Chapter # 03
(Topics in Differentiation)

3.3 Derivatives of Exponential and Inverse Trigonometric Functions: In this section we will show how the derivative of a one-to-one function can be used to obtain the derivative of its inverse function. This will provide the tools we need to obtain derivative formulas for exponential functions from the derivative formulas for logarithmic functions and to obtain derivative formulas for inverse trigonometric functions from the derivative formulas for trigonometric functions.

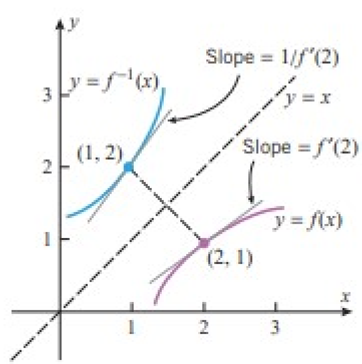
Our first goal in this section is to obtain a formula relating the derivative of the inverse function $\boldsymbol{f^{-1}}$ to the derivative of the function $\boldsymbol{f}$.

Example 1: Suppose that $\boldsymbol{f}$ is a one-to-one differentiable function such that $f(2) = 1$ and $f'(2) = \frac{3}{4}$. Then the tangent line to $\boldsymbol{y = f(x)}$ at the point (2, 1) has equation

$$y - 1 = \tfrac{3}{4}(x - 2)$$

The tangent line to $\boldsymbol{y = f^{-1}(x)}$ at the point (1, 2) is the reflection about the line $\boldsymbol{y = x}$ of the tangent line to $\boldsymbol{y = f(x)}$ at the point (2, 1), and its equation can be obtained by interchanging $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$x - 1 = \tfrac{3}{4}(y - 2) \quad \text{or} \quad y - 2 = \tfrac{4}{3}(x - 1)$$



Notice that the slope of the tangent line to $\boldsymbol{y = f^{-1}(x)}$ $\boldsymbol{at}$ $\boldsymbol{x = 1}$ is the reciprocal of the slope of the tangent line to $\boldsymbol{y = f(x)}$ $\boldsymbol{at}$ $\boldsymbol{x = 2}$. That is

$$(f^{-1})'(1) = \frac{1}{f'(2)} = \frac{4}{3}$$

Since $\boldsymbol{2 = f^{-1}(1)}$ for the function $\boldsymbol{f}$ in Example 1, it follows that $f'(2) = f'(f^{-1}(1))$. Thus, above formula can also be expressed as

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

In general, if $\boldsymbol{f}$ is a differentiable and one-to-one function, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided $f'(f^{-1}(x)) \neq 0$.

The equation $\boldsymbol{y = f^{-1}(x)}$ is equivalent to $\boldsymbol{x = f(y)}$. Differentiating with respect to $\boldsymbol{x}$ we obtain

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$$

so that

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Alternatively,

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Increasing or Decreasing Functions are One-to-One:

Theorem: Suppose that the domain of a function $\boldsymbol{f}$ is an open interval on which $\boldsymbol{f'(x)} > 0$ or on which $\boldsymbol{f'(x)} < 0$. Then $\boldsymbol{f}$ is one-to-one, $\boldsymbol{f^{-1}(x)}$ is differentiable at all values of $\boldsymbol{x}$ in the range of $\boldsymbol{f}$.

Example 2: Consider the function $\boldsymbol{f(x) = x^5 + x + 1}$.

(***a***) Show that $\boldsymbol{f}$ is one-to-one on the interval $(-\infty, \infty)$.

(***b***) Find a formula for the derivative of f^{-1}.

(***c***) Compute $(f^{-1})'(1)$.

Solution: (***a***) Given, $f(x) = x^5 + x + 1$

$$\therefore \quad f'(x) = 5x^4 + 1$$

Since $\boldsymbol{f'(x) = 5x^4 + 1} > 0$ for all real values of $\boldsymbol{x}$, it follows that $\boldsymbol{f}$ is one-to-one on the interval $(-\infty, \infty)$.

(***b***) Let $\boldsymbol{y = f^{-}(x)}$. Differentiating $\boldsymbol{x = f(y) = y^5 + y + 1}$ implicitly with respect to $\boldsymbol{x}$ yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1}$$

We cannot solve $\boldsymbol{x = y^5 + y + 1}$ for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$, so we leave the expression for $\frac{dy}{dx}$ in the above equation in terms of y.

(*c*) From **(*b*)**,

$$(f^{-1})'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{5y^4 + 1}\right|_{x=1}$$

Thus, we need to know the value of $\boldsymbol{y = f^{-1}(x)\ at\ x = 1}$, which we can obtain by solving the equation

$$f(y) = x$$

$$\Rightarrow f(y) = 1$$

$$\Rightarrow\ y^5 + y + 1 = 1 \quad \Rightarrow y^5 + y = 0 \quad \Rightarrow y(y^4 + 1) = 0 \quad \therefore y = 0$$

Thus,

$$(f^{-1})'(1) = \left.\frac{1}{5y^4 + 1}\right|_{y=0} = 1$$

Derivative of Exponential Functions: Our next objective is to show that the general exponential function $\boldsymbol{b^x}$ $(\boldsymbol{b} > 0, b \neq 1)$ is differentiable everywhere and to find its derivative.

To obtain a derivative formula for $\boldsymbol{b^x}$ we rewrite $\boldsymbol{y = b^x}$ as

$$x = \log_b y$$

and differentiate implicitly to obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ and replacing $\boldsymbol{y}$ by $\boldsymbol{b^x}$ we have

$$\frac{dy}{dx} = y \ln b = b^x \ln b$$

In the special case where $\boldsymbol{b = e}$ we have $\boldsymbol{\ln e = 1}$, so that above equation becomes

$$\frac{d}{dx}[e^x] = e^x$$

Moreover, if ***u*** is a differentiable function of ***x***, then

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

Example 3:

$$\frac{d}{dx}[2^x] = 2^x \ln 2$$
$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$
$$\frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$$
$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x) e^{\cos x}$$

Example 4: Use logarithmic differentiation to find

$$\frac{d}{dx}[(x^2+1)^{\sin x}].$$

Solution:

Setting $y = (x^2+1)^{\sin x}$ we have

$$\ln y = \ln[(x^2+1)^{\sin x}] = (\sin x)\ln(x^2+1)$$

Differentiating both sides with respect to ***x*** yields

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}[(\sin x)\ln(x^2+1)] \\ &= (\sin x)\frac{1}{x^2+1}(2x) + (\cos x)\ln(x^2+1)\end{aligned}$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right] \\ &= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right]\end{aligned}$$

Derivative of the Inverse Trigonometric Functions:

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Example 5: Find $\frac{dy}{dx}$ if

(a) $y = \sin^{-1}(x^3)$ (b) $y = \sec^{-1}(e^x)$

Solution: (***a***)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2) = \frac{3x^2}{\sqrt{1-x^6}}$$

(***b***)

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}}$$

Home Work: Exercise 3.3: Problem No. 5, 6, 15-26, 31 – 35, 37-51, 65-67