MAT 350
ENGINEERING MATHEMATICS

Laplace and Inverse Laplace

Lecture: Handworks. Here only few problems

Lecture: 9

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Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \tag{2}$$

is said to be the **Laplace transform** of f, provided that the integral converges.

Evaluate $\mathscr{L}\{t\}$.

SOLUTION From Definition 7.1.1 we have $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$. Integrating by parts and using $\lim_{t\to\infty} te^{-st} = 0$, $s \quad 0$, along with the result from Example 1, we obtain

$$\mathscr{L}\{t\} = \frac{-te^{-st}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}.$$

Evaluate **(a)** $\mathscr{L}\{e^{-3t}\}$ **(b)** $\mathscr{L}\{e^{5t}\}$

SOLUTION

(a)

$$\mathscr{L}\{e^{-3t}\} = \int_0^\infty e^{-3t}e^{-st}\,dt = \int_0^\infty e^{-(s+3)t}\,dt$$

$$= \left.\frac{-e^{-(s+3)t}}{s+3}\right|_0^\infty$$

$$= \frac{1}{s+3}.$$

The last result is valid for $s > -3$ because in order to have $\lim_{t\to\infty} e^{-(s+3)t} = 0$ we must require that $s + 3 > 0$ or $s > -3$.

More Exercises on board

THEOREM 7.1.1 Transforms of Some Basic Functions

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

DEFINITION : **Exponential Order**

A function *f is said to be of* ***exponential order*** *if there exist constants c, M> 0, and T> 0 such*

$$|f(t)| \leq Me^{ct}$$

for all t> T.

Properties of Laplace Transform :

Inverse Laplace Transforms:

$$\mathscr{L}\{f(t)\} = F(s),$$

$$f(t) = \mathscr{L}^{-1}\{F(s)\}.$$

Transform	Inverse Transform
$\mathscr{L}\{1\} = \dfrac{1}{s}$	$1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$
$\mathscr{L}\{t\} = \dfrac{1}{s^2}$	$t = \mathscr{L}^{-1}\left\{\dfrac{1}{s^2}\right\}$
$\mathscr{L}\{e^{-3t}\} = \dfrac{1}{s+3}$	$e^{-3t} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$.

SOLUTION **(a)** To match the form given in part (b) of Theorem 7.2.1, we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) To match the form given in part (d) of Theorem 7.2.1, we identify $k^2 = 7$, so $k = \sqrt{7}$. We fix up the expression by multiplying and dividing b $\sqrt{7}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

$\mathscr{L}^{-1}$ is a Linear Transform

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\},$$

for constants α and β

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s + 6}{s^2 + 4}\right\}$.

$$\mathscr{L}^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2 + 4} + \frac{6}{s^2 + 4}\right\}$$

$$= -2\,\mathscr{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + \frac{6}{2}\,\mathscr{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}$$

$$= -2\cos 2t + 3\sin 2t.$$

Problem:

$$\boxed{\text{Solve } y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.}$$

Solution: Hints

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s+4)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}.$$

(for partial fraction, see the class lecture).

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$= \frac{A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)}{(s-1)(s-2)(s+4)}.$$

Since the denominators are identical, the numerators are identical:

$$s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2).$$

and so $A = -\frac{16}{5}$, $B = \frac{25}{6}$, and $C = \frac{1}{30}$.

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4},$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

Exercise 7.2 (9th Edition)

35. $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

Laplace transform of the initial-value problem is

$$s^2 \mathscr{L}\{y\} - sy(0) - y'(0) + 5\left[s\,\mathscr{L}\{y\} - y(0)\right] + 4\,\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+5}{s^2+5s+4} = \frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}.$$

The Inverse Laplace hence gives

$$y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}.$$

36. $y'' - 4y' = 6e^{3t} - 3e^{-t}, \quad y(0) = 1, \quad y'(0) = -1$

Laplace transform of the initial-value problem is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 4[s\mathscr{L}\{y\} - y(0)] = \frac{6}{s-3} - \frac{3}{s+1}$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)} + \frac{s-5}{s^2-4s}$$

$$= \frac{5}{2}\cdot\frac{1}{s} - \frac{2}{s-3} - \frac{3}{5}\cdot\frac{1}{s+1} + \frac{11}{10}\cdot\frac{1}{s-4}.$$

The Inverse Laplace hence gives

$$y = \frac{5}{2} - 2e^{3t} - \frac{3}{5}e^{-t} + \frac{11}{10}e^{4t}.$$

TRANSLATION ON THE s-AXIS

THEOREM 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a).$$

PROOF The proof is immediate, since by Definition 7.1.1

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a).$$

$$\boxed{\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}\big|_{s\to s-a},}$$

EXAMPLE 1 Using the First Translation Theorem

Evaluate **(a)** $\mathscr{L}\{e^{5t}t^3\}$ **(b)** $\mathscr{L}\{e^{-2t}\cos 4t\}$.

SOLUTION The results follow from Theorems 7.1.1 and 7.3.1.

(a) $$\mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}\big|_{s\to s-5} = \left.\frac{3!}{s^4}\right|_{s\to s-5} = \frac{6}{(s-5)^4}$$

(b) $$\mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}\big|_{s\to s-(-2)} = \left.\frac{s}{s^2+16}\right|_{s\to s+2} = \frac{s+2}{(s+2)^2+16} \quad \equiv$$

Inverse Form of Theorem 7.3.1

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t),$$

EXAMPLE 2 Partial Fractions: Repeated Linear Factors

Evaluate (a) $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$ (b) $\mathscr{L}^{-1}\left\{\dfrac{s/2+5/3}{s^2+4s+6}\right\}$.

Sol. (a)

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}.$$

$$\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2}$$

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}.$$

Now $1/(s-3)^2$ is $F(s) = 1/s^2$ shifted three units to the right. Since $\mathscr{L}^{-1}\{1/s^2\} = t$, it follows from (1) that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\bigg|_{s \to s-3}\right\} = e^{3t}t.$$

Finally,

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2e^{3t} + 11e^{3t}t.$$

EXAMPLE 3 An Initial-Value Problem

Solve $y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2, \quad y'(0) = 17.$

$$\mathscr{L}\{y''\} - 6\mathscr{L}\{y'\} + 9\mathscr{L}\{y\} = \mathscr{L}\{t^2e^{3t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$(s-3)^2Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

Using partial fraction:

$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

Thus $$y(t) = 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}.$$

We know,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\bigg|_{s\to s-3}\right\} = te^{3t} \qquad \text{and} \qquad \mathscr{L}^{-1}\left\{\frac{4!}{s^5}\bigg|_{s\to s-3}\right\} = t^4e^{3t}.$$

Hence: $$y(t) = 2e^{3t} + 11te^{3t} + \tfrac{1}{12}t^4e^{3t}.$$

7.3.2 TRANSLATION ON THE t-AXIS

DEFINITION 7.3.1 Unit Step Function

The **unit step function** $\mathcal{U}(t-a)$ is defined to b

$$\mathcal{U}(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a.\end{cases}$$

THEOREM 7.3.2 Second Translation Theorem

If $F(s)=\mathscr{L}\{f(t)\}$ and $a>0$, then

$$\mathscr{L}\{f(t-a)\,\mathcal{U}(t-a)\}=e^{-as}F(s).$$

For inverse:

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}=f(t-a)\,\mathcal{U}(t-a).$$

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+9}e^{-\quad s/2}\right\}$.

SOLUTION **(a)** With the three identifications $a = 2$, $F(s) = 1/(s-4)$, and $\mathscr{L}^{-1}\{F(s)\} = e^{4t}$, we have from (15)

$$\mathscr{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}\,\mathscr{U}(t-2).$$

(b) With $a = \pi/2$, $F(s) = s/(s^2+9)$, and $\mathscr{L}^{-1}\{F(s)\} = \cos 3t$, (15) yields

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\quad s/2}\right\} = \cos 3\left(t-\frac{\quad}{2}\right)\mathscr{U}\left(t-\frac{\quad}{2}\right).$$

Exercise 7.3

Use the Laplace transform to solve the given initial-value problem

24. $y'' - 4y' + 4y = t^3e^{2t}, \quad y(0) = 0, y'(0) = 0$

Solution:

The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 4\,[s\,\mathscr{L}\{y\} - y(0)] + 4\,\mathscr{L}\{y\} = \frac{6}{(s-2)^4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{20}\,\frac{5!}{(s-2)^6}\,.$$

Thus, $y = \dfrac{1}{20}t^5e^{2t}$.

25. $y'' - 6y' + 9y = t, \quad y(0) = 0, y'(0) = 1$

Solution:

The Laplace transform of the differential equation is

$$s^2 \mathscr{L}\{y\} - sy(0) - y'(0) - 6\left[s\mathscr{L}\{y\} - y(0)\right] + 9\mathscr{L}\{y\} = \frac{1}{s^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1+s^2}{s^2(s-3)^2}$$
$$= \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{10}{9}\frac{1}{(s-3)^2}$$

The Inverse Laplace then gives

$$y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}.$$

29. $y'' - y' = e^t \cos t, \quad y(0) = 0, y'(0) = 0$

Solution:

The Laplace transform of the differential equation is

$$s^2 \mathscr{L}\{y\} - sy(0) - y'(0) - [s\mathscr{L}\{y\} - y(0)] = \frac{s-1}{(s-1)^2+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s^2-2s+2)}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s-1}{(s-1)^2+1} + \frac{1}{2}\frac{1}{(s-1)^2+1}$$

The Inverse Laplace then gives

$$y = \frac{1}{2} - \frac{1}{2}e^t \cos t + \frac{1}{2}e^t \sin t.$$

34. Recall that the differential equation for the instantaneous charge $q(t)$ on the capacitor in an *LRC*-series circuit is given by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{20}$$

See Section 5.1. Use the Laplace transform to find $q(t)$ when $L = 1$ h, $R = 20\ \Omega$, $C = 0.005$ f, $E(t) = 150$ V, t =0, $q(0) = 0$, and $i(0) = 0$. What is the current $i(t)$?

Solution:

Note: dq/dt = i(t), i(0)=0 implies dq/dt at t=0 is 0 (zero).

The differential equation is

$$\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 200q = 150, \quad q(0) = q'(0) = 0.$$

The Laplace transform of this equation is

$$s^2\mathscr{L}\{q\} + 20s\mathscr{L}\{q\} + 200\mathscr{L}\{q\} = \frac{150}{s}.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \frac{150}{s(s^2+20s+200)}$$

$$= \frac{3}{4}\frac{1}{s} - \frac{3}{4}\frac{s+10}{(s+10)^2+10^2} - \frac{3}{4}\frac{10}{(s+10)^2+10^2}$$

$$q(t) = \frac{3}{4} - \frac{3}{4}e^{-10t}\cos 10t - \frac{3}{4}e^{-10t}\sin 10t$$

and

$$i(t) = q'(t) = 15e^{-10t}\sin 10t.$$

Thank You.