Spring 2019

MAT 350
Engineering mathematics

Lecture -1
Introduction to ODEs

Mohammad Sahadet Hossain, PhD
Associate Professor, Department of Mathematics and Physics
Additional Director, IQAC
North South University
mohammad.hossain@northsouth.edu

Why should we learn

Ordinary Differential Equations (ODEs)

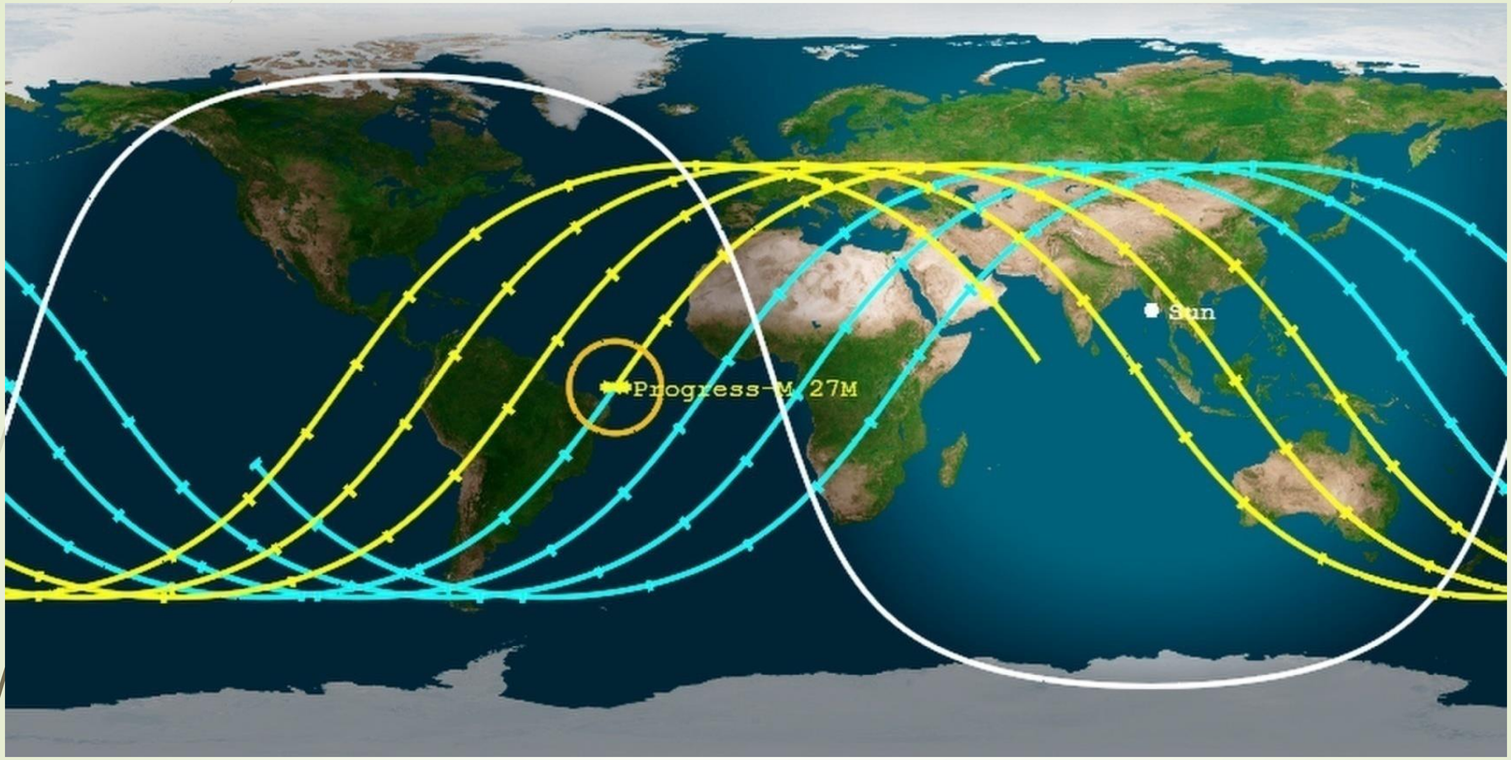
The BIG Picture of Differential Equations

Motion of a Satellite ISS on the earth surface

The BIG Picture of Differential Equations

Orientation of Satellite ISS on Earth

The BIG Picture of Differential Equations



Ground Track View of ISS orientation

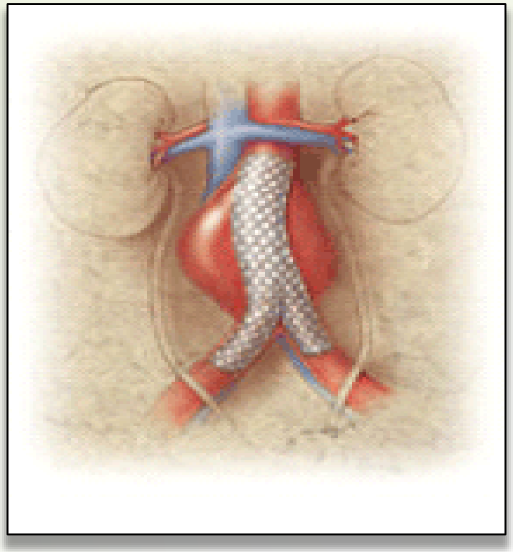
The BIG Picture of Differential Equations

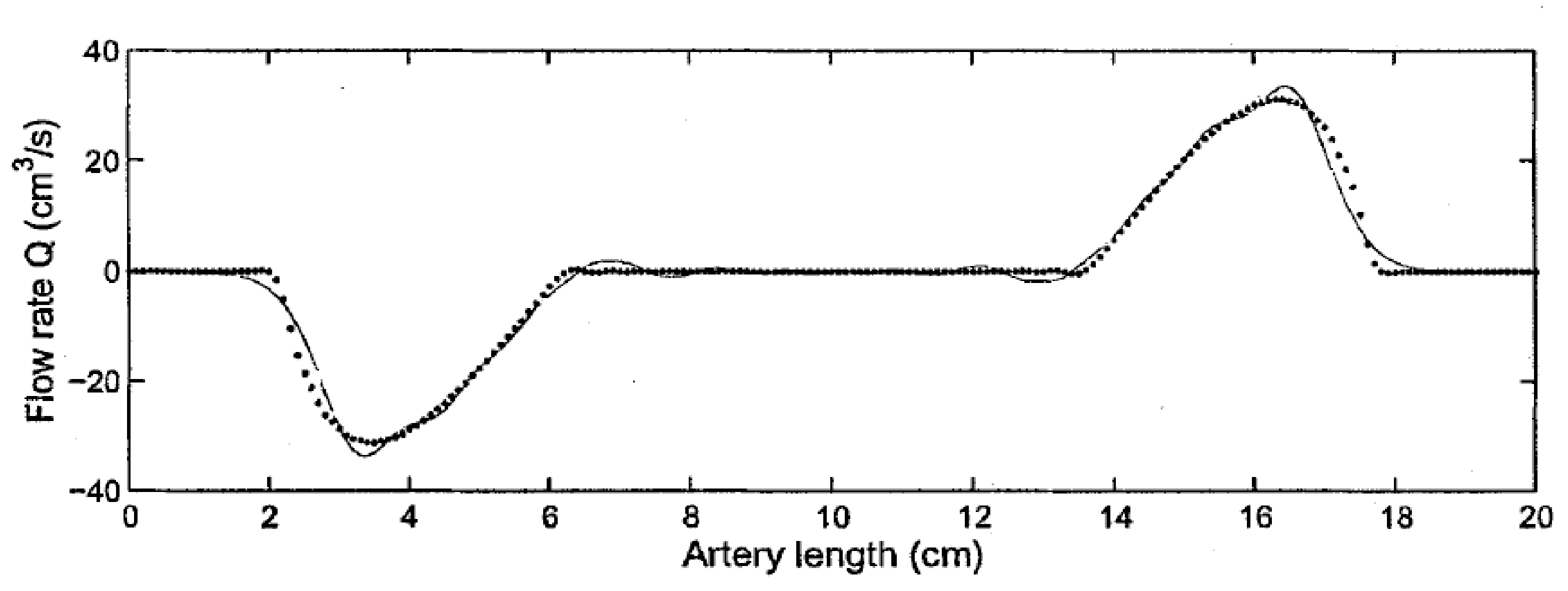
Path of a guided Missile

The BIG Picture of Differential Equations

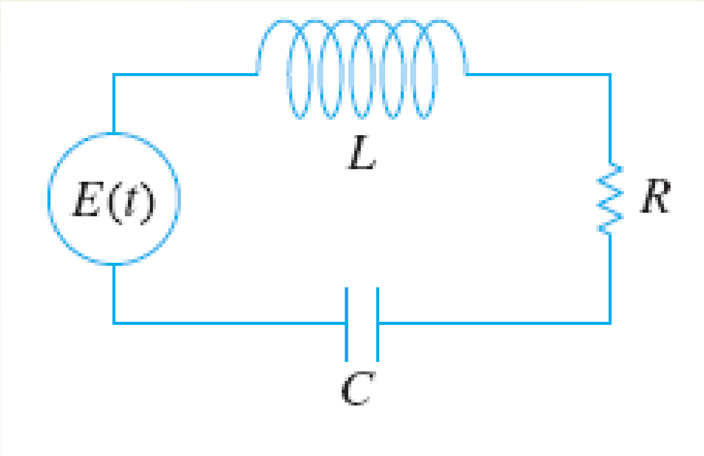
Endograft placement in the surgical treatment of abdominal aortic aneurysms.

The BIG Picture of Differential Equations

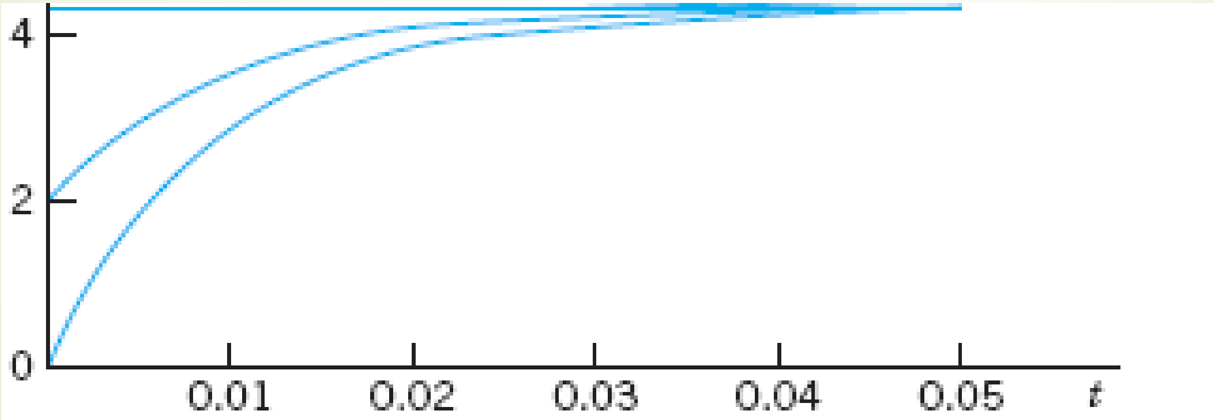




The BIG Picture of Differential Equations

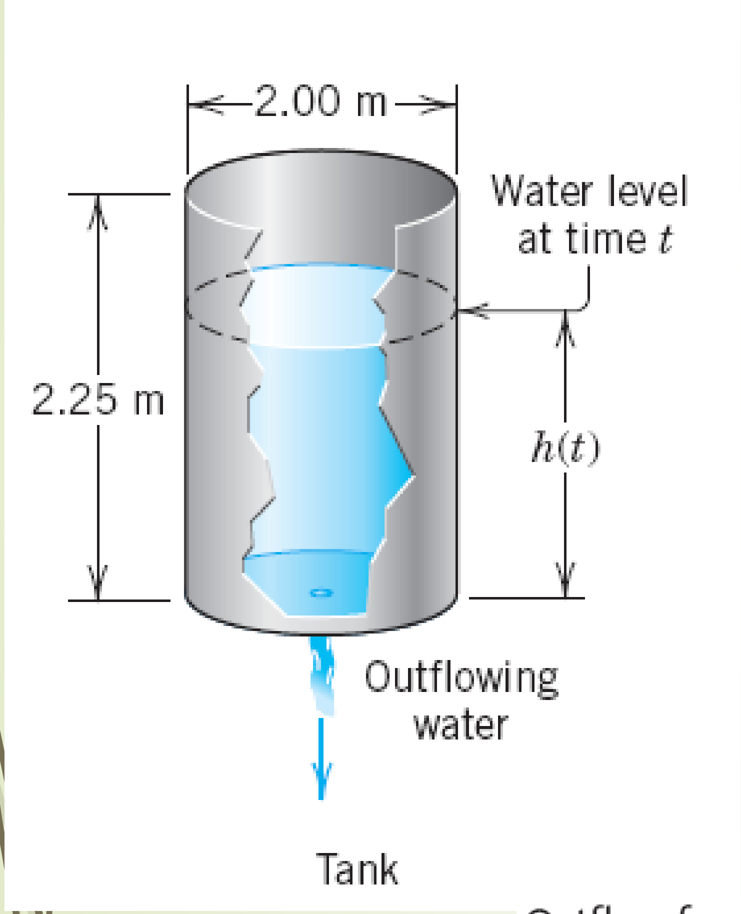


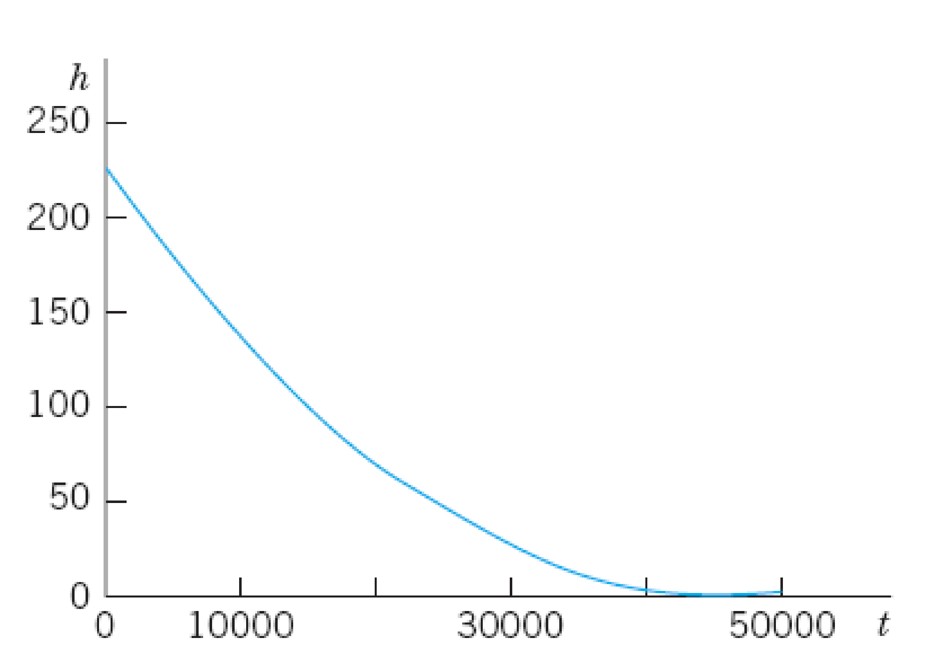
(a) *LRC*-series circuit



Current $I(t)$

The BIG Picture of Differential Equations





Outflow from a cylindrical tank ("leaking tank").
Torricelli's law

The BIG Picture of Differential Equations

Many other applications:

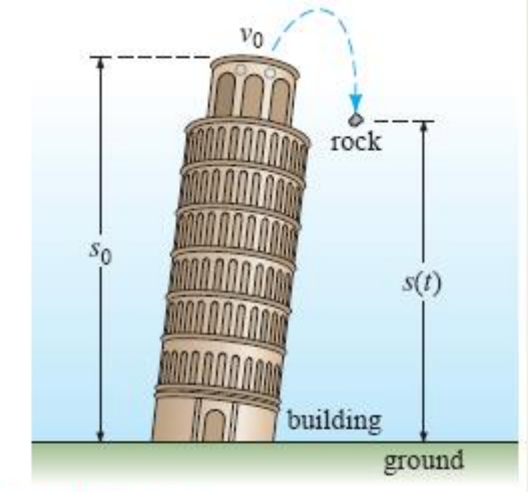
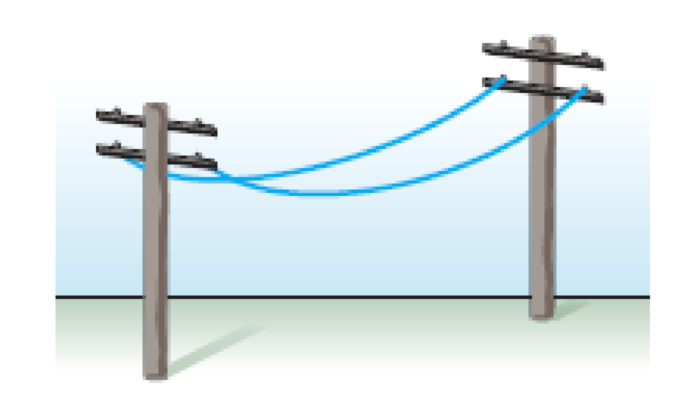


FIGURE 1.3.5 Position of rock measured from ground level


(b) telephone wires

Differential Equations

- **Definitions:**

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

$$\frac{dy}{dx} + 5y = e^x,$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0,$$

Classification of differential equations: Classify by type, order, and linearity.

Differential Equations(DE)

CLASSIFICATION BY TYPE

A DE containing only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).**

For example,

$$\frac{dy}{dx} + 5y = e^x, \qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0,$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

A DE can contain more than one dependent variable

- An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE).**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- Leibniz notation: *dy/dx, d²y/dx², ... and* prime notation: y, y´, y´´,...

Differential Equations

CLASSIFICATION BY ORDER

- The **order** of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation

$$\underbrace{\frac{d^2y}{dx^2}}_{\text{second order}} + 5\underbrace{\left(\frac{dy}{dx}\right)^3}_{\text{first order}} - 4y = e^x$$

is a **second-order** ordinary differential equation.

- In symbols we can express an *nth-order ordinary differential equation in one* dependent variable by the general form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \qquad -(1)$$

where ***F*** *is a real-valued* function of *n*+2 v

The **degree of the DE** is the power of its

$$\underbrace{\frac{d^2y}{dx^2}}_{\text{second order}} + 5\underbrace{\left(\frac{dy}{dx}\right)^3}_{\text{first order}} - 4y = e^x$$

Classification by linearity:

- **An n-th order ordinary differential equation (1) is said to be linear if F is linear in** $y, y', y'', \ldots\ldots\ldots\ldots, y^{n}$. **That means, that is of the form**

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdot \quad \cdot + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \qquad \ldots(2)$$

Examples: Linear first order:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Linear second order:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Two properties of a linear ODEs:

- The dependent variable y and all its derivatives $y, y', y'', \ldots\ldots\ldots\ldots, y^{n}$. are of the first degree.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', y'', \ldots\ldots\ldots\ldots, y^{n}$. depend at most on the independent variable x.

A **nonlinear ordinary differential equation is simply one that is not linear.**

Differential Equations

Examples:

First order linear ODE: $(y - x)dx + 4xy\,dy = 0,$

Second order linear ODE: $y'' - 2y + y = 0,$

Third order linear ODE: $x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$

Nonlinear first, second, and fourth order ODEs:

nonlinear term: coefficient depends on y

$$(1 - y)y' + 2y = e^x,$$

nonlinear term: nonlinear function of y

$$\frac{d^2y}{dx^2} + \sin y = 0,$$

and

nonlinear term: power not 1

$$\frac{d^4y}{dx^4} + y^2 = 0$$

Solutions of ODE:

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I.$$

where φ, is continuous function, satisfies the differential equation in (Eq-1) on some interval *I*.

Note: The domain of the solution of an ODE is valid in an open interval *I*=(a,b), closed interval *I*=[a,b], or, infinite interval (a, ∞) .

Verification of solutions: Verify the second eqn. is a solution of the first one.

$$dy/dx = xy^{1/2}; \quad y = \tfrac{1}{16}x^4$$

left-hand side: $\dfrac{dy}{dx} = \dfrac{1}{16}(4 \cdot x^3) = \dfrac{1}{4}x^3,$

right-hand side: $xy^{1/2} = x \cdot \left(\dfrac{1}{16}x^4\right)^{1/2} = x \cdot \left(\dfrac{1}{4}x^2\right) = \dfrac{1}{4}x^3,$

we see that each side of the equation is the same for every real number x.

Differential Equations

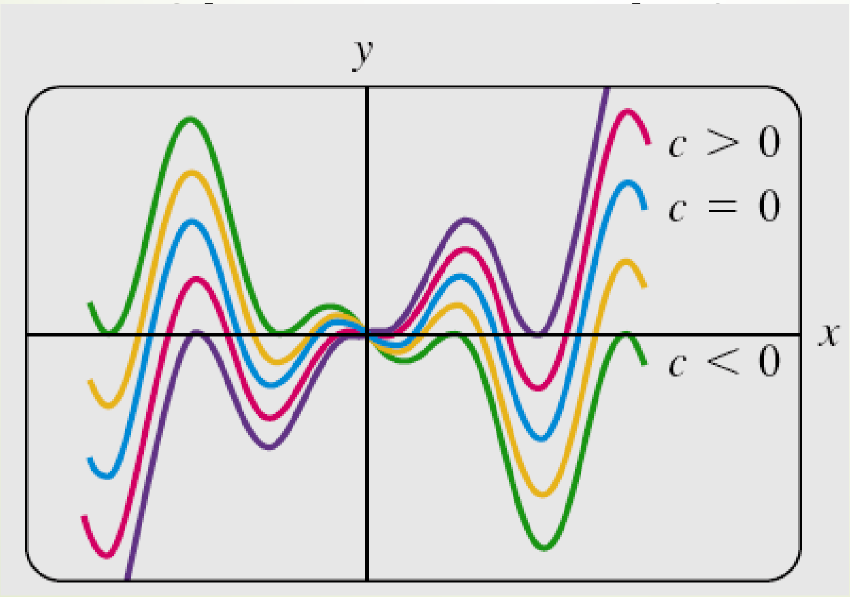
Families of Solutions:

Solving a first-order differential equation **F(x, y, y´)= 0**, we usually obtain a solution containing a single arbitrary constant or parameter c, i.e.,

$$\mathbf{G(x, y, c)= 0}$$

This is called **one-parameter family of solutions.**

When solving an nth-order differential equation $F(x, y, y', \ldots, y^{n})=0$, we seek an **n-parameter family of solutions**



Systems of Differential Equations:

A system of ordinary differential equations is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y).$$

A solution of the above system is a pair of differentiable functions **$x=\varphi_1(t)$, $y=\varphi_2(t)$,** defined on a common interval I, that satisfy each equation of the system on this interval.

Differential Equations

Quiz Questions…..

- Specify the order and degree of the ODE

$$x\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^4 + y = 0$$

- Is that Linear/nonlinear

$$\frac{d^2R}{dt^2} = -\frac{k}{R^2}$$

- Verify y is a solution of the ODE in left side

$$\frac{dy}{dt} + 20y = 24; \quad y = \frac{6}{5} - \frac{6}{5}e^{-20t}$$

Initial Value Problems(IVPs):

Solve: $$\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$$

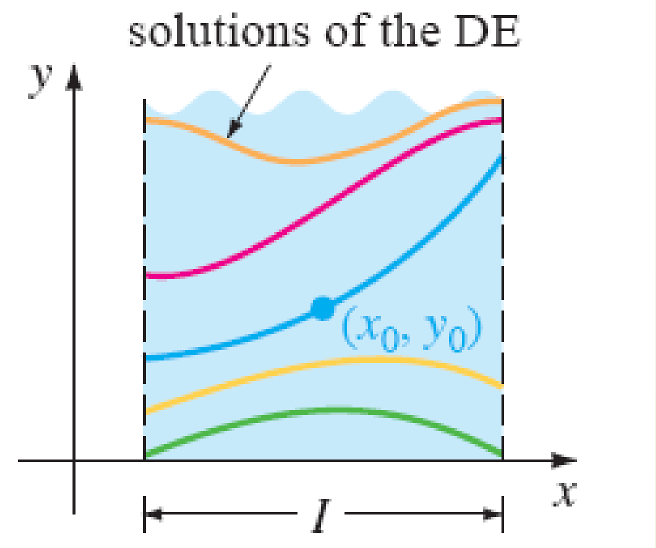
Subject to: $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary real constants, is called **an nth-order initial-value problem (IVP).** The values of y(x) and its first n-1 derivatives at x_0, $y(x_0) = y_0$, $y'(x_0) = y1, \ldots,$ $y^{n-1}(x_0) = y_{n-1}$ are called **initial conditions (IC).**

Geometric Interpretation of IVPs:
For n=1,

Solve: $$\frac{dy}{dx} = f(x, y)$$

Subject to: $$y(x_0) = y_0$$



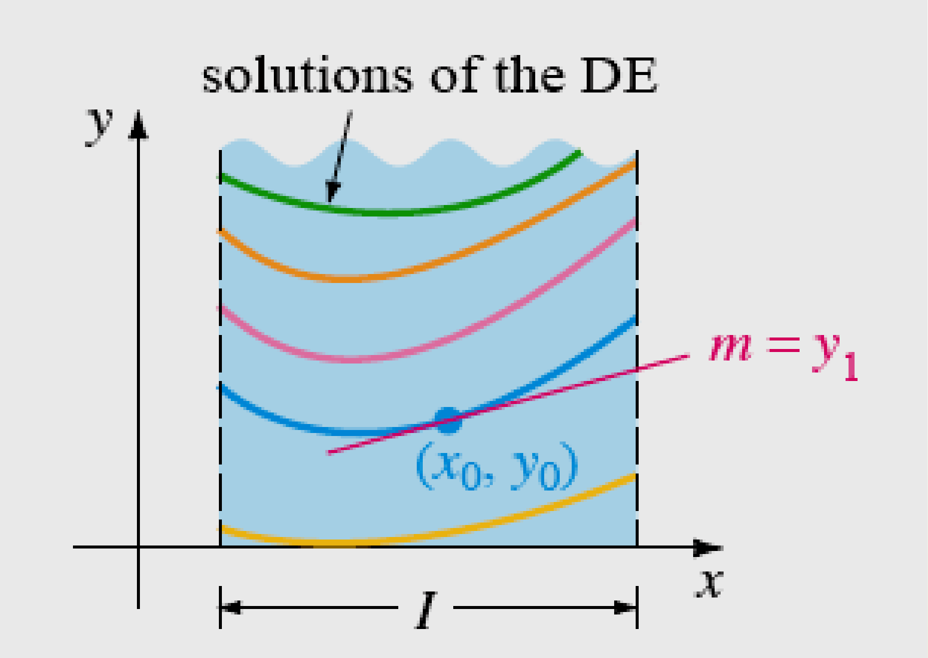
Geometric Interpretation of IVPs:

For n=2,

Solve: $$\frac{d^2y}{dx^2} = f(x, y, y')$$

Subject to: $$y(x_0) = y_0,\ y'(x_0) = y_1$$

Two boundary conditions



Differential Equations as Mathematical Models:

Mathematical Models:

The mathematical description of a system of phenomenon is called a **mathematical model.**

Construction of a mathematical model of a system starts with

(a) identification of the variables that are responsible for changing the system.

(b) make a set of reasonable assumptions, or hypotheses, about the system we are trying to describe. These assumptions will also include any empirical laws that may be applicable to the system.

Differential Equations as Mathematical Models

The Modeling Process with ODE

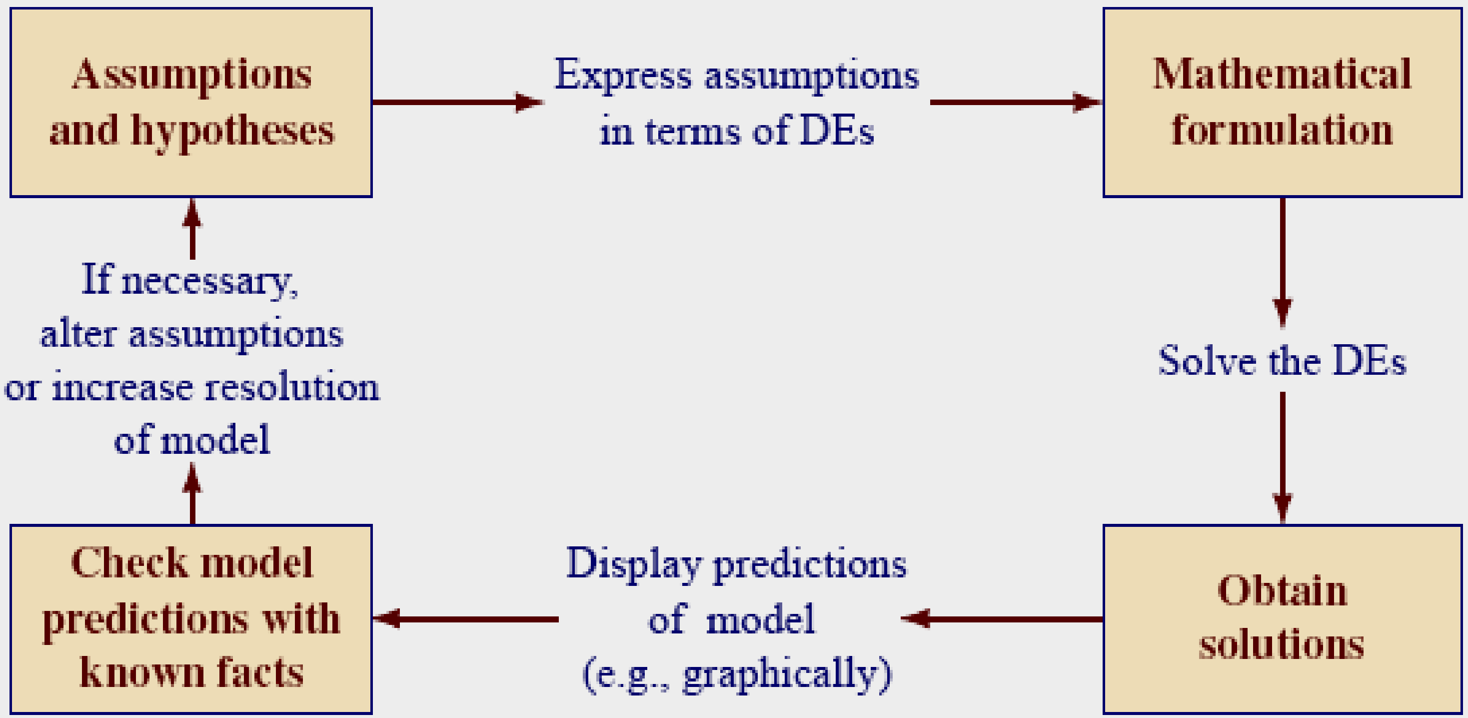


FIGURE 1.3.1 Steps in the modeling process with differential equations

Source: Dennis Zill. 10th edi. pp-21

Differential Equations as Mathematical Models

Population Dynamics

Malthusian model is the assumption that the rate at which the population of a country grows at a certain time is proportional to the total population of the country at that time. In other words, the more people there are at time t, the more there are going to be in the future.

In mathematical terms, if **P(t)** denotes the total population at time **t**, then this assumption can be expressed as

$$\frac{dP}{dt} \propto P \qquad \text{or} \qquad \frac{dP}{dt} = kP,$$

where k is a constant of proportionality.

Newton's Law of Cooling/Warming:

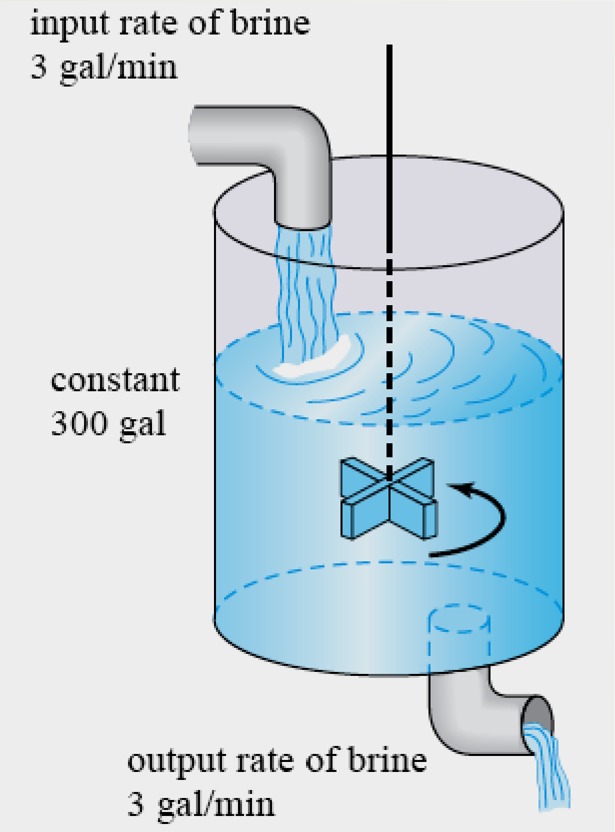
According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature.

If **T(t)** represents the temperature of a body at time **t,** $\mathbf{T_m}$ the temperature of the surrounding medium, and ***dT/dt*** *the rate at which the temperature of the body changes, then* Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m \qquad \text{or} \qquad \frac{dT}{dt} = k(T - T_m),$$

where k is a constant of proportionality. In either case, cooling or warming, if $\mathbf{T_m}$ is a constant, it stands to reason that k<0.

Differential Equations as Mathematical Models:



Mixture Problem

The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture.

Problem statement:

Let us suppose that a large mixing tank initially holds 300 gallons of brine. Another brine solution is pumped into the large tank at a rate of 3 gallons per minute; the concentration of the salt in this inflow is 2 pounds per gallon.

When the solution in the tank is well stirred, it is pumped out at the same rate as the entering solution.

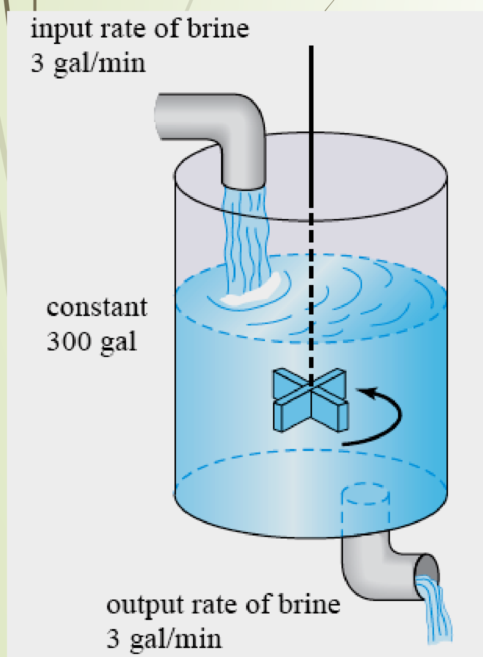
Form a mathematical model for the above phenomena.

Differential Equations as Mathematical Models

If A(t) denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which A(t) changes is a net rate:

$$\frac{dA}{dt} = \textbf{(input rate of salt)} - \textbf{(output rate of salt)}$$
$$= \mathbf{R_{in}} - \mathbf{R_{out}}$$

The input rate R_{in} at which salt enters the tank is the product of the inflow concentration of salt and the inflow rate of fluid



$$R_{in} = \underbrace{(2\ \text{lb/gal})}_{\text{concentration of salt in inflo}} \cdot \underbrace{(3\ \text{gal/min})}_{\text{input rate of brine}} = \underbrace{(6\ \text{lb/min})}_{\text{input rate of salt}}.$$

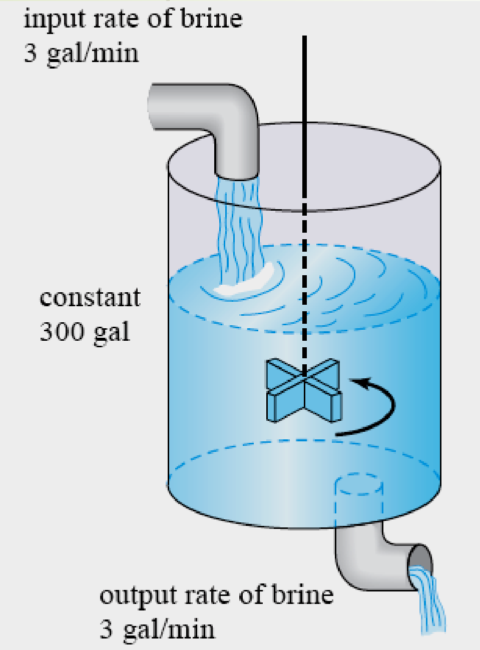
Since, the input and output rates are same, 300 gal brine remains constant at time t in the Tank

$$R_{out} = \underbrace{\left(\frac{A(t)}{300}\ \text{lb/gal}\right)}_{\text{concentration of salt in outflo}} \cdot \underbrace{(3\ \text{gal/min})}_{\text{output rate of brine}} = \underbrace{\frac{A(t)}{100}\ \text{lb/min}}_{\text{output rate of salt}}.$$

Differential Equations as Mathematical Models

- The net rate of change then becomes

$$\frac{dA}{dt} = 6 - \frac{A}{100} \quad \text{or} \quad \frac{dA}{dt} + \frac{1}{100}A = 6.$$

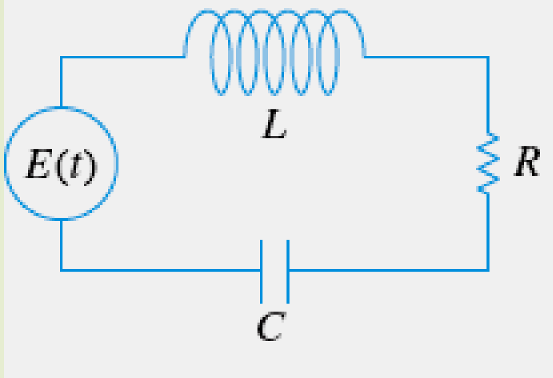


If $\mathbf{R}_{in}$ and $\mathbf{R}_{out}$ is the input and output rates of the brine solutions, then

case 1: If $\mathbf{R}_{in} = \mathbf{R}_{out}$, brine level in Tank is unchanged.
case 2: If $\mathbf{R}_{in} > \mathbf{R}_{out}$, the brine level increases.
case 3: If $\mathbf{R}_{in} < \mathbf{R}_{out}$, the brine level decreases.

Differential Equations as Mathematical Models

Current passes through electric circuits:



(a) *LRC*-series circuit

Consider the single-loop LRC-series circuit. L, R, and C are known as inductance, resistance, and capacitance, respectively, and are generally constants.
The current in a circuit after a switch is closed is denoted by i(t); the charge on a capacitor at time t is denoted by q(t). E(t) is impressive voltage drop at time t.

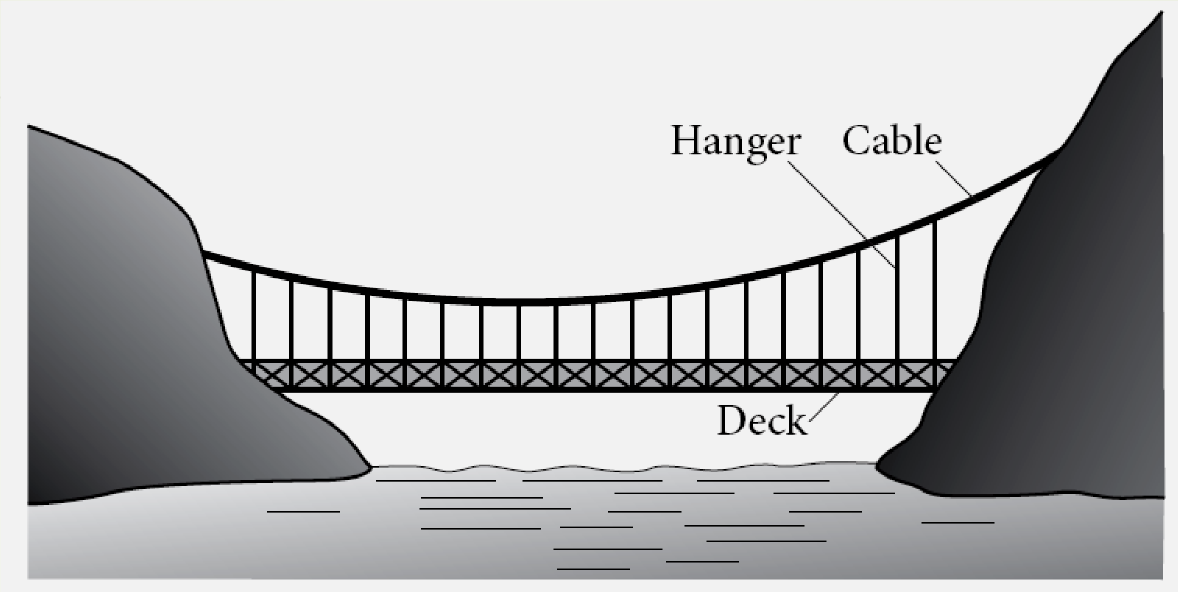
According to **Kirchhoff's second law,** respective voltage drops across an inductor, a capacitor, and a resistor, i.e.,

$$\underbrace{L\frac{di}{dt} = L\frac{d^2q}{dt^2}}_{\text{inductor}}, \qquad \underbrace{iR = R\frac{dq}{dt}}_{\text{resistor}}, \qquad \text{and} \qquad \underbrace{\frac{1}{C}q}_{\text{capacitor}}$$

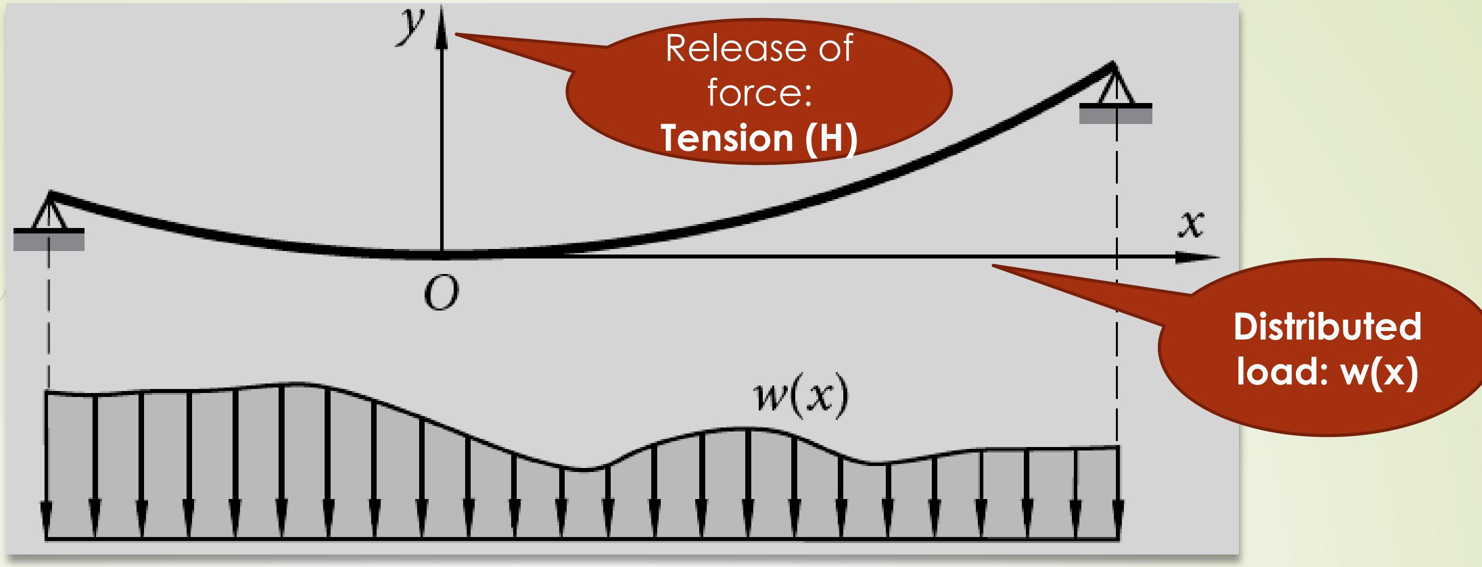
satisfy

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Motivating Example: Civil Engineering problem



Consider the suspension bridge as shown, which consists of the main cable, the hangers, and the deck. The self-weight of the deck and the loads applied on the deck are transferred to the cable through the hangers.



Set up the Cartesian coordinate system by placing the origin O at the lowest point of the cable. The cable can be modeled as subjected to a distributed load $w(x)$. The equation governing the shape of the cable is given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{w(x)}{H},$$

where H is the tension in the cable at the lowest point O. This is a second-order ordinary differential equation.

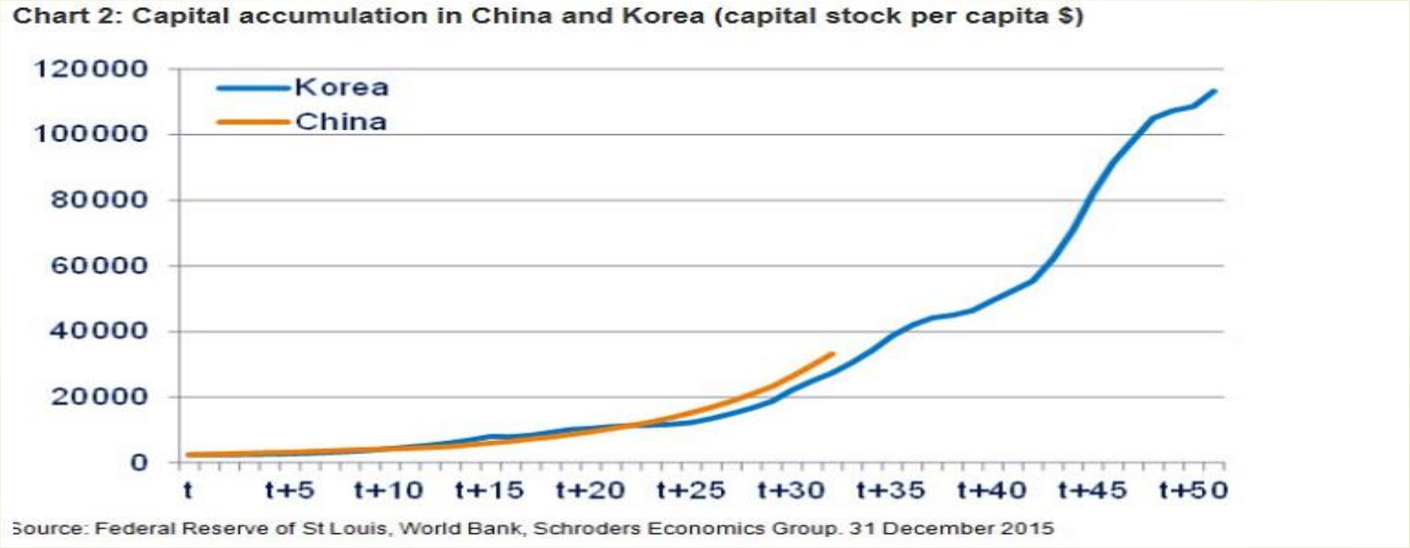
Motivating Example: Economics Problem-

The capital stock and investm

Capital accumulation:

Capital Stock= K(t)
Investment = I(t)

$$\frac{dK(t)}{dt} = I(t)$$



Motivating Example: Economics Problem-

Simple gross domestic product (GDP) model

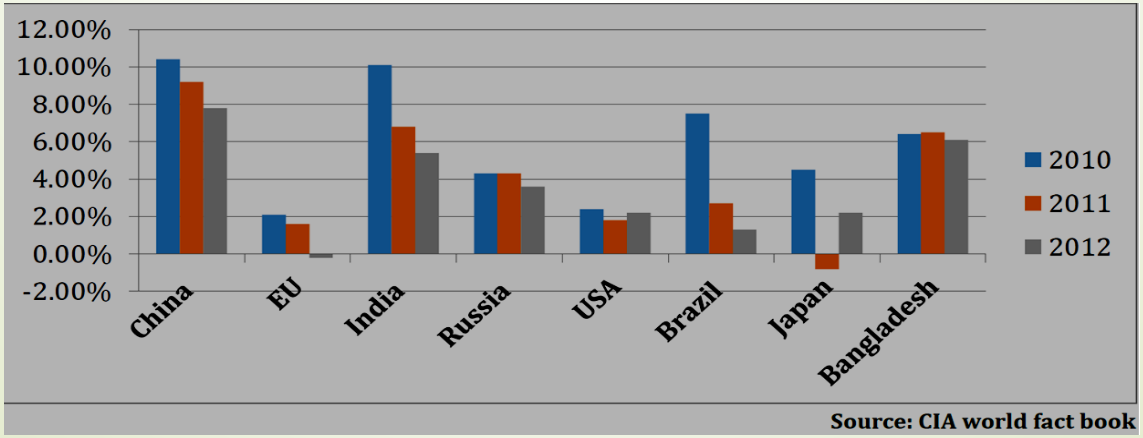
The rate of change of the gross domestic product (GDP), x(t), over time

$$\dot{x}(t) = gx(t),$$

where t stands for time and g is the growth rate, at any point time t .

If g = g(x, t), then

$$\dot{x}(t) = g(x(t), t)x(t).$$



Motivating Example: Economics Problem-
Demand and supply model

$$Q_d = a_1 - b_1 P, \;\; Q_s = -a_2 + b_2 P, \;\; a_j, b_j > 0,$$

where **Q_d** and **Q_s** are respectively the demand and supply for price **P** and **a_j** and **b_j** are parameters.

Assume that the rate of price change with regard to time at t is proportional to the excess demand, **Q_d - Q_s**, *that is*

$$\frac{dP}{dt} \propto \mathbf{Q}_d(t) - \mathbf{Q}_s(t)$$

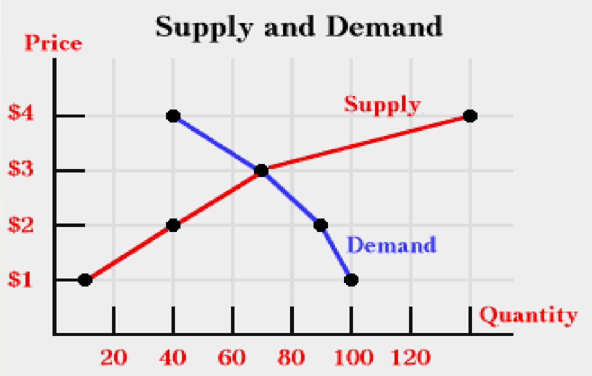
$$\dot{P}(t) = m(Q_d(t) - Q_s(t)), \quad m > 0$$

$$\dot{P}(t) + m(b_1 + b_2)P = m(a_1 + a_2).$$

$$P(t) = \left(P(0) - P^*\right)e^{-m_0 t} + P^*,$$

where,

$$P^* \equiv \frac{a_1 + a_2}{b_1 + b_2}, \quad m_0 \equiv m(a_2 + b_2) > 0.$$



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