MAT 350
ENGINEERING MATHEMATICS

Power Series Solutions of ODEs

Lecture: 13

Dr. M. Sahadet Hossain (Mth)
Associate Professor
Department of Mathematics and Physics, NSU.

Power Series Recall from calculus that **power series** in $x - a$ is an infinit series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

Such a series also said to be a **power series centered at *a*.**

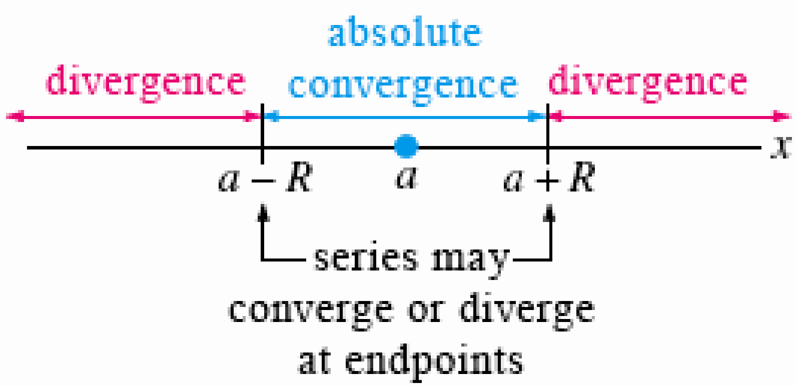
$$\sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + \cdots$$

- **Convergence** A power series is **convergent** at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, that is, $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists. If the limit does not exist at x, then the series is said to be **divergent**.

- **Interval of Convergence** Every power series has an **interval of convergence.** The interval of convergence is the set of *all* real numbers x for which the series converges. The center of the interval of convergence is the center a of the series.

- **Radius of Convergence** The radius R of the interval of convergence of a power series is called its **radius of convergence.** If $R > 0$, then a power series converges for $|x - a| < R$ and diverges for $|x - a| > R$. If the series converges only at its center a, then $R = 0$. If the series converges for all x, then we write $R = \infty$. Recall, the absolute-value inequality $|x - a| < R$ is equivalent to the simultaneous inequality $a - R < x < a + R$. A power series may or may not converge at the endpoints $a - R$ and $a + R$ of this interval.

- **Absolute Convergence** Within its interval of convergence a power series **converges absolutely.** In other words, if x is in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty} |c_n(x - a)^n|$ converges. See Figure 6.1.1.



EXAMPLE 1 Interval of Convergence

Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$.

SOLUTION The ratio test gives

$$\lim_{n\to\infty} \left| \frac{\dfrac{(x-3)^{n+1}}{2^{n+1}(n+1)}}{\dfrac{(x-3)^n}{2^n n}} \right| = |x-3| \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}|x-3|.$$

The series converges absolutely for $\frac{1}{2}|x-3| < 1$ or $|x-3| < 2$ or $1 < x < 5$. This last inequality defines the *open* interval of convergence. The series diverges for $|x-3| > 2$, that is, for $x > 5$ or $x < 1$. At the left endpoint $x = 1$ of the open interval of convergence, the series of constants $\sum_{n=1}^{\infty} ((-1)^n/n)$ is convergent by the alternating series test. At the right endpoint $x = 5$, the series $\sum_{n=1}^{\infty} (1/n)$ is the divergent harmonic series. The interval of convergence of the series is $[1, 5)$, and the radius of convergence is $R = 2$.

- **Analytic at a Point** A function f is said to be **analytic at a point** a if it can be represented by a power series in $x - a$ with either a positive or an infinite radius of conve gence.

Maclaurin Series	Interval of Convergence
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$[-1, 1]$ (2)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	$(-1, 1]$
$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$

EXAMPLE 3 Addition of Power Series

Write

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

as one power ser

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

series starts with x for $n = 3$

series starts with x for $n = 0$

$$= 2 \cdot 1 c_2 x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \tag{3}$$

Now to get the same summation index we are inspired by the exponents of x; we let $k = n - 2$ in the first series and at the same time let $k = n + 1$ in the second series. For $n = 3$ in $k = n - 2$ we get $k = 1$, and for $n = 0$ in $k = n + 1$ we get $k = 1$, and so the right-hand side of (3) becomes

same

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k. \tag{4}$$

same

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} + c_{k-1}]x^k. \quad (5)$$

EXAMPLE 4 A Power Series Solution

Find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the differential equation $y' + y = 0$.

SOLUTION We break down the solution into a sequence of steps.

(*i*) First calculate the derivative of the assumed solution:

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \leftarrow \text{see the first line in (1)}$$

(*ii*) Then substitute y and y' into the given DE:

$$y' + y = \sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n.$$

(*iii*) Now shift the indices of summation. When the indices of summation have the same starting point and the powers of x agree, combine the summations:

$$y' + y = \underbrace{\sum_{n=1}^{\infty} c_n n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} [c_{k+1}(k+1) + c_k]x^k.$$

(*iv*) Because we want $y' + y = 0$ for all x in some interval,

$$\sum_{k=0}^{\infty} [c_{k+1}(k+1) + c_k]x^k = 0$$

is an identity and so we must have $c_{k+1}(k+1) + c_k = 0$, or

$$c_{k+1} = -\frac{1}{k+1}c_k, \quad k = 0, 1, 2, \ldots.$$

(v) By letting k take on successive integer values starting with $k = 0$, we fin

$$c_1 = -\frac{1}{1}c_0 = -c_0$$

$$c_2 = -\frac{1}{2}c_1 = -\frac{1}{2}(-c_0) = \frac{1}{2}c_0$$

$$c_3 = -\frac{1}{3}c_2 = -\frac{1}{3}\left(\frac{1}{2}c_0\right) = -\frac{1}{3\cdot 2}c_0$$

$$c_4 = -\frac{1}{4}c_2 = -\frac{1}{4}\left(-\frac{1}{3\cdot 2}c_0\right) = \frac{1}{4\cdot 3\cdot 2}c_0$$

and so on, where c_0 is arbitrary.

(*vi*) Using the original assumed solution and the results in part (*v*) we obtain a formal power series solution

$$
\begin{aligned}
y &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots \\
&= c_0 - c_0x + \frac{1}{2}c_0x^2 - c_0\frac{1}{3\cdot 2}x^3 + c_0\frac{1}{4\cdot 3\cdot 2}x^4 - \cdots \\
&= c_0\left[1 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 - \cdots\right].
\end{aligned}
$$

It should be fairly obvious that the pattern of the coefficients in part (*v*) is $c_k = c_0(-1)^k/k!$, $k = 0, 1, 2, \ldots$. so that in summation notation we can write

$$y = c_0\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^k. \tag{8}$$

6.2 SOLUTIONS ABOUT ORDINARY POINTS

A Definitio If we divide the homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

by the lead coefficient $a_2(x)$ we obtain the standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

DEFINITION 6.2.1 Ordinary and Singular Points

A point $x = x_0$ is said to be an **ordinary point** of the differential of the differential equation (1) if both coefficients $P(x)$ and $Q(x)$ in the standard form (2) are analytic at x_0. A point that is *not* an ordinary point of (1) is said to be a **singular point** of the DE.

Example-1

$$y'' + xy' + (\ln x)y = 0$$

$$P(x) = x \quad \text{and} \quad Q(x) = \ln x.$$

Now $P(x) = x$ is analytic at every real number, and $Q(x) = \ln x$ is analytic at every *positive* real number. However, since $Q(x) = \ln x$ is discontinuous at $x = 0$ it cannot be represented by a power series in x, that is, a power series centered at 0. We conclude that $x = 0$ is a singular point of the DE.

Example-2

$$xy'' + y' + xy = 0$$

$$y'' + \frac{1}{x}y' + y = 0,$$

we see that $P(x) = 1/x$ fails to be analytic at $x = 0$. Hence $x = 0$ is a singular point of the equation.

Example-3

(a) The only singular points of the differential equation

$$(x^2 - 1)y'' + 2xy' + 6y = 0$$

are the solutions of $x^2 - 1 = 0$ or $x = \pm 1$. All other values of x are ordinary points.

THEOREM 6.2.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (1), we can always find two linearly independent solutions in the form of a power series centered at x_0, that is,

$$y = \sum_{n=0}^{\infty} c_n(x - x_0)^n.$$

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

A solution of the form $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$ is said to be a **solution about the ordinary point x_0.** The distance R in Theorem 6.2.1 is the *minimum* value or *lower bound* for the radius of convergence.

Solve $y'' + xy = 0$.

SOLUTION Since there are no singular points, Theorem 6.2.1 guarantees two power series solutions centered at 0 that converge for $|x| < \infty$. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ and the second derivative $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$ (see (1) in Section 6.1) into the differential equation give

$$y'' + xy'' + xy = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} c = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}. \qquad (3)$$

$$y'' + xy = 2c_2 + \sum_{k=1}^{\infty}[(k+1)(k+2)c_{k+2} + c_{k-1}]x^k = 0. \qquad (4)$$

Equating coefficient of x^0: $2c_2 = 0$

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0, \qquad k = 1, 2, 3, \ldots . \tag{5}$$

the expression in (5), called a **recurrence relation.**

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, \qquad k = 1, 2, 3, \ldots . \tag{6}$$

$$k = 1, \qquad c_3 = -\frac{c_0}{2 \cdot 3}$$

$$k = 2, \qquad c_4 = -\frac{c_1}{3 \cdot 4}$$

$$k = 3, \qquad c_5 = -\frac{c_2}{4 \cdot 5} = 0 \qquad \leftarrow c_2 \text{ is zero}$$

$$k = 4, \qquad c_6 = -\frac{c_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} c_0$$

$$k = 5, \qquad c_7 = -\frac{c_4}{6 \cdot 7} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1$$

$$k = 6, \qquad c_8 = -\frac{c_5}{7 \cdot 8} = 0 \qquad \leftarrow c_5 \text{ is zero}$$

$$k = 7, \qquad c_9 = -\frac{c_6}{8 \cdot 9} = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0$$

$$k = 8, \qquad c_{10} = -\frac{c_7}{9 \cdot 10} = -\frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1$$

$$k = 9, \qquad c_{11} = -\frac{c_8}{10 \cdot 11} = 0 \qquad \leftarrow c_8 \text{ is zero}$$

Now substituting the coefficients just obtained into the original assumption

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 +$$
$$c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9 + c_{10}x^{10} + c_{11}x^{11} + \cdots,$$

we get

$$y = c_0 + c_1x + 0 - \frac{c_0}{2\cdot 3}x^3 - \frac{c_1}{3\cdot 4}x^4 + 0 + \frac{c_0}{2\cdot 3\cdot 5\cdot 6}x^6$$
$$+ \frac{c_1}{3\cdot 4\cdot 6\cdot 7}x^7 + 0 - \frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 - \frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + 0 + \cdot$$

After grouping the terms containing c_0 and the terms containing c_1, we obtain $y = c_0y_1(x) + c_1y_2(x)$, where

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \cdots$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + \cdots$$

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2 \cdot 3 \cdots (3k-1)(3k)} x^{3k}$$

$$y_2(x) = x + \sum_{k=1}^{\infty} \frac{(-1)^k}{3 \cdot 4 \cdots (3k)(3k+1)} x^{3k+1}.$$