Course Name : Physics – I

Course # PHY 107

Notes-9 : Linear Momentum, Impulse and Collision

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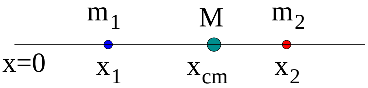
Topics to be studied

- Center of Mass: Discrete and Continuous Distribution
- Linear Momentum and Newton's 2nd Law
- Linear Momentum for a system of Particles
- The Total Momentum conservation law
- Impulse: Definition and Properties
- Collision: Elastic and Inelastic
- One dimensional Elastic collision
- Collision in two dimensions
- Examples
- Suggested Problems

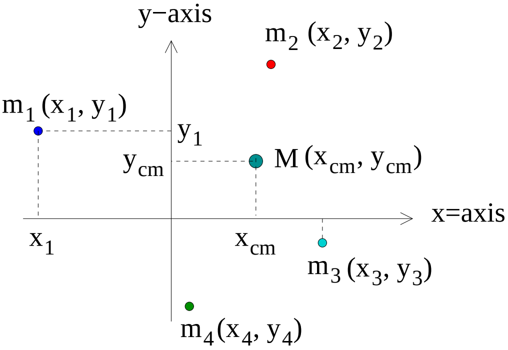
Center of Mass

- Let's consider one dimension. Suppose mass m_1, m_2 are located at x_1 and x_2 respectively.
- The center of mass of these two masses is defines by $x_{\mathrm{cm}} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$.



- From mathematical point of view, the 'Center of Mass' is the weighted average of discrete mass distribution.
- Using the concept of center of mass, a system of mass (discrete mass distribution) can be replaced by a single mass M, which is equal to the total mass of the system, and located at x_{cm}.

- Let's consider two dimension as shown below:



- The coordinates of the masses is also noted in the diagram.

- The total mass is $M = m_1 + m_2 + m_3 + m_4$. The center of mass is given by

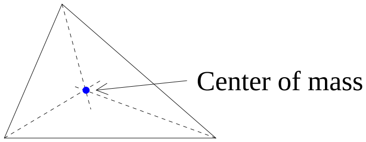
$$x_{\mathrm{cm}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_3 x_3}{m_1 + m_2 + m_3 + m_4} = \frac{\sum_{i=1}^{i=4} m_i x_i}{\sum_{i=1}^{i=4} m_i}.$$

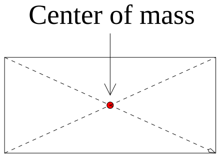
$$y_{\mathrm{cm}} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_3 y_3}{m_1 + m_2 + m_3 + m_4} = \frac{\sum_{i=1}^{i=4} m_i y_i}{\sum_{i=1}^{i=4} m_i}.$$

- From mathematical point of view, the 'Center of Mass' is the weighted average of discrete mass distribution.
- Using the concept of center of mass, a system of mass (discrete mass distribution) can be replaced by a single mass M, which is equal to the total mass of the system, and located at x_{cm}.
- For three dimension, we need to add z_{cm} component.
- For more than four masses, we just need to the corresponding terms.

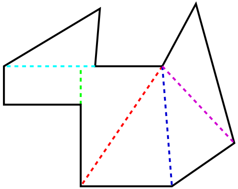
Continuous Mass Distribution:

- For extended object (object is NOT point size), the geometric center is the center of mass (if the mass density is uniform).
- For an object of triangular and rectangular shapes, the center of masses are shown below:





- For an arbitrary shaped object, it can be treated as collection different regular shaped objects, like circular, spherical, triangular, rectangular, cubic, parallelepiped, etc. as shown in the adjacent diagram.
- Each of these have its own center of mass.
- using discrete distribution, the center of mass these the center of masses can be calculated.



Linear Momentum:

- The linear momentum of an object of mass m and moving with velocity $\vec{v}$ is defined as
$$\vec{p} = m\vec{v} \, .$$
- It is parallel with the velocity.
- Taking the derivative of Momentum, we get, (for constant mass)
$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} = \sum \vec{F} \, .$$
- The rate of change of linear momentum is the net Force acting on the object of mass m.
- If $\vec{P}$ is constant, the object os obeying the 1st law, otherwise obeying the second law.
- For the p-t graph, the slope at any point is the net force at that moment acting on the object of mass m.

System of particles:

- If there are n particles with mass m_i and velocity $\vec{v}_i$, the total momentum of the system is
$$\vec{P} \;=\; M\vec{v}_{\rm cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots = \vec{p}_1 + \vec{p}_2 + \cdots \;.$$
- If the net force on the system is zero, then we get:
$$\sum \vec{F} = \frac{d\vec{P}}{dt} = 0 \quad \Longrightarrow \boxed{\vec{P} = \vec{p}_1 + \vec{p}_2 + \cdots = \text{Constant}}\;.$$
This is known as the'Conservation Law' of the total linear momentum.
- The net force is the net 'External force acting on the system of particles.
- Therefore, for a system of particles, if the net external force is zero, then we get,
$$\sum_{\rm final} \vec{p}_i \;=\; \sum_{\rm initial} \vec{p}_i \;.$$
Here $i = 1, 2, \cdots, n$ is the number of particles in the system.

Impulse: Definition and Properties

- If a net force $\sum \vec{F}$ acts on an object fort a time interval Δt, then the impulse, $\vec{J}$, on that object is defined as

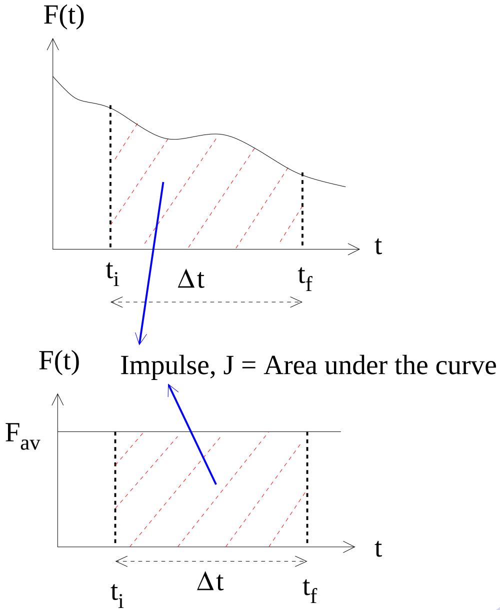
$$\begin{aligned} \vec{J} &= \sum \vec{F}\, \Delta t \ . \quad \text{(if net force is constant)} \\ &= \int_i^f \vec{F}(t)\, dt \ . \quad \text{(if the net force is not constant)} \end{aligned}$$

- By using Newton;s second law, we can write,

$$\vec{J} = \int_i^f \frac{d\vec{P}}{dt}\, dt = \int_{p_i}^{p_f} d\vec{P} = \vec{P}_f - \vec{P}_i = \Delta \vec{P} \ .$$

- That is, the impulse gives how much momentum has been changed.
- If an object obey's Newton's 1st law, the linear momentum will be constant and the impulse will be zero.

- Geometrically, the impulse is the area under the F vs. t graph as shown below:



- Let's consider free fall of a tennis ball without any air resistance. The ball hit the floor and then bounces up.
- During the collision with the floor, the force applied by the ball on to the floor id $w = mg$.
- The force applied by the floor on to ball is n, the normal force. That is why the ball bounces upward. By the 3rd law, these two forces are equal, and hence the net force on the floor-ball system is zero.
- Let Δt is the duration of contact (collision time). The impulse on the ball by the floor is
$$\vec{J}_{\text{on the ball by the floor}} = \ = \vec{n}\,\Delta t = n\,\Delta t\,\hat{j}\ .$$
- Similarly the impulse on the floor by the ball is
$$\vec{J}_{\text{on the floor by the ball}} = \ = \vec{w}\,\Delta t = -mg\,\Delta t\,\hat{j}\ .$$

Collision:

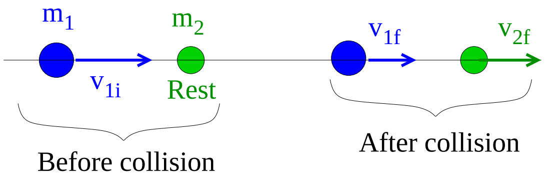
- When two objects collides, their linear momentum changes. This implies that during the collision, the objects exchanges impulses, and hence they apply force on each other.
- These forces during collision are equal in magnitude but opposite in direction. These forces obey Newton's 3rd law.
- The net force on the system of objects involved in collision is zero.
- This implies that the total linear momentum during collision is conserved:

$$\sum_{\text{before collision}} \vec{P} = \sum_{\text{after collision}} \vec{P} \quad \text{or simply written:} \quad \sum_i \vec{P}_i = \sum_f \vec{P}_f \ .$$

- But the kinetic energy may or may not conserved. If the total kinetic energy is also conserved, it is known as the 'Elastic Collision'.
- If the total kinetic energy is not conserved, it is an 'Inelastic Collision'.
- If the objects sticks together after collision, it is completely inelastic collision.

One dimensional elastic collision:

- Let's consider one dimensional elastic collision where an object of mass m_1 moving with velocity v_{1i} collides with an object of mass m_2 at rest as shown below.



- After the collision mass m_1 moves with velocity v_{1f} and mass m_2 moves with velocity v_{2f}.
- Note that after the collision, m_2 must move to the right because it has been hit from the left.
- But the mass m_1 may move to the right, or left, or may stop moving. In the diagram the direction of m_1 is chosen arbitrarily.
- Since the collision is elastic, both the total linear momentum and the total kinetic energy will conserve.

- The total momentum conservation implies that

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \quad \Longrightarrow \quad m_1(v_{1i} - v_{1f}) = m_2 v_{2f} \ . \tag{1}$$

- The total kinetic energy conservation implies that

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \Longrightarrow \quad m_1(v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2 \ . \tag{2}$$

- Divide Eq.(2) by Eq.(1) gives: $v_{1i} + v_{1f} = v_{2f}$.
- Substituting v_{2f} in Eq.(1) yields:

$$m_1(v_{1i} - v_{1f}) = m_2(v_{1i} + v_{1f}) \quad \Longrightarrow \quad v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \ .$$

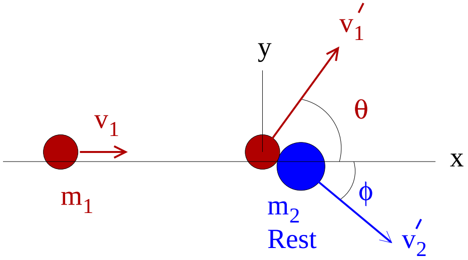
- Putting v_{1f} back into v_{2f}, and rearranging gives:

$$v_{1i} + \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} = v_{2f} \quad \Longrightarrow \quad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \ .$$

- Clearly, after collision, mass m_2 always moves to the right, because it was at rest before collision.
- But mass m_1 three possibilities after the collision depending on the value of masses:
  - If $m_1 > m_2$, $v_{1f} > 0$. So mass m_1 will move to the right after collision. It is partial momentum transfer from m_1 to m_2.
  - If $m_1 = m_2$, $v_{1f} = 0$. It is complete momentum transfer from m_1 tp m_2. m_1 comes to a rest, and m_2 moves with the velocity of m_1, *i.e.*. $v_{2f} = v_{1i}$.
  - If $m_1 < m_2$, $v_{1f} < 0$. So m_1 bounces backward because it has negative velocity.
- The above clearly explain why a tennis ball bounces off the floor, because the tennis ball's mass is very small compare to the floor.
- But a huge boulder falls on the floor does not bounces up. It breaks the floor and go forward, because it's mass is large compare to the floor.
- Note that for inelastic collision, the above results are not valid. Only the momentum conservation can be used for inelastic collision.

Collision in two dimensions:

- Suppose the target is at rest and the projectile is moving with velocity $\vec{v}_1$ and collide with the target and scatters off as shown below:



- In this case, only the total momentum conservation law can be used, because the collision is not elastic.
- The x- and y-components of the momentum gives the following:

$$\begin{aligned} m_1 v_1 &= m_1 v_1' \cos\theta + m_2 v_2' \cos\phi \ , \\ \text{and} \quad 0 &= m_1 v_1' \sin\theta - m_2 v_2' \sin\phi \ . \end{aligned}$$

Examples
Problem # 9.37

A soccer player kicks a soccer ball of mass 0.45 kg that is initially at rest. The foot of the player is in contact with the ball for $3.0 \times 10^{-3}\,\mathrm{s}$, and the force of the kick is given by

$$F(t) = \left[\left(6.0 \times 10^{6}\right) t - \left(2.0 \times 10^{9}\right) t^{2}\right] \mathrm{N} \;,$$

for $0 \leq t \leq 3.0 \times 10^{-3}\,\mathrm{s}$, where t is in seconds. Find the magnitudes of

- (a) the impulse on the ball due to the kick,
- (b) the average force on the ball from the player's foot during the period of contact,
- (c) the maximum force on the ball from the player's foot during the period of contact, and
- (d) the ball's velocity immediately after it loses contact with the player's foot.

Solution # 9.37:

- (a) The magnitude of the impulse is

$$\begin{aligned}
J &= \int_{t_i}^{t_f} F(t)\, dt \ , \\
&= \int_0^{3.0\times10^{-3}} \left[\left(6.0\times10^6\right)t - \left(2.0\times10^9\right)t^2\right] dt \ , \\
&= \left[\left(6.0\times10^6\right)\frac{t^2}{2} - \left(2.0\times10^9\right)\frac{t^3}{3}\right]_{t=0}^{t=3.0\times10^{-3}} \ , \\
&= 9.0\,\mathrm{N.s} \ .\checkmark
\end{aligned}$$

- (b) The average force is the impulse per unit time. Therefore,

$$\begin{aligned}
F_{\mathrm{av}} &= \frac{J}{\Delta t} \ , \\
&= \frac{9.0\,\mathrm{N.s}}{3.0\times10^{-3}\mathrm{s}} \ , \\
&= 3.0\times10^3\,\mathrm{N} \ = \ 3.0\,\mathrm{kN} \ .\checkmark
\end{aligned}$$

- (c) The maximum force $F_{\max}$ when $\dfrac{dF}{dt} = 0$ and $\dfrac{d^2F}{dt^2} \neq 0$. Solving $dF/dt = 0$ for t, we find

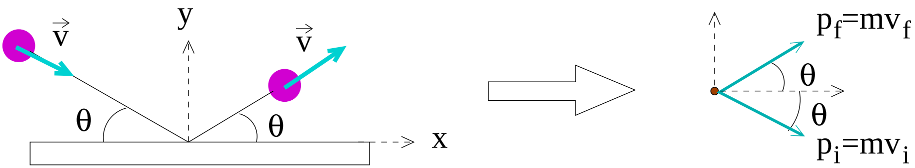
$$\begin{aligned}
\frac{dF}{dt} &= 6.0 \times 10^6 - (2.0 \times 10^9)2t = \; = 0 \, , \\
t &= 1.5 \times 10^{-3} \, \mathrm{sec} \, . \\
\text{Also } \frac{d^2F}{dt^2} &< 0 \, . \quad \text{(check it out yourself)} \\
\therefore \quad F_{\max} &= \left[\left(6.0 \times 10^6\right)\left(1.5 \times 10^{-3}\right) - \left(2.0 \times 10^9\right)\left(1.5 \times 10^{-3}\right)^2 \right] \mathrm{N} \, , \\
&= 4500 \, \mathrm{N} = 4.5 \, \mathrm{kN} \, . \checkmark
\end{aligned}$$

- (d) Here $v_i = 0$ (because the ball was at rest). So $p_i = 0$. Therefore,

$$J = \Delta p = p_f - \cancelto{0}{p_i} = m v_f \, . \quad \Longrightarrow \quad v_f = \frac{J}{m} = \frac{9.0}{0.45} \, \mathrm{m/s} = 20 \, \mathrm{m/s} \, . \checkmark$$

Problem # 9.38:

In the overhead view of the figure below, a 300 g ball with a speed v of 6.0 $\mathrm{m/s}$ strikes a wall at an angle θ of 30° and then rebounds with the same speed and angle.



It is in contact with the wall for 10 ms. In unit vector notation, what are

- (a) the impulse on the ball from the wall and
- (b) the average force on the wall from the ball?

Solution # 9.38:

- (a) The impulse on the ball from the floor is

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i = (p_{fx} - p_{ix})\hat{i} + (p_{fy} - p_{iy})\hat{j} \ .$$

Now the x- and y-components of the momentum are

$$p_{fx} = mv\cos\theta, \ p_{ix} = mv\cos(360-\theta) = mv\cos\theta. \quad \therefore \ \Delta p_x = p_{fx} - p_{ix} = 0 \ .$$

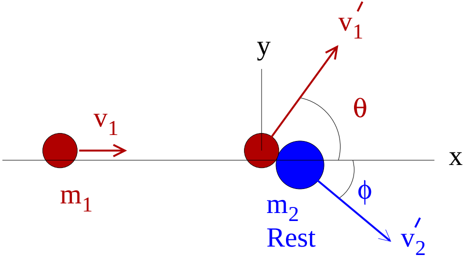
$$p_{fy} = mv\sin\theta, \ p_{iy} = mv\sin(360-\theta) = -mv\sin\theta. \quad \therefore \ \Delta p_y = p_{fy} - p_{iy} = 2mv\sin\theta \ .$$

Substituting, we finally obtain,

$$\vec{J} \ = \ 2mv\sin\theta = (0.300)(6.0)\sin 30^\circ \,\mathrm{kg.m/s}\,\hat{j} \ = \ 1.80\,\mathrm{kg.m/s}\,\hat{j} \ .$$

Problem # 9.71:

In the figure below, projectile particle 1 is an alpha particle and target particle 2 is an oxygen nucleus. The alpha particle is scattered at angle $\theta = 64.0^\circ$ and the oxygen nucleus recoils with speed $1.20 \times 10^5\ \mathrm{m/s}$



and at angle $\phi = 51.0^\circ$. In atomic mass units, the mass of the alpha particle is $m_1 = 4.00u$ and the mass of the oxygen nucleus is $m_2 = 16.0u$. What are the (a) final and (b) initial speeds of the alpha particle?

Solution # 9.71:

- The total momentum is conserved (whether the collision is elastic or inelastic). The y-component of the total momentum gives

$$\begin{aligned} 0 &= m_1 v_1' \sin\theta - m_2 v_2' \sin\phi \ , \\ v_1' &= \frac{m_2 v_2' \sin 51^\circ}{m_1 \sin 64^\circ} \ , \\ &= \frac{(16u)(1.2\times 10^5)(sin52^\circ)}{(4u)(\sin 64^\circ)} \, \mathrm{m/s} \ = \ 4.15\times 10^5 \, \mathrm{m/s} \ .\checkmark \end{aligned}$$

- Substituting v_1' in to the x-component of the total momentum, we find,

$$\begin{aligned} m_1 v_1 &= m_1 v_1' \cos\theta + m_2 v_2' \cos\phi \ , \\ v_1 &= \frac{m_1 v_1' \cos\theta + m_2 v_2' \cos\phi}{m_1} \ , \\ &= \frac{(4u)(4.15\times 10^5)(\cos 64^\circ) + (16u)(1.2\times 10^5)(\cos 51^\circ)}{4u} \, \mathrm{m/s} \ , \\ &= 4.84\times 10^5 \, \mathrm{m/s} \ .\checkmark \end{aligned}$$

Problem # 9.74:

Two 2.0 kg bodies, a and b, collide. The velocities before the collision are $v_a = (15\hat{i} + 30\hat{j})\mathrm{m/s}$ and $v_b = (-10\hat{i} + 5.0\hat{j})\mathrm{m/s}$. After the collision, $v_a' = (-5.0\hat{i} + 20\hat{j})\mathrm{m/s}$. What are

- (i) the final velocity of b and
- (ii) the change in the total kinetic energy (including sign)?

Solution # 9.74:

- (i) The conservation law of total momentum implies

$$\begin{aligned} m(\vec{v}_a + \vec{v}_b) &= m(\vec{v}'_a + \vec{v}'_b) \ , \\ \vec{v}'_b &= \vec{v}_a + \vec{v}_b - \vec{v}'_a \ , \\ &= \left[(15\hat{i} + 30\hat{j}) + (-10\hat{i} + 5.0\hat{j}) - (-5.0\hat{i} + 20\hat{j})\right]\mathrm{m/s} \ , \\ &= (10\hat{i} - 15\hat{j})\mathrm{m/s} \ .\checkmark \end{aligned}$$

- The change in total kinetic energy is

$$\begin{aligned} \Delta K &= \sum K_f - \sum K_i = \frac{1}{2}m\left[\left(v_a'^2 + v_b'^2\right) - \left(v_a^2 + v_b^2\right)\right] \ , \\ &= \frac{1}{2} \times 2 \times \left[(5^2 + 20^2) + (10^2 + 15^2) - (15^2 + 30^2) - (10^2 + 5^2)\right] \mathrm{J} \ , \\ &= -500\,\mathrm{J} \ .\checkmark \end{aligned}$$

Suggested Problems:

Chapter 9: 2, 3, 5, 19, 22, 25, 27, 36, 37, 38, 46, 47, 50, 58, 64 and 74.