(Jointly distributed random variable)

- **Joint Probability Distribution**

The function $f(x, y)$ is a **joint probability density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dx\ dy = 1$,

- **Marginal probability distribution**

The marginal distribution of a random variable X is obtained from the joint probability distribution of two random variables X and Y by integrating over the values of the random variable Y. The marginal distribution is the individual probability distribution of the random variable X considered alone. Similarly, The marginal distribution of a random variable Y is obtained from the joint probability distribution of two random variables X and Y by integrating over the values of the random variable X. The marginal distribution is the individual probability distribution of the random variable Y considered alone.

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\ dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x, y)\ dx$$

- **Conditional probability**

Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$f(y|x) = \frac{f(x, y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x, y)}{h(y)}, \text{ provided } h(y) > 0.$$

- **Statistical Independence**

Let X and Y be two random variables, discrete or continuous, with joint probability distribution $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be **statistically independent** if and only if

$$f(x, y) = g(x)h(y)$$

Covariance

The **covariance** of two random variables X and Y is defined to be

$$\mathrm{Cov}(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$$

The covariance can be any positive or negative number, and independent random variables have a covariance of 0.

Correlation

The **correlation** between two random variables X and Y is defined to be

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

The correlation takes values between -1 and 1, and independent random variables have a correlation of 0.

- **Classwork**
  Suppose that two continuous random variables X and Y have a joint probability density function

$$f(x, y) = A(x - 3)y$$

for $-2 \leq x \leq 3$ and $4 \leq y \leq 6$
a) What is the value of A?
b) What is $P(0 \leq x \leq 1 \text{ and } 4 \leq y \leq 5)$?
c) Construct the marginal probability density functions.
d) Are the random variables X and Y independent?
e) If Y = 5, what is the conditional probability density function of X?
f) What are the expectations and variances of the random variables X and Y ?
g) What is the covariance of X and Y?
h) What is the correlation between X and Y?