac{r}{2} \cos \theta (\omega_r + \omega_l)$$

$$\dot{y} = \frac{r}{2} \sin \theta (\omega_r + \omega_l)$$

$$\dot{\theta} = \frac{r}{L} (\omega_r - \omega_l)$$

Summary of the Euler Method Steps

1. Calculate \dot{x} , \dot{y} , and $\dot{\theta}$ at the current state (x_k, y_k, θ_k) using the control inputs ω_r and ω_l .
2. Update each state variable using the Euler method:
 - $x_{k+1} = x_k + \dot{x} \Delta t$
 - $y_{k+1} = y_k + \dot{y} \Delta t$
 - $\theta_{k+1} = \theta_k + \dot{\theta} \Delta t$
3. Repeat this process at each time step.



solve all problems in question no 5 briefly

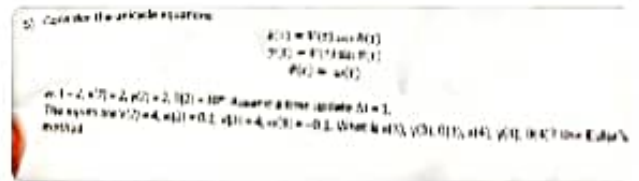


Let's solve the question step-by-step.

Problem 5 - Unicycle Equations and Euler Method Integration



The unicycle model is given by the following differential equations:



solve all problems in question no 5 briefly



Let's solve the question step-by-step.

Problem 5 - Unicycle Equations and Euler Method Integration

The unicycle model is given by the following differential equations:

$$\dot{x}(t) = V(t) \cos \theta(t)$$

$$\dot{y}(t) = V(t) \sin \theta(t)$$

$$\dot{\theta}(t) = \omega(t)$$

Given Initial Conditions:

- At $t = 2$:
 - $x(2) = 2$
 - $y(2) = 2$
 - $\theta(2) = 30^\circ$ (which we need to convert to radians: $\theta(2) = \frac{\pi}{6} \approx 0.5236$ radians)

Time Update:

- $\Delta t = 1$

Inputs for Each Step:

- At $t = 2$:
 - $V(2) = 4$
 - $\omega(2) = 0.1$
- At $t = 3$:
 - $V(3) = 4$
 - $\omega(3) = -0.1$

Goal: Calculate:

- $x(3), y(3), \theta(3)$
- $x(4), y(4), \theta(4)$



Goal: Calculate:

- $x(3), y(3), \theta(3)$
- $x(4), y(4), \theta(4)$

Step-by-Step Calculation Using Euler's Method

We will use the Euler method to find the values of x , y , and θ at each time step.

The Euler update equations are:

$$x_{k+1} = x_k + \dot{x}_k \Delta t$$

$$y_{k+1} = y_k + \dot{y}_k \Delta t$$

$$\theta_{k+1} = \theta_k + \dot{\theta}_k \Delta t$$

where $\dot{x}_k = V(k) \cos \theta(k)$, $\dot{y}_k = V(k) \sin \theta(k)$, and $\dot{\theta}_k = \omega(k)$.

Step 1: Calculate $x(3)$, $y(3)$, $\theta(3)$

1. At $t = 2$:

- $x(2) = 2$
- $y(2) = 2$
- $\theta(2) = 0.5236$ radians
- $V(2) = 4$
- $\omega(2) = 0.1$

2. Compute $\dot{x}(2)$, $\dot{y}(2)$, and $\dot{\theta}(2)$:

$$\dot{x}(2) = V(2) \cos \theta(2) = 4 \cos(0.5236) \approx 4 \times 0.866 = 3.464$$

$$\dot{y}(2) = V(2) \sin \theta(2) = 4 \sin(0.5236) \approx 4 \times 0.5 = 2$$

$$\dot{\theta}(2) = \omega(2) = 0.1$$

3. Update $x(3)$, $y(3)$, and $\theta(3)$:

$$x(3) = x(2) + \dot{x}(2) \times \Delta t = 2 + 3.464 \times 1 = 5.464$$

$$y(3) = y(2) + \dot{y}(2) \times \Delta t = 2 + 2 \times 1 = 4$$

$$\theta(3) = \theta(2) + \dot{\theta}(2) \times \Delta t = 0.5236 + 0.1 \times 1 = 0.6236 \text{ radians}$$

Step 2: Calculate $x(4)$, $y(4)$, $\theta(4)$

1. At $t = 3$:

- $x(3) = 5.464$



3. Update $x(3)$, $y(3)$, and $\theta(3)$:

$$x(3) = x(2) + \dot{x}(2) \times \Delta t = 2 + 3.464 \times 1 = 5.464$$

$$y(3) = y(2) + \dot{y}(2) \times \Delta t = 2 + 2 \times 1 = 4$$

$$\theta(3) = \theta(2) + \dot{\theta}(2) \times \Delta t = 0.5236 + 0.1 \times 1 = 0.6236 \text{ radians}$$

Step 2: Calculate $x(4)$, $y(4)$, $\theta(4)$

1. At $t = 3$:

- $x(3) = 5.464$
- $y(3) = 4$
- $\theta(3) = 0.6236 \text{ radians}$
- $V(3) = 4$
- $\omega(3) = -0.1$

2. Compute $\dot{x}(3)$, $\dot{y}(3)$, and $\dot{\theta}(3)$:

$$\dot{x}(3) = V(3) \cos \theta(3) = 4 \cos(0.6236) \approx 4 \times 0.812 = 3.248$$

$$\dot{y}(3) = V(3) \sin \theta(3) = 4 \sin(0.6236) \approx 4 \times 0.569 = 2.276$$

$$\dot{\theta}(3) = \omega(3) = -0.1$$

3. Update $x(4)$, $y(4)$, and $\theta(4)$:

$$x(4) = x(3) + \dot{x}(3) \times \Delta t = 5.464 + 3.248 \times 1 = 8.712$$

$$y(4) = y(3) + \dot{y}(3) \times \Delta t = 4 + 2.276 \times 1 = 6.276$$

$$\theta(4) = \theta(3) + \dot{\theta}(3) \times \Delta t = 0.6236 - 0.1 \times 1 = 0.5236 \text{ radians}$$

Final Answers

- $x(3) = 5.464$
- $y(3) = 4$
- $\theta(3) = 0.6236 \text{ radians}$
- $x(4) = 8.712$
- $y(4) = 6.276$
- $\theta(4) = 0.5236 \text{ radians}$

Midterm
October 10, 2023

- 1) Consider the differential equations for extended unicycle model:

$$\begin{aligned}\dot{x}(t) &= V(t) \cos \theta(t) \\ \dot{y}(t) &= V(t) \sin \theta(t) \\ \dot{V}(t) &= a(t) \\ \dot{\theta}(t) &= \omega(t)\end{aligned}$$

Show that this system is differentially flat with flat output $z = (x, y)$.

- 2) For the system in problem 1, assume the initial conditions are:

$$\begin{aligned}x(0) &= 0, y(0) = 0, V(0) = 0.5, \theta(0) = -\pi/2 \\ x(T) &= 5, y(T) = 5, V(T) = 0.5, \theta(T) = -\pi/2 \text{ where } T = 10\end{aligned}$$

Parametrize the flat output trajectory using five basis functions, $\psi_1 = 1, \psi_2 = t, \psi_3 = t^2, \psi_4 = t^3, \psi_5 = t^4$.

- a) Write the matrix-vector system, as $Ax = b$. Clearly write all the elements for the matrix A and vectors x, b .
 - b) Write the formula for the pseudo-inverse of A .
- 3) Assume there is no noise in the system of Problem 1. Show how you can implement an open-loop controller for the mission in Problem 2. Specifically, show the generated control inputs.
- 4) I) Consider the differential equations for the (a) simple car and (b) differential drive car model. Identify the states, controls and constant parameters in each model.

$$\begin{aligned}\dot{x} &= v \cos \theta, \\ \dot{y} &= v \sin \theta, \\ \dot{\theta} &= \frac{v}{L} \tan \phi,\end{aligned}$$

(a)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} \cos \theta & \frac{r}{2} \cos \theta \\ \frac{r}{2} \sin \theta & \frac{r}{2} \sin \theta \\ \frac{r}{L} & -\frac{r}{L} \end{bmatrix} \begin{bmatrix} \omega_r \\ \omega_l \end{bmatrix}$$

(b)

- II) Setup a numerical integration scheme using the Euler method for the differential drive car model. Assume a time update Δt .

- 5) Consider the unicycle equations:

$$\begin{aligned}\dot{x}(t) &= V(t) \cos \theta(t) \\ \dot{y}(t) &= V(t) \sin \theta(t) \\ \dot{\theta}(t) &= \omega(t)\end{aligned}$$

At $t = 2$, $x(2) = 2$, $y(2) = 2$, $\theta(2) = 30^\circ$. Assume a time update $\Delta t = 1$.

The inputs are $V(2) = 4$, $\omega(2) = 0.1$, $V(3) = 4$, $\omega(3) = -0.1$. What is $x(3)$, $y(3)$, $\theta(3)$, $x(4)$, $y(4)$, $\theta(4)$? Use Euler's method.