Chapter 5.3

Exponential Functions

5.3.1 Evaluate Exponential Functions

EXAMPLE 1 Using a Calculator to Evaluate Powers of 2

THEOREM Laws of Exponents

If s, t, a and b are real numbers with $a > 0$ and $b > 0$, then

(a) $a^s . a^t = a^{s+t}$ (b) $(a^s)^t = a^{st}$ (c) $(ab)^s = a^s . b^s$

(d) $1^s = 1$ (e) $a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s$ (f) $a^0 = 1$

Introduction to Exponential Growth

Definition

An **exponential function** is a function of the form

$$f(x) = Ca^x$$

where a is a positive real number, $a > 0$, $a \neq 1$ and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor** and the number C is called the **initial value**.

Theorem

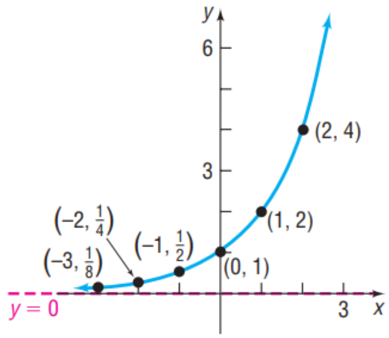
For an exponential function $f(x) = Ca^x$, where $a > 0$ and $a \neq 1$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)$$

5.3.2 Graph Exponential Functions

EXAMPLE 3 Graphing an Exponential Function

Figure 18



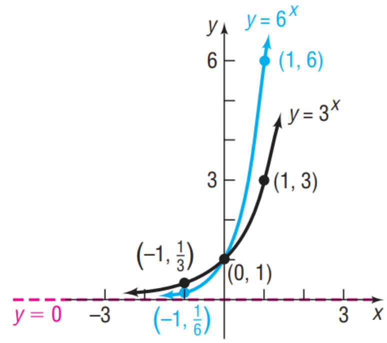
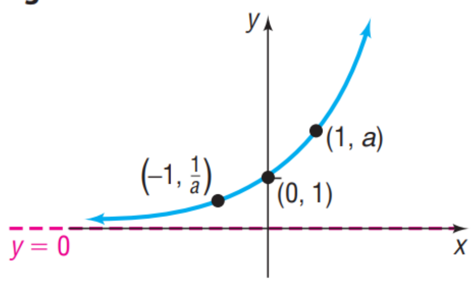


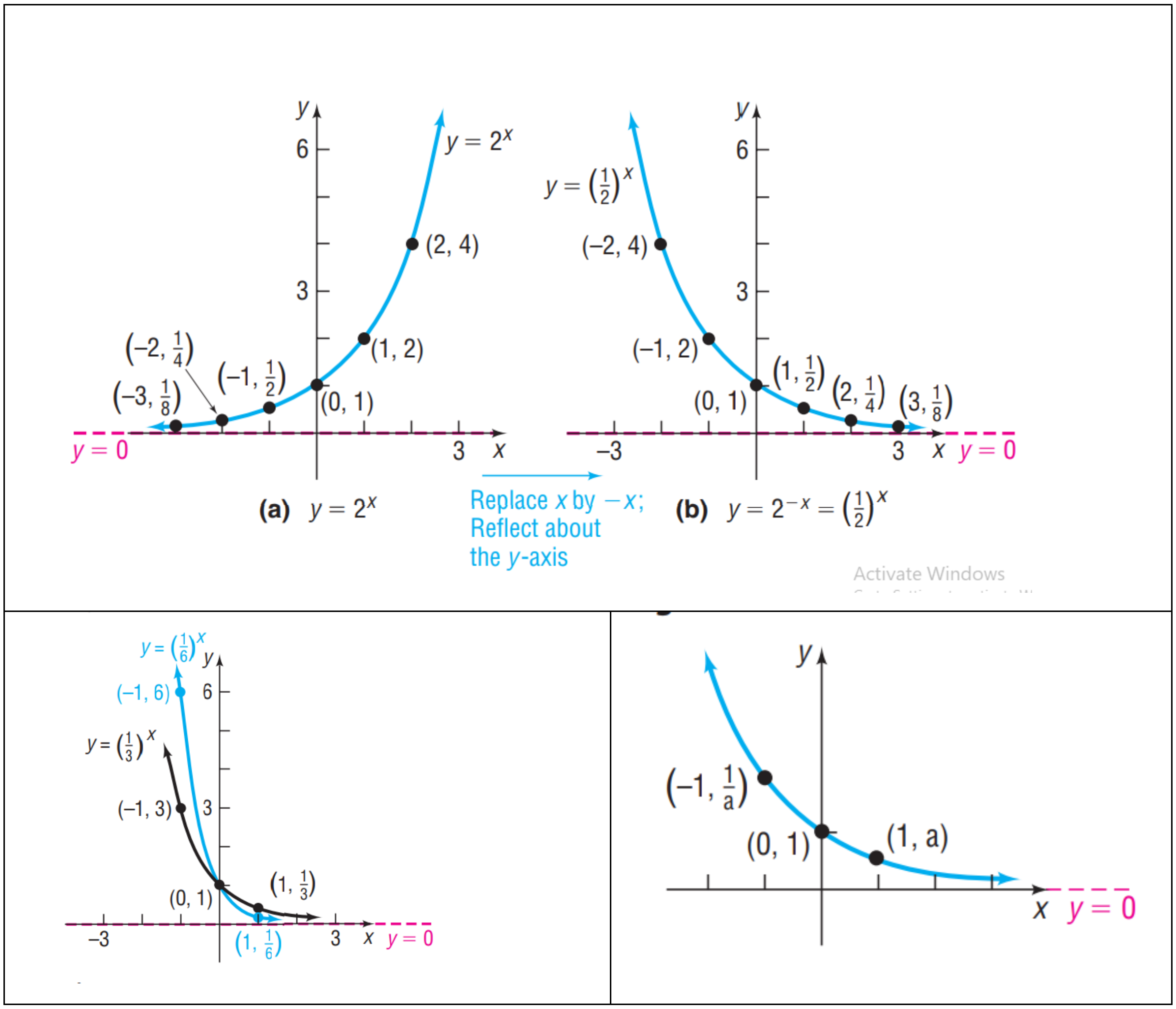
Figure 21



Properties of the Exponential Function $f(x) = a^x,\ a > 1$

1. The domain of f is the set of all real numbers and the range is the set of positive real numbers.
2. There are no x-intercepts and the y-intercept is 1.
3. The x-axis ($y = 0$) is a horizontal asymptote as $x \to -\infty$, i.e. $\lim\limits_{x \to -\infty} a^x = 0$.
4. $f(x) = a^x$, for $a > 1$, is an increasing function and is one-to-one.
5. The graph of f contains the points $\left(-1, \dfrac{1}{a}\right)$, $(0,1)$ and $(1, a)$.
6. The graph of f is smooth and continuous with no corners or gaps.

EXAMPLE 4 Graphing an Exponential Function



Properties of the Exponential Function $f(x)=a^x,\ 0<a<1$

1. The domain of f is the set of all real numbers and the range is the set of positive real numbers.
2. There are no x-intercepts and the y-intercept is 1.
3. The x-axis ($y=0$) is a horizontal asymptote as $x\to\infty$, i.e. $\lim\limits_{x\to\infty} a^x=0$.
4. $f(x)=a^x$, for $0<a<1$, is a decreasing function and is one-to-one.
5. The graph of f contains the points $\left(-1,\dfrac{1}{a}\right)$, $(0,1)$ and $(1,a)$.
6. The graph of f is smooth and continuous with no corners or gaps.

EXAMPLE 5 Graphing Exponential Functions Using Transformations

5.3.3 Define the Number *e*

Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter *e*. The letter *e* was chosen to represent this irrational number in honor of the Swiss mathematician Leonhard Euler (pronounced "oiler") (1707-1783).

Definition

The number *e* is defined as the number that the expression

$$\left(1+\frac{1}{n}\right)^n$$

approaches as $n \to \infty$.

In calculus, this is expressed using limit notation as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

EXAMPLE 6 Graphing Exponential Functions Using Transformations

5.3.4 Solve Exponential Equations

Definition

Equations that involve terms of the form a^x, where $a > 0$ and $a \neq 1$, are referred to as **exponential equations.** Such equations can sometimes be solved by appropriately applying the Laws of Exponents with the property given by

$$\boxed{\text{If } a^u = a^v \text{ then } u = v}$$

EXAMPLE 7 Solving Exponential Equations

Solve each exponential equation

(a) $3^{x+1} = 81$ (b) $4^{2x-1} = 8^{x+3}$

Solution:

(a) $3^{x+1} = 81 \Rightarrow 3^{x+1} = 3^4 \Rightarrow x+1 = 4 \Rightarrow x = 4-1 \Rightarrow x = 3$

Hence the solution is $x = 3.$

(b) $4^{2x-1} = 8^{x+3} \Rightarrow (2^2)^{2x-1} = (2^3)^{x+3} \Rightarrow 2^{4x-2} = 2^{3x+9}$

Using formula, we get

$$4x-2=3x+9 \Rightarrow 4x-3x=9+2 \Rightarrow x=11$$

Hence the solution is $x=11$.

EXAMPLE 8 Solving an Exponential Equation

Solve the exponential equation $e^{-x^2}=(e^x)^2.\dfrac{1}{e^3}$

Solution: Using Laws of Exponents, we can write

$$e^{-x^2}=(e^x)^2.\frac{1}{e^3}=e^{2x}.e^{-3} \Rightarrow e^{-x^2}=e^{2x-3}$$

$$\Rightarrow -x^2=2x-3 \Rightarrow x^2+2x-3=0$$

$$\Rightarrow (x+3)(x-1)=0 \Rightarrow \quad x=-3 \text{ or } \quad x=1$$

Hence the solution is $(x, y)=(-3,1)$.

SUMMARY Properties of the Exponential Function

5.3 Assess Your Understanding

Skill Building

In Problems 41–52, use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

In Problems 53–60, begin with the graph of [Figure 27] and use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

Solve the following exponential equations:

63. $2^{-x}=16$ **65.** $\left(\dfrac{1}{5}\right)^x=\dfrac{1}{25}$ **67.** $2^{2x-1}=4$ **69.** $3^{x^3}=9^x$

71. $8^{-x+14}=16^x$ **73.** $3^{x^2-7}=27^{2x}$ **75.** $4^x.2^{x^2}=16^2$ **77.** $e^x=e^{3x+8}$

79. $e^{x^2}=e^{3x}.\dfrac{1}{e^2}$

CHAPTER 5.4

Logarithmic Functions

The exponential function $y = a^x$, where $a > 0$, and $a \neq 1$, is one-to-one and hence has an inverse function that is defined implicitly by the equation

$$x = a^y, \quad (a > 0,\ a \neq 1)$$

This inverse function is known as the **logarithmic function**.

Definition

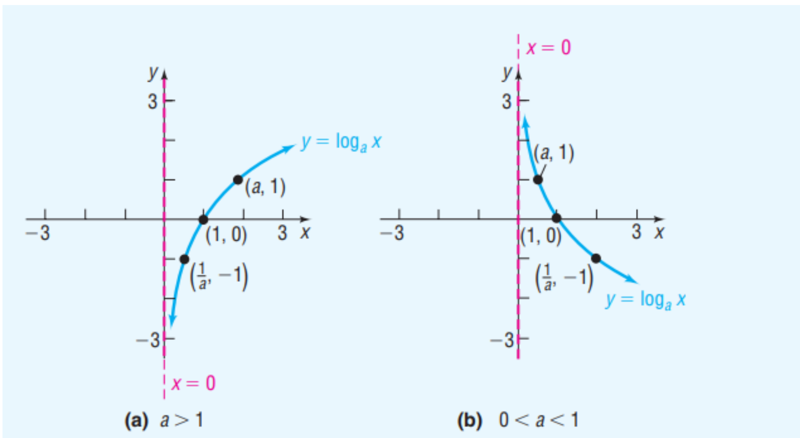
The **logarithmic function to the base *a***, where $a > 0$, and $a \neq 1$, is denoted by $\log_a x$ (read as "*y* is the logarithm of *x* to the base *a*") and is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

The definition illustrates that $\log_a x$ represents the exponent to which *a* must be raised to obtain *x*.

If $s = \log_a r$, then the power to which *a* must be raised to get *r* is *s*, that is, $r = a^s$. So that



EXAMPLE

The chart below shows several pairs of equivalent statements:

Exponential form	Logarithmic form
1. $2^3 = 8$	1. $\log_2 8 = 3$
2. $\left(\frac{1}{2}\right)^{-4} = 16$	2. $\log_{1/2} 16 = -4$
3. $10^5 = 100{,}000$	3. $\log_{10} 100000 = 5$
4. $5^1 = 5$	4. $\log_5 5 = 1$
5. $\left(\frac{3}{4}\right)^0 = 1$	5. $\log_{3/4} 1 = 0$

EXAMPLE

Find the exact value of

(a) $\log_2 16$ (b) $\log_3 \frac{1}{27}$

Solution: (a) Let $y = \log_2 16$

$\Rightarrow 2^y = 16$

$\Rightarrow 2^y = 2^4$

$\Rightarrow y = 4$

Therefore, $\log_2 16 = 4.$

(b) Let $y = \log_3 \frac{1}{27}$

$\Rightarrow 3^y = \frac{1}{27}$

$\Rightarrow 3^y = 3^{-3}$

$\Rightarrow y = -3$

Therefore, $\log_3 \frac{1}{27} = -3.$

Domain

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x.$ We already have

Domain of f^{-1} = Range of f and Range of f^{-1} = Domain of f

Consequently, it follows that

Domain of the logarithmic function = Range of the exponential function = $(0, \infty)$

Range of the logarithmic function = Domain of the exponential function = $(-\infty, \infty)$

In the next box, we summarize some properties of the logarithmic function:

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

EXAMPLE

Find the domain of each logarithmic function:

(a) $F(x) = \log_2(x+3)$ (b) $g(x) = \log_5 \dfrac{1+x}{1-x}$ (c) $h(x) = \log_{1/2}|x|$

Solution: (a) The domain of F consists of all x for which $x+3>0$, that is, $x>-3$.

Using interval notation, domain of F is $(-3, \infty)$.

(b) The domain of g consists of all x for which $g(x) = \dfrac{1+x}{1-x} > 0$

Solving this inequality, we find that the domain of g consists of all x between -1 and 1, that is $-1 < x < 1$, or using interval notation, domain of g is $(-1, 1)$.

Check

Zero of the denominator is 1 and the zero of the numerator is -1.

These zeros divide the number line into three intervals:

$$(-\infty, -1), \quad (-1, 1), \quad (1, +\infty)$$

To draw the conclusion, prepare the following table:

Interval	Number picked	Value of *g*
$(-\infty, -1)$	-2	$g(-2) = -\dfrac{1}{3} < 0$
$(-1, 1)$	0	$g(0) = 1 > 0$
$(1, +\infty)$	2	$g(2) = -3 < 0$

(c) Since $|x| > 0$ if $x \neq 0$, the domain of h consists of all real numbers except 0, or using interval notation, domain of g is $(-\infty, 0) \cup (0, \infty)$.

Graphing Logarithmic Functions

Since exponential functions and logarithmic functions are inverses of each other, the graph of the logarithmic function $y = \log_a x$ is the reflection of the graph of the exponential function $y = a^x$ about the line $y = x$ as shown in FIGURE 30.

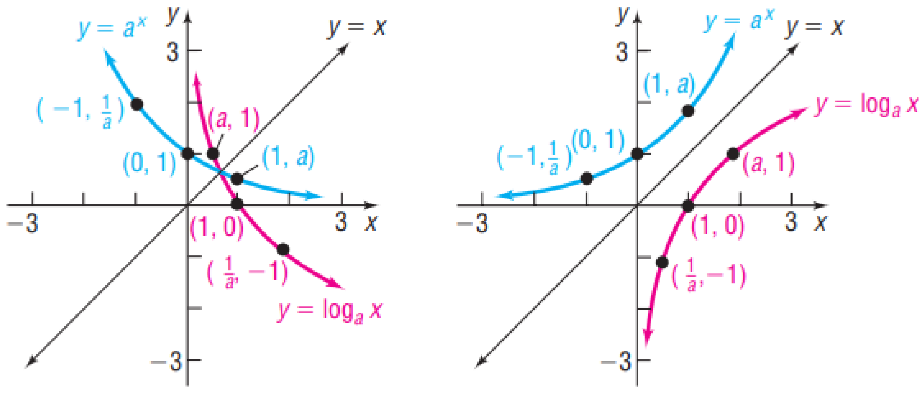


FIGURE 30

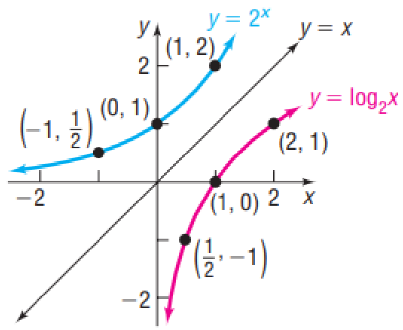


FIGURE 31

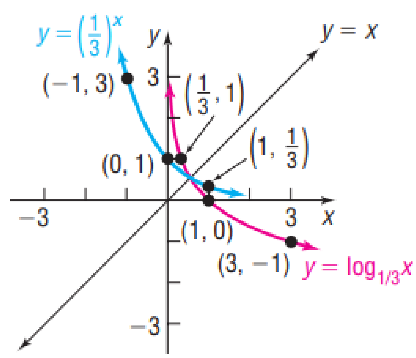


FIGURE 32

Examples

(i) To graph $y = \log_2 x$, graph $y = 2^x$ and reflect it about the line $y = x$. See FIGURE 31,

(ii) To graph $y = \log_{1/3} x$, graph $y = \left(\dfrac{1}{3}\right)^x$ and reflect it about the line $y = x$. See FIGURE 32.

The graphs of $y = \log_a x$ in FIGURES 30(a) and (b) lead to the following properties.

Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain of f is the set of positive real numbers and the range is the set of all real numbers.
2. The x-intercept of the graph is 1. There is no y-intercept.
3. The y-axis ($x = 0$) is a vertical asymptote of the graph.
4. A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$ and $\left(\dfrac{1}{a}, -1\right)$,
6. The graph of f is smooth and continuous with no corners or gaps.

Properties of Logarithms

If x and y are any positive real numbers, r is any real number and a is any positive real number such that $a \neq 1$, then the following properties are satisfied:

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
3. $\log_a x^r = r.\log_a x$
4. $\log_a a = 1$
5. $\log_a 1 = 0$

The properties of logarithms were developed over a twenty-five-year period by the Scottish mathematician John Napier (1550-1617).

EXAMPLE

Assuming all variables represent positive real numbers, use the properties of logarithms to write each of the following in a different form:

(a) $\log_6(7\times 9) = \log_6 7 + \log_6 9$

(b) $\log\left(\frac{15}{7}\right) = \log 15 - \log 7$

(c) $\ln\sqrt{8} = \ln 8^{1/2} = \frac{1}{2}\ln 8$

(d) $\log_a\left(\frac{mnq}{p^2}\right) = \log_a(mnq) - \log_a p^2$

$= \log_a m + \log_a n + \log_a q - 2\log_a p$

(e) $\log_a\left(\sqrt[3]{m^2}\right) = \log_a m^{2/3} = \frac{2}{3}\log_a m$

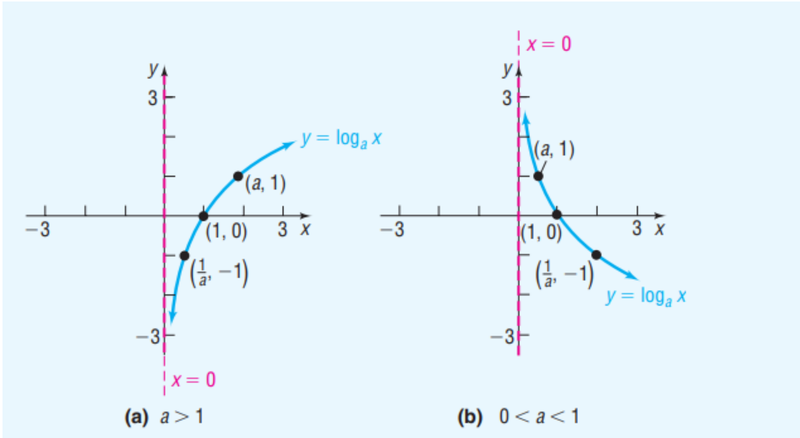
(f) $\log\sqrt[n]{\frac{x^3y^5}{z^5}} = \log\left(\frac{x^3y^5}{z^5}\right)^{1/n}$

$= \frac{1}{n}\left[\log x^3y^5 - \log z^5\right]$

$= \frac{1}{n}\left[\log x^3 + \log y^5 - \log z^5\right]$

$= \frac{1}{n}\left[3\log x + 5\log y - 5\log z\right]$

$= \frac{3}{n}\log x + \frac{5}{n}\log y - \frac{5}{n}\log z$



5.4 Assess Your Understanding

Skill Building

Find the domain of each function:

41. $f(x) = 3 - 2\log_4\left[\dfrac{x}{2} - 5\right]$

Solution: The domain of f consists of all x for which $\dfrac{x}{2} - 5 > 0$ or $\dfrac{x}{2} > 5$ or $x > 10.$

Solving this inequality, we find that the domain of f consists of all x for which $x > 10$ or using interval notation, domain of f is $(10,\ \infty).$

43. $f(x) = \ln\left(\dfrac{1}{x+1}\right)$

Solution: The domain of f consists of all x for which $\dfrac{1}{x+1} > 0.$

We see that the zero of the denominator is $-1.$

This zero divides the number line into two intervals:

$$(-\infty, -1), \qquad (-1, +\infty)$$

To draw the conclusion, prepare the following table:

Interval	Number picked	Value of *f*
$(-\infty, -1)$	-2	$f(-2) = -1 < 0$
$(-1, +\infty)$	0	$f(0) = 1 > 0$

From the table, we can conclude that the domain of *f* consists of all *x* satisfying $x > -1$, or using the interval notation, the domain of *f* is $(-1, +\infty)$.

45. $g(x) = \ln_5\left(\dfrac{x+1}{x}\right)$

Solution: The domain of *g* consists of all *x* for which $\dfrac{x+1}{x} > 0.$

We see that the zero of the denominator is 0 and that of the numerator is -1.
These zeros divide the number line into three intervals:

$$(-\infty, -1), \qquad (-1, 0), \qquad (0, \infty)$$

To draw the conclusion, prepare the following table:

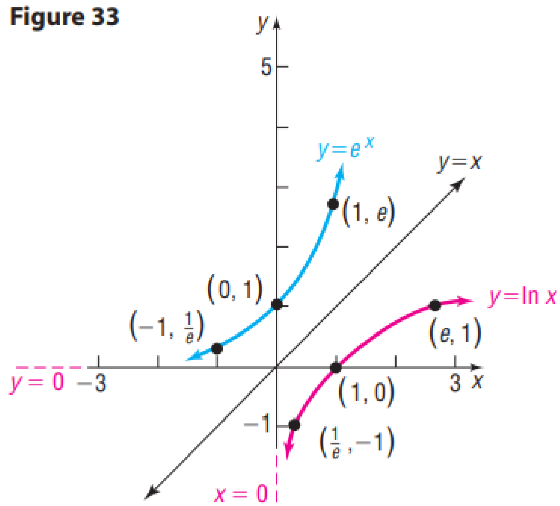
Interval	Number picked	Value of *g*
$(-\infty, -1)$	-2	$g(-2) = \frac{1}{2} > 0$
$(-1, 0)$	$-\frac{1}{2}$	$g\left(-\frac{1}{2}\right) = -1 < 0$
$(0, +\infty)$	1	$g(1) = 2 > 0$

From the table, we can conclude that the domain of *g* consists of all *x* satisfying $x < -1$ or $x > 0$, or using the interval notation, the domain of *g* is $(-\infty, -1) \cup (0, +\infty)$.

47. $f(x) = \sqrt{\ln x}$

Solution: The domain of *f* consists of all *x* for which $\ln x \geq 0$

Figure 33



From the graph, we see that $\ln x \geq 0$ for $x \geq 1$. Therefore, we can conclude that the domain of *f* consists of all *x* for which $x \geq 1$, or using interval notation, domain of *f* is $[1, \infty)$.