Chapter # 02
(The Derivative)

2.5 Derivative of Trigonometric Functions: The main objective of this section is to obtain formulas for the derivatives of the six basic trigonometric functions.

Formula:

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \quad \text{and} \quad \lim_{h\to 0}\frac{1-\cos h}{h}=0$$

Question: Using the definition, find $f'(x)$ for $f(x)=\sin x$.

Solution: Given $f(x)=\sin x \quad \therefore\ f(x+h)=\sin(x+h)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right]\\
&= \lim_{h\to 0}\left[\cos x\left(\frac{\sin h}{h}\right)-\sin x\left(\frac{1-\cos h}{h}\right)\right]\\
&= \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}-\lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{1-\cos h}{h}\\
&= \left(\lim_{h\to 0}\cos x\right)(1)-\left(\lim_{h\to 0}\sin x\right)(0)\\
&= \lim_{h\to 0}\cos x=\cos x
\end{aligned}$$

Similarly:

(i) $\frac{d}{dx}[\cos x]=-\sin x$

(ii) $\frac{d}{dx}[\tan x]=sec^2 x$

(iii) $\frac{d}{dx}[\sec x]=\sec x\tan x$

(iv) $\frac{d}{dx}[\cot x]=-cosec^2 x$

(v) $\frac{d}{dx}[cosec\ x]=-cosec\ x\cot x$

Example 2: Find $\frac{dy}{dx}\ \ if\ \ y=\frac{\mathrm{si}}{1+\cos x}$

Solution: Given, $y=\frac{\sin x}{1+\cos x}$

$$\begin{aligned}\frac{dy}{dx}&=\frac{(1+\cos x)\cdot\frac{d}{dx}[\sin x]-\sin x\cdot\frac{d}{dx}[1+\cos x]}{(1+\cos x)^2}\\&=\frac{(1+\cos x)(\cos x)-(\sin x)(-\sin x)}{(1+\cos x)^2}\\&=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}=\frac{\cos x+1}{(1+\cos x)^2}=\frac{1}{1+\cos x}\end{aligned}$$

Example 3: Find $f''\left(\frac{\pi}{4}\right)\ \ if\ \ f(x)=\sec x.$

Solution: Given, $f(x)=\sec x$

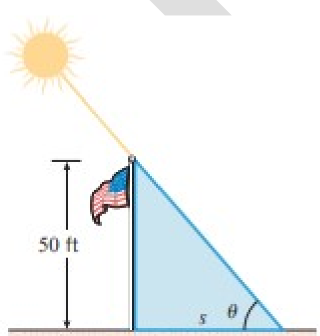
$$\therefore\ \ f'(x)=\sec x\tan x$$

$$\begin{aligned}f''(x)&=\sec x\cdot\frac{d}{dx}[\tan x]+\tan x\cdot\frac{d}{dx}[\sec x]\\&=\sec x\cdot\sec^2 x+\tan x\cdot\sec x\tan x\\&=\sec^3 x+\sec x\tan^2 x\end{aligned}$$

Thus,

$$\begin{aligned}f''(\pi/4)&=\sec^3(\pi/4)+\sec(\pi/4)\tan^2(\pi/4)\\&=(\sqrt{2})^3+(\sqrt{2})(1)^2=3\sqrt{2}\ \blacktriangleleft\end{aligned}$$

Example 4: On a sunny day, a **50** ft flagpole casts a shadow that changes with the angle of elevation of the Sun. Let ***s*** be the length of the shadow and **θ** the angle of elevation of the Sun (following figure). Find the rate at which the length of the shadow is changing with respect to **θ** when $\boldsymbol{\theta}=\mathbf{45^0}$. Express your answer in units of **feet/degree**.



Solution: The variables $\boldsymbol{s}$ and $\boldsymbol{\theta}$ are related by $\mathbf{tan}\,\boldsymbol{\theta}=\frac{\mathbf{50}}{\boldsymbol{s}}$ or, equivalently,

$s = 50 \cot\theta$, here **θ** is measured in radians

Therefore,

$$\frac{ds}{d\theta} = -50 \csc^2\theta$$

which is the rate of change of shadow length with respect to the elevation angle **θ** in units of feet/radian.

When $\theta = 45°$ (or equivalently $\theta = \pi/4$ radians), we obtain

$$\left.\frac{ds}{d\theta}\right|_{\theta=\pi/4} = -50\csc^2(\pi/4) = -100 \text{ feet/radian}$$

Converting radians (rad) to degrees (deg) yields

$$-100\,\frac{\text{ft}}{\text{rad}} \cdot \frac{\pi}{180}\frac{\text{rad}}{\text{deg}} = -\frac{5}{9}\pi\,\frac{\text{ft}}{\text{deg}} \approx -1.75 \text{ ft/deg}$$

Home Work: Exercise 2.5: Problem No. 1-28, 31, 32

2.6 The Chain Rules: In this section we will derive a formula that expresses the derivative of a composition $\boldsymbol{f \circ g}$ in terms of the derivatives of $\boldsymbol{f}$ and $\boldsymbol{g}$. This formula will enable us to differentiate complicated functions using known derivatives of simpler functions.

Theorem (The Chain Rule): If $\boldsymbol{g}$ is differentiable at $\boldsymbol{x}$ and $\boldsymbol{f}$ is differentiable at $\boldsymbol{g(x)}$, then the composition $\boldsymbol{f \circ g}$ is differentiable at $\boldsymbol{x}$. Moreover, if

$$y = f(g(x)) \quad and \quad u = g(x)$$

then $y = f(u)$ *and*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1: Find $\frac{dy}{dx}$ if $y = \cos(x^3)$.

Solution: Let $\boldsymbol{u = x^3}$ and express $\boldsymbol{y}$ as $\boldsymbol{y = \cos u}$.

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \frac{d}{du}[\cos u]\cdot\frac{d}{dx}[x^3]\\ &= (-\sin u)\cdot(3x^2)\\ &= (-\sin(x^3))\cdot(3x^2) = -3x^2\sin(x^3)\end{aligned}$$

Example 2: Find $\frac{dw}{dt}$ $\;if\;\; w = \tan x \quad and\;\; x = 4t^3 + t.$

Solution: In this case the chain rule computations take the form

$$\begin{aligned}\frac{dw}{dt} &= \frac{dw}{dx}\cdot\frac{dx}{dt}\\ &= \frac{d}{dx}[\tan x]\cdot\frac{d}{dt}[4t^3+t]\\ &= (\sec^2 x)\cdot(12t^2+1)\\ &= [\sec^2(4t^3+t)]\cdot(12t^2+1) = (12t^2+1)\sec^2(4t^3+t)\end{aligned}$$

An Alternative Version of The Chain Rule: The derivative of $\boldsymbol{f(g(x))}$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{Derivative of the outside function evaluated at the inside function}}\cdot\underbrace{g'(x)}_{\text{Derivative of the inside function}}$$

Example 4:

$$\frac{d}{dx}[\tan^2 x] = \frac{d}{dx}[(\tan x)^2] = \underbrace{(2\tan x)}\cdot\underbrace{(\sec^2 x)} = 2\tan x\sec^2 x$$

Generalized Derivative Formulas:

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

GENERALIZED DERIVATIVE FORMULAS

$$\frac{d}{dx}[u^r] = ru^{r-1}\frac{du}{dx}$$

$$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} \qquad \frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = -\csc u \cot u\frac{du}{dx}$$

Example 5: Find

(a) $\frac{d}{dx}[\sin(2x)]$ (b) $\frac{d}{dx}[\tan(x^2+1)]$ (c) $\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right]$

(d) $\frac{d}{dx}[x^2-x+2]^{3/4}$ (e) $\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right]$

Solution: (*a*) Taking $\boldsymbol{u = 2x}$ in the generalized derivative formula for $\boldsymbol{\sin u}$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2 = 2\cos 2x$$

(*b*) Taking $\boldsymbol{u = x^2 + 1}$ in the generalized derivative formula for $\boldsymbol{\tan u}$ yields

$$\begin{aligned}\frac{d}{dx}[\tan(x^2+1)] &= \frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}\\ &= \sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1] = \sec^2(x^2+1)\cdot 2x\\ &= 2x\sec^2(x^2+1)\end{aligned}$$

(*c*) Taking $\boldsymbol{u = x^3 + \csc x}$ in the generalized derivative formula for $\boldsymbol{\sqrt{u}}$ yields

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right] &= \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{x^3+\csc x}}\cdot\frac{d}{dx}[x^3+\csc x]\\ &= \frac{1}{2\sqrt{x^3+\csc x}}\cdot(3x^2-\csc x\cot x) = \frac{3x^2-\csc x\cot x}{2\sqrt{x^3+\csc x}}\end{aligned}$$

Home Work: Exercise 2.6: Problem No. 7-40, 43-56