MAT 350 Engineering mathematics

Higher Order ODEs
UNDETERMINED COEFFICIENTS

Lecture: 6

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To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x), \qquad (1)$$

The solution y has two parts:

- the complementary function y_c *and*
- any particular solution y_p of the nonhomogeneous equation (1).

The general solution of (1) is $\mathbf{y = y_c + y_p}$.

* y_c is obtained from the homogeneous part:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

- y_p can be obtained by many methods. Few familiar approaches:
  -Superposition approach, Annihilator method, Variation of parameters, and more...

Method of Undetermined Coefficient: Superposition approach

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$

Solution: For y_c we find the solution of

$$y'' + 4y' - 2y = 0.$$

The auxiliary equation is

$$m^2 + 4m - 2 = 0$$

Roots are:

$$m_1 = -2 - \sqrt{6} \text{ and } m_2 = -2 + \sqrt{6}.$$

The complementary function is:

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Method of Undetermined Coefficient: Superposition approach

We assume that y_p has the same form as g(x).

Now, because the function g(x) is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

$$y_p' = 2Ax + B \qquad \text{and} \qquad y_p'' = 2A$$

Substitute into the original ODE:

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6.$$

Equating the coefficients from both sides, gives

$$A = -1,\ B = -\tfrac{5}{2},\ \text{and } C = -9.$$

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

The general solution of the given equation is

$$\boxed{y = y_c + y_p = c_1e^{-(2+\sqrt{6})x} + c_2e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.}$$

Method of Undetermined Coefficient

Find a particular solution of $y'' - y' + y = 2 \sin 3x$.

Try yourself

Method of Undetermined Coefficient: Superposition approach

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.

Hints for solution:

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

$$g(x) = g_1(x) + g_2(x) = polynomial + exponentials.$$

$$y_p = y_{p_1} + y_{p_2},$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + Ee^{2x}$. Substituting

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x}$$
$$= 4x - 5 + 6xe^{2x}.$$

$A = -\frac{4}{3}, B = \frac{23}{9}, C = -2,$ $\quad y_p = -\dfrac{4}{3}x + \dfrac{23}{9} - 2xe^{2x} - \dfrac{4}{3}e^{2x}.$

$$\boxed{y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}.}$$

TABLE 4.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

Differential operator: D

Example: D(2x) = 2, D(*sin*x) = *cosx*, etc.

Inverse Differential operator: 1/D

$$\frac{1}{D}(x) = \frac{x^2}{2} \qquad \frac{1}{D}(\sin x) = -\cos x$$

Solution Technique:

$y = y_c + y_p$ where

Yc= from homogeneous form

Yp= write the ODE as

$f(D)y_p = g(x)$

$y_p = g(x)/f(D) = [f(D)]^{-1} g(x)$

Few inversion formulae:

$$(1-D)^{-1}=1+D+D^{2}+D^{3}+$$
$$(1+D)^{-1}=1-D+D^{2}-D^{3}+$$
$$(1-D)^{-2}=1-2D+3D^{2}-4D^{3}$$
$$(1+D)^{-2}=1+2D+3D^{2}+4D^{3}$$

Example:

$(1\text{-}D)^{-1}(x^2) = ?$

Type -1: g(x) is algebraic function.

$$Solve \quad y'' - 3y' + 2y = 4x$$

Solution: Auxiliary equation is:

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow m = 2,1$$

$$y_p = c_1 e^x + c_2 e^{2x}$$

For y_p:

$$(D^2 - 3D + 2) y_p = 4x$$

$$y_p = \frac{1}{(D^2 - 3D + 2)}(4x)$$

$$= \frac{1}{2(1 - \frac{3D}{2} + \frac{D^2}{2})}(4x)$$

$$y_p = \frac{1}{2}\left[1-\left(\frac{3D}{2}-\frac{D^2}{2}\right)\right]^{-1}(4x)$$

$$= \frac{1}{2}\left[1+\left(\frac{3D}{2}-\frac{D^2}{2}\right)+\left(\frac{3D}{2}-\frac{D^2}{2}\right)^2+\cdots\right](4x)$$

$$= \frac{1}{2}\left(4x+\frac{3}{2}.4+0+\cdots+0\right)$$

= 2x+3

The general solution is y= yc + yp

$$y = c_1 e^x + c_2 e^{2x} + 2x + 3$$

Type -2: g(x) is exponential function.

$$f(D)\,y_p = e^{ax}$$

$$y_p = \frac{1}{f(D)}e^{ax}$$

If f(a) ≠ 0, then

$$y_p = \frac{1}{f(a)}e^{ax}$$

If f(a) = 0, then

$$y_p = e^{ax}\frac{1}{f(D+a)}(1)$$

$$Solve \quad y'' - y' - 2y = e^{x}$$

For comp. solution:

$$y_c = c_1 e^{2x} + c_2 e^{-x}$$

For part. solution:

$$(D^2 - D - 2)\, y_p = e^{x}$$

$$y_p = \frac{1}{(D^2 - D - 2)} e^{-x}$$

Note that, a=1 *and f*(a) ≠ 0.

$$= \frac{1}{(D+1)(D-2)} e^{x}$$

$$= \frac{1}{(1+1)(1-2)} e^{x} = -\frac{1}{2} e^{x}$$

The general solution is y= yc + yp

$$y = c_1 e^{2x} + c_2 e^{-x} - \frac{1}{2} e^{x}$$

$$Solve \quad y'' - y' - 2y = e^{-x}$$

For comp. solution:

$$y_c = c_1 e^{2x} + c_2 e^{-x}$$

For part. solution:

$$(D^2 - D - 2) y_p = e^{-x}$$

$$y_p = \frac{1}{(D^2 - D - 2)} e^{-x}$$

Note that, a= -1 *and f*(a) = 0.

$$= \frac{1}{(D+1)(D-2)} e^{-x}$$

$$= e^{-x} \frac{1}{(D+1-1)(-1-2)} (1)$$

Note: (D+1)=0, for a= -1. Hence D it is replaced there by (D+a).

$$= e^{-x} \frac{1}{-3D} (1)$$

$$= e^{-x} (-\frac{1}{3})(x) = -\frac{1}{3} x e^{-x}$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-x} - \frac{1}{3} x e^{-x}}$$

Exercise: **On Superposition approach**

$$y'' - 10y' + 25y = 30x + 3$$

$$y'' + 3y = -48x^2e^{3x}$$

$$4y'' - 4y' - 3y = \cos 2x$$

$$y'' + y = 2x \sin x$$

$$y'' - 2y' + 5y = e^x \cos 2x$$

$$y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, y'(0) = 1$$

$$y'' + 4y = g(x), \quad y(0) = 1, y'(0) = 2, \quad \text{where}$$

$$g(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & x > \pi/2 \end{cases}$$

Exercise: On **Inverse Operator Method/Substitution method**

$$y'' + 3y' + 2y = 4x^2.$$

$$y''' + 10y'' + 25y' = e^x$$

$$y'' - y' - 12y = e^{4x}$$

$$y'' - 2y' - 3y = 4e^x - 9$$

$$y'' + 2y' + y = x^2e^{-x}$$

$$y'' + 5y' - 6y = 10e^{2x}, \quad y(0) = 1, y'(0) = 1$$

$$y'' + y' = x, \quad y(0) = 1, y'(0) = 0$$

End