MAT 350
ENGINEERING MATHEMATICS

System of Linear ODEs: 2nd lecture
NONHOMOGENEOUS LINEAR SYSTEMS

Lecture: 12

Dr. M. Sahadet Hossain (Mth)
Associate Professor
Department of Mathematics and Physics, NSU.

NONHOMOGENEOUS LINEAR SYSTEMS

The general solution of a nonhomogeneous linear system

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$$

on an interval I is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$,

where $\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$ is the **complementary function** and $\mathbf{X}_p$ is any **particular solution**

The methods of **undetermined coefficient and variation of parameters are used to** find the particular solutions of nonhomogeneous linear ODEs. We only focus on of **undetermined coefficient.**

UNDETERMINED COEFFICIENTS

EXAMPLE 1 **Undetermined Coefficient**

Solve the system $\mathbf{X}' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$ on $(-\infty, \infty)$.

SOLUTION We first solve the associated homogeneous syste

$$\mathbf{X}' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\mathbf{X}.$$

The characteristic equation of the coefficient matrix **A**,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

yields the complex eigenvalues $\lambda_1 = i$ and $\lambda_2 = \overline{\lambda_1} = -i$.

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Note that α= 0, and β=1

$$\mathbf{X}_c = c_1\begin{pmatrix}\cos t + \sin t\\ \cos t\end{pmatrix} + c_2\begin{pmatrix}\cos t - \sin t\\ -\sin t\end{pmatrix}.$$

Now since F(t) is a constant vector, we assume a constant particular solution vector

$$\mathbf{X}_p = \begin{pmatrix}a_1\\ b_1\end{pmatrix}.$$

Substituting this latter assumption into the original system and equating entries leads to

$$\begin{aligned}0 &= -a_1 + 2b_1 - 8\\ 0 &= -a_1 + b_1 + 3.\end{aligned}$$

Solving this algebraic system gives $a_1 = 14$ and $b_1 = 11$, and so a particular solution is $\mathbf{X}_p = \begin{pmatrix} 14 \\ 11 \end{pmatrix}$

The general solution of the original system of DEs on the interval $(-\infty, \infty)$ is then $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$ or

$$\mathbf{X} = c_1\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t - \sin t \\ -\sin t \end{pmatrix} + \begin{pmatrix} 14 \\ 11 \end{pmatrix}.$$

EXAMPLE 2 Undetermined Coefficient

Solve the system $\mathbf{X}' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}$ on $(-\infty, \infty)$.

SOLUTION The eigenvalues and corresponding eigenvectors of the associated homogeneous system $\mathbf{X}' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\mathbf{X}$ are found to be $\lambda_1 = 2$, $\lambda_2 = 7$, $\mathbf{K}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, and $\mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence the complementary function is

$$\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t}.$$

Now because $\mathbf{F}(t)$ can be written $\mathbf{F}(t) = \begin{pmatrix} 6 \\ -10 \end{pmatrix}t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$,

we shall try to find a particular solution of the system that possesses the *same* form:

$$\mathbf{X}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}.$$

Substituting this last assumption into the given system yields

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\left[\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\right] + \begin{pmatrix} 6 \\ -10 \end{pmatrix}t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (6a_2 + b_2 + 6)t + 6a_1 + b_1 - a_2 \\ (4a_2 + 3b_2 - 10)t + 4a_1 + 3b_1 - b_2 + 4 \end{pmatrix}.$$

From the last identity we obtain four algebraic equations in four unknowns

$$\begin{aligned} 6a_2 + b_2 + 6 &= 0 \\ 4a_2 + 3b_2 - 10 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 6a_1 + b_1 - a_2 &= 0 \\ 4a_1 + 3b_1 - b_2 + 4 &= 0. \end{aligned}$$

Solving the first two equations simultaneously yields $a_2 = -2$ and $b_2 = 6$.

We then substitute these values into the last two equations

and solve for a_1 and b_1.

The results are $a_1 = -\frac{4}{7}$, $b_1 = \frac{10}{7}$.

It follows, therefore, that a particular solution vector is

$$\mathbf{X}_p = \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}.$$

The general solution of the system on $(-\infty, \infty)$ is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$ or

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}.$$

$$\mathbf{8.}\quad \mathbf{X}' = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 5 \\ -10 \\ 40 \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 5 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & -\lambda \end{vmatrix} = -(\lambda-5)^2(\lambda+5) = 0$$

we obtain the eigenvalues $\lambda_1 = 5$, $\lambda_2 = 5$, and $\lambda_3 = -5$.

Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t} + C_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-5t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

into the system yields

$$5c_1 = -5$$

$$5b_1 = 10$$

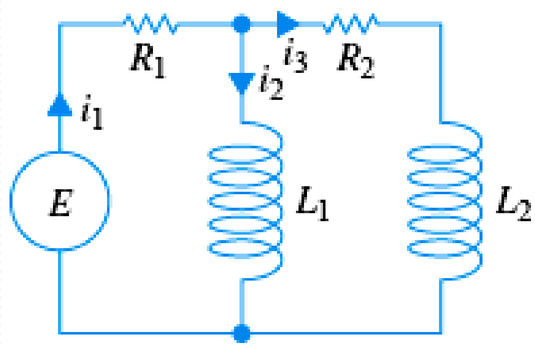
$$5a_1 = -40$$

from which we obtain $c_1 = -1$, $b_1 = 2$, and $a_1 = -8$. Then

$$\mathbf{X}(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t} + C_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} -8 \\ 2 \\ -1 \end{pmatrix}.$$

10. **(a)** The system of differential equations for the currents $i_2(t)$ and $i_3(t)$ in the electrical network shown in Figure 8.3.1 is

$$\frac{d}{dt}\begin{pmatrix} i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -R_1/L_1 & -R_1/L_1 \\ -R_1/L_2 & -(R_1 + R_2)/L_2 \end{pmatrix}\begin{pmatrix} i_2 \\ i_3 \end{pmatrix} + \begin{pmatrix} E/L_1 \\ E/L_2 \end{pmatrix}.$$



Use the method of undetermined coefficients to solve the system if $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, $L_1 = 1$ h, $L_2 = 1$ h, $E = 60$ V, $i_2(0) = 0$, and $i_3(0) = 0$.

(b) Determine the current $i_1(t)$.

a) Let $\mathbf{I} = \begin{pmatrix} i_2 \\ i_3 \end{pmatrix}$ so that

$$\mathbf{I}' = \begin{pmatrix} -2 & -2 \\ -2 & -5 \end{pmatrix} \mathbf{I} + \begin{pmatrix} 60 \\ 60 \end{pmatrix}$$

and

$$\mathbf{I}_c = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t}.$$

If $\mathbf{I}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ then $\mathbf{I}_p = \begin{pmatrix} 30 \\ 0 \end{pmatrix}$ so that

$$\mathbf{I} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t} + \begin{pmatrix} 30 \\ 0 \end{pmatrix}.$$

For $\mathbf{I}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we find $c_1 = -12$ and $c_2 = -6$.

(b) $i_1(t) = i_2(t) + i_3(t) = -12e^{-t} - 18e^{-6t} + 30.$