Chapter # 05
(Integration)

5.9 Evaluating Definite Integrals by Substitution: In this section we will discuss two methods for evaluating definite integrals in which a substitution is required.

Two Methods for Making Substitutions in Definite Integrals: Indefinite integrals of the form

$$\int f(g(x))g'(x)\,dx$$

can sometimes be evaluated by making the $\boldsymbol{u}$-substitution

$$u = g(x), \quad du = g'(x)\,dx \tag{1}$$

which converts the integral to the form

$$\int f(u)\,du$$

To apply this method to a definite integral of the form

$$\int_a^b f(g(x))g'(x)\,dx$$

we need to account for the effect that the substitution has on the $\boldsymbol{x}$-limits of integration. There are two ways of doing this.

Method 1: First evaluate the indefinite integral

$$\int f(g(x))g'(x)\,dx$$

by substitution, and then use the relationship

$$\int_a^b f(g(x))g'(x)\,dx = \left[\int f(g(x))g'(x)\,dx\right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the **x**-limits of integration.

Method 2: Make the substitution **(1)** directly in the definite integral, and then use the relationship $\boldsymbol{u = g(x)}$ to replace the $\boldsymbol{x}$-limits, $\boldsymbol{x = a \text{ and } x = b}$, by corresponding $\boldsymbol{u}$-limits, $\boldsymbol{u = g(a) \text{ and } u = g(b)}$. This produces a new definite integral

$$\int_{g(a)}^{g(b)} f(u)\,du$$

that is expressed entirely in terms of ***u***.

Example 1: Use the two methods above to evaluate

$$\int_0^2 x(x^2+1)^3\,dx.$$

Solution (By Method 1): If we let

$$u = x^2 + 1 \quad \text{so that} \quad du = 2x\,dx \qquad (2)$$

then we obtain

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_{x=0}^{2}$$
$$= \frac{(x^2+1)^4}{8}\Bigg]_{x=0}^{2} = \frac{625}{8} - \frac{1}{8} = 78$$

Solution (By Method 2): If we make the substitution $\boldsymbol{u = x^2 + 1}$ in **(2)**, then

$$\begin{aligned} &\text{if} \quad x = 0, \quad u = 1 \\ &\text{if} \quad x = 2, \quad u = 5 \end{aligned}$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du$$
$$= \frac{u^4}{8}\Bigg]_{u=1}^{5} = \frac{625}{8} - \frac{1}{8} = 78$$

Theorem: If $\boldsymbol{g'}$ is continuous on **[*a*, *b*]** and ***f*** is continuous on an interval containing the values of $\boldsymbol{g(x)}$ for $\boldsymbol{a \le x \le b}$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 2: Evaluate

$$\text{(a)} \int_0^{\pi/8} \sin^5 2x \cos 2x \, dx \qquad \text{(b)} \int_2^5 (2x-5)(x-3)^9 \, dx$$

Solution: (*a*) Let

$$u = \sin 2x \quad \text{so that} \quad du = 2\cos 2x \, dx \quad \left(\text{or } \tfrac{1}{2}\, du = \cos 2x \, dx\right)$$

With this substitution

$$\begin{aligned} &\text{if} \quad x = 0, \quad u = \sin(0) = 0 \\ &\text{if} \quad x = \pi/8, \quad u = \sin(\pi/4) = 1/\sqrt{2} \end{aligned}$$

So

$$\begin{aligned} \int_0^{\pi/8} \sin^5 2x \cos 2x \, dx &= \frac{1}{2} \int_0^{1/\sqrt{2}} u^5 \, du \\ &= \frac{1}{2} \cdot \frac{u^6}{6} \bigg]_{u=0}^{1/\sqrt{2}} = \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96} \end{aligned}$$

Solution: (*b*) Let

$$u = x - 3 \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x - 5$ unresolved in the integrand. However

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u+3) - 5 = 2u + 1$$

With this substitution,

$$\begin{aligned} &\text{if} \quad x = 2, \quad u = 2 - 3 = -1 \\ &\text{if} \quad x = 5, \quad u = 5 - 3 = 2 \end{aligned}$$

So

$$\begin{aligned} \int_2^5 (2x-5)(x-3)^9 \, dx &= \int_{-1}^{2} (2u+1)u^9 \, du = \int_{-1}^{2} (2u^{10} + u^9) \, du \\ &= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10}\right]_{u=-1}^{2} = \left(\frac{2^{12}}{11} + \frac{2^{10}}{10}\right) - \left(-\frac{2}{11} + \frac{1}{10}\right) \\ &= \frac{52{,}233}{110} \approx 474.8 \end{aligned}$$

Example 3: Evaluate

$$\text{(a)} \int_0^{3/4} \frac{dx}{1-x} \qquad \text{(b)} \int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx$$

Solution: (*a*) Let

$$u = 1 - x \;\; so\; that \;\; du = -dx$$

With this substitution,

$$\begin{aligned} &\text{if} \quad x = 0, \quad u = 1 \\ &\text{if} \quad x = \tfrac{3}{4}, \quad u = \tfrac{1}{4} \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{3/4} \frac{dx}{1-x} &= -\int_1^{1/4} \frac{du}{u} \\ &= -\ln|u|\Big]_{u=1}^{1/4} = -\left[\ln\left(\frac{1}{4}\right) - \ln(1)\right] = \ln 4 \end{aligned}$$

Solution: (*b*) Make the ***u***-substitution

$$u = 1 + e^x, \quad du = e^x dx$$

and change the ***x***-limits of integration ($\boldsymbol{x = 0}$, $\boldsymbol{x = \ln x}$) to the ***u***-limits

$$u = 1 + e^0 = 2, \quad u = 1 + e^{\ln 3} = 1 + 3 = 4$$

This yields

$$\begin{aligned} \int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx &= \int_2^4 u^{1/2}\,du \\ &= \frac{2}{3}u^{3/2}\Big]_{u=2}^{4} = \frac{2}{3}[4^{3/2} - 2^{3/2}] = \frac{16 - 4\sqrt{2}}{3} \end{aligned}$$

Home Work: Exercise 5.9: Problem No. 5-18, 29-40, 52, 54 and 56

Q54: Given that ***m*** and ***n*** are positive integers, show that

$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx$$

by making a substitution.

Solution: Let $u = 1 - x \quad \therefore du = -dx \quad \Rightarrow dx = -du$

If $x = 0 \; then \;\; u = 1$

If $x = 1 \; then \; u = 0$

$$L.H.S = \int_0^1 x^m(1-x)^m dx = \int_1^0 (1-u)^m u^n(-du) = -\int_1^0 u^n(1-u)^m du = \int_0^1 u^n(1-u)^m du$$

$$= \int_0^1 x^n(1-x)^m dx = R.H.S \text{ (Proved)}$$

Q40: Evaluate

$$\int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}} \sin\sqrt{x}\, dx$$

Solution: Let, $u = \sqrt{x} \quad \therefore du = \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \frac{dx}{\sqrt{x}} = 2du$

If $x = \pi^2 \; then \;\; u = \pi$

If $x = 4\pi^2 \; then \; u = 2\pi$

$$\therefore \int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}} \sin\sqrt{x}\; dx$$

$$= \int_{\pi}^{2\pi} \sin u \; 2du$$

$$= 2\int_{\pi}^{2\pi} \sin u \; du = -2\cos u|_{\pi}^{2\pi} = -2[\cos 2\pi - \cos\pi] = -2[1+1] = -4$$

Q33: Evaluate

$$\int_{-1}^{1} \frac{x^2\, dx}{\sqrt{x^3+9}}$$

Solution: Let, $u = x^3 + 9 \quad \therefore du = 3x^2 dx \quad \Rightarrow \; x^2 dx = \frac{du}{3}$

If $x = -1 \; then \;\; u = 8$

If $x = 1 \; then \; u = 10$

$$\int_{-1}^{1} \frac{x^2 dx}{\sqrt{x^3+9}} = \int_8^{10} \frac{du}{3\sqrt{u}} = \frac{1}{3}\int_8^{10} u^{-\frac{1}{2}} du = \frac{1}{3}\frac{u^{1/2}}{1/2}\Big|_8^{10} = \frac{2}{3}\left(\sqrt{10} - \sqrt{8}\right) \quad \text{(Ans.)}$$

Q18: Evaluate

$$\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}\,dx}{\sqrt{1-e^{-2x}}}$$

Solution: Let, $u = e^{-x} \quad \therefore du = -e^{-x}dx \;\Rightarrow\; e^{-x}dx = -du$

If $x = ln2\;\; then \;\; u = \frac{1}{2}$

If $x = \ln\left(\frac{2}{\sqrt{3}}\right)\; then\; u = e^{-\ln\left(\frac{2}{\sqrt{3}}\right)} = \frac{\sqrt{3}}{2}$

$$\int_{\ln 2}^{ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^{-x}dx}{\sqrt{1-e^{-2x}}}$$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{-du}{\sqrt{1-u^2}}$$

$$= -\int_{1/2}^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = -sin^{-1}u\Big|_{1/2}^{\sqrt{3}/2} = -\left(sin^{-1}\frac{\sqrt{3}}{2} - sin^{-1}\frac{1}{2}\right) = -\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = -\frac{\pi}{12}$$