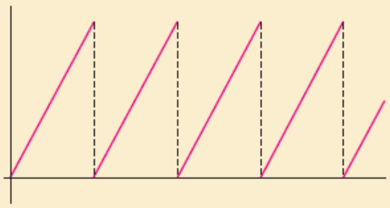
7 The Laplace Transform

In the linear mathematical models for a physical system such as a spring/mass system or a series electrical circuit, the right-hand member, or input, of the differential equations

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t) \quad \text{or} \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$



is a driving function and represents either an external force $f(t)$ or an impressed voltage $E(t)$. In Section 5.1 we considered problems in which the functions f and E were continuous. However, discontinuous driving functions are not uncommon. For example, the impressed voltage on a circuit could be piecewise continuous and periodic, such as the "sawtooth" function shown on the left. Solving the differential

$$\frac{d}{dx}x^2 = 2x \qquad \text{and} \qquad \int x^2\,dx = \frac{1}{3}x^3 + c.$$

Integral Transform If $f(x, y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable. For example, by holding y constant, we see that $\int_1^2 2xy^2\,dx = 3y^2$. Similarly, a definite integral such as $\int_a^b K(s, t)\,f(t)\,dt$ transforms a function f of the variable t into a function F of the variable s. We are particularly interested in an **integral transform,** where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_0^\infty K(s, t)\,f(t)\,dt$ is defined as a limit

$$\int_0^\infty K(s, t)\,f(t)\,dt = \lim_{b\to\infty}\int_0^b K(s, t)\,f(t)\,dt. \tag{1}$$

DEFINITION 7.1.1 Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \tag{2}$$

is said to be the **Laplace transform** of f, provided that the integral converges.

EXAMPLE 1 Applying Definition 7.1.

Evaluate $\mathscr{L}\{1\}$.

SOLUTION From (2),

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \frac{-e^{-st}}{s}\bigg|_0^b = \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

provided that s 0. In other words, when s 0, the exponent $-sb$ is negative, and $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$.

EXAMPLE 2 Applying Definition 7.1.

Evaluate $\mathscr{L}\{t\}$.

SOLUTION From Definition 7.1.1 we have $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\,t\,dt$. Integrating by parts and using $\lim_{t\to\infty} te^{-st} = 0$, s 0, along with the result from Example 1, we obtain

$$\mathscr{L}\{t\} = \frac{-te^{-st}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}.$$

EXAMPLE 3 Applying Definition 7.1.

Evaluate **(a)** $\mathscr{L}\{e^{-3t}\}$ **(b)** $\mathscr{L}\{e^{5t}\}$

SOLUTION In each case we use Definition 7.1.1.

(a)
$$\mathscr{L}\{e^{-3t}\} = \int_0^\infty e^{-3t}e^{-st}\,dt = \int_0^\infty e^{-(s+3)t}\,dt$$
$$= \left.\frac{-e^{-(s+3)t}}{s+3}\right|_0^\infty$$
$$= \frac{1}{s+3}.$$

The last result is valid for $s > -3$ because in order to have $\lim_{t\to\infty} e^{-(s+3)t} = 0$ we must require that $s + 3 > 0$ or $s > -3$.

EXAMPLE 4 Applying Definition 7.1.

Evaluate $\mathscr{L}\{\sin 2t\}$.

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4}, \qquad s \quad 0.$$

THEOREM 7.1.1 Transforms of Some Basic Functions

$$\textbf{(a)}\ \mathscr{L}\{1\} = \frac{1}{s}$$

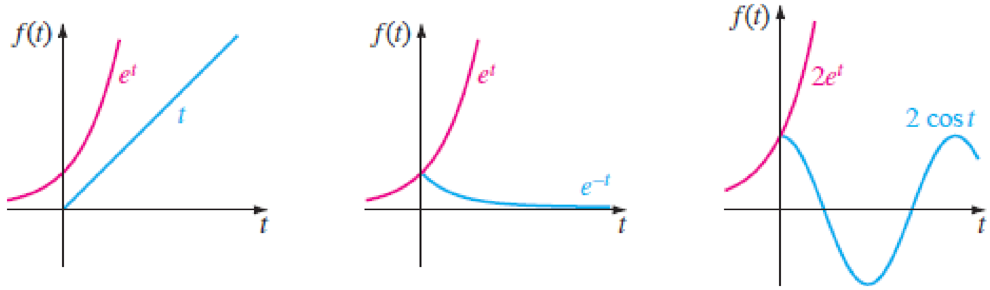
$$\textbf{(b)}\ \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots \qquad \textbf{(c)}\ \mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\textbf{(d)}\ \mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2} \qquad \textbf{(e)}\ \mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$\textbf{(f)}\ \mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2} \qquad \textbf{(g)}\ \mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

DEFINITION 7.1.2 Exponential Order

A function f is said to be of **exponential order** if there exist constants c, M 0, and T 0 such that $|f(t)| \leq Me^{ct}$ for all t T.



(a) **(b)** **(c)**

FIGURE 7.1.3 Three functions of exponential order

THEOREM 7.1.2 Sufficient Conditions fo Existence

If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for s c.

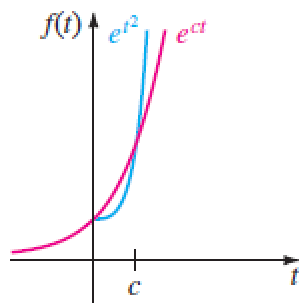


FIGURE 7.1.4 e^{t^2} is not of exponential order

EXAMPLE 6 Transform of a Piecewise Continuous Function

Evaluate $\mathscr{L}\{f(t)\}$ where $f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3. \end{cases}$

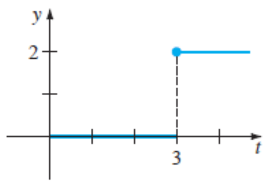


FIGURE 7.1.5 Piecewise continuous function in Example 6

SOLUTION The function f, shown in Figure 7.1.5, is piecewise continuous and of exponential order for t 0. Since f is defined in two pieces, $\mathscr{L}\{f(t)\}$ is expressed as the sum of two integrals:

$$\begin{aligned} \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt &= \int_0^3 e^{-st}(0)\, dt + \int_3^\infty e^{-st}(2)\, dt \\ &= 0 + \frac{2e^{-st}}{-s}\bigg|_3^\infty \\ &= \frac{2e^{-3s}}{s}, \qquad s \quad 0. \end{aligned}$$

THEOREM 7.1.3 Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0, \infty)$ and of exponential order and $F(s) = \mathscr{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

EXERCISES 7.1

In Problems 1–18 use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

9.

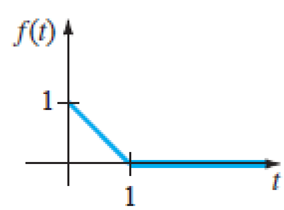


FIGURE 7.1.8 Graph for Problem 9

10.

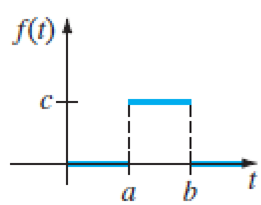


FIGURE 7.1.9 Graph for Problem 10

11. $f(t) = e^{t+7}$

12. $f(t) = e^{-2t-5}$

13. $f(t) = te^{4t}$

14. $f(t) = t^2e^{-2t}$

15. $f(t) = e^{-t}\sin t$

16. $f(t) = e^{t}\cos t$

17. $f(t) = t\cos t$

18. $f(t) = t\sin t$

7.2 INVERSE TRANSFORMS AND TRANSFORMS OF DERIVATIVES

THEOREM 7.2.1 Some Inverse Transforms

$$\text{(a)}\ \ 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\text{(b)}\ \ t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\},\quad n = 1, 2, 3, \ldots \qquad \text{(c)}\ \ e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\text{(d)}\ \ \sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad \text{(e)}\ \ \cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$\text{(f)}\ \ \sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad \text{(g)}\ \ \cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

EXAMPLE 1 Applying Theorem 7.2.1

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$.

SOLUTION **(a)** To match the form given in part (b) of Theorem 7.2.1, we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) To match the form given in part (d) of Theorem 7.2.1, we identify $k^2 = 7$, so $k = \sqrt{7}$. We fix up the expression by multiplying and dividing b $\sqrt{7}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

EXAMPLE 2 Termwise Division and Linearity

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

SOLUTION We first rewrite the given function of s as two expressions by means of termwise division and then use (1):

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\} \overset{\text{linearity and fixing up constants}}{=} -2\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\,\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \tag{2}$$

$$= -2\cos 2t + 3\sin 2t. \quad \leftarrow \text{parts (e) and (d) of Theorem 7.2.1 with } k=2$$

EXAMPLE 3 Partial Fractions: Distinct Linear Factors

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$.

SOLUTION There exist unique real constants A, B, and C so that

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}. \tag{5}$$

7.2.2 TRANSFORMS OF DERIVATIVES

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0).$$

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

THEOREM 7.2.2 Transform of a Derivative

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

EXAMPLE 5 Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s+4)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}. \quad (14)$$

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4},$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

EXERCISES 7.2

7.2.1 INVERSE TRANSFORMS

In Problems 1–30 use appropriate algebra and Theorem 7.2.1 to find the given inverse Laplace transform

24. $\mathscr{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\}$

29. $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$ **30.** $\mathscr{L}^{-1}\left\{\frac{6s+3}{s^4+5s^2+4}\right\}$

In Problems 31–40 use the Laplace transform to solve the given initial-value problem.

36. $y'' - 4y' = 6e^{3t} - 3e^{-t}, \quad y(0) = 1, \quad y'(0) = -1$

37. $y'' + y = \sqrt{2}\sin\sqrt{2}t, \quad y(0) = 10, \quad y'(0) = 0$

What does the Laplace Transform really tell us?

7.3 OPERATIONAL PROPERTIES I

7.3.1 TRANSLATION ON THE s-AXIS

THEOREM 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s - a).$$

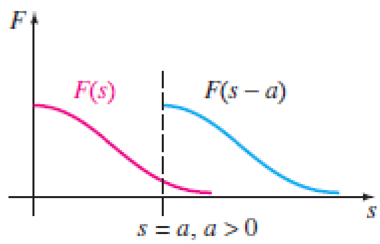


FIGURE 7.3.1 Shift on s-axis

For emphasis it is sometimes useful to use the symbolism

$$\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}|_{s \to s-a},$$

where $s \to s - a$ means that in the Laplace transform $F(s)$ of $f(t)$ we replace the symbol s wherever it appears by $s - a$.

EXAMPLE 1 Using the First Translation Theorem

Evaluate **(a)** $\mathscr{L}\{e^{5t}t^3\}$ **(b)** $\mathscr{L}\{e^{-2t}\cos 4t\}$.

SOLUTION The results follow from Theorems 7.1.1 and 7.3.1.

$$\text{(a)}\ \mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}\big|_{s\to s-5} = \frac{3!}{s^4}\bigg|_{s\to s-5} = \frac{6}{(s-5)^4}$$

$$\text{(b)}\ \mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}\big|_{s\to s-(-2)} = \frac{s}{s^2+16}\bigg|_{s\to s+2} = \frac{s+2}{(s+2)^2+16}$$

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at}f(t), \tag{1}$$

EXAMPLE 2 Partial Fractions: Repeated Linear Factors

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{s/2+5/3}{s^2+4s+6}\right\}$.

$$\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2} \tag{2}$$

and
$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}. \tag{3}$$

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2e^{3t} + 11e^{3t}t. \tag{4}$$

EXAMPLE 3 An Initial-Value Problem

Solve $y'' - 6y' + 9y = t^2e^{3t}$, $y(0) = 2$, $y'(0) = 17$.

$$\mathscr{L}\{y''\} - 6\mathscr{L}\{y'\} + 9\mathscr{L}\{y\} = \mathscr{L}\{t^2e^{3t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

Thus
$$y(t) = 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}. \tag{8}$$

$$y(t) = 2e^{3t} + 11te^{3t} + \tfrac{1}{12}t^4e^{3t}.$$

EXAMPLE 4 An Initial-Value Problem

Solve $y'' + 4y' + 6y = 1 + e^{-t}$, $\quad y(0) = 0$, $\quad y'(0) = 0$.

7.3.2 TRANSLATION ON THE t-AXIS

DEFINITION 7.3.1 Unit Step Function

The **unit step function** $\mathscr{U}(t - a)$ is defined to b

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$

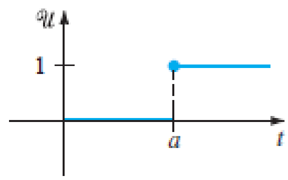


FIGURE 7.3.2 Graph of unit step function

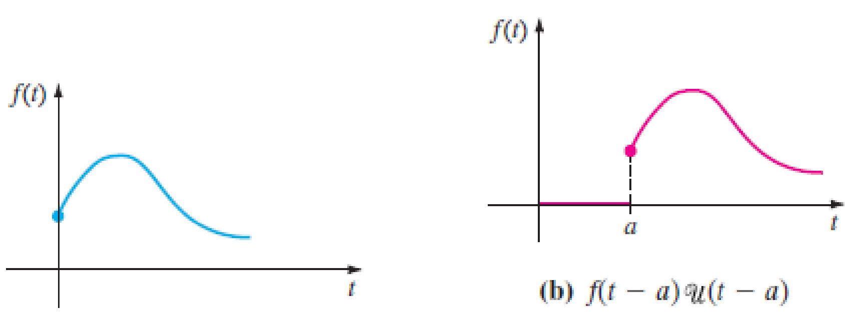
$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \tag{9}$$

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases} \tag{11}$$

THEOREM 7.3.2 Second Translation Theorem

If $F(s) = \mathscr{L}\{f(t)\}$ and a 0, then

$$\mathscr{L}\{f(t-a)\,\mathscr{U}(t-a)\} = e^{-as}F(s).$$



(a) $f(t)$, $t \ge 0$

(b) $f(t-a)\,\mathscr{U}(t-a)$

FIGURE 7.3.6 Shift on t-axis

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathscr{U}(t-a). \tag{15}$$

We often wish to find the Laplace transform of just a unit step function. This can be from either Definition 7.1.1 or Theorem 7.3.2. If we identify $f(t) = 1$ in Theorem 7.3.2, then $f(t - a) = 1$, $F(s) = \mathscr{L}\{1\} = 1/s$, and so

$$\mathscr{L}\{\mathscr{U}(t - a)\} = \frac{e^{-as}}{s}. \tag{14}$$

EXERCISES 7.3

7.3.1 TRANSLATION ON THE s-AXIS

15. $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 + 4s + 5}\right\}$

16. $\mathscr{L}^{-1}\left\{\dfrac{2s + 5}{s^2 + 6s + 34}\right\}$

17. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s + 1)^2}\right\}$

18. $\mathscr{L}^{-1}\left\{\dfrac{5s}{(s - 2)^2}\right\}$

In Problems 21–30 use the Laplace transform to solve the given initial-value problem.

24. $y'' - 4y' + 4y = t^3e^{2t}, \quad y(0) = 0, y'(0) = 0$

25. $y'' - 6y' + 9y = t, \quad y(0) = 0, y'(0) = 1$

26. $y'' - 4y' + 4y = t^3, \quad y(0) = 1, y'(0) = 0$

27. $y'' - 6y' + 13y = 0, \quad y(0) = 0, y'(0) = -3$

28. $2y'' + 20y' + 51y = 0, \quad y(0) = 2, y'(0) = 0$

29. $y'' - y' = e^t \cos t, \quad y(0) = 0, y'(0) = 0$

30. $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, y'(0) = 4$

7.3.2 TRANSLATION ON THE t-AXIS

In Problems 37–48 find either $F(s)$ or $f(t)$, as indicated.

43. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\}$

44. $\mathscr{L}^{-1}\left\{\dfrac{(1 + e^{-2s})^2}{s + 2}\right\}$

45. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\ s}}{s^2 + 1}\right\}$

46. $\mathscr{L}^{-1}\left\{\dfrac{se^{-\ s/2}}{s^2 + 4}\right\}$

7.4.1 DERIVATIVES OF A TRANSFORM

THEOREM 7.4.1 Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

EXAMPLE 2 An Initial-Value Problem

Solve $x'' + 16x = \cos 4t, \quad x(0) = 0, \quad x'(0) = 1.$

$$X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}.$$

$$x(t) = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\}$$

$$= \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t.$$

7.4.2 TRANSFORMS OF INTEGRALS

Convolution If functions f and g are piecewise continuous on the interval $[0, \infty)$, then a special product, denoted by $f * g$, is defined by the integra

$$f * g = \int_0^t f(\tau)\, g(t - \tau)\, d\tau \tag{2}$$

and is called the **convolution** of f and g. The convolution $f * g$ is a function of t.

THEOREM 7.4.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

EXAMPLE 3 Transform of a Convolution

Evaluate $\mathscr{L}\left\{\int_0^t e^{\tau}\sin(t - \tau)\, d\tau\right\}.$

SOLUTION With $f(t) = e^t$ and $g(t) = \sin t$, the convolution theorem states that the Laplace transform of the convolution of f and g is the product of their Laplace transforms:

$$\mathscr{L}\left\{\int_0^t e^{\tau}\sin(t - \tau)\, d\tau\right\} = \mathscr{L}\{e^t\} \cdot \mathscr{L}\{\sin t\} = \frac{1}{s - 1} \cdot \frac{1}{s^2 + 1} = \frac{1}{(s - 1)(s^2 + 1)}.$$