Chapter 4.1

Polynomial and Rational Functions

Example 6

Graphing a polynomial using its x-intercepts

For the polynomial

$$f(x) = x^2(x-2)$$

(a) Find the x- and y-intercepts of the graph of f.

(b) Use the x-intercepts to find the intervals on which the graph of f is above the x-axis and the intervals on which the graph of f is below the x-axis.

(c) Locate other points on the graph and connect all the points plotted with a smooth, continuous curve.

Solution: (a) The y-intercept is $f(0)=0$. To find the x-intercepts, solve the equation $f(x)=0$. Therefore, we get $f(x)=x^2(x-2)=0$ giving $x=0$ or $x=2$.

(b) The two x-intercepts divide the real number line into three intervals:

$$(-\infty,0), \quad (0,2), \quad (2,+\infty)$$

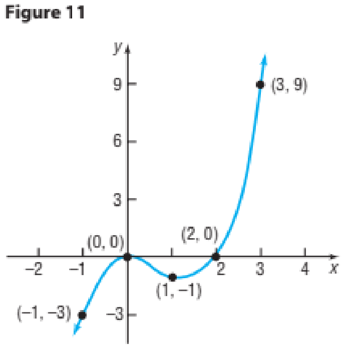
Since the graph of f crosses or touches the x-axis at $x=0$ and $x=2$, it follows that the graph of f is either above the x-axis $(f(x)>0)$ or below the x-axis $(f(x)<0)$ on each of these three intervals. To see where the graph lies, we only need to pick a number in each interval, evaluate f there and see whether the value is positive (above the x-axis) or negative (below the x-axis). To draw the conclusion, prepare the following table:

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty,0)$	-1	$f(-1)=-3$	$(-1,-3)$	Below the x-axis
$(0,2)$	1	$f(1)=-1$	$(1,-1)$	Below the x-axis
$(2,+\infty)$	3	$f(3)=9$	$(3, 9)$	Above the x-axis

(c) From the above Table, we can see that the points on the graph are $(-1,-3)$, $(1,-1)$ and $(3, 9)$.

Figure 11 illustrates these points, the intercepts, etc.

Figure 11



Conclusions

Since the points $(-1, -3)$ and $(1, -1)$ lies below the *x*-axis on both sides of 0, the graph of *f* touches the *x*-axis at $x = 0$ which is a *zero of multiplicity* 2.
Since the graph of f lies below the *x*-axis for $x < 2$ and above *x*-axis for $x < 2$, the graph of f crosses x-axis at $x = 2$ which is a *zero of multiplicity* 1.

The problem suggests the following results:

If *r* is a Zero of Even Multiplicity

Algebra	**Geometry**
The sign of *f*(*x*) does not change from one side to the other side of *r*.	The graph of *f* **touches** the *x*-axis at *r*.

If *r* is a Zero of Odd Multiplicity

Algebra	**Geometry**
The sign of *f*(*x*) changes from one side to the other side of *r*.	The graph of *f* **crosses** the *x*-axis at *r*.

Behavior Near a Zero

Now we see that the graph of $f(x) = x^2(x-2)$ behaves like the graph of $f(x) = -2x^2$ near $x = 0$.
Since the zero 0 comes from the factor x^2, we evaluate all factors in the function f at 0 with the exception of x^2. Therefore, we get

$$f(x) = x^2(x-2) \approx x^2(0-2) = -2x^2$$

(We keep the factor x^2 fixed and let $x = 0$ in the remaining factors)

Next we see that the graph of $f(x) = x^2(x-2)$ behaves like the graph of $f(x) = 4(x-2)$ near $x = 2$.
Since the zero 2 comes from the factor $x-2$, we evaluate all factors in the function f at 2 with the exception of $x-2$. Therefore, we get

$$f(x) = x^2(x-2) \approx 2^2(x-2) = 4(x-2)$$

(We keep the factor x^2 fixed and let $x = 0$ in the remaining factors)

Figure 14 illustrates how we would use this information to begin to graph $f(x) = x^2(x-2)$.

Turning Points

The points at which a graph changes its direction are called **turning points**.

The following Theorem tells us the maximum number of turning points that the graph of a polynomial function can have.

Theorem

If f is a polynomial function of degree n, then the graph of f has at most $n-1$ turning points.

If the graph of a polynomial function f has $n-1$ turning points, the degree of f is at least n.

For very large values of x, either positive or negative, the graph of $f(x)=x^2(x-2)$ looks like the graph of $f(x)=x^3$.
To see why, rewrite f in the form

$$f(x)=x^2(x-2)=x^3-2x^2=x^3\left(1-\frac{2}{x}\right)$$

Now, for large values of x, either positive or negative, the term $\frac{2}{x}$ is close to 0, so for large values of x,

$$f(x)=x^3-2x^2=x^3\left(1-\frac{2}{x}\right)\approx x^3.$$

The behavior of the graph of a function for LARGE values of x, either positive or negative, is referred as its **end behavior**.

This suggests the following theorem,

Theorem (End Behavior)

For large values of x, either positive or negative, the graph of the polynomial function

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

resembles the graph of the power function

$$y=a_nx^n$$

For example, if $f(x)=-2x^3+5x^2+x-4$, then the graph of f will behave like the graph of $y=-2x^3$ for LARGE values of x, either positive or negative.

We describe the behavior of the graph of a function using notation.

We can symbolize "the value of f becomes a larger and larger negative number as x becomes a larger and larger positive number" by writing $f(x)\to-\infty$ as $f(x)\to\infty$ (read as **the values of *f* approach negative infinity as *x* approaches infinity**) where we use the symbolism

$$\lim_{x\to\infty} f(x)=-\infty$$

When the value of a limit equals infinity, we mean that the values of the function are unbounded in the positive or negative direction and we call the limit an **infinite limit**.

Based on the preceding theorem and above discussion on power functions, the end behavior of a polynomial function are of FOUR types. See **Figure 18**.
For example, if $f(x)=-2x^4+x^3+4x^2-7x+1$, the graph of f will resemble the graph of the power function $y=-2x^4$ for large $|x|$. The graph of f will behave like **Figure 18(b)** for large $|x|$.

SUMMARY

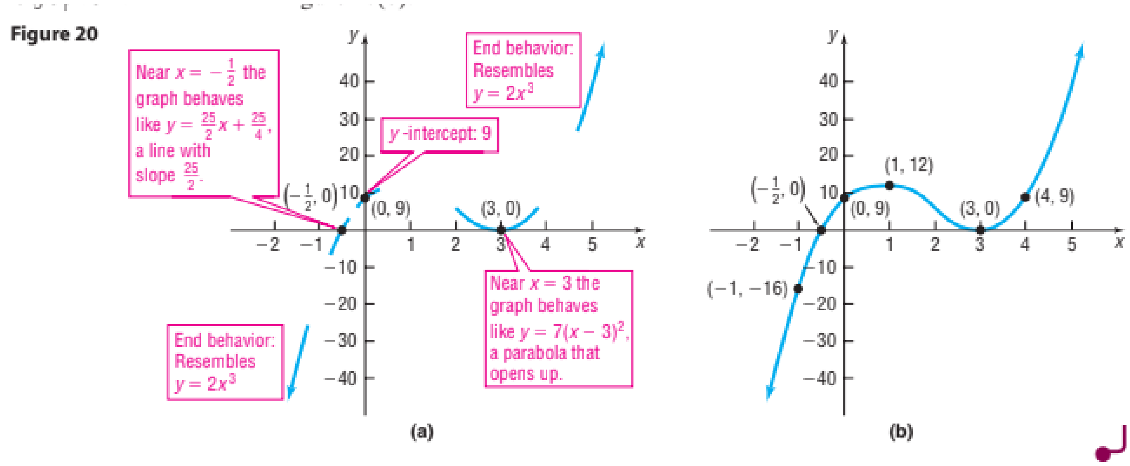
For the graph of a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ \ a_n \neq 0$$

(a) Degree of the polynomial function f is n;
(b) The graph f is smooth and continuous;
(c) There are $n-1$ turning points;
(d) At a zero of even multiplicity the graph f touches the x-axis;
(e) At a zero of odd multiplicity the graph f crosses the x-axis;
(f) Between zeros, the graph of f is either above or below the x-axis;
(g) End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

Example 9 Analyzing the Graph of a Polynomial Function

Analyze the graph of the polynomial function f(x) = (2x+1)(x-3)^2.



Example 10 Analyzing the Graph of a Polynomial Function

Analyze the graph of the polynomial function

$$f(x) = x^2(x-4)(x+1)$$

Solution

Step 1: End behavior: the graph of f resembles that of the power function $y = x^4$ for large values of $|x|$, i.e.

$$\lim_{x\to-\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty$$

Step 2: The y-intercept is $f(0) = 0$. To find the x-intercepts, solve the equation $f(x) = 0$.

Therefore, we get $f(x) = x^2(x-4)(x+1) = 0$ giving $x = -1$ or $x = 0$ or $x = 4$.

Thus, y-intercept is 0 and the x-intercepts are $x = -1,\ 0$ and 4.

Step 3: The intercept 0 is a zero of multiplicity 2, so the graph of f touches the x-axis at 0.
The other intercepts -1 and 4 are zeros of multiplicity 1, so the graph of f crosses the x-axis at -1 and 4.
Step 4: Since f is a polynomial function of degree 4, the graph of f contains at most three turning points.

Step 5: Behavior near zeros:

Near -1: $f(x) = x^2(x-4)(x+1) \approx (-1)^2(-1-4)(x+1) = -5(x+1)$ which is a line with slope -5

(We keep the factor $x+1$ fixed and let $x=-1$ in the remaining factors)

Near 0: $f(x) = x^2(x-4)(x+1) \approx x^2(0-4)(0+1) = -4x^2$ which is a parabola that opens down

(We keep the factor x^2 fixed and let $x=0$ in the remaining factors)

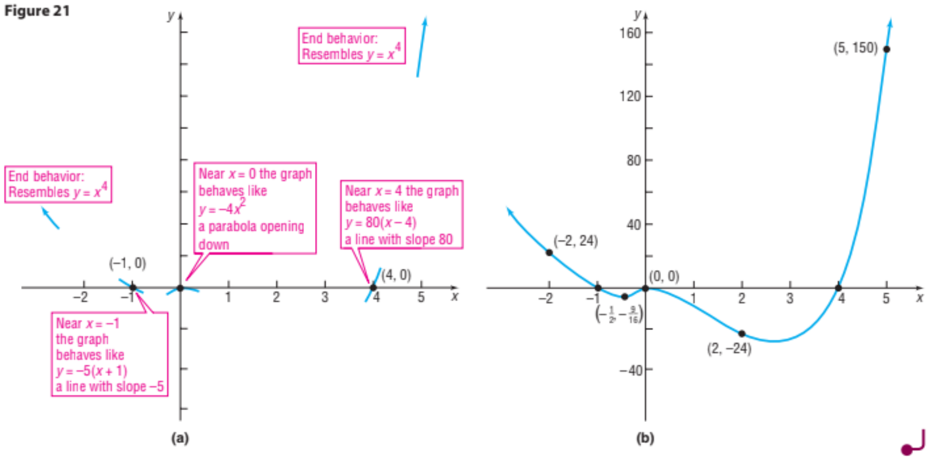
Near 4: $f(x) = x^2(x-4)(x+1) \approx 4^2(x-4)(4+1) = 80(x-4)$ which is a line with slope 80

(We keep the factor $x-4$ fixed and let $x=4$ in the remaining factors)

Step 6: **Figure 21(a)** illustrates the information obtained from Step 1 to Step 5.

The graph of f is given in **Figure 21(b)**.

Figure 21



To draw the conclusion, prepare the following table:

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty, -1)$	-2	$f(-2) = 24$	$(-2,\ 24)$	Above the x-axis
$(-1, 0)$	$-\frac{1}{2}$	$f\left(-\frac{1}{2}\right) = -\frac{9}{16}$	$\left(-\frac{1}{2}, -\frac{9}{16}\right)$	Below the x-axis
$(0, 4)$	2	$f(2) = -24$	$(2,\ -24)$	Below the x-axis
$(4, +\infty)$	5	$f(5) = 150$	$(5,\ 150)$	Above the x-axis

We evaluated f at $-2, -\frac{1}{2}, 2$ and 5 to help establish the scale on the y-axis.

49. $y = 3(x-7)(x+3)^2$

(a) Zeros: 7, multiplicity 1; -3, multiplicity 2;

(b) Graph touches the x-axis at -3 and crosses it at 7.

(c) Near -3: $f(x) \approx -30(x+3)^2$ which is a parabola that opens down

Near 7: $f(x) \approx 300(x-7)$ which is a line with slope 300

(d) 2

(e) $y = 3x^3$

To draw the conclusion, prepare the following table:

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty,-3)$	-4	$f(-4)=-33$	$(-4,\ -33)$	Below the x-axis
$(-3,7)$	0	$f(0)=-189$	$(0,-189)$	Below the x-axis
$(7,+\infty)$	8	$f(8)=363$	$(8,\ 363)$	Above the x-axis

51. $y=4(x^2+1)(x-2)^3$

(a) Zero 2, multiplicity 3;

(b) Graph crosses the x-axis at 2;

(c) Near 2: $f(x)\approx 20(x-2)^3$ which is a cubic function with its turning point at $(2,0)$.

(d) 4

(e) $y=4x^5$

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty,2)$	1	$f(1)=-8$	$(1,\ -8)$	Below the x-axis
$(2,+\infty)$	3	$f(3)=40$	$(3,40)$	Above the x-axis

53. $y=-2\left(x+\dfrac{1}{2}\right)^2(x+4)^3$

(a) Zeros: $-\dfrac{1}{2}$, multiplicity 2; -4, multiplicity 3;

(b) Graph touches the x-axis at $-\dfrac{1}{2}$, and crosses the x-axis at -4;

(c) Near $-\dfrac{1}{2}$: $f(x)\approx -85.75\left(x+\dfrac{1}{2}\right)^2$ which is a parabola that opens down

Near -4: $f(x)\approx -24.5(x+4)^3$ which is a cubic function lying in the second and fourth quadrant

(d) 4

(e) $y=-2x^5$

Interval	Number picked	Value of f	Point on graph	Location of graph
$(-\infty,-4)$	-5	$f(-4)=40.5$	$(-4,\ 40.5)$	Above the x-axis
$\left(-4,-\dfrac{1}{2}\right)$	-2	$f(-2)=-36$	$(-2,-36)$	Below the x-axis
$\left(-\dfrac{1}{2},+\infty\right)$	1	$f(1)=-562.5$	$(1,-562.5)$	Below the x-axis

55. $y=(x-5)^3(x+4)^2$

(a) Zeros: 5, multiplicity 3; -4, multiplicity 2;

(b) Graph touches the x-axis at -4, and crosses the x-axis at 5;

(c) Near -4: $f(x)\approx -729(x+4)^2$ which is a parabola that opens down

Near 5: $f(x)\approx 81(x-5)^3$ which is a cubic function lying in the first and third quadrant

(d) 4

(e) $y=x^5$

Analyze each polynomial function:

69. $y=x^2(x-3)$

69. Step 1: $y = x^3$

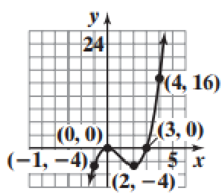
Step 2: x-intercepts: 0, 3; y-intercept: 0

Step 3: 0: multiplicity 2, touches; 3: multiplicity 1, crosses

Step 4: At most 2 turning points

Step 5: Near 0: $f(x) \approx -3x^2$; Near 3: $f(x) \approx 9(x-3)$

Step 6:



71. $y=(x+4)(x-2)^2$

71. Step 1: $y = x^3$

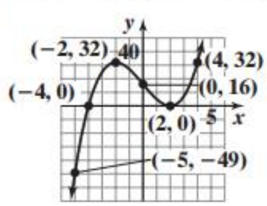
Step 2: x-intercepts: -4, 2; y-intercept: 16

Step 3: -4: multiplicity 1, crosses; 2: multiplicity 2, touches

Step 4: At most 2 turning points

Step 5: Near -4: $f(x) \approx 36(x+4)$; Near 2: $f(x) \approx 6(x-2)^2$

Step 6:



73. $y = -2(x+2)(x-2)^3$

73. Step 1: $y = -2x^4$

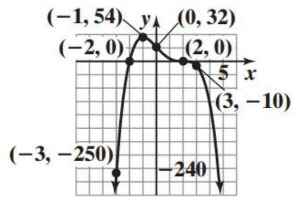
Step 2: x-intercepts: $-2, 2$; y-intercept: 32

Step 3: -2: multiplicity 1, crosses; 2: multiplicity 3, crosses

Step 4: At most 3 turning points

Step 5: Near -2: $f(x) \approx 128(x+2)$; Near 2: $f(x) \approx -8(x-2)^3$

Step 6:



75. $y = (x+1)(x-2)(x+4)$

75. Step 1: $y = x^3$

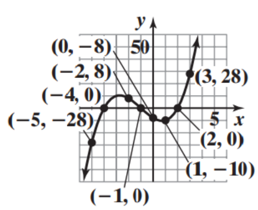
Step 2: x-intercepts: $-4, -1, 2$; y-intercept: -8

Step 3: $-4, -1, 2$: multiplicity 1, crosses

Step 4: At most 2 turning points

Step 5: Near -4: $f(x) \approx 18(x+4)$; Near -1: $f(x) \approx -9(x+1)$;
Near 2: $f(x) \approx 18(x-2)$

Step 6:



77. $y = x^2(x+2)(x-2)$

77. Step 1: $y = x^4$

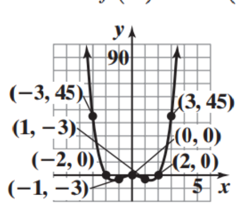
Step 2: x-intercepts: $-2, 0, 2$; y-intercept: 0

Step 3: $-2, 2$, multiplicity 1, crosses; 0, multiplicity 2, touches

Step 4: At most 3 turning points

Step 5: Near -2: $f(x) \approx -16(x+2)$; Near 0: $f(x) \approx -4x^2$;
Near 2: $f(x) \approx 16(x-2)$

Step 6:



79. $y=(x+1)^2(x-2)^2$

79. Step 1: $y = x^4$

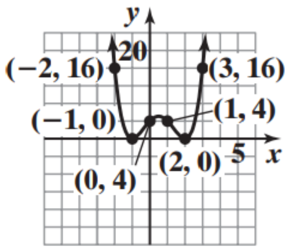
Step 2: x-intercepts: $-1, 2$; y-intercept: 4

Step 3: $-1, 2$: multiplicity 2, touches

Step 4: At most 3 turning points

Step 5: Near -1: $f(x) \approx 9(x+1)^2$; Near 2: $f(x) \approx 9(x-2)^2$

Step 6:



81. $y=x^2(x-3)(x+1)$

81. Step 1: $y = x^4$

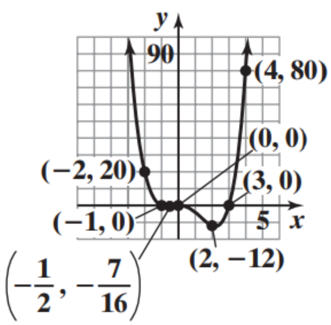
Step 2: x-intercepts: $-1, 0, 3$; y-intercept: 0

Step 3: $-1, 3$: multiplicity 1, crosses; 0: multiplicity 2, touches

Step 4: At most 3 turning points

Step 5: Near -1: $f(x) \approx -4(x+1)$; Near 0: $f(x) \approx -3x^2$;
Near 3: $f(x) \approx 36(x-3)$

Step 6:



83. $y=(x+2)^2(x-4)^2$

83. Step 1: $y = x^4$

Step 2: x-intercepts: $-2, 4$; y-intercept: 64

Step 3: $-2, 4$, multiplicity 2, touches

Step 4: At most 3 turning points

Step 5: Near -2: $f(x) \approx 36(x+2)^2$; Near 4: $f(x) \approx 36(x-4)^2$

Step 6: