Laplace Transforms

Definition.

The Laplace transform, named after Pierre-Simon De **Laplace**, is an **integral transform** from a function of a real variable t (often time) to a function of a complex variable s (complex frequency). The transform has many applications in engineering and sciences as it is used to solve linear ODEs as well as system of linear ODEs. The transform is written as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty k(s,t)\, f(t) dt = \int_0^\infty e^{-st}\, f(t)\, dt,$$

where,

$f(t)$: function of $t \geq 0$, which must be integrable on $[0, \infty]$ as for a necessary condition

$s = \sigma + i\omega$ with real numbers σ and ω

$\mathcal{L}$: The Laplace transform operator.

$k(s,t) = e^{-st}$ is called the kernel of the transform

The original function $f(t)$ is called the inverse Laplace transform or inverse of $F(s)$ and is denoted by $\mathcal{L}^{-1}\{F(s)\}$ and is defined by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Therefore, $\mathcal{L}^{-1}\{\mathcal{L}(f)\} = f$ and $\mathcal{L}\{\mathcal{L}^{-1}(F)\} = F$

Laplace Transforms

Example-1. If $f(t) = 1$ for $t \geq 0$ then

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{b\to\infty} \int_0^b e^{-st}\, dt = \lim_{b\to\infty} \left[\frac{1}{-s}\, e^{-st}\Big|_0^b\right]$$

$$= \lim_{b\to\infty} \left[-\frac{1}{s}(e^{-bs} - 1)\right] = -\frac{1}{s}[(e^{-\infty} - 1)] = \frac{1}{s} \qquad (s > 0)$$

Example-2. If $f(t) = t$ for $t \geq 0$ then

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = \lim_{b\to\infty} \int_0^b t\, e^{-st}\, dt = \lim_{b\to\infty} -\left(\frac{t}{s} + \frac{1}{s^2}\right) e^{-st}\Big|_0^b$$

$$= \lim_{b\to\infty} \left[-\left(\frac{b}{s} + \frac{1}{s^2}\right) e^{-bs} + \frac{1}{s^2}\right] = \frac{1}{s^2} \qquad (s > 0)$$

Example-3. If $f(t) = e^t$ for $t \geq 0$ then

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^t\} = \int_0^\infty e^{-st} \cdot e^t\, dt = \lim_{b\to\infty} \int_0^b e^{-(s-1)t}\, dt$$

$$= \lim_{b\to\infty} -\left(\frac{1}{s-1}\right) e^{-(s-1)t}\Big|_0^b = \lim_{b\to\infty} \left[-\left(\frac{1}{s-1}\right)\left(e^{-(s-1)b} - 1\right)\right] = \frac{1}{s-1} \qquad (s > 1)$$

Laplace Transforms

Linearity of the Laplace Transform:

$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$, where a & b are constants.

First Shifting Theorem or s-Shifting:

If $\mathcal{L}\{f(t)\}=F(s)$, then $\mathcal{L}\{e^{at}f(t)\}=F(s-a)$ and hence $\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)$

Multiplication by t^n: If $\mathcal{L}\{f(t)\}=F(s)$, then $\mathcal{L}\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}[F(s)]$

Example-4. If $f(t)=e^{iat}$ for $t\geq 0$ then

$$F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\left\{e^{iat}\right\}=\frac{1}{s-ia}=\frac{s+ia}{(s-ia)(s+ia)}=\frac{s+ia}{s^2+a^2} \qquad (s^2+a^2>0)$$

But, $e^{iat}=\cos(at)+i\sin(at)$, then $\mathcal{L}\left\{e^{iat}\right\}=\frac{s+ia}{s^2+a^2} \Rightarrow \mathcal{L}\left\{e^{iat}\right\}=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}$

$$\Rightarrow \mathcal{L}\{\cos(at)+i\cdot\sin(at)\}=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}$$

$$\Rightarrow \mathcal{L}\{\cos(at)\}+i\cdot\mathcal{L}\{\sin(at)\}=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}$$

$$Thus, \mathcal{L}\{\cos(at)\}=\frac{s}{s^2+a^2} \text{ and } \mathcal{L}\{\sin(at)\}=\frac{a}{s^2+a^2} \quad [s>0]$$

Laplace Transforms

Linearity of the Laplace Transform:

$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$, where a & b are constants.

First Shifting Theorem or s-Shifting:

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ and hence $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Multiplication by t^n: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Example-5. If $f(t) = e^{at}\cos\omega t$ for $t \geq 0$ then

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\cos\omega t\} = \int_0^\infty e^{-st}e^{at}\cos\omega t\,dt = \lim_{b\to\infty}\int_0^b e^{-(s-a)t}\cos\omega t\,dt$$

$$\Rightarrow \int_0^\infty e^{-st}e^{at}\cos\omega t\,dt = \lim_{b\to\infty}\left[\left\{\frac{\omega\sin\omega t}{(s-a)^2} - \frac{\cos\omega t}{s-a}\right\}e^{-(s-a)t}\right]_0^b - \lim_{b\to\infty}\frac{\omega^2}{(s-a)^2}\int_0^b e^{-(s-a)t}\cos\omega t\,dt$$

$$\Rightarrow \left\{1+\frac{\omega^2}{(s-a)^2}\right\}\int_0^\infty e^{-st}e^{at}\cos\omega t\,dt = \lim_{b\to\infty}\left[\left\{\frac{\omega\sin\omega b}{(s-a)^2} - \frac{\cos\omega b}{s-a}\right\}e^{-(s-a)b} + \frac{1}{s-a}\right] = \frac{1}{s-a}$$

$$\Rightarrow \int_0^\infty e^{-st}e^{at}\cos\omega t\,dt = \frac{(s-a)^2}{(s-a)^2+\omega^2}\cdot\frac{1}{(s-a)} \qquad [s>a]$$

$$\therefore \mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2} = F(s-a) \quad \text{where} \quad F(s) = \frac{s}{s^2+\omega^2} = \mathcal{L}\{\cos\omega t\}$$

Laplace Transforms

❖ Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

1. $\mathcal{L}\{c\} = \frac{c}{s}$, c is any constant, $(s > 0)$

2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, when $n = 0,1,2,3,\dots$

3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $(s > \mathrm{a})$

4. $\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$ $(s > 0)$

5. $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$ $(s > 0)$

6. $\mathcal{L}\{\cosh\omega t\} = \frac{s}{s^2-\omega^2}$ $(s > 0)$

7. $\mathcal{L}\{\sinh\omega t\} = \frac{\omega}{s^2-\omega^2}$ $(s > 0)$

8. $\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$

9. $\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$

10. $\mathcal{L}\{e^{at}\cosh\omega t\} = \frac{s-a}{(s-a)^2-\omega^2}$

11. $\mathcal{L}\{e^{at}\sinh\omega t\} = \frac{\omega}{(s-a)^2-\omega^2}$

Laplace Transforms

Example-7.

$$\mathcal{L}\{(t^2+1)^2\} = \mathcal{L}\{t^4+2t^2+1\} = \mathcal{L}\{t^4\}+2\mathcal{L}\{t^2\}+\mathcal{L}\{1\} = \frac{4!}{s^{4+1}}+2\frac{2!}{s^{2+1}}+\frac{1}{s} = \frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s}$$

Example-8.

$$\mathcal{L}\{\mathrm{e}^{-3t}+5\cosh t\} = \mathcal{L}\{\mathrm{e}^{-3t}\}+5\,\mathcal{L}\{\cosh\ t\} = \frac{1}{\mathrm{s}-(-3)}+5\frac{\mathrm{s}}{\mathrm{s}^2-1^2} = \frac{1}{\mathrm{s}+3}+\frac{5\mathrm{s}}{\mathrm{s}^2-1}$$

Example-9.

$$\mathcal{L}\{(\sin t-\cos t)^2+\cos^2 3t\} = \mathcal{L}\left\{\sin^2 t-2\sin t\cos t+\cos^2 t+\frac{1}{2}(1+\cos 6t)\right\}$$

$$= \mathcal{L}\left\{1-\sin 2t+\frac{1}{2}+\frac{1}{2}\cos 6t)\right\} = \mathcal{L}\left\{\frac{3}{2}\right\}-\mathcal{L}\{\sin 2t\}+\frac{1}{2}\mathcal{L}\{\cos 6t\}$$

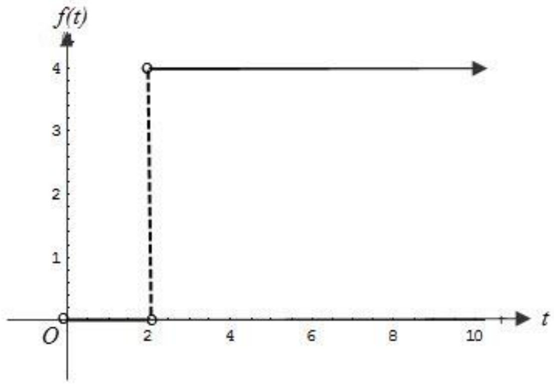
$$= \frac{3}{2}\frac{1}{s}-\frac{2}{s^2+2^2}+\frac{1}{2}\frac{s}{s^2+6^2} = \frac{3}{2s}-\frac{2}{s^2+4}+\frac{1}{2}\left(\frac{s}{s^2+36}\right)$$

Example-10.

If $f(t)=\begin{cases}0, & 0<t<2\\ 4, & t>2\end{cases}$ then

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \int_0^2 e^{-st}.0\,dt+\int_2^\infty e^{-st}\,4\,dt$$

$$= 4\left[\frac{e^{-st}}{-s}\right]\Bigg|_2^\infty = 4\frac{e^{-2s}}{s}$$



Inverse Laplace Transforms

❖ Some Functions $F(s)$ and Their Inverse Laplace Transforms $\mathcal{L}^{-1}(F)$

1. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \quad (s > 0)$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$, when $n = 0,1,2,3,\ldots$

3. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad (s - \mathrm{a} > 0)$

4. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos \omega t$

5. $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin \omega t$

6. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-\omega^2}\right\} = \cosh \omega t$

7. $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2-\omega^2}\right\} = \sinh \omega t$

8. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+\omega^2}\right\} = e^{at} \cos \omega t$

9. $\mathcal{L}^{-1}\left\{\frac{\omega}{(s-a)^2+\omega^2}\right\} = e^{at} \sin \omega t$

10. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2-\omega^2}\right\} = e^{at} \cosh \omega t$

11. $\mathcal{L}^{-1}\left\{\frac{\omega}{(s-a)^2-\omega^2}\right\} = e^{at} \sinh \omega t$

Laplace Transforms

Example-11.

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{s^3}\right\} = 1+\frac{t^2}{2!} = 1+\frac{t^2}{2}$$

Example-12.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)}-\frac{1}{(s-2)}\right\} = e^{3t}-e^{2t}$$

Example-13.

$$\mathcal{L}^{-1}\left\{\frac{4}{s-2}-\frac{s}{s^2-16}+\frac{4}{s^2-4}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-\mathcal{L}^{-1}\left\{\frac{s}{s^2-4^2}\right\}+2\mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\}$$

$$= 4e^{2t}-\cosh 4t+2\sinh 2t$$

Example-14.

$$\mathcal{L}^{-1}\left\{\frac{5}{s}-\frac{3s}{s^2+16}+\frac{2}{s^2+4}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}+\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 5-3\cos 4t+\sin 2t$$

Laplace Transforms

Exercise problems:

Find the Laplace transform. Show the details of your work. Assume that a, b, ω, θ are constants

1. $3t + 12$

2. $(a - bt)^2$

3. $\cos \pi t$

4. $\cos^2 \omega t$

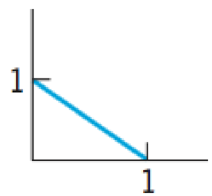
5. $e^{2t} \sinh t$

6. $e^{-t} \sinh 4t$

7. $\sin(\omega t + \theta)$

8. $1.5 \sin(3t - \pi/2)$

9.



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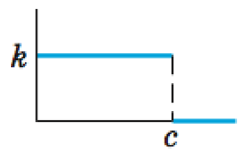

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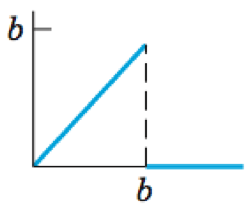

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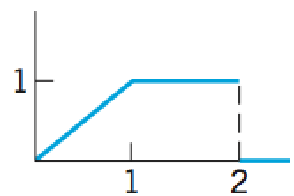

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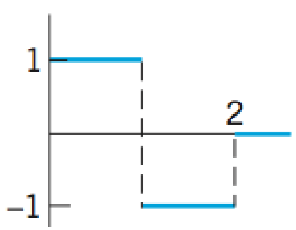

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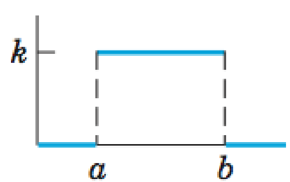


33. $t^2 e^{-3t}$

34. $ke^{-at} \cos \omega t$

35. $0.5e^{-4.5t} \sin 2\pi t$

36. $\sinh t \cos t$

Given $F(s) = \mathcal{L}(f)$, find $f(t)$. a, b, L, n are constants. Show the details of your work.

25. $\dfrac{0.2s + 1.8}{s^2 + 3.24}$

26. $\dfrac{5s + 1}{s^2 - 25}$

27. $\dfrac{s}{L^2 s^2 + n^2 \pi^2}$

28. $\dfrac{1}{(s + \sqrt{2})(s - \sqrt{3})}$

29. $\dfrac{12}{s^4} - \dfrac{228}{s^6}$

30. $\dfrac{4s + 32}{s^2 - 16}$

31. $\dfrac{s + 10}{s^2 - s - 2}$

32. $\dfrac{1}{(s + a)(s + b)}$

37. $\dfrac{\pi}{(s + \pi)^2}$

38. $\dfrac{6}{(s + 1)^3}$

39. $\dfrac{21}{(s + \sqrt{2})^4}$

40. $\dfrac{4}{s^2 - 2s - 3}$

41. $\dfrac{\pi}{s^2 + 10\pi s + 24\pi^2}$

42. $\dfrac{a_0}{s + 1} + \dfrac{a_1}{(s + 1)^2} + \dfrac{a_2}{(s + 1)^3}$

Laplace Transforms to solve IVPs

The process of solving an ODE using the Laplace transform method consists of three steps;

***Step 1*.** The given ODE is transformed into an algebraic equation (**subsidiary equation**).

***Step 2*.** The subsidiary equation is solved by purely algebraic manipulations.

***Step 3*.** The solution in Step 2 is transformed back, resulting in the required solution.

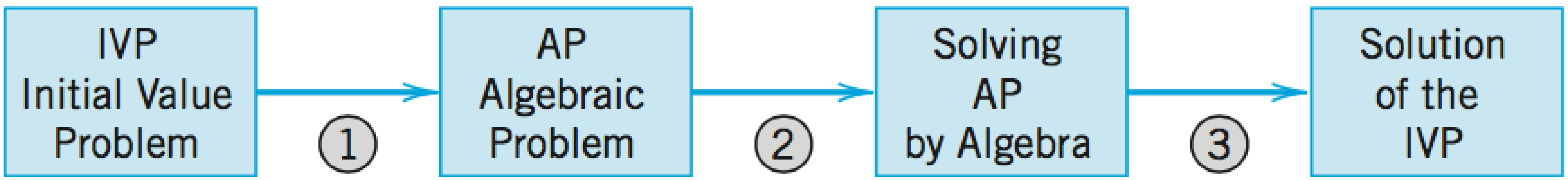


Figure. Solving an IVP by Laplace transforms

Laplace Transform of Derivatives

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = s\mathcal{L}(y) - y(0) = sY(s) - y(0).$$

$$\mathcal{L}\{y''(t)\} = \mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0)$$

where $y(0)$, and $y'(0)$ are the initial conditions of $y(t)$.

Laplace Transforms to solve IVPs

1. $\mathcal{L}\{c\} = \frac{c}{s}$, c is any constant, $(s > 0)$	1. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \quad (s > 0)$
2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, when $n = 0,1,2,3,\ldots$	2. $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$, when $n = 0,1,2,3,\ldots$
3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (s > \mathrm{a})$	3. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad (s - \mathrm{a} > 0)$
4. $\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2} \quad (s > 0)$	4. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos \omega t$
5. $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2} \quad (s > 0)$	5. $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin \omega t$
6. $\mathcal{L}\{\cosh \omega t\} = \frac{s}{s^2-\omega^2} \quad (s > 0)$	6. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-\omega^2}\right\} = \cosh \omega t$
7. $\mathcal{L}\{\sinh \omega t\} = \frac{\omega}{s^2-\omega^2} \quad (s > 0)$	7. $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2-\omega^2}\right\} = \sinh \omega t$
8. $\mathcal{L}\{e^{at} \cos \omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$	8. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+\omega^2}\right\} = e^{at} \cos \omega t$
9. $\mathcal{L}\{e^{at} \sin \omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$	9. $\mathcal{L}^{-1}\left\{\frac{\omega}{(s-a)^2+\omega^2}\right\} = e^{at} \sin \omega t$
10. $\mathcal{L}\{e^{at} \cosh \omega t\} = \frac{s-a}{(s-a)^2-\omega^2}$	10. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2-\omega^2}\right\} = e^{at} \cosh \omega t$
11. $\mathcal{L}\{e^{at} \sinh \omega t\} = \frac{\omega}{(s-a)^2-\omega^2}$	11. $\mathcal{L}^{-1}\left\{\frac{\omega}{(s-a)^2-\omega^2}\right\} = e^{at} \sinh \omega t$

Laplace Transforms to solve IVPs

Example. Solve the IVP: $y''(t) + y(t) = e^t,\ y(0) = 1, y'(0) = -2.$

Solution. Let, $\mathcal{L}\{y(t)\} = Y(s)$. Applying Laplace transform to the given ODE, we have

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\}$$

$$\Rightarrow s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s-1}$$

$$\Rightarrow s^2Y(s) - s(1) - (-2) + Y(s) = \frac{1}{s-1}$$

$$\Rightarrow (s^2+1)Y(s) = \frac{1}{s-1} + (s-2) \Rightarrow Y(s) = \frac{s^2-3s+3}{(s^2+1)(s-1)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2-3s+3}{(s^2+1)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-1} + \frac{Bs+C}{s^2+1}\right\}$$

$$\frac{s^2-3s+3}{(s^2+1)\,(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \Rightarrow s^2-3s+3 = A(s^2+1) + (Bs+C)(s-1)$$

$$\Rightarrow s^2-3s+3 \equiv (A+B)s^2 + (C-B)s + A - C$$

Equating coefficients: $A + B = 1, C - B = -3, A - C = 3$

Solving we get, $A = \frac{1}{2},\ B = \frac{1}{2}, C = -\frac{5}{2}$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s-1} + \frac{\frac{s}{2}}{s^2+1} - \frac{\frac{5}{2}}{s^2+1}\right\} \Rightarrow y(t) = \frac{e^t}{2} + \frac{1}{2}\cos t - \frac{5}{2}\sin t$$

Laplace Transforms to solve IVPs

Example. Solve the IVP: $y''(t) - y(t) = t,\ \ y(0) = 1, y'(0) = 1.$

Solution. Let, $\mathcal{L}\{y(t)\} = Y(s)$. Now, applying Laplace transform to the given ODE, we have

$$\mathcal{L}\{y''(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

$$\Rightarrow s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s^2}$$

$$\Rightarrow s^2Y(s) - s - 1 - Y(s) = \frac{1}{s^2}$$

$$\Rightarrow (s^2 - 1)Y(s) = \frac{1}{s^2} + (s+1) = (s+1) + \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{(s+1)}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\therefore y(t) = e^t + \sinh t - t$$

Laplace Transforms to solve IVPs

Example. Solve the IVP: $y''(t) + y(t) = 2t,\ y\left(\frac{\pi}{4}\right) = \frac{\pi}{2},\quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}.$

Solution. Let, $t = \bar{t} + \frac{\pi}{4}$ so that as $t = \pi/4$ implies $\bar{t} = 0$. Then the problem becomes,

$$\bar{y}'' + \bar{y} = 2\left(\bar{t} + \frac{\pi}{4}\right), \qquad \bar{y}(0) = \frac{\pi}{2},\ \bar{y}'(0) = 2 - \sqrt{2}\ \text{ where }\ \bar{y}(\bar{t}) = y(t)$$

Now, applying Laplace transform to the given ODE and denoting $\mathcal{L}\{\bar{y}(\bar{t})\} = \bar{Y}(s)$, we get

$$\mathcal{L}\{\bar{y}''\} + \mathcal{L}\{\bar{y}(\bar{t})\} = 2\mathcal{L}\left\{\bar{t} + \frac{\pi}{4}\right\}$$

$$\Rightarrow s^2\bar{Y}(s) - s\bar{y}(0) - \bar{y}'(0) + \bar{Y}(s) = 2\left[\mathcal{L}\{\bar{t}\} + \mathcal{L}\left\{\frac{\pi}{4}\right\}\right] = 2\left[\frac{1}{s^2} + \frac{\pi}{4s}\right] = \frac{2}{s^2} + \frac{\pi}{2s}$$

$$\Rightarrow (s^2 + 1)\bar{Y}(s) = \frac{2}{s^2} + \frac{\pi}{2s} + \frac{\pi s}{2} + (2 - \sqrt{2})$$

$$\Rightarrow \bar{Y}(s) = \frac{2}{s^2(s^2+1)} + \frac{\pi}{2s(s^2+1)} + \frac{\pi}{2}\frac{s}{s^2+1} + \left(2 - \sqrt{2}\right)\frac{1}{s^2+1}$$

$$\Rightarrow \bar{Y}(s) = \left[\frac{2}{s^2} - \frac{2}{s^2+1}\right] + \frac{\pi}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{\pi}{2}\frac{s}{s^2+1} + \left(2 - \sqrt{2}\right)\frac{1}{s^2+1}$$

Laplace Transforms to solve IVPs

Example. Solve the IVP: $y''(t) + y(t) = 2t,\ y\left(\frac{\pi}{4}\right) = \frac{\pi}{2},\quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}.$

Solution.

$$\Rightarrow \bar{Y}(s) = \left[\frac{2}{s^2} - \frac{2}{s^2+1}\right] + \frac{\pi}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{\pi}{2}\frac{s}{s^2+1} + \left(2-\sqrt{2}\right)\frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{\bar{Y}(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2} + \frac{\pi}{2}\frac{1}{s} - \sqrt{2}\frac{1}{s^2+1}\right\}$$

$$\Rightarrow \bar{y}(\bar{t}) = 2\bar{t} - \sqrt{2}\sin\bar{t} + \frac{\pi}{2}$$

$$\Rightarrow y(t) = 2\left(t - \frac{\pi}{4}\right) - \sqrt{2}\sin\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2} = 2t - \sqrt{2}\left(\sin t\cos\frac{\pi}{4} - cos\,t\sin\frac{\pi}{4}\right)$$

$$\therefore y(t) = 2t - \sin t + \cos t.$$

Advantages of the Laplace Method

- *Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE as we have seen in the above example.*
- *Initial values are automatically taken care of after the Laplace transform of a derivative term.*
- *Complicated inputs* $r(t)$ *(right sides of linear ODEs) can be handled very efficiently.*

Laplace Transforms of Integral

Laplace Transform of Integral:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s) \quad \text{thus} \quad \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{f(t)\}\right\}$$

Example. Find $F(t)$ if $\mathcal{L}\{F\} = \frac{3}{s^2+s/4}$.

Solution. Here,

$$\frac{1}{s^2+s/4} = \frac{1}{s}.\frac{1}{s+1/4} = \frac{1}{s}\mathcal{L}\left\{e^{-\frac{1}{4}t}\right\}$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+s/4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+s/4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{e^{-\frac{1}{4}t}\right\}\right\} = 3\int_0^t e^{-\tau/4}\,d\tau$$

$$\therefore F(t) = 12\left[-e^{-\frac{\tau}{4}}\right]_0^t = 12\left(1-e^{-t/4}\right)$$

Laplace Transforms

Exercise problems:

Solve the following IVPs by the Laplace transform showing the procedure in details

1. $y' + 5.2y = 19.4 \sin 2t, \quad y(0) = 0$

2. $y' + 2y = 0, \quad y(0) = 1.5$

3. $y'' - y' - 6y = 0, \quad y(0) = 11, \quad y'(0) = 28$

4. $y'' + 9y = 10e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$

5. $y'' - \frac{1}{4}y = 0, \quad y(0) = 12, \quad y'(0) = 0$

6. $y'' - 6y' + 5y = 29 \cos 2t, \quad y(0) = 3.2,$
$y'(0) = 6.2$

7. $y'' + 7y' + 12y = 21e^{3t}, \quad y(0) = 3.5,$
$y'(0) = -10$

8. $y'' - 4y' + 4y = 0, \quad y(0) = 8.1, \quad y'(0) = 3.9$

9. $y'' - 4y' + 3y = 6t - 8, \quad y(0) = 0, \quad y'(0) = 0$

10. $y'' + 0.04y = 0.02t^2, \quad y(0) = -25, \quad y'(0) = 0$

11. $y'' + 3y' + 2.25y = 9t^3 + 64, \quad y(0) = 1,$
$y'(0) = 31.5$

Solve the shifted data IVPs by the Laplace transform showing the procedure in details

12. $y'' - 2y' - 3y = 0, \quad y(4) = -3,$
$y'(4) = -17$

13. $y' - 6y = 0, \quad y(-1) = 4$

14. $y'' + 2y' + 5y = 50t - 100, \quad y(2) = -4,$
$y'(2) = 14$

15. $y'' + 3y' - 4y = 6e^{2t-3}, \quad y(1.5) = 4,$
$y'(1.5) = 5$

Using Laplace transform for integrals, find $F(t)$ if $\mathcal{L}\{F\}$ equals:

23. $\dfrac{3}{s^2 + s/4}$

24. $\dfrac{20}{s^3 - 2\pi s^2}$

25. $\dfrac{1}{s(s^2 + \omega^2)}$

26. $\dfrac{1}{s^4 - s^2}$

27. $\dfrac{s + 1}{s^4 + 9s^2}$

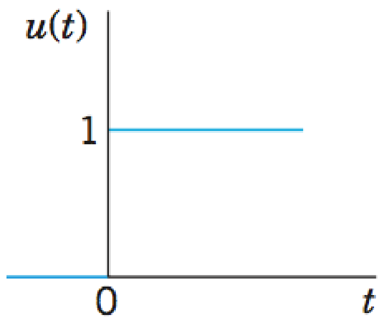
28. $\dfrac{3s + 4}{s^4 + k^2 s^2}$

29. $\dfrac{1}{s^3 + as^2}$

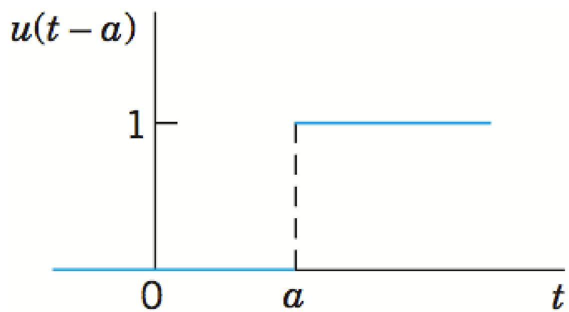
Laplace Transforms

❑ Unit Step Function or Heaviside Function:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geq 0)$$

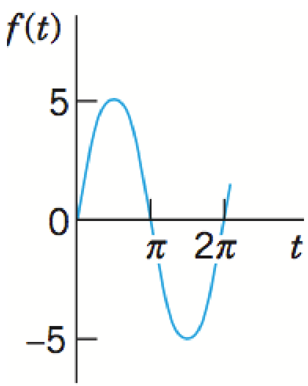


Unit step function $u(t)$

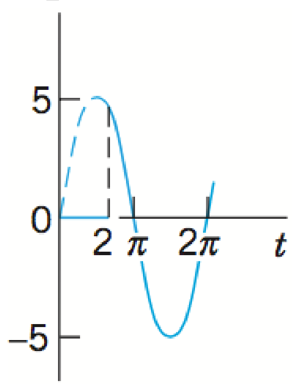


Unit step function $u(t-a)$

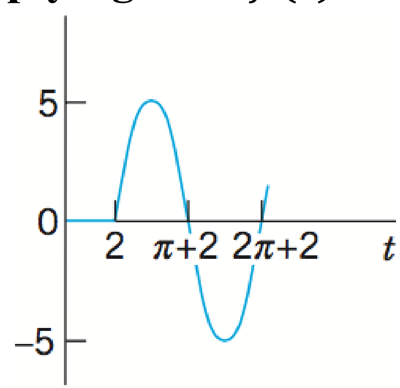
❑ Effect of unit step function multiplying with $f(t)$:



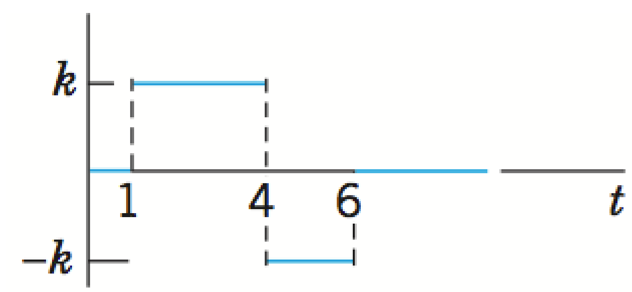
$f(t) = 5\sin t$



$f(t)\,u(t-2)$



$f(t-2)\,u(t-2)$



$k[u(t-1) - 2u(t-4) + u(t-6)]$

Laplace Transforms

- **Laplace transform of Unit Step Function or Heaviside Function:**

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt = \int_a^\infty e^{-st}\cdot 1\, dt = -\left.\frac{e^{-st}}{s}\right|_a^\infty = \frac{e^{-as}}{s} \qquad (s>a)$$

- **Second shifting theorem: t-shifting**

If $\mathcal{L}\{f(t)\} = F(s)$ and $f(t-a)\,u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$ then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \qquad i.e., \qquad f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

Example. Write the following function using unit step functions and find its Laplace transform

$$f(t) = \begin{cases} 1 & \text{if } 0<t<a \\ 0 & \text{if } t>a \end{cases}$$

Solution. We can write, $f(t) = \begin{cases} 1-0 & \text{if } 0<t<a \\ 1-1 & \text{if } t>a \end{cases} = \begin{cases} 1 & \text{if } 0<t<a \\ 1 & \text{if } t>a \end{cases} - \begin{cases} 0 & \text{if } 0<t<a \\ 1 & \text{if } t>a \end{cases}$

$$= \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t>0 \end{cases} - \begin{cases} 0 & \text{if } 0<t<a \\ 1 & \text{if } t>a \end{cases}$$

$$= u(t) - u(t-a)$$

Now, $\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-a)\} = \frac{1}{s} - \frac{e^{-as}}{s} = \frac{1}{s}(1-e^{-as})$.

Laplace Transforms

Example. Write the following function using unit step functions and find its Laplace transform

$$f(t) = \begin{cases} 0 & \text{if } t < 3 \\ 2t+8 & \text{if } 3 < t < 5 \\ 0 & \text{if } t > 5 \end{cases}$$

Solution. We can write, $f(t) = \begin{cases} 0 & \text{if } t < 3 \\ 2t+8 & \text{if } 3 < t < 5 \\ 0 & \text{if } t > 5 \end{cases} = (2t+8)\begin{cases} 0 & \text{if } t < 3 \\ 1 & \text{if } 3 < t < 5 \\ 0 & \text{if } t > 5 \end{cases}$

$$= (2t+8)\left[\begin{cases} 0 & \text{if } t < 3 \\ 1 & \text{if } t > 3 \end{cases} - \begin{cases} 0 & \text{if } t < 5 \\ 1 & \text{if } t > 5 \end{cases}\right]$$

$$= (2t+8)[u(t-3) - u(t-5)]$$

Now, $\mathcal{L}\{f(t)\} = \mathcal{L}\{(2t+8)[u(t-3)-u(t-5)]\}$

$$= \mathcal{L}\{2t\,u(t-3) - 2t\,u(t-5) + 8\,u(t-3) - 8\,u(t-5)\}$$

$$= 2\mathcal{L}\{t\,u(t-3)\} - 2\,\mathcal{L}\{t\,u(t-5)\} + 8\,\mathcal{L}\{u(t-3)\} - 8\,\mathcal{L}\{u(t-5)\}$$

$$= 2\,e^{-3s}\mathcal{L}\{t+3\} - 2\,e^{-5s}\mathcal{L}\{t+5\} + 8\frac{e^{-3s}}{s} - 8\frac{e^{-5s}}{s}$$

$$\boxed{\begin{aligned} \mathcal{L}\{f(t)u(t-a)\} &= \mathcal{L}\{f(t+a-a)u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\} \\ f(t)u(t-a) &= f(t+a-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}\,\mathcal{L}\{f(t+a)\}\} \end{aligned}}$$

Laplace Transforms

Example. Write the following function using unit step functions and find its Laplace transform

$$f(t) = \begin{cases} 0 & \text{if } t < 3 \\ 2t + 8 & \text{if } 3 < t < 5 \\ 0 & \text{if } \ t > 5 \end{cases}$$

Solution.

$$\mathcal{L}\{f(t)\} = 2\, e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) - 2e^{-5s}\left(\frac{1}{s^2} + \frac{5}{s}\right) + 8\frac{e^{-3s}}{s} - 8\frac{e^{-5s}}{s}$$

$$= \left(\frac{2}{s^2} + \frac{6}{s} + \frac{8}{s}\right)e^{-3s} - \left(\frac{2}{s^2} + \frac{10}{s} + \frac{8}{s}\right)e^{-5s}$$

$$= 2\left(\frac{1}{s^2} + \frac{7}{s}\right)e^{-3s} - 2\left(\frac{1}{s^2} + \frac{9}{s}\right)e^{-3s}.$$

Example.

$$\mathcal{L}\{e^{-2t}u(t-2)\} = e^{-2s}\mathcal{L}\{f(t+2)\} = e^{-2s}\mathcal{L}\left\{e^{-2(t+2)}\right\} = e^{-(2s+4)}\mathcal{L}\{e^{-2t}\} = \frac{e^{-2(s+2)}}{s+2}$$

Example.

$$\mathcal{L}\{t\, u(t-\pi)\} = e^{-\pi s}\mathcal{L}\{f(t+\pi)\} = e^{-\pi s}\mathcal{L}\{t+\pi\} = \left(\frac{1}{s^2} + \frac{\pi}{s}\right)e^{-\pi s}$$

Laplace Transforms

Example. Write the following function using unit step functions and find its Laplace transform

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ 0.5\,t^2 & \text{if } 1 < t < \pi/2 \\ \cos t & \text{if } \ t > \pi/2 \end{cases}$$

Solution. We can write $f(t)$ as,

$$f(t) = 2[u(t) - u(t-1)] + 0.5t^2\,[u(t-1) - u(t-\pi/2)] + \cos t\ \ u(t-\pi/2)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = 2\,[\mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-1)\}] + 0.5\left[\mathcal{L}\{t^2 u(t-1)\} - \mathcal{L}\left\{t^2 u\left(t-\frac{\pi}{2}\right)\right\}\right]$$

$$+\mathcal{L}\{\cos t\ \ u(t-\pi/2)\,\}$$

$$= 2\left[\frac{1}{s} - \frac{e^{-s}}{s}\right] + \frac{1}{2}\left[e^{-s}\mathcal{L}\{(t+1)^2\} - e^{-\frac{\pi}{2}s}\mathcal{L}\left\{\left(t+\frac{\pi}{2}\right)^2\right\}\right] + e^{-\frac{\pi}{2}s}\mathcal{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$$

$$= \frac{2}{s} - \frac{2}{s}e^{-s} + \frac{1}{2}\left[e^{-s}\mathcal{L}\{t^2 + 2t + 1\} - e^{-\frac{\pi}{2}s}\mathcal{L}\left\{t^2 + \pi t + \frac{\pi^2}{4}\right\}\right] - e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin t\}$$

$$= \frac{2}{s} - \frac{2}{s}e^{-s} + \frac{1}{2}\left[e^{-s}\left\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right\} - e^{-\frac{\pi}{2}s}\left\{\frac{2}{s^3} + \frac{\pi}{s^2} + \frac{\pi^2}{4s}\right\}\right] - e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}$$

$$= \frac{2}{s} + \left(\frac{1}{s^3} + \frac{1}{s^2} - \frac{3}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s} + \frac{1}{s^2+1}\right)e^{-\frac{\pi s}{2}}$$

Laplace Transforms

Exercise problems:

Sketch and represent the following function using unit step functions and find its Laplace transform

2. $t\ (0 < t < 2)$

3. $t - 2\ (t > 2)$

4. $\cos 4t\ (0 < t < \pi)$

5. $e^t\ (0 < t < \pi/2)$

6. $\sin \pi t\ (2 < t < 4)$

7. $e^{-\pi t}\ (2 < t < 4)$

8. $t^2\ (1 < t < 2)$

9. $t^2\ (t > \frac{3}{2})$

10. $\sinh t\ (0 < t < 2)$

11. $\sin t\ (\pi/2 < t < \pi)$

Find and sketch or graph $f(t)$ if $\mathcal{L}(f)$ equals

12. $e^{-3s}/(s-1)^3$

13. $6(1 - e^{-\pi s})/(s^2 + 9)$

14. $4(e^{-2s} - 2e^{-5s})/s$

15. e^{-3s}/s^4

16. $2(e^{-s} - e^{-3s})/(s^2 - 4)$

17. $(1 + e^{-2\pi(s+1)})(s+1)/((s+1)^2 + 1)$

Using the Laplace transform and showing the details, solve

18. $9y'' - 6y' + y = 0, \quad y(0) = 3, y'(0) = 1$

19. $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, \quad y(0) = 0, y'(0) = 0$

20. $y'' + 10y' + 24y = 144t^2, \quad y(0) = 19/12,$
$y'(0) = -5$

21. $y'' + 9y = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$;
$y(0) = 0, y'(0) = 4$

22. $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$;
$y(0) = 0, y'(0) = 0$

23. $y'' + y' - 2y = 3 \sin t - \cos t$ if $0 < t < 2\pi$ and
$3 \sin 2t - \cos 2t$ if $t > 2\pi$; $\quad y(0) = 1, y'(0) = 0$

24. $y'' + 3y' + 2y = 1$ if $0 < t < 1$ and 0 if $t > 1$;
$y(0) = 0, y'(0) = 0$

25. $y'' + y = t$ if $0 < t < 1$ and 0 if $t > 1$; $\quad y(0) = 0,$
$y'(0) = 0$