Course Name : Physics – I
Course # PHY 107

Notes-7 : Work and Eenergy

Abu Mohammad Khan
Department of Mathematics and Physics
North South University
https://abukhan.weebly.com

Topics to be studied

- Work: Definition and it's properties
- Total work done: Work-Energy Theorem in on dimension
- Work done by variable forces in vector form in any dimensions
- 2^{nd} Fundamental Theorem of Calculus and the Path independance
- Conservation law of total mechanical energy
- Work done by Gravitational force and the total energy
- Work done by Elastic force and the total energy
- Examples

Definition of Work:

Let's start from what we already learnt.

- Definition: If a force F acts on a body of mass and as a result the body moves a distance d, we say that the force F has done 'Work' on the body. Mathematically, it is expressed as: $W = Fd$, where W is the amount of work done by the force F on the body.
- The above formula is only valid when
  - the force F remains constant over the distance d.
  - the force and distance are in the same direction. This means that the distance is the magnitude of the displacement, not the actual path.
- In reality, a force may not remain constant. Also the displacement and the force may not be in the same direction.
- To take into account the general nature of force and displacement, the work is now defined as

$$\boxed{W = \int_i^f \vec{F} \cdot d\vec{r} = \int_i^f F \cos\theta \, dr}\ .$$

Here θ is the angle between the directions of force and displacement. i and f stand for the initial and final values respectively.

- For simplicity, we consider only one dimension. The variable force can be replaced by the average value for a fixed interval. Unless the interval changes, the average force remains fixed, and hence the integral becomes simple. So, the work done formula becomes

$$W = \int_i^f F(x)\,dx = F_{\rm av}(x_f - x_i) = F_{\rm av}\,\Delta x \ .$$

- If there are more forces acts on a body, then the total work done is given by the net force, *i.e.* if $\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$, then $F_{\rm av}$ is the average of the total force, and by using the $2^{\rm nd}$ law we can write,

$$\begin{aligned} W_{\rm tot} &= F_{\rm av}\Delta x = ma \times \frac{v_f^2 - v_i^2}{2a} \quad \text{(By the } 4^{\rm th} \text{ equation of motion)} \ , \\ &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \ , \\ &= \left(\frac{1}{2}mv^2\right)_f - \left(\frac{1}{2}mv^2\right)_i \ . \end{aligned}$$

- We define the kinetic energy as $K = (1/2)mv^2 \equiv (1/2)m\vec{v}\cdot\vec{v}$, so that, we can write:

$$K_f = \left(\frac{1}{2}mv^2\right)_f \equiv \frac{1}{2}mv_f^2 \quad \text{and} \quad K_i = \left(\frac{1}{2}mv^2\right)_i \equiv \frac{1}{2}mv_i^2 \ .$$

- In terms of kinetic energy, the total work done is

$$\boxed{W_{\text{tot}} = K_F - K_i = \Delta K} \ .$$

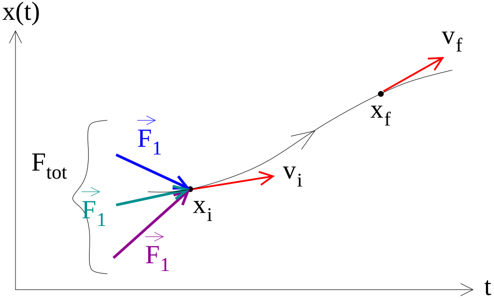
- This is the Work-Energy Theorem which states that the total work done on a body is equal to the change in kinetic energy of the body.
- It does not depend on the path.
- It is a scalar quantity.

Now, The total force on a body is $\sum \vec{F} \equiv \vec{F}_{\rm tot} = \vec{F}_1 + \vec{F}_2 \cdots$. The force here are five type of forces. Under this total force if a body of mass m undergoes displacement, then the total work can be written as:

$$\begin{aligned}
W_{\rm tot} &= \vec{F}_{\rm tot} \cdot \Delta\vec{x} \ = \ \int_{x_{\rm i}}^{x_{\rm f}} \vec{F}_{\rm tot} \cdot d\vec{x} \ = \ K_f - K_i \equiv \Delta K \ , \quad \text{(Work-Energy Theorem)} \\
&= \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) \cdot \Delta\vec{x} \equiv \int_{x_i}^{x_f} \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) \cdot d\vec{x} \\
&= \int_{x_i}^{x_f} \vec{F}_1 \cdot d\vec{x} + \int_{x_i}^{x_f} \vec{F}_2 \cdot d\vec{x} + cdots \ , \quad \text{(Work by each force)} \\
\therefore \quad W_{\rm tot} &= w_1 + w_2 + \cdots \ = \ K_f - K_i \ .
\end{aligned}$$

In the following, we now compute individual work done by any of the five forces.

- Suppose an object of mass m has velocity v_i at position x_i, and under total force $\vec{F}_{\text{tot}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ it moved to a position x_f with velocity v_f.



- The total work done is

$$W_{\text{tot}} = \int_{x_i}^{x_f} \vec{F}_{\text{tot}} \cdot d\vec{x} = \int_{x_i}^{x_f} \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) \cdot d\vec{x}\ ,$$

$$\underbrace{\int_{x_i}^{x_f} m\frac{d\vec{v}}{dt} \cdot d\vec{x}}_{2^{\text{nd}}\text{ law}} = \underbrace{\int_{x_i}^{x_f} \vec{F}_1 \cdot d\vec{x}}_{w_1} + \underbrace{\int_{x_i}^{x_f} \vec{F}_2 \cdot d\vec{x}}_{w_2} + \underbrace{\int_{x_i}^{x_f} \vec{F}_3 \cdot d\vec{x}}_{w_3}\ .$$

Now, we compute the left-hand side. The total work done is,

$$
\begin{aligned}
W_{\text{total}} &= \int_{x_i}^{x_f} \vec{F_{\text{tot}}} \cdot d\vec{x} \;=\; \int_{x_i}^{x_f} m\frac{d\vec{v}}{dt} \cdot d\vec{x} \ , \\
&= \int_{x_i}^{x_f} m\frac{dv_x}{dt}\, dx + \int_{y_i}^{y_f} m\frac{dv_y}{dt}\, dy + \int_{z_i}^{z_f} m\frac{dv_z}{dt}\, dz \ , \\
&= \int_{x_i}^{x_f} m\frac{dv_x}{dt}(v_x dt) + \text{similarly for the } y\text{- and } z\text{-components} \ , \\
&= \int_{v_{xi}}^{v_{xf}} mv_x\, dv_x + \cdots \ , \quad \text{(change of variable from } x \text{ to } v_x) \ , \\
&= \frac{1}{2}mv_{xf}^2 - \frac{1}{2}mv_{xi}^2 + \frac{1}{2}mv_{yf}^2 - \frac{1}{2}mv_{yi}^2 + \frac{1}{2}mv_{zf}^2 - \frac{1}{2}mv_{zi}^2 \ , \\
&= \left(\frac{1}{2}\, m \left[v_x^2 + v_y^2 + v_z^2\right]\right)_f - \left(\frac{1}{2}\, m \left[v_x^2 + v_y^2 + v_z^2\right]\right)_i = \left(\frac{1}{2}mv^2\right)_f - \left(\frac{1}{2}mv^2\right)_i \ , \\
\therefore \ W_{\text{tot}} &= K_f - k_i \;\equiv\; \Delta K \ . \qquad \Leftarrow \text{Work-Energy Theorem}
\end{aligned}
$$

Let's now compute the right-hand side.

- There are three integrals. Consider the first integral which is

$$w_1 \;=\; \int_{x_i}^{x_f} \vec{F}_1 \cdot \vec{x}\ .$$

- If the force is the gradient of a scalr function, then we write,

$$\vec{F}_1 \;=\; -\nabla U_1\ .$$

- Hence the work done is

$$\begin{aligned} w_1 &= \int_{x_i}^{x_f} \vec{F}_1 \cdot d\vec{x}\ , \\ &= -\int_{x_i}^{x_f} \nabla U_1 \cdot d\vec{x} \;=\; \int_{U_{1i}}^{U_{1f}} dU_1\ , \quad (2^{\text{nd}} \text{ Fundamental Theorem of Calculcus}) \\ &= -\Big(U_{1f} - U_{1i}\Big) \;=\; -\Delta U_1\ . \end{aligned}$$

- So What is the 2^{nd} Fundamental Theorem of Calculus!! Let's review it !!!

2^{nd} Fundamental Theorem of Calculus:

We need to understand and use the 2^{nd} fundamental theorem of Calculus. The Theorem states that

$$\underbrace{\int_a^b f(x)dx \; = \; F(b) - F(a) \; \equiv \; \Delta F}_{\text{Net change is given by the integral}} \quad \Longleftrightarrow \quad \underbrace{F'(x) \; = \; f(x)}_{\text{Anti-derivative}} \; .$$

- Here $f(x)$ is any of the five forces.
- ΔF is Path independent iff $F' = f$.
- The function $F(x)$ is called the potential function, and in this case it will be called potential energy function and denoted by $U(x)$.
- The function $f(x)$ is called the conservative function.
- It is very important to note that if the $\Longleftrightarrow$ condition does not hold, then $f(x)$ is NOT a conservative function and ΔF is also NOT Path independent.

Let's now use this concept into our evaluation of the first integral on the right-hand side.

- We can apply the Theorem for each component.
- For the x-component, we identify: $f(x) \rightarrow F_x$ and $F(x) \rightarrow U(x)$.
- The function $U(x)$ is known as the 'Potential Energy' of the body.
- Similarly for the y- and z-components.
- Therefore, the integral becomes:

$$\int_a^b f(x)dx \;=\; \Delta F \equiv F(b) - F(a) \;\Longrightarrow\; \int_{x_i}^{x_f} \vec{F}_1 \cdot d\vec{x} \;=\; -\,\Delta U_1 \equiv -(U_{1f} - U_{1i})$$

$$\text{iff} \quad f(x) \;=\; F' \;\Longrightarrow\; \vec{F}_1 \;=\; -\,\nabla U_1' \,.$$

- Similarly for any other integral:

$$\int_{x_i}^{x_f} \vec{F}_2 \cdot d\vec{x} \;=\; -\,\Delta U_2 \quad \text{iff} \quad \vec{F}_2 \;=\; -\,U_2' \qquad \text{and so on.}$$

- Putting all these together, it is easily found that (for any number of forces):

$$\begin{aligned} W_{\text{tot}} &= w_1 + w_2 + \cdots \ , \\ \Delta K &= -\Delta U_1 - \Delta U_2 - \cdots \ , \\ K_f - K_i &= -(U_{1f} - U_{1i}) - (U_{2f} - U_{2i}) - \cdots \ , \\ K_f + U_{1f} + U_{2f} + \cdots &= K_i + U_{1i} + U_{2i} + \cdots \ , \\ \Big(K + U_1 + U_2 + \cdots\Big)_f &= \Big(K + U_1 + U_2 + \cdots\Big)_i \ . \end{aligned}$$

- We now define

$$\text{Total Energy, } E = K + U_1 + U_2 + \cdots = K + \sum U \ .$$

Therefore, we find one of the most inportant equation:

$$E_f = E_i \ .$$

This is the Conservation Law of Total Mechanical Energy.

Short Summary:

From the previous analysis and derivation it is clear that:

- The total energy is conserved if and only if the force on the body is a derivative of a scalar function.
- The scalar function is known as the 'Potential Energy'.
- Therefore, the system is called a 'Conservative System', and the corresonding force is called the conservative force.
- The work done (that is, the integral) is Path independent, and is given by the Net Change in Potential energy.
- In short we can say that: if the work done is path independent then it also implies that the force is conservative and also the total energy is conserved.
- If any of three properties is NOT satisfied, then all are violated, and hence the total energy is NOT conserved or Work is path dependent, or the force is not conservative.