Mat-116 | lec: 21

The inverse sine, cosine and Tangent functions

The Inverse Sine function:

Sine function: $y = \sin x$ Domain: $-\infty < x < \infty$ Range: $-1 \le y \le 1$

Restricted domain: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ Range: $-1 \le y \le 1$

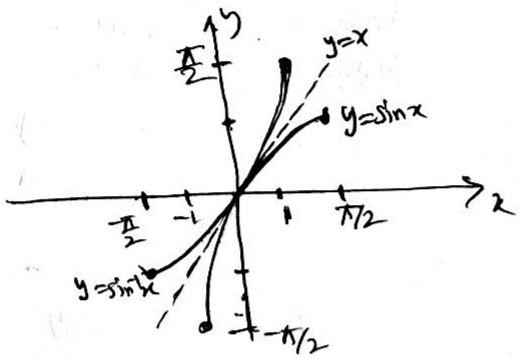
Inverse Sin function: Interchange x and y.

$x = \sin y$ Domain: $-1 \le x \le 1$, Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\therefore\ y = \sin^{-1} x$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	1

Ex: Find the exact value of $\sin^{-1} 1$.



Sol^n: $\theta = \sin^{-1} 1$

$\Rightarrow 1 = \sin\theta$

$\Rightarrow \sin\frac{\pi}{2} = \sin\theta$

$\therefore\ \theta = \frac{\pi}{2}$

Ex: Find the exact value of $\sin^{-1}\left(\frac{-1}{2}\right)$

Sol^n: $\theta = \sin^{-1}\left(\frac{-1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right)$ $[\because \sin(-\theta) = -\sin\theta]$

$\Rightarrow \sin\theta = \frac{-1}{2} = -\sin\left(\frac{\pi}{6}\right)$

$\therefore\ \theta = -\frac{\pi}{6}$.

Since $\sin x$ and $\sin^{-1}x$ are Inverse functions. so they follow the Property $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

$\therefore$ $\sin(\sin^{-1}x) = x$ and $\sin^{-1}(\sin x) = x$ $\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$-1 \le x \le 1$

Ex: $\sin^{-1}\left(\sin\frac{\pi}{8}\right) = \pi/8$

$\sin^{-1}\left(\sin\frac{5\pi}{8}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{3\pi}{8}\right)\right] = \sin^{-1}\left[\sin\left(\frac{3\pi}{8}\right)\right] = \frac{3\pi}{8}$.

Note: Here $\frac{5\pi}{8}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The inverse cosine function:

Cosine function $\quad y = \cos x \quad$ Domain: $-\infty < x < \infty$ $\quad$ Range: $-1 \le y \le 1$

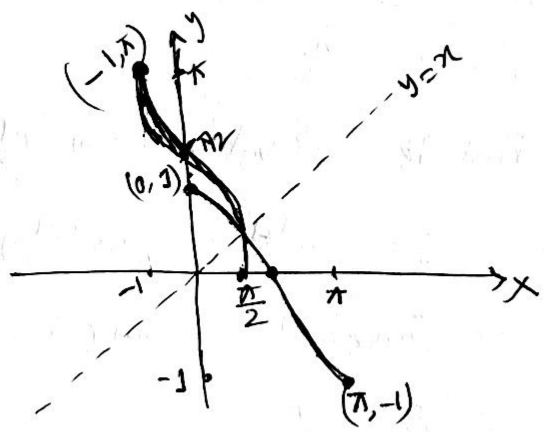
Restricted domain: $0 \le x \le \pi$

Inverse Cosine function $\quad$ interchange x and y. $\quad$ Domain: $-1 \le x \le 1$ $\quad$ Range: $0 \le y \le \pi$

$x = \cos y$

$\therefore y = \cos^{-1} x$

θ	$\cos\theta$
0	1
$\frac{\pi}{6}$	$\sqrt{3}/2$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$5\pi/6$	$-\sqrt{3}/2$
π	-1



Ex: Find the exact value of $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Soln: $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$

$\Rightarrow \cos\theta = \cos\frac{3\pi}{4}$

$\therefore \theta = \frac{3\pi}{4}$.

$f^{-1}(f(x)) = \cos^{-1}(\cos x) = x \qquad 0 \le x \le \pi$

$f(f^{-1}(x)) = \cos(\cos^{-1}x) = x \qquad -1 \le x \le 1$

Ex: # $\cos^{-1}\left(\cos\frac{\pi}{12}\right) = \frac{\pi}{12}$

$\cos^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$: Here the angle $-\frac{2\pi}{3}$ is not in the interval $[0, \pi]$. But cosine is even function, so $\cos\left(-\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3}$. and $\frac{2\pi}{3}$ lies in the interval $[0, \pi]$.

$\therefore \cos^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right] = \cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$.

$\cos(\cos^{-1}\pi)$

Because π is not in the interval $[-1, 1]$, the domain of $\cos^{-1}\pi$ is not defined, which means $\cos(\cos^{-1}\pi)$ is also not defined.

Solve $3\sin^{-1}x = \pi$

Soln: Given $3\sin^{-1}x = \pi$

$\Rightarrow \sin^{-1}x = \pi/3$

$\Rightarrow x = \sin\pi/3 = \frac{\sqrt{3}}{2}$

$\therefore$ The solution set is $\left\{\frac{\sqrt{3}}{2}\right\}$

The inverse tangent function:

tangent function $\quad y = \tan x \quad$ Domain: $-\infty < x < \infty$, x not equal to odd multiples of $\pi/2$

Range, $-\infty < y < \infty$

Restricted domain : $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Inverse tangent function $\quad x = \tan y$

$\therefore y = \tan^{-1} x \quad$ Domain : $-\infty < x < \infty$, Range : $-\frac{\pi}{2} < y < \frac{\pi}{2}$

θ	$\tan\theta$
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{3}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{1}{\sqrt{3}}$
0	0
$\frac{\pi}{6}$	$1/\sqrt{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	undefined

Ex: Find exact value of $\tan^{-1}(-\sqrt{3})$

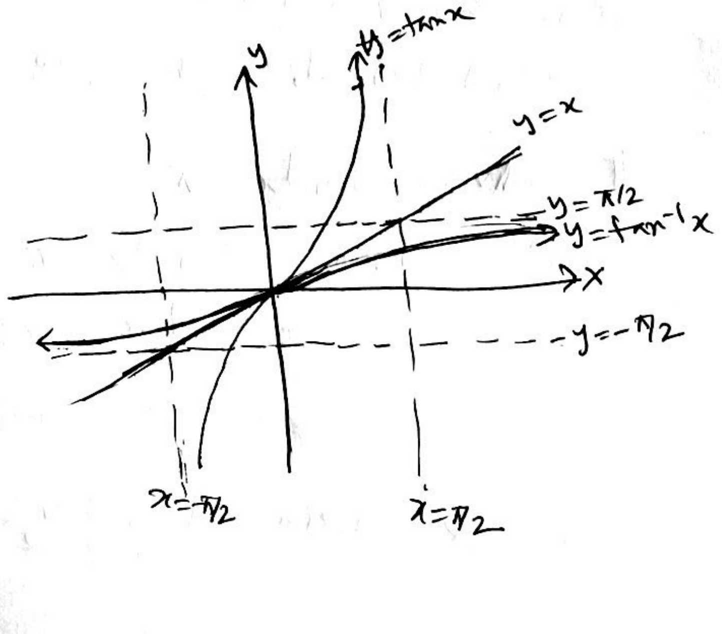
Soln: $\theta = \tan^{-1}(-\sqrt{3})$

$\Rightarrow \tan\theta = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$

$\therefore \theta = -\frac{\pi}{3}$

$\tan^{-1}(\tan x) = x \quad ; \quad -\frac{\pi}{2} < x < \pi/2$

$\tan(\tan^{-1} x) = x \quad ; \quad -\infty < x < \infty$



Ex1 Find the inverse function f^{-1} of $f(x) = 2\sin x - 1$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Find the range of f and the domain and range of f^{-1}.

Soln: Given $y = 2\sin x - 1$

To find f^{-1} let's interchange x and y and solve for y.

$$x = 2\sin y - 1$$

$$\Rightarrow 2\sin y = x+1$$

$$\Rightarrow \sin y = \frac{x+1}{2} \quad \therefore y = \sin^{-1}\left(\frac{x+1}{2}\right)$$

Thus the inverse function is $f^{-1}(x) = \sin^{-1}\frac{x+1}{2}$

To find the range of f, solve $y = 2\sin x - 1$ for $\sin x$ and use the fact that $-1 \le \sin x \le 1$.

$$\therefore y = 2\sin x - 1$$

$$\Rightarrow \sin x = \frac{y+1}{2}$$

$$\therefore -1 \le \frac{y+1}{2} \le 1$$

$$\Rightarrow -2 \le y+1 \le 2$$

$$\Rightarrow -3 \le y \le 1$$

$\therefore$ The range of f is $\{y \mid -3 \le y \le 1\}$ or $[-3, 1]$.

Domain of f^{-1} is the range of $f = [3, 1]$

Range of f^{-1} is the domain of $f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

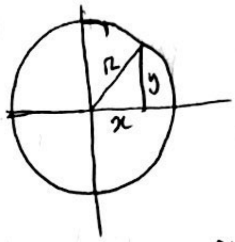
Ex: Find the exact value of $\sin\left(\tan^{-1}\frac{1}{2}\right)$

Soln: Given $\sin\left(\tan^{-1}\frac{1}{2}\right)$

Let $\tan^{-1}\frac{1}{2} = \theta$

$\Rightarrow \tan\theta = \frac{1}{2} = \frac{y}{x}$

Here $x = 2,\ y = 1$ and $r = \sqrt{5}$



$x^2+y^2=r^2$

$\Rightarrow r = \sqrt{x^2+y^2}$

$= \sqrt{4+1} = \sqrt{5}$

We know $\sin\theta = \frac{y}{r}$

$\therefore \sin\left(\tan^{-1}\frac{1}{2}\right) = \sin\theta = \frac{y}{r} = \frac{1}{\sqrt{5}}$

$\therefore \sin\left(\tan^{-1}\frac{1}{2}\right) = \frac{1}{\sqrt{5}}$

Ex: Find the exact value of $\cos\left[\sin^{-1}\left(-\frac{1}{3}\right)\right]$

Soln: Given $\cos\left[\sin^{-1}\left(-\frac{1}{3}\right)\right]$

Let $\theta = \sin^{-1}\left(-\frac{1}{3}\right)$

$\Rightarrow \sin\theta = -\frac{1}{3} = \frac{y}{r}$

Here $y = -1$ and $r = 3$

Now $x^2+y^2=r^2$

$\Rightarrow x^2 = 9-1 = 8$

$\therefore x = \sqrt{8} = 2\sqrt{2},\ x>0$

$\therefore \cos\left[\sin^{-1}\left(-\frac{1}{3}\right)\right] = \cos\theta = \frac{x}{r} = \frac{2\sqrt{2}}{3}$

$\therefore \cos\left[\sin^{-1}\left(-\frac{1}{3}\right)\right] = \frac{2\sqrt{2}}{3}$

Ex: $\tan\left[\cos^{-1}\left(\frac{1}{-3}\right)\right]$