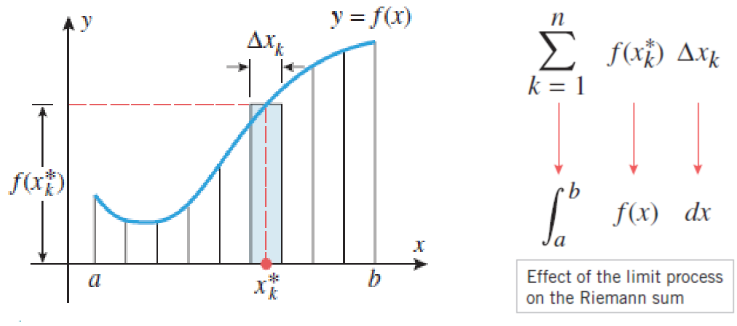
MAT-130, Lecture-9

Chapter 6: Applications of Integration

Section 6.1 Area Between Two Curves

MAT-120: A REVIEW OF RIEMANN SUMS (AREA UNDER THE CURVE $y = f(x)$ on $[a, b]$)



$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

Here $n \to \infty$.

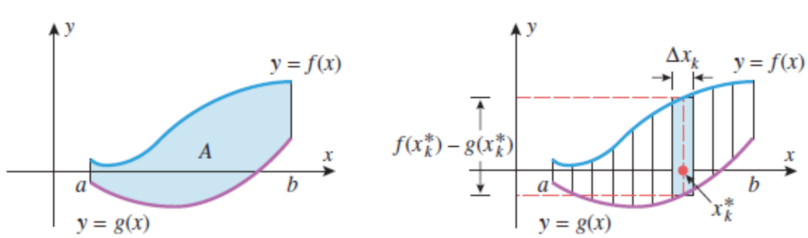
Note that $g(x) = 0$ is the lower boundary. Hence, $A = \int_a^b [f(x) - 0]dx = \int_a^b f(x)\,dx$

AREA BETWEEN $y = f(x)$ AND $y = g(x)$

6.1.1 FIRST AREA PROBLEM Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$

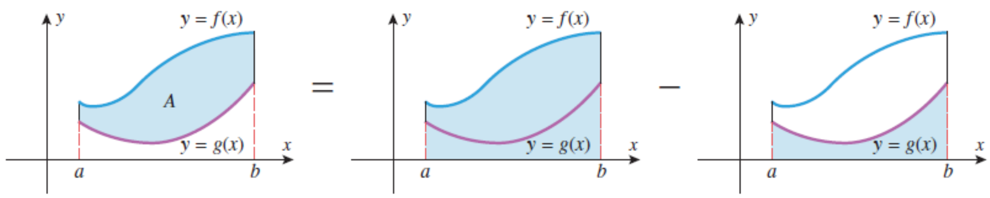


(a) (b)

6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

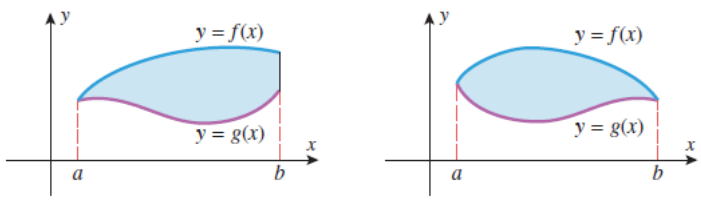
$$A = \int_a^b [f(x) - g(x)]\,dx \qquad (1)$$

GEOMERTICAL INTERPRETATION



Area between f and g | Area below f | Area below g

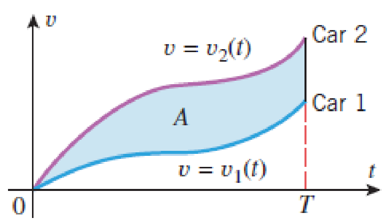
POINT AS A BOUNDARY



The left-hand boundary reduces to a point. | Both side boundaries reduce to points.

Example 1 [Worked out example from book]

The figure below shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area A between the curves over the interval $0 \leq t \leq T$.



Note: Graphing will help you a lot to know the boundaries.

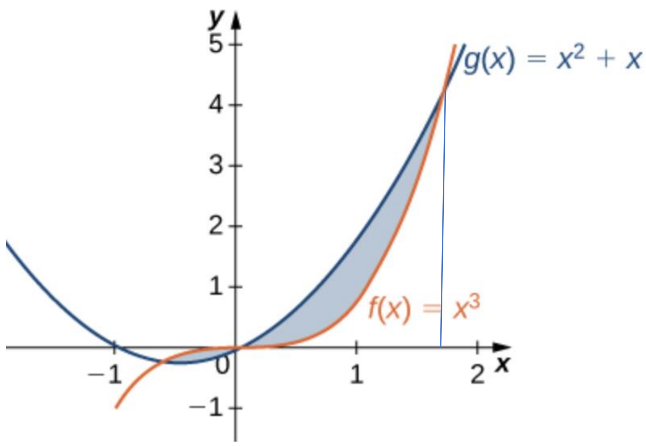
Formula: Area $= \int_a^b [\textbf{Upper boundary} - \textbf{Lower boundary}]\,dx$

Example 2

Find the area of the region bounded by the curves $f(x) = x^3$ and $g(x) = x^2 + x$ in the **first quadrant.**

Solution: Given $f(x) = x^3$ and $g(x) = x^2 + x$.

Note that $g(x) = x^2 + x$ is a parabola opening upward, with $x-$intercepts $x = 0, -1$ since $g(x) = x(x + 1)$.



To find the points of intersection, set $f(x) = g(x) \Rightarrow x^3 - x^2 - x = 0$

$$\Rightarrow x(x^2 - x - 1) = 0$$

$$\Rightarrow x = 0, \quad x^2 - x - 1 = 0$$

$$x^2 - x - 1 = 0 \ \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

That is, $x = 0, \ \frac{1+\sqrt{5}}{2}, \ \frac{1-\sqrt{5}}{2}$

Interval: $I = [a, b] = \left[0, \ \frac{1+\sqrt{5}}{2}\right]$

The area of the region is

$$A = \int_0^{\frac{1+\sqrt{5}}{2}} [g(x) - f(x)] \ dx = \int_0^{\frac{1+\sqrt{5}}{2}} [x^2 + x - x^3] dx, \qquad \text{please complete}$$

Example 3

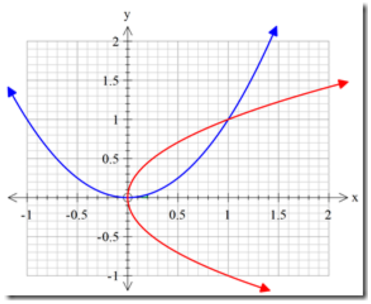
Find the area of the region **bounded** by the curves $y = x^2$ and $y^2 = x$.

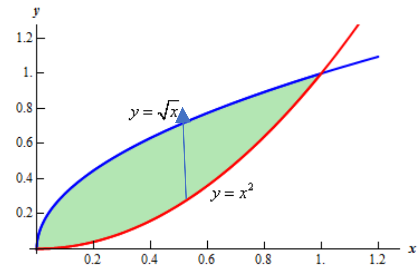
Solution: Given $y = x^2$ and $y^2 = x$

From $y^2 = x$, we get $y = \pm\sqrt{x}$.

Here we get $y = \sqrt{x}$, the half-portion of the red parabola which is above the $x-$axis [in first quadrant]

And $y = -\sqrt{x}$, the half-portion of the red parabola which is below the $x-$axis [in forth quadrant]





To find the interval which is given by the points of intersection,
set $x^2 = \sqrt{x} \quad \Rightarrow x^4 = x$

$\Rightarrow x^4 - x = 0 \quad \Rightarrow x(x^3 - 1) = 0$

We get $x = 0, 1$. Interval $= [0, 1]$
The area of the region is

$$A = \int_0^1 [\sqrt{x} - x^2] dx$$

$$= \int_0^1 \left[x^{\frac{1}{2}} - x^2\right] dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$$

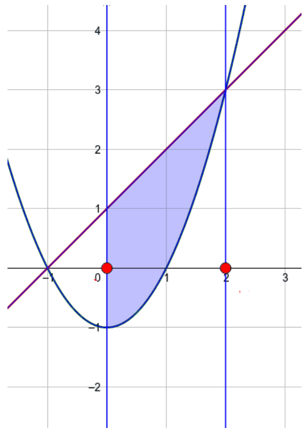
$$= \frac{2}{3} - \frac{1}{3} - 0 = \frac{1}{3}\ unit^2$$

Example 4

Find the area of the region **bounded** by the curves $y = x^2 - 1$ and $y = x + 1$ on the interval $[0,2]$.
Solution: Given $y = x^2 - 1$ and $y = x + 1$ on the interval $[0,2]$.

To find the points of intersection:

Set $x^2 - 1 = x + 1 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow \quad x = -1, 2.$



The area of the region is

$$A = \int_0^2 [(x+1) - (x^2 - 1)]dx$$

$$= \int_0^2 [-x^2 + x + 2]dx = -\frac{8}{3} + 2 + 4 = -\frac{8}{3} + 6 = \frac{10}{3}\ unit^2$$

Example 5

Find the area of the region **bounded** by the curves $y = \sin x$ and $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution:

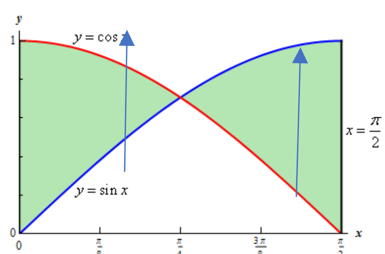
Given $y = \sin x$ and $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

To find the point of intersection, set $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$, which belongs to the given interval.

Intervals are $I_1 = \left[0, \frac{\pi}{4}\right]$ and $I_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

On $I_1 = \left[0, \frac{\pi}{4}\right]$, $y = \cos x$ is the upper boundary

On $I_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $y = \sin x$ is the upper boundary.



$$\text{Area A} = \int_0^{\frac{\pi}{4}} [\cos x - \sin x]\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x]\, dx = ??$$

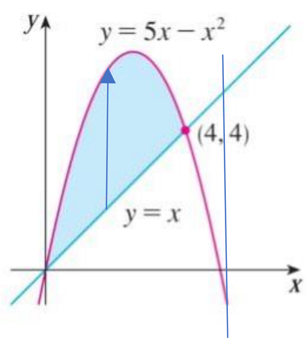
Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x –axis, connecting the top and bottom boundaries.

Step 2. The y –coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$.

Example 6 (a) Find the area of the region bounded by the functions $y = 5x - x^2$ and $y = x$ on the interval $[0,5]$.



Points of intersection $x = 0, 4$

Note that $[0,5] = [0,4] \cup [4,5]$

On the interval $[0,4]$: Upper boundary $y = 5x - x^2$ and lower bounbdary $y = x$

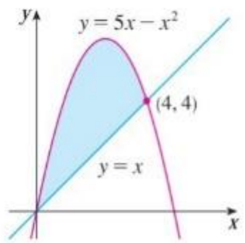
On the interval $[4,5]$: Upper boundary $y = x$ and lower bounbdary $y = 5x - x^2$

$$\text{Area } A = \int_0^4 [5x - x^2 - x]dx + \int_4^5 [x - (5x - x^2)]\, dx$$

Complete!

Note: $\int_4^4 F(x)\, dx = 0$ for any function $F(x)$.

Example 6 (b) Find the area of the shaded region given below.



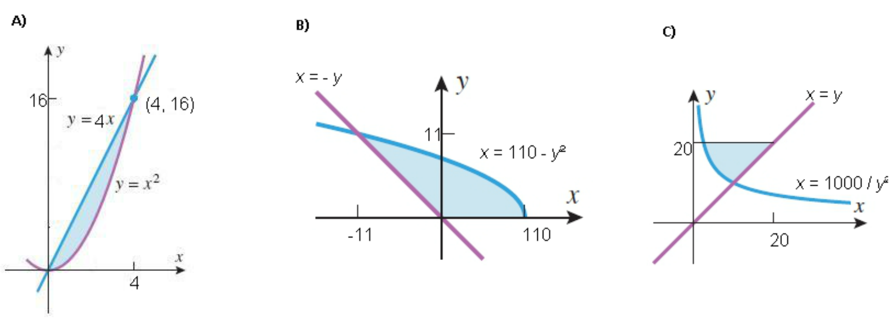
Solution:

$$A = \int_{0}^{4} [(5x - x^2) - x] \; dx.$$

Please submit Example 7 by tomorrow, 23rd March, 2021.

Example 7

Find the area of the given graphs in the shaded regions.

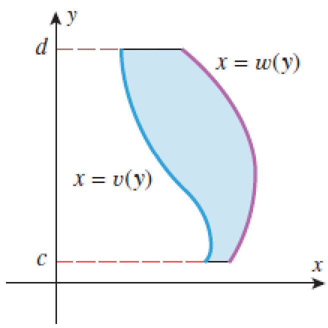


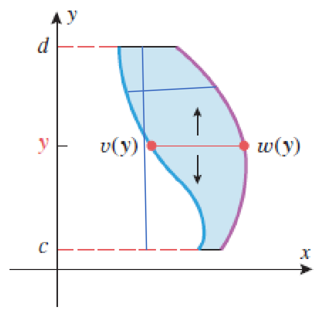
<u>**Lecture-10**</u>

REVERSING THE ROLES OF x AND y : $x = w(y)$, $x = v(y)$

6.1.4 AREA FORMULA If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$





<u>Formula:</u> If the boundaries are given by functions of y, then Area =$\int_c^d [\text{Right boundary} - \text{Left boundary}]dy$

<u>Finding the Limits of Integration for the Area Between Two Curves</u>

Step 1. Sketch the region and then draw a horizontal line segment through the region at an arbitrary point y on the y —axis, connecting the left and right boundaries.

Step 2. The x —coordinate of the right endpoint of the line segment sketched in Step 1 will be $w(y)$, the left one $v(y)$, and the length of the line segment will be $w(y) - v(y)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment top and then bottom. The bottommost position at which the line segment intersects the region is $y = c$ and the topmost is $y = d$.

Example 8

Find the area of the region **bounded** by the curves $y^2 = x$ and $y = x - 2$.

[NOTE: If we consider these curves as functions of x, then we get three curves given by

From the equation $y^2 = x \quad \Rightarrow \quad y = \pm\sqrt{x}$

$y = x - 2, \;\; y = \sqrt{x}\,, \;\; y = -\sqrt{x}, \;\; INTERVAL = [0, 4]$.

$y = \sqrt{x} \rightarrow$ Upper part of the parabola which is above the x-axis

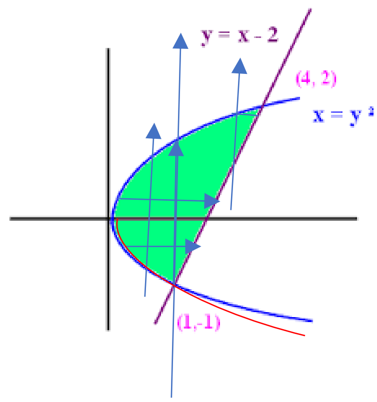
$y = -\sqrt{x} \rightarrow$ Lower part of the parabola which is below the x-axis

$[0, 4] = [0, 1] \cup [1,4]$

On the $[0,1]$: Upper boundary $y = \sqrt{x}$,   Lower boundary $y = -\sqrt{x}$

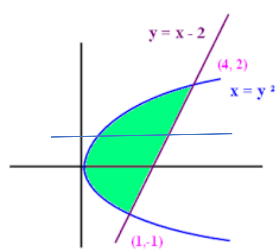
On the $[1,4]$: Upper boundary $y = \sqrt{x}$,   Lower boundary $y = x - 2$

$$A = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\, dx + \int_1^4 [\sqrt{x} - (x-2)] dx \; = ???$$



Alternative method:

Find the area of the region **bounded** by the curves $y^2 = x$ and $y = x - 2$.



Note that $y^2 = x$ is not a function of x, but $x = y^2$ and $x = y + 2$ are functions of y.

The area of the region is

$$A = \int_{-1}^{2} [(y+2) - y^2]\; dy$$

Example 9

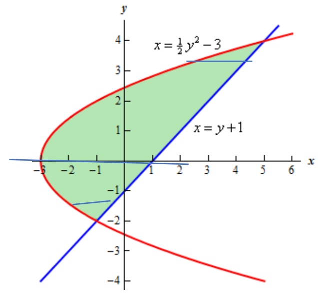
Find the area of the region **bounded** by the curves $x = \frac{1}{2}y^2 - 3$ and $x = y + 1$.

Solution: The region is bounded on the left by $x = v(y) = \frac{1}{2}y^2 - 3$ and on the right by $x = w(y) = y + 1$.

To find the interval, left find the points of intersection.

Set $\frac{1}{2}y^2 - 3 \; = y + 1 \;\; \Rightarrow y^2 - 2y - 8 = 0 \;\; \Rightarrow (y-4)(y+2) = 0$, $i.e.$, $y = -2, 4$.

Interval $= [c, d] = [-2, 4]$.

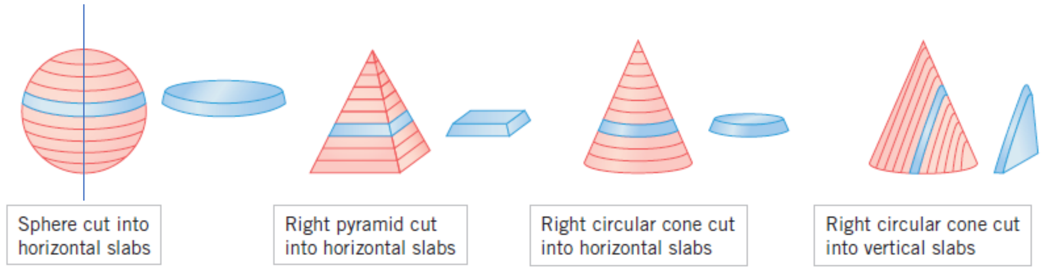


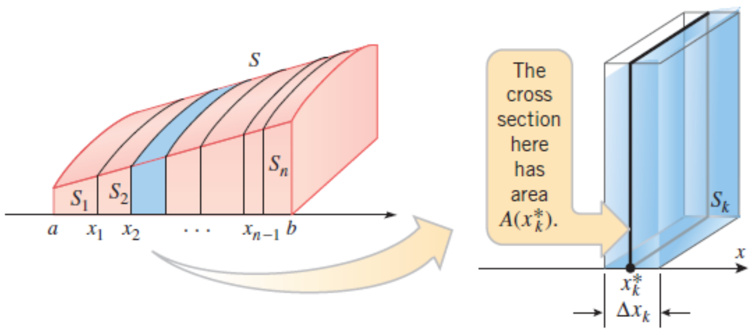
Area $=\int_c^d[\textbf{Right boundary} - \textbf{Left boundary}]\,dy$

$$=\int_{-2}^{4}\left[(y+1)-\left(\frac{1}{2}y^2-3\right)\right]\,dy = \int_{-2}^{4}\left[-\frac{1}{2}y^2+y+4\right]\,dy$$

$$=\left[-\frac{1}{6}y^3+\frac{1}{2}y^2+4y\right]_{-2}^{4}=\frac{1}{6}\left[-y^3+3y^2+24y\right]_{-2}^{4}=\frac{1}{6}[-72+36+144]=-12+6+24=18\,unit^2.$$

SECTION 6.2: VOLUME BY SLICING: DISC AND WASHER METHOD





Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

In summary, we have the following result.

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

Summary:

The volume of the solid S is given by

$V = \int_a^b (\text{Area of the cross} - \text{section})\ dx$ Or $V = \int_c^d (\text{Area of the cross} - \text{section})\ dy$.

DAY-11

Practice problems:

6.1: 4, 6, all odds: 7 - 25, Analyze true-false: 27 - 30

6.2: Odds: 1- 25, 31, 33

6.3: Odds: 1- 20, 27, 29, 34*, 35**

6.4: Odds: 3-9, 15, 17(a) - 17(c), odds: 27 - 32, 34***(Challenging)

6.5: Odds: 1 - 17, 30**, 35*, 37*

Note: For 35 and 37, you need to use the formula for the surface area for parametric function. Please see the attached file slide_33.jpg for formula. Also see the example for Surface Area.

6.9: 11-43 (odds), 67, 69

→7.8: 3, 4, 5, 10, 15, 17, 23

Assignment:

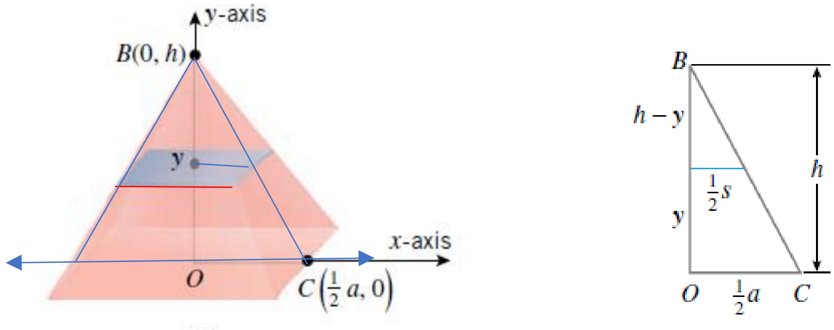
6.1: 4, 6, all odds: 7 - 25, Analyze true-false: 27 - 30

Due: On 1st April, Wednesday, 2021

Quiz-2: Syllabus 6.1-6.3, Date: 12th April, 2021.

Example 1

Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length $\boldsymbol{a}$.



(a) (b)

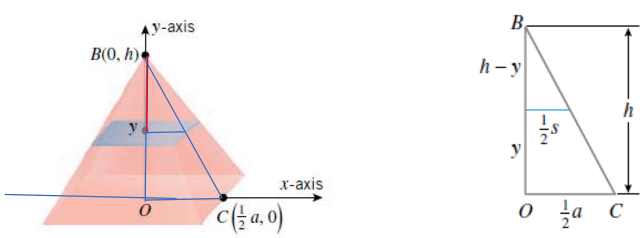
Project the Pyramid along the y-axis, placing the height of the pyramid along the axis with the center of the base at the origin.

Now, take any cross-section of the pyramid at any $y,\ 0 \le y \le h$. The cross-section is a square that is perpendicular to the y −axis.

$$\text{Volume} \quad V = \int_0^h (\text{Area of the cross} - \text{section})\ dy \ldots\ldots\ldots (1)$$

Let the length of a side of the cross-section be s. Then the area of the cross-section is

$$A(y) = s^2 \ldots\ldots\ldots (2)$$



(a) (b)

By the similar triangle property on the triangles

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \Rightarrow \frac{s}{a} = \frac{h-y}{h}. \quad \text{Hence} \quad s = \frac{a}{h}\,(h-y)$$

From equation (2): Area of the cross-section is $A(y) = s^2 = \left[\frac{a}{h}\,(h-y)\right]^2 = \frac{a^2}{h^2}(h-y)^2$

Then from (1): volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$

$$= \int_0^h \frac{a^2}{h^2}(h-y)^2 \; dy$$

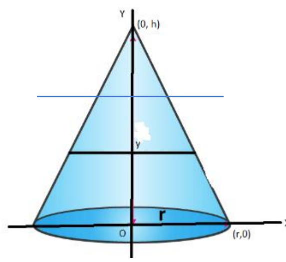
$$= \frac{a^2}{h^2} \int_0^h [h^2 - 2hy + y^2] \; dy$$

$$= \frac{a^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$

$$V = \frac{1}{3}a^2 h \quad unit^3.$$

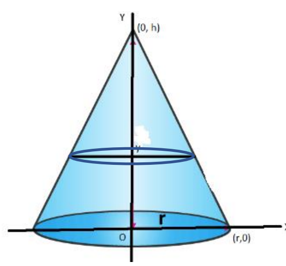
Example 2

Derive the formula for the volume of a **right circular cone** whose altitude is h and whose base is a circle of radius r.



Solution: Project the right circular cone placing the height of the cone along the y −axis with the center of the base at the origin.

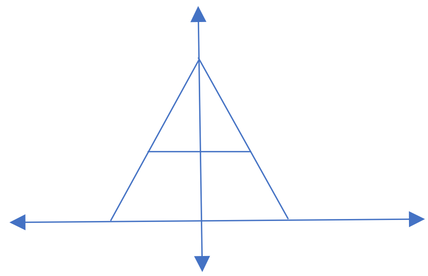
Now, take any cross-section of the cone at any $y,\ 0 \le y \le h.$ The cross-section is a disk that is perpendicular to the y −axis.



$$\text{Volume } V = \int_0^h (\text{Area of the crosssection}) \; dy \; \ldots\ldots \; \ldots \; (1)$$

Let the radius of the cross-section be r_1. Then the area of the cross-section is

$$A(y) = \pi r_1^2 \ldots\ldots\ldots (2)$$



By the similar triangle property on the triangles

$$\frac{2r_1}{2r} = \frac{h-y}{h} \quad \Rightarrow \frac{r_1}{r} = \frac{h-y}{h}. \quad \text{Hence} \quad r_1 = \frac{r}{h}\,(h-y)$$

From equation (2): Area of the cross-section is $A(y) = \pi r_1^2 = \pi\left[\frac{r}{h}\,(h-y)\right]^2 = \pi\frac{r^2}{h^2}(h-y)^2$

Then volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$

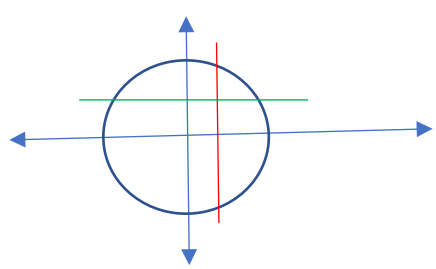
$$= \pi\int_0^h \frac{r^2}{h^2}(h-y)^2\ dy$$

$$= \frac{\pi r^2}{h^2}\int_0^h [h^2 - 2hy + y^2]\ dy$$

$$= \frac{\pi r^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$

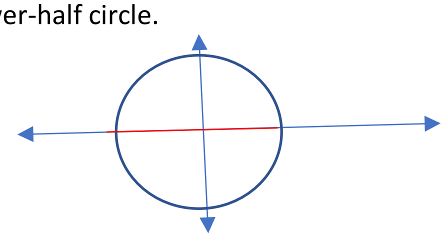
$$V = \frac{1}{3}\pi r^2 h \quad unit^3.$$

Note: Note that $x^2 + y^2 = 1$ is neither a function of x nor a function of y.



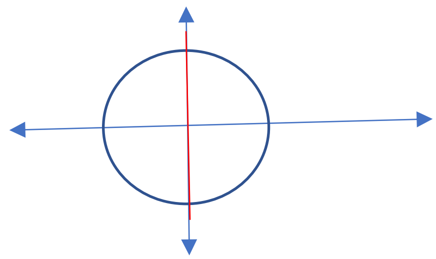
If we solve the equation $x^2 + y^2 = 1$ for y, then we get $y = \pm\sqrt{1-x^2}$

Here $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ are two functions of x, where $y = \sqrt{1-x^2}$ is the upper-half circle and $y = -\sqrt{1-x^2}$ is the lower-half circle.



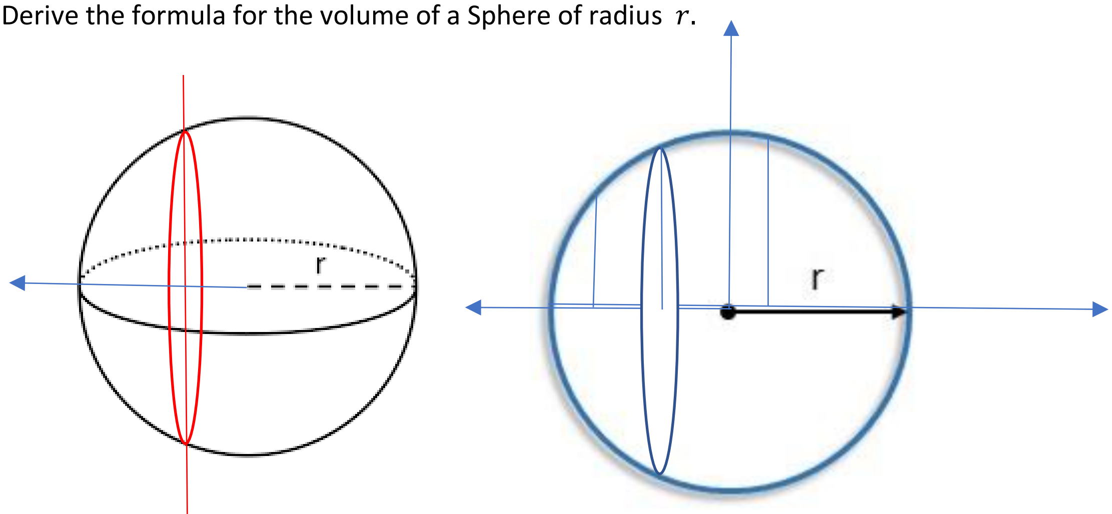
Similarly, if we solve the equation $x^2 + y^2 = 1$ for x, then we get $x = \pm\sqrt{1 - y^2}$

Here $x = \sqrt{1 - y^2}$ and $x = -\sqrt{1 - y^2}$ are two functions of y, where $x = \sqrt{1 - y^2}$ is the right-half circle and $x = -\sqrt{1 - y^2}$ is the left-half circle.

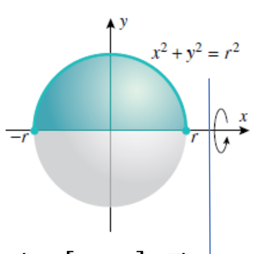


Example 3

Derive the formula for the volume of a Sphere of radius r.



The projection of the sphere on the xy −plane is a disk of radius r, bounded by the circle $x^2 + y^2 = r^2$. But $x^2 + y^2 = r^2$ is not a function. The upper-half circle represents a function of x given by $y = \sqrt{r^2 - x^2}$.



Interval $= [-r, r]$, The cross-section at any x, $-r \le x \le r$, is a disk of radius, **say r_1 .**

Here the radius is $r_1 = \sqrt{r^2 - x^2} \;\; - 0 = \;\; \sqrt{r^2 - x^2}$.

Area of the cross-section $\quad A(x) = \pi r_1^2 = \pi(r^2 - x^2)$.

volume $\;V = \int_a^b (\text{Area of the cross} - \text{section})\; dx$

$$V = \int_{-r}^{r} \pi(r^2 - x^2)\ dx = \frac{4}{3}\pi r^3 \quad \text{unit}^3.$$

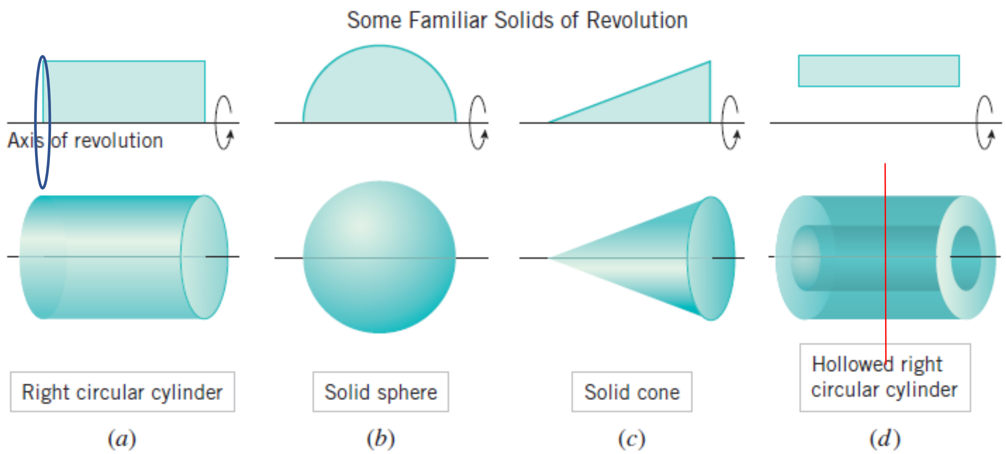
[Note: $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$

Lower-half circle $y = -\sqrt{r^2 - x^2}$,

Upper-half circle $y = \sqrt{r^2 - x^2}$]

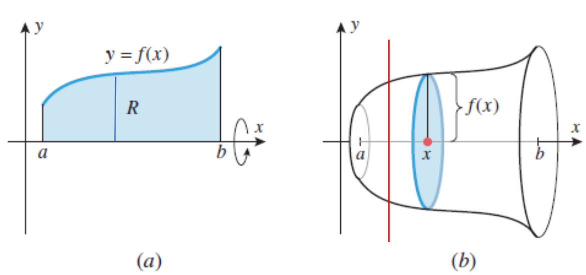
SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type



VOLUMES BY DISKS PERPENDICULAR TO THE x –AXIS

Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x –axis, and on the sides by the vertical lines $x = a$ and $x = b$. Then the volume of the solid of revolution that is generated by revolving the region R about the x –axis is given by



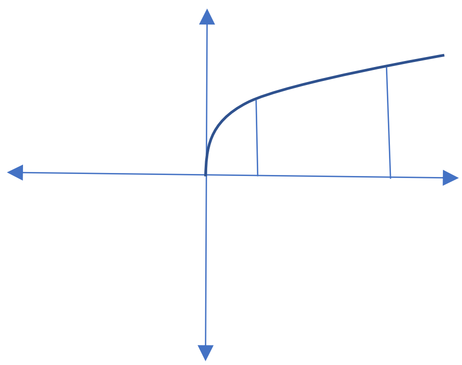
The cross-section is a disk of radius $r = f(x)$, and the cross-section is perpendicular to the x —axis.

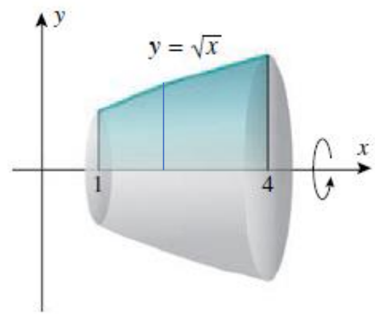
Hence, the volume is

$$V = \int_a^b \pi[f(x)]^2 dx.$$

Example 4 Find the volume of the solid that is obtained when the region **under the curve** $y = \sqrt{x}$ **over** the interval $[\mathbf{1}, \mathbf{4}]$ is revolved about the x —axis.

<u>Solution:</u>





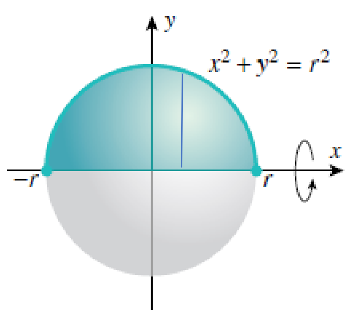
The cross-section at any x, $1 \le x \le 4$, is a disk of radius $r = \sqrt{x}$.

The Volume

$$V = \int_a^b \pi[f(x)]^2 dx = \int_1^4 \pi[\sqrt{x}]^2 dx = \int_1^4 \pi x dx = \left[\frac{\pi x^2}{2}\right]_1^4 = \frac{15\pi}{2} \text{ unit}^3$$

Homework

Example 5 Find the volume of the solid generated by revolving the circle $x^2 + y^2 = r^2$ about the x —axis.



The upper-half disk, which is the region, is bounded above by the function $y = \sqrt{r^2 - x^2}$, below by the x —axis, and on the left and right by the vertical lines $x = -r$ and $x = r$.

Here the cross-section at any $x, -r \leq x \leq r$, is a disk of radius $r_1 = \sqrt{r^2 - x^2}$.

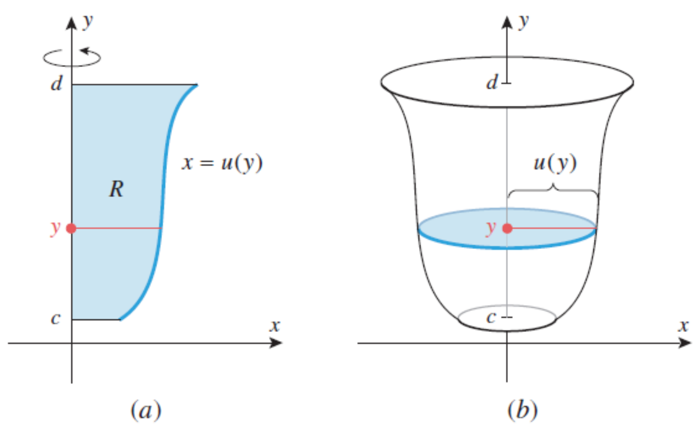
Complete this exercise!

VOLUMES BY DISKS PERPENDICULAR TO THE y –AXIS

Let $x = u(y)$ be continuous and nonnegative on $[c, d]$, and let R be the region that is bounded on the right by $x = u(y)$, on the left by the y –axis, and at the bottom and top by the horizontal lines $y = c$ and $y = d$. Then the volume of the solid of revolution that is generated by revolving the region R about the y –axis is given by

$$V = \int_c^d \pi[u(y)]^2 dy$$

Note that the cross-section is a disk of radius $r = u(y)$ that is perpendicular to the y –axis.



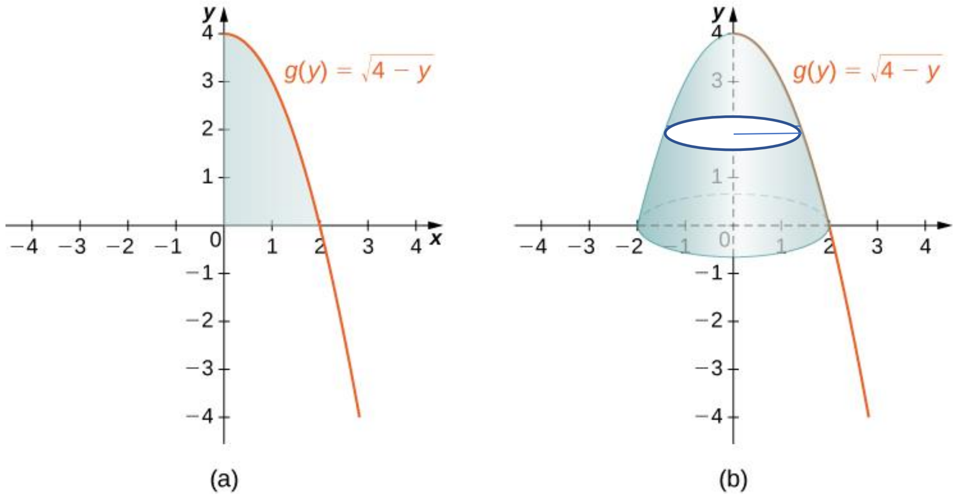
Disks

Information comes from the origin R . Interval, radius, height come from the region.

Example 6 **[Complete !!!]**

Find the volume of the solid that is obtained when the region R is revolved about the $\boldsymbol{y}$ **–axis,** where R is bounded by the curve $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{y}) = \sqrt{4 - y}$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]



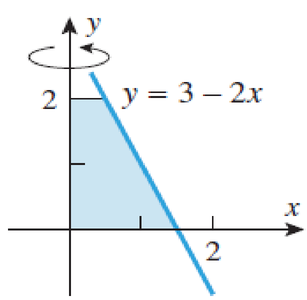
$$V = \int_c^d \pi[u(y)]^2 dy = \int_0^4 \pi\left[\sqrt{4-y}\right]^2 dy$$

complete!

Example 7

Find the volume of the solid that is obtained when the region R revolved about the y —axis, where R is bounded by the curve $y = 3 - 2x$, $y = 2$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]



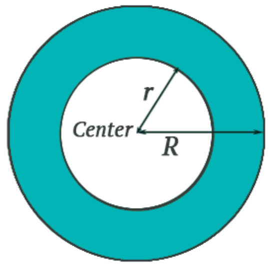
Interval, Shape of the cross-section, Area of the cross-section, Variable of the function that gives you the area of the cross-section.

$$\boldsymbol{y = 3 - 2x}\ \textbf{. That is,} \qquad \boldsymbol{x = \frac{1}{2}(3 - y)}$$

Radius of the cross-section is $= \frac{1}{2}(3 - y)$

$$I = [0, 2], \text{Cross} - \text{section is a disk}, \qquad A(y) = \pi \left(\frac{3}{2} - \frac{y}{2}\right)^2$$

<u>Annulus or ring or washer</u>



Inner radius = Radius of the inner circle $= r$

Outer radius = Radius of the outer circle $= R$

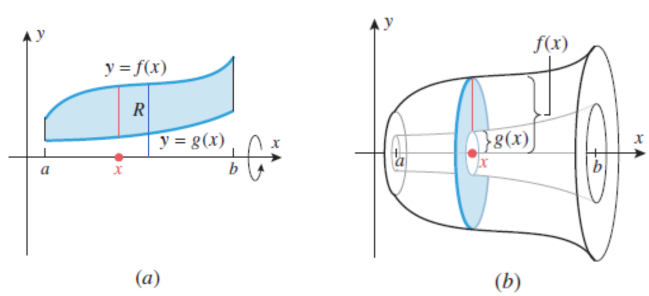
Area of the Annulus $= \pi R^2 - \pi r^2$.

In, Washer method by slicing, the cross-section would be an annulus.

VOLUMES BY WASHERS PERPENDICULAR TO THE x −AXIS

Let f and g be continuous and **non-negative** on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. The volume of the solid of revolution that is generated by revolving the region R about the x −axis is given by

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\, dx$$

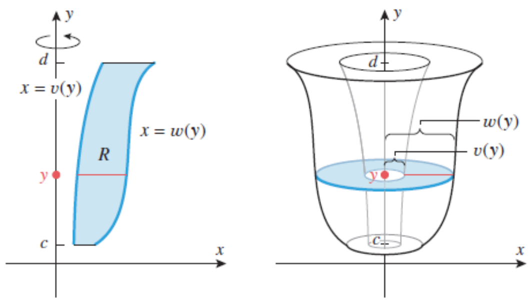


VOLUMES BY WASHERS PERPENDICULAR TO THE y −AXIS

Let w and v be continuous and nonnegative on $[c, d]$, and suppose $w(y) \geq v(y)$ for all y in the interval $[c, d]$. Let R be the region that is bounded on the right by $x = w(y)$, on the left by $x = v(y)$, and at the bottom and top by the lines $y = c$ and $y = d$. The volume of the solid of revolution that is generated by revolving the region R about the y −axis is given by

$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy$$

Washers

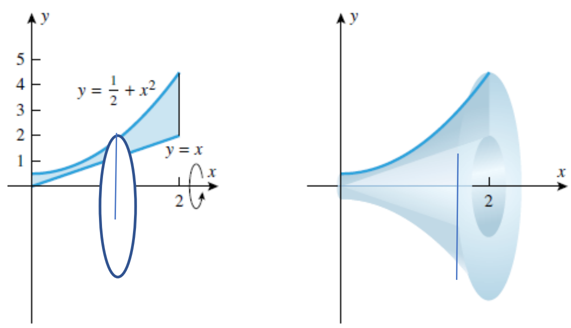


(a) (b)

Washers

Examples

Example 1 Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x −axis.



Unequal scales on axes

Region defined by f and g

(a)

The resulting solid of revolution

(b)

The interval $I = [0, 2]$

The inner radius $r = x$

The outer radius $R = \frac{1}{2} + x^2$

The area of the cross section at any x: $A(x) = \pi R^2 - \pi r^2 = \pi\left[\left(\frac{1}{2} + x^2\right)^2 - x^2\right]$

The volume of the solid is

$$V = \int_0^2 A(x)\ dx\ = \int_0^2 \pi \left[\left(\frac{1}{2} + x^2\right)^2 - x^2\right]\ dx$$

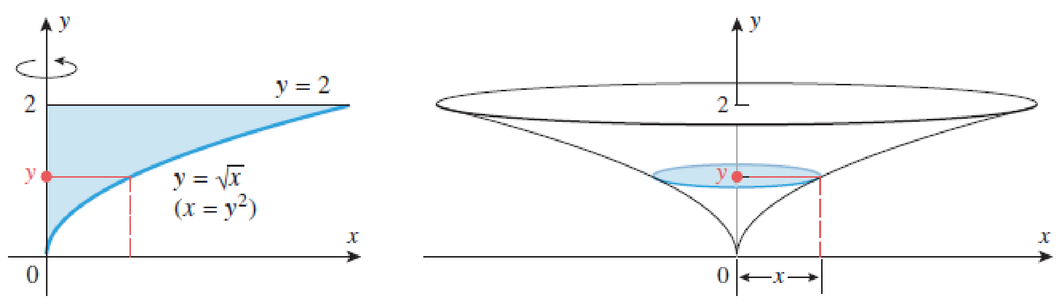
$$= \pi \int_0^2 \left[\frac{1}{4} + x^2 + x^4 - x^2\right]\ dx$$

$$= \pi \int_0^2 \left[\frac{1}{4} + x^4\right]\ dx\ = \pi \left[\frac{1}{4}(2) + \frac{1}{5}(2)^5\right]$$

$$= \frac{1}{20}[10 + 128]\pi = \frac{69}{10}\pi \quad unit^3$$

Example 2

(a) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x},\ \ y = 2$, and $x = 0$ is evolved about the y −axis.



The cross-section is a disk (revolving about a boundary).

Radius of the disk= $R = y^2$

Interval $I = [0, 2]$

Area of the cross-section $A(y) = \pi R^2 = \pi (y^2)^2 = \pi y^4$

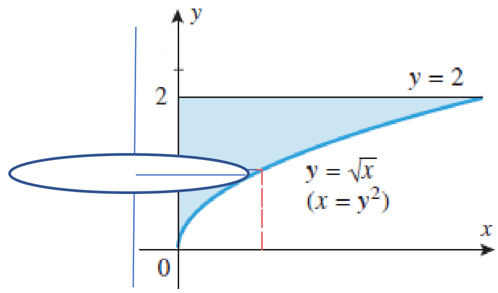
The volume of the solid is

$$V = \int_0^2 A(y)\ dy\ = \int_0^2 \pi y^4\ dy \quad \text{Complete!!}$$

Example 2

(b) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is evolved about the line $\boldsymbol{x = -1}$.

Solution: R: $y = \sqrt{x} \Rightarrow x = y^2$, $x = 0$, $y = 2$. Axis of the solid: $\boldsymbol{x = -1}$.



Interval $= [0, 2]$

Inner radius $r = 0 - (-1) = 1$

Outer radius $R = y^2 - (-1) = y^2 + 1$

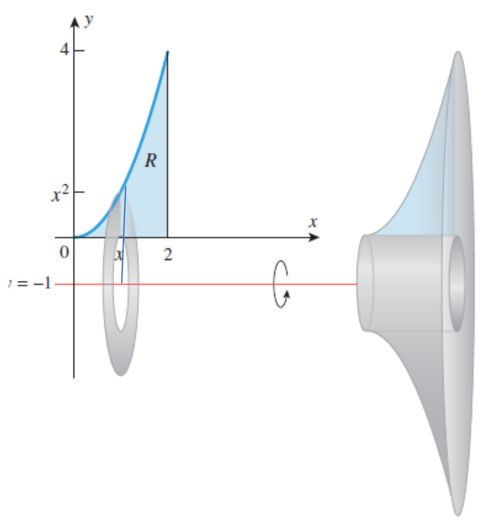
$V = \int_0^2 A(y)\, dy = \int_0^2 \pi[(y^2+1)^2 - 1^2]\, dy$; Please complete!!

Example 3

Find the volume of the solid generated when the region **under** the curve $y = x^2$ **over** the interval $[0, 2]$ is rotated about the line $\boldsymbol{y = -1}$.

Solution: The region under the curve $y = x^2$ over the interval $[0, 2]$, that is, R is bounded by

$$R: \quad y = x^2, \qquad y = 0, \quad x = 0, \qquad x = 2.$$



Interval $= [0, 2]$

Inner radius $r = 0 - (-1) = 1$

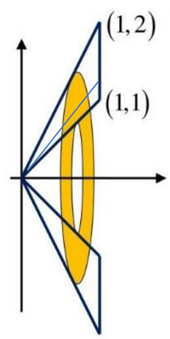
Outer radius $R = x^2 - (-1) = x^2 + 1$

$V = \int_0^2 A(x)\ dx\ = \int_0^2 \pi[(x^2+1)^2 - 1^2]\ dx$; Please complete!!

Example 4

Find the volume of the solid generated when the region R revolves about the x −axis, where R is bounded by the lines $y = 2x$, $y = x$ **over** the interval $[0, 1]$.

Solution:



$$V = \int_0^1 A(x)\ dx\ = \int_0^1 \pi[(2x)^2 - x^2]\ dx$$

$$= \pi \int_0^1 [4x^2 - x^2]\ dx$$

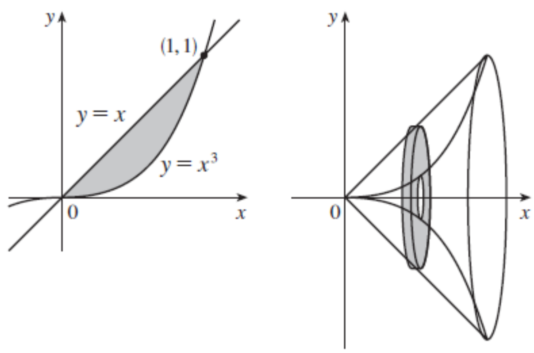
$$= \pi \int_0^1 3x^2\ dx$$

$$= \pi\ unit^3$$

Example 5

Find the volume of the solid generated when the region R revolves about the x −axis, where R is the region in the first quadrant bounded by the lines $y = x^3$ and $y = x$.

Solution: Given region R: $y = x^3$ and $y = x$.



To find the interval, set : $x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = 0, 1, -1.$

Interval $I = [0, 1]$.

Inner Radius $r = x^3$

Outer Radius $R = x$

Area of the cross section $A(x) = \pi[x^2 - (x^3)^2] = \pi\,[x^2 - x^6]$

$$V = \int_0^1 A(x)\ dx = \int_0^1 \pi[x^2 - x^6]\,dx = \frac{4}{21}\pi\ \ unit^3$$

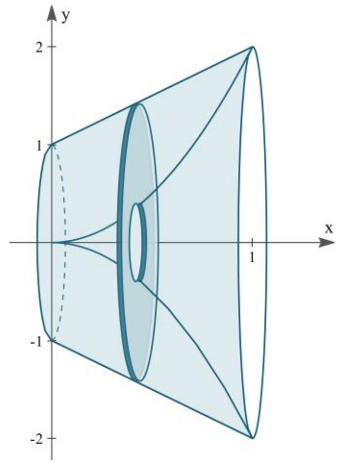
Example 6

Find the volume of the solid generated when the region R revolves about the x –axis, where R is the region **in the first quadrant** bounded by the lines $y = 2x^2$, $x = 0$ and $y = x + 1$.

Solution: Given region R: $y = x^2$, $y = x + 1$ and $x = 0$.

For the interval, find the point of intersection in first quadrant. Set $2x^2 = x + 1 \Rightarrow 2x^2 - x - 1 = 0$.

Hence, $x = -\frac{1}{2}$, 1. Interval $I = [0,\ 1]$



Complete!

Example 7

Find the volume of the solid generated when the region R revolves about the y −axis, where R is the region bounded by $y = \sqrt{x}$ and $y = x$.

Solution: Given region R: $y = \sqrt{x}$ and $y = x$.

For the interval, set $\sqrt{x} = x \Rightarrow x^2 - x = 0$.

Hence, $x = 0,\ 1$. Interval $I = [0,\ 1]$