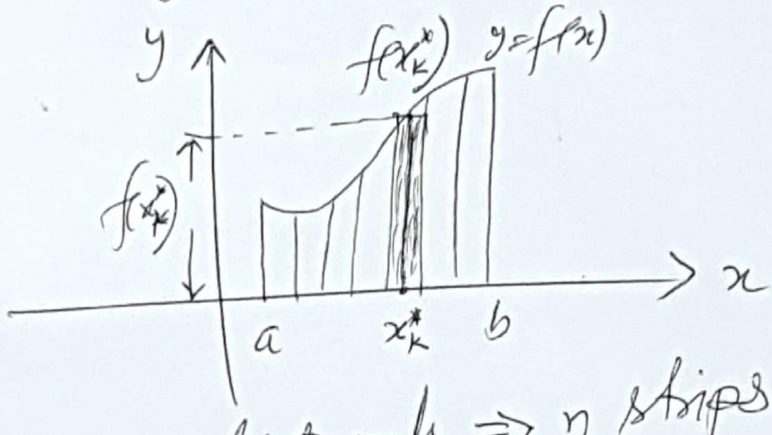
6.1

f is continuous and nonnegative on $[a,b]$, then the area A under $y=f(x)$ over the interval $[a,b]$ is:



* $[a,b] \longrightarrow n$ subintervals $\Rightarrow n$ strips
* Area of a strip (kth strip) : $f(x_k^*)\Delta x_k$; (Δx_k = width)
* $A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$
* $A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$

$x_k^* \longrightarrow x$, $\Delta x_k \longrightarrow dx$, $[a,b] \xrightarrow{\Delta x_k} \int_a^b$

$\therefore$ $\boxed{A = \int_a^b f(x)\,dx}$

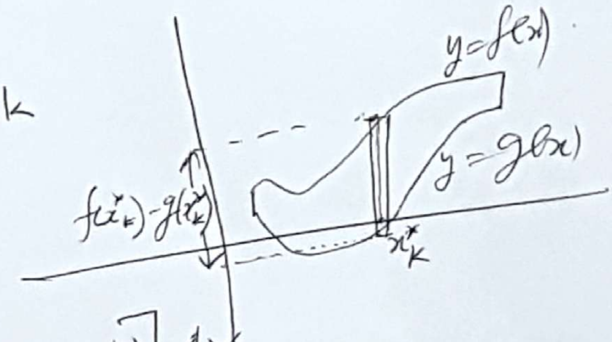
Area between $y = f(x)$ and $y = g(x)$

f and g continuous on $[a, b]$ and $f(x) \geq g(x)$ for $a \leq x \leq b$

($y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross).

Area between the curve; (bounded above by $y = f(x)$, bounded below by $y = g(x)$) and on the sides the lines $x = a$ and $x = b$

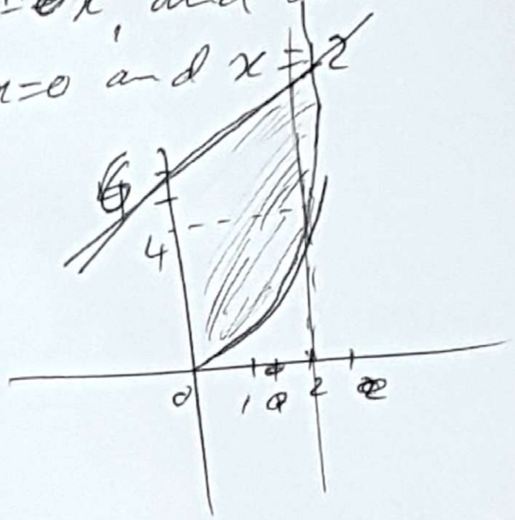
$$A \approx \sum_{k=1}^{n} \left(f(x_k^*) - g(x_k^*) \right) \Delta x_k$$



$$\Rightarrow A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \left[f(x_k^*) - g(x_k^*) \right] \Delta x_k$$

$$\Rightarrow \quad A = \int_a^b [f(x) - g(x)] \, dx$$

Ex.1 Find the area of the region bounded above by $y = x+6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x=0$ and $x=2$



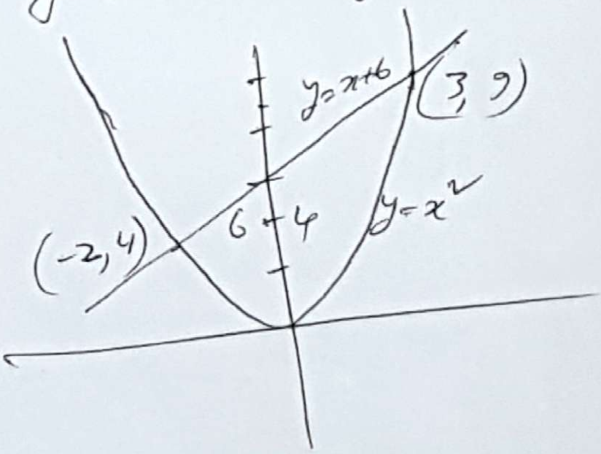
$$A = \int_0^2 \left[(x+6) - x^2\right] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{2^2}{2} + 6.2 - \frac{2^3}{3} - 0$$

$$= \frac{34}{3}$$

Ex.2 Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x+6$



We have $x^2 = x+6$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3, \ x = -2$

$$\therefore A = \int_{-2}^{3} \left[(x+6) - x^2\right] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{9}{2} + 18 - \frac{27}{3} - \frac{4}{2} + 12 - \frac{8}{3}$$

$$= \frac{27 + 108 - 54 - 12 + 72 - 16}{6}$$

$$= \frac{125}{6}$$

Ex. 3 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

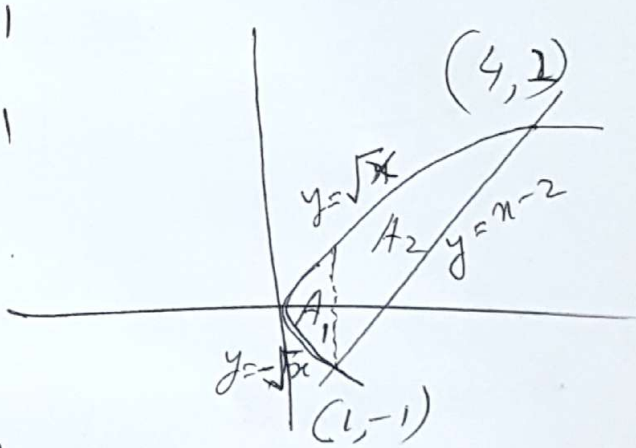
(integrating with respect to x)

$x = y^2$, $\quad y = x - 2 \Rightarrow x = y + 2$

$\Rightarrow y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2$ or $y = -1$

$\Rightarrow x = 4$ or $x = 1$



$$A_1 = \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right] dx$$

$$= \int_0^1 2\sqrt{x}\, dx = 2\int_0^1 x^{1/2} dx = 2 \cdot x^{3/2} \cdot \frac{2}{3}\Big]_0^1$$

$$= \frac{4}{3}[1 - 0] = \frac{4}{3}$$

$$A_2 = \int_1^4 \left[\sqrt{x} - (x-2)\right] dx = \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)\Big]_1^4$$

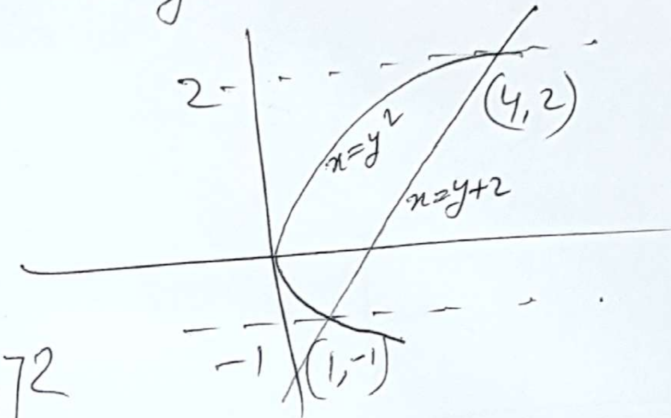
$$= \frac{2}{3}\cdot 8 - \frac{16}{2} + 8 - \frac{2}{3} + \frac{1}{2} - 2 = \frac{32 - 48 + 48 - 4 + 3 - 12}{6}$$

$$= \frac{19}{6}$$

$$\therefore A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{8 + 19}{6} = \frac{27}{6} = \frac{9}{2}$$

OR

OR Ex3 integrating w.r.to y.



$$A=\int_{-1}^{2}\left[(y+2)-y^2\right]dy$$

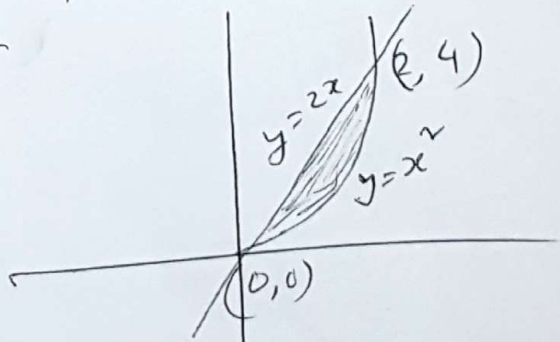
$$=\left(\frac{y^2}{2}+2y-\frac{y^3}{3}\right)\Big]_{-1}^{2}$$

$$=\left(\frac{4}{2}+2.2-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$

$$=2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}=\frac{12+24-16-3+12-2}{6}$$

$$=\frac{27}{6}=\frac{9}{2}$$

Ex4 find the area by (a) integrating with respect to x and (b) integrating with respect to y.



$x^2=y \Rightarrow x=\pm\sqrt{y}$
$\Rightarrow x=\sqrt{y}$

$y=2x \Rightarrow x=\frac{y}{2}$

(a) $\displaystyle\int_0^2 (2x-x^2)\,dx=\left(2\frac{x^2}{2}-\frac{x^3}{3}\right)\Big]_0^2$

$$=4-\frac{8}{3}$$

$$=\frac{4}{3}$$

(b) $\displaystyle\int_0^4\left(\sqrt{y}-\frac{y}{2}\right)dy=\int_0^4\left(y^{\frac{1}{2}}-\frac{y}{2}\right)dy$

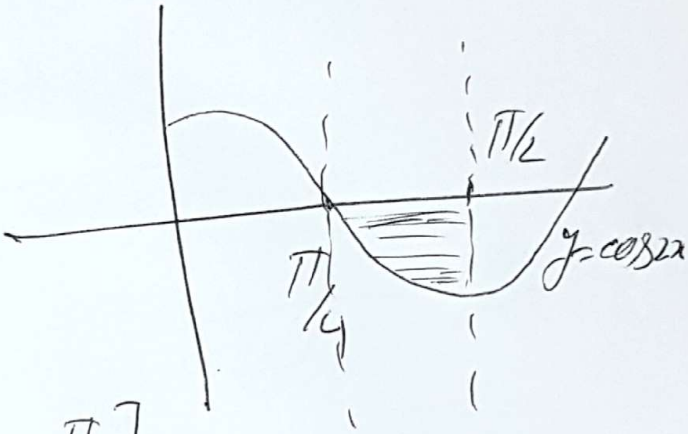
$$=\left(\frac{2}{3}y^{3/2}-\frac{y^2}{2.2}\right)\Big]_0^4$$

$$=\frac{2}{3}.8-\frac{16}{4}=\frac{16-12}{3}=\frac{4}{3}$$

6.1

Ex 5 Sketch the region ~~in~~ enclosed by the curves $y=\cos 2x$, $y=0$, $x=\frac{\pi}{4}$, $x=\frac{\pi}{2}$ and find its area.

$$A=\int_{\pi/4}^{\pi/2}(0-\cos 2x)\,dx$$



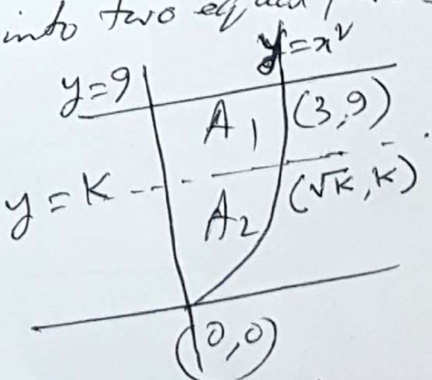
$$=-\frac{\sin 2x}{2}\Big]_{\pi/4}^{\pi/2}$$

$$=-\frac{1}{2}\left[\sin \cancel{2}\cdot\frac{\pi}{\cancel{2}}-\sin \cancel{2}\cdot\frac{\pi}{\cancel{4}\,2}\right]$$

$$=-\frac{1}{2}[0-1]$$

$$=\frac{1}{2}$$

Ex 6 Find a horizontal ~~bake~~ line $y=k$ that divides the area between $y=x^2$ and $y=9$ into two equal parts.



$A_1=A_2$

$A=A_1+A_2$

$$A=\int_0^3(9-x^2)\,dx=\left(9x-\frac{x^3}{3}\right)\Big]_0^3=18$$

$$\Rightarrow A_1=9=A_2$$

$$A_2=\int_0^{\sqrt{k}}(k-x^2)\,dx$$

$$\Rightarrow 9=\left(kx-\frac{x^3}{3}\right)\Big]_0^{\sqrt{k}}$$

$$\Rightarrow 9=k\sqrt{k}-\frac{k\sqrt{k}}{3}\Rightarrow 9=\frac{2}{3}k\sqrt{k}$$

$$\Rightarrow k\sqrt{k}=\frac{9\cdot3}{2}\Rightarrow k^{3/2}=\frac{9\cdot3}{2}\Rightarrow k^3=\frac{9\cdot9\cdot3\cdot3}{2\cdot2}$$

$$\Rightarrow k=\frac{9}{\sqrt[3]{4}}\Rightarrow \boxed{\text{The line } y=k=\frac{9}{\sqrt[3]{4}}}$$

Arc length

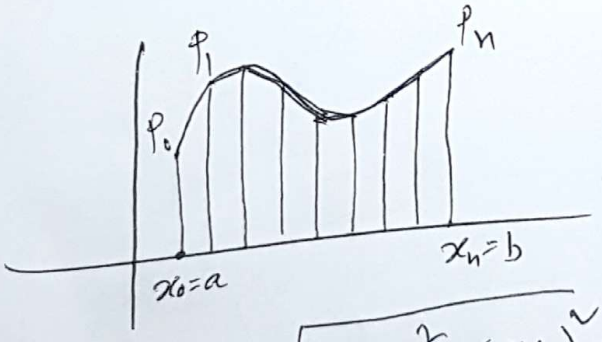
If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

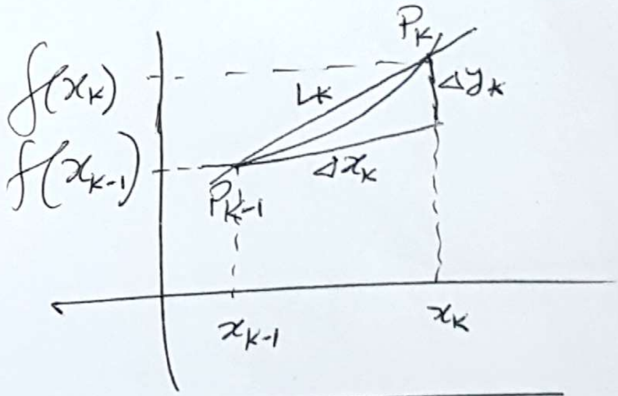
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If $x = g(y)$, $y = c$ and $y = d$, Then

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$





$$L_K = \sqrt{(\Delta x_K)^2 + (\Delta y_K)^2} = \sqrt{(\Delta x_K)^2 + [f(x_K) - f(x_{K-1})]^2}$$

$$L \approx \sum_{K=1}^{n} L_K = \sum_{K=1}^{n} \sqrt{(\Delta x_K)^2 + [f(x_K) - f(x_{K-1})]^2}$$

$$\Rightarrow L \approx \sum_{K=1}^{n} \sqrt{(\Delta x_K)^2 + [f'(x_K^*)]^2 (\Delta x_K)^2} = \sum_{K=1}^{n} \sqrt{1 + [f'(x_K^*)]^2}\, \Delta x_K$$

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k$$

$$= \int_a^b \sqrt{1 + [f'(x_k)]^2}\, dx$$

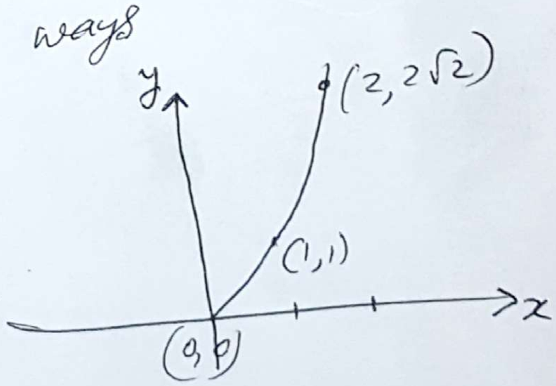
$\boxed{x_k - x_{k-1} = \Delta x_k}$

By Mean-Value theorem

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$$

$$\Rightarrow f(x_k) - f(x_{k-1}) = f'(x_k^*)\, \Delta x_k$$

Ex 1 Find the arc length of the curve $y = x^{3/2}$ from $(1,1)$ to $(2, 2\sqrt{2})$ in two ways



(a) $y = x^{3/2}$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$$

$$L = \int_1^2 \sqrt{1 + \frac{9}{4} x}\, dx$$

put $u = 1 + \frac{9}{4} x \Rightarrow \frac{4\,du}{9} = dx$

$$\Rightarrow du = \frac{9}{4} dx$$

6.4

$x=1 \Rightarrow u = 1+\frac{9}{4}\cdot 1 = \frac{13}{4}$

$x=2 \Rightarrow u = 1+\frac{9}{4}\cdot 2 = \frac{4+18}{4} = \frac{22}{4}$

$$\therefore L = \int_{13/4}^{22/4} u^{1/2}\cdot\frac{4du}{9} = \frac{4}{9}\cdot u^{3/2}\cdot\frac{2}{3}\Big]_{13/4}^{22/4}$$

$$= \frac{8}{27}\left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] = \frac{8}{27}\left[\frac{22\sqrt{22}}{8} - \frac{13\sqrt{13}}{8}\right]$$

$$\approx 2.09$$

(b) $y = x^{3/2} \Rightarrow x = y^{2/3}$

$$\Rightarrow \frac{dx}{dy} = \frac{2}{3}y^{-\frac{1}{3}}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4}{9}y^{-\frac{2}{3}}$$

$$L = \int_1^{2\sqrt{2}} \sqrt{1+\frac{4}{9}y^{-\frac{2}{3}}}\,dy = \int_1^{2\sqrt{2}} \sqrt{\frac{y^{-2/3}}{9}\left(9y^{2/3}+4\right)}\,dy$$

$$= \int_1^{2\sqrt{2}} \frac{y^{-\frac{1}{3}}}{3}\sqrt{9y^{2/3}+4}\,dy$$

put $u = 9y^{2/3}+4 \Rightarrow du = \frac{3}{\cancel{9}}\cdot\frac{2}{\cancel{3}}y^{-\frac{1}{3}}dy$

$\Rightarrow \frac{du}{6} = y^{-\frac{1}{3}}dy$

$y=1 \Rightarrow u=13$

$y=2\sqrt{2} \Rightarrow u = 22$

$$= \int_{13}^{22} \frac{1}{3}\cdot u^{1/2}\cdot\frac{du}{6}$$

$$= \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big]_{13}^{22}$$

$$\approx 2.09$$

Ex.2 Find the exact ~~value~~ arc length of the curve over the interval

$y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$

$y = 3x^{3/2} - 1$

$\frac{dy}{dx} = 3 \cdot \frac{3}{2} \cdot x^{\frac{1}{2}}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{81}{4} x$

$L = \int_0^1 \sqrt{1 + \frac{81}{4} x} \, dx$

put $u = 1 + \frac{81}{4} x$

$\Rightarrow du = \frac{81}{4} dx$

$\Rightarrow \frac{4}{81} du = dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = \frac{85}{4}$

$= \int_1^{\frac{85}{4}} u^{\frac{1}{2}} \cdot \frac{4}{81} du$

$= \frac{4}{81} \cdot x^{\frac{3}{2}} \cdot \frac{2}{3} \Big]_1^{\frac{85}{4}}$

$= \frac{8}{243} \left[\left(\frac{85}{4}\right)^{3/2} - 1 \right]$

$= \frac{8}{243} \left[\frac{85\sqrt{85}}{8} - 1 \right]$

$= \frac{\cancel{8}}{243} \left[\frac{85\sqrt{85} - 8}{\cancel{8}} \right]$

$= \frac{85\sqrt{85} - 8}{243}$

Ex. 3

$24xy = y^4 + 48$ from $y=2$ to $y=4$

$$\Rightarrow x = \frac{y^4}{24y} + \frac{48}{24y} \Rightarrow x = \frac{y^3}{24} + \frac{2}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cancel{3}y^2}{\cancel{24}_8} - \frac{2}{y^2}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \left(\frac{y^2}{8} - \frac{2}{y^2}\right)^2$$

$$= \frac{y^4}{64} - \cancel{2}\cdot\frac{\cancel{y^2}}{\cancel{8}}\cdot\frac{\cancel{2}}{\cancel{y^2}} + \frac{4}{y^4}$$

$$= \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}$$

$$\therefore L = \int_2^4 \sqrt{1 + \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}}\, dy$$

$$= \int_2^4 \sqrt{\frac{64y^4 + y^8 - 32y^4 + 256}{64y^4}}\, dy$$

$$= \int_2^4 \sqrt{\frac{y^8 + 32y^4 + 256}{(8y^2)^2}}\, dy$$

$$= \int_2^4 \sqrt{\frac{(y^4+16)^2}{(8y^2)^2}}\, dy = \int_2^4 \frac{y^4+16}{8y^2}\, dy$$

$$= \int_2^4 \left(\frac{y^2}{8} + \frac{2}{y^2}\right) dy = \left(\frac{y^3}{8\cdot 3} - \frac{2}{y}\right)\Bigg]_2^4$$

$$= \frac{4\cdot 4\cdot 4}{8\cdot 3} - \frac{2}{4} - \frac{2\cdot 2\cdot 2}{8\cdot 3} + \frac{2}{2}$$

$$= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 = \frac{16 - 3 - 2 + 6}{6}$$

$$= \frac{17}{6}$$