Chapter 4.5

The Real Zeros of a Polynomial Function

In Section 4.1, we were able to identify the real zeros of a polynomial function because either the polynomial function was in factored form or it could be easily factored. But how do we find the real zeros of a polynomial function if it is not factored or cannot be easily factored?

4.5.1 Uses of the Remainder and Factor Theorems

When we divide one polynomial (the dividend) by another (the divisor), we obtain a quotient polynomial and a remainder, the remainder being either the zero polynomial or a polynomial whose degree is less than the degree of the divisor. To check our work, we verify that

(Quotient) (Divisor) + Remainder = Dividend

This checking routine is the basis for a famous theorem called the **division algorithm for polynomials**.

THEOREM Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad f(x) = q(x)\,g(x) + r(x) \qquad (1)$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

REMAINDER THEOREM

Let f be a polynomial function. If $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

EXAMPLE 1 Uses of the Remainder Theorem

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x-3$ (b) $x+2$

An important and useful consequence of the Remainder Theorem is the **Factor Theorem**.

FACTOR THEOREM

Let f be a polynomial function. Then $x-c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

This Theorem consists of two separate statements:

1. If $f(c) = 0$, then $x-c$ is a factor of $f(x)$.
2. If $x-c$ is a factor of $f(x)$, then $f(c) = 0$.

EXAMPLE 2 Uses of the Factor Theorem

Use the Factor Theorem to determine whether the function $f(x) = 2x^3 - x^2 + 2x - 3$ has the factor

(a) $x-1$ (b) $x+3$

THEOREM Number of Real Zeros

A polynomial function cannot have more real zeros than its degree.

4.5.2 Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function

The next result, called the **Rational Zeros Theorem,** provides information about the rational zeros of a polynomial *with integer coefficients*.

THEOREM Rational Zeros Theorem

Let f be a polynomial function of degree 1or higher of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0, \quad a_0 \neq 0$$

where each coefficient is an integer. If $\dfrac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

EXAMPLE 3 Listing Potential Rational Zeros

List the potential rational zeros of $f(x) = 2x^3 + 11x^2 - 7x - 16.$

4.5.3 Find the Real Zeros of a Polynomial Function

EXAMPLE 4 How to Find the Real Zeros of a Polynomial Function

Find the real zeros of the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6.$ Write f is factored form.

Step-by-Step Solution

Step 1: Use the degree of the polynomial to determine the maximum number of zeros.

Step 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros. Use the Factor Theorem to determine if each potential rational zero is a zero. If it is, use synthetic division or long division to factor the polynomial function. Repeat Step 2 until all the zeros of the polynomial function have been identified and the polynomial function is completely factored.

SUMMARY Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of real zeros.

STEP 2:

(a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 2 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

EXAMPLE 5 Finding the Real Zeros of a Polynomial Function

Find the real zeros of $f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16.$ Write f is factored form.

4.5.4 Solve Polynomial Equations

EXAMPLE 6 Solving a Polynomial Equation

Find the real solutions of the equation $f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16 = 0.$

THEOREM

Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible (prime) quadratic factors.

THEOREM

A polynomial function of odd degree that has real coefficients has at least one real zero.

4.5 Assess your understanding

Skill Building

In Problems 11-20, use the Remainder Theorem to find the remainder when $f(x)$ is divided by $x - c$. Then use the Factor Theorem to determine whether $x - c$ is a factor of $f(x)$.

11. $f(x) = 4x^3 - 3x^2 - 8x + 4;\ x - 2$

13. $f(x) = 3x^4 - 6x^3 - 5x + 10;\ x - 2$

15. $f(x) = 3x^6 + 82x^3 + 27;\ x + 3$

17. $f(x) = 4x^6 - 64x^4 + x^2 - 15;\ x + 4$

19. $f(x) = 2x^4 - x^3 + 2x - 1;\ x - \frac{1}{2}$

In Problems 21–32, tell the maximum number of real zeros that each polynomial function may have. Do not attempt to find the zeros.

21. $f(x) = -4x^7 + x^3 - x^2 + 2$

23. $f(x) = 2x^6 - 3x^2 - x + 1$

25. $f(x) = 3x^3 - 2x^2 + x + 2$

27. $f(x) = -x^4 + x^2 - 1$

29. $f(x) = x^5 + x^4 + x^2 + x + 1$

31. $f(x) = x^6 - 1$

In Problems 33–44, list the potential rational zeros of each polynomial function. Do not attempt to find the zeros.

33. $f(x)=3x^4-3x^3+x^2-x+1$

35. $f(x)=x^5-6x^2+9x-3$

37. $f(x)=-4x^3-x^2+x+2$

39. $f(x)=6x^4-x^2+9$

41. $f(x)=2x^5-x^3+2x^2+12$

43. $f(x)=6x^4+2x^3-x^2+20$

In Problems 45–56, use the Rational Zeros Theorem to find all the real zeros of each polynomial function. Use the zeros to factor f over the real numbers.

45. $f(x)=x^3+2x^2-5x-6$

Since f has integer coefficients, we may use the Rational Zeros Theorem.
Denote p by the integer factors of the constant term -6 and denote q by the integer factors of the leading coefficient 1.
Since the leading coefficient is 1, we can list the potential rational zeros:

$$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 6$$

Since $f(1)=-8,$ and $f(-1)=0,$ we use the Synthetic division technique:

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

Therefore, the depressed equation is $x^2+x-6=0.$ Now using grouping technique, we get

$$x^2+x-6=0 \Rightarrow x^2+3x-2x-6=0 \Rightarrow x(x+3)-2(x+3)=0$$
$$\Rightarrow (x+3)(x-2)=0 \Rightarrow x=-3 \text{ or } x=2$$

The zeros of f are $-3, -1, 2$

Hence $f(x)=x^3+2x^2-5x-6=(x+3)(x+1)(x-2).$

47. $f(x)=2x^3-x^2+2x-1$

Since f has integer coefficients, we may use the Rational Zeros Theorem.
Denote p by the integer factors of the constant term -1 and denote q by the integer factors of the leading coefficient 2:

$$p: \pm 1 \qquad\qquad q: \pm 1, \pm 2$$

List the potential rational zeros:

$$\frac{p}{q}: \pm 1, \pm \frac{1}{2}$$

Since $f(1) = 2,\ \ f(-1) = -6,\ \text{and}\ \ f\left(\frac{1}{2}\right) = 0,$ we use Synthetic division technique:

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & 2 & -1 \\ & & 1 & 0 & 1 \\ \hline & 2 & 0 & 2 & 0 \end{array}$$

Therefore, the depressed equation is $2x^2 + 2 = 0 \Rightarrow 2(x^2 + 1) = 0 \Rightarrow x^2 + 1 = 0$ which has no real solution.

The only real zero of f is $\frac{1}{2}$.

Hence $f(x) = 2x^3 - x^2 + 2x - 1 = 2\left(x - \frac{1}{2}\right)(x^2 + 1).$

48. $f(x) = 2x^3 + x^2 + 2x + 1$

49. $f(x) = 2x^3 - 4x^2 - 10x + 20$

Since f has integer coefficients, we may use the Rational Zeros Theorem.

Denote p by the integer factors of the constant term 20 and denote q by the integer factors of the leading coefficient 2:

$$p: \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20 \qquad\qquad q: \pm 1, \pm 2$$

List the potential rational zeros:

$$\frac{p}{q}:\ \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm \frac{1}{2}, \pm \frac{5}{2}$$

Since $f(1) = 8,\ \ f(-1) = 28,\ \text{and}\ \ f(2) = 0,$ we use Synthetic division technique:

$$\begin{array}{r|rrrr} 2 & 2 & -4 & -10 & 20 \\ & & 4 & 0 & -20 \\ \hline & 2 & 0 & -10 & 0 \end{array}$$

Therefore, the depressed equation is

$$2x^2 - 10 = 0 \Rightarrow 2(x^2 - 5) = 0$$

$$\Rightarrow x^2 - 5 = 0 \Rightarrow (x + \sqrt{5})(x - \sqrt{5}) = 0$$

The real zeros of f are $2, -\sqrt{5}$ and $\sqrt{5}.$

Hence $f(x) = 2x^3 - 4x^2 - 10x + 20 = (x - 2)(x + \sqrt{5})(x - \sqrt{5}).$

51. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

Since f has integer coefficients, we may use the Rational Zeros Theorem.

Denote p by the integer factors of the constant term 3 and denote q by the integer factors of the leading coefficient 2:

$$p: \pm 1, \pm 3 \qquad\qquad q: \pm 1, \pm 2$$

List the potential rational zeros:

$$\frac{p}{q}: \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

Since $f(1) = -4$, and $f(-1) = 0$, we use Synthetic division technique:

$$\begin{array}{r|rrrrr} -1 & 2 & 1 & -7 & -3 & 3 \\ & & -2 & 1 & 6 & -3 \\ \hline & 2 & -1 & -6 & 3 & 0 \end{array}$$

Therefore, the depressed equation is

$$2x^3 - x^2 - 6x + 3 = 0$$

Now using the grouping technique, we get

$$2x^3 - x^2 - 6x + 3 = 0 \Rightarrow x^2(2x-1) - 3(2x-1) = 0$$

$$\Rightarrow (2x-1)(x^2-3) = 0 \Rightarrow 2x-1 = 0 \text{ or } x^2 - 3 = 0$$

$$\Rightarrow x = \frac{1}{2} \text{ or } x = -\sqrt{3} \text{ or } x = \sqrt{3}$$

The real zeros of f are $-1, \frac{1}{2}, -\sqrt{3}, \sqrt{3}$.

Hence $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3 = 2(x+1)\left(x - \frac{1}{2}\right)(x+\sqrt{3})(x-\sqrt{3})$.

53. $f(x) = x^4 + x^3 - 3x^2 - x + 2$

Since f has integer coefficients, we may use the Rational Zeros Theorem.

Denote p by the integer factors of the constant term 2 and denote q by the integer factors of the leading coefficient 1:

$$p: \pm 1, \pm 2 \qquad\qquad q: \pm 1$$

List the potential rational zeros:

$$\frac{p}{q}: \pm 1, \pm 2$$

Since $f(1) = 0$, we use Synthetic division technique:

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & -3 & -1 & 2 \\ & & 1 & 2 & -1 & -2 \\ \hline & 1 & 2 & -1 & -2 & 0 \end{array}$$

So that $f(x) = (x-1)(x^3 + 2x^2 - x - 2) = (x-1)\,q_1(x)$ and the depressed equation is $q_1(x) = 0$.

Again, since $q_1(1) = 0$, we use Synthetic division technique:

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

So that $f(x)=(x-1)(x-1)(x^2+3x+2)=(x-1)^2\,q_2(x)$ and the depressed equation is $q_2(x)=0$.

Now using the grouping technique, we get

$$x^2+3x+2=0 \Rightarrow (x+2)(x+1)=0 \;\Rightarrow\; x=-2 \text{ or } x=-1$$

The real zeros of f are 1 with multiplicity 2; -2, -1.

Hence $f(x)=x^4+x^3-3x^2-x+2=(x-1)^2(x+2)(x+1)$.

55. $f(x)=4x^4+5x^3+9x^2+10x+2$

Since *f* has integer coefficients, we may use the Rational Zeros Theorem.

Denote p by the integer factors of the constant term 2 and denote q by the integer factors of the leading coefficient 4:

$$p:\pm1,\pm2 \qquad\qquad q:\pm1,\pm2,\pm4$$

List the potential rational zeros:

$$\frac{p}{q}:\ \pm1,\ \pm2,\ \pm\frac{1}{2},\ \pm\frac{1}{4}$$

Since $f(1)=30$, $f(-1)=0$, we use Synthetic division technique:

$$\begin{array}{r|rrrrr} -1 & 4 & 5 & 9 & 10 & 2 \\ & & -4 & -1 & -8 & -2 \\ \hline & 4 & 1 & 8 & 2 & 0 \end{array}$$

So that $f(x)=(x-1)(4x^3+x^2+8x+2)=(x+1)\,q_1(x)$ and the depressed equation is $q_1(x)=0$.

Now using the grouping technique, we get

$$4x^3+x^2+8x+2=0 \Rightarrow x^2(4x+1)+2(4x+1)=0$$

$$\Rightarrow (4x+1)(x^2+2)=0 \Rightarrow 4x+1=0 \text{ or } x^2+2=0 \text{ which has no real solution}$$

$$\Rightarrow\; x=-\frac{1}{4}$$

The real zeros of f are $-1,-\frac{1}{4}$.

Hence $f(x)=4x^4+5x^3+9x^2+10x+2=4(x+1)\left(x-\frac{1}{4}\right)(x^2+2)$.

Chapter 4.6

Complex Zeros; Fundamental Theorem of Algebra

In this section, we will find the *complex zeros* of polynomial functions of degree 3 or higher.

Definition

A variable in the complex number system is referred to as a **complex variable.** A **complex polynomial function *f*** of degree n is a function of the form

In most of our work the coefficients in (1) will be real numbers. We have learned that some quadratic equations have no real solutions, but that in the complex number system every quadratic equation has a solution, either real or complex. The next result, proved by Karl Friedrich Gauss (1777–1855) when he was 22 years old, gives an extension to complex polynomial equations. In fact, this result is so important and useful that it has become known as the **Fundamental Theorem of Algebra.**

FUNDAMENTAL THEOREM OF ALGEBRA

However, using the Fundamental Theorem of Algebra and the Factor Theorem, we can prove the following result:

THEOREM

4.6.1 Use the Conjugate Pairs Theorem

We can use the Fundamental Theorem of Algebra to obtain valuable information about the complex zeros of polynomial functions whose coefficients are real numbers.

CONJUGATE PAIRS THEOREM

In other words, for polynomial functions whose coefficients are real numbers, the complex zeros occur in conjugate pairs.

COROLLARY

A polynomial function f of odd degree with real coefficients has at least one real zero.

EXAMPLE 1 Using the Conjugate Pairs Theorem

4.6.2 Find a Polynomial Function with Specified Zeros

EXAMPLE 2 Finding a Polynomial Function Whose Zeros Are Given

THEOREM

Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible quadratic factors.

4.6.3 Find the Complex Zeros of a Polynomial Function

The steps for finding the complex zeros of a polynomial function are the same as those for finding the real zeros.

EXAMPLE 3 Finding the Complex Zeros of a Polynomial Function

4.6 Assess Your Understanding

Skill Building

In Problems 17–22, form a polynomial function f(x) with real coefficients having the given degree and zeros. Answers will vary depending on the choice of the leading coefficient.

17. Degree 4; Zeros: $3+2i; 4$, multiplicity 2

By the Conjugate Pairs Theorem, $3-2i$ is a complex zero of f.
Therefore, $f(x)=a[x-(3+2i)][x-(3-2i)](x-4)^2=a(x^2-6x+13)(x^2-8x+16)$
$=a(x^4-14x^3+77x^2-200x+208)=x^4-14x^3+77x^2-200x+208$ when $a=1$.

19. Degree 5; Zeros: 2; $-i$; $1+i$.

By the Conjugate Pairs Theorem, i and $1-i$ are the other complex zeros of f.
Therefore, $f(x)=a(x-2)(x+i)(x-i)[x-(1+i)][x-(1-i)]$
$=a(x-2)(x^2+1)(x^2-2x+2)=x^5-4x^4+7x^3-8x^2+6x-4$ when $a=1$.

21. Degree 4; Zeros: 3, multiplicity 2; $-i$
$f(x)=a(x-3)^2(x+i)(x-i)$

In Problems 31–40, find the complex zeros of each polynomial function. Write f in factored form.

31. $f(x)=x^3-1$
Using formula, we get $f(x)=(x-1)(x^2+x+1)=(x-1)q_1(x)$.
Therefore, the depressed equation is $x^2+x+1=0$. By using the general quadratic formula, we get

$$x^2+x+1=0 \Rightarrow x=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

So that $x=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ or $x=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

The complex zeros of f are 1, $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

Hence, $f(x)=x^3-1=(x-1)\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$.

33. $f(x)=x^3-8x^2+25x-26$
Step I: Since degree of the complex polynomial function is 3, f will have 3 complex zeros.
Step II: By Rational Zeros Theorem, the potential rational zeros of f are

$$\pm1, \pm2, \pm13, \pm26$$

Since $f(1)=-8$, $f(-1)=-60$, $f(2)=0$, we use Synthetic division technique:

$$\begin{array}{r|rrrr} 2 & 1 & -8 & 25 & -26 \\ & & 2 & -12 & 26 \\ \hline & 1 & -6 & 13 & 0 \end{array}$$

So that $f(x)=(x-2)(x^2-6x+13)=(x-2)\,q_1(x)$ and the depressed equation is $q_1(x)=0.$

By using the general quadratic formula, we get

$$x^2-6x+13=0 \Rightarrow x=\frac{6\pm\sqrt{36-52}}{2}=3\pm 2i$$

So that $x=3+2i$ or $x=3-2i.$
The complex zeros of f are $2,\ 3+2i,\ 3-2i.$

Hence $f(x)=x^3-8x^2+25x-26=(x-2)(x-3-2i)(x-3+2i).$

35. $f(x)=x^4+5x^2+4$

Step I: Since degree of the complex polynomial function is 4, f will have 4 complex zeros.
Step II: Compute $f(i)=0.$ Since i is zero of f, by the Conjugate Pairs Theorem, $-i$ must also be a complex zero of $f.$
Therefore, we can write $f(x)=(x-i)(x+i)\,q_1(x),$ where the depressed equation is $q_1(x)=0.$ To find $q_1(x),$ we use long division to get

$$f(x)=(x^2+1)(x^2+4)$$

So, the complex zeros of f are $i,\ -i,\ 2i,\ -2i.$

Hence $f(x)=x^4+5x^2+4=(x-i)(x+i)(x-2i)(x+2i).$