2(b)/2(c)

— Open-loop controller for a differentially flat system:

Step-1:- Create the trajectory.

Since system is differentially flat.

$$\boxed{z = \sum \alpha_{i,j} \psi_i(t)} \to z_1 = \alpha_{11} + \alpha_{12} t + \alpha_{13} t^2 + \alpha_{14} t^3$$

$$\to z_2 = \alpha_{21} + \alpha_{22} t + \alpha_{23} t^2 + \alpha_{24} t^3$$

Take this into a matrix-vector form for easy computation. where-

$$\psi_i(t) = \begin{bmatrix} 1\,0\,0\,0 & 0\,0\,0\,0 \\ 0\,1\,0\,0 & 0\,0\,0\,0 \\ \text{——} & 1\,0\,0\,0 \\ \text{——} & \text{——} \\ 1\,T\,T^2\,T^3 & \text{——} \\ \text{——} & \text{——} \\ \text{——} & 0\,1\,2T\,3T^2 \end{bmatrix} \qquad \alpha_{ij} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{24} \end{bmatrix} \qquad z = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

Consider,

$\psi_i(t) = A$ $\qquad \psi(t) = A$

$\alpha_{ij} = X$ $\qquad \alpha = X$

$z = b$ $\qquad z = b$

So the equation $z = \psi(t)\alpha$. is basically

$b = AX$

or, $AX = b$.

We already have $\Psi(t)$. and for the α matrix we use the initial and final conditions given in the question or by calculating them

$$\begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix} = \begin{bmatrix} x(0) \\ 0 \\ y(0) \\ -1/2 \\ x(T) \\ 0 \\ y(T) \\ -1/2 \end{bmatrix}$$

So we can find the α matrix by simply using-

$$AX = b \rightarrow \Psi\alpha = z$$

$$X = A^{-1}b \rightarrow \alpha = \Psi^{-1}z$$

↳ Use pseudo inverse

Ψ^{-1} = np.linalg.pinv(Ψ).

and then,

np.dot(Ψ^{-1}, z)

or

np.linalg.solve(Ψ, z).

directly.

This α and Ψ are what we need to make the trajectory equation

$$z_1 = \alpha_{11} + \alpha_{12}t + \alpha_{13}t^2 + \alpha_{14}t^3$$

$$z_2 = \alpha_{21} + \alpha_{22}t + \alpha_{13}t^2 + \alpha_{14}t^3$$

2(c)

Step-2: Move your robot along the trajectory

How do we move a robot?

- Use control vectors (a, ω)

Given, $[\dot{x} = v\sin\theta;\ \dot{y} = v\cos\theta$ for this example$]$

$x = z_1$

$\dot{x} = \dot{z}_1 = \cancel{v\cos\theta}\, v\sin\theta$

$\cancel{\dot{y}} = \ddot{x} = \ddot{z}_1 = \dot{v}\sin\theta + v\cos\theta \cdot \dot{\theta}$

$= a\sin\theta + v\omega\cos\theta \quad [\dot{v} = a;\ \dot{\theta} = \omega]$.

$y = z_2$

$\dot{y} = \dot{z}_2 = v\cos\theta$

$\ddot{y} = \ddot{z}_2 = \dot{v}\cos\theta \cancel{+} - v\sin\theta \cdot \dot{\theta}$

$= a\cos\theta \cancel{- v\omega\sin\theta}\, v\omega\sin\theta$

So,

$\ddot{x} = a\sin\theta + v\omega\cos\theta$

$\ddot{y} = a\cos\theta - v\omega\sin\theta$

Writing this in ~~vector~~ ~~ma~~ matrix-vector form:-

$$\underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} \cancel{\cos\theta}\ \sin\theta & v\cos\theta \\ \cos\theta & -v\sin\theta \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ \omega \end{bmatrix}}_{X} \rightarrow$$

This is given in the question step-2 starts here.

$AX = b$

$X = A^{-1}b$

$$\begin{bmatrix} a \\ \omega \end{bmatrix} = \begin{bmatrix} \sin\theta & v\cos\theta \\ \cos\theta & -v\sin\theta \end{bmatrix}^{-1} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \sin\theta & \cos\theta \\ \frac{1}{v}\cos\theta & -\frac{1}{v}\sin\theta \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{a_{11}\cdot a_{22} - a_{21}\cdot a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$$

$$\begin{bmatrix} a \\ \omega \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ \frac{1}{v}\cos\theta & -\frac{1}{v}\sin\theta \end{bmatrix}$$

<u>How to use this in your code:</u>

for (i=0, i< T; dt)

$$\underbrace{\begin{bmatrix} a \\ \omega \end{bmatrix}}_{X_2} = \underbrace{\begin{bmatrix} \sin\theta[i] & \cos\theta[i] \\ \frac{1}{V[i]} \cos\theta[i] & -\frac{1}{V[i]} \sin\theta[i] \end{bmatrix}}_{A_2\text{-inv.}} \underbrace{\begin{bmatrix} \ddot{z}_1[i] \\ \ddot{z}_2[i] \end{bmatrix}}_{b_2}$$

$\cancel{\ddot{z}_1[i] = 2\alpha_{13} + 6\alpha_{14} t}$

$$\ddot{z}_1 = 2\alpha_{13} + 6\alpha_{14} t = \ddot{x}$$

$$\ddot{z}_2 = 2\alpha_{23} + 6\alpha_{24} t = \ddot{y}$$

Since, $X_2 = \begin{bmatrix} a \\ \omega \end{bmatrix}$.

So, $a = X_2[0]$

$w = X_2[1]$.

Step-3: Update state variables using integration (euler).

$$x(t) = x(t-1) + dt \times \dot{x}(t-1)$$

Cannot usually use this when the for ~~loop~~ loop starts at 0, instead, we do this

$$t \to t+1$$

$$t-1 \to t$$

So, the equation will be,

$$x(t+1) = x(t) + dt \times \dot{x}(t)$$

$$x(t+1) = x(t) + dt \times v\sin\theta(t)$$

$$y(t+1) = y(t) + dt \times v\cos\theta(t)$$

$$v(t+1) = v(t) + dt \times a(t)$$

$$\theta(t+1) = \theta(t) + dt \times \omega(t).$$

Step-4: Plot the graph

Use matplotlib. documentations as necessary.