First Order Ordinary Differential Equations(ODEs)

- ***First order ODEs :*** $\frac{dy}{dx} = f(x, y)$
- ***First order separable ODEs :*** $\frac{dy}{dx} = g(x)h(y) \Rightarrow \frac{1}{h(y)}dy = g(x)dx$
- ***Solution by integrating:***

$$\int \frac{1}{h(y)} dy = \int g(x)\, dx + C$$

Example. *Population Dynamics,* $\frac{dP(t)}{dt} \propto P(t),$

Malthus's law which can also applies to humans for small populations in a large country (e.g., the United States in early times)

$\frac{dP(t)}{dt} = r\,P(t),$ where r is the intrinsic rate of natural increase

$$\Rightarrow \frac{1}{P}dP = r\,dt \Rightarrow \int \frac{1}{P}dP = r\int dt \Rightarrow \ln P = r\,t + \ln(c)$$

$\Rightarrow P(t) = ce^{rt}$, *where c is a constant.*

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Example. $\frac{dy}{dt} = 0.2\,y$

$$\Rightarrow \frac{1}{y}dy = 0.2\,dt \Rightarrow \int \frac{1}{y}dy = 0.2\int dt$$

$$\Rightarrow \ln y = 0.2\,t + \ln(c) \Rightarrow y(t) = ce^{0.2\,t}, \textit{ where } c \textit{ is a constant.}$$

$\mathit{Cooling\ case}: k < 0$
$\mathit{Heating\ case}: k > 0$

Example. **Newton's Law of cooling/heating,** $\frac{dT(t)}{dt} \propto T - T_m$

$$\frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{1}{(T - T_m)}dT = k\,dt \Rightarrow \int \frac{1}{(T - T_m)}dT = k\int dt$$

$$\Rightarrow \ln(T - T_m) = kt + \ln(c) \Rightarrow T(t) = T_m + ce^{kt}, \textit{ where } c \textit{ is a constant.}$$

Example. ***Radioactivity*-Exponential decay** $(k > 0)$

$$\frac{dC}{dt} = -kC \Rightarrow \frac{1}{C}dC = -k\,dt \Rightarrow \int \frac{1}{C}dC = -k\int dt$$

$$\Rightarrow \ln C = -kt + \ln(c) \Rightarrow C(t) = ce^{-kt}, \textit{ where } c \textit{ is a constant.}$$

First Order Ordinary Differential Equations(ODEs)

Example. $\frac{dy}{dx} = y^{-1}xe^{3x+4y}$

$$\Rightarrow \frac{dy}{dx} = y^{-1}xe^{3x}e^{4y} = xe^{3x} \cdot y^{-1}e^{4y}$$

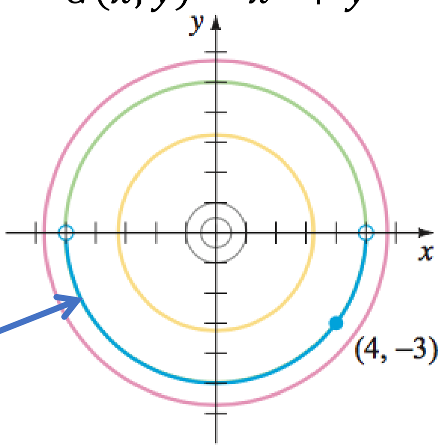
$$\Rightarrow \frac{1}{y^{-1}e^{4y}}dy = xe^{3x}dx \Rightarrow \int ye^{-4y}dy = \int xe^{3x}\,dx$$

$$\Rightarrow -\frac{(4y+1)}{16}e^{-4y} = \frac{(3x-1)}{9}e^{3x} + c$$

Example. $(1+x)dy - ydx = 0 \Rightarrow (1+x)dy = ydx \Rightarrow \frac{1}{y}dy = \frac{1}{1+x}dx$

$$\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{1+x}dx \Rightarrow \ln y = \ln(1+x) + \ln c \Rightarrow y = c(1+x).$$

Level curves of $G(x,y) = x^2 + y^2$



Example. $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow ydy + xdx = 0$

$$\Rightarrow \int ydy + \int x\,dx = 0 \Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = \frac{c}{2} \Rightarrow x^2 + y^2 = c.$$

If $y(4) = -3$, i.e. when $x = 4, y = -3$, then $4^2 + 3^2 = c \Rightarrow c = 25$

The solution of the corresponding IVP yields, $x^2 + y^2 = 25$

$$\Rightarrow y = -\sqrt{25 - x^2}$$

First Order Ordinary Differential Equations(ODEs)

First Order Initial Value Problem (IVP):

Example. Solve the following initial value problem,

$$(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y}\sin 2x, \qquad y(0) = 0$$

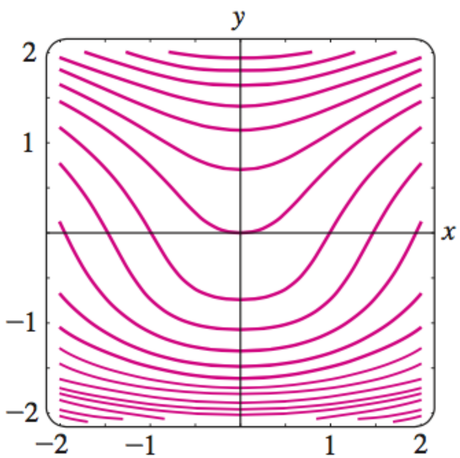
Solution. The ODE is given by,

$$(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y}\sin 2x \Rightarrow \frac{(e^{2y} - y)}{e^{y}}dy = \frac{\sin 2x}{\cos x}dx$$

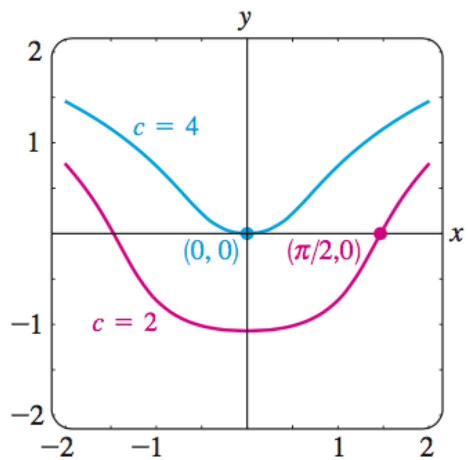
$$\Rightarrow \int (e^{y} - ye^{-y})\,dy = 2\int \sin x\,dx \Rightarrow e^{y} + ye^{-y} + e^{-y} + 2\cos x = c$$

Since, $y(0) = 0 \Rightarrow 1 + 0 + 1 + 2 = c \Rightarrow \mathrm{c} = 4$

Thus, the solution of the given IVP becomes, $e^{y} + (y + 1)e^{-y} + 2\cos x = 4.$



Level curves of $G(x,y) = e^{y} + (y+1)e^{-y} + 2\cos x$



Level curves of $G(x,y) = 2$ and $G(x,y) = 4$

First Order Ordinary Differential Equations(ODEs)

First Order Initial Value Problem (IVP):

2nd Fundamental theorem of calculus:

$$\frac{d}{dx}\left[\int_3^x e^{-t^2}\,dt\right] = e^{-x^2}$$

Example. Solve the following initial value problem,

$$\frac{dy}{dx} = e^{-x^2}, \qquad y(3) = 5$$

Solution. The ODE is given by,

$$\frac{dy}{dx} = e^{-x^2} \Rightarrow dy = e^{-x^2}dx \Rightarrow \frac{dy}{dt}dt = e^{-t^2}dt$$

$$\Rightarrow \int_{t=3}^{x} \frac{dy}{dt}dt = \int_{t=3}^{x} e^{-t^2}\,dt \Rightarrow y(t)\Big|_{t=3}^{t=x} = \int_3^x e^{-t^2}\,dt$$

$$\Rightarrow y(x) = y(3) + \int_3^x e^{-t^2}\,dt$$

Thus, the solution of the given ODE yields, $y(x) = 5 + \int_3^x e^{-t^2}\,dt$

First Order Ordinary Differential Equations(ODEs)

Exercises 2.2

H.W. from the text book

Solve the following differential equation by separation of variables:

1. $\frac{dy}{dx} = \sin 5x$
2. $\frac{dy}{dx} = (x + 1)^2$
3. $dx + e^{3x}dy = 0$
4. $dy - (y - 1)^2 dx = 0$
5. $x\frac{dy}{dx} = 4y$
6. $\frac{dy}{dx} + 2xy^2 = 0$
7. $\frac{dy}{dx} = e^{3x+2y}$
8. $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$
9. $y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$
10. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$
11. $\csc y\, dx + \sec^2 x\, dy = 0$

15. $\frac{dS}{dr} = kS$
16. $\frac{dQ}{dt} = k(Q - 70)$
17. $\frac{dP}{dt} = P - P^2$
18. $\frac{dN}{dt} + N = Nte^{t+2}$
21. $\frac{dy}{dx} = x\sqrt{1 - y^2}$
22. $(e^x + e^{-x})\frac{dy}{dx} = y^2$

Find an explicit solution of the following IVPs:

23. $\frac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1$
24. $\frac{dy}{dx} = \frac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$
25. $x^2\frac{dy}{dx} = y - xy, \quad y(-1) = -1$
26. $\frac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$
27. $\sqrt{1 - y^2}\, dx - \sqrt{1 - x^2}\, dy = 0, \quad y(0) = \frac{\sqrt{3}}{2}$
28. $(1 + x^4)\, dy + x(1 + 4y^2)\, dx = 0, \quad y(1) = 0$
29. $\frac{dy}{dx} = ye^{-x^2}, \quad y(4) = 1$
30. $\frac{dy}{dx} = y^2 \sin x^2, \quad y(-2) = \frac{1}{3}$

First Order Ordinary Differential Equations(ODEs)

Modeling and Applications

1. **Exponential growth.** If the growth rate of the number of bacteria at any time t is proportional to the number present at t and doubles in 1 week, how many bacteria can be expected after 2 weeks? After 4 weeks?
2. **Another population model.**
   (a) If the birth rate and death rate of the number of bacteria are proportional to the number of bacteria present, what is the population as a function of time.
   (b) What is the limiting situation for increasing time? Interpret it.
3. **Radiocarbon dating.** What should be the ${}^{14}_{6}C$ content (in percent of y_0) of a fossilized tree that is claimed to be 3000 years old?
4. **Newton's law of cooling.** A thermometer, reading 5° C, is brought into a room whose temperature is 22° C. One minute later the thermometer reading is 12° C. How long does it take until the reading is practically 22°C , say, 21.9°C ?
5. **Rocket.** A rocket is shot straight up from the earth, with a net acceleration (acceleration by the rocket engine minus gravitational pullback) of $7t$ m/sec^2 during the initial stage of flight until the engine cut out at $t = 10$ sec. How high will it go, air resistance neglected?