# Homework 2

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Course : CSE495A

Section : 1

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Date : 18-04-2024

Ans to the Q.NO-1

(a) Given,
the equations for the nonholonomic integratri

$\dot{x}_1 = u_1$ . . . . . (i)

$\dot{x}_2 = u_2$ . . . . . (ii)

$\dot{x}_3 = x_2 u_1$ . . . . . (iii)

Now,

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

So,

$x_1 = z_1$

$x_3 = z_2$

$\therefore \dot{z}_1 = \dot{x}_1 = u_1$ $\quad\therefore u_1 = \dot{z}_1$

$\therefore \dot{z}_2 = \dot{x}_3 = x_2 u_1$

$\Rightarrow x_2 = \dfrac{\dot{x}_3}{u_1}$

$\Rightarrow x_2 = \dfrac{\dot{x}_3}{\dot{z}_1}$ $\quad$ So, $x_2 = \dfrac{\dot{z}_2}{\dot{z}_1}$

$\therefore u_2 = \dot{x}_2$

$= \dfrac{d}{dt}\left(\dfrac{\dot{z}_2}{\dot{z}_1}\right) = \dfrac{1}{\dot{z}_1}\ddot{z}_2 + \dot{z}_2\left(-\dfrac{1}{\dot{z}_1^2}\right).\ddot{z}_1$

$= \dfrac{\ddot{z}_2}{\dot{z}_1} - \dfrac{\dot{z}_2\ddot{z}_1}{\dot{z}_1^2}$

Therefore, the systems states and inputs can be expressed in term of the flat output $z = (x_1, x_3)$,

$u_1 = \dot{z}_1$

$u_2 = \frac{\ddot{z}_2}{\dot{z}_1} - \frac{\dot{z}_2 \ddot{z}_1}{\dot{z}_1^2}$

$x_1 = z_1$,

$x_2 = \frac{\dot{z}_2}{\dot{z}_1}$

$x_3 = z_2$

As $x$ and $u$ can be written as linear combination of $z$ and $z$'s derivatives, the system is differentially flat for $z = (x_1, x_3)$

(Ans)

(b) Given,

$t_i = 0$

$t_f = T$

and the following initial conditions:

At $t_i = 0$, $x_1(0)$, $x_2(0)$, $x_3(0)$, $\dot{x}_1(0) = 1$

At $t_f = T$, $x_1(T)$, $x_2(T)$, $x_3(T)$, $\dot{x}_1(T) = 1$

In addition there are four basis functions,

$\psi_1(t) = 1$, $\psi_2(t) = t$, $\psi_3(t) = t^2$, $\psi_4(t) = t^3$

Now,

$$z_1(t) = \alpha_{11}.1 + \alpha_{12}.t + \alpha_{13} t^2 + \alpha_{14} t^3$$

$$\dot{z}_1(t) = \alpha_{11}.0 + \alpha_{12}.1 + \alpha_{13}.2t + \alpha_{14}.3t^2$$

$$= \alpha_{12} + 2\alpha_{13} t + 3\alpha_{14} t^2$$

$$z_2(t) = \alpha_{21}.1 + \alpha_{22} t + \alpha_{23} t^2 + \alpha_{24} t^3$$

$$\dot{z}_2(t) = \alpha_{21}.0 + \alpha_{22}.1 + \alpha_{23}.2t + \alpha_{24}.3t^2$$

$$= \alpha_{22} + 2\alpha_{23} t + 3\alpha_{24} t^2$$

Therefore, the matrix-vector equations for the system is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & T & T^2 & T^3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T & T^2 & T^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2T & 3T^2 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{24} \end{bmatrix} = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(Ans)

(c) Given, six basis functions

$$\psi_1(t) = 1,\ \psi_2(t) = t,\ \psi_3(t) = t^2,\ \psi_4(t) = t^3,$$

$$\psi_5(t) = t^4,\ \psi_6(t) = t^5$$

Now,

$$z_1(t) = \alpha_{11}.1 + \alpha_{12}.t + \alpha_{13}t^2 + \alpha_{14}.t^3 + \alpha_{15}t^4 + \alpha_{16}t^5$$

$$\dot{z}_1(t) = \alpha_{11}.0 + \alpha_{12}.1 + \alpha_{13}.2t + \alpha_{14}.3t^2 + \alpha_{15}.4t^3 + \alpha_{16}5t^4$$

$$= \alpha_{12} + 2\alpha_{13}t + 3\alpha_{14}t^2 + 4\alpha_{15}t^3 + 5\alpha_{16}t^4$$

$$z_2(t) = \alpha_{21}.1 + \alpha_{22}.t + \alpha_{23}t^2 + \alpha_{24}.t^3 + \alpha_{25}t^4 + \alpha_{26}t^5$$

$$\dot{z}_2(t) = \alpha_{21}.0 + \alpha_{22}.1 + \alpha_{23}2t + \alpha_{24}3t^2 + \alpha_{25}4t^3 + 5\alpha_{26}t^4$$

$$= \alpha_{22} + 2\alpha_{23}t + 3\alpha_{24}t^2 + 4\alpha_{25}t^3 + 5\alpha_{26}t^4$$

Therefore, the matrix-vector equations for the system is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & T & T^2 & T^3 & T^4 & T^5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 & 4T^3 & 5T^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & T & T^2 & T^3 & T^4 & T^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2T & 3T^2 & 4T^3 & 5T^4 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{15} \\ \alpha_{16} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{24} \\ \alpha_{25} \\ \alpha_{26} \end{bmatrix} = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(Ans).

Ans to the QNO-2

(a) Given,

the equations for dynamically extend unicycle robot:

$$\dot{x}(t) = v(t)\cos\theta(t) \quad \ldots \text{(i)}$$

$$\dot{y}(t) = v(t)\sin\theta(t) \quad \ldots \text{(ii)}$$

$$\dot{v}(t) = a(t) \quad \ldots \text{(iii)}$$

$$\dot{\theta}(t) = \omega(t) \quad \ldots \text{(iv)}$$

with initial conditions:

$x(0)=0$, $y(0)=0$, $v(0)=0.5$, $\theta(0) = -\frac{\pi}{2}$

and final conditions:

$x(t_f)=5$, $y(t_f)=5$, $v(t_f)=0.5$, $\theta(t_f) = -\frac{\pi}{2}$ where $t_f=15$

The flat outputs are defined as $z=(x,y)$

$$\Rightarrow \dot{z} = (\dot{x}(t), \dot{y}(t))$$

$$= (v(t)\cos\theta(t),\ v(t)\sin\theta(t)) \quad \ldots \text{(v)}$$

$$\Rightarrow \ddot{z} = (\ddot{x}(t), \ddot{y}(t))$$

$$= (a(t)\cos\theta(t) - v(t)\omega(t)\sin\theta(t),\ a(t)\sin\theta(t) + v(t)\omega(t)\cos\theta(t))$$

$$\therefore a(t) = \frac{\ddot{z}_x \cos\theta(t) + \ddot{z}_y \sin\theta(t)}{v(t)}$$

$$\therefore \omega(t) = \frac{\dot{z}_y \cos\theta(t) - \dot{z}_x \sin\theta(t)}{v(t)}$$

From (v), using pythagorean theorem:

$$v(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} = \sqrt{(\dot{z}_x)^2 + (\dot{z}_y)^2}$$

From (v),

$$\tan\theta = \frac{\dot{y}(t)}{\dot{x}(t)}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right)$$

$$\therefore \theta = \tan^{-1}\left(\frac{\dot{z}_y}{\dot{z}_x}\right).$$

whereas $x(t) = z_x(t)$

$y(t) = z_y(t)$

Therefore, the state variables and control inputs can be fully determined from the outputs $z = (x, y)$ and their derivatives.

(Ans)