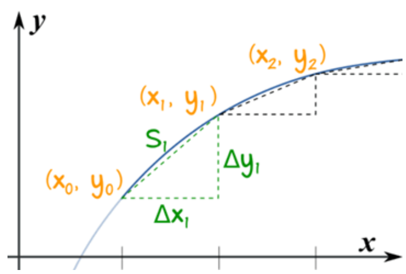
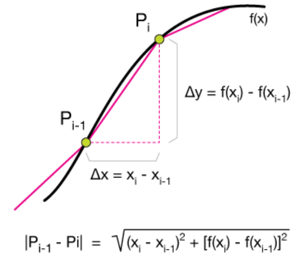
DAY-14

Quiz-2 on 12th April, 2021

Study: 6.1, 6.2, 6.3

6.4: Length of a Plane Curve





The length of a **curve** $\boldsymbol{y = f(x)}$ from $x = a$ to $x = b$, that is, the length of a curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$:

$L = \lim_{n\to\infty} \sum_1^n L_k$

$$= \lim_{n\to\infty} \sum_1^n \sqrt{\Delta x_k^2 + \Delta y_k^2} \quad ; \quad \Delta x_k, \qquad \Delta y_k \to 0$$

$$= \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_a^b \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2}$$

$$L = \int_a^b \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\ dx$$

Definitions:

Definition 1: If $\boldsymbol{y = f(x)}$ is a **smooth curve** on the interval $[a, b]$, then the length of the curve over the interval $[a, b]$ is defined by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx \quad = \quad \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\ dx$$

Steps: Given a smooth function $\boldsymbol{y = f(x)}$; $\quad \boldsymbol{x = a}$ **to** $\boldsymbol{x = b}$

Step 1: Find $\boldsymbol{\frac{dy}{dx} = f'(x)}$

Step 2: Find $\boldsymbol{\left(\frac{dy}{dx}\right)^2 = [f'(x)]^2}$

Step 3: Find $\boldsymbol{1 + \left(\frac{dy}{dx}\right)^2 = 1 + [f'(x)]^2}$

Step 4: Find $\boldsymbol{\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + [f'(x)]^2}}$.

$\boldsymbol{HINT}$: Here we should try, if possible, to write $1 + [f'(x)]^2$ as a perfect square so that we can cancel the square root.

Step 5: Evaluate the integral to find the length L:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx \quad = \quad \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\ dx$$

Exercise :1 Find the length of the curve $\boldsymbol{24xy = x^4 + 48}$ **from** $\boldsymbol{x = 1}$ **to** $\boldsymbol{x = 4}$.

Solution: Given curve $24xy = x^4 + 48$ from $x = 1$ to $x = 4$. That is, $y = \frac{x^4+48}{24x}$ for $1 \le x \le 4$.

We know that the length of thew curve is given by

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\ dx \ \ldots\ldots\ldots (1)$$

Here $y = \frac{x^4+48}{24x} = \frac{x^4}{24x} + \frac{48}{24x}$

$$y = \frac{1}{24}x^3 + 2\,x^{-1}$$

Differentiating:

$$\frac{dy}{dx} = \frac{1}{24}(3x^2) + 2\,(-x^{-2}) = \frac{1}{8}x^2 - 2x^{-2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{8}x^2 - 2x^{-2}\right)^2 \quad \text{; compare with } (a-b)^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{8}x^2\right)^2 - 2 \cdot \frac{1}{8}x^2 \cdot 2x^{-2} + (2x^{-2})^2 \;;$$

$[\text{don't simplify } a^2 \text{ and } b^2 \text{ in the expressaion } a^2 - 2ab + b^2]$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{8}x^2\right)^2 - \frac{1}{2} + (2x^{-2})^2$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{8}x^2\right)^2 - \frac{1}{2} + (2x^{-2})^2$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{8}x^2\right)^2 + \frac{1}{2} + (2x^{-2})^2$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{8}x^2\right)^2 + 2 \cdot \frac{1}{8}x^2 \cdot 2x^{-2} + (2x^{-2})^2$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{8}x^2 + 2x^{-2}\right)^2$$

$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{1}{8}x^2 + 2x^{-2}\right)^2}$ for $1 \le x \le 4$, Note: $\sqrt{m^2} = |m|$ for any real number m.

$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left|\frac{1}{8}x^2 + 2x^{-2}\right|$ for $1 \le x \le 4$.

$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{8}x^2 + 2x^{-2}$, for $1 \le x \le 4$.

Now, from equation (1):

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\; dx$$

$$= \int_1^4 \left(\frac{1}{8}x^2 + 2x^{-2}\right) dx \;=\; \left[\frac{1}{8}\frac{x^3}{3} + 2\frac{x^{-1}}{-1}\right]_1^4$$

$$= \left[\frac{1}{24}x^3 - 2\frac{1}{x}\right]_1^4$$

$$= \frac{1}{24}(4^3 - 1^3) - 2\left(\frac{1}{4} - \frac{1}{1}\right)$$

$$= \frac{1}{24}(64 - 1) - 2\left(-\frac{3}{4}\right)$$

$$= \frac{63}{24} + \frac{3}{2}$$

$$= \frac{21}{8} + \frac{12}{8}$$

$$= \frac{33}{8} \ \ unit$$

Definition: $|x| = \begin{cases} x & ; \quad x \geq 0 \\ -x & ; \quad x < 0 \end{cases}$

Exercise :2 (a) Find the length of the curve $\boldsymbol{y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}}$ **from** $\boldsymbol{x = -2}$ **to** $\boldsymbol{x = -1}$.

Hint: $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left|\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right| = -\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)$ for $-2 \leq x \leq -1$. Complete!

Exercise :2 (b) Find the length of the curve $\boldsymbol{y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}}$ **from** $\boldsymbol{x = -2}$ **to** $\boldsymbol{x = 1}$.

Hint: $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left|\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right| = \begin{cases} -\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right) & ; \quad -2 \leq x \leq 0 \\ \frac{1}{2}x^3 + \frac{1}{2}x^{-3} & ; \quad 0 \leq x \leq 1 \end{cases}$

Exercise :3 Find the length of the curve $\boldsymbol{y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}}$ **from** $\boldsymbol{x = 1}$ **to** $\boldsymbol{x = 4}$.

Solution:

We know that the length of the curve is given by

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} \ dx \quad \dots\dots\dots \ (1)$$

Given function $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ over the interval $[1, 4]$.

Then, $\frac{dy}{dx} = \frac{1}{8}(4x^3) + \frac{1}{4}(-2x^{-3}) = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2 = \left(\frac{1}{2}x^3\right)^2 - \mathbf{2 \cdot \frac{1}{2}x^3 \cdot \frac{1}{2}x^{-3}} + \left(\frac{1}{2}x^{-3}\right)^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^3\right)^2 - \frac{1}{2} + \left(\frac{1}{2}x^{-3}\right)^2$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{2}x^3\right)^2 - \frac{1}{2} + \left(\frac{1}{2}x^{-3}\right)^2$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^3\right)^2 + \frac{1}{2} + \left(\frac{1}{2}x^{-3}\right)^2$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^3\right)^2 + \mathbf{2 \cdot \frac{1}{2}x^3 \cdot \frac{1}{2}x^{-3}} + \left(\frac{1}{2}x^{-3}\right)^2$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2$$

$$\Rightarrow \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2}$$

$$\Rightarrow \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left|\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right| \quad \text{for} \quad 1 \le x \le 4.$$

$$\therefore \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}x^3 + \frac{1}{2}x^{-3} \quad \text{for} \quad 1 \le x \le 4.$$

Now, from formula (1):

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\ dx$$

$$= \int_1^4 \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right) dx$$

$$= \left[\frac{1}{8}x^4 - \frac{1}{4}x^{-2}\right]_1^4$$

$$= \left[\frac{1}{8}x^4 - \frac{1}{4}\frac{1}{x^2}\right]_1^4 = \frac{1}{8}(4^4 - 1^4) - \frac{1}{4}\left(\frac{1}{4^2} - \frac{1}{1^2}\right)$$

$$= \frac{1}{8}(256 - 1) - \frac{1}{4}\left(\frac{1}{16} - 1\right)$$

$$= \frac{255}{8} + \frac{15}{64}$$

$$= \frac{2055}{64}\ unit$$

Exercise :4 Find the length of the curve $y = \frac{x^6+8}{16x^2}$ **from** $x = 2$ **to** $x = 3$.

Solution: Given

$$y = \frac{x^6+8}{16x^2} = \frac{x^6}{16x^2} + \frac{8}{16x^2} = \frac{1}{16}x^4 + \frac{1}{2}x^{-2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4}x^3 - x^{-3}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = \left(\frac{1}{4}x^3\right)^2 - 2 \cdot \frac{1}{4}x^3 \cdot x^{-3} + (x^{-3})^2 = \left(\frac{1}{4}x^3\right)^2 - \frac{1}{2} + (x^{-3})^2 \quad ;$$

[Note: $x^n \cdot x^m = x^{n+m}$ and $x^n \cdot x^{-n} = x^{n-n} = x^0 = 1$]

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{4}x^3\right)^2 - \frac{1}{2} + (x^{-3})^2 = \left(\frac{1}{4}x^3\right)^2 + \frac{1}{2} + (x^{-3})^2 = \left(\frac{1}{4}x^3 + x^{-3}\right)^2$$

$$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{1}{4}x^3 + x^{-3}\right)^2} = \left|\frac{1}{4}x^3 + x^{-3}\right| = \frac{1}{4}x^3 + x^{-3} \qquad \text{for } 2 \le x \le 3.$$

So the length of the curve

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_2^3 \left(\frac{1}{4}x^3 + x^{-3}\right) dx = \frac{595}{144}\ unit$$

Exercise: 5 [Similar to Exercise 1]

(a) Find the length of the curve $y = \frac{e^x + e^{-x}}{2}$ from $x = 0$ to $x = 1$.

Answer: $L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_0^1 \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right) dx = \left[\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right]_0^1$

$$= \frac{1}{2}e - \frac{1}{2}\frac{1}{e} - \frac{1}{2} + \frac{1}{2}$$

$$= \frac{1}{2e}(e^2 - 1) \quad unit$$

(b) Find the length of the curve $x = g(y) = \frac{e^{2y} + e^{-2y}}{4} = \frac{e^{2y}}{4} + \frac{e^{-2y}}{4}$ from $y = 0$ to $y = 3$.

Homework **Similar to Exercise 1**

Hint: $\sqrt{1 + \left[\frac{dx}{dy}\right]^2} = \sqrt{\left(\frac{e^{2y}}{2} + \frac{e^{-2y}}{2}\right)^2} = \left|\frac{e^{2y}}{2} + \frac{e^{-2y}}{2}\right| = \frac{e^{2y}}{2} + \frac{e^{-2y}}{2} \quad ; \quad 0 \le y \le 3$

Definition 2: If $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the length of the curve over the interval $[c, d]$ is defined by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy \quad = \quad \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\ dy$$

Steps: Given a smooth function $\boldsymbol{x = g(y)}$; $\quad \boldsymbol{y = c}$ **to** $\boldsymbol{y = d}$

Step 1: Find $\boldsymbol{\frac{dx}{dy} = g'(y)}$

Step 2: Find $\boldsymbol{\left(\frac{dx}{dy}\right)^2}$

Step 3: Find $\boldsymbol{1 + \left(\frac{dx}{dy}\right)^2}$

Step 4: Find $\boldsymbol{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}$ **. Here we should try, if possible, to write** $\boldsymbol{1 + \left[\frac{dx}{dy}\right]^2}$ **as a perfect square so that we can cancel the square root.**

Step 5: Evaluate the integral to find the length L:

$$L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\ dy$$

Exercise :6 Find the length of the curve $\boldsymbol{x = y^{\frac{3}{2}}}$ **from** $\boldsymbol{y = 1}$ **to** $\boldsymbol{y = 2}$.

Solution:

$$L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\ dy \quad \dots\dots\dots (1)$$

Given $x = y^{\frac{3}{2}}$, $\quad 1 \le y \le 2.$

$$\Rightarrow \frac{dx}{dy} = \frac{3}{2} y^{\frac{1}{2}}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \left(\frac{3}{2} y^{\frac{1}{2}}\right)^2 = \frac{9}{4}\ y$$

$$\Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{9}{4}\ y$$

$$\Rightarrow \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{9}{4}\ y} = \sqrt{\frac{4+9y}{4}} = \frac{1}{2}\sqrt{4 + 9y} \qquad \text{for } 1 \le y \le 2.$$

So, the length of the curve

$$L \;=\; \int_{c}^{d} \sqrt{1+\left[\frac{dx}{dy}\right]^2}\; dy = \int_{1}^{2} \frac{1}{2}\sqrt{4+9y}\; dy$$

Now, set $u = 4 + 9y$. Then $\frac{du}{dy} = 9$, that is, $\frac{1}{9}\, du = dy$.

$If\;\; y = 1, then\, u = 13\;\; and\;\; if\; y = 2, then\, u = 22,$ so we get $13 \le u \le 22$.

Hence, Length

$$L \;=\; \int_{13}^{22} \frac{1}{18}\, u^{\frac{1}{2}}\, du \;=\; \frac{1}{18}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{13}^{22} \;=\; \frac{1}{18}\,\frac{2}{3}\;\left[u^{\frac{3}{2}}\right]_{13}^{22}$$

$$= \frac{1}{27}\left[\left(\sqrt{u}\right)^{3}\right]_{13}^{22} = \frac{1}{27}\;\left[22\sqrt{22}\; - 13\sqrt{13}\right]\; unit$$

Exercise: 4 [Similar to exercise 3] Homework

Find the length of the curve $y = \sqrt{x} + 2$ from $x = 0$ to $x = 2$.

$$L = \int_{0}^{2} \sqrt{1 + \frac{1}{4x}}\; dx = ?$$

Definition 3: If no segment of the parametric curve

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

traced more than once as t increases from a to b, then the length of the parametric curve is defined by

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;\; dt$$

Step: Given a parametric curve $x = x(t),\; y = y(t), \qquad a \le t \le b$

Step 1: Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

Step 2: Find $\left(\frac{dx}{dt}\right)^2$ and $\left(\frac{dy}{dt}\right)^2$

Step 3: Find $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2$

Step 4: Find $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Here we should try, if possible, to write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ as a perfect square so that we can cancel the square root.

Step 5: Evaluate the integral to find the length L:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Exercise:7 Find the length of the curve $x = 3\cos(2\theta)$, $y = 3\sin(2\theta)$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.

__Solution:__ Given $x = 3\cos(2\theta)$, $y = 3\sin(2\theta)$

Then $\frac{dx}{d\theta} = -3\ \sin(2\theta)\ (2) = -6\ \sin(2\theta)$ and $\frac{dy}{d\theta} = 6\ cos(2\theta)$

$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 = 36\sin^2 2\theta$ and $\left(\frac{dy}{d\theta}\right)^2 = 36\cos^2 2\theta$

Then $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 36$

Length of the curve:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{36}\ d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 6\ d\theta = [6\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{3\pi}{2}\ \ unit$$

Exercise : 8 Find the length of the curve $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$ for $0 \leq t \leq 1$.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^1 \sqrt{(t^2)^2 + (t)^2}\ dt$$

$$= \int_0^1 \sqrt{t^4 + t^2}\ dt$$

$$= \int_0^1 t\sqrt{t^2+1}\ dt = \frac{1}{2}\int_0^1 2t\sqrt{t^2+1}\ dt = \frac{1}{2}\frac{2}{3}\left[(t^2+1)^{\frac{3}{2}}\right]_0^1 =$$

Exercise : 9 Find the length of the curve $x = e^t \cos t$, $y = e^t \sin t$ for $0 \le t \le \frac{\pi}{2}$.

Solution: We know that the length of a parametric curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \quad \dots\dots\dots \ (1)$$

Given $x = e^t \cos t$, $y = e^t \sin t$ for $0 \le t \le \frac{\pi}{2}$.

$$\frac{dx}{dt} = e^t(-\sin t) + e^t \cos t = -e^t \sin t + e^t \cos t$$

$$\frac{dy}{dt} = e^t \cos t + e^t \sin t$$

Then $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$$= (-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2$$

$$= e^{2t}\sin^2 t + e^{2t}\cos^2 t - 2\,e^{2t}\sin t\ \cos t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2\,e^{2t}\sin t\ \cos t$$

$$= 2e^{2t}(\sin^2 t + \cos^2 t)$$

That is, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2e^{2t}}$$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2}\ e^t$$

Now,

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_{0}^{\frac{\pi}{2}} \sqrt{2}\, e^{t} \; dt$$

$$= \sqrt{2} \left(e^{\frac{\pi}{2}} - 1\right) \; unit$$

Exercise: 10 Find the length of the curve $x = 2\cos t$, $y = 2\sin t$ for $0 \le t \le \frac{3\pi}{2}$.