Textbook: Precalculus by Howard anton, Irl Bivens and Stephen Davis

Chapter 3.5: Inequalities Involving Quadratic Functions

Solution techniques of inequalities involving quadratic functions

To solve a quadratic inequality like

$$ax^2 + bx + c > 0 \text{ with } a \neq 0,$$

we first graph the function $f(x) = ax^2 + bx + c$ and then from the graph determine where it is above the *x*-axis.

Similarly, to solve a quadratic inequality like

$$ax^2 + bx + c < 0 \text{ with } a \neq 0,$$

we first graph the function $f(x) = ax^2 + bx + c$ and then from the graph determine where it is below the *x*-axis.

If the inequality is not strict, include the *x*-intercepts, if any, in the solution.

Example 1

Solve the inequality $x^2 - 4x - 12 \leq 0$ and graph the solution set.

Solution: The *y*-intercept is $f(0) = -12$ and for *x*-intercepts solve the equation

$$x^2 - 4x - 12 = 0 \Rightarrow x = -2,\ 6$$

Therefore, *y*-intercept is -12 and *x*-intercepts are -2 and 6.

Next the axis of symmetry of the parabola is $x = -\dfrac{b}{2a} = -\dfrac{-4}{2} = 2.$ Compute $f(2) = -16$ and so the vertex of the parabola is $(2, -16)$.

Since $a = 1 > 0,$ the parabola opens up.

Using the above information, graph the function $f(x) = x^2 - 4x - 12$ which represents a parabola (Figure 33, page 156).

Finally, since the given inequality is not strict, include the *x*-intercepts in the solution.

Hence the solution set is $S = \{x : -2 \leq x \leq 6\} = [-2, 6].$

See Figure 34, page 156, for the graph of the solution set.

Example 2

Solve the inequality $2x^2 < x + 10$ and graph the solution set.

Solution: First of all, rewrite the inequality $2x^2 < x + 10 \Rightarrow 2x^2 - x - 10 < 0$

The *y*-intercept is $f(0) = -10$ and for *x*-intercepts solve the equation

$$2x^2 - x - 10 = 0 \Rightarrow x = -2,\ \frac{5}{2}$$

Therefore, *y*-intercept is -10 and *x*-intercepts are -2 and $\dfrac{5}{2}$.

Next the axis of symmetry of the parabola is $x = -\dfrac{b}{2a} = -\dfrac{-1}{2 \times 2} = \dfrac{1}{4}.$

Compute $f(1/4) = -10.125$ and so the vertex of the parabola is $(1/4, -10.125).$

Since $a = 2 > 0,$ the parabola opens up.

Using the above information, graph the function $f(x)=2x^2-x-10$ which represents a parabola (Figure 35, page 156).
Finally, since the given inequality is strict, the solution set is $S=\{x:-2<x<5/2\}=(-2,5/2)$
See Figure 37, page 157, for the graph of the solution set.

Example 3

Solve the inequality $x^2+x+1>0$ and graph the solution set.
Solution: The y-intercept is $f(0)=1.$ Since discriminant has the value $b^2-4ac=1-4=-3<0,$ there are no x-intercepts.
Therefore, y-intercept is 1 and there are no x-intercepts.
Next the axis of symmetry of the parabola is $x=-\dfrac{b}{2a}=-\dfrac{1}{2}.$ Compute $f(-1/2)=3/4$ and so the vertex of the parabola is $(-1/2,3/4).$
Since $a=1>0,$ the parabola opens up.
Using the above information, graph the function $f(x)=x^2+x+1$ which represents a parabola (Figure 38, page 157).
Finally, since the graph of f lies above the x-axis for all x, the solution set is the set of all real numbers.
See Figure 39, page 157, for the graph of the solution set.

Assess Your Understanding for Chapter 3.5

(Exercise Set 3.5)

7. Solve the inequality $x^2-3x-10<0$ and graph the solution set.
Solution: To graph the function $f(x)=x^2-3x-10,$ list the following properties of $f(x)$:

(i) y-intercept: $f(0)=-10.$

(ii) To find the x-intercepts, we solve $f(x)=0 \Rightarrow x^2-3x-10=0$ giving $x=-2,\ 5$

(iii) Axis of symmetry: $x=-\dfrac{-3}{2}=\dfrac{3}{2}=1.5.$ Compute $f(3/2)=-49/4$ and so the vertex of the parabola is $(3/2,-49/4).$

(iv) Since $a=1>0,$ the parabola opens up.

Using the above information, graph the function $f(x)=x^2-3x-10$ which represents a parabola.
Finally, since the given inequality is strict, the solution set is $S=\{x:-2<x<5\}=(-2,5).$
Draw a figure for the graph of the solution set.

9. Solve the inequality $x^2-4x>0$ and graph the solution set.
Solution: To graph the function $f(x)=x^2-4x,$ list the following properties of $f(x)$:

(i) y-intercept: $f(0)=0.$

(ii) To find the x-intercepts, we solve $f(x)=0 \Rightarrow x^2-4x=0$ giving $x=0,\ 4$

(iii) Axis of symmetry: $x = -\dfrac{-4}{2} = 2.$

Compute $f(2) = -4$ and so the vertex of the parabola is $(2,-4).$

(iv) Since $a = 1 > 0,$ the parabola opens up.

Using the above information, graph the function $f(x) = x^2 - 4x$ which represents a parabola.
Finally, since the given inequality is strict, the solution set is

$$S = \{x : x < 0 \text{ or } x > 4\} = (-\infty, 0) \cup (4, \infty).$$

Draw a figure for the graph of the solution set.

11. Solve the inequality $x^2 - 9 < 0$ and graph the solution set.

Solution: To graph the function $f(x) = x^2 - 9,$ list the following properties of $f(x)$:

(i) y-intercept: $f(0) = -9.$

(ii) To find the x-intercepts, we solve $f(x) = 0 \Rightarrow x^2 - 9 = 0$ giving $x = -3,\ 3$

(iii) Axis of symmetry: $x = -\dfrac{0}{2} = 0.$

Compute $f(0) = -9$ and so the vertex of the parabola is $(0,-9).$

(iv) Since $a = 1 > 0,$ the parabola opens up.

Using the above information, graph the function $f(x) = x^2 - 9$ which represents a parabola.
Finally, since the given inequality is strict, the solution set is

$$S = \{x : -3 < x < 3\} = (-3, 3).$$

Draw a figure for the graph of the solution set.

15. Solve the inequality $2x^2 < 5x + 3$ and graph the solution set.

Solution: First of all, rewrite the inequality $2x^2 < 5x + 3 \Rightarrow 2x^2 - 5x - 3 < 0$

To graph the function $f(x) = 2x^2 - 5x - 3,$ list the following properties of $f(x)$:

(i) y-intercept: $f(0) = -3.$

(ii) To find the x-intercepts, we solve $f(x) = 0 \Rightarrow 2x^2 - 5x - 3 = 0$ giving $x = -1/2,\ 3$

(iii) Axis of symmetry: $x = -\dfrac{-5}{2 \times 2} = \dfrac{5}{4} = 1.25.$ Compute $f(5/4) = -49/8$ and so the vertex of the parabola is $(5/4, -49/8).$

(iv) Since $a = 2 > 0,$ the parabola opens up.

Using the above information, graph the function $f(x) = 2x^2 - 5x - 3$ which represents a parabola.
Finally, since the given inequality is strict, the solution set is $S = \{x : -1/2 < x < 3\} = (-1/2, 3).$
Draw a figure for the graph of the solution set.

21. Solve the inequality $6(x^2 - 1) > 5x$ and graph the solution set.

Solution: First of all, rewrite the inequality $6(x^2 - 1) > 5x \Rightarrow 6x^2 - 5x - 6 > 0$

To graph the function $f(x) = 6x^2 - 5x - 6,$ list the following properties of $f(x)$:

(i) y-intercept: $f(0) = -6.$

(ii) To find the *x*-intercepts, we solve $f(x)=0 \Rightarrow 6x^2-5x-6=0$ giving $x=-2/3,\ 3/2$

(iii) Axis of symmetry: $x=-\dfrac{-5}{2\times 6}=\dfrac{5}{12}$. Compute $f(5/12)=-169/24$ and so the vertex of the parabola is $(5/12,-169/24)$.

(iv) Since $a=6>0$, the parabola opens up.

Using the above information, graph the function $f(x)=6x^2-5x-6$ which represents a parabola.

Finally, since the given inequality is strict, the solution set is

$S=\{x: x<-2/3 \ \ or \ \ x>3/2\}=(-\infty,\ -2/3)\cup(3/2,+\infty)$.

Draw a figure for the graph of the solution set.

Chapter 4: Polynomial and Rational Functions

We have the following functions:

(i) Polynomial functions: $p_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are all real numbers and the number *n* is a nonnegative integer.

(ii) Rational functions: $R(x)=\dfrac{p(x)}{q(x)}$ if $q(x)\neq 0$, i.e. every rational function is a ratio of polynomial functions.

In this chapter, we will consider two general classes of functions, namely, polynomial functions and rational functions and examine their properties.

4.1 Polynomial Functions and Models

Consider a *linear function*

$$p_1(x)=f(x)=a_1x+a_0 \qquad (1)$$

since we know the slope-intercept form of a line is $y=mx+b$ where the number *b* is the *y*-intercept and *m* is the slope of the line defined by $m=\tan\theta$.

Then we consider a *quadratic function*

$$f(x)=a_2x^2+a_1x+a_0 \qquad (2)$$

Equations (1) and (2) are examples of *polynomial functions*.

Definition

A **polynomial function** is a function of the form

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are all real numbers and the number *n* is a nonnegative integer.

The domain of a polynomial function is the set of all real numbers.

A polynomial function is a function whose rule is given by a polynomial in one variable.

The **degree** of a polynomial function is the largest power of *x* that appears.

The zero polynomial function

$$f(x) = 0x^n + 0x^{n-1} + \cdots + 0x + 0$$

is not assigned a degree.

Example 1

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

(a) $f(x) = 2 - 3x^4$ (b) $g(x) = \sqrt{x}$ (c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

(d) $F(x) = 0$ (e) $G(x) = 8$ (f) $h(x) = -2x^3(x-1)^2$

Solution: (a) The given function f is a polynomial function of degree 4.

(b) The given function g is not a polynomial function because in $g(x)$, power of x is a rational number not a nonnegative integer.

(c) The given function h is not a polynomial function. It is the ratio of two distinct polynomials, and the polynomial in the denominator is of positive degree.

(d) The given function F is the zero polynomial function; it is not assigned a degree.

(e) The given function G is a nonzero constant function. It is a polynomial function of degree 0 since $G(x) = 8 = 8x^0$.

(f) The given function $h(x) = -2x^3(x-1)^2$ can be simplified in the form as

$$h(x) = -2x^3(x-1)^2 = -2x^3(x^2 - 2x + 1) = -2x^5 + 4x^4 - 2x^3$$

So h is a polynomial function of degree 5.

We have already discussed in detail polynomial functions of different degrees. Now let us give the summary of the properties of the graphs of these polynomial functions.

(i) The *zero function* $f(x) = 0$ is a polynomial function with no degree whose graph is the x-axis.

(ii) The *constant function* $f(x) = a_0$ with $a_0 \neq 0$, is a polynomial function of degree zero whose graph is the horizontal line with y-intercept a_0.

(iii) The *linear function* $f(x) = a_1 x + a_0$ with $a_1 \neq 0$, is a polynomial function of degree one whose graph is a nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0.

(iv) The *quadratic function* $f(x) = a_2 x^2 + a_1 x + a_0$ with $a_2 \neq 0$, is a polynomial function of degree two whose graph is a parabola which opens up if $a_2 > 0$ and the graph opens down if $a_2 < 0$.

(v) The *power function* $f(x) = ax^n$ is a *monomial function* of degree n where a is a nonzero real number and n is a positive integer.

Examples of power functions are

$$f(x) = 3x,\ g(x) = -5x^2,\ h(x) = 8x^3,\ p(x) = -5x^4$$

The graph of a power function of degree 1, $f(x) = ax$, is a straight line, with slope a, that passes through the origin.

The graph of a power function of degree 2, $f(x) = ax^2$, is a parabola, with vertex at the origin, that opens up if $a > 0$ and opens down if $a < 0$.

To graph a power function of the form $f(x)=x^n$, we need to use a compression or a stretching and a reflection about the *x*-axis which will enable us to obtain the graph of $g(x)=ax^n$.

Now we begin with power functions of even degree of the form $f(x)=x^n$, $n \geq 2$ and *n* even. The domain of f is the set of all real numbers, and the range is the set of nonnegative real numbers. Such a power function is an even function and so its graph is symmetric with respect to the *y*-axis. Its graph always contains the origin and the points $(-1,1)$ and $(1,1)$.