Higher Order Linear ODEs: Variable coefficients

Cauchy-Euler equations:

A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

where a_n, a_{n-1}, a_{n-2},…a_0 are constants.

Example. $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = 0$

Solution. Let, $y = x^m$ is trial solution of the given homogeneous ODE. Then we have,

$x^2 \cdot m(m-1)x^{m-2} - 2x \cdot m x^{m-1} - 4x^m = 0$

$\Rightarrow \{m(m-1) - 2m - 4\}x^m = 0$

$\Rightarrow m(m-1) - 2m - 4 = 0 \quad [\because x^m \neq 0]$

$\Rightarrow m^2 - 3m - 4 = 0 \Rightarrow m = -1, 4$

The general solution of the ODE becomes,

$$y = c_1 x^{-1} + c_2 x^4$$

For real and distinct roots m_1 and m_2:
$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Higher Order Linear ODEs: Variable coefficients

Cauchy-Euler equations: Homogeneous form

Example. $4x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+y=0$

Solution. Let, $y=x^m$ is trial solution of the given homogeneous ODE. Then we have,

$4x^2\cdot m(m-1)x^{m-2}+8x\cdot mx^{m-1}+x^m=0$

$\Rightarrow \{4m(m-1)+8m+1\}x^m=0$

$\Rightarrow 4m(m-1)+8m+1=0 \quad [\because x^m\neq 0]$

$\Rightarrow 4m^2+4m+1=0$

$\Rightarrow (2m+1)^2=0\Rightarrow m=-\frac{1}{2},-\frac{1}{2}$

For real and equal roots m_1 and m_1:
$y=(c_1+c_2\ln x)x^{m_1}$

The general solution of the ODE becomes,

$$y=(c_1+c_2\ln x)x^{-\frac{1}{2}}$$

Higher Order Linear ODEs: Variable coefficients

Cauchy-Euler equations: Homogeneous form

Example. $x^2 \frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0$

Solution. Let, $y = x^m$ is trial solution of the given homogeneous ODE. Then we have,

$x^2 \cdot m(m-1)x^{m-2} - 5x \cdot mx^{m-1} + 9x^m = 0$

$\Rightarrow \{m(m-1) - 5m + 9\}x^m = 0$

$\Rightarrow m(m-1) - 5m + 9 = 0 \quad [\because x^m \neq 0]$

$\Rightarrow m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$

For real and equal roots m_1 and m_1:
$y = (c_1 + c_2 \ln x)x^{m_1}$

The general solution of the ODE becomes,

$$y = c_1x^3 + c_2x^3 \ln x = x^3(c_1 + c_2 \ln x)$$

Higher Order Linear ODEs: Variable coefficients

Cauchy-Euler equations: Homogeneous form

Example. $4x^2\frac{d^2y}{dx^2} + 17y = 0,\ y(1) = -1,\ y'(1) = -\frac{1}{2}$

Solution. Let, $y = x^m$ is trial solution of the given homogeneous ODE. Then we have,

$4x^2 \cdot m(m-1)x^{m-2} + 17x^m = 0$

$\Rightarrow \{4m(m-1) + 17\}x^m = 0$

$\Rightarrow 4m(m-1) + 17 = 0 \quad [\because x^m \neq 0]$

$\Rightarrow 4m^2 - 4m + 17 = 0$

$\Rightarrow (2m-1)^2 = -16 \Rightarrow m = \frac{1}{2} \pm 2i$

For complex roots $m_1 \pm im_2$:
$y_c = x^{m_1}[c_1\cos(m_2\ln x) + c_2\sin(m_2\ln x)]$

The general solution of the ODE becomes,

$$y = x^{\frac{1}{2}}[c_1\cos(2\ln x) + c_2\sin(2\ln x)]$$

Applying the boundary conditions, $y(1) = -1,\ y'(1) = -\frac{1}{2}$, we obtain $c_1 = -1$ and $c_2 = 0$

The required solution of the given IVP, $y = -x^{\frac{1}{2}}\cos(2\ln x)$.

Higher Order Linear ODEs: Variable coefficients

Cauchy-Euler equations: Homogeneous form

Example. $x^3 \frac{d^3y}{dx^3} + 5x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 8y = 0$

Solution. Let, $y = x^m$ is trial solution of the given homogeneous ODE. Then we have,

$x^3 \cdot m(m-1)(m-2)x^{m-3} + 5x^2 \cdot m(m-1)x^{m-2} + 7x \cdot mx^{m-1} + 8x^m = 0$

$\Rightarrow \{m(m-1)(m-2) + 5m(m-1) + 7m + 8\}x^m = 0$

$\Rightarrow m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0 \quad [\because x^m \neq 0]$

$\Rightarrow m^3 + 2m^2 + 4m + 8 = 0$

$\Rightarrow m^2(m+2) + 4(m+2) = 0 \Rightarrow (m+2)(m^2+4) = 0 \Rightarrow m = -2, \pm 2i$

The general solution of the ODE becomes,

$$y = c_1 x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x)$$

Higher Order Linear ODEs: Variable coefficients

Cauchy-Euler equations: Non-homogeneous form

Example. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 2x^4e^x$

Solution. Let, $y = x^m$ is trial solution of the homogeneous form of given ODE.

Then we have,

$x^2 \cdot m(m-1)x^{m-2} - 3x \cdot mx^{m-1} + 3x^m = 0$

$\Rightarrow \{m(m-1) - 3m + 3\}x^m = 0 \Rightarrow m(m-1) - 3m + 3 = 0 \quad [\because x^m \neq 0]$

$\Rightarrow m^2 - 4m + 3 = 0 \Rightarrow (m-1)(m-3) = 0 \Rightarrow m = 1, 3$

The complementary solution of the ODE becomes, $y_c = c_1\, x + c_2\, x^3$

The standard form of the given ODE becomes, $\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + 3\frac{y}{x^2} = 2x^2e^x$

for which the particular solution can be written as, $y_p = A(x)\, x + B(x)x^3$

Here, the **Wronskian** of the solutions $y_1 = x$ and $y_2 = x^3$ yields,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3 \neq 0$$

Higher Order Linear ODEs: Variable coefficients

Cauchy-Euler equations: Non-homogeneous form

Example. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 2x^4e^x \Rightarrow \frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + 3\frac{y}{x^2} = 2x^2e^x$

Solution. Therefore, the solutions y_1 and y_2 are linearly independent. We also compute,

$$W_1(y_1, y_2) = \begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x,$$

$$W_2(y_1, y_2) = \begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix} = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x.$$

Now we find,

$$A(x) = \int \frac{W_1}{W}dx = -\int \frac{2x^5e^x}{2x^3}dx = -\int x^2e^x\,dx = -(x^2e^x - 2xe^x + 2e^x),$$

$$B(x) = \int \frac{W_2}{W}dx = \int \frac{2x^3e^x}{2x^3}dx = \int e^x\,dx = e^x.$$

Thus the particular solution becomes, $y_p = -(x^2e^x - 2xe^x + 2e^x)\,x + (e^x)\,x^3$

$$\Rightarrow y_p = 2x^2e^x - 2xe^x$$

Finally, the general solution of the given problem yields,

$$y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x.$$

Higher Order Linear ODEs: Variable coefficients

Exercise 4.7

Solve the following differential equations

1. $x^2y'' - 2y = 0$

2. $4x^2y'' + y = 0$

3. $xy'' + y' = 0$

4. $xy'' - 3y' = 0$

5. $x^2y'' + xy' + 4y = 0$

6. $x^2y'' + 5xy' + 3y = 0$

7. $x^2y'' - 3xy' - 2y = 0$

8. $x^2y'' + 3xy' - 4y = 0$

9. $25x^2y'' + 25xy' + y = 0$

10. $4x^2y'' + 4xy' - y = 0$

11. $x^2y'' + 5xy' + 4y = 0$

12. $x^2y'' + 8xy' + 6y = 0$

13. $3x^2y'' + 6xy' + y = 0$

14. $x^2y'' - 7xy' + 41y = 0$

15. $x^3y''' - 6y = 0$

16. $x^3y''' + xy' - y = 0$

17. $xy^{(4)} + 6y''' = 0$

18. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$

Higher Order Linear ODEs: Variable coefficients

Exercise 4.7

Solve the following differential equations

19. $xy'' - 4y' = x^4$

20. $2x^2y'' + 5xy' + y = x^2 - x$

21. $x^2y'' - xy' + y = 2x$ **22.** $x^2y'' - 2xy' + 2y = x^4e^x$

23. $x^2y'' + xy' - y = \ln x$ **24.** $x^2y'' + xy' - y = \dfrac{1}{x+1}$

25. $x^2y'' + 3xy' = 0, \quad y(1) = 0, y'(1) = 4$

26. $x^2y'' - 5xy' + 8y = 0, \quad y(2) = 32, y'(2) = 0$

27. $x^2y'' + xy' + y = 0, \quad y(1) = 1, y'(1) = 2$

28. $x^2y'' - 3xy' + 4y = 0, \quad y(1) = 5, y'(1) = 3$

29. $xy'' + y' = x, \quad y(1) = 1, y'(1) = -\frac{1}{2}$

30. $x^2y'' - 5xy' + 8y = 8x^6, \quad y\left(\frac{1}{2}\right) = 0, y'\left(\frac{1}{2}\right) = 0$