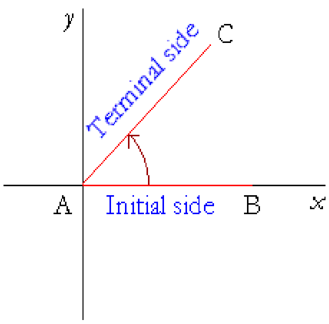
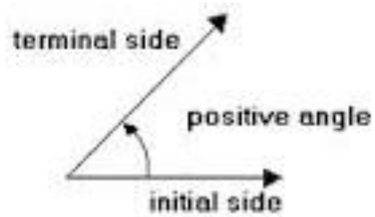
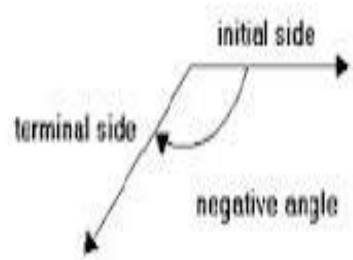
Chapter 6: Trigonometric Functions

Section: 6.1 Angles and Their Measures

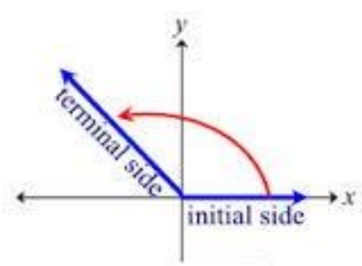
How to measure angles?

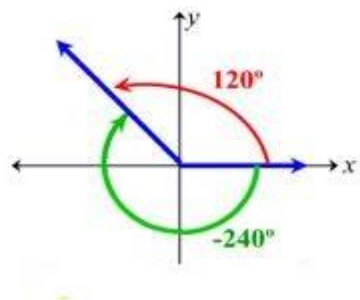


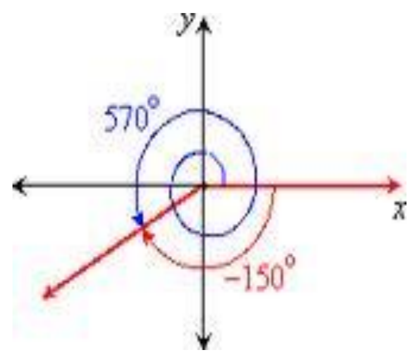


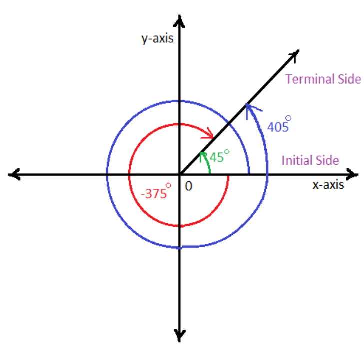


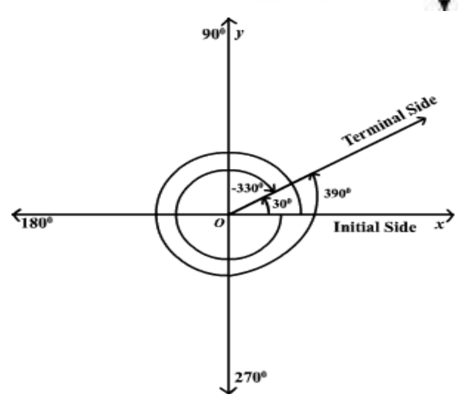
Examples of positive and negative angles:











Example:1 Draw each of the following angles:

(a) $45°$ (c) $-135°$ (e) $450°$ (g) $-645°$

(b) $645°$ (d) $-765°$ (f) $-450°$ (h) $765°$

Relation between "Radian" and "Degree":

1) $1 \text{ Revolution} = 360^{\circ} = 2\pi \text{ radians}$

2) $180° = \pi \text{ radians}$

3) $1 \text{ degree} = \frac{\pi}{180} \text{ radians}$

4) $1 \text{ radian} = \frac{180}{\pi}$ degrees

Note:

a) To convert an angle from degrees to radian, you multiply the angle by $\frac{\pi}{180}$ [relation (3)]

b) To convert an angle from radians to degrees, you multiply the angle by $\frac{180}{\pi}$ [relation (4)]

Example: 2

i) Converts from degrees to radians: $80°$, $47°$, $-71°$

ii) Converts from radians to degrees: $\frac{3\pi}{8}$ $-\frac{4\pi}{7}$ $\frac{9\pi}{11}$

Convert between Decimals and Degrees, Minutes, Seconds Measures for Angles

$$1 \text{ counterclockwise revolution} = 360^\circ$$

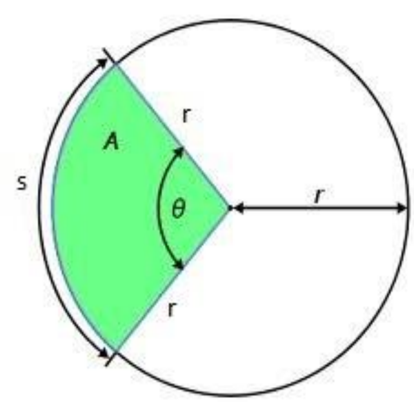
$$1^\circ = 60' \qquad \text{that is} \qquad \left(\frac{1}{60}\right)^\circ = 1'$$

$$1' = 60'' \qquad \text{that is} \qquad \left(\frac{1}{60}\right)' = 1''$$

Example 3: (a) Convert the angle 86.45° to degrees, minutes, and seconds.

(b) Convert the angle $42^\circ 9' 12''$ to a decimal in degrees.

How to find the length and the area of an Arc of a circle?



Arc Length: For a circle of radius r, a central angle of θ radians subtends an arc whose length s is given by $s = r\theta$

Sector Area: The area A of the sector of a circle of radius r formed by a central angle of θ radian is $A = \frac{1}{2}r^2\theta$

Note: To find the length of an Arc s or the area of a Sector A, you MUST convert the angle θ to radians if it is given in degrees.

Example: 4

(a) Find the length of the arc of a circle of radius 7 meters subtended by a central angle of 2.25 radians.

(b) Find the length of the arc of a circle of radius 5 feet subtended by a central angle of 74°.

(c) Find the length of the arc of a circle of radius 16 inches subtended by a central angle of $25°$.

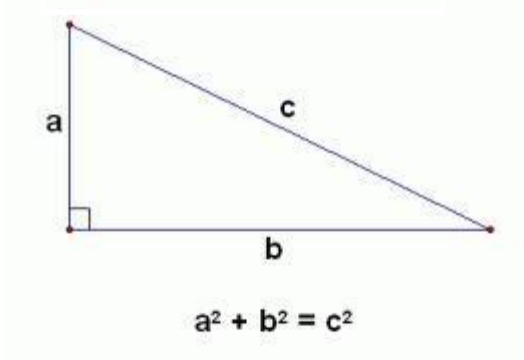
Example:5

(a) Find the area of the sector of a circle of radius 7 feet formed by an angle of $30°$. Round the answer to two decimal places.

(b) Find the area of the sector of a circle of radius $23\ meters$ formed by an angle of $60°$. Round the answer to two decimal places.

Right Triangle Trigonometry

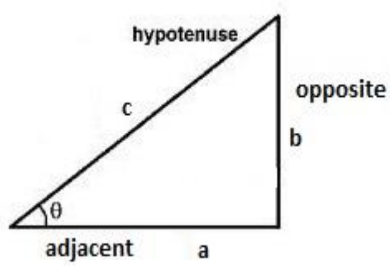
Pythagorean Theorem:



Definition: The six ratios of the lengths of the sides of a right triangle are called **trigonometric functions of acute angles** and are defined as follows:

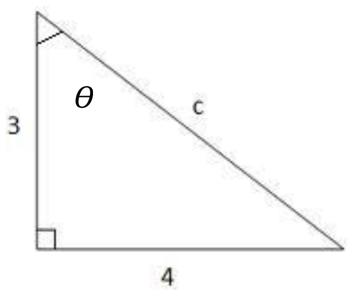
Functions Name	Abbreviation	Functions Name	Abbreviation
sine of θ	$\sin\theta$	cosecant of θ	$\csc\theta$
cosine of θ	$\cos\theta$	secant of θ	$\sec\theta$
tangent of θ	$\tan\theta$	cotangent of θ	$\cot\theta$

From a right triangle, the six **trigonometric functions of acute angles a**re defined as follows:



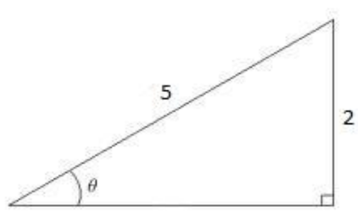
$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

Example:1 Find the value of each of the trigonometric functions of the angle θ in the figure given below



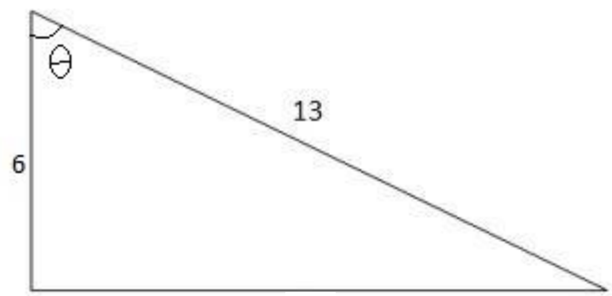
$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

Example:2 Find the value of each of the trigonometric functions of the angle θ in the figure given below



$\sin\theta = \dfrac{opposite}{hypotenuse}$	$\cos\theta = \dfrac{adjacent}{hypotenuse}$	$\tan\theta = \dfrac{opposite}{adjacent}$
$\csc\theta = \dfrac{hypotenuse}{opposite}$	$\sec\theta = \dfrac{hypotenuse}{adjacent}$	$\cot\theta = \dfrac{adjacent}{opposite}$

Example:3 Find the value of each of the trigonometric functions of the angle θ in the figure given below



$\sin\theta = \dfrac{opposite}{hypotenuse}$	$\cos\theta = \dfrac{adjacent}{hypotenuse}$	$\tan\theta = \dfrac{opposite}{adjacent}$
$\csc\theta = \dfrac{hypotenuse}{opposite}$	$\sec\theta = \dfrac{hypotenuse}{adjacent}$	$\cot\theta = \dfrac{adjacent}{opposite}$

Example: 4 If $\sin\theta = \frac{3}{7}$, then find all trigonometric functions of the angle θ.

<u>Example: 5</u> If $\cot\theta = \frac{2}{5}$, then find all trigonometric functions of the angle θ.

Fundamental Identities:

Reciprocal Identities:

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

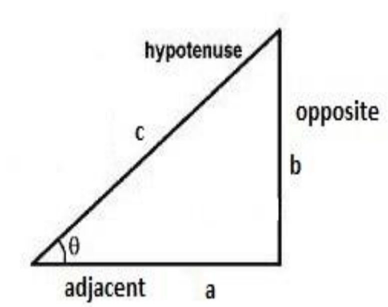
Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Proof of Pythagorean Identities:



Pythagorean Theorem: $a^2 + b^2 = c^2$

$$\sin\theta = \frac{b}{c} \quad and \quad \cos\theta = \frac{a}{c}$$

$$sin^2\theta + cos^2\theta = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{b^2 + a^2}{c^2} = \frac{c^2}{c^2} = 1$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad (\div\cos^2\theta)$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \qquad (\div\sin^2\theta)$$

$$\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

Even-Odd Identities:

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Example: 6 Given $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$. Use identities to find the exact value of each of the four remaining trigonometric functions of the acute angle θ.

Example: 7 Given $\sin\theta = \frac{2}{3}$, $\cos\theta = \frac{\sqrt{5}}{3}$. Use identities to find the exact value of each of the four remaining trigonometric functions of the acute angle θ.

Example: 8 Given $\sin\theta = \frac{5}{7}$, $\cos\theta = \frac{-2\sqrt{6}}{7}$. Use identities to find the exact value of each of the four remaining trigonometric functions of the angle θ.

Example: 9 Given $\cos\theta = \frac{1}{3}$. Use identities to find the exact value of each of the four remaining trigonometric functions of the acute angle θ.

Example :10 Use the fundamental identities to find the exact values of the followings:

(a) $sin^2\, 31^{\circ} + cos^2\, 31^{\circ} =$

(b) $sec^2\, 12^{\circ} - tan^2\, 12^{\circ} =$

(c) $tan\, 31^{\circ} - \frac{sin\, 31^{\circ}}{cos\, 31^{\circ}} =$

Definition: Complementary Angle: Two acute angles are called complementary if their sum is a right angle, or 90°.

Complementary Angle Theorem: Co-functions of complementary angles are equal.

θ in Degrees:

$\cos(90^{\circ} - \theta) = \sin\theta$ $\qquad$ $\sin(90^{\circ} - \theta) = \cos\theta$

$\cot(90^{\circ} - \theta) = \tan\theta$ $\qquad$ $\tan(90^{\circ} - \theta) = \cot\theta$

$\csc(90^{\circ} - \theta) = \sec\theta$ $\qquad$ $\sec(90^{\circ} - \theta) = \csc\theta$

θ in Radians: $90^\circ\ degrees = \frac{\pi}{2}\ Radians$

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta$$

$$\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$$

For example:

$$\sin\theta = \cos(90^\circ - \theta) \quad \rightarrow \sin 40^\circ = \cos 50^\circ$$
$$\cos\theta = \sin(90^\circ - \theta) \quad \rightarrow \cos 15^\circ = \sin 75^\circ$$
$$\tan\theta = \cot(90^\circ - \theta) \quad \rightarrow \tan 30^\circ = \cot 60^\circ$$

Example :11 Use the fundamental identities to find the exact values of the followings:

(a) $sin\ 17^\circ - cos\ 73^\circ =$

(b) $\frac{cos\ 9^\circ}{sin\ 81^\circ} =$

(c) $tan\ 21^\circ - \frac{cos\ 69^\circ}{sin\ 69^\circ} =$

Example: 12 Given that $\sin 30^\circ = \frac{1}{2}$. Use trigonometric identities to find the exact values of the followings:

(a) $\cos 60^\circ$ (b) $\cos^2 30^\circ$

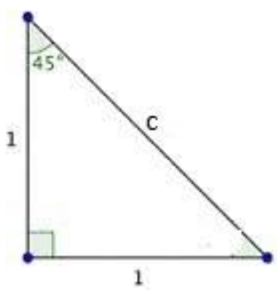
(c) $\csc\frac{\pi}{6}$ (d) $\sec\frac{\pi}{3}$

Computing the Values of Trigonometric Functions of acute Angles

degrees	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0°$	0	0	1	0	–	1	–
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$90°$	$\frac{\pi}{2}$	1	0	–	1	–	0

Finding exact values of the Trigonometric Functions of $\frac{\pi}{4} = 45^{\circ}$:

Use the right triangle given bellow:



<u>Example: 1</u> Do not use a Calculator: Find the exact value of the expression

(a) $\csc 45^0 - \cos 45^0$

(b) $\sec 30^0 - \cot 30^0$

(c) $\sin 30^0 - \cos 60^0$

Example: 2 Do not use a Calculator: Find the exact value of the expression

(a) $\sin\frac{\pi}{3} - tan\frac{\pi}{3}$

(b) $\csc\frac{\pi}{6} - sec\frac{\pi}{6}$

(a) $\cos\frac{\pi}{4} - cot\frac{\pi}{4}$

Example: 3 Use a Calculator to approximate the value of the expression

(a) $\sin\frac{2\pi}{5} - tan\frac{\pi}{7}$

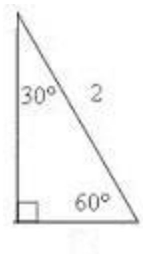
(b) $sin12^{\circ} + cos12^{\circ} =$

Example: 4 A 22 $-$foot extension ladder leaning against a building makes a 70^{0} angle with the ground. How far up the building does the ladder touch?

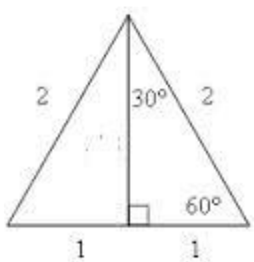
[Please come to my office if you want to learn how to find the exact values of the Trigonometric Functions of $\frac{\pi}{6} = 30^{\circ}$ and $\frac{\pi}{3} = 60^{\circ}$]

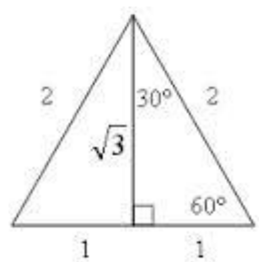
Hint: Finding exact values of the Trigonometric Functions of $\frac{\pi}{6} = 30^{\circ}$ and $\frac{\pi}{3} = 60^{\circ}$

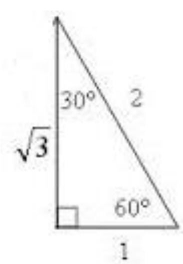
Use the following right Triangle:



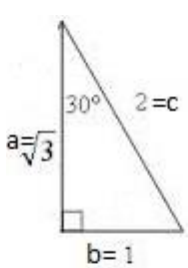
Place a triangle next to the triangle above that is congruent to it. The resulting triangle is an equilateral triangle, so each side is of length 2.



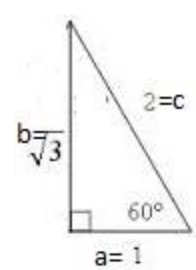




For $\frac{\pi}{6} = 30^{\circ}$:

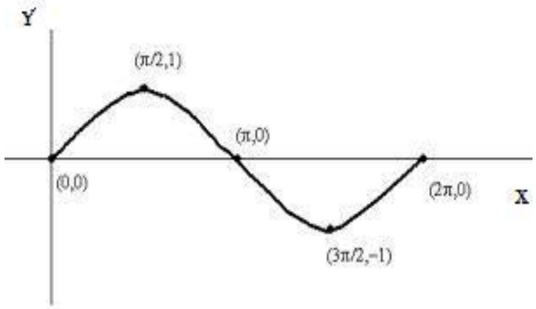


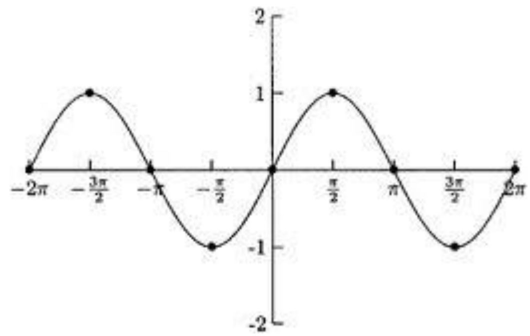
For $\frac{\pi}{3} = 60^{\circ}$:



Section: 6.4 Graph of Sine and Cosine Functions

The Graph of The Sine function: $y = sinx$





Properties of the sine function $\boldsymbol{y = sinx}$:

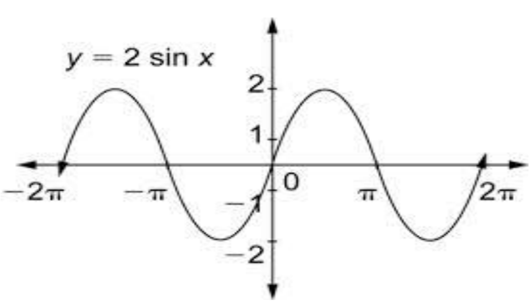
1. Domain= All real numbers, Range =$[-1, 1]$.

2. The Sine function is an odd function, as the symmetry of the graph with respect to the origin indicates, that is , $\boldsymbol{sin(-x) = -sinx.}$
3. The Sine function is periodic with period $T = 2\pi$.

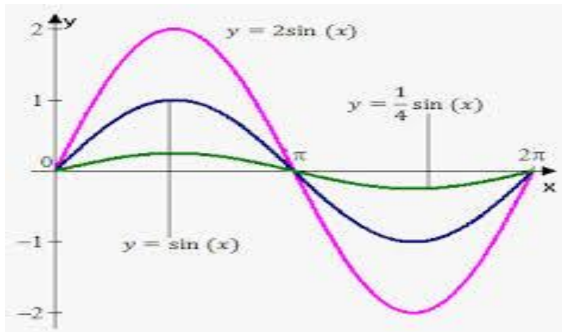
4. The x −intercepts are $\cdots, -3\pi, -2\pi, -\pi, \ 0, \pi, 2\pi, 3\pi, \cdots$; and the y −intercept is 0.
5. The maximum value is 1 and occurs at $x = \cdots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2},$ $\cdots$; and the minimum value is -1 and occurs at $x = \cdots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2},$ $\frac{11\pi}{2}, \cdots$.

Graph Functions of the form: $y = A\sin(\omega x)$

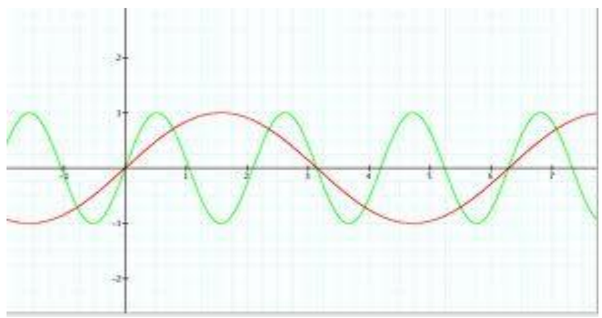
1. **The Graph of the Function $\boldsymbol{y = 2\sin x}$**



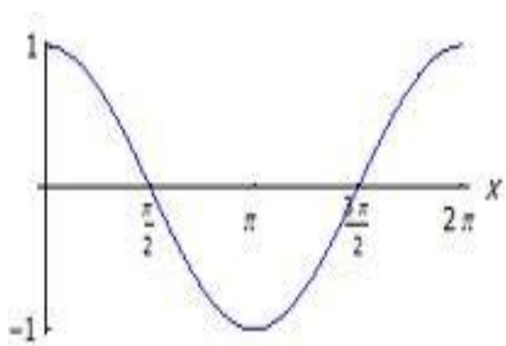
2. **<u>Comparison of the graphs</u>:** $y = 2\sin x,\ y = \sin x$ and $y = \frac{1}{4}\sin x$

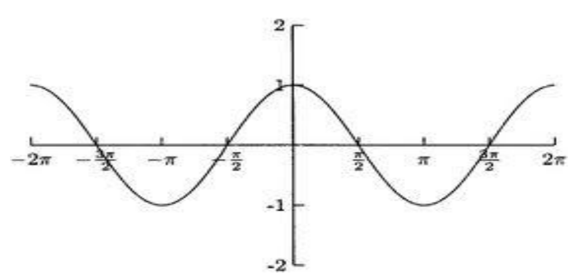


3. **<u>Comparison of the graphs</u>:** $y = \sin x$ and $y = \sin(3x)$



<u>The Graph of The Cosine function:</u> $y = \cos x$





Properties of the sine function $y = cos\ x$:

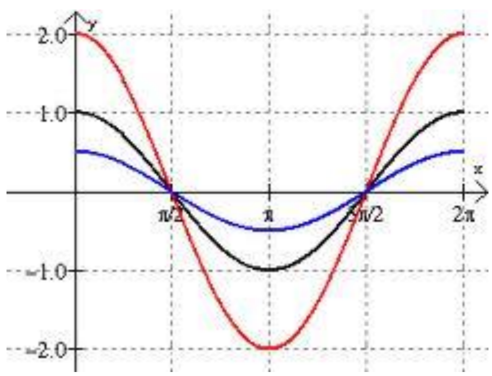
1. Domain= All real numbers, Range =$[-1, 1]$.

2. The Sine function is an even function, as the symmetry of the graph with respect to the y −axis indicates, that is , $\boldsymbol{cos(-x) = cos\ x.}$

3. The Cosine function is periodic with period $T = 2\pi$.
4. The x −intercepts are $\cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$; and the y −intercept is 1.

5. The maximum value is 1 and occurs at are $x = \cdots, -2\pi,\ 0,\ 2\pi,\ 4\pi,\ 6\pi, \cdots$; and the minimum value is -1 and occurs at $x = \cdots, -\pi,\ \pi,\ 3\pi,\ 5\pi, \cdots$..

<u>Comparison of the graphs</u>: $y = 2\ cosx,\ y = cos\ x$ and $y = \frac{1}{2}\cos x$



1. **<u>Definition:</u>** The number $|A|$ is called the **amplitude** of the functions $y = A\sin x$ or $y = A\cos x$

2. **<u>Definition:</u>** If $\omega > 0$, the **amplitude and period** of the functions $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ are

3. **Amplitude $= |A|$ and Period $= T = \frac{2\pi}{\omega}$**

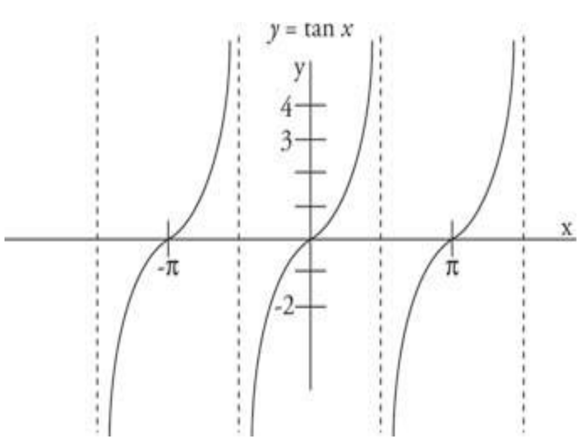
Example: 1 Find the Amplitude and the period of the function $y = 3\sin(5x)$.

Example: 2 Find the Amplitude and the period of the function $y = -4\cos(3x)$.

Section: 6.5 Graphs of the Tangent, Cotangent, Cosecant and Secant Functions

1. **Graph of the tangent function:** $y = \tan x = \dfrac{\sin x}{\cos x}$

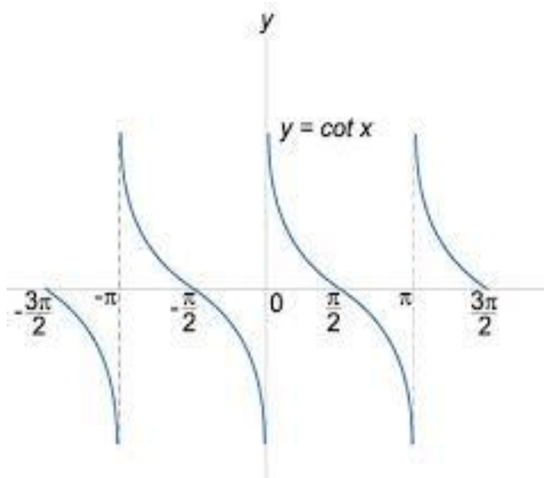
Note: Vertical asymptotes for the graph of the tangent functions are the vertical line where $\cos x = 0$, that is, for $x = \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$. So, the tangent graph does not intersect the vertical lines given by $\cdots, x = -\frac{3\pi}{2},\ x = -\frac{\pi}{2},\ x = \frac{\pi}{2},\ x = \frac{3\pi}{2},\ x = \frac{5\pi}{2},\ \cdots$.



1. Domain = All real numbers, except odd integer multiples of $\frac{\pi}{2}$
2. Range = $(-\infty, \infty)$
3. Tangent function is an odd function
4. Period T = π
5. x –intercepts: $\cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

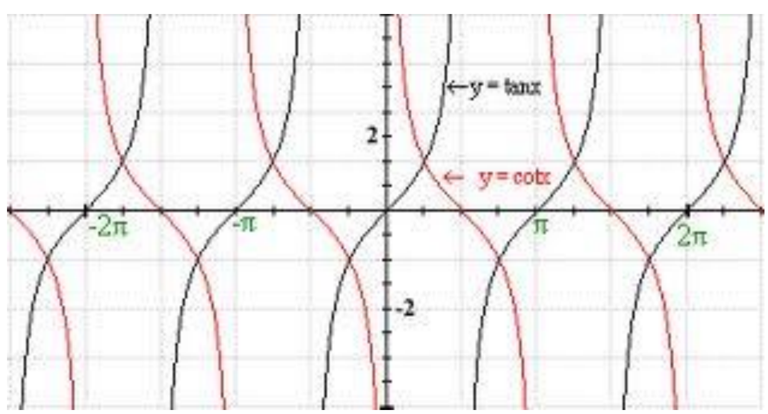
2. <u>Graph of the Cotangent function:</u> $y = \cot x = \frac{\cos x}{\sin x}$

<u>Note:</u> Vertical asymptotes for the graph of the cotangent functions are the vertical line where $\sin x = 0$, that is, for $x = \cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$. So, the cotangent graph does not intersect the vertical lines given by $\cdots,\ x = -2\pi,\ x = -\pi,\ x = 0,\ x = \pi,\ x = 2\pi,\ \cdots$.



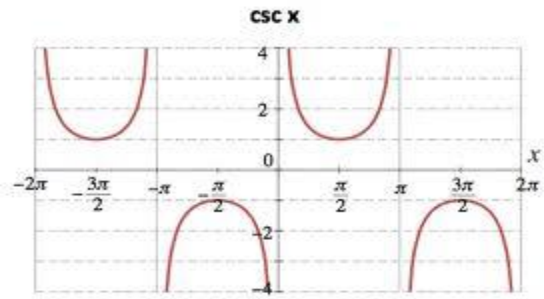
1. Domain = All real numbers, except integer multiples of π
2. Range = $(-\infty, \infty)$
3. Cotangent function is an odd function
4. Period T = π
5. x –intercepts are $\cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$.

3. <u>Comparison of the graphs</u>: $\boldsymbol{y = tan\ x}$ **and** $\boldsymbol{y = \cot x}$



4. Graph of the Cosecant function: $y = \csc x = \dfrac{1}{\sin x}$

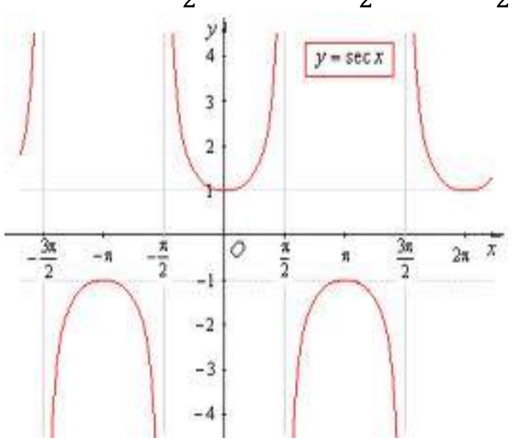
Note: Vertical asymptotes for the graph of the cosecant functions are the vertical line where $\sin x = 0$, that is, for $x = \cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$. So, the cosecant graph does not intersect the vertical lines given by $\cdots,\ x = -2\pi,\ x = -\pi,\ x = 0,\ x = \pi,\ x = 2\pi,\ \cdots$.



1. Domain = All real numbers, except integer multiples of π
2. Range = $(-\infty, -1] \cup [1, \infty)$
3. Cosecant function is an odd function
4. Period T = 2π
5. No x –intercepts

5. Graph of the Secant function: $y = \sec x = \dfrac{1}{\cos x}$

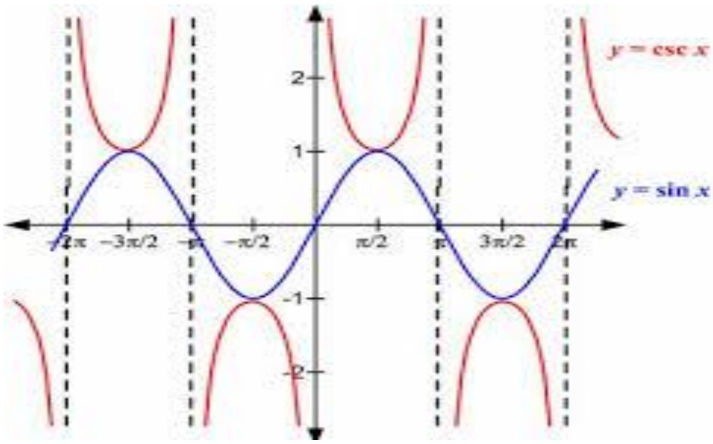
Note: Vertical asymptotes for the graph of the secant functions are the vertical line where $\cos x = 0$, that is, for $x = \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$. So, the secant graph does not intersect the vertical lines given by $\cdots, x = -\frac{3\pi}{2},\ x = -\frac{\pi}{2},\ x = \frac{\pi}{2},\ x = \frac{3\pi}{2},\ x = \frac{5\pi}{2},\ \cdots$.



1. Domain = All real numbers, except odd integer multiples of $\frac{\pi}{2}$
2. Range = $(-\infty, -1] \cup [1, \infty)$
3. Secant function is an even function
4. Period T = 2π
5. No x –intercepts

6. <u>Comparison of the graphs</u>:

(a) $y = sin\ x$ **and** $y = \csc x$



(b) $y = cos\ x$ **and** $y = \sec x$