Chapter # 05
(Integration)

In this chapter we will begin with an overview of the problem of finding areas—we will discuss what the term “area” means, and we will outline two approaches to defining and calculating areas. Following this overview, we will discuss the ***Fundamental Theorem of Calculus***, which is the theorem that relates the problems of finding tangent lines and areas, and we will discuss techniques for calculating areas. We will then use the ideas in this chapter to define the average value of a function, to continue our study of rectilinear motion, and to examine some consequences of the chain rule in integral calculus. We conclude the chapter by studying functions defined by integrals, with a focus on the natural logarithm function.

5.2 The Indefinite Integral:

Antiderivative: A function $\boldsymbol{F}$ is called an antiderivative of a function $\boldsymbol{f}$ on a given open interval if $\boldsymbol{F'(x) = f(x)}$ for all $\boldsymbol{x}$ in the interval.

Example: $\boldsymbol{F(x) = \frac{1}{3}x^3}$ is an antiderivative of $\boldsymbol{f(x) = x^2}$ on the interval $(-\infty, \infty)$ because for each $\boldsymbol{x}$ in this interval

$$F'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2 = f(x)$$

However, the function $\boldsymbol{G(x) = \frac{1}{3}x^3 + C}$ is also an antiderivative of $\boldsymbol{f}$ on $(-\infty, \infty)$, since

$$G'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative.

Theorem: If $\boldsymbol{F(x)}$ is any antiderivative of $\boldsymbol{f(x)}$ on an open interval, then for any constant $\boldsymbol{C}$ the function $\boldsymbol{F(x) + C}$ is also an antiderivative on that interval. Moreover, each antiderivative of $\boldsymbol{f(x)}$ on the interval can be expressed in the form $\boldsymbol{F(x) + C}$ by choosing the constant $\boldsymbol{C}$ appropriately.

The Indefinite Integral: The process of finding antiderivatives is called ***antidifferentiation or integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x)$$

then integrating (or antidifferentiating) the function $\boldsymbol{f(x)}$ produces an antiderivative of the form $\boldsymbol{F(x) + C}$. To emphasize this process, the above equation is recast using integral notation,

$$\int f(x)\,dx = F(x) + C$$

where C is understood to represent an arbitrary constant.

For example,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

The expression $\int f(x)dx$ is called an indefinite integral. The "elongated s" is called an integral sign, the function $f(x)$ is called the integrand, and the constant C is called the constant of integration.

Integration Formulas:

INTEGRATION FORMULAS

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$	9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x\,dx = e^x + C$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b, b \neq 1)$	$\int b^x\,dx = \frac{b^x}{\ln b} + C \quad (0 < b, b \neq 1)$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$	11. $\frac{d}{dx}[\ln\lvert x\rvert] = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	14. $\frac{d}{dx}[\sec^{-1}\lvert x\rvert] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}\lvert x\rvert + C$

Example 1:

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \boxed{r=2}$$

$$\int x^3\,dx = \frac{x^4}{4} + C \qquad \boxed{r=3}$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C \qquad \boxed{r=-5}$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C = \tfrac{2}{3}(\sqrt{x})^3 + C \qquad \boxed{r=\tfrac{1}{2}}$$

Theorem: Suppose that $\boldsymbol{F(x)}$ and $\boldsymbol{G(x)}$ are antiderivatives of $\boldsymbol{f(x)}$ and $\boldsymbol{g(x)}$, respectively, and that $\boldsymbol{c}$ is a constant. Then:

(*a*) A constant factor can be moved through an integral sign; that is

$$\int cf(x)\,dx = cF(x) + C$$

(*b*) An antiderivative of a sum is the sum of the antiderivatives; that is,

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(*c*) An antiderivative of a difference is the difference of the antiderivatives; that is

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

Example 2: Evaluate

$$\text{(a)}\ \int 4\cos x\,dx \qquad \text{(b)}\ \int (x + x^2)\,dx$$

Solution: (*a*)

$$\int 4\cos x\,dx = 4\int \cos x\,dx = 4\sin x + C$$

(*b*)

$$\int (x + x^2)\,dx = \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C$$

Example 3:

$$\int (3x^6 - 2x^2 + 7x + 1)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

Example 4: Evaluate

$$\text{(a)} \int \frac{\cos x}{\sin^2 x}\,dx \qquad \text{(b)} \int \frac{t^2 - 2t^4}{t^4}\,dt \qquad \text{(c)} \int \frac{x^2}{x^2+1}\,dx$$

Solution: (*a*)

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx = -\csc x + C$$

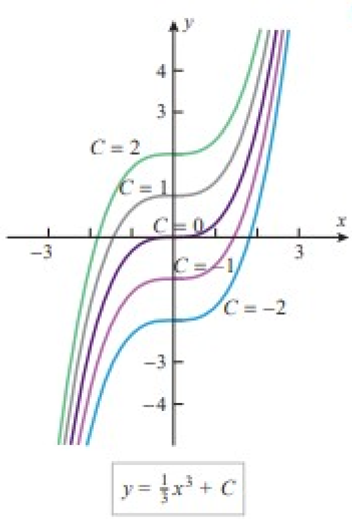
(*b*)

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt = \int (t^{-2} - 2)\,dt$$
$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

(*c*)

$$\int \frac{x^2}{x^2+1}\,dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)dx$$
$$= \int \left(1 - \frac{1}{x^2+1}\right)dx = x - \tan^{-1} x + C$$

Integral Curves: Graphs of antiderivatives of a function $\boldsymbol{f}$ are called integral curves of $\boldsymbol{f}$. We know that if $\boldsymbol{y = F(x)}$ is any integral curve of $\boldsymbol{f(x)}$, then all other integral curves are vertical translations of this curve, since they have equations of the form $\boldsymbol{y = F(x) + C}$. For example, $\boldsymbol{y = \frac{1}{3}x^3}$ is one integral curve for $\boldsymbol{f(x) = x^2}$, so all the other integral curves have equations of the form $\boldsymbol{y = \frac{1}{3}x^3 + C}$; conversely, the graph of any equation of this form is an integral curve (following figure).



Example 5: Suppose that a curve $y = f(x)$ in the ***xy***-plane has the property that at each point ***(x, y)*** on the curve, the tangent line has slope x^2. Find an equation for the curve given that it passes through the point **(2, 1)**.

Solution: Since the slope of the line tangent to $y = f(x)$ is $\frac{dy}{dx}$, we have $\frac{dy}{dx} = x^2$, and

$$y = \int x^2\,dx = \tfrac{1}{3}x^3 + C$$

Since the curve passes through (2, 1), a specific value for C can be found by using the fact that $y = 1 \text{ if } x = 2$. Substituting these values in the above equation yields

$$1 = \tfrac{1}{3}(2^3) + C \quad \text{or} \quad C = -\tfrac{5}{3}$$

so an equation of the curve is

$$y = \tfrac{1}{3}x^3 - \tfrac{5}{3}$$

Integration from the Viewpoint of Differential Equations: Suppose that $f(x)$ is a known function and we are interested in finding a function $F(x)$ such that $y = F(x)$ satisfies the equation

$$\frac{dy}{dx} = f(x) \dots\dots\dots\dots\dots\dots\dots (i)$$

The solutions of this equation are the antiderivatives of $f(x)$, and we know that these can be obtained by integrating $f(x)$.

Equation **(*i*)** is called a differential equation because it involves a derivative of an unknown function.

For simplicity, it is common in the study of differential equations to denote a solution of $\frac{dy}{dx} = f(x)$ as $y(x)$ rather than $F(x)$, as earlier. With this notation, the problem of finding a function $y(x)$ whose derivative is $f(x)$ and whose graph passes through the point (x_0, y_0) is expressed as

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

This is called an ***initial-value problem***, and the requirement that $y(x_0) = y_0$ is called the **initial condition** for the problem.

Example 6: Solve the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

Solution: The solution of the differential equation is

$$y = \int \cos x \, dx = \sin x + C \qquad \textbf{(}\boldsymbol{i}\textbf{)}$$

The initial condition $\boldsymbol{y(0) = 1}$ implies that $\boldsymbol{y = 1 \; if \; x = 0}$; substituting these values in **(*i*)** yields

$$1 = \sin(0) + C \quad or \;\; C = 1$$

Thus, the solution of the initial-value problem is

$$\boldsymbol{y = \sin x + 1} \quad \textbf{(Ans.)}$$

Home Work: Exercise 5.2: Problem No. 9-36, 43- 45 and 53-55