Exercise- 5.3

1. Explain why the following are linearly dependent set of vectors ~~sto~~ (solve this problem by inspection)

(a) $u_1 = (-1,2,4)$ and $u_2 = (5,-10,-20)$ in $\mathbb{R}^3$.

There are only two vectors and $u_2 = -5u_1$. Since u_2 is ~~scac~~ scalar multiple of u_1, both the vectors lies on the same line.

Hence u_1 and u_2 are linearly dependent.

(b) $u_1 = (3,-1)$, $u_2 = (4,5)$, $u_3 = (-4,7)$ in $\mathbb{R}^2$.

Number of vectors is 3 and each vector has only 2-components. If number of vectors is greater than the number of component in ~~vec~~ each vector then they are linearly dependent.

(c) $p_1 = 3-2x+x^2$ and $p_2 = 6-4x+2x^2$ in P_2.

There ary only two polynomials and $p_2 = 2p_1$. One is scalar multiple of other, hence linearly dependent.

(d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

There are only two matrices and $B = -A$. That is one is scalar multiple of other, hence linearly dependent.

2. Which of the following set of vectors in $\mathbb{R}^3$ are linearly dependent ?

(a) $(4,-1,2)$, $(-4,10,2)$

Two vectors and none of these is scalar multiple of other. They are linearly independent.

(b) $(-3,0,4)$, $(5,-1,2)$, $(1,1,3)$

Let, $u_1 = (-3,0,4)$, $u_2 = (5,-1,2)$ and $u_3 = (1,1,3)$

$$k_1\vec{u}_1 + k_2\vec{u}_2 + k_3\vec{u}_3 = \vec{0}$$

$$k_1(-3,0,4) + k_2(5,-1,2) + k_3(1,1,3) = (0,0,0)$$

$$\left.\begin{aligned} -3k_1 + 5k_2 + k_3 &= 0 \\ 0.k_1 - k_2 + k_3 &= 0 \\ 4k_1 + 2k_2 + 3k_3 &= 0 \end{aligned}\right\}$$ System of Homogeneous equations.

The coefficient matrix, $A = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{vmatrix} = -3\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} - 5\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 4 & 2 \end{vmatrix}$$

$$= -3(-3-2) - 5(0-4) + 1(0+4)$$

$$= 15 + 20 + 4$$

$$= 39 \neq 0.$$

$\therefore$ A is invertible. Hence the system is consistent and has unique solution/ trivial solution.

i.e. $k_1 = k_2 = k_3 = 0$.

$\therefore$ Given vectors are linearly independent.

(c) $(8,-1,3)$, $(4,0,1)$

two vectors which are not scalar multiple of each other, hence linearly independent.

(d) $(-2,0,1)$, $(3,2,5)$, $(6,-1,1)$, $(7,0,-2)$

No. of vector $= 4$.

No. of component in each vector $= 3$.

since, $4 > 3$, hence given vector are linearly dependent.

Use : Theorem : 5.3.3

Let $S = \{v_1, v_2, \dots, v_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

3. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly dependent?

(a) $(3,8,7,-3)$, $(1,5,3,-1)$, $(2,-1,2,6)$, $(1,4,0,3)$

Set a linear combination of these 4-vectors equal to zero

$$k_1(3,8,7,-3)+k_2(1,5,3,-1)+k_3(2,-1,2,6)+k_4(1,4,0,3) = (0,0,0,0)$$

Equating the corresponding components,

$$3k_1+k_2+2k_3+k_4=0$$
$$8k_1+5k_2-k_3+4k_4=0$$
$$7k_1+3k_2+2k_3+0\cdot k_4=0$$
$$-3k_1-k_2+6k_3+3\cdot k_4=0$$

Homogeneous system of Linear equation.

The coefficient matrix, $A = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 2 & 1 \\ 0 & 7 & -19 & 4 \\ 0 & 2 & -8 & -7 \\ 0 & 0 & 8 & 4 \end{vmatrix} \quad \begin{aligned} R_2' &= 3R_2 - 8R_1 \\ R_3' &= 3R_3 - 7R_1 \\ R_4' &= R_3 + R_1 \end{aligned}$$

$$= 3\begin{vmatrix} 7 & -19 & 4 \\ 2 & -8 & -7 \\ 0 & 8 & 4 \end{vmatrix} = 3[7(-32+56)-2(-76-32)]$$

$$= 3(7\cdot 24 + 216) = \neq 0$$

[A is invertible System has only trivial solution]

$\therefore$ The vectors are linearly independent.

4. Which of the following sets of vectors in P_2 are linearly dependent?

(a) $2 - x + 4x^2, \quad 3 + 6x + 2x^2, \quad 2 + 10x - 4x^2$

Set zero as a linear combination of ~~these~~ given polynomials,

$$0 + 0 \cdot x + 0 \cdot x^2 = k_1(2 - x + 4x^2) + k_2(3 + 6x + 2x^2) + k_3(2 + 10x - 4x^2)$$

~~Equating the different po~~

Equating the coefficients of different power of x,

$$2k_1 + 3k_2 + 2k_3 = 0$$
$$-k_1 + 6k_2 + 10k_3 = 0$$
$$4k_1 + 2k_2 - 4k_3 = 0$$

Coefficient matrix, $A = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{vmatrix} = 2\begin{vmatrix} 6 & 10 \\ 2 & -4 \end{vmatrix} - 3\begin{vmatrix} -1 & 10 \\ 4 & -4 \end{vmatrix} + 2\begin{vmatrix} -1 & 6 \\ 4 & 2 \end{vmatrix}$$

$$= 2(-24 - 20) - 3(4 - 40) + 2(-2 - 24)$$

$$= -88 + 108 - 52$$

$$= -32 \neq 0.$$

A is invertible, hence the system has unique solution. And the solution is trivial solution.

i.e. $k_1 = k_2 = k_3 = 0$.

$\therefore$ Given vectors are linearly independent.

5. Assume that v_1, v_2 and v_3 are vectors in $\mathbb{R}^3$ that have their initial points at the origin. ~~Detter~~ Determine whether the three veetors lie in a plane.

(a) $v_1 = (2, -2, 0)$, $v_2 = (6, 1, 4)$, $v_3 = (2, 0, -4)$

Three vectors lie in a plane if the vectors are linearly dependent.

Set zero as a linear combination of the given vectors

$$(0,0,0) = k_1(2,-2,0) + k_2(6,1,4) + k_3(2,0,-4)$$

$$2k_1 + 6k_2 + 2k_3 = 0$$
$$-2k_1 + k_2 + 0 \cdot k_3 = 0$$
$$0 \cdot k_1 + 4k_2 - 4k_3 = 0$$

The coefficient matrix is, $A = \begin{bmatrix} 2 & 6 & 2 \\ -2 & 1 & 0 \\ 0 & 4 & -4 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 2 & 6 & 2 \\ -2 & 1 & 0 \\ 0 & 4 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 4 & -4 \end{vmatrix} + 2\begin{vmatrix} 6 & 2 \\ 4 & -4 \end{vmatrix} + 0\begin{vmatrix} 6 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= 2(-4-0) + 2(-24-8)$$

$$= -8 - 64 = -72 \neq 0$$

Since $\det(A) \neq 0$, the A is invertible and the homogeneous system has unique solution, i.e.

$$k_1 = k_2 = k_3 = 0$$

So the vectors are linearly independent.

Hence $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ do not lie in a plane.

6. Assume that v_1, v_2, and v_3 are vectors in $\mathbb{R}^3$ that have their initial points at the origin. Determine whether the three vectors lie on the same line.

(c) $v_1 = (4, 6, 8)$, $v_2 = (2, 3, 4)$, $v_3 = (-2, -3, -4)$

These three vector lie on the same plane, since each of them are scalar multiple of others. Hence they are linearly dependent and lie on the same plane.

$$v_1 = 2v_2, \quad v_2 = \frac{1}{2}v_1, \quad v_3 = -v_2, \quad v_3 = -\frac{1}{2}(v_1)$$

(a) $v_1 = (-1, 2, 3)$, $v_2 = (2, -4, -6)$, $v_3 = (-3, 6, 0)$

Here, $v_2 = -2v_1$ hence v_1 and v_2 lies on the same ~~plane~~ line. If v_3 can be expressed as a linear combination of v_1 and v_3 then they lies on the same line.

$$v_3 = k_1 v_1 + k_2 v_2$$

$$(-3, 6, 0) = k_1(-1, 2, 3) + k_2(2, -4, -6)$$

$$= (-k_1 + 2k_2, 2k_1 - 4k_2, 3k_1 - 6k_2)$$

$$\therefore -k_1 + 2k_2 = -3$$
$$2k_1 - 4k_2 = 6$$
$$3k_1 - 6k_2 = 0$$

Reduced by Gaussian elimination

$$\begin{aligned} k_1 - 2k_2 &= 3 \\ 2k_1 - 4k_2 &= 6 \\ 3k_1 - 6k_2 &= 0 \end{aligned} \Rightarrow \begin{aligned} k_1 - 2k_3 &= 3 \\ 0 - 0 &= 0 \\ 0 - 0 &= -3 \end{aligned} \Rightarrow \begin{aligned} k_1 - 2k_3 &= 3 \\ 0 &= -3 \end{aligned}$$

Inconsistent.

Hence v_3 is not in the plane that containts v_1 and v_2. Hence they are not in the same line.

21. Use Wronskian to show that the following sets of vectors are linearly independent.

(a) $1, x, e^x$.

The Wronskian is, $W = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$

$= 1 \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}$

$= e^x - 0$

$= e^x \neq 0$

Since the Wronskian is not zero, the vectors are linearly independent.

21 (b) $\sin x$, $\cos x$, $x \sin x$

The Wronskian is,

$$W = \begin{vmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & x\cos x + \sin x \\ -\sin x & -\cos x & -x\sin x + 2\cos x \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & x\cos x + \sin x \\ 0 & 0 & 2\cos x \end{vmatrix} \quad R_3' = R_3 + R_1$$

$$= 2\cos x \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= 2\cos x \left(-\sin^2 x - \cos^2 x\right)$$

$$= -2\cos x \neq 0$$

Since the Wronskian is not zero the vectors are linearly independent.

21.(c) e^x, xe^x, x^2e^x

The Wronskian is

$$W = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x+e^x & x^2e^x+2xe^x \\ e^x & xe^x+e^x+e^x & x^2e^x+2xe^x+2xe^x+2e^x \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x(x+1) & (x^2+2x)e^x \\ e^x & e^x(x+2) & (x^2+4x+2)e^x \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & x^2e^x \\ 0 & e^x & 2xe^x \\ 0 & 2e^x & (4x+2)e^x \end{vmatrix} \quad \begin{matrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{matrix}$$

$$= e^x \begin{vmatrix} e^x & 2xe^x \\ 2e^x & (4x+2)e^x \end{vmatrix}$$

$$= e^x \left[(4x+2)e^{2x} - 4xe^{2x}\right]$$

$$= e^x \cdot 2e^{2x}$$

$$= 2e^{3x} \neq 0.$$

Since the Wronskian is not zero, the vectors are linearly independent.