Active Calculus - Activities Workbook

Matt Boelkins
Department of Mathematics
Grand Valley State University

December 30, 2013

Contents

Chapter 1

Understanding the Derivative

1.1 How do we measure velocity?

Preview Activity 1.1. Suppose that the height s of a ball (in feet) at time t (in seconds) is given by the formula $s(t) = 64 - 16(t-1)^2$.

(a) Construct an accurate graph of $y = s(t)$ on the time interval $0 \le t \le 3$. Label at least six distinct points on the graph, including the three points that correspond to when the ball was released, when the ball reaches its highest point, and when the ball lands.

(b) In everyday language, describe the behavior of the ball on the time interval $0 < t < 1$ and on time interval $1 < t < 3$. What occurs at the instant $t = 1$?

(c) Consider the expression

$$AV_{[0.5,1]} = \frac{s(1) - s(0.5)}{1 - 0.5}.$$

Compute the value of $AV_{[0.5,1]}$. What does this value measure geometrically? What does this value measure physically? In particular, what are the units on $AV_{[0.5,1]}$?

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Activity 1.1.

The following questions concern the position function given by $s(t) = 64 - 16(t-1)^2$, which is the same function considered in Preview Activity 1.1.

(a) Compute the average velocity of the ball on each of the following time intervals: $[0.4, 0.8]$, $[0.7, 0.8]$, $[0.79, 0.8]$, $[0.799, 0.8]$, $[0.8, 1.2]$, $[0.8, 0.9]$, $[0.8, 0.81]$, $[0.8, 0.801]$. Include units for each value.

(b) On the provided graph in Figure 1.1, sketch the line that passes through the points $A = (0.4, s(0.4))$ and $B = (0.8, s(0.8))$. What is the meaning of the slope of this line? In light of this meaning, what is a geometric way to interpret each of the values computed in the preceding question?

(c) Use a graphing utility to plot the graph of $s(t) = 64-16(t-1)^2$ on an interval containing the value $t = 0.8$. Then, zoom in repeatedly on the point $(0.8, s(0.8))$. What do you observe about how the graph appears as you view it more and more closely?

(d) What do you conjecture is the velocity of the ball at the instant $t = 0.8$? Why?

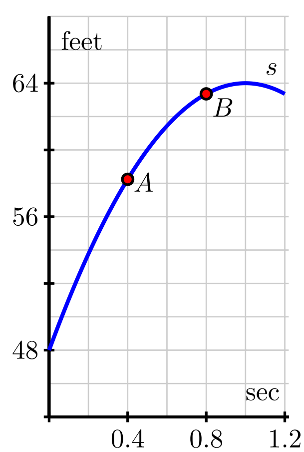

Figure 1.1: A partial plot of $s(t) = 64 - 16(t-1)^2$.

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Activity 1.2.

Each of the following questions concern $s(t) = 64 - 16(t-1)^2$, the position function from Preview Activity 1.1.

(a) Compute the average velocity of the ball on the time interval $[1.5, 2]$. What is different between this value and the average velocity on the interval $[0, 0.5]$?

(b) Use appropriate computing technology to estimate the instantaneous velocity of the ball at $t = 1.5$. Likewise, estimate the instantaneous velocity of the ball at $t = 2$. Which value is greater?

(c) How is the sign of the instantaneous velocity of the ball related to its behavior at a given point in time? That is, what does positive instantaneous velocity tell you the ball is doing? Negative instantaneous velocity?

(d) Without doing any computations, what do you expect to be the instantaneous velocity of the ball at $t = 1$? Why?

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Activity 1.3.

For the function given by $s(t) = 64 - 16(t-1)^2$ from Preview Activity 1.1, find the most simplified expression you can for the average velocity of the ball on the interval $[2, 2+h]$. Use your result to compute the average velocity on $[1.5, 2]$ and to estimate the instantaneous velocity at $t = 2$. Finally, compare your earlier work in Activity 1.1.

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1.2 The notion of limit

Preview Activity 1.2. Suppose that g is the function given by the graph below. Use the graph to answer each of the following questions.

(a) Determine the values $g(-2)$, $g(-1)$, $g(0)$, $g(1)$, and $g(2)$, if defined. If the function value is not defined, explain what feature of the graph tells you this.

(b) For each of the values $a = -1$, $a = 0$, and $a = 2$, complete the following sentence: "As x gets closer and closer (but not equal) to a, $g(x)$ gets as close as we want to ____."

(c) What happens as x gets closer and closer (but not equal) to $a = 1$? Does the function $g(x)$ get as close as we would like to a single value?

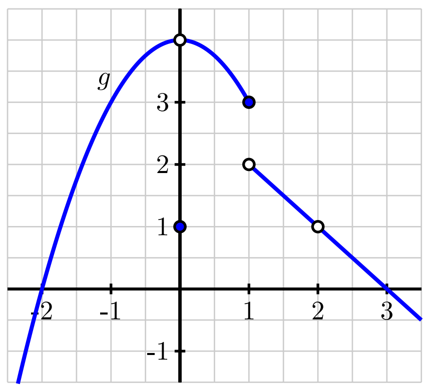

Figure 1.2: Graph of $y = g(x)$ for Preview Activity 1.2.

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Activity 1.4.

Estimate the value of each of the following limits by constructing appropriate tables of values. Then determine the exact value of the limit by using algebra to simplify the function. Finally, plot each function on an appropriate interval to check your result visually.

(a) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

(b) $\lim_{x \to 0} \frac{(2 + x)^3 - 8}{x}$

(c) $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

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Activity 1.5.

Consider a moving object whose position function is given by $s(t) = t^2$, where s is measured in meters and t is measured in minutes.

(a) Determine the most simplified expression for the average velocity of the object on the interval $[3, 3+h]$, where $h > 0$.

(b) Determine the average velocity of the object on the interval $[3, 3.2]$. Include units on your answer.

(c) Determine the instantaneous velocity of the object when $t = 3$. Include units on your answer.

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Activity 1.6.

For the moving object whose position s at time t is given by the graph below, answer each of the following questions. Assume that s is measured in feet and t is measured in seconds.

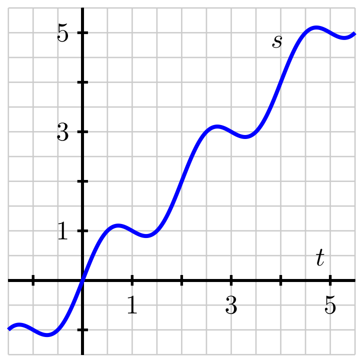

Figure 1.3: Plot of the position function $y = s(t)$ in Activity 1.6.

(a) Use the graph to estimate the average velocity of the object on each of the following intervals: $[0.5, 1]$, $[1.5, 2.5]$, $[0, 5]$. Draw each line whose slope represents the average velocity you seek.

(b) How could you use average velocities or slopes of lines to estimate the instantaneous velocity of the object at a fixed time?

(c) Use the graph to estimate the instantaneous velocity of the object when $t = 2$. Should this instantaneous velocity at $t = 2$ be greater or less than the average velocity on $[1.5, 2.5]$ that you computed in (a)? Why?

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1.3 The derivative of a function at a point

Preview Activity 1.3. Suppose that f is the function given by the graph below and that a and $a+h$ are the input values as labeled on the x-axis. Use the graph in Figure 1.4 to answer the following questions.

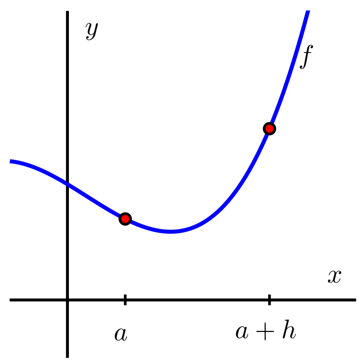

Figure 1.4: Plot of $y = f(x)$ for Preview Activity 1.3.

(a) Locate and label the points $(a, f(a))$ and $(a+h, f(a+h))$ on the graph.

(b) Construct a right triangle whose hypotenuse is the line segment from $(a, f(a))$ to $(a+h, f(a+h))$. What are the lengths of the respective legs of this triangle?

(c) What is the slope of the line that connects the points $(a, f(a))$ and $(a+h, f(a+h))$?

(d) Write a meaningful sentence that explains how the average rate of change of the function on a given interval and the slope of a related line are connected.

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Activity 1.7.

Consider the function f whose formula is $f(x) = 3 - 2x$.

(a) What familiar type of function is f? What can you say about the slope of f at every value of x?

(b) Compute the average rate of change of f on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$; simplify each result as much as possible. What do you notice about these quantities?

(c) Use the limit definition of the derivative to compute the exact instantaneous rate of change of f with respect to x at the value $a = 1$. That is, compute $f'(1)$ using the limit definition. Show your work. Is your result surprising?

(d) Without doing any additional computations, what are the values of $f'(2)$, $f'(\pi)$, and $f'(-\sqrt{2})$? Why?

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Activity 1.8.

A water balloon is tossed vertically in the air from a window. The balloon's height in feet at time t in seconds after being launched is given by $s(t) = -16t^2 + 16t + 32$. Use this function to respond to each of the following questions.

(a) Sketch an accurate, labeled graph of s on the axes provided in Figure 1.5. You should be able to do this without using computing technology.

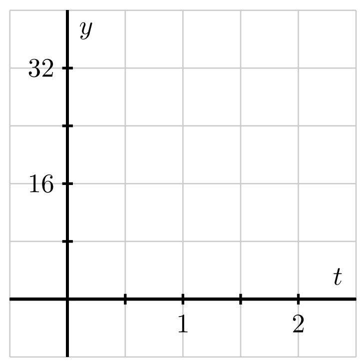

Figure 1.5: Axes for plotting $y = s(t)$ in Activity 1.8.

(b) Compute the average rate of change of s on the time interval $[1, 2]$. Include units on your answer and write one sentence to explain the meaning of the value you found.

(c) Use the limit definition to compute the instantaneous rate of change of s with respect to time, t, at the instant $a = 1$. Show your work using proper notation, include units on your answer, and write one sentence to explain the meaning of the value you found.

(d) On your graph in (a), sketch two lines: one whose slope represents the average rate of change of s on $[1, 2]$, the other whose slope represents the instantaneous rate of change of s at the instant $a = 1$. Label each line clearly.

(e) For what values of a do you expect $s'(a)$ to be positive? Why? Answer the same questions when "positive" is replaced by "negative" and "zero."

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Activity 1.9.

A rapidly growing city in Arizona has its population P at time t, where t is the number of decades after the year 2010, modeled by the formula $P(t) = 25000e^{t/5}$. Use this function to respond to the following questions.

(a) Sketch an accurate graph of P for $t = 0$ to $t = 5$ on the axes provided in Figure 1.6. Label the scale on the axes carefully.

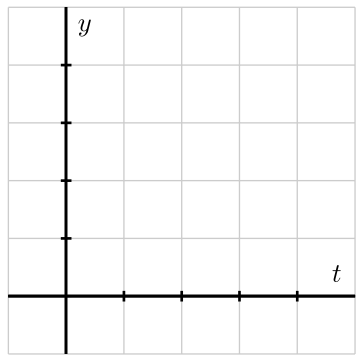

Figure 1.6: Axes for plotting $y = P(t)$ in Activity 1.9.

(b) Compute the average rate of change of P between 2030 and 2050. Include units on your answer and write one sentence to explain the meaning (in everyday language) of the value you found.

(c) Use the limit definition to write an expression for the instantaneous rate of change of P with respect to time, t, at the instant $a = 2$. Explain why this limit is difficult to evaluate exactly.

(d) Estimate the limit in (c) for the instantaneous rate of change of P at the instant $a = 2$ by using several small h values. Once you have determined an accurate estimate of $P'(2)$, include units on your answer, and write one sentence (using everyday language) to explain the meaning of the value you found.

(e) On your graph above, sketch two lines: one whose slope represents the average rate of change of P on $[2, 4]$, the other whose slope represents the instantaneous rate of change of P at the instant $a = 2$.

(f) In a carefully-worded sentence, describe the behavior of $P'(a)$ as a increases in value. What does this reflect about the behavior of the given function P?

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1.4 The derivative function

Preview Activity 1.4. Consider the function $f(x) = 4x - x^2$.

(a) Use the limit definition to compute the following derivative values: $f'(0)$, $f'(1)$, $f'(2)$, and $f'(3)$.

(b) Observe that the work to find $f'(a)$ is the same, regardless of the value of a. Based on your work in (a), what do you conjecture is the value of $f'(4)$? How about $f'(5)$? (Note: you should *not* use the limit definition of the derivative to find either value.)

(c) Conjecture a formula for $f'(a)$ that depends only on the value a. That is, in the same way that we have a formula for $f(x)$ (recall $f(x) = 4x - x^2$), see if you can use your work above to guess a formula for $f'(a)$ in terms of a.

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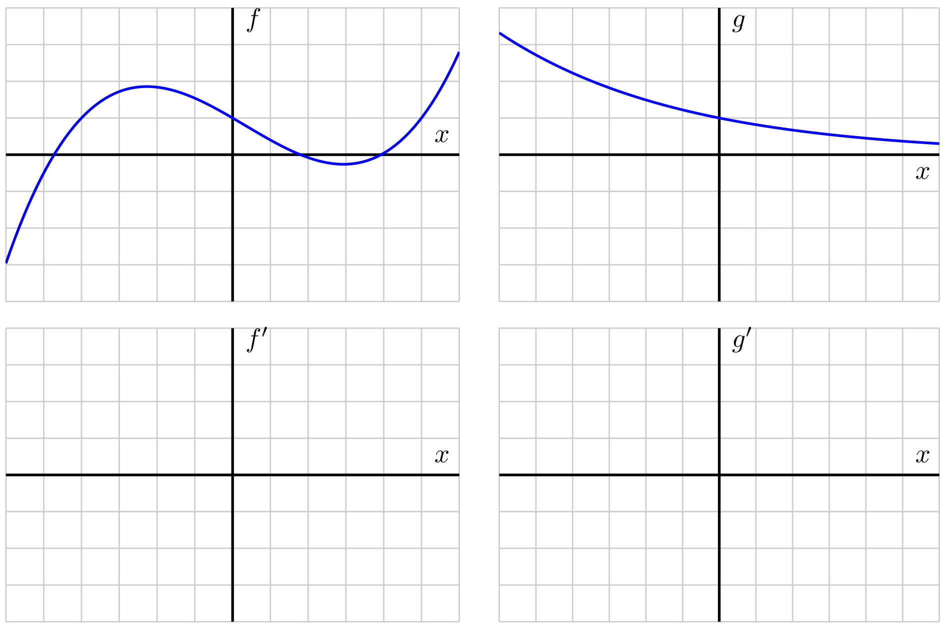
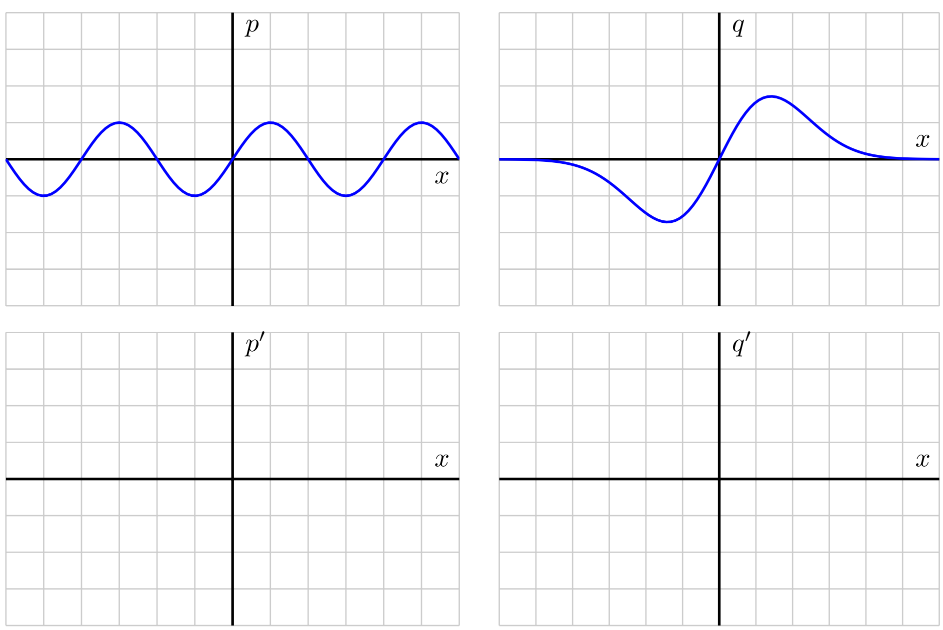
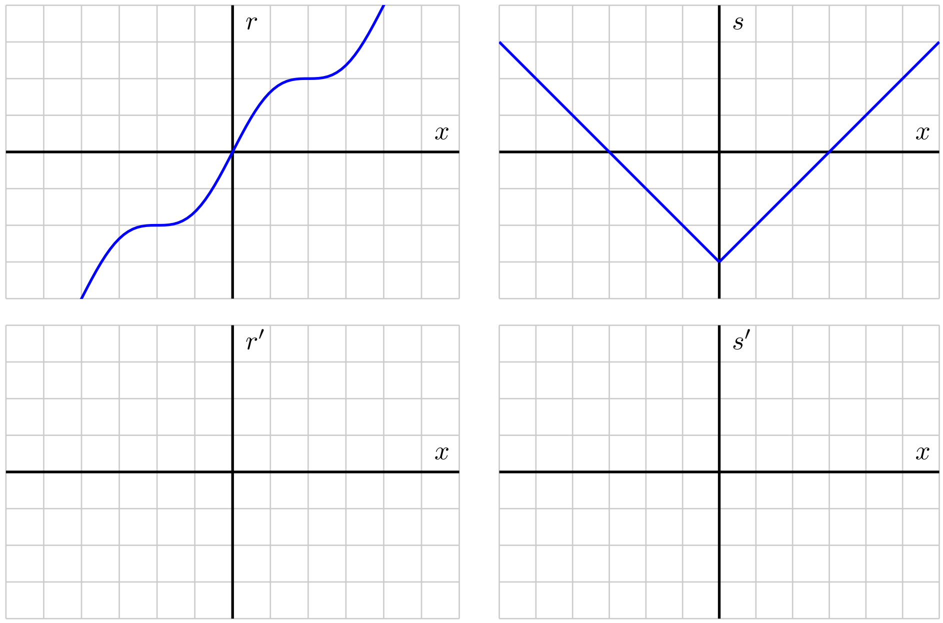
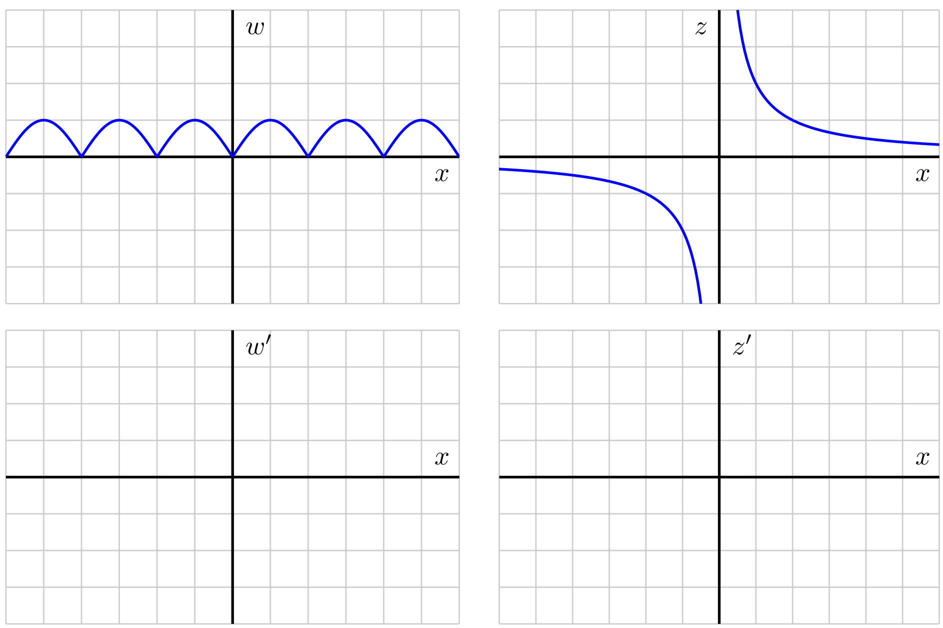
Activity 1.10.

For each given graph of $y = f(x)$, sketch an approximate graph of its derivative function, $y = f'(x)$, on the axes immediately below. The scale of the grid for the graph of f is 1×1; assume the horizontal scale of the grid for the graph of f' is identical to that for f. If necessary, adjust and label the vertical scale on the axes for the graph of f'.









Write several sentences that describe your overall process for sketching the graph of the derivative function, given the graph the original function. What are the values of the derivative function that you tend to identify first? What do you do thereafter? How do key traits of the graph of the derivative function exemplify properties of the graph of the original function?

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Activity 1.11.

For each of the listed functions, determine a formula for the derivative function. For the first two, determine the formula for the derivative by thinking about the nature of the given function and its slope at various points; do not use the limit definition. For the latter four, use the limit definition. Pay careful attention to the function names and independent variables. It is important to be comfortable with using letters other than f and x. For example, given a function $p(z)$, we call its derivative $p'(z)$.

(a) $f(x) = 1$

(b) $g(t) = t$

(c) $p(z) = z^2$

(d) $q(s) = s^3$

(e) $F(t) = \frac{1}{t}$

(f) $G(y) = \sqrt{y}$

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1.5 Interpreting the derivative and its units

Preview Activity 1.5. One of the longest stretches of straight (and flat) road in North America can be found on the Great Plains in the state of North Dakota on state highway 46, which lies just south of the interstate highway I-94 and runs through the town of Gackle. A car leaves town (at time $t = 0$) and heads east on highway 46; its position in miles from Gackle at time t in minutes is given by the graph of the function in Figure 1.7. Three important points are labeled on the graph; where the curve looks linear, assume that it is indeed a straight line.

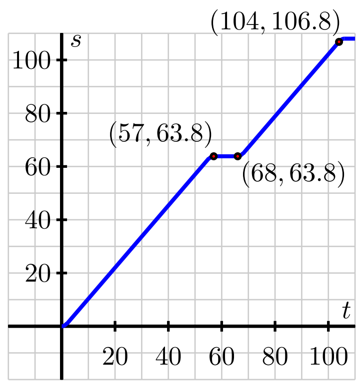

Figure 1.7: The graph of $y = s(t)$, the position of the car along highway 46, which tells its distance in miles from Gackle, ND, at time t in minutes.

(a) In everyday language, describe the behavior of the car over the provided time interval. In particular, discuss what is happening on the time intervals $[57, 68]$ and $[68, 104]$.

(b) Find the slope of the line between the points $(57, 63.8)$ and $(104, 106.8)$. What are the units on this slope? What does the slope represent?

(c) Find the average rate of change of the car's position on the interval $[68, 104]$. Include units on your answer.

(d) Estimate the instantaneous rate of change of the car's position at the moment $t = 80$. Write a sentence to explain your reasoning and the meaning of this value.

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Activity 1.12.

A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	324.5
75	342.8
90	354.5

(a) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 30$. Include units on your answer.

(b) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 60$. Include units on your answer.

(c) Without doing any calculation, which do you expect to be greater: $F'(75)$ or $F'(90)$? Why?

(d) Suppose it is given that $F(64) = 330.28$ and $F'(64) = 1.341$. What are the units on these two quantities? What do you expect the temperature of the potato to be when $t = 65$? when $t = 66$? Why?

(e) Write a couple of careful sentences that describe the behavior of the temperature of the potato on the time interval $[0, 90]$, as well as the behavior of the instantaneous rate of change of the temperature of the potato on the same time interval.

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Activity 1.13.

A company manufactures rope, and the total cost of producing r feet of rope is $C(r)$ dollars.

(a) What does it mean to say that $C(2000) = 800$?

(b) What are the units of $C'(r)$?

(c) Suppose that $C(2000) = 800$ and $C'(2000) = 0.35$. Estimate $C(2100)$, and justify your estimate by writing at least one sentence that explains your thinking.

(d) Which of the following statements do you think is true, and why?

- $C'(2000) < C'(3000)$
- $C'(2000) = C'(3000)$
- $C'(2000) > C'(3000)$

(e) Suppose someone claims that $C'(5000) = -0.1$. What would the practical meaning of this derivative value tell you about the approximate cost of the next foot of rope? Is this possible? Why or why not?

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Activity 1.14.

Researchers at a major car company have found a function that relates gasoline consumption to speed for a particular model of car. In particular, they have determined that the consumption C, in **liters per kilometer**, at a given speed s, is given by a function $C = f(s)$, where s is the car's speed in **kilometers per hour**.

(a) Data provided by the car company tells us that $f(80) = 0.015$, $f(90) = 0.02$, and $f(100) = 0.027$. Use this information to estimate the instantaneous rate of change of fuel consumption with respect to speed at $s = 90$. Be as accurate as possible, use proper notation, and include units on your answer.

(b) By writing a complete sentence, interpret the meaning (in the context of fuel consumption) of "$f(80) = 0.015$."

(c) Write at least one complete sentence that interprets the meaning of the value of $f'(90)$ that you estimated in (a).

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1.6 The second derivative

Preview Activity 1.6. The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.8. The car's position function has units measured in thousands of feet. For instance, the point $(2, 4)$ on the graph indicates that after 2 minutes, the car has traveled 4000 feet.

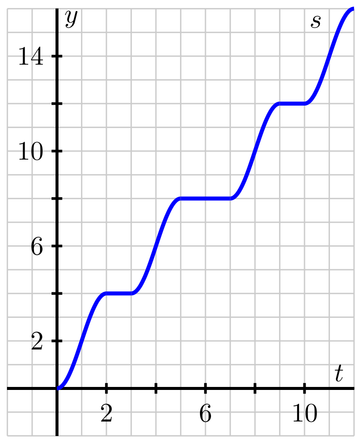

Figure 1.8: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

(a) In everyday language, describe the behavior of the car over the provided time interval. In particular, you should carefully discuss what is happening on each of the time intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, 5]$, plus provide commentary overall on what the car is doing on the interval $[0, 12]$.

(b) On the lefthand axes provided in Figure 1.9, sketch a careful, accurate graph of $y = s'(t)$.

(c) What is the meaning of the function $y = s'(t)$ in the context of the given problem? What can we say about the car's behavior when $s'(t)$ is positive? when $s'(t)$ is zero? when $s'(t)$ is negative?

(d) Rename the function you graphed in (b) to be called $y = v(t)$. Describe the behavior of v in words, using phrases like "v is increasing on the interval ..." and "v is constant on the interval ...."

(e) Sketch a graph of the function $y = v'(t)$ on the righthand axes provide in Figure 1.8. Write at least one sentence to explain how the behavior of $v'(t)$ is connected to the graph of $y = v(t)$.

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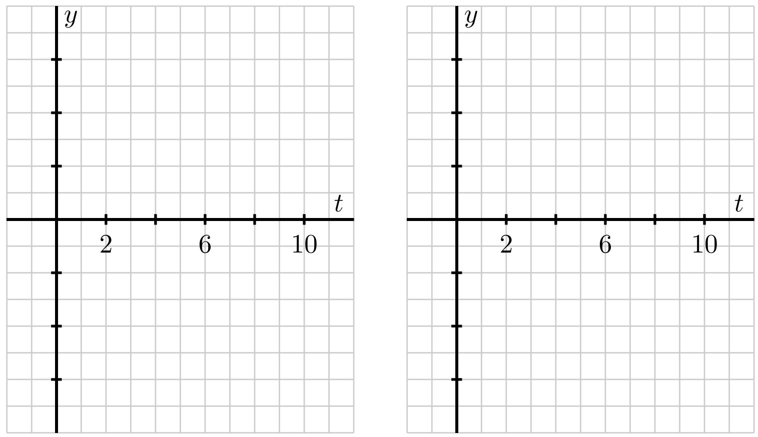

Figure 1.9: Axes for plotting $y = v(t) = s'(t)$ and $y = v'(t)$.

Activity 1.15.

The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.10. The car's position function has units measured in thousands of feet. Remember that you worked with this function and sketched graphs of $y = v(t) = s'(t)$ and $y = v'(t)$ in Preview Activity 1.6.

(a) On what intervals is the position function $y = s(t)$ increasing? decreasing? Why?

(b) On which intervals is the velocity function $y = v(t) = s'(t)$ increasing? decreasing? neither? Why?

(c) *Acceleration* is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car's acceleration function is named $a(t)$. How is $a(t)$ computed from $v(t)$? How is $a(t)$ computed from $s(t)$? Explain.

(d) What can you say about s'' whenever s' is increasing? Why?

(e) Using only the words *increasing*, *decreasing*, *constant*, *concave up*, *concave down*, and *linear*, complete the following sentences. For the position function s with velocity v and acceleration a,

- on an interval where v is positive, s is ____________________.
- on an interval where v is negative, s is ____________________.
- on an interval where v is zero, s is ____________________.
- on an interval where a is positive, v is ____________________.
- on an interval where a is negative, v is ____________________.
- on an interval where a is zero, v is ____________________.

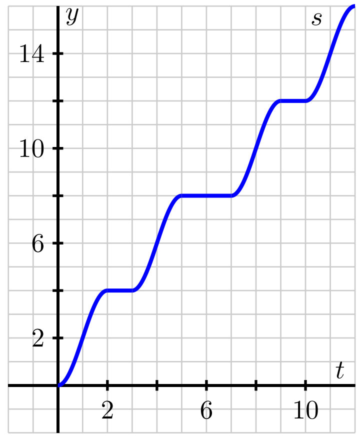

Figure 1.10: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

- on an interval where a is positive, s is ____________________.
- on an interval where a is negative, s is ____________________.
- on an interval where a is zero, s is ____________________.

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Activity 1.16.

This activity builds on our experience and understanding of how to sketch the graph of f' given the graph of f. Below, given the graph of a function f, sketch f' on the first axes below, and then sketch f'' on the second set of axes. In addition, for each, write several careful sentences in the spirit of those in Activity 1.15 that connect the behaviors of f, f', and f''. For instance, write something such as

> f' is ____________________ on the interval _______, which is connected to the fact that f is ____________________ on the same interval _______, and f'' is ____________________ on the interval as well

but of course with the blanks filled in. Throughout, view the scale of the grid for the graph of f as being 1×1, and assume the horizontal scale of the grid for the graph of f' is identical to that for f. If you need to adjust the vertical scale on the axes for the graph of f' or f'', you should label that accordingly.

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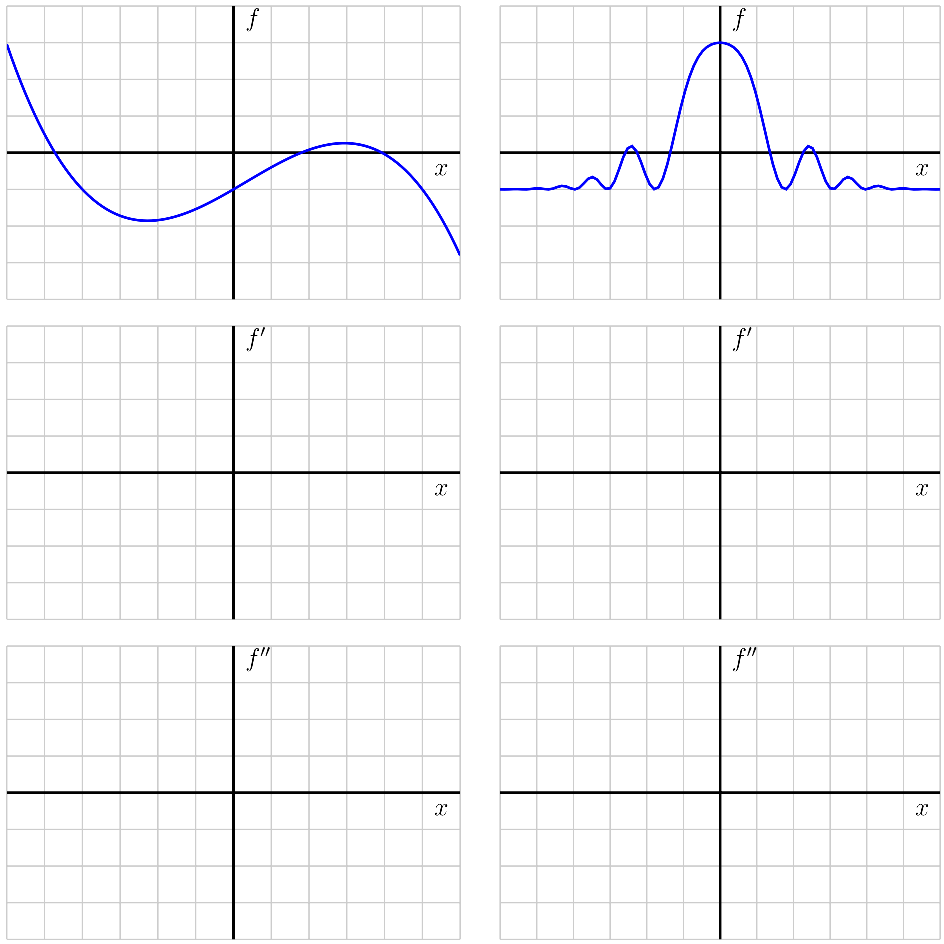

Figure 1.11: Two given functions f, with axes provided for plotting f' and f'' below.

Activity 1.17.

A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes. In Activity 1.12, we computed approximations to $F'(30)$ and $F'(60)$ using central differences. Those values and more are provided in the second table below, along with several others computed in the same way.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	324.5
75	342.8
90	354.5

t	$F'(t)$
0	NA
15	6.03
30	3.85
45	2.45
60	1.56
75	1.00
90	NA

(a) What are the units on the values of $F'(t)$?

(b) Use a central difference to estimate the value of $F''(30)$.

(c) What is the meaning of the value of $F''(30)$ that you have computed in (b) in terms of the potato's temperature? Write several careful sentences that discuss, with appropriate units, the values of $F(30)$, $F'(30)$, and $F''(30)$, and explain the overall behavior of the potato's temperature at this point in time.

(d) Overall, is the potato's temperature increasing at an increasing rate, increasing at a constant rate, or increasing at a decreasing rate? Why?

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1.7 Limits, Continuity, and Differentiability

Preview Activity 1.7. A function f defined on $-4 < x < 4$ is given by the graph in Figure 1.12. Use the graph to answer each of the following questions. Note: to the right of $x = 2$, the graph of f is exhibiting infinite oscillatory behavior similar to the function $\sin(\frac{\pi}{x})$ that we encountered in the key example early in Section 1.2.

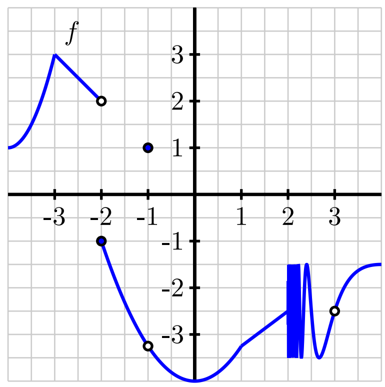

Figure 1.12: The graph of $y = f(x)$.

(a) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether or not $\lim_{x \to a} f(x)$ exists. If the function has a limit L at a given point, state the value of the limit using the notation $\lim_{x \to a} f(x) = L$. If the function does not have a limit at a given point, write a sentence to explain why.

(b) For each of the values of a from part (a) where f has a limit, determine the value of $f(a)$ at each such point. In addition, for each such a value, does $f(a)$ have the same value as $\lim_{x \to a} f(x)$?

(c) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether or not $f'(a)$ exists. In particular, based on the given graph, ask yourself if it is reasonable to say that f has a tangent line at $(a, f(a))$ for each of the given a-values. If so, visually estimate the slope of the tangent line to find the value of $f'(a)$.

$\bowtie$

Activity 1.18.

Consider a function that is piecewise-defined according to the formula

$$f(x) = \begin{cases} 3(x+2)+2 & \text{for } -3 < x < -2 \\ \frac{2}{3}(x+2)+1 & \text{for } -2 \leq x < -1 \\ \frac{2}{3}(x+2)+1 & \text{for } -1 < x < 1 \\ 2 & \text{for } x = 1 \\ 4-x & \text{for } x > 1 \end{cases}$$

Use the given formula to answer the following questions.

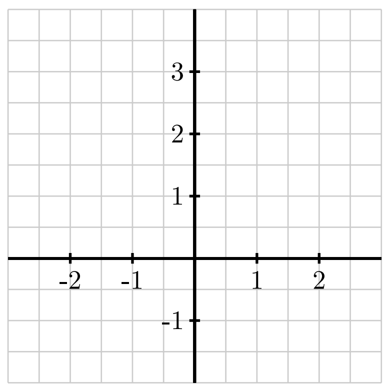

Figure 1.13: Axes for plotting the function $y = f(x)$ in Activity 1.18.

(a) For each of the values $a = -2, -1, 0, 1, 2$, compute $f(a)$.

(b) For each of the values $a = -2, -1, 0, 1, 2$, determine $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

(c) For each of the values $a = -2, -1, 0, 1, 2$, determine $\lim_{x \to a} f(x)$. If the limit fails to exist, explain why by discussing the left- and right-hand limits at the relevant a-value.

(d) For which values of a is the following statement true?

$$\lim_{x \to a} f(x) \neq f(a)$$

(e) On the axes provided in Figure 1.13, sketch an accurate, labeled graph of $y = f(x)$. Be sure to carefully use open circles (○) and filled circles (●) to represent key points on the graph, as dictated by the piecewise formula.

◁

Activity 1.19.

This activity builds on your work in Preview Activity 1.7, using the same function f as given by the graph that is repeated in Figure 1.14

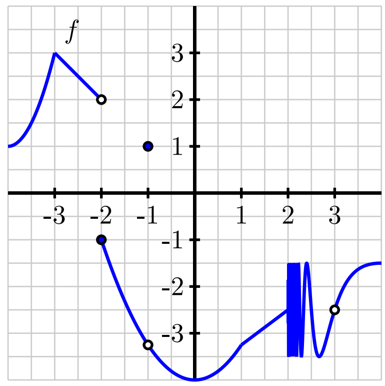

Figure 1.14: The graph of $y = f(x)$ for Activity 1.19.

(a) At which values of a does $\lim_{x\to a} f(x)$ not exist?

(b) At which values of a is $f(a)$ not defined?

(c) At which values of a does f have a limit, but $\lim_{x\to a} f(x) \neq f(a)$?

(d) State all values of a for which f is not continuous at $x = a$.

(e) Which condition is stronger, and hence implies the other: f has a limit at $x = a$ or f is continuous at $x = a$? Explain, and hence complete the following sentence: "If f ____________________ at $x = a$, then f ____________________ at $x = a$," where you complete the blanks with *has a limit* and *is continuous*, using each phrase once.

◁

Activity 1.20.

In this activity, we explore two different functions and classify the points at which each is not differentiable. Let g be the function given by the rule $g(x) = |x|$, and let f be the function that we have previously explored in Preview Activity 1.7, whose graph is given again in Figure 1.15.

(a) Reasoning visually, explain why g is differentiable at every point x such that $x \neq 0$.

(b) Use the limit definition of the derivative to show that $g'(0) = \lim_{h \to 0} \frac{|h|}{h}$.

(c) Explain why $g'(0)$ fails to exist by using small positive and negative values of h.

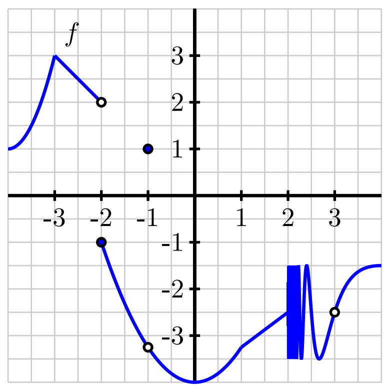

Figure 1.15: The graph of $y = f(x)$ for Activity 1.20.

(d) State all values of a for which f is not differentiable at $x = a$. For each, provide a reason for your conclusion.

(e) True or false: if a function p is differentiable at $x = b$, then $\lim_{x \to b} p(x)$ must exist. Why?

◁

1.8 The Tangent Line Approximation

Preview Activity 1.8. Consider the function $y = g(x) = -x^2 + 3x + 2$.

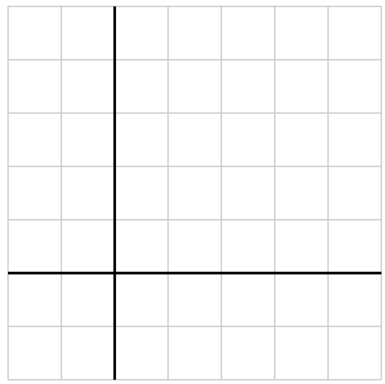

Figure 1.16: Axes for plotting $y = g(x)$ and its tangent line to the point $(2, g(2))$.

(a) Use the limit definition of the derivative to compute a formula for $y = g'(x)$.

(b) Determine the slope of the tangent line to $y = g(x)$ at the value $x = 2$.

(c) Compute $g(2)$.

(d) Find an equation for the tangent line to $y = g(x)$ at the point $(2, g(2))$. Write your result in point-slope form[1].

(e) On the axes provided in Figure 1.16, sketch an accurate, labeled graph of $y = g(x)$ along with its tangent line at the point $(2, g(2))$.

⋈

[1]Recall that a line with slope m that passes through (x_0, y_0) has equation $y - y_0 = m(x - x_0)$, and this is the *point-slope form* of the equation.

Activity 1.21.

Suppose it is known that for a given differentiable function $y = g(x)$, its local linearization at the point where $a = -1$ is given by $L(x) = -2 + 3(x + 1)$.

(a) Compute the values of $L(-1)$ and $L'(-1)$.

(b) What must be the values of $g(-1)$ and $g'(-1)$? Why?

(c) Do you expect the value of $g(-1.03)$ to be greater than or less than the value of $g(-1)$? Why?

(d) Use the local linearization to estimate the value of $g(-1.03)$.

(e) Suppose that you also know that $g''(-1) = 2$. What does this tell you about the graph of $y = g(x)$ at $a = -1$?

(f) For x near -1, sketch the graph of the local linearization $y = L(x)$ as well as a possible graph of $y = g(x)$ on the axes provided in Figure 1.17.

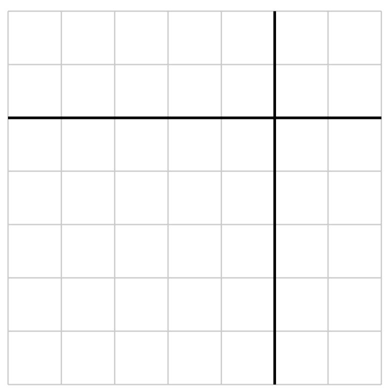

Figure 1.17: Axes for plotting $y = L(x)$ and $y = g(x)$.

◁

Activity 1.22.

This activity concerns a function $f(x)$ about which the following information is known:

- f is a differentiable function defined at every real number x
- $f(2) = -1$
- $y = f'(x)$ has its graph given in Figure 1.18

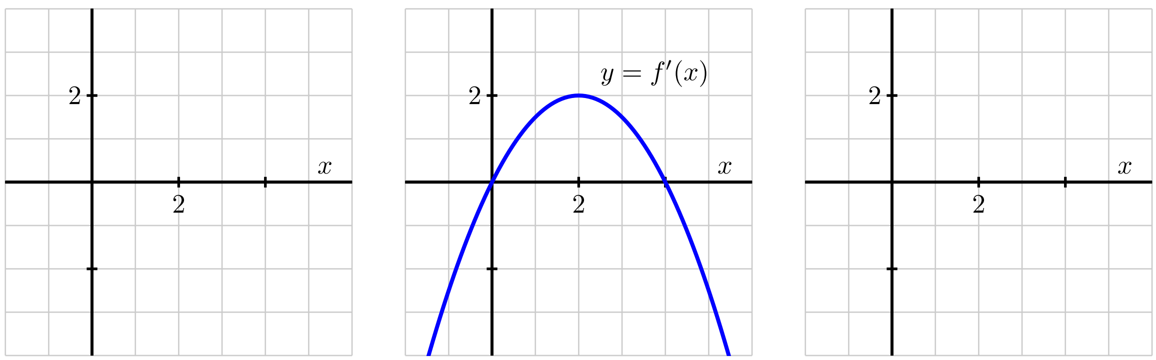

Figure 1.18: At center, a graph of $y = f'(x)$; at left, axes for plotting $y = f(x)$; at right, axes for plotting $y = f''(x)$.

Your task is to determine as much information as possible about f (especially near the value $a = 2$) by responding to the questions below.

(a) Find a formula for the tangent line approximation, $L(x)$, to f at the point $(2, -1)$.

(b) Use the tangent line approximation to estimate the value of $f(2.07)$. Show your work carefully and clearly.

(c) Sketch a graph of $y = f''(x)$ on the righthand grid in Figure 1.18; label it appropriately.

(d) Is the slope of the tangent line to $y = f(x)$ increasing, decreasing, or neither when $x = 2$? Explain.

(e) Sketch a possible graph of $y = f(x)$ near $x = 2$ on the lefthand grid in Figure 1.18. Include a sketch of $y = L(x)$ (found in part (a)). Explain how you know the graph of $y = f(x)$ looks like you have drawn it.

(f) Does your estimate in (b) over- or under-estimate the true value of $f(2)$? Why?

◁

Chapter 2

Computing Derivatives

2.1 Elementary derivative rules

Preview Activity 2.1. Functions of the form $f(x) = x^n$, where $n = 1, 2, 3, \ldots$, are often called *power functions*. The first two questions below revisit work we did earlier in Chapter 1, and the following questions extend those ideas to higher powers of x.

(a) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^2$.

(b) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^3$.

(c) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^4$. (Hint: $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Apply this rule to $(x + h)^4$ within the limit definition.)

(d) Based on your work in (a), (b), and (c), what do you conjecture is the derivative of $f(x) = x^5$? Of $f(x) = x^{13}$?

(e) Conjecture a formula for the derivative of $f(x) = x^n$ that holds for any positive integer n. That is, given $f(x) = x^n$ where n is a positive integer, what do you think is the formula for $f'(x)$?

$\bowtie$

Activity 2.1.

Use the three rules above to determine the derivative of each of the following functions. For each, state your answer using full and proper notation, labeling the derivative with its name. For example, if you are given a function $h(z)$, you should write "$h'(z) =$" or "$\frac{dh}{dz} =$" as part of your response.

(a) $f(t) = \pi$

(b) $g(z) = 7^z$

(c) $h(w) = w^{3/4}$

(d) $p(x) = 3^{1/2}$

(e) $r(t) = (\sqrt{2})^t$

(f) $\frac{d}{dq}[q^{-1}]$

(g) $m(t) = \frac{1}{t^3}$

◁

Activity 2.2.

Use only the rules for constant, power, and exponential functions, together with the Constant Multiple and Sum Rules, to compute the derivative of each function below with respect to the given independent variable. Note well that we do not yet know any rules for how to differentiate the product or quotient of functions. This means that you may have to do some algebra first on the functions below before you can actually use existing rules to compute the desired derivative formula. In each case, label the derivative you calculate with its name using proper notation such as $f'(x)$, $h'(z)$, dr/dt, etc.

(a) $f(x) = x^{5/3} - x^4 + 2^x$

(b) $g(x) = 14e^x + 3x^5 - x$

(c) $h(z) = \sqrt{z} + \frac{1}{z^4} + 5^z$

(d) $r(t) = \sqrt{53}\, t^7 - \pi e^t + e^4$

(e) $s(y) = (y^2 + 1)(y^2 - 1)$

(f) $q(x) = \dfrac{x^3 - x + 2}{x}$

(g) $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$

◁

Activity 2.3.

Each of the following questions asks you to use derivatives to answer key questions about functions. Be sure to think carefully about each question and to use proper notation in your responses.

(a) Find the slope of the tangent line to $h(z) = \sqrt{z} + \frac{1}{z}$ at the point where $z = 4$.

(b) A population of cells is growing in such a way that its total number in millions is given by the function $P(t) = 2(1.37)^t + 32$, where t is measured in days.

  i. Determine the instantaneous rate at which the population is growing on day 4, and include units on your answer.

  ii. Is the population growing at an increasing rate or growing at a decreasing rate on day 4? Explain.

(c) Find an equation for the tangent line to the curve $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$ at the point where $a = -1$.

(d) What is the difference between being asked to find the *slope* of the tangent line (asked in (a)) and the *equation* of the tangent line (asked in (c))?

$\lhd$

2.2 The sine and cosine functions

Preview Activity 2.2. Consider the function $g(x) = 2^x$, which is graphed in Figure 2.1.

(a) At each of $x = -2, -1, 0, 1, 2$, use a straightedge to sketch an accurate tangent line to $y = g(x)$.

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point in (a).

(c) Use the limit definition of the derivative to estimate $g'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at $x = 0$ in (b).

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.

(e) Write at least one sentence that explains why it is reasonable to think that $g'(x) = cg(x)$, where c is a constant. In addition, calculate $\ln(2)$, and then discuss how this value, combined with your work above, reasonably suggests that $g'(x) = 2^x \ln(2)$.

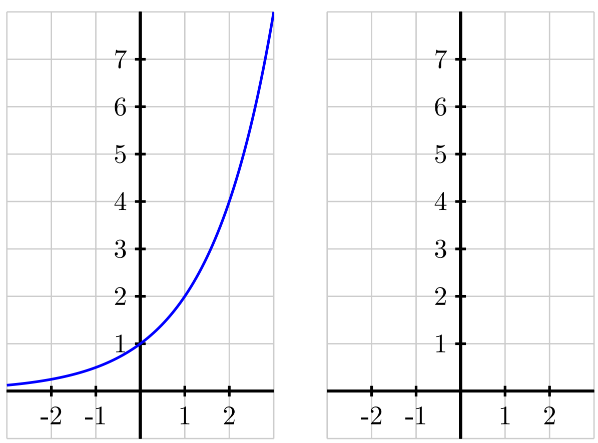

Figure 2.1: At left, the graph of $y = g(x) = 2^x$. At right, axes for plotting $y = g'(x)$.

$\bowtie$

Activity 2.4.

Consider the function $f(x) = \sin(x)$, which is graphed in Figure 2.2 below. Note carefully that the grid in the diagram does not have boxes that are 1×1, but rather approximately 1.57×1, as the horizontal scale of the grid is $\pi/2$ units per box.

(a) At each of $x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, use a straightedge to sketch an accurate tangent line to $y = f(x)$.

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Pay careful attention to the scale of the grid.

(c) Use the limit definition of the derivative to estimate $f'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = f(x)$ at $x = 0$ in (b). Using periodicity, what does this result suggest about $f'(2\pi)$? about $f'(-2\pi)$?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = f'(x)$ on the axes adjacent to the graph of $y = f(x)$.

(e) What familiar function do you think is the derivative of $f(x) = \sin(x)$?

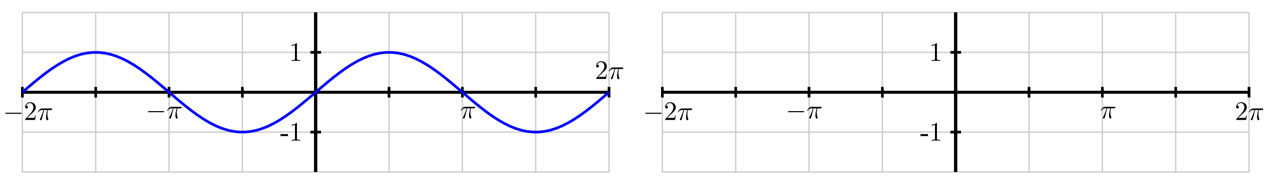

Figure 2.2: At left, the graph of $y = f(x) = \sin(x)$.

◁

Activity 2.5.

Consider the function $g(x) = \cos(x)$, which is graphed in Figure 2.3 below. Note carefully that the grid in the diagram does not have boxes that are 1×1, but rather approximately 1.57×1, as the horizontal scale of the grid is $\pi/2$ units per box.

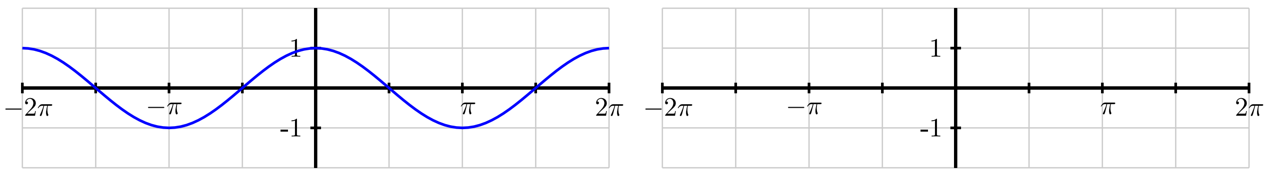

Figure 2.3: At left, the graph of $y = g(x) = \cos(x)$.

(a) At each of $x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, use a straightedge to sketch an accurate tangent line to $y = g(x)$.

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Again, note the scale of the axes and grid.

(c) Use the limit definition of the derivative to estimate $g'(\frac{\pi}{2})$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at $x = \frac{\pi}{2}$ in (b). Using periodicity, what does this result suggest about $g'(-\frac{3\pi}{2})$? can symmetry on the graph help you estimate other slopes easily?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.

(e) What familiar function do you think is the derivative of $g(x) = \cos(x)$?

$\triangleleft$

Activity 2.6.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Determine the derivative of $h(t) = 3\cos(t) - 4\sin(t)$.

(b) Find the exact slope of the tangent line to $y = f(x) = 2x + \frac{\sin(x)}{2}$ at the point where $x = \frac{\pi}{6}$.

(c) Find the equation of the tangent line to $y = g(x) = x^2 + 2\cos(x)$ at the point where $x = \frac{\pi}{2}$.

(d) Determine the derivative of $p(z) = z^4 + 4^z + 4\cos(z) - \sin(\frac{\pi}{2})$.

(e) The function $P(t) = 24 + 8\sin(t)$ represents a population of a particular kind of animal that lives on a small island, where P is measured in hundreds and t is measured in decades since January 1, 2010. What is the instantaneous rate of change of P on January 1, 2030? What are the units of this quantity? Write a sentence in everyday language that explains how the population is behaving at this point in time.

$\lhd$

2.3 The product and quotient rules

Preview Activity 2.3. Let f and g be the functions defined by $f(t) = 2t^2$ and $g(t) = t^3 + 4t$.

(a) Determine $f'(t)$ and $g'(t)$.

(b) Let $p(t) = 2t^2(t^3 + 4t)$ and observe that $p(t) = f(t) \cdot g(t)$. Rewrite the formula for p by distributing the $2t^2$ term. Then, compute $p'(t)$ using the sum and constant multiple rules.

(c) True or false: $p'(t) = f'(t) \cdot g'(t)$.

(d) Let $q(t) = \dfrac{t^3 + 4t}{2t^2}$ and observe that $q(t) = \dfrac{g(t)}{f(t)}$. Rewrite the formula for q by dividing each term in the numerator by the denominator and simplify to write q as a sum of constant multiples of powers of t. Then, compute $q'(t)$ using the sum and constant multiple rules.

(e) True or false: $q'(t) = \dfrac{g'(t)}{f'(t)}$.

$\bowtie$

Activity 2.7.

Use the product rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you're given a formula for $f(x)$, clearly label the formula you find for $f'(x)$. It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let $m(w) = 3w^{17}4^w$. Find $m'(w)$.

(b) Let $h(t) = (\sin(t) + \cos(t))t^4$. Find $h'(t)$.

(c) Determine the slope of the tangent line to the curve $y = f(x)$ at the point where $a = 1$ if f is given by the rule $f(x) = e^x \sin(x)$.

(d) Find the tangent line approximation $L(x)$ to the function $y = g(x)$ at the point where $a = -1$ if g is given by the rule $g(x) = (x^2 + x)2^x$.

$\triangleleft$

Activity 2.8.

Use the quotient rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you're given a formula for $f(x)$, clearly label the formula you find for $f'(x)$. It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let $r(z) = \dfrac{3^z}{z^4+1}$. Find $r'(z)$.

(b) Let $v(t) = \dfrac{\sin(t)}{\cos(t)+t^2}$. Find $v'(t)$.

(c) Determine the slope of the tangent line to the curve $R(x) = \dfrac{x^2-2x-8}{x^2-9}$ at the point where $x = 0$.

(d) When a camera flashes, the intensity I of light seen by the eye is given by the function

$$I(t) = \frac{100t}{e^t},$$

where I is measured in candles and t is measured in milliseconds. Compute $I'(0.5)$, $I'(2)$, and $I'(5)$; include appropriate units on each value; and discuss the meaning of each.

$\triangleleft$

Activity 2.9.

Use relevant derivative rules to answer each of the questions below. Throughout, be sure to use proper notation and carefully label any derivative you find by name.

(a) Let $f(r) = (5r^3 + \sin(r))(4^r - 2\cos(r))$. Find $f'(r)$.

(b) Let $p(t) = \dfrac{\cos(t)}{t^6 \cdot 6^t}$. Find $p'(t)$.

(c) Let $g(z) = 3z^7 e^z - 2z^2 \sin(z) + \dfrac{z}{z^2+1}$. Find $g'(z)$.

(d) A moving particle has its position in feet at time t in seconds given by the function $s(t) = \dfrac{3\cos(t) - \sin(t)}{e^t}$. Find the particle's instantaneous velocity at the moment $t = 1$.

(e) Suppose that $f(x)$ and $g(x)$ are differentiable functions and it is known that $f(3) = -2$, $f'(3) = 7$, $g(3) = 4$, and $g'(3) = -1$. If $p(x) = f(x) \cdot g(x)$ and $q(x) = \dfrac{f(x)}{g(x)}$, calculate $p'(3)$ and $q'(3)$.

$\triangleleft$

2.4 Derivatives of other trigonometric functions

Preview Activity 2.4. Consider the function $f(x) = \tan(x)$, and remember that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

(a) What is the domain of f?

(b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}.$$

(c) What is the Fundamental Trigonometric Identity? How can this identity be used to find a simpler form for $f'(x)$?

(d) Recall that $\sec(x) = \frac{1}{\cos(x)}$. How can we express $f'(x)$ in terms of the secant function?

(e) For what values of x is $f'(x)$ defined? How does this set compare to the domain of f?

$\bowtie$

Activity 2.10.

Let $h(x) = \sec(x)$ and recall that $\sec(x) = \frac{1}{\cos(x)}$.

(a) What is the domain of h?

(b) Use the quotient rule to develop a formula for $h'(x)$ that is expressed completely in terms of $\sin(x)$ and $\cos(x)$.

(c) How can you use other relationships among trigonometric functions to write $h'(x)$ only in terms of $\tan(x)$ and $\sec(x)$?

(d) What is the domain of h'? How does this compare to the domain of h?

$\triangleleft$

Activity 2.11.

Let $p(x) = \csc(x)$ and recall that $\csc(x) = \frac{1}{\sin(x)}$.

(a) What is the domain of p?

(b) Use the quotient rule to develop a formula for $p'(x)$ that is expressed completely in terms of $\sin(x)$ and $\cos(x)$.

(c) How can you use other relationships among trigonometric functions to write $p'(x)$ only in terms of $\cot(x)$ and $\csc(x)$?

(d) What is the domain of p'? How does this compare to the domain of p?

$\triangleleft$

Activity 2.12.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Let $f(x) = 5\sec(x) - 2\csc(x)$. Find the slope of the tangent line to f at the point where $x = \frac{\pi}{3}$.

(b) Let $p(z) = z^2 \sec(z) - z\cot(z)$. Find the instantaneous rate of change of p at the point where $z = \frac{\pi}{4}$.

(c) Let $h(t) = \dfrac{\tan(t)}{t^2+1} - 2e^t\cos(t)$. Find $h'(t)$.

(d) Let $g(r) = \dfrac{r\sec(r)}{5^r}$. Find $g'(r)$.

(e) When a mass hangs from a spring and is set in motion, the object's position oscillates in a way that the size of the oscillations decrease. This is usually called a *damped oscillation*. Suppose that for a particular object, its displacement from equilibrium (where the object sits at rest) is modeled by the function

$$s(t) = \frac{15\sin(t)}{e^t}.$$

Assume that s is measured in inches and t in seconds. Sketch a graph of this function for $t \geq 0$ to see how it represents the situation described. Then compute ds/dt, state the units on this function, and explain what it tells you about the object's motion. Finally, compute and interpret $s'(2)$.

$\triangleleft$

2.5 The chain rule

Preview Activity 2.5. For each function given below, identify its fundamental algebraic structure. In particular, is the given function a sum, product, quotient, or composition of basic functions? If the function is a composition of basic functions, state a formula for the inner function g and the outer function f so that the overall composite function can be written in the form $f(g(x))$. If the function is a sum, product, or quotient of basic functions, use the appropriate rule to determine its derivative.

(a) $h(x) = \tan(2^x)$

(b) $p(x) = 2^x \tan(x)$

(c) $r(x) = (\tan(x))^2$

(d) $m(x) = e^{\tan(x)}$

(e) $w(x) = \sqrt{x} + \tan(x)$

(f) $z(x) = \sqrt{\tan(x)}$

$\bowtie$

Activity 2.13.

For each function given below, identify an inner function g and outer function f to write the function in the form $f(g(x))$. Then, determine $f'(x)$, $g'(x)$, and $f'(g(x))$, and finally apply the chain rule to determine the derivative of the given function.

(a) $h(x) = \cos(x^4)$

(b) $p(x) = \sqrt{\tan(x)}$

(c) $s(x) = 2^{\sin(x)}$

(d) $z(x) = \cot^5(x)$

(e) $m(x) = (\sec(x) + e^x)^9$

$\triangleleft$

Activity 2.14.

For each of the following functions, find the function's derivative. State the rule(s) you use, label relevant derivatives appropriately, and be sure to clearly identify your overall answer.

(a) $p(r) = 4\sqrt{r^6 + 2e^r}$

(b) $m(v) = \sin(v^2)\cos(v^3)$

(c) $h(y) = \dfrac{\cos(10y)}{e^{4y} + 1}$

(d) $s(z) = 2^{z^2 \sec(z)}$

(e) $c(x) = \sin(e^{x^2})$

$\triangleleft$

Activity 2.15.

Use known derivative rules, including the chain rule, as needed to answer each of the following questions.

(a) Find an equation for the tangent line to the curve $y = \sqrt{e^x + 3}$ at the point where $x = 0$.

(b) If $s(t) = \frac{1}{(t^2+1)^3}$ represents the position function of a particle moving horizontally along an axis at time t (where s is measured in inches and t in seconds), find the particle's instantaneous velocity at $t = 1$. Is the particle moving to the left or right at that instant?

(c) At sea level, air pressure is 30 inches of mercury. At an altitude of h feet above sea level, the air pressure, P, in inches of mercury, is given by the function

$$P = 30e^{-0.0000323h}.$$

Compute dP/dh and explain what this derivative function tells you about air pressure, including a discussion of the units on dP/dh. In addition, determine how fast the air pressure is changing for a pilot of a small plane passing through an altitude of 1000 feet.

(d) Suppose that $f(x)$ and $g(x)$ are differentiable functions and that the following information about them is known:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	2	-5	-3	4
2	-3	4	-1	2

If $C(x)$ is a function given by the formula $f(g(x))$, determine $C'(2)$. In addition, if $D(x)$ is the function $f(f(x))$, find $D'(-1)$.

◁

2.6 Derivatives of Inverse Functions

Preview Activity 2.6. The equation $y = \frac{5}{9}(x-32)$ relates a temperature given in x degrees Fahrenheit to the corresponding temperature y measured in degrees Celcius.

(a) Solve the equation $y = \frac{5}{9}(x - 32)$ for x to write x (Fahrenheit temperature) in terms of y (Celcius temperature).

(b) Let $C(x) = \frac{5}{9}(x - 32)$ be the function that takes a Fahrenheit temperature as input and produces the Celcius temperature as output. In addition, let $F(y)$ be the function that converts a temperature given in y degrees Celcius to the temperature $F(y)$ measured in degrees Fahrenheit. Use your work in (a) to write a formula for $F(y)$.

(c) Next consider the new function defined by $p(x) = F(C(x))$. Use the formulas for F and C to determine an expression for $p(x)$ and simplify this expression as much as possible. What do you observe?

(d) Now, let $r(y) = C(F(y))$. Use the formulas for F and C to determine an expression for $r(y)$ and simplify this expression as much as possible. What do you observe?

(e) What is the value of $C'(x)$? of $F'(y)$? How do these values appear to be related?

$\bowtie$

Activity 2.16.

For each function given below, find its derivative.

(a) $h(x) = x^2 \ln(x)$

(b) $p(t) = \dfrac{\ln(t)}{e^t + 1}$

(c) $s(y) = \ln(\cos(y) + 2)$

(d) $z(x) = \tan(\ln(x))$

(e) $m(z) = \ln(\ln(z))$

$\triangleleft$

Activity 2.17.

The following prompts in this activity will lead you to develop the derivative of the inverse tangent function.

(a) Let $r(x) = \arctan(x)$. Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.

(b) Differentiate both sides of the equation you found in (a). Solve the resulting equation for $r'(x)$, writing $r'(x)$ as simply as possible in terms of a trigonometric function evaluated at $r(x)$.

(c) Recall that $r(x) = \arctan(x)$. Update your expression for $r'(x)$ so that it only involves trigonometric functions and the independent variable x.

(d) Introduce a right triangle with angle θ so that $\theta = \arctan(x)$. What are the three sides of the triangle?

(e) In terms of only x and 1, what is the value of $\cos(\arctan(x))$?

(f) Use the results of your work above to find an expression involving only 1 and x for $r'(x)$.

$\triangleleft$

Activity 2.18.

Determine the derivative of each of the following functions.

(a) $f(x) = x^3 \arctan(x) + e^x \ln(x)$

(b) $p(t) = 2^{t \arcsin(t)}$

(c) $h(z) = (\arcsin(5z) + \arctan(4 - z))^{27}$

(d) $s(y) = \cot(\arctan(y))$

(e) $m(v) = \ln(\sin^2(v) + 1)$

(f) $g(w) = \arctan\left(\dfrac{\ln(w)}{1 + w^2}\right)$

◁

2.7 Derivatives of Functions Given Implicitly

Preview Activity 2.7. Let f be a differentiable function of x (whose formula is not known) and recall that $\frac{d}{dx}[f(x)]$ and $f'(x)$ are interchangeable notations. Determine each of the following derivatives of combinations of explicit functions of x, the unknown function f, and an arbitrary constant c.

(a) $\dfrac{d}{dx}\left[x^2 + f(x)\right]$

(b) $\dfrac{d}{dx}\left[x^2 f(x)\right]$

(c) $\dfrac{d}{dx}\left[c + x + f(x)^2\right]$

(d) $\dfrac{d}{dx}\left[f(x^2)\right]$

(e) $\dfrac{d}{dx}\left[xf(x) + f(cx) + cf(x)\right]$

$\bowtie$

Activity 2.19.

Consider the curve defined by the equation $x = y^5 - 5y^3 + 4y$, whose graph is pictured in Figure 2.4.

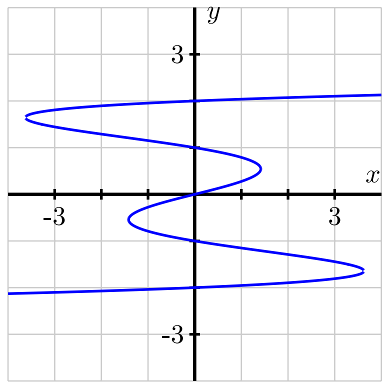

Figure 2.4: The curve $x = y^5 - 5y^3 + 4y$.

(a) Explain why it is not possible to express y as an explicit function of x.

(b) Use implicit differentiation to find a formula for dy/dx.

(c) Use your result from part (b) to find an equation of the line tangent to the graph of $x = y^5 - 5y^3 + 4y$ at the point $(0, 1)$.

(d) Use your result from part (b) to determine all of the points at which the graph of $x = y^5 - 5y^3 + 4y$ has a vertical tangent line.

$\triangleleft$

Activity 2.20.

Consider the curve defined by the equation $y(y^2-1)(y-2) = x(x-1)(x-2)$, whose graph is pictured in Figure 2.5. Through implicit differentiation, it can be shown that

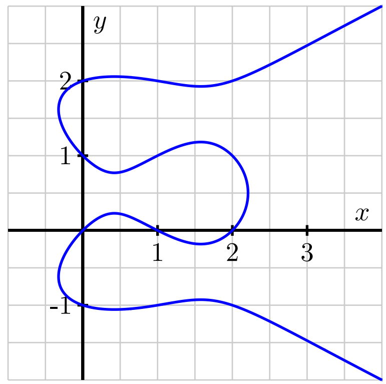

Figure 2.5: The curve $y(y^2-1)(y-2) = x(x-1)(x-2)$.

$$\frac{dy}{dx} = \frac{(x-1)(x-2) + x(x-2) + x(x-1)}{(y^2-1)(y-2) + 2y^2(y-2) + y(y^2-1)}.$$

Use this fact to answer each of the following questions.

(a) Determine all points (x, y) at which the tangent line to the curve is horizontal. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)

(b) Determine all points (x, y) at which the tangent line is vertical. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)

(c) Find the equation of the tangent line to the curve at one of the points where $x = 1$.

◁

Activity 2.21.

For each of the following curves, use implicit differentiation to find dy/dx and determine the equation of the tangent line at the given point.

(a) $x^3 - y^3 = 6xy, \quad (-3, 3)$

(b) $\sin(y) + y = x^3 + x, \quad (0, 0)$

(c) $xe^{-xy} = y^2, \quad (0.571433, 1)$

◁

2.8 Using Derivatives to Evaluate Limits

Preview Activity 2.8. Let h be the function given by $h(x) = \dfrac{x^5 + x - 2}{x^2 - 1}$.

(a) What is the domain of h?

(b) Explain why $\displaystyle\lim_{x\to 1} \frac{x^5 + x - 2}{x^2 - 1}$ results in an indeterminate form.

(c) Next we will investigate the behavior of both the numerator and denominator of h near the point where $x = 1$. Let $f(x) = x^5 + x - 2$ and $g(x) = x^2 - 1$. Find the local linearizations of f and g at $a = 1$, and call these functions $L_f(x)$ and $L_g(x)$, respectively.

(d) Explain why $h(x) \approx \dfrac{L_f(x)}{L_g(x)}$ for x near $a = 1$.

(e) Using your work from (c) and (d), evaluate

$$\lim_{x\to 1} \frac{L_f(x)}{L_g(x)}.$$

What do you think your result tells us about $\displaystyle\lim_{x\to 1} h(x)$?

(f) Investigate the function $h(x)$ graphically and numerically near $x = 1$. What do you think is the value of $\displaystyle\lim_{x\to 1} h(x)$?

$\bowtie$

Activity 2.22.

Evaluate each of the following limits. If you use L'Hopital's Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x}$

(b) $\displaystyle\lim_{x\to \pi} \frac{\cos(x)}{x}$

(c) $\displaystyle\lim_{x\to 1} \frac{2\ln(x)}{1-e^{x-1}}$

(d) $\displaystyle\lim_{x\to 0} \frac{\sin(x)-x}{\cos(2x)-1}$

$\triangleleft$

Activity 2.23.

In this activity, we reason graphically to evaluate limits of ratios of functions about which some information is known.

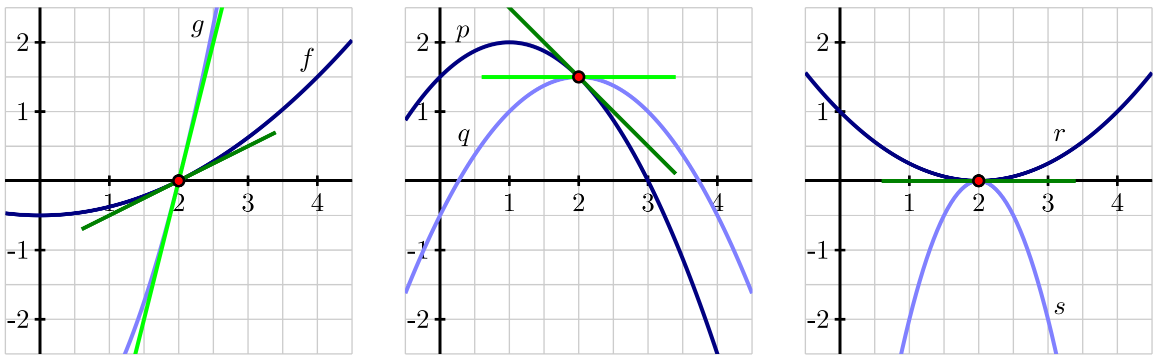

Figure 2.6: Three graphs referenced in the questions of Activity 2.23.

(a) Use the left-hand graph to determine the values of $f(2)$, $f'(2)$, $g(2)$, and $g'(2)$. Then, evaluate

$$\lim_{x \to 2} \frac{f(x)}{g(x)}.$$

(b) Use the middle graph to find $p(2)$, $p'(2)$, $q(2)$, and $q'(2)$. Then, determine the value of

$$\lim_{x \to 2} \frac{p(x)}{q(x)}.$$

(c) Use the right-hand graph to compute $r(2)$, $r'(2)$, $s(2)$, $s'(2)$. Explain why you cannot determine the exact value of

$$\lim_{x \to 2} \frac{r(x)}{s(x)}$$

without further information being provided, but that you can determine the sign of $\lim_{x \to 2} \frac{r(x)}{s(x)}$. In addition, state what the sign of the limit will be, with justification.

$\triangleleft$

Activity 2.24.

Evaluate each of the following limits. If you use L'Hopital's Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) $\lim_{x \to \infty} \frac{x}{\ln(x)}$

(b) $\lim_{x \to \infty} \frac{e^x + x}{2e^x + x^2}$

(c) $\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}}$

(d) $\lim_{x \to \frac{\pi}{2}^-} \frac{\tan(x)}{x - \frac{\pi}{2}}$

(e) $\lim_{x \to \infty} xe^{-x}$

$\triangleleft$

Chapter 3

Using Derivatives

3.1 Using derivatives to identify extreme values of a function

Preview Activity 3.1. Consider the function h given by the graph in Figure 3.1. Use the graph to answer each of the following questions.

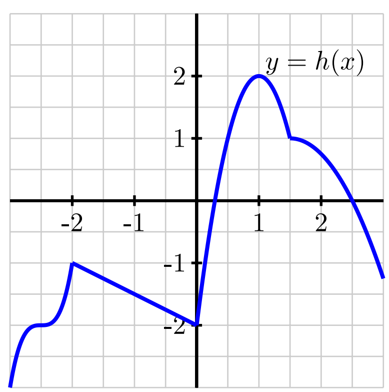

Figure 3.1: The graph of a function h on the interval $[-3, 3]$.

(a) Identify all of the values of c for which $h(c)$ is a local maximum of h.

(b) Identify all of the values of c for which $h(c)$ is a local minimum of h.

(c) Does h have a global maximum? If so, what is the value of this global maximum?

(d) Does h have a global minimum? If so, what is its value?

(e) Identify all values of c for which $h'(c) = 0$.

(f) Identify all values of c for which $h'(c)$ does not exist.

(g) True or false: every relative maximum and minimum of h occurs at a point where $h'(c)$ is either zero or does not exist.

(h) True or false: at every point where $h'(c)$ is zero or does not exist, h has a relative maximum or minimum.

$\bowtie$

Activity 3.1.

Suppose that $g(x)$ is a function continuous for every value of $x \neq 2$ whose first derivative is $g'(x) = \dfrac{(x+4)(x-1)^2}{x-2}$. Further, assume that it is known that g has a vertical asymptote at $x = 2$.

(a) Determine all critical values of g.

(b) By developing a carefully labeled first derivative sign chart, decide whether g has as a local maximum, local minimum, or neither at each critical value.

(c) Does g have a global maximum? global minimum? Justify your claims.

(d) What is the value of $\lim_{x\to\infty} g'(x)$? What does the value of this limit tell you about the long-term behavior of g?

(e) Sketch a possible graph of $y = g(x)$.

$\triangleleft$

Activity 3.2.

Suppose that g is a function whose second derivative, g'', is given by the following graph.

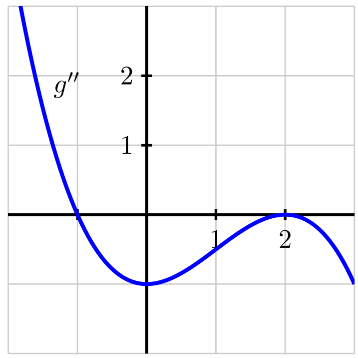

Figure 3.2: The graph of $y = g''(x)$.

(a) Find all points of inflection of g.

(b) Fully describe the concavity of g by making an appropriate sign chart.

(c) Suppose you are given that $g'(-1.67857351) = 0$. Is there is a local maximum, local minimum, or neither (for the function g) at this critical value of g, or is it impossible to say? Why?

(d) Assuming that $g''(x)$ is a polynomial (and that all important behavior of g'' is seen in the graph above, what degree polynomial do you think $g(x)$ is? Why?

◁

Activity 3.3.

Consider the family of functions given by $h(x) = x^2 + \cos(kx)$, where k is an arbitrary positive real number.

(a) Use a graphing utility to sketch the graph of h for several different k-values, including $k = 1, 3, 5, 10$. Plot $h(x) = x^2 + \cos(3x)$ on the axes provided below. What is the smallest

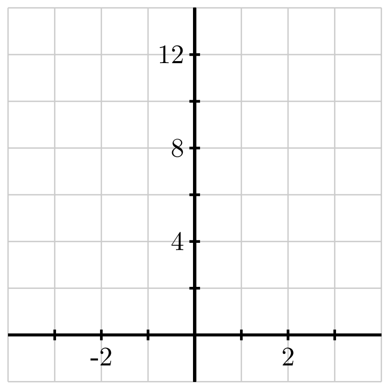

Figure 3.3: Axes for plotting $y = h(x)$.

value of k at which you think you can see (just by looking at the graph) at least one inflection point on the graph of h?

(b) Explain why the graph of h has no inflection points if $k \le \sqrt{2}$, but infinitely many inflection points if $k > \sqrt{2}$.

(c) Explain why, no matter the value of k, h can only have a finite number of critical values.

$\triangleleft$

3.2 Using derivatives to describe families of functions

Preview Activity 3.2. Let a, h, and k be arbitrary real numbers with $a \neq 0$, and let f be the function given by the rule $f(x) = a(x-h)^2 + k$.

(a) What familiar type of function is f? What information do you know about f just by looking at its form? (Think about the roles of a, h, and k.)

(b) Next we use some calculus to develop familiar ideas from a different perspective. To start, treat a, h, and k as constants and compute $f'(x)$.

(c) Find all critical values of f. (These will depend on at least one of a, h, and k.)

(d) Assume that $a < 0$. Construct a first derivative sign chart for f.

(e) Based on the information you've found above, classify the critical values of f as maxima or minima.

$\bowtie$

Activity 3.4.

Consider the family of functions defined by $p(x) = x^3 - ax$, where $a \neq 0$ is an arbitrary constant.

(a) Find $p'(x)$ and determine the critical values of p. How many critical values does p have?

(b) Construct a first derivative sign chart for p. What can you say about the overall behavior of p if the constant a is positive? Why? What if the constant a is negative? In each case, describe the relative extremes of p.

(c) Find $p''(x)$ and construct a second derivative sign chart for p. What does this tell you about the concavity of p? What role does a play in determining the concavity of p?

(d) Without using a graphing utility, sketch and label typical graphs of $p(x)$ for the cases where $a > 0$ and $a < 0$. Label all inflection points and local extrema.

(e) Finally, use a graphing utility to test your observations above by entering and plotting the function $p(x) = x^3 - ax$ for at least four different values of a. Write several sentences to describe your overall conclusions about how the behavior of p depends on a.

◁

Activity 3.5.

Consider the two-parameter family of functions of the form $h(x) = a(1 - e^{-bx})$, where a and b are positive real numbers.

(a) Find the first derivative and the critical values of h. Use these to construct a first derivative sign chart and determine for which values of x the function h is increasing and decreasing.

(b) Find the second derivative and build a second derivative sign chart. For which values of x is a function in this family concave up? concave down?

(c) What is the value of $\lim_{x\to\infty} a(1 - e^{-bx})$? $\lim_{x\to-\infty} a(1 - e^{-bx})$?

(d) How does changing the value of b affect the shape of the curve?

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function h and how this behavior depends on a and b.

$\lhd$

Activity 3.6.

Let $L(t) = \dfrac{A}{1+ce^{-kt}}$, where A, c, and k are all positive real numbers.

(a) Observe that we can equivalently write $L(t) = A(1+ce^{-kt})^{-1}$. Find $L'(t)$ and explain why L has no critical values. Is L always increasing or always decreasing? Why?

(b) Given the fact that

$$L''(t) = Ack^2e^{-kt}\frac{ce^{-kt}-1}{(1+ce^{-kt})^3},$$

find all values of t such that $L''(t) = 0$ and hence construct a second derivative sign chart. For which values of t is a function in this family concave up? concave down?

(c) What is the value of $\displaystyle\lim_{t\to\infty}\frac{A}{1+ce^{-kt}}$? $\displaystyle\lim_{t\to-\infty}\frac{A}{1+ce^{-kt}}$?

(d) Find the value of $L(x)$ at the inflection point found in (b).

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function h and how this behavior depends on a and b.

(f) Explain why it is reasonable to think that the function $L(t)$ models the growth of a population over time in a setting where the largest possible population the surrounding environment can support is A.

◁

3.3 Global Optimization

Preview Activity 3.3. Let $f(x) = 2 + \dfrac{3}{1+(x+1)^2}$.

(a) Determine all of the critical values of f.

(b) Construct a first derivative sign chart for f and thus determine all intervals on which f is increasing or decreasing.

(c) Does f have a global maximum? If so, why, and what is its value and where is the maximum attained? If not, explain why.

(d) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(e) Explain why $f(x) > 2$ for every value of x.

(f) Does f have a global minimum? If so, why, and what is its value and where is the minimum attained? If not, explain why.

$\bowtie$

Activity 3.7.

Let $g(x) = \frac{1}{3}x^3 - 2x + 2$.

(a) Find all critical values of g that lie in the interval $-2 \leq x \leq 3$.

(b) Use a graphing utility to construct the graph of g on the interval $-2 \leq x \leq 3$.

(c) From the graph, determine the x-values at which the absolute minimum and absolute maximum of g occur on the interval $[-2, 3]$.

(d) How do your answers change if we instead consider the interval $-2 \leq x \leq 2$?

(e) What if we instead consider the interval $-2 \leq x \leq 1$?

$\triangleleft$

Activity 3.8.

Find the *exact* absolute maximum and minimum of each function on the stated interval.

(a) $h(x) = xe^{-x}$, $[0, 3]$

(b) $p(t) = \sin(t) + \cos(t)$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$

(c) $q(x) = \frac{x^2}{x-2}$, $[3, 7]$

(d) $f(x) = 4 - e^{-(x-2)^2}$, $(-\infty, \infty)$

(e) $h(x) = xe^{-ax}$, $[0, \frac{2}{a}]$ $(a > 0)$

(f) $f(x) = b - e^{-(x-a)^2}$, $(-\infty, infty)$, $a, b > 0$

$\lhd$

Activity 3.9.

A piece of cardboard that is 10×15 (each measured in inches) is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least 1 inch deep and no more than 3 inches deep, what is the maximum possible volume of the box? what is the minimum volume? Justify your answers using calculus.

(a) Draw a labeled diagram that shows the given information. What variable should we introduce to represent the choice we make in creating the box? Label the diagram appropriately with the variable, and write a sentence to state what the variable represents.

(b) Determine a formula for the function V (that depends on the variable in (a)) that tells us the volume of the box.

(c) What is the domain of the function V? That is, what values of x make sense for input? Are there additional restrictions provided in the problem?

(d) Determine all critical values of the function V.

(e) Evaluate V at each of the endpoints of the domain and at any critical values that lie in the domain.

(f) What is the maximum possible volume of the box? the minimum?

$\triangleleft$

3.4 Applied Optimization

Preview Activity 3.4. According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by "girth" we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

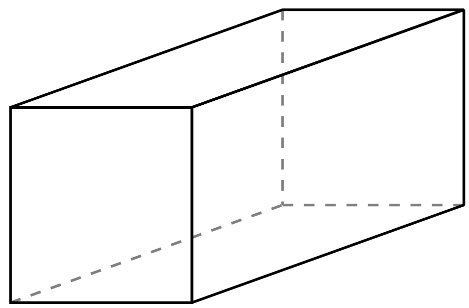

Figure 3.4: A rectangular parcel with a square end.

(a) Let x represent the length of one side of the square end and y the length of the longer side. Label these quantities appropriately on the image shown in Figure 3.4.

(b) What is the quantity to be optimized in this problem? Find a formula for this quantity in terms of x and y.

(c) The problem statement tells us that the parcel's girth plus length may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length to equal 108 inches. What equation does this produce involving x and y?

(d) Solve the equation you found in (c) for one of x or y (whichever is easier).

(e) Now use your work in (b) and (d) to determine a formula for the volume of the parcel so that this formula is a function of a single variable.

(f) Over what domain should we consider this function? Note that both x and y must be positive; how does the constraint that girth plus length is 108 inches produce intervals of possible values for x and y?

(g) Find the absolute maximum of the volume of the parcel on the domain you established in (f) and hence also determine the dimensions of the box of greatest volume. Justify that you've found the maximum using calculus.

⋈

Activity 3.10.

A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs \$$0.015$ per square inch and the material for the lids costs \$$0.027$ per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

(a) Draw a picture of the can and label its dimensions with appropriate variables.

(b) Use your variables to determine expressions for the volume, surface area, and cost of the can.

(c) Determine the total cost function as a function of a single variable. What is the domain on which you should consider this function?

(d) Find the absolute minimum cost and the dimensions that produce this value.

◁

Activity 3.11.

A hiker starting at a point P on a straight road walks east towards point Q, which is on the road and 3 kilometers from point P. Two kilometers due north of point Q is a cabin. The hiker will walk down the road for a while, at a pace of 8 kilometers per hour. At some point Z between P and Q, the hiker leaves the road and makes a straight line towards the cabin through the woods, hiking at a pace of 3 kph, as pictured in Figure 3.5. In order to minimize the time to go from P to Z to the cabin, where should the hiker turn into the forest?

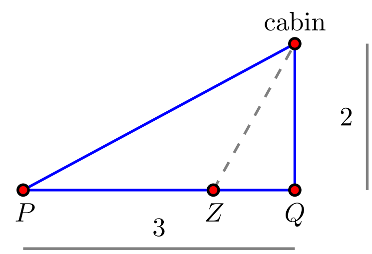

Figure 3.5: A hiker walks from P to Z to the cabin, as pictured.

◁

Activity 3.12.

Consider the region in the x-y plane that is bounded by the x-axis and the function $f(x) = 25 - x^2$. Construct a rectangle whose base lies on the x-axis and is centered at the origin, and whose sides extend vertically until they intersect the curve $y = 25 - x^2$. Which such rectangle has the maximum possible area? Which such rectangle has the greatest perimeter? Which has the greatest combined perimeter and area? (Challenge: answer the same questions in terms of positive parameters a and b for the function $f(x) = b - ax^2$.)

$\lhd$

Activity 3.13.

A trough is being constructed by bending a 4×24 (measured in feet) rectangular piece of sheet metal. Two symmetric folds 2 feet apart will be made parallel to the longest side of the rectangle so that the trough has cross-sections in the shape of a trapezoid, as pictured in Figure 3.6. At what angle should the folds be made to produce the trough of maximum volume?

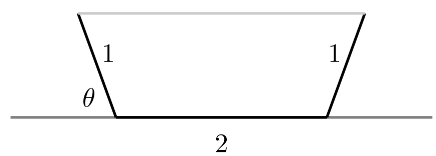

Figure 3.6: A cross-section of the trough formed by folding to an angle of θ.

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3.5 Related Rates

Preview Activity 3.5. A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why?

(a) Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change.

(b) Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Note well that in the setting of this problem, *both* V and r are changing as time t changes, and thus both V and r may be viewed as implicit functions of t, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$.

(c) At this point in the problem, by differentiating we have "related the rates" of change of V and r. Recall that we are given in the problem that the balloon is being inflated at a constant *rate* of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why?

(d) From part (c), we know the value of $\frac{dV}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?

(e) How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

$\bowtie$

Activity 3.14.

A water tank has the shape of an inverted circular cone (point down) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water's surface and the water's depth in the tank, and label them on your figure.

(b) Say that r is the radius and h the depth of the water at a given time, t. What equation relates the radius and height of the water, and why?

(c) Determine an equation that relates the volume of water in the tank at time t to the depth h of the water at that time.

(d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time t.

(e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.

(f) When is the water rising most rapidly: at $h = 3$, $h = 4$, or $h = 5$?

$\triangleleft$

Activity 3.15.

A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. In addition, the auto-focus of the camera has to take into account the increasing distance between the camera and the rocket. We assume that the rocket rises vertically. (A similar problem is discussed and pictured dynamically at http://gvsu.edu/s/9t. Exploring the applet at the link will be helpful to you in answering the questions that follow.)

(a) Draw a figure that summarizes the given situation. What parts of the picture are changing? What parts are constant? Introduce appropriate variables to represent the quantities that are changing.

(b) Find an equation that relates the camera's angle of elevation to the height of the rocket, and then find an equation that relates the instantaneous rate of change of the camera's elevation angle to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

(c) Find an equation that relates the distance from the camera to the rocket to the rocket's height, as well as an equation that relates the instantaneous rate of change of distance from the camera to the rocket to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

(d) Suppose that the rocket's speed is 600 ft/sec at the instant it has risen 3000 feet. How fast is the distance from the television camera to the rocket changing at that moment? If the camera is following the rocket, how fast is the camera's angle of elevation changing at that same moment?

(e) If from an elevation of 3000 feet onward the rocket continues to rise at 600 feet/sec, will the rate of change of distance with respect to time be greater when the elevation is 4000 feet than it was at 3000 feet, or less? Why?

$\triangleleft$

Activity 3.16.

As pictured in the applet at http://gvsu.edu/s/9q, a skateboarder who is 6 feet tall rides under a 15 foot tall lamppost at a constant rate of 3 feet per second. We are interested in understanding how fast his shadow is changing at various points in time.

(a) Draw an appropriate right triangle that represents a snapshot in time of the skateboarder, lamppost, and his shadow. Let x denote the horizontal distance from the base of the lamppost to the skateboarder and s represent the length of his shadow. Label these quantities, as well as the skateboarder's height and the lamppost's height on the diagram.

(b) Observe that the skateboarder and the lamppost represent parallel line segments in the diagram, and thus similar triangles are present. Use similar triangles to establish an equation that relates x and s.

(c) Use your work in (b) to find an equation that relates $\frac{dx}{dt}$ and $\frac{ds}{dt}$.

(d) At what rate is the length of the skateboarder's shadow increasing at the instant the skateboarder is 8 feet from the lamppost?

(e) As the skateboarder's distance from the lamppost increases, is his shadow's length increasing at an increasing rate, increasing at a decreasing rate, or increasing at a constant rate?

(f) Which is moving more rapidly: the skateboarder or the tip of his shadow? Explain, and justify your answer.

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Activity 3.17.

A baseball diamond is $90'$ square. A batter hits a ball along the third base line runs to first base. At what rate is the distance between the ball and first base changing when the ball is halfway to third base, if at that instant the ball is traveling 100 feet/sec? At what rate is the distance between the ball and the runner changing at the same instant, if at the same instant the runner is 1/8 of the way to first base running at 30 feet/sec?

$\triangleleft$

Chapter 4

The Definite Integral

4.1 Determining distance traveled from velocity

Preview Activity 4.1. Suppose that a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

(a) On the left-hand axes provided in Figure 4.1, sketch a labeled graph of the velocity function $v(t) = 3$. Note that while the scale on the two sets of axes is the same, the units on the right-

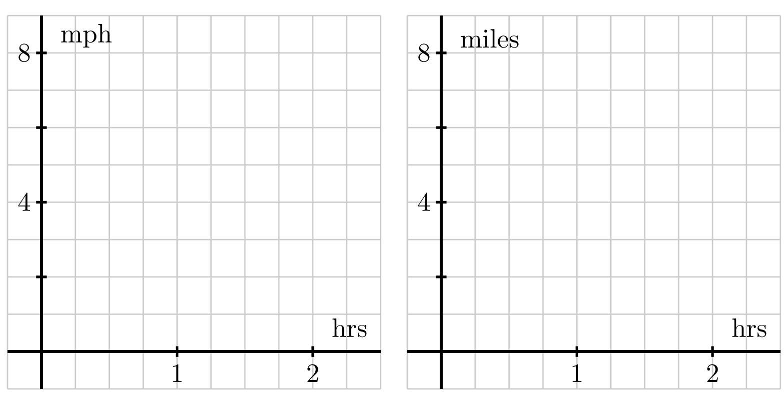

Figure 4.1: At left, axes for plotting $y = v(t)$; at right, for plotting $y = s(t)$.

hand axes differ from those on the left. The right-hand axes will be used in question (d).

(b) How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of $y = v(t)$?

(c) Find an algebraic formula, $s(t)$, for the position of the person at time t, assuming that $s(0) = 0$. Explain your thinking.

(d) On the right-hand axes provided in Figure 4.1, sketch a labeled graph of the position function $y = s(t)$.

(e) For what values of t is the position function s increasing? Explain why this is the case using relevant information about the velocity function v.

⋈

Activity 4.1.

Suppose that a person is walking in such a way that her velocity varies slightly according to the information given in the table below and graph given in Figure 4.2.

t	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
$v(t)$	1.5000	1.7891	1.9375	1.9922	2.0000	2.0078	2.0625	2.2109	2.5000

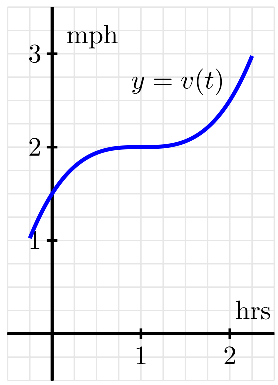

Figure 4.2: The graph of $y = v(t)$.

(a) Using the grid, graph, and given data appropriately, estimate the distance traveled by the walker during the two hour interval from $t = 0$ to $t = 2$. You should use time intervals of width $\triangle t = 0.5$, choosing a way to use the function consistently to determine the height of each rectangle in order to approximate distance traveled.

(b) How could you get a better approximation of the distance traveled on $[0, 2]$? Explain, and then find this new estimate.

(c) Now suppose that you know that v is given by $v(t) = 0.5t^3 - 1.5t^2 + 1.5t + 1.5$. Remember that v is the derivative of the walker's position function, s. Find a formula for s so that $s' = v$.

(d) Based on your work in (c), what is the value of $s(2) - s(0)$? What is the meaning of this quantity?

$\triangleleft$

Activity 4.2.

A ball is tossed vertically in such a way that its velocity function is given by $v(t) = 32 - 32t$, where t is measured in seconds and v in feet per second. Assume that this function is valid for $0 \le t \le 2$.

(a) For what values of t is the velocity of the ball positive? What does this tell you about the motion of the ball on this interval of time values?

(b) Find an antiderivative, s, of v that satisfies $s(0) = 0$.

(c) Compute the value of $s(1) - s(\frac{1}{2})$. What is the meaning of the value you find?

(d) Using the graph of $y = v(t)$ provided in Figure 4.3, find the exact area of the region under the velocity curve between $t = \frac{1}{2}$ and $t = 1$. What is the meaning of the value you find?

(e) Answer the same questions as in (c) and (d) but instead using the interval $[0, 1]$.

(f) What is the value of $s(2) - s(0)$? What does this result tell you about the flight of the ball? How is this value connected to the provided graph of $y = v(t)$? Explain.

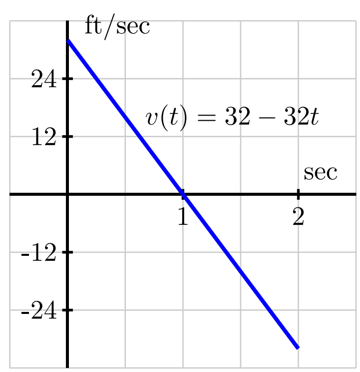

Figure 4.3: The graph of $y = v(t)$.

◁

Activity 4.3.

Suppose that an object moving along a straight line path has its velocity v (in meters per second) at time t (in seconds) given by the piecewise linear function whose graph is pictured in Figure 4.4. We view movement to the right as being in the positive direction (with positive velocity), while movement to the left is in the negative direction. Suppose further that the object's

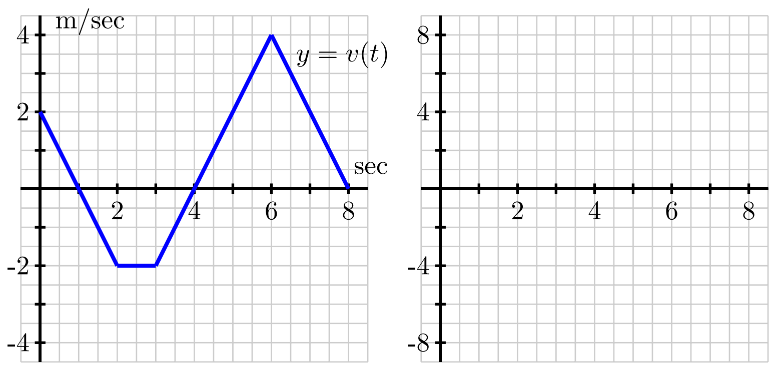

Figure 4.4: The velocity function of a moving object.

initial position at time $t = 0$ is $s(0) = 1$.

(a) Determine the total distance traveled and the total change in position on the time interval $0 \le t \le 2$. What is the object's position at $t = 2$?

(b) On what time intervals is the moving object's position function increasing? Why? On what intervals is the object's position decreasing? Why?

(c) What is the object's position at $t = 8$? How many total meters has it traveled to get to this point (including distance in both directions)? Is this different from the object's total change in position on $t = 0$ to $t = 8$?

(d) Find the exact position of the object at $t = 1, 2, 3, \ldots, 8$ and use this data to sketch an accurate graph of $y = s(t)$ on the axes provided at right. How can you use the provided information about $y = v(t)$ to determine the concavity of s on each relevant interval?

$\triangleleft$

4.2 Riemann Sums

Preview Activity 4.2. A person walking along a straight path has her velocity in miles per hour at time t given by the function $v(t) = 0.25t^3 - 1.5t^2 + 3t + 0.25$, for times in the interval $0 \leq t \leq 2$. The graph of this function is also given in each of the three diagrams in Figure 4.5. Note that in

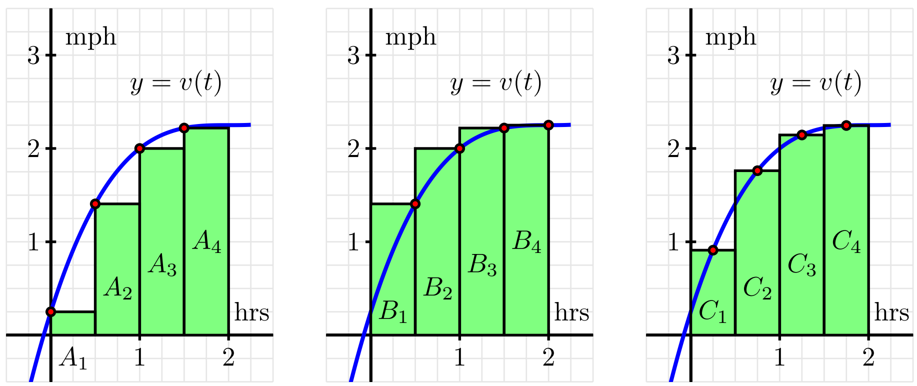

Figure 4.5: Three approaches to estimating the area under $y = v(t)$ on the interval $[0, 2]$.

each diagram, we use four rectangles to estimate the area under $y = v(t)$ on the interval $[0, 2]$, but the method by which the four rectangles' respective heights are decided varies among the three individual graphs.

(a) How are the heights of rectangles in the left-most diagram being chosen? Explain, and hence determine the value of
$$S = A_1 + A_2 + A_3 + A_4$$
by evaluating the function $y = v(t)$ at appropriately chosen values and observing the width of each rectangle. Note, for example, that
$$A_3 = v(1) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1.$$

(b) Explain how the heights of rectangles are being chosen in the middle diagram and find the value of
$$T = B_1 + B_2 + B_3 + B_4.$$

(c) Likewise, determine the pattern of how heights of rectangles are chosen in the right-most diagram and determine
$$U = C_1 + C_2 + C_3 + C_4.$$

(d) Of the estimates S, T, and U, which do you think is the best approximation of D, the total distance the person traveled on $[0, 2]$? Why?

$\bowtie$

Activity 4.4.

For each sum written in sigma notation, write the sum long-hand and evaluate the sum to find its value. For each sum written in expanded form, write the sum in sigma notation.

(a) $\sum_{k=1}^{5}(k^2+2)$

(b) $\sum_{i=3}^{6}(2i-1)$

(c) $3+7+11+15+\cdots+27$

(d) $4+8+16+32\cdots+256$

(e) $\sum_{i=1}^{6}\frac{1}{2^i}$

◁

Activity 4.5.

Suppose that an object moving along a straight line path has its velocity in feet per second at time t in seconds given by $v(t) = \frac{2}{9}(t-3)^2 + 2$.

(a) Carefully sketch the region whose exact area will tell you the value of the distance the object traveled on the time interval $2 \leq t \leq 5$.

(b) Estimate the distance traveled on $[2, 5]$ by computing L_4, R_4, and M_4.

(c) Does averaging L_4 and R_4 result in the same value as M_4? If not, what do you think the average of L_4 and R_4 measures?

(d) For this question, think about an arbitrary function f, rather than the particular function v given above. If f is positive and increasing on $[a, b]$, will L_n over-estimate or under-estimate the exact area under f on $[a, b]$? Will R_n over- or under-estimate the exact area under f on $[a, b]$? Explain.

$\triangleleft$

Activity 4.6.

Suppose that an object moving along a straight line path has its velocity v (in feet per second) at time t (in seconds) given by

$$v(t) = \frac{1}{2}t^2 - 3t + \frac{7}{2}.$$

(a) Compute M_5, the middle Riemann sum, for v on the time interval $[1, 5]$. Be sure to clearly identify the value of $\triangle t$ as well as the locations of t_0, t_1, $\cdots$, t_5. In addition, provide a careful sketch of the function and the corresponding rectangles that are being used in the sum.

(b) Building on your work in (a), estimate the total change in position of the object on the interval $[1, 5]$.

(c) Building on your work in (a) and (b), estimate the total distance traveled by the object on $[1, 5]$.

(d) Use appropriate computing technology[1] to compute M_{10} and M_{20}. What exact value do you think the middle sum eventually approaches as n increases without bound? What does that number represent in the physical context of the overall problem?

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[1] For instance, consider the applet at http://gvsu.edu/s/a9 and change the function and adjust the locations of the blue points that represent the interval endpoints a and b.

4.3 The Definite Integral

Preview Activity 4.3. Consider the applet found at http://gvsu.edu/s/aw[2]. There, you will initially see the situation shown in Figure 4.6. Observe that we can change the window in which

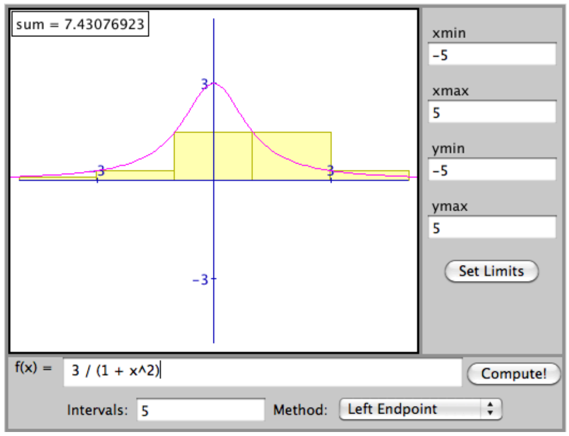

Figure 4.6: A left Riemann sum with 5 subintervals for the function $f(x) = \frac{3}{1+x^2}$ on the interval $[-5, 5]$. The value of the sum is $L_5 = 7.43076923$.

the function is viewed, as well as the function itself. Set the minimum and maximum values of x and y so that we view the function on the window where $1 \le x \le 4$ and $-1 \le y \le 12$, where the function is $f(x) = 2x + 1$ (note that you need to enter "2*x+1" as the function's formula). You should see the updated figure shown in Figure 4.7. Note that the value of the Riemann sum of our choice is displayed in the upper left corner of the window. Further, by updating the value in the "Intervals" window and/or the "Method", we can see the different value of the Riemann sum that arises by clicking the "Compute!" button.

(a) Update the applet so that the function being considered is $f(x) = 2x+1$ on $[1, 4]$, as directed above. For this function on this interval, compute L_n, M_n, R_n for $n = 10$, $n = 100$, and $n = 1000$. What do you conjecture is the exact area bounded by $f(x) = 2x + 1$ and the x-axis on $[1, 4]$?

(b) Use basic geometry to determine the exact area bounded by $f(x) = 2x + 1$ and the x-axis on $[1, 4]$.

(c) Based on your work in (a) and (b), what do you observe occurs when we increase the number of subintervals used in the Riemann sum?

(d) Update the applet to consider the function $f(x) = x^2 + 1$ on the interval $[1, 4]$ (note that you will want to increase the maximum value of y to at least 17, and you need to enter "x^2 +

[2]David Eck of Hobart and William Smith Colleges, author of Java Components for Mathematics, http://gvsu.edu/s/av.

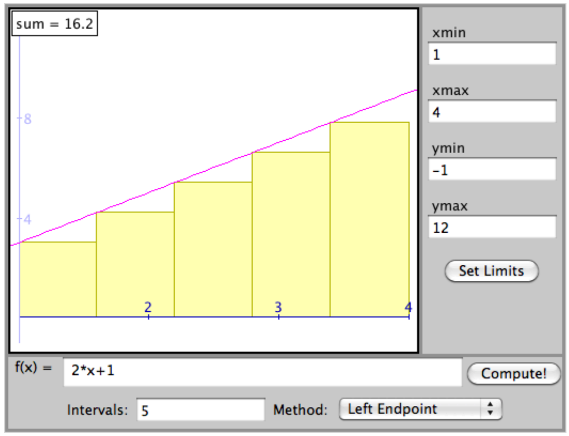

Figure 4.7: A left Riemann sum with 5 subintervals for the function $f(x) = 2x + 1$ on the interval $[1, 4]$. The value of the sum is $L_5 = 16.2$.

1" for the function formula). Use the applet to compute L_n, M_n, R_n for $n = 10$, $n = 100$, and $n = 1000$. What do you conjecture is the exact area bounded by $f(x) = x^2 + 1$ and the x-axis on $[1, 4]$?

(e) Why can we not compute the exact value of the area bounded by $f(x) = x^2 + 1$ and the x-axis on $[1, 4]$ using a formula like we did in (b)?

$\bowtie$

Activity 4.7.

Use known geometric formulas and the net signed area interpretation of the definite integral to evaluate each of the definite integrals below.

(a) $\displaystyle\int_0^1 3x\,dx$

(b) $\displaystyle\int_{-1}^4 (2-2x)\,dx$

(c) $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$

(d) $\displaystyle\int_{-3}^4 g(x)\,dx$, where g is the function pictured in Figure 4.8. Assume that each portion of g is either part of a line or part of a circle.

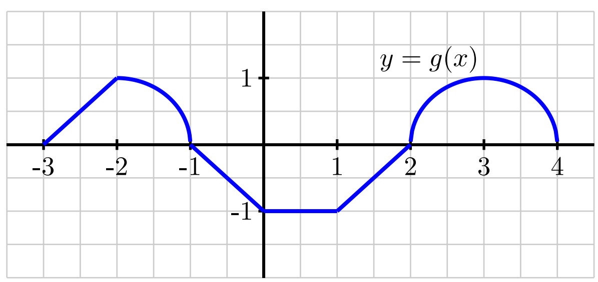

Figure 4.8: A function g that is piecewise defined; each piece of the function is part of a circle or part of a line.

◁

Activity 4.8.

Suppose that the following information is known about the functions f, g, x^2, and x^3:

- $\int_0^2 f(x)\,dx = -3$; $\int_2^5 f(x)\,dx = 2$
- $\int_0^2 g(x)\,dx = 4$; $\int_2^5 g(x)\,dx = -1$
- $\int_0^2 x^2\,dx = \frac{8}{3}$; $\int_2^5 x^2\,dx = \frac{117}{3}$
- $\int_0^2 x^3\,dx = 4$; $\int_2^5 x^3\,dx = \frac{609}{4}$

Use the provided information and the rules discussed in the preceding section to evaluate each of the following definite integrals.

(a) $\int_5^2 f(x)\,dx$

(b) $\int_0^5 g(x)\,dx$

(c) $\int_0^5 (f(x) + g(x))\,dx$

(d) $\int_2^5 (3x^2 - 4x^3)\,dx$

(e) $\int_5^0 (2x^3 - 7g(x))\,dx$

$\triangleleft$

Activity 4.9.

Suppose that $v(t) = \sqrt{4-(t-2)^2}$ tells us the instantaneous velocity of a moving object on the interval $0 \leq t \leq 4$, where t is measured in minutes and v is measured in meters per minute.

(a) Sketch an accurate graph of $y = v(t)$. What kind of curve is $y = \sqrt{4-(t-2)^2}$?

(b) Evaluate $\int_0^4 v(t)\,dt$ exactly.

(c) In terms of the physical problem of the moving object with velocity $v(t)$, what is the meaning of $\int_0^4 v(t)\,dt$? Include units on your answer.

(d) Determine the exact average value of $v(t)$ on $[0, 4]$. Include units on your answer.

(e) Sketch a rectangle whose base is the line segment from $t = 0$ to $t = 4$ on the t-axis such that the rectangle's area is equal to the value of $\int_0^4 v(t)\,dt$. What is the rectangle's exact height?

(f) How can you use the average value you found in (d) to compute the total distance traveled by the moving object over $[0, 4]$?

$\triangleleft$

4.4 The Fundamental Theorem of Calculus

Preview Activity 4.4. A student with a third floor dormitory window 32 feet off the ground tosses a water balloon straight up in the air with an initial velocity of 16 feet per second. It turns out that the instantaneous velocity of the water balloon is given by the velocity function $v(t) = -32t + 16$, where v is measured in feet per second and t is measured in seconds.

(a) Let $s(t)$ represent the height of the water balloon above the ground at time t, and note that s is an antiderivative of v. That is, v is the derivative of s: $s'(t) = v(t)$. Find a formula for $s(t)$ that satisfies the initial condition that the balloon is tossed from 32 feet above ground. In other words, make your formula for s satisfy $s(0) = 32$.

(b) At what time does the water balloon reach its maximum height? At what time does the water balloon land?

(c) Compute the three differences $s(\frac{1}{2}) - s(0)$, $s(2) - s(\frac{1}{2})$, and $s(2) - s(0)$. What do these differences represent?

(d) What is the total vertical distance traveled by the water balloon from the time it is tossed until the time it lands?

(e) Sketch a graph of the velocity function $y = v(t)$ on the time interval $[0, 2]$. What is the total net signed area bounded by $y = v(t)$ and the t-axis on $[0, 2]$? Answer this question in two ways: first by using your work above, and then by using a familiar geometric formula to compute areas of certain relevant regions.

$\bowtie$

Activity 4.10.

Use the Fundamental Theorem of Calculus to evaluate each of the following integrals exactly. For each, sketch a graph of the integrand on the relevant interval and write one sentence that explains the meaning of the value of the integral in terms of the (net signed) area bounded by the curve.

(a) $\displaystyle\int_{-1}^{4} (2 - 2x)\,dx$

(b) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin(x)\,dx$

(c) $\displaystyle\int_{0}^{1} e^x\,dx$

(d) $\displaystyle\int_{-1}^{1} x^5\,dx$

(e) $\displaystyle\int_{0}^{2} (3x^3 - 2x^2 - e^x)\,dx$

◁

given function, $f(x)$	antiderivative, $F(x)$
k, (k is constant)	
x^n, $n \neq -1$	
$\frac{1}{x}$, $x > 0$	
$\sin(x)$	
$\cos(x)$	
$\sec(x)\tan(x)$	
$\csc(x)\cot(x)$	
$\sec^2(x)$	
$\csc^2(x)$	
e^x	
a^x $(a > 1)$	
$\frac{1}{1+x^2}$	
$\frac{1}{\sqrt{1-x^2}}$	

Table 4.1: Familiar basic functions and their antiderivatives.

Activity 4.11.

Use your knowledge of derivatives of basic functions to complete the above table of antiderivatives. For each entry, your task is to find a function F whose derivative is the given function f. When finished, use the FTC and the results in the table to evaluate the three given definite integrals.

(a) $\displaystyle\int_0^1 \left(x^3 - x - e^x + 2\right)\, dx$

(b) $\displaystyle\int_0^{\pi/3} (2\sin(t) - 4\cos(t) + \sec^2(t) - \pi)\, dt$

(c) $\displaystyle\int_0^1 (\sqrt{x} - x^2)\, dx$

◁

Activity 4.12.

During a 30-minute workout, a person riding an exercise machine burns calories at a rate of c calories per minute, where the function $y = c(t)$ is given in Figure 4.9. On the interval $0 \leq t \leq 10$, the formula for c is $c(t) = -0.05t^2 + t + 10$, while on $20 \leq t \leq 30$, its formula is $c(t) = -0.05t^2 + 2t - 5$.

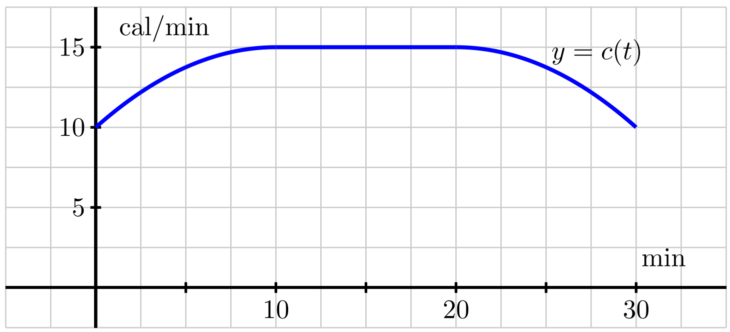

Figure 4.9: The rate $c(t)$ at which a person exercising burns calories, measured in calories per minute.

(a) What is the exact total number of calories the person burns during the first 10 minutes of her workout?

(b) Let $C(t)$ be an antiderivative of $c(t)$. What is the meaning of $C(30) - C(0)$ in the context of the person exercising? Include units on your answer.

(c) Determine the exact average rate at which the person burned calories during the 30-minute workout.

(d) At what time(s), if any, is the instantaneous rate at which the person is burning calories equal to the average rate at which she burns calories, on the time interval $0 \leq t \leq 30$?

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Chapter 5

Finding Antiderivatives and Evaluating Integrals

5.1 Constructing Accurate Graphs of Antiderivatives

Preview Activity 5.1. Suppose that the following information is known about a function f: the graph of its derivative, $y = f'(x)$, is given in Figure 5.1. Further, assume that f' is piecewise linear (as pictured) and that for $x \leq 0$ and $x \geq 6$, $f'(x) = 0$. Finally, it is given that $f(0) = 1$.

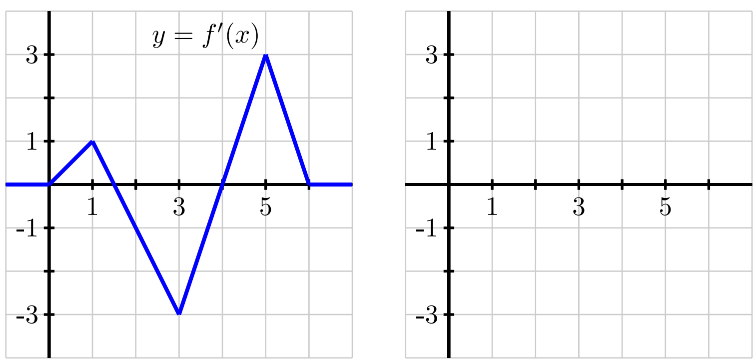

Figure 5.1: At left, the graph of $y = f'(x)$; at right, axes for plotting $y = f(x)$.

(a) On what interval(s) is f an increasing function? On what intervals is f decreasing?

(b) On what interval(s) is f concave up? concave down?

(c) At what point(s) does f have a relative minimum? a relative maximum?

(d) Recall that the Total Change Theorem tells us that

$$f(1) - f(0) = \int_0^1 f'(x)\, dx.$$

What is the exact value of $f(1)$?

(e) Use the given information and similar reasoning to that in (d) to determine the exact value of $f(2)$, $f(3)$, $f(4)$, $f(5)$, and $f(6)$.

(f) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = f(x)$ on the axes provided, being sure to indicate the behavior of f for $x < 0$ and $x > 6$.

$\bowtie$

Activity 5.1.

Suppose that the function $y = f(x)$ is given by the graph shown in Figure 5.2, and that the pieces of f are either portions of lines or portions of circles. In addition, let F be an antiderivative of f and say that $F(0) = -1$. Finally, assume that for $x \leq 0$ and $x \geq 7$, $f(x) = 0$.

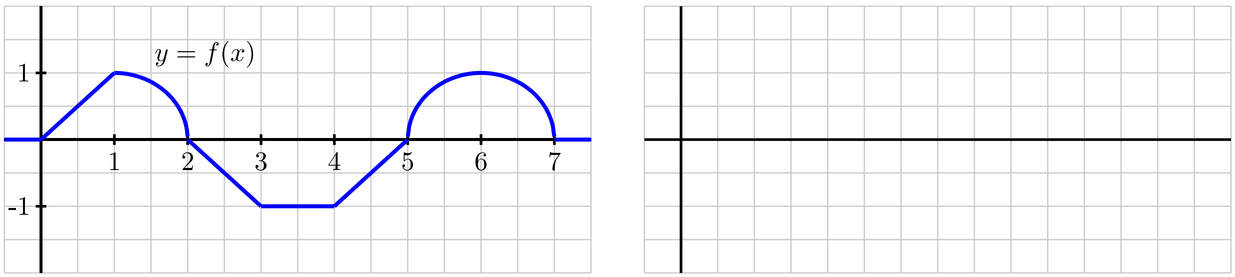

Figure 5.2: At left, the graph of $y = f(x)$.

(a) On what interval(s) is F an increasing function? On what intervals is F decreasing?

(b) On what interval(s) is F concave up? concave down? neither?

(c) At what point(s) does F have a relative minimum? a relative maximum?

(d) Use the given information to determine the exact value of $F(x)$ for $x = 1, 2, \ldots, 7$. In addition, what are the values of $F(-1)$ and $F(8)$?

(e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = F(x)$ on the axes provided, being sure to indicate the behavior of F for $x < 0$ and $x > 7$. Clearly indicate the scale on the vertical and horizontal axes of your graph.

(f) What happens if we change one key piece of information: in particular, say that G is an antiderivative of f and $G(0) = 0$. How (if at all) would your answers to the preceding questions change? Sketch a graph of G on the same axes as the graph of F you constructed in (e).

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Activity 5.2.

For each of the following functions, sketch an accurate graph of the antiderivative that satisfies the given initial condition. In addition, sketch the graph of two additional antiderivatives of the given function, and state the corresponding initial conditions that each of them satisfy. If possible, find an algebraic formula for the antiderivative that satisfies the initial condition.

(a) original function: $g(x) = |x| - 1$;
initial condition: $G(-1) = 0$;
interval for sketch: $[-2, 2]$

(b) original function: $h(x) = \sin(x)$;
initial condition: $H(0) = 1$;
interval for sketch: $[0, 4\pi]$

(c) original function: $p(x) = \begin{cases} x^2, & \text{if } 0 < x \leq 1 \\ -(x-2)^2, & \text{if } 1 < x < 2; \\ 0 & \text{otherwise} \end{cases}$
initial condition: $P(0) = 1$;
interval for sketch: $[-1, 3]$

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Activity 5.3.

Suppose that g is given by the graph at left in Figure 5.3 and that A is the corresponding integral function defined by $A(x) = \int_1^x g(t)\,dt$.

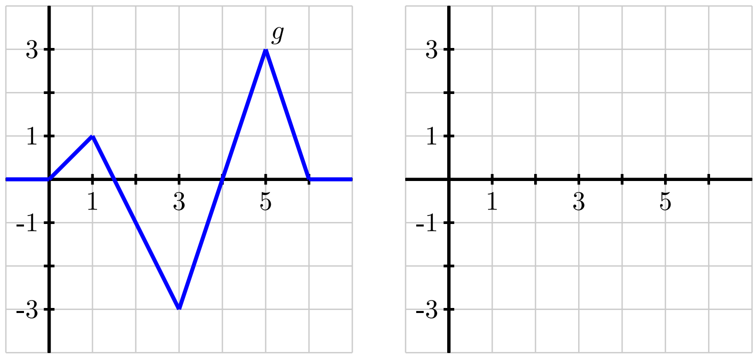

Figure 5.3: At left, the graph of $y = g(t)$; at right, axes for plotting $y = A(x)$, where A is defined by the formula $A(x) = \int_1^x g(t)\,dt$.

(a) On what interval(s) is A an increasing function? On what intervals is A decreasing? Why?

(b) On what interval(s) do you think A is concave up? concave down? Why?

(c) At what point(s) does A have a relative minimum? a relative maximum?

(d) Use the given information to determine the exact values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, and $A(6)$.

(e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = A(x)$ on the axes provided, being sure to indicate the behavior of A for $x < 0$ and $x > 6$.

(f) How does the graph of B compare to A if B is instead defined by $B(x) = \int_0^x g(t)\,dt$?

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5.2 The Second Fundamental Theorem of Calculus

Preview Activity 5.2. Consider the function A defined by the rule

$$A(x) = \int_1^x f(t)\,dt,$$

where $f(t) = 4 - 2t$.

(a) Compute $A(1)$ and $A(2)$ exactly.

(b) Use the First Fundamental Theorem of Calculus to find an equivalent formula for $A(x)$ that does not involve integrals. That is, use the first FTC to evaluate $\int_1^x (4 - 2t)\,dt$.

(c) Observe that f is a linear function; what kind of function is A?

(d) Using the formula you found in (b) that does not involve integrals, compute $A'(x)$.

(e) While we have defined f by the rule $f(t) = 4 - 2t$, it is equivalent to say that f is given by the rule $f(x) = 4 - 2x$. What do you observe about the relationship between A and f?

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Activity 5.4.

Suppose that f is the function given in Figure 5.4 and that f is a piecewise function whose parts are either portions of lines or portions of circles, as pictured. In addition, let A be the function

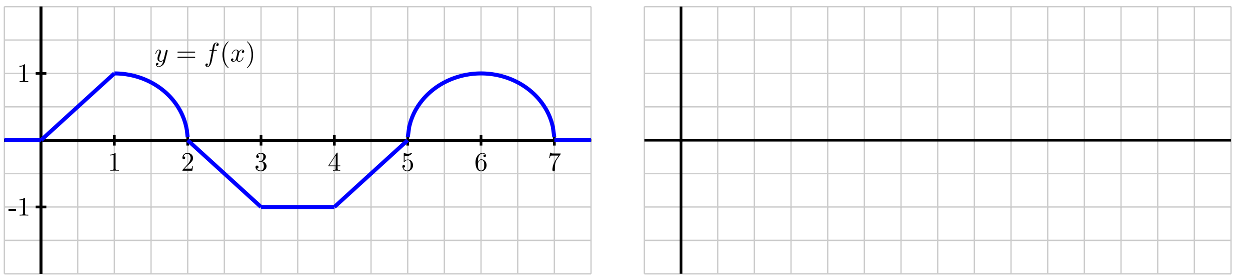

Figure 5.4: At left, the graph of $y = f(x)$. At right, axes for sketching $y = A(x)$.

defined by the rule $A(x) = \int_2^x f(t)\,dt$.

(a) What does the Second FTC tell us about the relationship between A and f?

(b) Compute $A(1)$ and $A(3)$ exactly.

(c) Sketch a precise graph of $y = A(x)$ on the axes at right that accurately reflects where A is increasing and decreasing, where A is concave up and concave down, and the exact values of A at $x = 0, 1, \ldots, 7$.

(d) How is A similar to, but different from, the function F that you found in Activity 5.1?

(e) With as little additional work as possible, sketch precise graphs of the functions $B(x) = \int_3^x f(t)\,dt$ and $C(x) = \int_1^x f(t)\,dt$. Justify your results with at least one sentence of explanation.

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Activity 5.5.

Suppose that $f(t) = \frac{t}{1+t^2}$ and $F(x) = \int_0^x f(t)\,dt$.

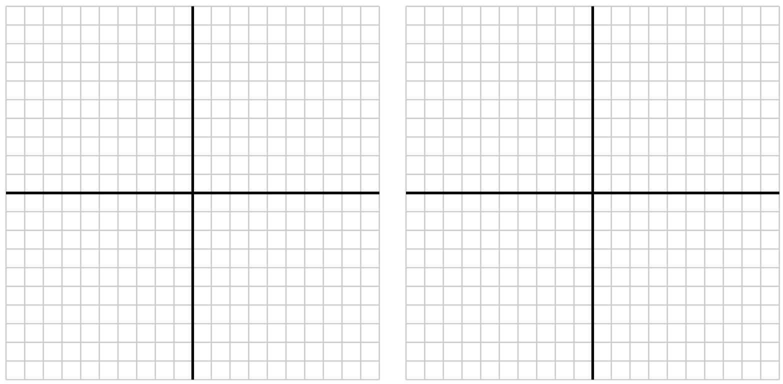

Figure 5.5: Axes for plotting f and F.

(a) On the axes at left in Figure 5.5, plot a graph of $f(t) = \frac{t}{1+t^2}$ on the interval $-10 \le t \le 10$. Clearly label the vertical axes with appropriate scale.

(b) What is the key relationship between F and f, according to the Second FTC?

(c) Use the first derivative test to determine the intervals on which F is increasing and decreasing.

(d) Use the second derivative test to determine the intervals on which F is concave up and concave down. Note that $f'(t)$ can be simplified to be written in the form $f'(t) = \frac{1-t^2}{(1+t^2)^2}$.

(e) Using technology appropriately, estimate the values of $F(5)$ and $F(10)$ through appropriate Riemann sums.

(f) Sketch an accurate graph of $y = F(x)$ on the righthand axes provided, and clearly label the vertical axes with appropriate scale.

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Activity 5.6.

Evaluate each of the following derivatives and definite integrals. Clearly cite whether you use the First or Second FTC in so doing.

(a) $\displaystyle \frac{d}{dx}\left[\int_4^x e^{t^2}\,dt\right]$

(b) $\displaystyle \int_{-2}^x \frac{d}{dt}\left[\frac{t^4}{1+t^4}\right]\,dt$

(c) $\displaystyle \frac{d}{dx}\left[\int_x^1 \cos(t^3)\,dt\right]$

(d) $\displaystyle \int_3^x \frac{d}{dt}\left[\ln(1+t^2)\right]\,dt$

(e) $\displaystyle \frac{d}{dx}\left[\int_4^{x^3} \sin(t^2)\,dt\right]$

(Hint: Let $F(x) = \int_4^x \sin(t^2)\,dt$ and observe that this problem is asking you to evaluate $\frac{d}{dx}\left[F(x^3)\right]$].

◁

5.3 Integration by Substitution

Preview Activity 5.3. In Section 2.5, we learned the Chain Rule and how it can be applied to find the derivative of a composite function. In particular, if u is a differentiable function of x, and f is a differentiable function of $u(x)$, then

$$\frac{d}{dx}\left[f(u(x))\right] = f'(u(x)) \cdot u'(x).$$

In words, we say that the derivative of a composite function $c(x) = f(u(x))$, where f is considered the "outer" function and u the "inner" function, is "the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function."

(a) For each of the following functions, use the Chain Rule to find the function's derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled $g'(x)$).

i. $g(x) = e^{3x}$

ii. $h(x) = \sin(5x + 1)$

iii. $p(x) = \arctan(2x)$

iv. $q(x) = (2 - 7x)^4$

v. $r(x) = 3^{4-11x}$

(b) For each of the following functions, use your work in (a) to help you determine the general antiderivative[1] of the function. Label each antiderivative by name (e.g., the antiderivative of m should be called M). In addition, check your work by computing the derivative of each proposed antiderivative.

i. $m(x) = e^{3x}$

ii. $n(x) = \cos(5x + 1)$

iii. $s(x) = \frac{1}{1+4x^2}$

iv. $v(x) = (2 - 7x)^3$

v. $w(x) = 3^{4-11x}$

(c) Based on your experience in parts (a) and (b), conjecture an antiderivative for each of the following functions. Test your conjectures by computing the derivative of each proposed antiderivative.

[1] Recall that the general antiderivative of a function includes "$+C$" to reflect the entire family of functions that share the same derivative.

i. $a(x) = \cos(\pi x)$

ii. $b(x) = (4x + 7)^{11}$

iii. $c(x) = xe^{x^2}$

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Activity 5.7.

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) $\int \sin(8-3x)\,dx$

(b) $\int \sec^2(4x)\,dx$

(c) $\int \frac{1}{11x-9}\,dx$

(d) $\int \csc(2x+1)\cot(2x+1)\,dx$

(e) $\int \frac{1}{\sqrt{1-16x^2}}\,dx$

(f) $\int 5^{-x}\,dx$

$\triangleleft$

Activity 5.8.

Evaluate each of the following indefinite integrals by using these steps:

- Find two functions within the integrand that form (up to a possible missing constant) a function-derivative pair;
- Make a substitution and convert the integral to one involving u and du;
- Evaluate the new integral in u;
- Convert the resulting function of u back to a function of x by using your earlier substitution;
- Check your work by differentiating the function of x. You should come up with the integrand originally given.

(a) $\displaystyle\int \frac{x^2}{5x^3+1}\,dx$

(b) $\displaystyle\int e^x \sin(e^x)\,dx$

(c) $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

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Activity 5.9.

Evaluate each of the following definite integrals exactly through an appropriate u-substitution.

(a) $\displaystyle\int_1^2 \frac{x}{1+4x^2}\,dx$

(b) $\displaystyle\int_0^1 e^{-x}(2e^{-x}+3)^9\,dx$

(c) $\displaystyle\int_{2/\pi}^{4/\pi} \frac{\cos\left(\frac{1}{x}\right)}{x^2}\,dx$

$\triangleleft$

5.4 Integration by Parts

Preview Activity 5.4. In Section 2.3, we developed the Product Rule and studied how it is employed to differentiate a product of two functions. In particular, recall that if f and g are differentiable functions of x, then

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + g(x)\cdot f'(x).$$

(a) For each of the following functions, use the Product Rule to find the function's derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled $g'(x)$).

  i. $g(x) = x\sin(x)$

  ii. $h(x) = xe^x$

  iii. $p(x) = x\ln(x)$

  iv. $q(x) = x^2\cos(x)$

  v. $r(x) = e^x\sin(x)$

(b) Use your work in (a) to help you evaluate the following indefinite integrals. Use differentiation to check your work.

  i. $\displaystyle\int xe^x + e^x\,dx$

  ii. $\displaystyle\int e^x(\sin(x) + \cos(x))\,dx$

  iii. $\displaystyle\int 2x\cos(x) - x^2\sin(x)\,dx$

  iv. $\displaystyle\int x\cos(x) + \sin(x)\,dx$

  v. $\displaystyle\int 1 + \ln(x)\,dx$

(c) Observe that the examples in (b) work nicely because of the derivatives you were asked to calculate in (a). Each integrand in (b) is precisely the result of differentiating one of the products of basic functions found in (a). To see what happens when an integrand is still a product but not necessarily the result of differentiating an elementary product, we consider how to evaluate

$$\int x\cos(x)\,dx.$$

i. First, observe that

$$\frac{d}{dx}\left[x\sin(x)\right] = x\cos(x) + \sin(x).$$

Integrating both sides indefinitely and using the fact that the integral of a sum is the sum of the integrals, we find that

$$\int \left(\frac{d}{dx}\left[x\sin(x)\right]\right)\, dx = \int x\cos(x)\, dx + \int \sin(x)\, dx.$$

In this last equation, evaluate the indefinite integral on the left side as well as the rightmost indefinite integral on the right.

ii. In the most recent equation from (i.), solve the equation for the expression $\int x\cos(x)\, dx$.

iii. For which product of basic functions have you now found the antiderivative?

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Activity 5.10.

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) $\int te^{-t}\,dt$

(b) $\int 4x\sin(3x)\,dx$

(c) $\int z\sec^2(z)\,dz$

(d) $\int x\ln(x)\,dx$

$\lhd$

Activity 5.11.

Evaluate each of the following indefinite integrals, using the provided hints.

(a) Evaluate $\int \arctan(x)\,dx$ by using Integration by Parts with the substitution $u = \arctan(x)$ and $dv = 1\,dx$.

(b) Evaluate $\int \ln(z)\,dz$. Consider a similar substitution to the one in (a).

(c) Use the substitution $z = t^2$ to transform the integral $\int t^3 \sin(t^2)\,dt$ to a new integral in the variable z, and evaluate that new integral by parts.

(d) Evaluate $\int s^5 e^{s^3}\,ds$ using an approach similar to that described in (c).

(e) Evaluate $\int e^{2t} \cos(e^t)\,dt$. You will find it helpful to note that $e^{2t} = e^t \cdot e^t$.

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Activity 5.12.

Evaluate each of the following indefinite integrals.

(a) $\displaystyle\int x^2 \sin(x)\, dx$

(b) $\displaystyle\int t^3 \ln(t)\, dt$

(c) $\displaystyle\int e^z \sin(z)\, dz$

(d) $\displaystyle\int s^2 e^{3s}\, ds$

(e) $\displaystyle\int t \arctan(t)\, dt$

(**Hint:** At a certain point in this problem, it is very helpful to note that $\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$.)

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5.5 Other Options for Finding Algebraic Antiderivatives

Preview Activity 5.5. For each of the indefinite integrals below, the main question is to decide whether the integral can be evaluated using u-substitution, integration by parts, a combination of the two, or neither. For integrals for which your answer is affirmative, state the substitution(s) you would use. It is not necessary to actually evaluate any of the integrals completely, unless the integral can be evaluated immediately using a familiar basic antiderivative.

(a) $\displaystyle\int x^2 \sin(x^3)\,dx,\quad \int x^2 \sin(x)\,dx,\quad \int \sin(x^3)\,dx,\quad \int x^5 \sin(x^3)\,dx$

(b) $\displaystyle\int \frac{1}{1+x^2}\,dx,\quad \int \frac{x}{1+x^2}\,dx,\quad \int \frac{2x+3}{1+x^2}\,dx,\quad \int \frac{e^x}{1+(e^x)^2}\,dx,$

(c) $\displaystyle\int x\ln(x)\,dx,\quad \int \frac{\ln(x)}{x}\,dx,\quad \int \ln(1+x^2)\,dx,\quad \int x\ln(1+x^2)\,dx,$

(d) $\displaystyle\int x\sqrt{1-x^2}\,dx,\quad \int \frac{1}{\sqrt{1-x^2}}\,dx,\quad \int \frac{x}{\sqrt{1-x^2}}\,dx,\quad \int \frac{1}{x\sqrt{1-x^2}}\,dx,$

⋈

Activity 5.13.

For each of the following problems, evaluate the integral by using the partial fraction decomposition provided.

(a) $\displaystyle\int \frac{1}{x^2 - 2x - 3}\,dx$, given that $\frac{1}{x^2-2x-3} = \frac{1/4}{x-3} - \frac{1/4}{x+1}$

(b) $\displaystyle\int \frac{x^2 + 1}{x^3 - x^2}\,dx$, given that $\frac{x^2+1}{x^3-x^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$

(c) $\displaystyle\int \frac{x - 2}{x^4 + x^2}\,dx$, given that $\frac{x-2}{x^4+x^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{-x+2}{1+x^2}$

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Activity 5.14.

For each of the following integrals, evaluate the integral using u-substitution and/or an entry from the table found in Appendix **??**.

(a) $\displaystyle\int \sqrt{x^2+4}\,dx$

(b) $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$

(c) $\displaystyle\int \frac{2}{\sqrt{16+25x^2}}\,dx$

(d) $\displaystyle\int \frac{1}{x^2\sqrt{49-36x^2}}\,dx$

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5.6 Numerical Integration

Preview Activity 5.6. As we begin to investigate ways to approximate definite integrals, it will be insightful to compare results to integrals whose exact values we know. To that end, the following sequence of questions centers on $\int_0^3 x^2\,dx$.

(a) Use the applet[2] at http://gvsu.edu/s/dP with the function $f(x) = x^2$ on the window of x values from 0 to 3 and y values from -1 to 10, to compute L_3, the left Riemann sum with three subintervals.

(b) Likewise, use the applet to compute R_3 and M_3, the right and middle Riemann sums with three subintervals, respectively.

(c) Use the Fundamental Theorem of Calculus to compute the exact value of $I = \int_0^3 x^2\,dx$.

(d) We define the *error* in an approximation of a definite integral to be the difference between the integral's exact value and the approximation's value. What is the error that results from using L_3? From R_3? From M_3?

(e) In what follows in this section, we will learn a new approach to estimating the value of a definite integral known as the Trapezoid Rule. For now, use the "Trapezoid" option in the applet in the pull-down menu for the "Method" of estimating the definite integral, and determine the approximation generated by 3 trapezoids. What is the error in this approximation? How does it compare to the errors you calculated in (d)?

(f) What is the formula for the area of a trapezoid with bases of length b_1 and b_2 and height h?

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[2]Mike May, St. Louis University, http://gvsu.edu/s/dQ.

Activity 5.15.

In this activity, we explore the relationships among the errors generated by left, right, midpoint, and trapezoid approximations to the definite integral $\int_1^2 \frac{1}{x^2}\, dx$

(a) Use the First FTC to evaluate $\int_1^2 \frac{1}{x^2}\, dx$ exactly.

(b) Use appropriate computing technology to compute the following approximations for $\int_1^2 \frac{1}{x^2}\, dx$: T_4, M_4, T_8, and M_8.

(c) Let the *error* of an approximation be the difference between the exact value of the definite integral and the resulting approximation. For instance, if we let $E_{T,4}$ represent the error that results from using the trapezoid rule with 4 subintervals to estimate the integral, we have

$$E_{T,4} = \int_1^2 \frac{1}{x^2}\, dx - T_4.$$

Similarly, we compute the error of the midpoint rule approximation with 8 subintervals by the formula

$$E_{M,8} = \int_1^2 \frac{1}{x^2}\, dx - M_8.$$

Based on your work in (a) and (b) above, compute $E_{T,4}$, $E_{T,8}$, $E_{M,4}$, $E_{M,8}$.

(d) Which rule consistently over-estimates the exact value of the definite integral? Which rule consistently under-estimates the definite integral?

(e) What behavior(s) of the function $f(x) = \frac{1}{x^2}$ lead to your observations in (d)?

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Activity 5.16.

A car traveling along a straight road is braking and its velocity is measured at several different points in time, as given in the following table. Assume that v is continuous, always decreasing, and always decreasing at a decreasing rate, as is suggested by the data.

seconds, t	0	0.3	0.6	0.9	1.2	1.5	1.8
Velocity in ft/sec, $v(t)$	100	99	96	90	80	50	0

(a) Plot the given data on the set of axes provided in Figure 5.6 with time on the horizontal axis and the velocity on the vertical axis.

(b) What definite integral will give you the exact distance the car traveled on $[0, 1.8]$?

(c) Estimate the total distance traveled on $[0, 1.8]$ by computing L_3, R_3, and T_3. Which of these under-estimates the true distance traveled?

(d) Estimate the total distance traveled on $[0, 1.8]$ by computing M_3. Is this an over- or under-estimate? Why?

(e) Use your results from (c) and (d) improve your estimate further by using Simpson's Rule.

(f) What is your best estimate of the average velocity of the car on $[0, 1.8]$? Why? What are the units on this quantity?

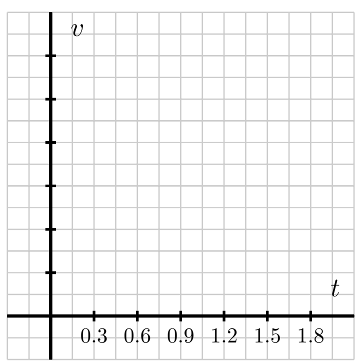

Figure 5.6: Axes for plotting the data in Activity 5.16.

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Activity 5.17.

Consider the functions $f(x) = 2 - x^2$, $g(x) = 2 - x^3$, and $h(x) = 2 - x^4$, all on the interval $[0, 1]$. For each of the questions that require a numerical answer in what follows, write your answer exactly in fraction form.

(a) On the three sets of axes provided in Figure 5.7, sketch a graph of each function on the interval $[0, 1]$, and compute L_1 and R_1 for each. What do you observe?

(b) Compute M_1 for each function to approximate $\int_0^1 f(x)\,dx$, $\int_0^1 g(x)\,dx$, and $\int_0^1 h(x)\,dx$, respectively.

(c) Compute T_1 for each of the three functions, and hence compute S_1 for each of the three functions.

(d) Evaluate each of the integrals $\int_0^1 f(x)\,dx$, $\int_0^1 g(x)\,dx$, and $\int_0^1 h(x)\,dx$ exactly using the First FTC.

(e) For each of the three functions f, g, and h, compare the results of L_1, R_1, M_1, T_1, and S_1 to the true value of the corresponding definite integral. What patterns do you observe?

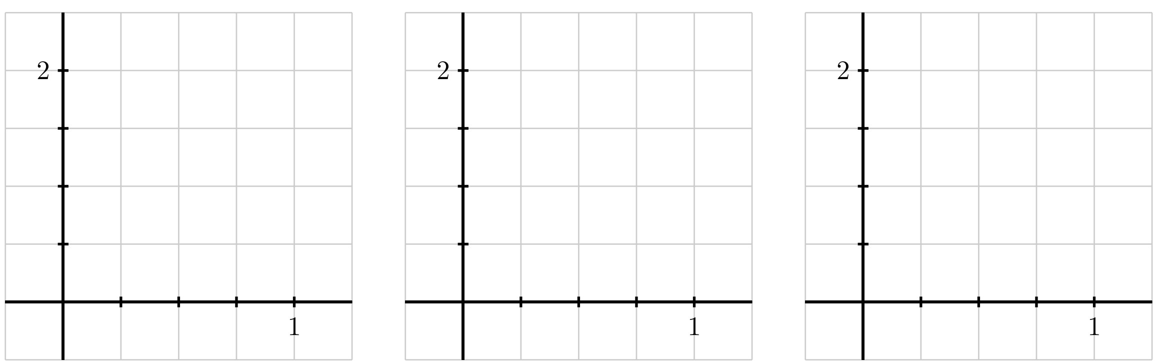

Figure 5.7: Axes for plotting the functions in Activity 5.17.

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Chapter 6

Using Definite Integrals

6.1 Using Definite Integrals to Find Area and Length

Preview Activity 6.1. Consider the functions given by $f(x) = 5 - (x-1)^2$ and $g(x) = 4 - x$.

(a) Use algebra to find the points where the graphs of f and g intersect.

(b) Sketch an accurate graph of f and g on the axes provided, labeling the curves by name and the intersection points with ordered pairs.

(c) Find and evaluate exactly an integral expression that represents the area between $y = f(x)$ and the x-axis on the interval between the intersection points of f and g.

(d) Find and evaluate exactly an integral expression that represents the area between $y = g(x)$ and the x-axis on the interval between the intersection points of f and g.

(e) What is the exact area between f and g between their intersection points? Why?

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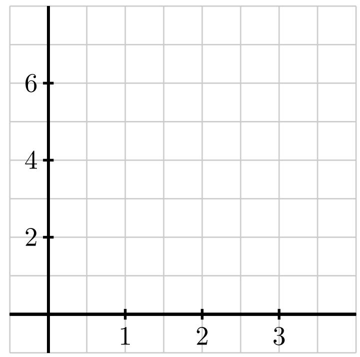

Figure 6.1: Axes for plotting f and g in Preview Activity 6.1

Activity 6.1.

In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

(a) The finite region bounded by $y = \sqrt{x}$ and $y = \frac{1}{4}x$.

(b) The finite region bounded by $y = 12 - 2x^2$ and $y = x^2 - 8$.

(c) The area bounded by the y-axis, $f(x) = \cos(x)$, and $g(x) = \sin(x)$, where we consider the region formed by the first positive value of x for which f and g intersect.

(d) The finite regions between the curves $y = x^3 - x$ and $y = x^2$.

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Activity 6.2.

In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area. **Note well:** At the step where you draw a representative slice, you need to make a choice about whether to slice vertically or horizontally.

(a) The finite region bounded by $x = y^2$ and $x = 6 - 2y^2$.

(b) The finite region bounded by $x = 1 - y^2$ and $x = 2 - 2y^2$.

(c) The area bounded by the x-axis, $y = x^2$, and $y = 2 - x$.

(d) The finite regions between the curves $x = y^2 - 2y$ and $y = x$.

$\triangleleft$

Activity 6.3.

Each of the following questions somehow involves the arc length along a curve.

(a) Use the definition and appropriate computational technology to determine the arc length along $y = x^2$ from $x = -1$ to $x = 1$.

(b) Find the arc length of $y = \sqrt{4 - x^2}$ on the interval $0 \le x \le 4$. Find this value in two different ways: (a) by using a definite integral, and (b) by using a familiar property of the curve.

(c) Determine the arc length of $y = xe^{3x}$ on the interval $[0, 1]$.

(d) Will the integrals that arise calculating arc length typically be ones that we can evaluate exactly using the First FTC, or ones that we need to approximate? Why?

(e) A moving particle is traveling along the curve given by $y = f(x) = 0.1x^2 + 1$, and does so at a constant rate of 7 cm/sec, where both x and y are measured in cm (that is, the curve $y = f(x)$ is the path along which the object actually travels; the curve is not a "position function"). Find the position of the particle when $t = 4$ sec, assuming that when $t = 0$, the particle's location is $(0, f(0))$.

$\triangleleft$

6.2 Using Definite Integrals to Find Volume

Preview Activity 6.2. Consider a circular cone of radius 3 and height 5, which we view horizontally as pictured in Figure 6.2. Our goal in this activity is to use a definite integral to determine the volume of the cone.

(a) Find a formula for the linear function $y = f(x)$ that is pictured in Figure 6.2.

(b) For the representative slice of thickness $\triangle x$ that is located horizontally at a location x (somewhere between $x = 0$ and $x = 5$), what is the radius of the representative slice? Note that the radius depends on the value of x.

(c) What is the volume of the representative slice you found in (b)?

(d) What definite integral will sum the volumes of the thin slices across the full horizontal span of the cone? What is the exact value of this definite integral?

(e) Compare the result of your work in (d) to the volume of the cone that comes from using the formula $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$.

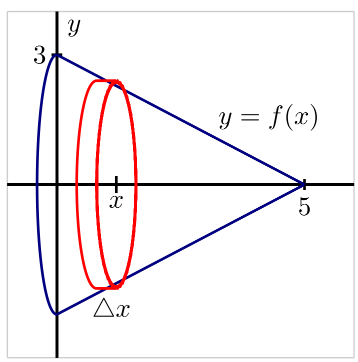

Figure 6.2: The circular cone described in Preview Activity 6.2

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Activity 6.4.

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

(a) The region S bounded by the x-axis, the curve $y = \sqrt{x}$, and the line $x = 4$; revolve S about the x-axis.

(b) The region S bounded by the y-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve S about the x-axis.

(c) The finite region S bounded by the curves $y = \sqrt{x}$ and $y = x^3$; revolve S about the x-axis.

(d) The finite region S bounded by the curves $y = 2x^2 + 1$ and $y = x^2 + 4$; revolve S about the x-axis

(e) The region S bounded by the y-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve S about the y-axis. How does the problem change considerably when we revolve about the y-axis?

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Activity 6.5.

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

(a) The region S bounded by the y-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve S about the y-axis.

(b) The region S bounded by the x-axis, the curve $y = \sqrt{x}$, and the line $x = 4$; revolve S about the y-axis.

(c) The finite region S in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$; revolve S about the x-axis.

(d) The finite region S in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$; revolve S about the y-axis.

(e) The finite region S bounded by the curves $x = (y-1)^2$ and $y = x - 1$; revolve S about the y-axis

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Activity 6.6.

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find. For each prompt, use the finite region S in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$.

(a) Revolve S about the line $y = -2$.

(b) Revolve S about the line $y = 4$.

(c) Revolve S about the line $x = -1$.

(d) Revolve S about the line $x = 5$.

$\triangleleft$

6.3 Density, Mass, and Center of Mass

Preview Activity 6.3. In each of the following scenarios, we consider the distribution of a quantity along an axis.

(a) Suppose that the function $c(x) = 200 + 100e^{-0.1x}$ models the density of traffic on a straight road, measured in cars per mile, where x is number of miles east of a major interchange, and consider the definite integral $\int_0^2 (200 + 100e^{-0.1x})\,dx$.

i. What are the units on the product $c(x) \cdot \triangle x$?

ii. What are the units on the definite integral and its Riemann sum approximation given by
$$\int_0^2 c(x)\,dx \approx \sum_{i=1}^{n} c(x_i)\triangle x?$$

iii. Evaluate the definite integral $\int_0^2 c(x)\,dx = \int_0^2 (200 + 100e^{-0.1x})\,dx$ and write one sentence to explain the meaning of the value you find.

(b) On a 6 foot long shelf filled with books, the function B models the distribution of the weight of the books, measured in pounds per inch, where x is the number of inches from the left end of the bookshelf. Let $B(x)$ be given by the rule $B(x) = 0.5 + \frac{1}{(x+1)^2}$.

i. What are the units on the product $B(x) \cdot \triangle x$?

ii. What are the units on the definite integral and its Riemann sum approximation given by
$$\int_{12}^{36} B(x)\,dx \approx \sum_{i=1}^{n} B(x_i)\triangle x?$$

iii. Evaluate the definite integral $\int_0^{72} B(x)\,dx = \int_0^{72} (0.5 + \frac{1}{(x+1)^2})\,dx$ and write one sentence to explain the meaning of the value you find.

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Activity 6.7.

Consider the following situations in which mass is distributed in a non-constant manner.

(a) Suppose that a thin rod with constant cross-sectional area of 1 cm^2 has its mass distributed according to the density function $\rho(x) = 2e^{-0.2x}$, where x is the distance in cm from the left end of the rod, and the units on $\rho(x)$ are g/cm. If the rod is 10 cm long, determine the exact mass of the rod.

(b) Consider the cone that has a base of radius 4 m and a height of 5 m. Picture the cone lying horizontally with the center of its base at the origin and think of the cone as a solid of revolution.

i. Write and evaluate a definite integral whose value is the volume of the cone.

ii. Next, suppose that the cone has uniform density of 800 kg/m^3. What is the mass of the solid cone?

iii. Now suppose that the cone's density is not uniform, but rather that the cone is most dense at its base. In particular, assume that the density of the cone is uniform across cross sections parallel to its base, but that in each such cross section that is a distance x units from the origin, the density of the cross section is given by the function $\rho(x) = 400 + \frac{200}{1+x^2}$, measured in kg/m^3. Determine and evaluate a definite integral whose value is the mass of this cone of non-uniform density. Do so by first thinking about the mass of a given slice of the cone x units away from the base; remember that in such a slice, the density will be *essentially constant*.

(c) Let a thin rod of constant cross-sectional area 1 cm^2 and length 12 cm have its mass be distributed according to the density function $\rho(x) = \frac{1}{25}(x - 15)^2$, measured in g/cm. Find the exact location z at which to cut the bar so that the two pieces will each have identical mass.

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Activity 6.8.

For quantities of equal weight, such as two children on a teeter-totter, the balancing point is found by taking the average of their locations. When the weights of the quantities differ, we use a weighted average of their respective locations to find the balancing point.

(a) Suppose that a shelf is 6 feet long, with its left end situated at $x = 0$. If one book of weight 1 lb is placed at $x_1 = 0$, and another book of weight 1 lb is placed at $x_2 = 6$, what is the location of $\overline{x}$, the point at which the shelf would (theoretically) balance on a fulcrum?

(b) Now, say that we place four books on the shelf, each weighing 1 lb: at $x_1 = 0$, at $x_2 = 2$, at $x_3 = 4$, and at $x_4 = 6$. Find $\overline{x}$, the balancing point of the shelf.

(c) How does $\overline{x}$ change if we change the location of the third book? Say the locations of the 1-lb books are $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 6$.

(d) Next, suppose that we place four books on the shelf, but of varying weights: at $x_1 = 0$ a 2-lb book, at $x_2 = 2$ a 3-lb book, and $x_3 = 4$ a 1-lb book, and at $x_4 = 6$ a 1-lb book. Use a weighted average of the locations to find $\overline{x}$, the balancing point of the shelf. How does the balancing point in this scenario compare to that found in (b)?

(e) What happens if we change the location of one of the books? Say that we keep everything the same in (d), except that $x_3 = 5$. How does $\overline{x}$ change?

(f) What happens if we change the weight of one of the books? Say that we keep everything the same in (d), except that the book at $x_3 = 4$ now weighs 2 lbs. How does $\overline{x}$ change?

(g) Experiment with a couple of different scenarios of your choosing where you move the location of one of the books to the left, or you decrease the weight of one of the books.

(h) Write a couple of sentences to explain how adjusting the location of one of the books or the weight of one of the books affects the location of the balancing point of the shelf. Think carefully here about how your changes should be considered relative to the location of the balancing point $\overline{x}$ of the current scenario.

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Activity 6.9.

Consider a thin bar of length 20 cm whose density is distributed according to the function $\rho(x) = 4 + 0.1x$, where $x = 0$ represents the left end of the bar. Assume that ρ is measured in g/cm and x is measured in cm.

(a) Find the total mass, M, of the bar.

(b) Without doing any calculations, do you expect the center of mass of the bar to be equal to 10, less than 10, or greater than 10? Why?

(c) Compute $\overline{x}$, the exact center of mass of the bar.

(d) What is the average density of the bar?

(e) Now consider a different density function, given by $p(x) = 4e^{0.020732x}$, also for a bar of length 20 cm whose left end is at $x = 0$. Plot both $\rho(x)$ and $p(x)$ on the same axes. Without doing any calculations, which bar do you expect to have the greater center of mass? Why?

(f) Compute the exact center of mass of the bar described in (e) whose density function is $p(x) = 4e^{0.020732x}$. Check the result against the prediction you made in (e).

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6.4 Physics Applications: Work, Force, and Pressure

Preview Activity 6.4. A bucket is being lifted from the bottom of a 50-foot deep well; its weight (including the water), B, in pounds at a height h feet above the water is given by the function $B(h)$. When the bucket leaves the water, the bucket and water together weigh $B(0) = 20$ pounds, and when the bucket reaches the top of the well, $B(50) = 12$ pounds. Assume that the bucket loses water at a constant rate (as a function of height, h) throughout its journey from the bottom to the top of the well.

(a) Find a formula for $B(h)$.

(b) Compute the value of the product $B(5)\triangle h$, where $\triangle h = 2$ feet. Include units on your answer. Explain why this product represents the approximate work it took to move the bucket of water from $h = 5$ to $h = 7$.

(c) Is the value in (b) an over- or under-estimate of the actual amount of work it took to move the bucket from $h = 5$ to $h = 7$? Why?

(d) Compute the value of the product $B(22)\triangle h$, where $\triangle h = 0.25$ feet. Include units on your answer. What is the meaning of the value you found?

(e) More generally, what does the quantity $W_{\text{slice}} = B(h)\triangle h$ measure for a given value of h and a small positive value of $\triangle h$?

(f) Evaluate the definite integral $\int_0^{50} B(h)\,dh$. What is the meaning of the value you find? Why?

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Activity 6.10.

Consider the following situations in which a varying force accomplishes work.

(a) Suppose that a heavy rope hangs over the side of a cliff. The rope is 200 feet long and weighs 0.3 pounds per foot; initially the rope is fully extended. How much work is required to haul in the entire length of the rope? (Hint: set up a function $F(h)$ whose value is the weight of the rope remaining over the cliff after h feet have been hauled in.)

(b) A leaky bucket is being hauled up from a 100 foot deep well. When lifted from the water, the bucket and water together weigh 40 pounds. As the bucket is being hauled upward at a constant rate, the bucket leaks water at a constant rate so that it is losing weight at a rate of 0.1 pounds per foot. What function $B(h)$ tells the weight of the bucket after the bucket has been lifted h feet? What is the total amount of work accomplished in lifting the bucket to the top of the well?

(c) Now suppose that the bucket in (b) does not leak at a constant rate, but rather that its weight at a height h feet above the water is given by $B(h) = 25 + 15e^{-0.05h}$. What is the total work required to lift the bucket 100 feet? What is the average force exerted on the bucket on the interval $h = 0$ to $h = 100$?

(d) From physics, *Hooke's Law* for springs states that the amount of force required to hold a spring that is compressed (or extended) to a particular length is proportionate to the distance the spring is compressed (or extended) from its natural length. That is, the force to compress (or extend) a spring x units from its natural length is $F(x) = kx$ for some constant k (which is called the *spring constant*.) For springs, we choose to measure the force in pounds and the distance the spring is compressed in feet.

Suppose that a force of 5 pounds extends a particular spring 4 inches (1/3 foot) beyond its natural length.

i. Use the given fact that $F(1/3) = 5$ to find the spring constant k.

ii. Find the work done to extend the spring from its natural length to 1 foot beyond its natural length.

iii. Find the work required to extend the spring from 1 foot beyond its natural length to 1.5 feet beyond its natural length.

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Activity 6.11.

In each of the following problems, determine the total work required to accomplish the described task. In parts (b) and (c), a key step is to find a formula for a function that describes the curve that forms the side boundary of the tank.

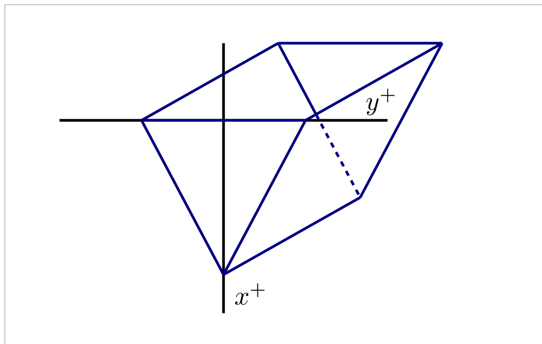

Figure 6.3: A trough with triangular ends, as described in Activity 6.11, part (c).

(a) Consider a vertical cylindrical tank of radius 2 meters and depth 6 meters. Suppose the tank is filled with 4 meters of water of mass density 1000 kg/m^3, and the top 1 meter of water is pumped over the top of the tank.

(b) Consider a hemispherical tank with a radius of 10 feet. Suppose that the tank is full to a depth of 7 feet with water of weight density 62.4 pounds/ft^3, and the top 5 feet of water are pumped out of the tank to a tanker truck whose height is 5 feet above the top of the tank.

(c) Consider a trough with triangular ends, as pictured in Figure 6.3, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3, and a pump is used to empty the tank until the water remaining in the tank is 1 foot deep.

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Activity 6.12.

In each of the following problems, determine the total force exerted by water against the surface that is described.

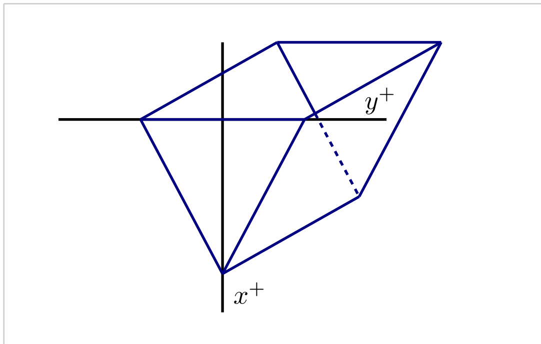

Figure 6.4: A trough with triangular ends, as described in Activity 6.12, part (c).

(a) Consider a rectangular dam that is 100 feet wide and 50 feet tall, and suppose that water presses against the dam all the way to the top.

(b) Consider a semicircular dam with a radius of 30 feet. Suppose that the water rises to within 10 feet of the top of the dam.

(c) Consider a trough with triangular ends, as pictured in Figure 6.4, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3. How much force does the water exert against one of the triangular ends?

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6.5 Improper Integrals

Preview Activity 6.5. A company with a large customer base has a call center that receives thousands of calls a day. After studying the data that represents how long callers wait for assistance, they find that the function $p(t) = 0.25e^{-0.25t}$ models the time customers wait in the following way: the fraction of customers who wait between $t = a$ and $t = b$ minutes is given by

$$\int_a^b p(t)\,dt.$$

Use this information to answer the following questions.

(a) Determine the fraction of callers who wait between 5 and 10 minutes.

(b) Determine the fraction of callers who wait between 10 and 20 minutes.

(c) Next, let's study how the fraction who wait up to a certain number of minutes:

  i. What is the fraction of callers who wait between 0 and 5 minutes?

  ii. What is the fraction of callers who wait between 0 and 10 minutes?

  iii. Between 0 and 15 minutes? Between 0 and 20?

(d) Let $F(b)$ represent the fraction of callers who wait between 0 and b minutes. Find a formula for $F(b)$ that involves a definite integral, and then use the First FTC to find a formula for $F(b)$ that does not involve a definite integral.

(e) What is the value of $\lim_{b\to\infty} F(b)$? Why?

$\bowtie$

Activity 6.13.

In this activity we explore the improper integrals $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$.

(a) First we investigate $\int_1^\infty \frac{1}{x}\,dx$.

i. Use the First FTC to determine the exact values of $\int_1^{10} \frac{1}{x}\,dx$, $\int_1^{1000} \frac{1}{x}\,dx$, and $\int_1^{100000} \frac{1}{x}\,dx$. Then, use your calculator to compute a decimal approximation of each result.

ii. Use the First FTC to evaluate the definite integral $\int_1^b \frac{1}{x}\,dx$ (which results in an expression that depends on b).

iii. Now, use your work from (ii.) to evaluate the limit given by

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx.$$

(b) Next, we investigate $\int_1^\infty \frac{1}{x^{3/2}}\,dx$.

i. Use the First FTC to determine the exact values of $\int_1^{10} \frac{1}{x^{3/2}}\,dx$, $\int_1^{1000} \frac{1}{x^{3/2}}\,dx$, and $\int_1^{100000} \frac{1}{x^{3/2}}\,dx$. Then, use your calculator to compute a decimal approximation of each result.

ii. Use the First FTC to evaluate the definite integral $\int_1^b \frac{1}{x^{3/2}}\,dx$ (which results in an expression that depends on b).

iii. Now, use your work from (ii.) to evaluate the limit given by

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}}\,dx.$$

(c) Plot the functions $y = \frac{1}{x}$ and $y = \frac{1}{x^{3/2}}$ on the same coordinate axes for the values $x = 0 \ldots 10$. How would you compare their behavior as x increases without bound? What is similar? What is different?

(d) How would you characterize the value of $\int_1^\infty \frac{1}{x}\,dx$? of $\int_1^\infty \frac{1}{x^{3/2}}\,dx$? What does this tell us about the respective areas bounded by these two curves for $x \geq 1$?

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Activity 6.14.

Determine whether each of the following improper integrals converges or diverges. For each integral that converges, find its exact value.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$

(b) $\displaystyle\int_0^{\infty} e^{-x/4}\, dx$

(c) $\displaystyle\int_2^{\infty} \frac{9}{(x+5)^{2/3}}\, dx$

(d) $\displaystyle\int_4^{\infty} \frac{3}{(x+2)^{5/4}}\, dx$

(e) $\displaystyle\int_0^{\infty} xe^{-x/4}\, dx$

(f) $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx$, where p is a positive real number

$\triangleleft$

Activity 6.15.

For each of the following definite integrals, decide whether the integral is improper or not. If the integral is proper, evaluate it using the First FTC. If the integral is improper, determine whether or not the integral converges or diverges; if the integral converges, find its exact value.

(a) $\int_0^1 \frac{1}{x^{1/3}}\,dx$

(b) $\int_0^2 e^{-x}\,dx$

(c) $\int_1^4 \frac{1}{\sqrt{4-x}}\,dx$

(d) $\int_{-2}^2 \frac{1}{x^2}\,dx$

(e) $\int_0^{\pi/2} \tan(x)\,dx$

(f) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

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Chapter 7

Differential Equations

7.1 An Introduction to Differential Equations

Preview Activity 7.1. The position of a moving object is given by the function $s(t)$, where s is measured in feet and t in seconds. We determine that the velocity is $v(t) = 4t + 1$ feet per second.

(a) How much does the position change over the time interval $[0, 4]$?

(b) Does this give you enough information to determine $s(4)$, the position at time $t = 4$? If so, what is $s(4)$? If not, what additional information would you need to know to determine $s(4)$?

(c) Suppose you are told that the object's initial position $s(0) = 7$. Determine $s(2)$, the object's position 2 seconds later.

(d) If you are told instead that the object's initial position is $s(0) = 3$, what is $s(2)$?

(e) If we only know the velocity $v(t) = 4t + 1$, is it possible that the object's position at all times is $s(t) = 2t^2 + t - 4$? Explain how you know.

(f) Are there other possibilities for $s(t)$? If so, what are they?

(g) If, in addition to knowing the velocity function is $v(t) = 4t+1$, we know the initial position $s(0)$, how many possibilities are there for $s(t)$?

$\bowtie$

Activity 7.1.

Express the following statements as differential equations. In each case, you will need to introduce notation to describe the important quantities in the statement so be sure to clearly state what your notation means.

(a) The population of a town grows at an annual rate of 1.25%.

(b) A radioactive sample loses 5.6% of its mass every day.

(c) You have a bank account that earns 4% interest every year. At the same time, you withdraw money continually from the account at the rate of $1000 per year.

(d) A cup of hot chocolate is sitting in a 70° room. The temperature of the hot chocolate cools by 10% of the difference between the hot chocolate's temperature and the room temperature every minute.

(e) A can of cold soda is sitting in a 70° room. The temperature of the soda warms at the rate of 10% of the difference between the soda's temperature and the room's temperature every minute.

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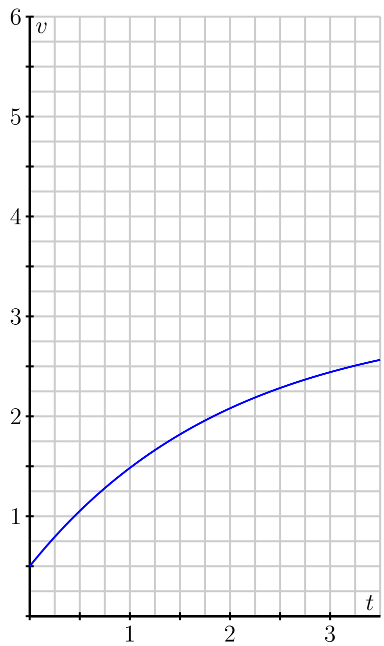
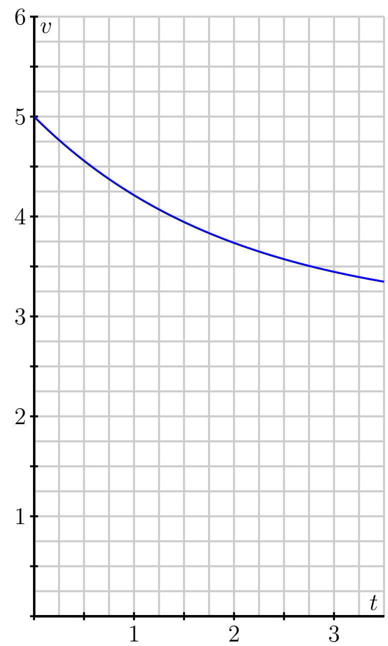
Activity 7.2.

Shown below are two graphs depicting the velocity of falling objects. On the left is the velocity of a skydiver, while on the right is the velocity of a meteorite entering the Earth's atmosphere.

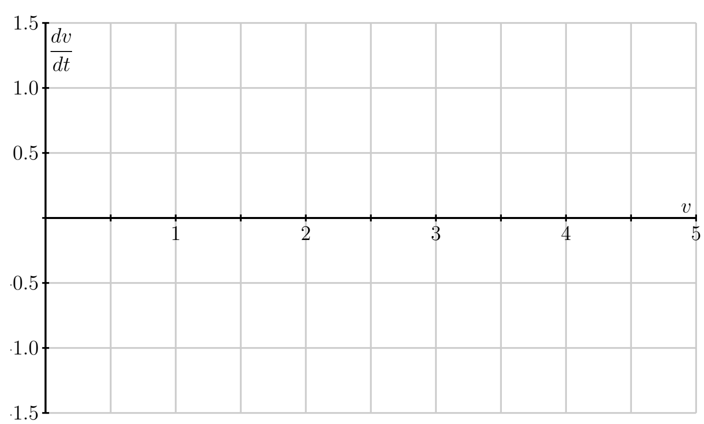


Skydiver's velocity



Meteorite's velocity

(a) Begin with the skydiver's velocity and use the given graph to measure the rate of change dv/dt when the velocity is $v = 0.5, 1.0, 1.5, 2.0$, and 2.5. Plot your values on the graph below. You will want to think carefully about this: you are plotting the derivative dv/dt as a function of *velocity*.



(b) Now do the same thing with the meteorite's velocity: use the given graph to measure the rate of change dv/dt when the velocity is $v = 3.5, 4.0, 4.5$, and 5.0. Plot your values on the graph above.

(c) You should find that all your points lie on a line. Write the equation of this line being careful to use proper notation for the quantities on the horizontal and vertical axes.

(d) The relationship you just found is a differential equation. Write a complete sentence that explains its meaning.

(e) By looking at the differential equation, determine the values of the velocity for which the velocity increases.

(f) By looking at the differential equation, determine the values of the velocity for which the velocity decreases.

(g) By looking at the differential equation, determine the values of the velocity for which the velocity remains constant.

$\triangleleft$

Activity 7.3.

Consider the differential equation

$$\frac{dv}{dt} = 1.5 - 0.5v.$$

Which of the following functions are solutions of this differential equation?

(a) $v(t) = 1.5t - 0.25t^2$.

(b) $v(t) = 3 + 2e^{-0.5t}$.

(c) $v(t) = 3$.

(d) $v(t) = 3 + Ce^{-0.5t}$ where C is any constant.

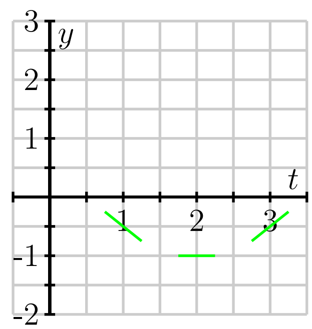
$\triangleleft$

7.2 Qualitative behavior of solutions to differential equations

Preview Activity 7.2. Let's consider the initial value problem

$$\frac{dy}{dt} = t - 2, \ \ y(0) = 1.$$

(a) Use the differential equation to find the slope of the tangent line to the solution $y(t)$ at $t = 0$. Then use the initial value to find the equation of the tangent line at $t = 0$. Sketch this tangent line over the interval $-0.25 \le t \le 0.25$ on the axes provided.



(b) Also shown in the given figure are the tangent lines to the solution $y(t)$ at the points $t = 1, 2,$ and 3 (we will see how to find these later). Use the graph to measure the slope of each tangent line and verify that each agrees with the value specified by the differential equation.

(c) Using these tangent lines as a guide, sketch a graph of the solution $y(t)$ over the interval $0 \le t \le 3$ so that the lines are tangent to the graph of $y(t)$.

(d) Use the Fundamental Theorem of Calculus to find $y(t)$, the solution to this initial value problem.

(e) Graph the solution you found in (d) on the axes provided, and compare it to the sketch you made using the tangent lines.

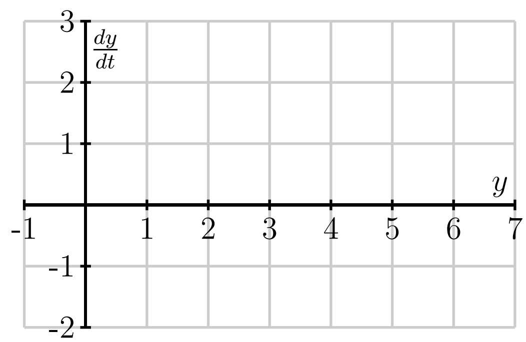
$\bowtie$

Activity 7.4.

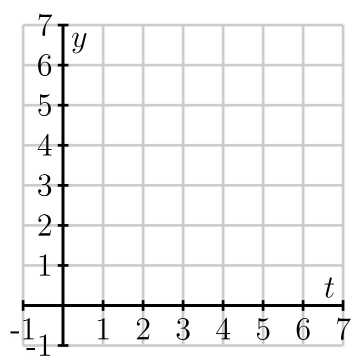
Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 4).$$

(a) Make a plot of $\frac{dy}{dt}$ versus y on the axes provided. Looking at the graph, for what values of y does y increase and for what values of y does y decrease?



(b) Next, sketch the slope field for this differential equation on the axes provided.



(c) Use your work in (b) to sketch the solutions that satisfy $y(0) = 0$, $y(0) = 2$, $y(0) = 4$ and $y(0) = 6$.

(d) Verify that $y(t) = 4 + 2e^{-t/2}$ is a solution to the given differential equation with the initial value $y(0) = 6$. Compare its graph to the one you sketched in (c).

(e) What is special about the solution where $y(0) = 4$?

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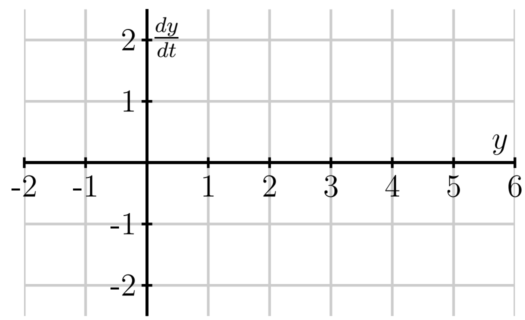
Activity 7.5.

Consider the autonomous differential equation

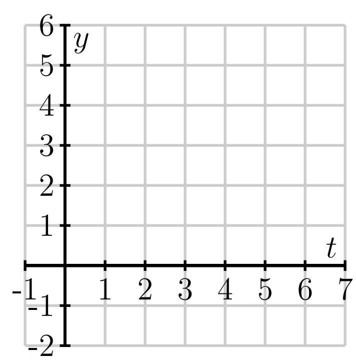
$$\frac{dy}{dt} = -\frac{1}{2}y(y-4).$$

(a) Make a plot of $\frac{dy}{dt}$ versus y. Looking at the graph, for what values of y does y increase and for what values of y does y decrease?



(b) Identify any equilibrium solutions of the given differential equation.

(c) Now sketch the slope field for the given differential equation.

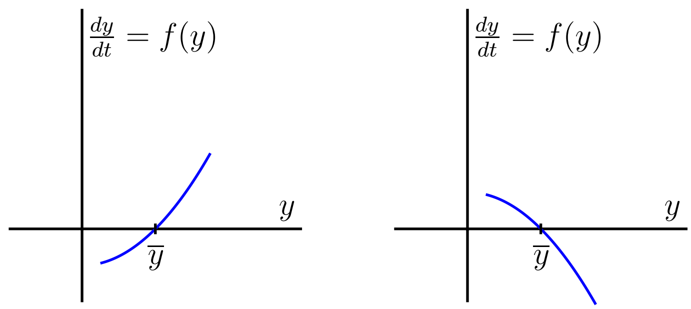


(d) Sketch the solutions to the given differential equation that correspond to initial values $y(0) = -1, 0, 1, \ldots, 5$.

(e) An equilibrium solution $\overline{y}$ is called *stable* if nearby solutions converge to $\overline{y}$. This means that if the inital condition varies slightly from $\overline{y}$, then $\lim_{t\to\infty} y(t) = \overline{y}$.
Conversely, an equilibrium solution $\overline{y}$ is called *unstable* if nearby solutions are pushed away from $\overline{y}$.
Using your work above, classify the equilibrium solutions you found in (b) as either stable or unstable.

(f) Suppose that $y(t)$ describes the population of a species of living organisms and that the initial value $y(0)$ is positive. What can you say about the eventual fate of this population?

(g) Remember that an equilibrium solution $\overline{y}$ satisfies $f(\overline{y}) = 0$. If we graph $dy/dt = f(y)$ as a function of y, for which of the following differential equations is $\overline{y}$ a stable equilibrium and for which is $\overline{y}$ unstable? Why?



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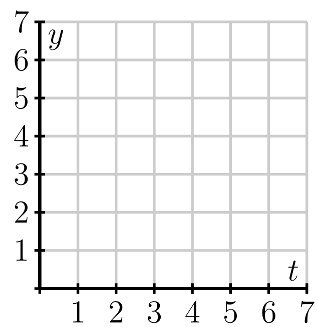
7.3 Euler's method

Preview Activity 7.3. Consider the initial value problem

$$\frac{dy}{dt} = \frac{1}{2}(y+1), \; y(0) = 0.$$

(a) Use the differential equation to find the slope of the tangent line to the solution $y(t)$ at $t = 0$. Then use the given initial value to find the equation of the tangent line at $t = 0$.

(b) Sketch the tangent line on the axes below on the interval $0 \leq t \leq 2$ and use it to approximate $y(2)$, the value of the solution at $t = 2$.



(c) Assuming that your approximation for $y(2)$ is the actual value of $y(2)$, use the differential equation to find the slope of the tangent line to $y(t)$ at $t = 2$. Then, write the equation of the tangent line at $t = 2$.

(d) Add a sketch of this tangent line to your plot on the axes above on the interval $2 \leq t \leq 4$; use this new tangent line to approximate $y(4)$, the value of the solution at $t = 4$.

(e) Repeat the same step to find an approximation for $y(6)$.

$\bowtie$

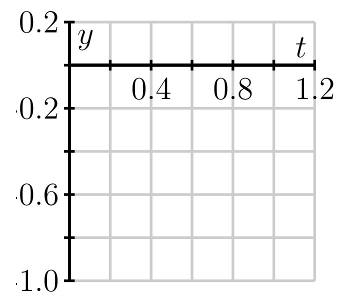
Activity 7.6.

Consider the initial value problem

$$\frac{dy}{dt} = 2t - 1,\ y(0) = 0$$

(a) Use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8$, and 1.0. Record your work in the following table, and sketch the points (t_i, y_i) on the axes provided at right.

t_i	y_i	dy/dt	Δy
0.0000	0.0000		
0.2000			
0.4000			
0.6000			
0.8000			
1.0000			



(b) Find the exact solution to the original initial value problem and use this function to find the error in your approximation at each one of the points t_i.

(c) Explain why the value y_5 generated by Euler's method for this initial value problem produces the same value as a left Riemann sum for the definite integral $\int_0^1 (2t - 1)\, dt$.

(d) How would your computations differ if the initial value was $y(0) = 1$? What does this mean about different solutions to this differential equation?

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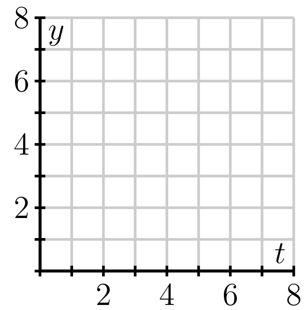
Activity 7.7.

Consider the differential equation:

$$\frac{dy}{dt} = 6y - y^2$$

(a) Sketch the slope field for this differential equation on the axes below.

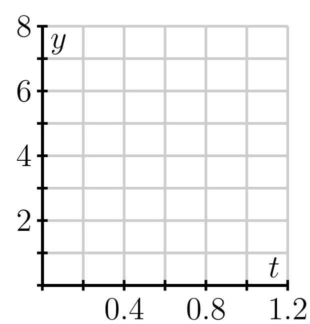


(b) Identify any equilibrium solutions and determine whether they are stable or unstable.

(c) What is the long-term behavior of the solution that satisfies the initial value $y(0) = 1$?

(d) Using the initial value $y(0) = 1$, use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8$, and 1.0. Sketch the points (t_i, y_i) on the axes provided.

t_i	y_i	dy/dt	Δy
0.0	1.0000		
0.2			
0.4			
0.6			
0.8			
1.0			



(e) What happens if we apply Euler's method to approximate the solution with $y(0) = 6$?

◁

Exercises

1. Newton's Law of Cooling says that the rate at which an object, such as a cup of coffee, cools is proportional to the difference in the object's temperature and room temperature. If $T(t)$ is the object's temperature and T_r is room temperature, this law is expressed at

$$\frac{dT}{dt} = -k(T - T_r),$$

where k is a constant of proportionality. In this problem, temperature is measured in degrees Fahrenheit and time in minutes.

(a) Two calculus students, Alice and Bob, enter a 70° classroom at the same time. Each has a cup of coffee that is 100°. The differential equation for Alice has a constant of proportionality $k = 0.5$, while the constant of proportionality for Bob is $k = 0.1$.

What is the initial rate of change for Alice's coffee? What is the initial rate of change for Bob's coffee?

(b) What feature of Alice's and Bob's cups of coffee could explain this difference?

(c) As the heating unit turns on and off in the room, the temperature in the room is

$$T_r = 70 + 10 \sin t.$$

Implement Euler's method with a step size of $\Delta t = 0.1$ to approximate the temperature of Alice's coffee over the time interval $0 \leq t \leq 50$. This will most easily be performed using a spreadsheet such as *Excel*. Graph the temperature of her coffee and room temperature over this interval.

(d) In the same way, implement Euler's method to approximate the temperature of Bob's coffee over the same time interval. Graph the temperature of his coffee and room temperature over the interval.

(e) Explain the similarities and differences that you see in the behavior of Alice's and Bob's cups of coffee.

2. We have seen that the error in approximating the solution to an initial value problem is proportional to Δt. That is, if $E_{\Delta t}$ is the Euler's method approximation to the solution to an initial value problem at $\bar{t}$, then

$$y(\bar{t}) - E_{\Delta t} \approx K\Delta t$$

for some constant of proportionality K.

In this problem, we will see how to use this fact to improve our estimates, using an idea called *accelerated convergence*.

(a) We will create a new approximation by assuming the error is *exactly* proportional to Δt, according to the formula

$$y(\bar{t}) - E_{\Delta t} = K\Delta t.$$

Using our earlier results from the initial value problem $dy/dt = y$ and $y(0) = 1$ with $\Delta t = 0.2$ and $\Delta t = 0.1$, we have

$$\begin{aligned} y(1) - 2.4883 &= 0.2K \\ y(1) - 2.5937 &= 0.1K. \end{aligned}$$

This is a system of two linear equations in the unknowns $y(1)$ and K. Solve this system to find a new approximation for $y(1)$. (You may remember that the exact value is $y(1) = e = 2.71828\ldots$.)

(b) Use the other data, $E_{0.05} = 2.6533$ and $E_{0.025} = 2.6851$ to do similar work as in (a) to obtain another approximation. Which gives the better approximation? Why do you think this is?

(c) Let's now study the initial value problem

$$\frac{dy}{dt} = t - y, \; y(0) = 0.$$

Approximate $y(0.3)$ by applying Euler's method to find approximations $E_{0.1}$ and $E_{0.05}$. Now use the idea of accelerated convergence to obtain a better approximation. (For the sake of comparison, you want to note that the actual value is $y(0.3) = 0.0408$.)

3. In this problem, we'll modify Euler's method to obtain better approximations to solutions of initial value problems. This method is called the *Improved Euler's method.*

In Euler's method, we walk across an interval of width Δt using the slope obtained from the differential equation at the left endpoint of the interval. Of course, the slope of the solution will most likely change over this interval. We can improve our approximation by trying to incorporate the change in the slope over the interval.

Let's again consider the initial value problem $dy/dt = y$ and $y(0) = 1$, which we will approximate using steps of width $\Delta t = 0.2$. Our first interval is therefore $0 \le t \le 0.2$. At $t = 0$, the differential equation tells us that the slope is 1, and the approximation we obtain from Euler's method is that $y(0.2) \approx y_1 = 1 + 1(0.2) = 1.2$.

This gives us some idea for how the slope has changed over the interval $0 \le t \le 0.2$. We know the slope at $t = 0$ is 1, while the slope at $t = 0.2$ is 1.2, trusting in the Euler's method approximation. We will therefore refine our estimate of the initial slope to be the average of these two slopes; that is, we will estimate the slope to be $(1 + 1.2)/2 = 1.1$. This gives the new approximation $y(1) = y_1 = 1 + 1.1(0.2) = 1.22$.

The first few steps look like this:

t_i	y_i	Slope at (t_{i+1}, y_{i+1})	Average slope
0.0	1.0000	1.2000	1.1000
0.2	1.2200	1.4640	1.3420
0.4	1.4884	1.7861	1.6372
$\vdots$	$\vdots$	$\vdots$	$\vdots$

(a) Continue with this method to obtain an approximation for $y(1) = e$.

(b) Repeat this method with $\Delta t = 0.1$ to obtain a better approximation for $y(1)$.

(c) We saw that the error in Euler's method is proportional to Δt. Using your results from parts (a) and (b), what power of Δt appears to be proportional to the error in the Improved Euler's Method?

7.4 Separable differential equations

Preview Activity 7.4. In this preview activity, we explore whether certain differential equations are separable or not, and then revisit some key ideas from earlier work in integral calculus.

(a) Which of the following differential equations are separable? If the equation is separable, write the equation in the revised form $g(y)\frac{dy}{dt} = h(t)$.

1. $\dfrac{dy}{dt} = -3y.$
2. $\dfrac{dy}{dt} = ty - y.$
3. $\dfrac{dy}{dt} = t + 1.$
4. $\dfrac{dy}{dt} = t^2 - y^2.$

(b) Explain why any autonomous differential equation is guaranteed to be separable.

(c) Why do we include the term "$+C$" in the expression

$$\int x\, dx = \frac{x^2}{2} + C?$$

(d) Suppose we know that a certain function f satisfies the equation

$$\int f'(x)\, dx = \int x\, dx.$$

What can you conclude about f?

⋈

Activity 7.8.

Suppose that the population of a town is increases by 3% every year.

(a) Let $P(t)$ be the population of the town in year t. Write a differential equation that describes the annual growth rate.

(b) Find the solutions of this differential equation.

(c) If you know that the town's population in year 0 is 10,000, find the population $P(t)$.

(d) How long does it take for the population to double? This time is called the *doubling time*.

(e) Working more generally, find the doubling time if the annual growth rate is k times the population.

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Activity 7.9.

Suppose that a cup of coffee is initially at a temperature of 105° F and is placed in a 75° F room. Newton's law of cooling says that

$$\frac{dT}{dt} = -k(T - 75),$$

where k is a constant of proportionality.

(a) Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Use the differential equation to determine the value of the constant k.

(b) Find all the solutions of this differential equation.

(c) What happens to all the solutions as $t \to \infty$? Explain how this agrees with your intuition.

(d) What is the temperature of the cup of coffee after 20 minutes?

(e) How long does it take for the coffee to cool to 80°?

$\triangleleft$

Activity 7.10.

Solve each of the following differential equations or initial value problems.

(a) $\dfrac{dy}{dt} - (2-t)y = 2-t$

(b) $\dfrac{1}{t}\dfrac{dy}{dt} = e^{t^2-2y}$

(c) $y' = 2y+2, \quad y(0) = 2$

(d) $y' = 2y^2, \quad y(-1) = 2$

(e) $\dfrac{dy}{dt} = \dfrac{-2ty}{t^2+1}, \quad y(0) = 4$

$\triangleleft$

7.5 Modeling with differential equations

Preview Activity 7.5. Any time that the rate of change of a quantity is related to the amount of a quantity, a differential equation naturally arises. In the following two problems, we see two such scenarios; for each, we want to develop a differential equation whose solution is the quantity of interest.

(a) Suppose you have a bank account in which money grows at an annual rate of 3%.

  (i) If you have $10,000 in the account, at what rate is your money growing?

  (ii) Suppose that you are also withdrawing money from the account at $1,000 per year. What is the rate of change in the amount of money in the account? What are the units on this rate of change?

(b) Suppose that a water tank holds 100 gallons and that a salty solution, which contains 20 grams of salt in every gallon, enters the tank at 2 gallons per minute.

  (i) How much salt enters the tank each minute?

  (ii) Suppose that initially there are 300 grams of salt in the tank. How much salt is in each gallon at this point in time?

  (iii) Finally, suppose that evenly mixed solution is pumped out of the tank at the rate of 2 gallons per minute. How much salt leaves the tank each minute?

  (iv) What is the total rate of change in the amount of salt in the tank?

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Activity 7.11.

Suppose you have a bank account that grows by 5% every year.

(a) Let $A(t)$ be the amount of money in the account in year t. What is the rate of change of A?

(b) Suppose that you are also withdrawing $10,000 per year. Write a differential equation that expresses the total rate of change of A.

(c) Sketch a slope field for this differential equation, find any equilibrium solutions, and identify them as either stable or unstable. Write a sentence or two that describes the significance of the stability of the equilibrium solution.

(d) Suppose that you initially deposit $100,000 into the account. How long does it take for you to deplete the account?

(e) What is the smallest amount of money you would need to have in the account to guarantee that you never deplete the money in the account?

(f) If your initial deposit is $300,000, how much could you withdraw every year without depleting the account?

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Activity 7.12.

A dose of morphine is absorbed from the bloodstream of a patient at a rate proportional to the amount in the bloodstream.

(a) Write a differential equation for $M(t)$, the amount of morphine in the patient's bloodstream, using k as the constant proportionality.

(b) Assuming that the initial dose of morphine is M_0, solve the initial value problem to find $M(t)$. Use the fact that the half-life for the absorption of morphine is two hours to find the constant k.

(c) Suppose that a patient is given morphine intraveneously at the rate of 3 milligrams per hour. Write a differential equation that combines the intraveneous administration of morphine with the body's natural absorption.

(d) Find any equilibrium solutions and determine their stability.

(e) Assuming that there is initially no morphine in the patient's bloodstream, solve the initial value problem to determine $M(t)$.

(f) What happens to $M(t)$ after a very long time?

(g) Suppose that a doctor asks you to reduce the intraveneous rate so that there is eventually 7 milligrams of morphine in the patient's bloodstream. To what rate would you reduce the intraveneous flow?

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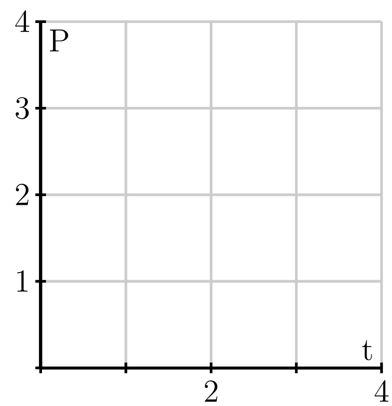
7.6 Population Growth and the Logistic Equation

Preview Activity 7.6. Recall that one model for population growth states that a population grows at a rate proportional to its size.

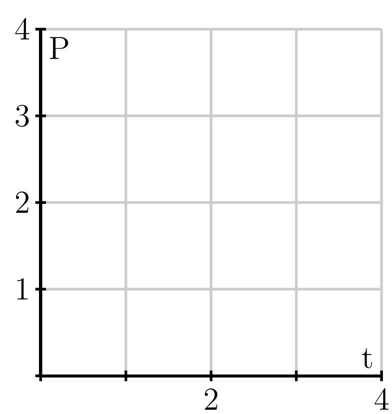
(a) We begin with the differential equation
$$\frac{dP}{dt} = \frac{1}{2}P.$$
Sketch a slope field below as well as a few typical solutions on the axes provided.



(b) Find all equilibrium solutions of the equation $\frac{dP}{dt} = \frac{1}{2}P$ and classify them as stable or unstable.

(c) If $P(0)$ is positive, describe the long-term behavior of the solution to $\frac{dP}{dt} = \frac{1}{2}P$.

(d) Let's now consider a modified differential equation given by
$$\frac{dP}{dt} = \frac{1}{2}P(3-P).$$
As before, sketch a slope field as well as a few typical solutions on the following axes provided.



(e) Find any equilibrium solutions and classify them as stable or unstable.

(f) If $P(0)$ is positive, describe the long-term behavior of the solution.

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Activity 7.13.

Our first model will be based on the following assumption:

The rate of change of the population is proportional to the population.

On the face of it, this seems pretty reasonable. When there is a relatively small number of people, there will be fewer births and deaths so the rate of change will be small. When there is a larger number of people, there will be more births and deaths so we expect a larger rate of change.

If $P(t)$ is the population t years after the year 2000, we may express this assumption as

$$\frac{dP}{dt} = kP$$

where k is a constant of proportionality.

(a) Use the data in the table to estimate the derivative $P'(0)$ using a central difference. Assume that $t = 0$ corresponds to the year 2000.

(b) What is the population $P(0)$?

(c) Use these two facts to estimate the constant of proportionality k in the differential equation.

(d) Now that we know the value of k, we have the initial value problem

$$\frac{dP}{dt} = kP,\ P(0) = 6.084.$$

Find the solution to this initial value problem.

(e) What does your solution predict for the population in the year 2010? Is this close to the actual population given in the table?

(f) When does your solution predict that the population will reach 12 billion?

(g) What does your solution predict for the population in the year 2500?

(h) Do you think this is a reasonable model for the earth's population? Why or why not? Explain your thinking using a couple of complete sentences.

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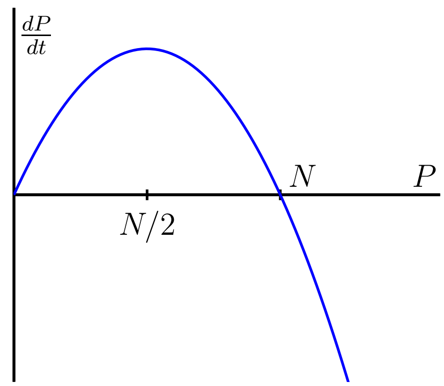
Activity 7.14.

Consider the logistic equation

$$\frac{dP}{dt} = kP(N-P)$$

with the graph of $\frac{dP}{dt}$ vs. P shown below.



(a) At what value of P is the rate of change greatest?

(b) Consider the model for the earth's population that we created. At what value of P is the rate of change greatest? How does that compare to the population in recent years?

(c) According to the model we developed, what will the population be in the year 2100?

(d) According to the model we developed, when will the population reach 9 billion?

(e) Now consider the general solution to the general logistic initial value problem that we found, given by

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt} + 1}.$$

Verify algebraically that $P(0) = P_0$ and that $\lim_{t\to\infty} P(t) = N$.

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Chapter 8

Sequences and Series

8.1 Sequences

Preview Activity 8.1. Suppose you receive $\$5000$ through an inheritance. You decide to invest this money into a fund that pays 8% annually, compounded monthly. That means that each month your investment earns $\frac{0.08}{12} \cdot P$ additional dollars, where P is your principal balance at the start of the month. So in the first month your investment earns

$$5000\left(\frac{0.08}{12}\right)$$

or $\$33.33$. If you reinvest this money, you will then have $\$5033.33$ in your account at the end of the first month. From this point on, assume that you reinvest all of the interest you earn.

(a) How much interest will you earn in the second month? How much money will you have in your account at the end of the second month?

(b) Complete Table 8.1 to determine the interest earned and total amount of money in this investment each month for one year.

(c) As we will see later, the amount of money P_n in the account after month n is given by

$$P_n = 5000\left(1 + \frac{0.08}{12}\right)^n.$$

Use this formula to check your calculations in Table 8.1. Then find the amount of money in the account after 5 years.

(d) How many years will it be before the account has doubled in value to $10000?

$\bowtie$

Month	Interest earned	Total amount of money in the account
0	\$0	\$5000.00
1	\$33.33	\$5033.33
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		

Table 8.1: Interest

Activity 8.1.

(a) Let s_n be the nth term in the sequence $1, 2, 3, \ldots$.

Find a formula for s_n and use appropriate technological tools to draw a graph of entries in this sequence by plotting points of the form (n, s_n) for some values of n. Most graphing calculators can plot sequences; directions follow for the TI-84.

- In the MODE menu, highlight SEQ in the FUNC line and press ENTER.
- In the Y= menu, you will now see lines to enter sequences. Enter a value for nMin (where the sequence starts), a function for $u(n)$ (the nth term in the sequence), and the value of $u_{n\text{Min}}$.
- Set your window coordinates (this involves choosing limits for n as well as the window coordinates XMin, XMax, YMin, and YMax.
- The GRAPH key will draw a plot of your sequence.

Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

(b) Let s_n be the nth term in the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Find a formula for s_n. Draw a graph of some points in this sequence. Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

(c) Let s_n be the nth term in the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$. Find a formula for s_n. Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

$\triangleleft$

Activity 8.2.

(a) Recall our earlier work with limits involving infinity in Section **??**. State clearly what it means for a continuous function f to have a limit L as $x \to \infty$.

(b) Given that an infinite sequence of real numbers is a function from the integers to the real numbers, apply the idea from part (a) to explain what you think it means for a sequence $\{s_n\}$ to have a limit as $n \to \infty$.

(c) Based on your response to (b), decide if the sequence $\left\{\frac{1+n}{2+n}\right\}$ has a limit as $n \to \infty$. If so, what is the limit? If not, why not?

$\lhd$

Activity 8.3.

Use graphical and/or algebraic methods to determine whether each of the following sequences converges or diverges.

(a) $\left\{\frac{1+2n}{3n-2}\right\}$

(b) $\left\{\frac{5+3^n}{10+2^n}\right\}$

(c) $\left\{\frac{10^n}{n!}\right\}$ (where ! is the *factorial* symbol and $n! = n(n-1)(n-2)\cdots(2)(1)$ for any positive integer n (as convention we define $0!$ to be 1)).

$\triangleleft$

8.2 Geometric Series

Preview Activity 8.2. Warfarin is an anticoagulant that prevents blood clotting; often it is prescribed to stroke victims in order to help ensure blood flow. The level of warfarin has to reach a certain concentration in the blood in order to be effective.

Suppose warfarin is taken by a particular patient in a 5 mg dose each day. The drug is absorbed by the body and some is excreted from the system between doses. Assume that at the end of a 24 hour period, 8% of the drug remains in the body. Let $Q(n)$ be the amount (in mg) of warfarin in the body before the $(n+1)$st dose of the drug is administered.

(a) Explain why $Q(1) = 5 \times 0.08$ mg.

(b) Explain why $Q(2) = (5 + Q(1)) \times 0.08$ mg. Then show that

$$Q(2) = (5 \times 0.08)\,(1 + 0.08) \text{ mg}.$$

(c) Explain why $Q(3) = (5 + Q(2)) \times 0.08$ mg. Then show that

$$Q(3) = (5 \times 0.08)\left(1 + 0.08 + 0.08^2\right) \text{ mg}.$$

(d) Explain why $Q(4) = (5 + Q(3)) \times 0.08$ mg. Then show that

$$Q(4) = (5 \times 0.08)\left(1 + 0.08 + 0.08^2 + 0.08^3\right) \text{ mg}.$$

(e) There is a pattern that you should see emerging. Use this pattern to find a formula for $Q(n)$, where n is an arbitrary positive integer.

(f) Complete Table 8.2 with values of $Q(n)$ for the provided n-values (reporting $Q(n)$ to 10 decimal places). What appears to be happening to the sequence $Q(n)$ as n increases?

$\bowtie$

$Q(1)$	0.40
$Q(2)$	
$Q(3)$	
$Q(4)$	
$Q(5)$	
$Q(6)$	
$Q(7)$	
$Q(8)$	
$Q(9)$	
$Q(10)$	

Table 8.2: Values of $Q(n)$ for selected values of n

Activity 8.4.

Let a and r be real numbers (with $r \neq 0$) and let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

In this activity we will find a shortcut formula for S_n that does not involve a sum of n terms.

(a) Multiply S_n by r. What does the resulting sum look like?

(b) Subtract rS_n from S_n and explain why

$$S_n - rS_n = a - ar^n. \tag{8.1}$$

(c) Solve equation (8.1) for S_n to find a simple formula for S_n that does not involve adding n terms.

◁

Activity 8.5.

Let $r > 0$ and a be real numbers and let

$$S = a + ar + ar^2 + \cdots ar^{n-1} + \cdots$$

be an infinite geometric series. For each positive integer n, let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Recall that

$$S_n = a\frac{1 - r^n}{1 - r}.$$

(a) What should we allow n to approach in order to have S_n approach S?

(b) What is the value of $\lim_{n\to\infty} r^n$ for

- $|r| > 1$?
- $|r| < 1$?

Explain.

(c) If $|r| < 1$, use the formula for S_n and your observations in (a) and (b) to explain why S is finite and find a resulting formula for S.

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Activity 8.6.

The formulas we have derived for the geometric series and its partial sum so far have assumed we begin indexing our sums at $n = 0$. If instead we have a sum that does not begin at $n = 0$, we can factor out common terms and use our established formulas. This process is illustrated in the examples in this activity.

(a) Consider the sum

$$\sum_{k=1}^{\infty}(2)\left(\frac{1}{3}\right)^k = (2)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{3}\right)^2 + (2)\left(\frac{1}{3}\right)^3 + \cdots .$$

Remove the common factor of $(2)\left(\frac{1}{3}\right)$ from each term and hence find the sum of the series.

(b) Next let a and r be real numbers with $-1 < r < 1$. Consider the sum

$$\sum_{k=3}^{\infty} ar^k = ar^3 + ar^4 + ar^5 + \cdots .$$

Remove the common factor of ar^3 from each term and find the sum of the series.

(c) Finally, we consider the most general case. Let a and r be real numbers with $-1 < r < 1$, let n be a positive integer, and consider the sum

$$\sum_{k=n}^{\infty} ar^k = ar^n + ar^{n+1} + ar^{n+2} + \cdots .$$

Remove the common factor of ar^n from each term to find the sum of the series.

◁

8.3 Series of Real Numbers

Preview Activity 8.3. Have you ever wondered how your calculator can produce a numeric approximation for complicated numbers like e, π or $\ln(2)$? After all, the only operations a calculator can really perform are addition, subtraction, multiplication, and division, the operations that make up polynomials. This activity provides the first steps in understanding how this process works. Throughout the activity, let $f(x) = e^x$.

(a) Find the tangent line to f at $x = 0$ and use this linearization to approximate e. That is, find a formula $L(x)$ for the tangent line, and compute $L(1)$, since $L(1) \approx f(1) = e$.

(b) The linearization of e^x does not provide a good approximation to e since 1 is not very close to 0. To obtain a better approximation, we alter our approach a bit. Instead of using a straight line to approximate e, we put an appropriate bend in our estimating function to make it better fit the graph of e^x for x close to 0. With the linearization, we had both $f(x)$ and $f'(x)$ share the same value as the linearization at $x = 0$. We will now use a quadratic approximation $P_2(x)$ to $f(x) = e^x$ centered at $x = 0$ which has the property that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$.

(i) Let $P_2(x) = 1 + x + \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. Then, use $P_2(x)$ to approximate e by observing that $P_2(1) \approx f(1)$.

(ii) We can continue approximating e with polynomials of larger degree whose higher derivatives agree with those of f at 0. This turns out to make the polynomials fit the graph of f better for more values of x around 0. For example, let $P_3(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{6}$. Show that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. Use $P_3(x)$ to approximate e in a way similar to how you did so with $P_2(x)$ above.

$\bowtie$

Activity 8.7.

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

While it is physically impossible to add an infinite collection of numbers, we can, of course, add any finite collection of them. In what follows, we investigate how understanding how to find the nth partial sum (that is, the sum of the first n terms) enables us to make sense of what the infinite sum.

(a) Sum the first two numbers in this series. That is, find a numeric value for

$$\sum_{k=1}^{2} \frac{1}{k^2}$$

(b) Next, add the first three numbers in the series.

(c) Continue adding terms in this series to complete the following table. Carry each sum to at least 8 decimal places.

$\sum_{k=1}^{1} \frac{1}{k^2}$	$= 1$	$\sum_{k=1}^{6} \frac{1}{k^2}$	$=$
$\sum_{k=1}^{2} \frac{1}{k^2}$	$=$	$\sum_{k=1}^{7} \frac{1}{k^2}$	$=$
$\sum_{k=1}^{3} \frac{1}{k^2}$	$=$	$\sum_{k=1}^{8} \frac{1}{k^2}$	$=$
$\sum_{k=1}^{4} \frac{1}{k^2}$	$=$	$\sum_{k=1}^{9} \frac{1}{k^2}$	$=$
$\sum_{k=1}^{5} \frac{1}{k^2}$	$=$	$\sum_{k=1}^{10} \frac{1}{k^2}$	$=$

Table 8.3: Sums of some of the first terms of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$

(d) The sums in the table in (c) form a sequence whose nth term is $S_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$. Based on your calculations in the table, do you think the sequence $\{S_n\}$ converges or diverges? Explain. How do you think this sequence $\{S_n\}$ is related to the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$?

◁

Activity 8.8.

If the *series* $\sum a_k$ converges, then an important result necessarily follows regarding the *sequence* $\{a_n\}$. This activity explores this result.

Assume that the series $\sum_{k=1}^{\infty} a_k$ converges and has sum equal to L.

(a) What is the nth partial sum S_n of the series $\sum_{k=1}^{\infty} a_k$?

(b) What is the $(n-1)$st partial sum S_{n-1} of the series $\sum_{k=1}^{\infty} a_k$?

(c) What is the difference between the nth partial sum and the $(n-1)$st partial sum of the series $\sum_{k=1}^{\infty} a_k$?

(d) Since we are assuming that $\sum_{k=1}^{\infty} a_k = L$, what does that tell us about $\lim_{n\to\infty} S_n$? Why? What does that tell us about $\lim_{n\to\infty} S_{n-1}$? Why?

(e) Combine the results of the previous two parts of this activity to determine $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$.

$\triangleleft$

Activity 8.9.

Determine if the Divergence Test applies to the following series. If the test does not apply, explain why. If the test does apply, what does it tell us about the series?

(a) $\sum \frac{k}{k+1}$

(b) $\sum (-1)^k$

(c) $\sum \frac{1}{k}$

$\triangleleft$

Activity 8.10.

Consider the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Recall that the harmonic series will converge provided that its sequence of partial sums converges. The nth partial sum S_n of the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= 1(1) + (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{3}\right) + \cdots + (1)\left(\frac{1}{n}\right). \end{aligned}$$

Through this last expression for S_n, we can visualize this partial sum as a sum of areas of rectangles with heights $\frac{1}{m}$ and bases of length 1, as shown in Figure 8.1, which uses the 9th partial sum. The graph of the continuous function f defined by $f(x) = \frac{1}{x}$ is overlaid on this

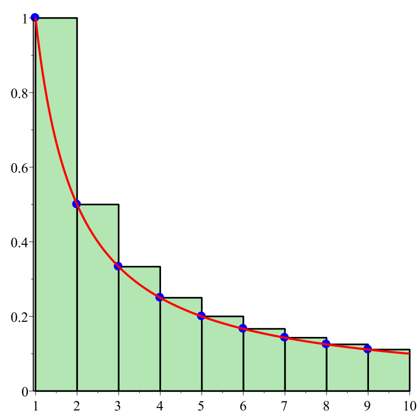

Figure 8.1: A picture of the 9th partial sum of the harmonic series as a sum of areas of rectangles.

plot.

(a) Explain how this picture represents a particular Riemann sum.

(b) What is the definite integral that corresponds to the Riemann sum you considered in (a)?

(c) Which is larger, the definite integral in (b), or the corresponding partial sum S_9 of the series? Why?

(d) If instead of considering the 9th partial sum, we consider the nth partial sum, and we let n go to infinity, we can then compare the series $\sum_{k=1}^{\infty} \frac{1}{k}$ to the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$.

Which of these quantities is larger? Why?

(e) Does the improper integral $\int_1^\infty \frac{1}{x}\, dx$ converge or diverge? What does that result, together with your work in (d), tell us about the series $\sum_{k=1}^{\infty} \frac{1}{k}$?

◁

Activity 8.11.

The series $\sum \frac{1}{k^p}$ are special series called p-series. We have already seen that the p-series with $p = 1$ (the harmonic series) diverges. We investigate the behavior of other p-series in this activity.

(a) Evaluate the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$. Does the series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^2}$ converge or diverge? Explain.

(b) Evaluate the improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ where $p > 1$. For which values of p can we conclude that the series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^p}$ converges?

(c) Evaluate the improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ where $p < 1$. What does this tell us about the corresponding p-series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^p}$?

(d) Summarize your work in this activity by completing the following statement.

The p-series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^p}$ converges if and only if ________________________.

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Activity 8.12.

Consider the series $\sum \frac{k+1}{k^3+2}$. Since the convergence or divergence of a series only depends on the behavior of the series for large values of k, we might examine the terms of this series more closely as k gets large.

(a) By computing the value of $\frac{k+1}{k^3+2}$ for $k = 100$ and $k = 1000$, explain why the terms $\frac{k+1}{k^3+2}$ are essentially $\frac{k}{k^3}$ when k is large.

(b) Let's formalize our observations in (a) a bit more. Let $a_k = \frac{k+1}{k^3+2}$ and $b_k = \frac{k}{k^3}$. Calculate

$$\lim_{k \to \infty} \frac{a_k}{b_k}.$$

What does the value of the limit tell you about a_k and b_k for large values of k? Compare your response from part (a).

(c) Does the series $\sum \frac{k}{k^3}$ converge or diverge? Why? What do you think that tells us about the convergence or divergence of the series $\sum \frac{k+1}{k^3+2}$? Explain.

$\triangleleft$

Activity 8.13.

Use the Limit Comparison Test to determine the convergence or divergence of the series

$$\sum \frac{3k^2+1}{5k^4+2k+2}.$$

by comparing it to the series $\sum \frac{1}{k^2}$.

◁

Activity 8.14.

Consider the series defined by

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}. \tag{8.2}$$

This series is not a geometric series, but this activity will illustrate how we might compare this series to a geometric one. Recall that a series $\sum a_k$ is geometric if the ratio $\frac{a_{k+1}}{a_k}$ is always the same. For the series in (8.2), note that $a_k = \frac{2^k}{3^k - k}$.

(a) To see if $\sum \frac{2^k}{3^k-k}$ is comparable to a geometric series, we analyze the ratios of successive terms in the series. Complete Table 1, listing your calculations to at least 8 decimal places.

k	$\frac{a_{k+1}}{a_k}$
5	
10	
20	
21	
22	
23	
24	
25	

Table 8.4: Ratios of successive terms in the series $\sum \frac{2^k}{3^k-k}$

(b) Based on your calculations in Table 1, what can we say about the ratio $\frac{a_{k+1}}{a_k}$ if k is large?

(c) Do you agree or disagree with the statement: "the series $\sum \frac{2^k}{3^k-k}$ is approximately geometric when k is large"? If not, why not? If so, do you think the series $\sum \frac{2^k}{3^k-k}$ converges or diverges? Explain.

◁

Activity 8.15.

Determine whether each of the following series convergens or diverges. Explicitly state which test you use.

(a) $\sum \frac{k}{2^k}$

(b) $\sum \frac{k^3+2}{k^2+1}$

(c) $\sum \frac{10^k}{k!}$

(d) $\sum \frac{k^3-2k^2+1}{k^6+4}$

$\triangleleft$

8.4 Alternating Series

Preview Activity 8.4. Preview Activity 8.3 showed how we can approximate the number e with linear, quadratic, and other polynomial approximations. We use a similar approach in this activity to obtain linear and quadratic approximations to $\ln(2)$. Along the way, we encounter a type of series that is different than most of the ones we have seen so far. Throughout this activity, let $f(x) = \ln(1 + x)$.

(a) Find the tangent line to f at $x = 0$ and use this linearization to approximate $\ln(2)$. That is, find $L(x)$, the tangent line approximation to $f(x)$, and use the fact that $L(1) \approx f(1)$ to estimate $\ln(2)$.

(b) The linearization of $\ln(1 + x)$ does not provide a very good approximation to $\ln(2)$ since 1 is not that close to 0. To obtain a better approximation, we alter our approach; instead of using a straight line to approximate $\ln(2)$, we use a quadratic function to account for the concavity of $\ln(1 + x)$ for x close to 0. With the linearization, both the function's value and slope agree with the linearization's value and slope at $x = 0$. We will now make a quadratic approximation $P_2(x)$ to $f(x) = \ln(1+x)$ centered at $x = 0$ with the property that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$.

(i) Let $P_2(x) = x - \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. Use $P_2(x)$ to approximate $\ln(2)$ by using the fact that $P_2(1) \approx f(1)$.

(ii) We can continue approximating $\ln(2)$ with polynomials of larger degree whose derivatives agree with those of f at 0. This makes the polynomials fit the graph of f better for more values of x around 0. For example, let $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. Show that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. Taking a similar approach to preceding questions, use $P_3(x)$ to approximate $\ln(2)$.

(iii) If we used a degree 4 or degree 5 polynomial to approximate $\ln(1 + x)$, what approximations of $\ln(2)$ do you think would result? Use the preceding questions to conjecture a pattern that holds, and state the degree 4 and degree 5 approximation.

$\bowtie$

Activity 8.16.

Remember that, by definition, a series converges if and only if its corresponding sequence of partial sums converges.

(a) Complete the following table by calculating the first few partial sums (to 10 decimal places) of the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}.$$

$\sum_{k=1}^{1}(-1)^{k+1}\frac{1}{k} =$	$\sum_{k=1}^{6}(-1)^{k+1}\frac{1}{k} =$
$\sum_{k=1}^{2}(-1)^{k+1}\frac{1}{k} =$	$\sum_{k=1}^{7}(-1)^{k+1}\frac{1}{k} =$
$\sum_{k=1}^{3}(-1)^{k+1}\frac{1}{k} =$	$\sum_{k=1}^{8}(-1)^{k+1}\frac{1}{k} =$
$\sum_{k=1}^{4}(-1)^{k+1}\frac{1}{k} =$	$\sum_{k=1}^{9}(-1)^{k+1}\frac{1}{k} =$
$\sum_{k=1}^{5}(-1)^{k+1}\frac{1}{k} =$	$\sum_{k=1}^{10}(-1)^{k+1}\frac{1}{k} =$

Table 8.5: Partial sums of the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$

(b) Plot the sequence of partial sums from part (a) in the plane. What do you notice about this sequence?

$\triangleleft$

Activity 8.17.

Which series converge and which diverge? Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+2}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}2k}{k+5}$

(c) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

$\triangleleft$

Activity 8.18.

Determine the number of terms it takes to approximate the sum of the convergent alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$$

to within 0.0001.

$\triangleleft$

Activity 8.19.

(a) Explain why the series

$$1-\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}-\frac{1}{81}-\frac{1}{100}+\cdots$$

must have a sum that is less than the series

$$\sum_{k=1}^{\infty}\frac{1}{k^2}.$$

(b) Explain why the series

$$1-\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}-\frac{1}{81}-\frac{1}{100}+\cdots$$

must have a sum that is greater than the series

$$\sum_{k=1}^{\infty}-\frac{1}{k^2}.$$

(c) Given that the terms in the series

$$1-\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}-\frac{1}{81}-\frac{1}{100}+\cdots$$

converge to 0, what do you think the previous two results tell us about the convergence status of this series?

◁

Activity 8.20.

(a) Consider the series $\sum(-1)^k \frac{\ln(k)}{k}$.

(i) Does this series converge? Explain.

(ii) Does this series converge absolutely? Explain what test you use to determine your answer.

(b) Consider the series $\sum(-1)^k \frac{\ln(k)}{k^2}$.

(i) Does this series converge? Explain.

(ii) Does this series converge absolutely? Hint: Use the fact that $\ln(k) < \sqrt{k}$ for large values of k and the compare to an appropriate p-series.

$\triangleleft$

Activity 8.21.

For (a)-(j), use appropriate tests to determine the convergence or divergence of the following series. Throughout, if a series is a convergent geometric series, find its sum.

(a) $\displaystyle\sum_{k=3}^{\infty} \frac{2}{\sqrt{k-2}}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1+2k}$

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{2k^2+1}{k^3+k+1}$

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{100^k}{k!}$

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{5^k}$

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3-1}{k^5+1}$

(g) $\displaystyle\sum_{k=2}^{\infty} \frac{3^{k-1}}{7^k}$

(h) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^k}$

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}}$

(j) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$

(k) Determine a value of n so that the nth partial sum S_n of the alternating series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ approximates the sum to within 0.001.

$\triangleleft$

8.5 Taylor Polynomials and Taylor Series

Preview Activity 8.5. Preview Activity 8.3 showed how we can approximate the number e using linear, quadratic, and other polynomial functions; we then used similar ideas in Preview Activity 8.4 to approximate $\ln(2)$. In this activity, we review and extend the process to find the "best" quadratic approximation to the exponential function e^x around the origin. Let $f(x) = e^x$ throughout this activity.

(a) Find a formula for $P_1(x)$, the linearization of $f(x)$ at $x = 0$. (We label this linearization P_1 because it is a first degree polynomial approximation.) Recall that $P_1(x)$ is a good approximation to $f(x)$ for values of x close to 0. Plot f and P_1 near $x = 0$ to illustrate this fact.

(b) Since $f(x) = e^x$ is not linear, the linear approximation eventually is not a very good one. To obtain better approximations, we want to develop a different approximation that "bends" to make it more closely fit the graph of f near $x = 0$. To do so, we add a quadratic term to $P_1(x)$. In other words, we let

$$P_2(x) = P_1(x) + c_2x^2$$

for some real number c_2. We need to determine the value of c_2 that makes the graph of $P_2(x)$ best fit the graph of $f(x)$ near $x = 0$.

Remember that $P_1(x)$ was a good linear approximation to $f(x)$ near 0; this is because $P_1(0) = f(0)$ and $P_1'(0) = f'(0)$. It is therefore reasonable to seek a value of c_2 so that

$$\begin{aligned} P_2(0) &= f(0), \\ P_2'(0) &= f'(0), \text{ and} \\ P_2''(0) &= f''(0). \end{aligned}$$

Remember, we are letting $P_2(x) = P_1(x) + c_2x^2$.

(i) Calculate $P_2(0)$ to show that $P_2(0) = f(0)$.

(ii) Calculate $P_2'(0)$ to show that $P_2'(0) = f'(0)$.

(iii) Calculate $P_2''(x)$. Then find a formula for c_2 so that $P_2''(0) = f''(0)$.

(iv) Explain why the condition $P_2''(0) = f''(0)$ will put an appropriate "bend" in the graph of P_2 to make P_2 fit the graph of f around $x = 0$.

$\bowtie$

Activity 8.22.

We have just seen that the nth order Taylor polynomial centered at $a = 0$ for the exponential function e^x is

$$\sum_{k=0}^{n} \frac{x^k}{k!}.$$

In this activity, we determine small order Taylor polynomials for several other familiar functions, and look for general patterns that will help us find the Taylor series expansions a bit later.

(a) Let $f(x) = \frac{1}{1-x}$.

  (i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\frac{1}{1-x}$ centered at 0.

  (ii) Based on your results from part (i), determine a general formula for $f^{(k)}(0)$.

(b) Let $f(x) = \cos(x)$.

  (i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\cos(x)$ centered at 0.

  (ii) Based on your results from part (i), find a general formula for $f^{(k)}(0)$. (Think about how k being even or odd affects the value of the kth derivative.)

(c) Let $f(x) = \sin(x)$.

  (i) Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\sin(x)$ centered at 0.

  (ii) Based on your results from part (i), find a general formula for $f^{(k)}(0)$. (Think about how k being even or odd affects the value of the kth derivative.)

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Activity 8.23.

In Activity 8.22 we determined small order Taylor polynomials for a few familiar functions, and also found general patterns in the derivatives evaluated at 0. Use that information to write the Taylor series centered at 0 for the following functions.

(a) $f(x) = \frac{1}{1-x}$

(b) $f(x) = \cos(x)$ (You will need to carefully consider how to indicate that many of the coefficients are 0. Think about a general way to represent an even integer.)

(c) $f(x) = \sin(x)$ (You will need to carefully consider how to indicate that many of the coefficients are 0. Think about a general way to represent an odd integer.)

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Activity 8.24.

(a) Plot the graphs of several of Taylor polynomials centered at 0 (of order at least 5) for e^x and convince yourself that these Taylor polynomials converge to e^x for every value of x.

(b) Draw the graphs of several of the Taylor polynomials centered at 0 (of order at least 6) for $\cos(x)$ and convince yourself that these Taylor polynomials converge to $\cos(x)$ for every value of x. Write the Taylor series centered at 0 for $\cos(x)$.

(c) Draw the graphs of several of the Taylor polynomials centered at 0 for $\frac{1}{1-x}$. Based on your graphs, for what values of x do these Taylor polynomials appear converge to $\frac{1}{1-x}$? How is this situation different from what we observe with e^x and $\cos(x)$? In addition, write the Taylor series centered at 0 for $\frac{1}{1-x}$.

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Activity 8.25.

(a) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ centered at $x = 0$.

(b) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \cos(x)$ centered at $x = 0$.

(c) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \sin(x)$ centered at $x = 0$.

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Activity 8.26.

Let $P_n(x)$ be the nth order Taylor polynomial for $\sin(x)$ centered at $x = 0$. Determine how large we need to choose n so that $P_n(2)$ approximates $\sin(2)$ to 20 decimal places.

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Activity 8.27.

(a) Show that the Taylor series centered at 0 for $\cos(x)$ converges to $\cos(x)$ for every real number x.

(b) Next we consider the Taylor series for e^x.

(i) Show that the Taylor series centered at 0 for e^x converges to e^x for every nonnegative value of x.

(ii) Show that the Taylor series centered at 0 for e^x converges to e^x for every negative value of x.

(iii) Explain why the Taylor series centered at 0 for e^x converges to e^x for every real number x. Recall that we earlier showed that the Taylor series centered at 0 for e^x converges for all x, and we have now completed the argument that the Taylor series for e^x actually converges to e^x for all x.

(c) Let $P_n(x)$ be the nth order Taylor polynomial for e^x centered at 0. Find a value of n so that $P_n(5)$ approximates e^5 correct to 8 decimal places.

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8.6 Power Series

Preview Activity 8.6. In Chapter 7, we learned some of the many important applications of differential equations, and learned some approaches to solve or analyze them. Here, we consider an important approach that will allow us to solve a wider variety of differential equations.

Let's consider the familiar differential equation from exponential population growth given by

$$y' = ky, \tag{8.3}$$

where k is the constant of proportionality. While we can solve this differential equation using methods we have already learned, we take a different approach now that can be applied to a much larger set of differential equations. For the rest of this activity, let's assume that $k = 1$. We will use our knowledge of Taylor series to find a solution to the differential equation (8.3).

To do so, we assume that we have a solution $y = f(x)$ and that $f(x)$ has a Taylor series that can be written in the form

$$y = f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where the coefficients a_k are undetermined. Our task is to find the coefficients.

(a) Assume that we can differentiate a power series term by term. By taking the derivative of $f(x)$ with respect to x and substituting the result into the differential equation (8.3), show that the equation

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=0}^{\infty} a_k x^k$$

must be satisfied in order for $f(x) = \sum_{k=0}^{\infty} a_k x^k$ to be a solution of the DE.

(b) Two series are equal if and only if they have the same coefficients on like power terms. Use this fact to find a relationship between a_1 and a_0.

(c) Now write a_2 in terms of a_1. Then write a_2 in terms of a_0.

(d) Write a_3 in terms of a_2. Then write a_3 in terms of a_0.

(e) Write a_4 in terms of a_3. Then write a_4 in terms of a_0.

(f) Observe that there is a pattern in (b)-(e). Find a general formula for a_k in terms of a_0.

(g) Write the series expansion for y using only the unknown coefficient a_0. From this, determine what familiar functions satisfy the differential equation (8.3). (**Hint**: Compare to a familiar Taylor series.)

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Activity 8.28.

Determine the interval of convergence of each power series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{3k}$

(b) $\displaystyle\sum_{k=1}^{\infty} kx^k$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2(x+1)^k}{4^k}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{(2k)!}$

(e) $\displaystyle\sum_{k=1}^{\infty} k!x^k$

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Activity 8.29.

Our goal in this activity is to find a power series expansion for $f(x) = \dfrac{1}{1+x^2}$ centered at $x = 0$.

While we could use the methods of Section 8.5 and differentiate $f(x) = \dfrac{1}{1+x^2}$ several times to look for patterns and find the Taylor series for $f(x)$, we seek an alternate approach because of how complicated the derivatives of $f(x)$ quickly become.

(a) What is the Taylor series expansion for $g(x) = \frac{1}{1-x}$? What is the interval of convergence of this series?

(b) How is $g(-x^2)$ related to $f(x)$? Explain, and hence substitute $-x^2$ for x in the power series expansion for $g(x)$. Given the relationship between $g(-x^2)$ and $f(x)$, how is the resulting series related to $f(x)$?

(c) For which values of x will this power series expansion for $f(x)$ be valid? Why?

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Activity 8.30.

Let f be the function given by the power series expansion

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

(a) Assume that we can differentiate a power series term by term, just like we can differentiate a (finite) polynomial. Use the fact that

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

to find a power series expansion for $f'(x)$.

(b) Observe that $f(x)$ and $f'(x)$ have familiar Taylor series. What familiar functions are these? What known relationship does our work demonstrate?

(c) What is the series expansion for $f''(x)$? What familiar function is $f''(x)$?

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Activity 8.31.

Find a power series expansion for $\ln(1 + x)$ centered at $x = 0$ and determine its interval of convergence. (**Hint**: Use the Taylor series expansion for $\dfrac{1}{1+x}$ centered at $x = 0$.)

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