Solution to ECE 302 Homework Assignment 1

1. Consider the experiment in which one tosses a fair die and counts the dots on the side facing up.

   (a) What is the sample space of this experiment?
   Solution: The sample space is the set of all possible outcomes so
   $$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}.$$

   (b) What is the event A corresponding to "an even number of dots were counted"?
   Solution: The event A corresponds to either a 2, a 4, or a 6 being thrown, so $A = \{2, 4, 6\}$.

   (c) List the outcomes contained in the event A^c. Also characterize the event A^c in words.
   Solution: The outcomes contained in the event A^c are the elements of $\mathcal{S}$ that are not in A, so $A^c = \{1, 3, 5\}$, the event that an odd number of dots is counted.

2. A fair die is tossed twice and the numbers n_i of dots facing up are noted, ($i \in \{1, 2\}$).

   (a) What is the sample space?
   Solution: The sample space is the set of ordered pairs (n_1, n_2) where $n_i \in \{1, 2, 3, 4, 5, 6\}$ where $i \in \{1, 2\}$. We can write this as
   $$\mathcal{S} = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$
   or
   $$\mathcal{S} = \{(n_1, n_2) : n_i \in \{1, 2, 3, 4, 5, 6\},\ \forall i \in \{1, 2\}\}.$$

   (b) What are the elements of the event A corresponding to "total number of dots is odd"?
   Solution: The elements of the event A are those ordered pairs where $n_1 + n_2$ is odd, so
   $$A = \{(1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), \ldots (6, 1), (6, 3), (6, 5)\}.$$

   (c) What are the elements of the event B corresponding to "both tosses are odd"?
   Solution:
   $$B = \{(1, 1), (1, 3), (1, 5), (3, 1), \ldots, (5, 5)\}$$

   (d) What are the elements of the event $A \cap B^c$. Also describe this event in words.
   Solution: $A \cap B^c = A$ because B^c is the set of ordered pairs at least one of which is even and A is precisely the set of ordered pairs exactly one of which is even, *i.e.* $A \subseteq B^c$.

(e) Let C correspond to the event "numbers of dots observed in the two tosses differ by 1". (Be sure to state precisely how you have defined C.) Find $A \cap C$.

Solution: We have two possibilities here. We can assume that "numbers of dots observed in the two tosses differ by 1" means either $n_2 - n_1 = 1$ or $|n_2 - n_1| = 1$.

Assuming it means $n_2 - n_1 = 1$ we have

$$C = \{(2,1), (3,2), (4,3), (5,4), (6,5)\}.$$

Assuming instead that it means $|n_2 - n_1| = 1$, we have

$$C = \{(2,1), (1,2), (3,2), (2,3), (4,3), (3,4), (5,4), (4,5), (6,5), (5,6)\}.$$

3. Consider the experiment in which one tosses two dice and records the total number of dots facing up.

(a) What is the sample space?

Solution:

$$\mathcal{S} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

(b) What are the elements of the event A corresponding to "total number of dots counted is even"?

Solution:

$$A = \{2, 4, 6, 8, 10, 12\}$$

(c) Express each of the outcomes in this sample space in terms of the elements of the sample space of the previous problem. (Think of the two throws of the one die in the previous problem as being throws of two separate dice.)

Solution:

2	$\longleftrightarrow$	$\{(1,1)\}$
3	$\longleftrightarrow$	$\{(1,2),(2,1)\}$
4	$\longleftrightarrow$	$\{(1,3),(2,2),(3,1)\}$
5	$\longleftrightarrow$	$\{(1,4),(2,3),(3,2),(4,1)\}$
6	$\longleftrightarrow$	$\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
7	$\longleftrightarrow$	$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
8	$\longleftrightarrow$	$\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$
9	$\longleftrightarrow$	$\{(3,6),(4,5),(5,4),(6,3)\}$
10	$\longleftrightarrow$	$\{(4,6),(5,5),(6,4)\}$
11	$\longleftrightarrow$	$\{(5,6),(6,5)\}$
12	$\longleftrightarrow$	$\{(6,6)\}$

4. Consider tossing a die and recording the number N_1 of dots facing up, then choosing an integer N_2 between 1 and N_1 at random (meaning that each integer is equally likely to be chosen).

(a) Find the sample space.
Solution:

$$\begin{aligned}\mathcal{S} &= \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4),(5,1),\\ &\quad (5,2),(5,3),(5,4),(5,5),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

(b) Find the set of outcomes in the event "die shows four dots facing up."
Solution:

$$\{(4,1),(4,2),(4,3),(4,4),(5,4),(6,4)\}$$

(c) Find the set of outcomes in the event "$N_2 = 3$".
Solution:

$$\{(3,3),(4,3),(5,3),(6,3)\}$$

(d) Find the set of outcomes in the event "$N_2 = 6$".
Solution:

$$\{(6,6)\}$$

5. A desk drawer contains five pens, three of which are dry.

(a) Suppose that one selects and tests pens at random, one by one, until a good pen is found and notes the sequence of test results. What is the sample space?
Solution:

$$\mathcal{S} = \{DDDG, DDG, DG, G\}$$

(b) Suppose that only the number of pens tested, as opposed to the sequence of test results, is noted. What is the sample space?
Solution:

$$\mathcal{S} = \{4,3,2,1\}$$

(c) Suppose that the pens are selected one by one and tested until both good pens have been identified, and the sequence of test results is noted. What is the sample space?
Solution:

$$\begin{aligned}\mathcal{S} &= \{DDDGG, DDGDG, DGDDG, GDDDG, DDGG, DGDG,\\ &\quad GDDG, DGG, GDG, GG\}\end{aligned}$$

(d) Suppose that the pens are selected one by one and tested until both good pens have been identified, and the number of pens tested is noted. What is the sample space.

Solution:

$$\mathcal{S} = \{5, 4, 3, 2\}$$

6. A student wakes up at time T_1 and goes to sleep at some later time T_2. Assume that $T_2 - T_1 \leq 24$.

(a) What is the sample space? Sketch it in the x-y plane. (Plot T_1 on the x-axis and T_2 on the y-axis.)

Solution:

$$\mathcal{S} = \{(T_1, T_2) : 0 \leq T_1 < T_2 \text{ and } T_2 \geq T_1 + 24\}.$$

(b) What are the outcomes contained in the event A "student is still awake 14 hours after waking up"? Sketch the region on the plane corresponding to this event.

Solution:

$$\{(T_1, T_2) : 0 \leq T_1 < T_2 \text{ and } T_2 \leq T_1 + 24 \text{ and } T_1 + 14 < T_2\}$$

or, simplifying,

$$\{(T_1, T_2) : 0 \leq T_1 < T_1 + 14 < T_2 \leq T_1 + 24\},$$

which is indicated in the graph by the region filled with circles.

(c) What are the outcomes in the event B "student is asleep more than awake"? Sketch the region in the plane corresponding to this event.

Solution: In order to define this region, we need more information, namely, the end of the interval of time that we are considering. (Otherwise we couldn't answer the question because there would be an infinitely long sequence of T_i where odd T_i correspond to waking up and even T_i correspond to falling asleep.) Let's call this upper limit on the times L for limit. For convenience, let's assume that $L > 24$.

Now, the student is asleep when $t \in [0, T_1]$ and when $t \in [T_2, L]$ for a total of $(T_1 - 0) + (L - T_2)$ hours. The student is awake for $T_2 - T_1$ hours. The student is thus asleep more hours than awake if $T_1 + L - T_2 > T_2 - T_1$. Adding $T_1 + T_2$ to both sides yields the inequality $2T_1 + L > 2T_2$ or $T_1 + L/2 > T_2$. Thus the line $T_1 + L/2 = T_2$ gives an upper bound on T_2 for this situation. (The lower bound is still $T_2 > T_1$.) This region is indicated in Figure 1 by diagonal stripes.)

(d) Sketch the region corresponding to $A^c \cap B$ and describe the corresponding event in words.

Solution: A^c corresponds to the event that the student is not awake 14 hours after waking up, *i.e.* $T_2 < T_1 + 14$.

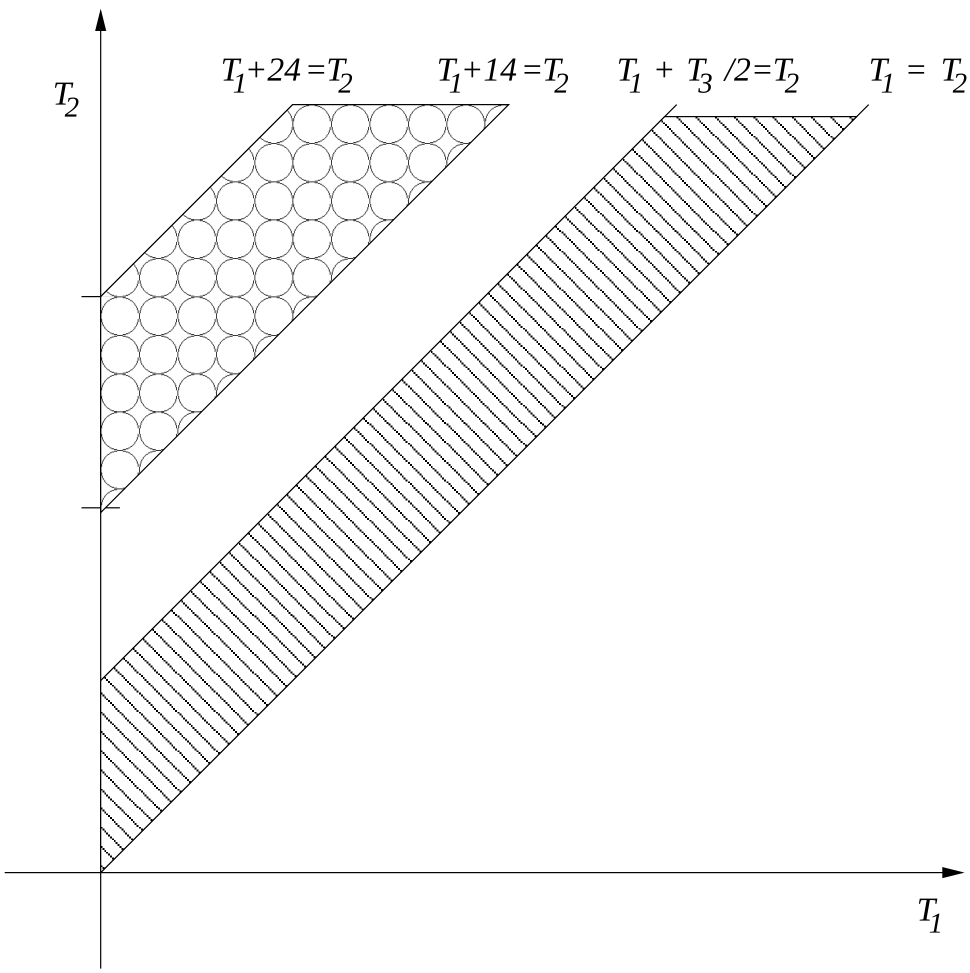

Figure 1: Graph for Problem 6

7. A die is tossed and the number of dots facing up is noted.

   (a) Assume the die is fair. Find the probability of each possible outcome in the sample space occurring.

   Solution: $P[s] = 1/6, \forall s \in \mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, by Theorem 1.6.

   (b) Now assume instead that the face with a single dot is twice as likely to be facing up as the rest of the faces. (Assume that the others are equally likely.) Find the probabilities as before.

   Solution: If $P[1] = 2(P[2] + P[3] + P[4] + P[5] + P[6])$, and $P[2] = P[3] = P[4] = P[5] = P[6]$, then let $p = P[2]$. Now the probabilities must sum to one, *i.e.* $\sum_{i=1}^{6} P[i] = 1$ so we have

$$\begin{aligned} 1 &= P[1] + P[2] + P[3] + P[4] + P[5] + P[6] \\ &= 2(P[2] + P[3] + P[4] + P[5] + P[6]) + P[2] + P[3] + P[4] + P[5] + P[6] \\ &= 2(5p) + 5p \\ &= 15p = 1 \end{aligned}$$

   or $p = 1/15$. Thus $P[1] = 10/15 = 2/3$.

   (c) Now assume instead that the face with a single dot is twice as likely to be facing up as any of the other faces. (Assume that the others are equally likely.) Find the probabilities as before.

   Solution:

   This time we have $P[1] = 2p$, where $P[2] = P[3] = P[4] = P[5] = P[6] = p$. Again the probabilities must sum to one, *i.e.* $\sum_{i=1}^{6} P[i] = 1$ so we have

$$P[1] + P[2] + P[3] + P[4] + P[5] + P[6] = 2p + 5p = 7p = 1$$

   or $p = 1/7$. Thus $P[1] = 2/7$.

   (d) Find the probabilities that the outcome of a toss is even under the three different assumptions above.

   Solution: If the die is fair, $P[\{2, 4, 6\}] = 3/6 = 1/2$.

   If the probability of getting a one is twice the probability of getting everything else (case (b)), then $P[\{2, 4, 6\}] = 3/15 = 1/5$.

   If the probability of getting a one is twice the probability of each other outcome (case (c)), then $P[\{2, 4, 6\}] = 3/7$.

8. A die is tossed twice and the number of dots facing up noted in the order of the tosses. Assuming that all outcomes are equally likely to occur, find the probabilities of the following events:

(a) A_k: "the sum of the two values noted is k", for each $k \in \{2, 3, ..., 12\}$.

(b) B: "the two values noted are different."

Solution: The problem statement indicates that order matters, so we have sample space

$$\begin{aligned}\mathcal{S} = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

- In order to determine the probabilities of the events A_k, we must first determine what subset of the sample space corresponds to each A_k. In fact, we already did this in problem 3(c). In particular, A_k is the set on the right hand side of the row that starts with the value k (which is actually the $k-1$st row of the table since there is no row corresponding to the values summing to one).

  Each element of the sample space is equally likely so to obtain $P[A_k]$ we simply sum the number of ordered pairs in event A_k and divide by the number of ordered pairs in $\mathcal{S}$. Thus from the table in problem 3(c),

$$\begin{aligned}P[A_2] &= P[A_{12}] = 1/36\\ P[A_3] &= P[A_{11}] = 2/36 = 1/18\\ P[A_4] &= P[A_{10}] = 3/36 = 1/12\\ P[A_5] &= P[A_9] = 4/36 = 1/9\\ P[A_6] &= P[A_8] = 5/36\\ P[A_7] &= 6/36 = 1/6\end{aligned}$$

- Event B corresponds to the set $B = \mathcal{S} \setminus \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$. Applying Axiom 2 and Theorem 1.6 we have that $P[B] = 1 - 6/36 = 5/6$.

9. An experiment has sample space $\mathcal{S} = \{a, b, c\}$. Given $P[\{a, c\}] = 5/8$ and $P[\{b, c\}] = 7/8$, use the axioms of probability to find the probability of each outcome in S occurring.

   Solution: By Axiom 2, the probability of the whole sample space is 1. Since $\mathcal{S} = \{a, b\} \cup \{c\}$ and $\{a, b\} \cap \{c\} = \emptyset$, this means that $1 = P[\{a, b\}] + P[\{c\}]$ so $P[\{c\}] = 1 - 7/8 = 1/8$.

   Again by Axiom 2, $\mathcal{S} = \{b\} \cup \{a, c\}$, this means that $1 = P[\{b\}] + P[\{a, c\}]$ so $P[\{b\}] = 1 - 5/8 = 3/8$.

   Finally, by Axiom 2, $1 = P[\{a\}] + P[\{b\}] + P[\{c\}]$ so $P[\{a\}] = 1 - 3/8 - 1/8 = 4/8 = 1/2$.

10. Explain why the probability that exactly one of events A or B (defined on the same sample space) occurs is $P[A] + P[B] - 2P[A \cap B]$.

Solution: First, let's characterize the events A and B in terms of each other. We can express the set corresponding to A (which, by conventional abuse of notation we will call A) as $A = (A \setminus B) \cup (A \cap B)$. Note that the sets $A \setminus B$ and $A \cap B$ are disjoint. Thus by Theorem 1.3,

$$P[A] = P[A \setminus B] + P[A \cap B].$$

Similarly we can express B as

$$B = (B \setminus A) \cup (B \cap A).$$

$B \setminus A$ and $B \cap A$ are also disjoint so $P[B] = P[B \setminus A] + P[B \cap A]$. Note that the set $A \cap B$ is the same set as $B \cap A$, that is the set of elements in both A and B.

The event that exactly one of A and B occurs corresponds to the set of outcomes in $A \setminus B \cup B \setminus A$, *i.e.* the set of outcomes in A but not B or B but not A. From above we have that $P[A \setminus B] = P[A] - P[A \cap B]$ and $P[B \setminus A] = P[B] - P[B \cap A]$. $(A \setminus B) \cap (B \setminus A) = \emptyset$ so

$$\begin{aligned} P[(A \setminus B) \cup (B \setminus A)] &= P[A \setminus B] + P[B \setminus A] \\ &= P[A] - P[A \cap B] + P[B] - P[B \cap A] \\ &= P[A] + P[B] - P[A \cap B] - P[B \cap A] \\ &= P[A] + P[B] - 2P[A \cap B]. \end{aligned}$$

11. Consider selecting two numbers at random from the interval $[0, 1]$. Find the probability that they differ by more than $1/2$.

    Solution: There are three possible interpretations of this problem. In the first and second, the sample space is the set of ordered pairs of values from the interval $[0, 1]$. This can be written as $\mathcal{S} = [0, 1] \times [0, 1]$ or $\mathcal{S} = \{(n_1, n_2) : 0 \leq n_1 \leq 1, 0 \leq n_2 \leq 1\}$. In the third, the sample space is $\mathcal{S} = \{n_1, n_2\}$, where it is assumed that $n_2 \neq n_1$.

    First lets consider the first two options in which order matters. Let n_1 correspond to a value on the x-axis and n_2 a value on the y-axis. Then (n_1, n_2) is a point in the unit square.

    We again have two choices regarding the meaning of "differ by more than 1". Suppose first that we mean $n_2 - n_1 > 1/2$. Then the region we are considering is the region above the line $n_2 = n_1 + 1/2$. If each point in the sample space is equally likely, then the probability that the ordered pair (n_1, n_2) is contained in a particular region of the graph corresponds to the area of the region over the area of the whole unit square. The region in question is bounded below by $n_2 = n_1 + 1/2$, at the left by $n_1 = 0$, and above by $n_2 = 1$. Thus its area is $1/8$, *i.e.* the probability that $n_2 - n_1 > 1/2$ is $1/8$.

    Now suppose instead that we mean $|n_2 - n_1| > 1/2$. This statement corresponds to either $n_2 - n_1 > 1/2$ or $n_1 - n_2 > 1/2$. The first is the region we just found and the second is the region below the line $n_2 < n_1 - 1/2$ By symmetry, the probability of being in this second region is also $1/8$, thus the probability that a randomly chosen value is in one or the other of these (disjoint) regions is $1/8 + 1/8 = 1/4$.

Now we suppose that $\mathcal{S} = \{\{n_1, n_2\} : 0 \leq n_1 \leq 1, 0 \leq n_2 \leq 1 \text{ and } n_2 \neq n_1\}$. We "justify" this restriction that the values not be equal by noting that the probability that a randomly chosen pair of values are equal is zero (the area of a line divided by the area of the area of the unit square). Now, the probability that the values differ by at least one half can have only one meaning. because we are considering an unordered pair of values, for them to differ by $1/2$ has to mean that $|n_2 - n_1| > 1/2$. Accordingly, if we arbitrarily assign either n_1 or n_2 to the x-axis and the other to the y-axis, we find that we are considering the same pair of regions as in the second option and the probability of the randomly chosen values being more than $1/2$ is again $1/4$ by geometric considerations.