DAY-5 : Section 7.4 Integration by Trogonometric Substitutions

Understanding:

- ➔ Trigonometric Functions [MAT 116]
- ➔ Inverse Trigonometric Functions. [From a different book]

TRIGONOMETRIC SUBSTITUTIONS: $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$

Formulas:

$1 - \sin^2 x = \cos^2 x$

$1 + \tan^2 x = \sec^2 x$

$\sec^2 x - 1 = \tan^2 x$

We need this method to integrate type (b) integrals.

Examples: (1) $\boldsymbol{a)}\ \int x\,(4 - x^2)^{44}\,dx \quad and \quad \boldsymbol{b)}\ \int (4 - x^2)^{44}\,dx$

$(2)\ \boldsymbol{a)}\ \int 2x\sqrt{1 - x^2}\,dx \quad and \quad \boldsymbol{b)}\ \int \sqrt{1 - x^2}\,dx$

Note: $\sqrt{9} = 3$ and $-\sqrt{9} = 3$ [Solve $x^2 = 9 \to x = \pm\sqrt{9}$, that is, $x = +\sqrt{9}$, $x = -\sqrt{9}$]

Definition: For any real number x, $\sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

Notes:

(1) $y = \sqrt{x}$: This is called **positive square root**, and we only get non-negative number y from this equation. Here, $y = 0$ if $x = 0$. Example: $\sqrt{4} = 2$.

(2) $y = -\sqrt{x}$: This is called **negative square root,** and we only get non-positive number y from this equation. Here, $y = 0$ if $x = 0$. Example: $-\sqrt{4} = -2$.

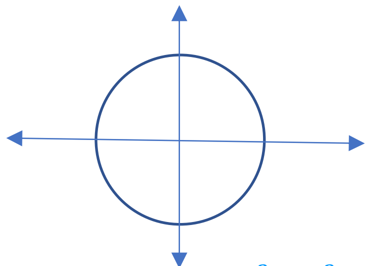
Recall: To solve $x^2 = 9 \Rightarrow x = \pm\sqrt{9} \Rightarrow x = \pm 3$. That is, $x = 3$ or $x = -3$

There are 3-cases

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \leq \theta \leq \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \leq \theta < \pi/2 & (\text{if } x \geq a) \\ \pi/2 < \theta \leq \pi & (\text{if } x \leq -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Recall: Consider the circle of radius r and with center at the origin.



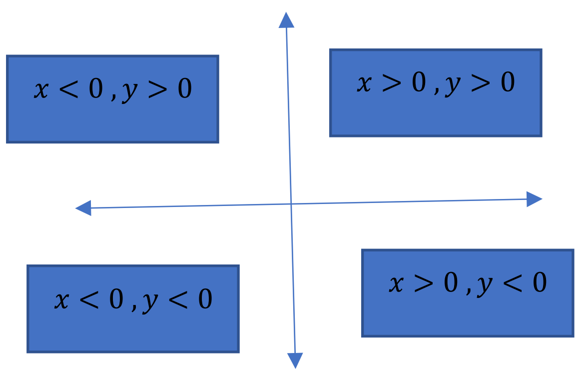
For any point (x, y) on the circle, we get $x^2 + y^2 = r^2$, and also, in polar coordinates,

$$x = r\cos\theta \quad , \quad y = r\sin\theta.$$

In particular, for any point (x, y) on the circle, we get $x^2 + y^2 = 1$, and also, in polar coordinates,

$$x = \cos\theta \quad , \quad y = \sin\theta.$$

Remember that $(x, y) = (\cos\theta \ , \sin\theta\ )$.



Case: 1 $\sqrt{a^2 - x^2}$; $a > 0$

Set $x = a\sin\theta$. Then $\frac{dx}{d\theta} = a\cos\theta \Rightarrow dx = a\cos\theta\, d\theta$

Also, $\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\, sin\,\theta)^2}$

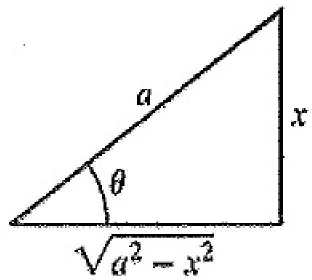
$= \sqrt{a^2 - a^2\, sin^2\,\theta} = \sqrt{a^2(1 - sin^2\theta\,)}$

$= \sqrt{a^2\, cos^2\theta} = |a| \sqrt{cos^2\theta} = a\ |\cos\theta|$

That is, $\sqrt{a^2 - x^2} = a\ \cos\theta$; when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Again, $x = a\sin\theta$ $\Rightarrow \frac{x}{a} = \sin\theta$. Hence $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

Now, $\sin\theta = \frac{x}{a} = \frac{Opposite}{Hypotenuse} = \frac{x \text{ is the opposite}}{a \text{ is the hypotenuse}}$



$x = a\sin\theta$

$cos\theta = \frac{\sqrt{a^2-x^2}}{a}$, $\sec\theta = \frac{a}{\sqrt{a^2-x^2}}$, $\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$, $\cot\theta = \frac{\sqrt{a^2-x^2}}{x}$, $\csc\theta = \frac{a}{x}$

Case: 2 $\sqrt{a^2 + x^2}$; $a > 0$

Set $x = a\tan\theta$. Then $\frac{dx}{d\theta} = a\sec^2\theta \Rightarrow \boldsymbol{dx = a\sec^2\theta \; d\theta}$

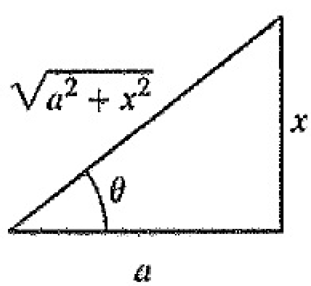
$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a\,tan\,\theta)^2}$

$= \sqrt{a^2(1 + \; tan^2\theta\,)} \; = \sqrt{a^2 sec^2\theta} \; = a\sqrt{\;sec^2\theta} \; = a|\sec\theta|$

$\sqrt{a^2 + x^2} = a\;\sec\theta$; when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Again, $\boldsymbol{x = a\tan\theta}$ $\Rightarrow \frac{x}{a} = \tan\theta$. Hence $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

Now, $\tan\theta = \frac{x}{a} = \frac{Opposite}{Adjecent} = \frac{x \text{ is the opposite}}{a \text{ is the Adjecent}}$



$x = a\tan\theta$

$cos\theta = \frac{a}{\sqrt{a^2+x^2}}$, $\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$, $\sin\theta = \frac{x}{\sqrt{a^2+x^2}}$, $\cot\theta = \frac{a}{x}$, $\csc\theta = \frac{\sqrt{a^2+x^2}}{x}$

Case: 3 $\sqrt{x^2-a^2}$; $a>0$

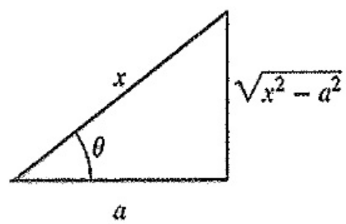
Set $x=a\sec\theta$. Then $\frac{dx}{d\theta}=a\sec\,\theta\tan\theta \;\Rightarrow\; dx=a\;\sec\,\theta\tan\theta\;\;d\theta$

$$\sqrt{x^2-a^2}=\sqrt{(a\,sec\,\theta)^2-a^2}\;=\sqrt{a^2(\,sec^2\theta-1)}\;=\sqrt{a^2tan^2\theta}\;=a\sqrt{\,tan^2\theta}\;=a|\tan\theta|$$

$\sqrt{x^2-a^2}=a\;\tan\theta$; when $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$

Again, $\boldsymbol{x=a\sec\theta}$ $\Rightarrow\frac{\boldsymbol{x}}{\boldsymbol{a}}=\mathbf{sec}\,\boldsymbol{\theta}$. Hence $\boldsymbol{\theta}=\mathbf{sec}^{-1}\left(\frac{\boldsymbol{x}}{\boldsymbol{a}}\right)$

Now, $\sec\theta=\frac{x}{a}=\frac{Hypotenuse}{Adjecent}=\frac{x\ is\ the\ Hypotenuse}{a\ is\ the\ Adjecent}$



$x=a\sec\theta$

$cos\theta=\frac{a}{x}$, $\cot\theta=\frac{a}{\sqrt{x^2-a^2}}$, $\sin\theta=\frac{\sqrt{x^2-a^2}}{x}$, $\tan\theta=\frac{\sqrt{x^2-a^2}}{a}$, $\csc\theta=\frac{x}{\sqrt{x^2-a^2}}$

Note: To evaluate an indefinite integral with this method, there are 3-steps.

- ➔ Use substitution and simplify, resulting an integral of trigonometric functions
- ➔ Evaluate the integral
- ➔ Re-substitute using a right triangle.

And for a definite integral, be very careful when you simplify the absolute value on the given closed interval.

Examples

1. $\int_{\frac{2\pi}{3}}^{\pi}|\tan\theta|\;d\theta=\int_{\frac{2\pi}{3}}^{\pi}(-\tan\theta\,)\;d\theta$; Here $\frac{2\pi}{3}\le\theta\le\pi$, hence $\theta\le0$.

$$[x=-2\,,\;|x|=|-2|=2=\;-(-2)=-x]$$

2. $\int_{0}^{\pi}|\cos\theta|\;d\theta=\int_{0}^{\frac{\pi}{2}}|\cos\theta|\;d\theta+\int_{\frac{\pi}{2}}^{\pi}|\cos\theta|\;d\theta$

$$= \int_{0}^{\frac{\pi}{2}} \cos\theta \ d\theta + \int_{\frac{\pi}{2}}^{\pi} (-\cos\theta\,)\ d\theta = \int_{0}^{\frac{\pi}{2}} \cos\theta \ d\theta - \int_{\frac{\pi}{2}}^{\pi} \cos\theta \ d\theta = 2$$

Section 7.4 Integration by Trogonometric Substitutions

EXERCISES

TRIGONOMETRIC SUBSTITUTIONS: $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$

Exercise: 1 (a) $\int \frac{\sqrt{1+t^2}}{t}\, dt$

Set $t = \tan\theta$. Then $\frac{dt}{d\theta} = \sec^2\theta$, that is, $dt = \sec^2\theta \ \ d\theta$

Now,

$$\int \frac{\sqrt{1+t^2}}{t}\, dt \ = \ \int \frac{\sqrt{1 + tan^2\,\theta}}{\tan\theta} \ \sec^2\theta \ \ d\theta \ = \ \int \frac{\sqrt{\sec^2\theta}}{\tan\theta} \ \sec^2\theta \ \ d\theta$$

$$= \ \int \frac{|\sec\theta|}{\tan\theta} \ \sec^2\theta \ \ d\theta$$

$$= \ \int \frac{\sec\theta}{\tan\theta} \ \sec^2\theta \ \ d\theta \ \ ; \quad \text{when} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \int \frac{\sec^3\theta}{\tan\theta} \ d\theta \ = \ \int \frac{\frac{1}{\cos^3\theta}}{\frac{(\sin\theta)}{\cos\theta}} d\theta$$

$$= \ \int \frac{\sin\theta}{\cos^2\theta}\, d\theta \ \ ; \ \ Set \ \ u = \ \cos\theta, \qquad then \quad -du = \ \sin\theta \ d\theta$$

$$= \ \int \frac{1}{u^2}\,(-1)\, du \ \ = \ - \int u^{-2}\, du$$

$$= \ - \ \frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{1}{u} + C \ = \frac{1}{\cos\theta} + C = \ \sec\theta + C \ = \ \sqrt{1+t^2} \ + C \ ; \ \ \text{if } t = \ \tan\theta, \ \text{then } \ \sec\theta = \ \sqrt{1+t^2}$$

$$\int \frac{\sqrt{1+t^2}}{t}\, dt \ = \ \sqrt{1+t^2} \ + C$$

$$(b) \int \frac{1}{x^2\sqrt{x^2+5}}\, dx$$

Set $x = \sqrt{5}\ \tan\theta$. $\quad dx = \sqrt{5}\sec^2\theta \ \ d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+5}}\, dx$$

$$= \int \frac{1}{(\sqrt{5}\ \tan\theta\,)^2\sqrt{(\sqrt{5}\ \tan\theta\,)^2+5}}\ \ \sqrt{5}\sec^2\theta\ \ d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\sqrt{5(\tan^2\theta+1)}}\ \ d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\ \sqrt{5\sec^2\theta}}\ \ d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\ |\sqrt{5}\sec\theta\,|}\ \ d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\,\sqrt{5}\sec\theta}\ \ d\theta \quad ; \quad \text{when} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \frac{1}{5}\int \frac{\sec\ \theta}{\tan^2\theta}\ d\theta$$

$$= \frac{1}{5}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\ d\theta$$

$$= \frac{1}{5}\int \frac{\cos\ \theta}{\sin^2\theta}\ d\theta \quad ; \quad set\ u = \sin\theta\,,\ \ du = \cos\theta\ d\theta$$

$$= \frac{1}{5}\int \frac{1}{u^2}\ du = \ = \frac{1}{5}\int u^{-2}\ du = \frac{1}{5}\ \ \frac{u^{-2+1}}{-2+1} = -\frac{1}{5}\frac{1}{u} + C$$

$$= -\frac{1}{5\sin\theta} + C\ = \ -\frac{\sqrt{x^2+5}}{5x} + C$$

Now, $x = \sqrt{5}\ \tan\theta\ \Rightarrow \tan\theta = \frac{x}{\sqrt{5}}$. Hence, $\sin\theta\ = \frac{x}{\sqrt{x^2+5}}$, that is, $\frac{1}{\sin\theta}\ \ \frac{x}{\sqrt{x^2+5}}$

$(c)\int \frac{1}{x^2\sqrt{x^2-4}}\ dx$ Set $x = 2\sec\theta$. Then $dx = \ 2\sec\theta\tan\theta\ \ d\theta$

Now,

$$\int \frac{1}{x^2\sqrt{x^2-4}}\ dx$$

$$= \int \frac{1}{(2\sec\theta)^2\ \sqrt{(2\sec\theta)^2-4}}\ \ 2\sec\theta\tan\theta\ \ d\theta$$

$$= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\ \sqrt{4\sec^2\theta - 4}}\ d\theta$$

$$= \int \frac{\tan\theta}{2\ \sec\theta\ \sqrt{4(\sec^2\theta - 1)}}\ d\theta$$

$$= \int \frac{\tan\theta}{4\ \sec\theta\ \sqrt{\tan^2\theta}}\ d\theta$$

$$= \int \frac{\tan\theta}{4\ \sec\theta\ |\tan\theta|}\ d\theta$$

$$= \int \frac{\tan\theta}{4\ \sec\theta\ \tan\theta}\ d\theta \ \ ; \text{ for } \ 0 \le \theta < \frac{\pi}{2} \quad \text{or } \pi \le \theta < \frac{3\pi}{2}.$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\ d\theta$$

$$= \frac{1}{4}\int \cos\theta \ \ d\theta$$

$$= \frac{1}{4}\ \sin\theta + C\ ; \qquad x = 2\sec\theta \to \ \sec\theta = \frac{x}{2} = \frac{hyp}{adj} \to opp. = \ \sqrt{x^2 - 4}$$

$$= \frac{1}{4}\frac{\sqrt{x^2 - 4}}{x} + C$$

$$= \frac{\sqrt{x^2 - 4}}{4x} + C$$

<u>**Exercise: 2**</u> $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\ dx$.

Solution: Set $x = \sin\theta$. Then, $dx = \cos\theta\ d\theta$. Now,

$$\int \frac{1}{(1 - x^2)^{\frac{3}{2}}}\ dx$$

$$= \int \frac{1}{(1-(\sin\theta)^2)^{\frac{3}{2}}} \cos\theta \; d\theta$$

$$= \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}} \cos\theta \; d\theta$$

$$= \int \frac{1}{(cos^2\theta)^{\frac{3}{2}}} \cos\theta \; d\theta$$

$$= \int \frac{1}{(cos^2\theta)^{\frac{3}{2}}} \cos\theta \; d\theta$$

$$= \int \frac{1}{\left(\sqrt{cos^2\theta}\right)^3} \cos\theta \; d\theta$$

$$= \int \frac{1}{(|\cos\theta|)^3} \cos\theta \; d\theta$$

$$= \int \frac{1}{\cos^3\theta} \cos\theta \; d\theta \; ; \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

$$= \int \frac{1}{\cos^2\theta} \; d\theta$$

$$= \int \sec^2\theta \; dx$$

$$= \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$$

$$; [\text{since } \frac{x}{1} = \sin\theta, \quad \text{then} \quad \tan\theta = \frac{x}{\sqrt{1-x^2}}]$$

DAY-6

Quiz-1

On Wednesday, 10th March

Study: 7.1 – 7.3

Sadia Afrin → 1 bonus point with Thanks

Exercise: 3 (A) Evaluate

$$\int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\ dx$$

Given $\int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\ dx = \int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{2^2-x^2}}\ dx$

Set $x = 2\ \sin y$. Then $dx = 2\cos y\ \ dy.$

Also,

x	1	$\sqrt{2}$
y	$\frac{\pi}{6}$	$\frac{\pi}{4}$

From, $x = 2\sin y$; *If* $x = 1$, *then* $1 = 2\sin y \Rightarrow \sin y = ½$, that is, $y = \frac{\pi}{6}$. *If* $x = \sqrt{2}$, *then* $y = \frac{\pi}{4}$.

Hence, $\int_{1}^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{(2\sin y)^2\ \sqrt{4-4\sin^2 y}}\ \ 2\cos y\ \ dy$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{4\ \sin^2 y\ \sqrt{4\cos^2 y}}\ \ 2\cos y\ \ dy = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2 y\ \sqrt{(2\cos y)^2}}\ \ 2\cos y\ \ dy$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2 y\ \ |2\cos y|}\ \ 2\cos y\ \ dy$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2 y\ \ 2\cos y}\ \ 2\cos y\ \ dy\ \ = \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2 y}\ \ dy$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc^2 y \;\; dy \; = \frac{1}{4}\,[-\cot y\,]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{4}\left[-\cot\frac{\pi}{4} \;+\; \cot\frac{\pi}{6}\right] \;= \frac{-1+\sqrt{3}}{4}$$

$$\int_{1}^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} \;\; = \;\; \frac{\sqrt{3}-1}{4}$$

<u>Exercise: 3</u> (B) Evaluate

$$\int_{2}^{2\sqrt{3}} \frac{1}{x^2\sqrt{4+x^2}} \;\; dx$$

Solution: Given

$$\int_{2}^{2\sqrt{3}} \frac{1}{x^2\sqrt{4+x^2}} \;\; dx = \int_{2}^{2\sqrt{3}} \frac{1}{x^2\sqrt{2^2+x^2}} \;\; dx$$

Set $x = 2\tan\theta$. Then $\frac{dx}{d\theta} = 2\sec^2\theta$, $i.e.,$ $dx = 2\sec^2\theta\, d\theta$.

If $x = 2,$ then $2\tan\theta = 2 \;\Rightarrow\; \tan\theta = 1 \Rightarrow\; \theta = \frac{\pi}{4}$ and

If $x = 2\sqrt{3},$ then $2\tan\theta = 2\sqrt{3} \;\Rightarrow \tan\theta = \sqrt{3} \Rightarrow\; \theta = \frac{\pi}{3}$.

Now,

$$\int_{2}^{2\sqrt{3}} \frac{1}{x^2\sqrt{4+x^2}} \;\; dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{(2\tan\theta)^2\sqrt{4+(2\tan\theta)^2}} \;\; 2\sec^2\theta\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sec^2\theta}{4\tan^2\theta\,\sqrt{4+4\tan^2\theta}} \;\; d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{2\tan^2\theta\sqrt{4(1+\tan^2\theta)}}\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{4\ \tan^2\theta\sqrt{sec^2\,\theta}}\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{4\ \tan^2\theta\ |\sec\theta\,|}\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{4\ \tan^2\theta\ \sec\theta}\ d\theta\ ;\quad \theta \text{ is in the first quadrant.}$$

$$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta}\ d\theta$$

$$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\ d\theta$$

$$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}\ d\theta$$

$$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin\ \theta}\ \frac{1}{\sin\theta}d\theta$$

$$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot\theta\ \csc\theta\, d\theta$$

$$= \frac{1}{4}\ [-\csc\theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \ \frac{1}{4}\left[-\csc\left(\frac{\pi}{3}\right)+\csc\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{2}{\sqrt{3}} + \sqrt{2}\right]$$

$$= \frac{1}{4}\left[-\frac{2\sqrt{3}}{3} + \sqrt{2}\right]$$

$$= \frac{1}{12}\left[\, 3\sqrt{2} - 2\sqrt{3}\,\right].$$

Exercise: 4 $\quad \int \frac{x^2}{(x^2-1)^{\frac{3}{2}}}\, dx$ Homework

Solution:

Set $x = \sec\theta$, then $dx = \sec\theta\tan\theta \; d\theta$. Now,

$$\int \frac{x^2}{(x^2-1)^{\frac{3}{2}}}\, dx$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta - 1)^{\frac{3}{2}}} \sec\theta\tan\theta \; d\theta$$

$$= \int \frac{\sec^3\theta \;\; \tan\theta}{(\tan^2\theta)^{\frac{3}{2}}}\, d\theta$$

$$= \int \frac{\sec^3\theta \;\; \tan\theta}{\left(\sqrt{\tan^2\theta}\right)^3}\, d\theta$$

$$= \int \frac{\sec^3\theta \;\; \tan\theta}{(|\tan\theta|)^3}\, d\theta$$

$$= \int \frac{\sec^3\theta \;\; \tan\theta}{\tan^3\theta}\, d\theta \;\; ; \quad \text{for } \; 0 \le \theta < \frac{\pi}{2} \quad \text{or } \; \pi \le \theta < \frac{3\pi}{2}.$$

$$= \int \frac{\sec^3\theta}{\tan^2\theta}\, d\theta$$

$$= \int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta$$

$$= \int \frac{1}{\cos\theta \sin^2\theta}\ d\theta$$

$$= \int \sec\theta \csc^2\theta\ \ d\theta$$

$$= \int \sec\theta\,[1+\cot^2\theta]\ d\theta$$

$$= \int \sec\theta\ d\theta\ + \int \sec\theta\ \cot^2\theta\ \ d\theta$$

$$= \int \sec\theta\ d\theta\ + \int \frac{\cos\theta}{\sin^2\theta}\ d\theta$$

$$= \int \sec\theta\ d\theta\ + \int \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\ \theta}\ d\theta$$

$$= \int \sec\theta\ d\theta\ + \int \csc\theta \cot\theta\ \ d\theta$$

$$= \ln|\tan\theta + \sec\theta\,| - \csc\theta + C$$

[From $\sec\theta\ = x = \frac{hyp}{adj}$, we get $\text{hyp} = x, \text{adje} = 1, \text{opp.} = \ \sqrt{x^2-1}$]

$$= \ln\left|\sqrt{x^2-1}+x\right| - \frac{x}{\sqrt{x^2-1}} + C$$

Exercise: 5 (Homework) Evaluate

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4+x^2}}\ \ dx$$

Exercise: 6 Evaluate

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{2x^2-4}}{x}\ \ dx$$

Solution: Given

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{2x^2-4}}{x}\ \ dx = \int_{\sqrt{2}}^{2} \frac{\sqrt{2(x^2-2)}}{x}\ \ dx = \sqrt{2}\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-2}}{x}\ \ dx = \sqrt{2}\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-\left(\sqrt{2}\right)^2}}{x}\ \ dx \ldots\ldots\ldots (1)$$

Set $x = \sqrt{2}\sec\theta$. Then $\frac{dx}{d\theta} = \sqrt{2}\sec\theta\tan\theta$, that is, $dx = \sqrt{2}\sec\theta\tan\theta\ d\theta$.

Also, if $x = \sqrt{2}$, then $\sqrt{2} = \sqrt{2}\sec\theta \Rightarrow 1 = \sec\theta$, that is, $\theta = 0$.

If $x = 2$, then $2 = \sqrt{2}\sec\theta \Rightarrow \sqrt{2} = \sec\theta$, that is, $\theta = \frac{\pi}{4}$.

Now, from equation (1):

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{2x^2-4}}{x}\ dx = \sqrt{2}\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-(\sqrt{2})^2}}{x}\ dx$$

$$= \sqrt{2}\int_{0}^{\frac{\pi}{4}} \frac{\sqrt{(\sqrt{2}\sec\theta)^2-2}}{\sqrt{2}\sec\theta}\ \sqrt{2}\sec\theta\tan\theta\ d\theta$$

$$= \sqrt{2}\int_{0}^{\frac{\pi}{4}} \sqrt{2\sec^2\theta-2}\ \tan\theta\ d\theta$$

$$= \sqrt{2}\int_{0}^{\frac{\pi}{4}} \sqrt{2(\sec^2\theta-1)}\ \tan\theta\ d\theta$$

$$= \sqrt{2}\int_{0}^{\frac{\pi}{4}} \sqrt{2}\ \sqrt{\tan^2\theta}\ \tan\theta\ d\theta$$

$$= 2\int_{0}^{\frac{\pi}{4}} |\tan\theta|\ \tan\theta\ d\theta$$

$$= 2\int_{0}^{\frac{\pi}{4}} \tan\theta \cdot \tan\theta\ d\theta$$

$$= 2\int_{0}^{\frac{\pi}{4}} \tan^2\theta\ d\theta$$

$$= 2\int_{0}^{\frac{\pi}{4}} [\,sec^2\theta-1\,]\ d\theta$$

$$= 2[\tan\theta - \theta]_0^{\frac{\pi}{4}}$$

$$= 2\left[\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - 2\tan 0 + 0\right]$$

$$= 2\left[1 - \frac{\pi}{4}\right]$$

$$= 2 \times 4[4 - \pi]$$

$$= \frac{1}{2}[4 - \pi].$$

Exercise: 7 Evaluate

$$\int \frac{\cos\theta}{\sqrt{2 - \sin^2\theta}}\, d\theta$$

Solution: Given integral

$$\int \frac{\cos\theta}{\sqrt{2 - \sin^2\theta}}\, dx \quad ; \quad Set\ u = \sin\theta\ \text{, then } du = \cos\theta\ d\theta.$$

$$= \int \frac{1}{\sqrt{2 - u^2}}\, du \quad ; \quad u = \sqrt{2}\ \sin x\ \text{. Then } du = \sqrt{2}\ \cos x\ dx$$

$$= \int \frac{1}{\sqrt{2 - \left(\sqrt{2}\ \sin x\right)^2}}\ \sqrt{2}\ \cos x\ dx$$

$$= \int \frac{1}{\sqrt{2 - 2\sin^2 x}}\ \sqrt{2}\ \cos x\ dx$$

$$= \int \frac{1}{\sqrt{2\,cos^2\,x}}\ \sqrt{2}\ \cos x\ dx$$

$$= \int \frac{1}{\sqrt{2}|\cos x|}\ \sqrt{2}\ \cos x\ dx$$

$$= \int \frac{1}{\sqrt{2}\ \cos x}\ \sqrt{2}\ \cos x\ dx\ ; \quad \text{For} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$= \int 1 \quad dx = x + C \quad ; \quad u = \sqrt{2}\ \sin x \to \sin x = \frac{u}{\sqrt{2}}$$

$$= \sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \sin^{-1}\left(\frac{\sin\theta}{\sqrt{2}}\right) + C$$

Exercise: 8 Evaluate the integral

$$\int e^x \sqrt{1 - e^{2x}}\ dx.$$

Please submit in 10 minutes.