Ch3: Discrete Probability Distributions ······ Ch4: Continuous Distributions

Motivation:

Lest us toss a coin 3 times and record the sequences of $2^3 = 8$ outcomes as

S = { HHH, HHT, HTH, THH, TTH, THT, HTT, TTT }. Also, Let X be the number of heads of the sequence of outcomes and consequently we obtain the following probability distribution:

X	0	1	2	3	Total
p(x)	1/8	3/8	3/8	1/8	1

Can we construct manually the probability distribution of the number of heads when a coin is tossed 100 times?

It's almost impossible to construct or analyze manually. In such cases mathematical/Statistical formula can be used to solve the problem. The mathematical/statistical formula can be treated as probability distribution.

Ch3: Discrete Probability Distribution (pp. 147-185)

AjM

Distribution (symbolic presentation)	RV: X	Section, Page #	pmf: $P(X=x)$	Range of X	Mean= μ	Variance= σ^2
Binomial $X \sim B(n,p)$	number of successes in ***n*** trials	S3,1,2, p148	$\binom{n}{x} p^x (1-p)^{n-x}$	$x = 0,1,2\cdots,n$	np	$np(1-p)$
Geometric $X \sim G(p)$	number of trials required to get **1st** success	S3.2, p160	$p(1-p)^{x-1}$	$x = 1,2,3,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial $X \sim NB(r,p)$	number of trials required to get the first ***r*** successes	S3.2, p162	$\binom{x-1}{r-1}(1-p)^{x-r} p^r$	$x = r, r+1, r+2,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson $X \sim P(\lambda)$	no. of events occur within certain specified boundaries	S3.4, p173	$\frac{e^{-\lambda}\lambda^x}{x!}$	$x = 0,1,2,3,\ldots$	λ	λ
Hypergeometric $X \sim H(N,r,n)$	no of defective items chosen from *r* items drawn without replacement from *N* items of which *n* are defective	S3.3, p169	$\frac{\binom{r}{x}\times\binom{N-r}{n-x}}{\binom{n}{N}}$	$\max\{0, n+r-N\} \le x \le \min(n,r)$	$\frac{nr}{N}$	$\left(\frac{N-n}{N-1}\right)\frac{nr}{N}\left(1-\frac{r}{N}\right)$

First Column: *symbols/letters in each parenthesis represent parameter(s) of the respective distribution*

P-1

3.1.1 Suppose that $X \sim B(10, 0.12)$. Calculate:

(a) $P(X=3)$; (b) $P(X=6)$; (c) $P(X \le 2)$

(d) $P(X \ge 7)$; **(e)** $E(X)$; **(f)** $\mathrm{Var}(X)$

Soln. $P(x) = \binom{10}{x} 0.12^{x} (1-.12)^{10-x}$, $x = 0, 1, 2, \cdots; n = 10$

(a) $P(X=3) = \binom{10}{3} 0.12^{3} (1-0.12)^{10-3} = \underline{0.0847}$

(b) $P(X=6) = \binom{10}{6} 0.12^{6} (1-0.12)^{10-6} = \underline{0.0004}$

(c) $P(X \le 2) = \sum_{x=0}^{2} P(x) = P(0) + P(1) + P(2) = 0.2785 + 0.3798 + 0.2330$
$= 0.8913$

(d) $P(X \ge 7) = P(X=7) + P(X=8) + P(9) + P(10) = 0.00003085$

(e) $E(X) = np = 10 \times 0.12 = 1.2$ $\underline{\text{Unit}}$

(f) $\mathrm{Var}(X) = np(1-p) = 1.2 \times (1-.12) = 1.056$ Unit^2

3.1.3 Draw line graphs of the probability mass functions of a $B(6, 0.5)$ distribution and a $B(6, 0.7)$ distribution. Mark the expected values of the distributions on the line graphs and calculate the standard deviations of the two distributions.

Soln.: $X \sim B(6, 0.5) \Rightarrow P(x) = \binom{6}{x} 0.5^{x}\, 0.5^{6-x} = \binom{6}{x} 0.5^{6};\ x = 0, 1, \dots, 6$

Prob. distribution of $X \sim B(6, .05)$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0937	.2344	.3125	.2344	.0937	.0156

$E(X) = np = 6 \times .5 = 3$
$V(X) = np(1-p) = 3 \times .5 = 1.5$
$\sigma = \sqrt{1.5} = 1.22$

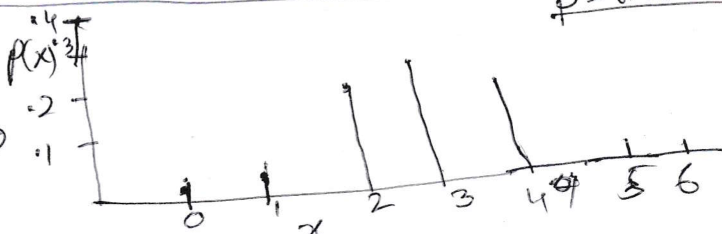
Similarly, $X \sim B(6, 0.7) \Rightarrow p(x) = \binom{6}{x} 0.7^{x}\, 0.3^{6-x},\ x = 0, 1, 2, \dots, 6$

x_i	0	1	2	3	4	5	6
$P(x_i)$	.0007	.0102	.0595	.1852	.3241	.3025	.1176

$E(X) = np = 6 \times .7 = 4.2$
$V(X) = 4.2 \times .3 = 1.26$
$\sigma = \sqrt{1.26} = 1.12$

Exercise graph Hints

$X \sim B(6, .5) \Rightarrow$



3.1.5 A fair die is rolled *eight* times. Calculate the probability that there are:

(a) Exactly five even numbers; **(b)** Exactly one 6; **(c)** No 4s

Soln (a) Prob(even no) = Prob(odd no) = 0.5

$\therefore$ X be an even number, $X \sim B(8, 0.5)$

$\therefore P(x) = \binom{8}{x} 0.5^{x}\, 0.5^{8-x} = \binom{8}{x} 0.5^{8}$, $x = 0, 1, 2, \ldots, 8$

$\therefore P(X=5) = \binom{8}{5} 0.5^{8} = 56 \times 0.003906 = 0.2187$

(b) Y be the number of times 6 occurs ~~any number~~, $P(y) = \binom{8}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{8-x}$, $x = 0, 1, 2, \ldots, 8$

$Y \sim B(8, \frac{1}{6})$ $\therefore P(Y=1) = \binom{8}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{7} = \underline{0.3721}$

(c) Z be the number of times 4 occurs, $Z \sim B(8, 1/6)$

$P(Z=0) = \binom{8}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{8} = 0.2326$

3.2.1 If X has a geometric distribution with parameter $p = 0.7$, calculate:

A. $P(X = 4)$ **B.** $P(X = 1)$ **C.** $P(X \leq 5)$ **D.** $P(X \geq 8)$

Soln $X \sim G(0.7) \Rightarrow P(x) = 0.7 \times 0.3^{x-1}, \; x = 1, 2, \ldots$

A. $P(X=4) = 0.7 \times 0.3^3 = \underline{0.0189}$

B. $P(X=1) = 0.7 \times 0.3^{1-1} = \underline{0.7}$

C. $P(X \leq 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 0.7[0.3^0 + 0.3^1 + 0.3^2 + 0.3^3 + 0.3^4] = 0.9975$

D. $P(X \geq 8) = 1 - P(X \leq 7) = 1 - 0.9975 - P(6) - P(7)$

$= 1 - 0.9975 - 0.7 \times .3^5 - 0.7 \times .3^6 = \underline{0.000289}$

$= 1 - .9975 - .001701 - 0.00051 = 1 - 0.999711 = 0.000289$

3.2.5 Recall Problem 3.1.4 where an archer hits a bull's-eye with a probability of 0.09, and the results of different attempts can be taken to be independent of each other. P-9

(a) If the archer shoots a series of arrows, what is the probability that the *first* bull's-eye is scored with the fourth arrow? X= ....

(b) What is the probability that the *third* bull's-eye is scored with the tenth arrow? Hw

(c) What is the expected number of arrows shot before the *first* bull's-eye is scored?

(d) What is the expected number of arrows shot before the *third* bull's-eye is scored?

Soln: (a) $X \sim G(.09)$, $P(X=4) = 0.09\,(0.91)^3 = \underline{0.0678}$

(b) $Y \sim NB(r=3, p=0.09) \Rightarrow P(x)$

$P(Y=3) = \binom{9}{2} \times (1-0.09)^7 \times 0.09^3 = 0.0136$

(c) $E(X) = \frac{1}{p} = \frac{1}{0.09} = 11.11$, (d) $E(Y) = \frac{r}{p} = \frac{3}{0.09} = 33.33$

Hyper geometric distribution

3.3.2 A committee consists of eight right-wing members and seven left-wing members. A subcommittee is formed by randomly choosing five of the committee members. Draw a line graph of the probability mass function of the number of right-wing members serving on the subcommittee. P-11

Soln $N =$, $r =$, $n =$

X be the number of right-wing members

$X \sim H($ $\Rightarrow p(x) =$

x_i	0	1	2	3	4	5
$P(x_i)$	$\frac{3}{429}$	$\frac{40}{429}$	$\frac{140}{429}$	$\frac{168}{426}$	$\frac{70}{429}$	$\frac{8}{429}$

HW **3.3.4** A jury of 12 people is selected at random from a group of 16 men and 18 women. What is the probability that the jury contains exactly 7 women? Suppose that the jury is selected at random from a group of 1600 men and 1800 women. Use the binomial approximation to the hypergeometric distribution to calculate the probability that in this case the jury contains exactly 7 women. P-13

3.3.4 A jury of 12 people is selected at random from a group of 16 men and 18 women. What is the probability that the jury contains exactly 7 women? Suppose that the jury is selected at random from a group of 1600 men and 1800 women. Use the binomial approximation to the hypergeometric distribution to calculate the probability that in this case the jury contains exactly 7 women.

P-13

HW

Poisson

(P-15)

3.4.2 If $X \sim P(2.1)$, calculate: **(a)** $P(X = 0)$ **(b)** $P(X \leq 2)$ **(c)** $P(X \geq 5)$

(d) $P(X = 1 | X \leq 2)$

Soln: $P(x) = e^{-2.1}\, 2.1^{x}/x!\,, \quad x = 0, 1, 2, \ldots$

(a) $P(X=0) = \frac{2.1^0}{0!} e^{-2.1} = e^{-2.1} = 0.1225$

(b) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= e^{-2.1}\left[\frac{2.1^0}{0!} + \frac{2.1^1}{1!} + \frac{2.1^2}{2!}\right] = e^{-2.1}\left(1 + 2.1 + \frac{2.1^2}{2}\right) = 0.6496$

(c) $P(X \geq 5) = 1 - P(X \leq 4) = 1 - e^{-2.1}\left[\frac{2.1^0}{0!} + \frac{2.1^1}{1!} + \frac{2.1^2}{2!} + \frac{2.1^3}{3!} + \frac{2.1^4}{4!}\right]$

$= 0.6496$

(d) $P(X = 1 / X \leq 2) = \frac{P(X = 1 \cap X \leq 2)}{P(X \leq 2)} = \frac{P(X=1)}{P(X \leq 2)} = \frac{0.2572}{0.6496} = 0.3959$

3.4.5 On average there are about 25 imperfections in 100 meters of (p-17) optical cable. Use the Poisson distribution to estimate the probability that there are no imperfections in 1 meter of cable. What is the probability that there is no more than one imperfection in 1 meter of cable?

Sol $X =$ number of imperfection,

$$X \sim P\left(\lambda = \frac{25}{100} = 0.25\right) \Rightarrow p(x) = \frac{e^{-0.25}\, 0.25^{x}}{x!}, \quad x = 0, 1, \ldots$$

$$P(X=0) = \frac{e^{-0.25}\, 0.25^{0}}{0!} = e^{-.25} = 0.7788$$

$$P(X \leq 1) = \frac{e^{-0.25}\, 0.25^{1}}{1!} + 0.7788 = 0.9735$$

$P(X=0) + P(X=1)$

HW
K5

P-19

3.4.7 Recall that the Poisson distribution with a parameter value of $\lambda =$ np can be used to approximate the $B(n, p)$ distribution when n is very large and the success probability p is very small.
A box contains 500 electrical switches, each one of which has a probability of 0.005 of being defective. Use the Poisson distribution to make an approximate calculation of the probability that the box contains no more than 3 defective switches.

Multinomial distribution (P-21)

3.5.2 A fair die is rolled 15 times. Calculate the probability that there are:

(a) Exactly three 6s and three 5s
(b) Exactly three 6s, three 5s, and four 4s
(c) Exactly two 6s
What is the expected number of 6s obtained?

K - Categories

$p_1 + \cdots + p_K = 1$

$P(X_1 = x_1) + \cdots + P(X_K = x_K) = 1$

$P(X_1 = x_1, X_2 = x_2, \ldots, X_K = x_K) = \dfrac{n!}{x_1! \cdots x_K!} p_1^{x_1} \times \cdots \times p_K^{x_K}$

Soln: $n = 15$, $X_1 =$ no. of 6's, $X_2 = \ldots$ 5's

$X_3 =$ no of others

$P(X_1) = P(X_2) = \frac{1}{6}$, $P(X_3) = \frac{4}{6} = \frac{2}{3}$

(a) $P(X_1 = 3, X_2 = 3, X_3 = 9) = \dfrac{8!}{3!\,3!\,9!} \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^3 \left(\frac{2}{3}\right)^9 = 0.0502$

(b) $\cancel{\dfrac{8!}{1 \times 5!\,2!} \times 0.0814}$ $\dfrac{15!}{3!\,3!\,4!\,5!} \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^4 \left(\frac{1}{2}\right)^4 = 0.0065$

(c) $\left(15! / (2! \times 13!)\right) \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^{13} = 0.2726$

$E(6\text{'s}) = \frac{15}{6} = 2.5$

A **probability distribution** in which the **random variable** X can take on any value (is continuous). Because there are infinite values that X could assume, the probability of X taking on any one specific value is zero.

Following are the most commonly used continuous probability distribution:

(For example check the web address:

https://www.itl.nist.gov/div898/handbook/eda/section3/eda366.htm)

• **Uniform distribution** • **Exponential Distribution** • **Normal Distribution**	• **Chi-Square distribution** • **T-distribution** • **F Distribution, etc.**

- Gamma Distribution, Beta Distribution, etc.

The Uiniform Distribution

The Uniform Distribution

A random variable X with a flat probability density function between two points a and b, so that

$$f(x) = \frac{1}{b-a}$$

for $a \leq x \leq b$ and $f(x) = 0$ elsewhere, is said to have a **uniform distribution**, which is written $X \sim U(a, b)$. The cumulative distribution function is

$$F(x) = \frac{x-a}{b-a}$$

for $a \leq x \leq b$, and the expectation and variance are

$$E(X) = \frac{a+b}{2} \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}.$$

4.1.2 A new battery supposedly with a charge of 1.5 volts actually has a voltage with a uniform distribution between 1.43 and 1.60 volts.

(a) What is the expectation of the voltage?

(b) What is the standard deviation of the voltage?

(c) What is the cumulative distribution function of the voltage?

(d) What is the probability that a battery has a voltage less than 1.48 volts?

(e) If a box contains 50 batteries, what are the expectation and variance of the number of batteries in the box with a voltage less than 1.5 volts?

$X \sim U(1.43, 1.60)$.¶

$$f(x) = \begin{cases} \frac{1}{1.60-1.43} = \frac{1}{0.17}, & 1.43 \le X \le 1.60 \\ 0, & \text{otherwise} \end{cases}$$ ¶

a) $E(X) = \frac{1.60+1.43}{2} = 1.15$ Volts¶

b) $\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(1.6-1.43)^2}{12}} = 0.0491$ Volts¶

¶

c) $F(x) = \int_{1.43}^{x} f(y)dy = \int_{1.43}^{x} \frac{1}{0.17}dy = \frac{x-1.43}{0.17}$

d) $P(X < 1.48) = F(1.48) = \frac{1.48-1.43}{0.17} = 0.2941$

e) $F(1.5) = \frac{1.5-1.43}{0.17} = 0.412$

Continuation….

- The **<u>number of batteries with a voltage less than 1.5 Volts</u>** has a binomial distribution with parameters n = 50 and p = 0.412.
- Therefore,

$$\mu = E(X) = np = 50 \times 0.412 = 20.6 \text{ Volts,}$$

- and the variance is

$$\sigma^2 = Var(X) = np(1-p) = 50 \times 0.412 \times 0.588 = 12.11 \text{ Volt}^2$$

4.2.2 Suppose that you are waiting for a friend to call you and that the time you wait in minutes has an exponential distribution with parameter λ = 0.1.

(a) What is the expectation of your waiting time? (b) What is the probability that you will wait longer than 10 minutes?

(c) What is the probability that you will wait less than 5 minutes?

Solution: $X \sim Exp(\lambda = 0.1)$

$$\therefore f(x) = 0.1 e^{-0.1x} \;;\; x \geq 0$$

(a) $$E(X) = \frac{1}{\lambda} = \frac{1}{0.1} = 10 \text{ minutes}$$

(b) $$P(X > 10) = 1 - P(X \leq 10) = 1 - F(10)$$
$$= 1 - (1 - e^{-0.1 \times 10}) = e^{-1} = 0.3679$$

(c) $$P(X \leq 5) = F(5) = 1 - e^{-0.1 \times 5}$$
$$= 1 - e^{-0.5} = 0.3935$$

(d) Suppose that after 5 minutes you are still waiting for the call. What is the distribution of your additional waiting time? In this case, what is the probability that your total waiting time is longer than 15 minutes?
(e) Suppose now that the time you wait in minutes for the call has a U(0, 20) distribution. What is the expectation of your waiting time? If after 5 minutes

(d) $Pr(X > 15 / X > 5) = P(X > 10+5 / X > 5)$

$$= \frac{P(X > 10+5 \cap X > 5)}{P(X > 5)} = \frac{P(X > 15)}{P(X > 5)}$$

$$= \frac{1 - F(15)}{1 - F(5)} = \frac{1 - (1 - e^{-0.1 \times 15})}{1 - (1 - e^{-0.1 \times 5})}$$

$$= e^{-0.15} / e^{-0.5} = e^{-0.10} = 0.3679$$

(e) $X \sim U(0,20) \Rightarrow f(x) = \frac{1}{20}, \; 0 \le x \le 20$

$E(X) = \frac{0+20}{2} = 10$ mins.

$X \sim U(0,20)$, after 5 minutes, $X \sim U(0,15)$

The additional waiting time has a U(0,15) distribution.

5.1.1 Suppose that $Z \sim N(0, 1)$. Find: (a) $P(Z \leq 1.34)$; (b) $P(Z \geq -0.22)$; (c) $P(-2.19 \leq Z \leq 0.43)$, d) $P(0.09 \leq Z \leq 1.76)$; (e) $P(|Z| \leq 0.38)$

Solution: (a) $P(Z \leq 1\cdot 34) = \Phi(1\cdot 34) = \underline{0\cdot 9099}$

Using table I, pp 787–788

(b) $P(Z \geq -0\cdot 22) = 1 - P(Z \leq -0\cdot 22)$

$= 1 - \Phi(-0\cdot 22) = 1 - [1 - \Phi(0\cdot 22)]$

$= \Phi(0\cdot 22) = 0\cdot 5871$ from table I

(c) $P(-2\cdot 19 \leq Z \leq 0\cdot 43) = \Phi(0\cdot 43) - \Phi(-2\cdot 19)$

$= \Phi(0\cdot 43) - 1 + \Phi(2\cdot 19)$ | Using table 1

$= 0\cdot 6664 - 1 + 0\cdot 9857 = \underline{0\cdot 6521}$

(d) $P(0\cdot 09 \leq Z \leq 1\cdot 76) = \Phi(1\cdot 76) - \Phi(0\cdot 09)$

$= 0\cdot 9608 - 0\cdot 5339 = \underline{0\cdot 4249}$

(e) $P(|Z| < 0\cdot 38) = P(-0\cdot 38 \leq Z \leq 0\cdot 38)$

$= \Phi(0\cdot 38) - \Phi(-0\cdot 38) = \Phi(0\cdot 38) - 1 + \Phi(0\cdot 38)$

$= 2(0\cdot 6480) - 1 = 0\cdot 2960$

For (f)-(h), find the value of *x* for which
(f) $P(Z \leq x) = 0.55$; (g) $P(Z \geq x) = 0.72$; (h) $P(|Z| \leq x) = 0.31$

(f) $\Phi(x) = 0.55$
From Table, we get $\Phi(0.126) \approx 0.55$
$\therefore x = 0.126$

(g) $P(Z \geq x) = 1 - P(Z < x) = 1 - \Phi(x)$
$= 0.72 \Rightarrow \Phi(x) = 1 - 0.72 = 0.28$
$\therefore$ From Table, $\Phi(-.583) \approx 0.28$
Thus, $x \approx -0.583$

(h) $P(|Z| \leq x) = 0.31$
$\Rightarrow P(-x \leq Z \leq x) = 0.31$
$\Rightarrow \Phi(x) - \Phi(-x) = 0.31$
$\Phi(x) - [1 - \Phi(x)] = 1.31$
$\therefore 2\Phi(x) = 1.31$, $\Phi(x) = 0.655$
From Table I, $\Phi(.40) = 0.655$, $\therefore x \approx 0.40$

5.1.3 Suppose that $X \sim N(10, 2)$. Find: (a) $P(X \leq 10.34)$; (b) $P(X \geq 11.98)$; (c) $P(7.67 \leq X \leq 9.90)$; (d) $P(10.88 \leq X \leq 13.22)$; (e) $P(|X - 10| \leq 3)$

Solution: (a) $P\left(\frac{X-\mu}{\sigma} \leq \frac{10.34-10}{\sqrt{2}}\right) = P(Z \leq 0.240)$

$= \Phi(0.240) = 0.5948$

(b) $P\left(\frac{X-\mu}{\sigma} \geq \frac{11.98-10}{\sqrt{2}}\right) = 1 - P(Z < 1.40) = \Phi(1.4)$

$= 1 - \Phi(1.4) = (1 - 0.9192) = .0808$

(c) $P(7.67 \leq X \leq 9.90) = P\left(\frac{7.67-10}{\sqrt{2}} \leq Z \leq \frac{9.90-10}{\sqrt{2}}\right)$

$= \Phi\left(\frac{9.90-10}{\sqrt{2}}\right) - \Phi\left(\frac{7.67-10}{\sqrt{2}}\right) = \Phi(-0.07) - \Phi(-1.65)$

$= 1 - \Phi(0.07) - 1 + \Phi(1.65) = \Phi(1.65) - \Phi(0.07)$

$= 0.9505 - 0.5279 = 0.4226$

(d) $P(10.88 \leq X \leq 13.22) = \Phi\left(\frac{13.22-10}{\sqrt{2}}\right) - \Phi\left(\frac{10.88-10}{\sqrt{2}}\right)$

$= \Phi(2.26) - \Phi(0.62) = 0.9887 - 0.7324 = 0.2563$

(e) $P(|X-10| \leq 3) = P(-3 \leq (X-10) \leq 3)$

$= P\left(\frac{-3}{\sqrt{2}} \leq \frac{X-10}{\sigma} \leq \frac{3}{\sqrt{2}}\right) = \Phi\left(\frac{3}{\sqrt{2}}\right) - \Phi\left(\frac{-3}{\sqrt{2}}\right)$

$= \Phi\left(\frac{3}{\sqrt{2}}\right) - 1 + \Phi\left(\frac{3}{\sqrt{2}}\right) = 2\Phi\left(\frac{3}{\sqrt{2}}\right) - 1 = 2(0.983) - 1$

$= 2\Phi\left(\frac{3}{2.12}\right) - 1 = 2 \times \cancel{0.8365}\ 0.9830 - 1 = 0.966$

For (f)-(h), find the value of *x* for which

(f) $P(X \le x) = 0.81$; (g) $P(X \ge x) = 0.04$; (h) $P(|X - 10| \ge x) = 0.63$

(f) $P(X \le x) = 0.81 \Rightarrow P\left(\frac{X-\mu}{\sigma} \le \frac{x-10}{\sqrt{2}}\right) = 0.81$

$\Rightarrow \Phi\left(\frac{x-10}{\sqrt{2}}\right) = 0.81$ || From Table I, $\Phi(0.88) \approx 0.81$

$\therefore \frac{x-10}{\sqrt{2}} = 0.88$, $x = 0.88 \times \sqrt{2} + 10$

$\therefore x = 11.2445$

(g) $P(X \ge x) = 0.04 \Rightarrow P\left(\frac{X-\mu}{\sigma} \ge \frac{x-10}{\sqrt{2}}\right) = 0.04$

$\Rightarrow 1 - \Phi\left(\frac{x-10}{\sqrt{2}}\right) = 0.04$

$\Phi\left(\frac{x-10}{\sqrt{2}}\right) = 0.96$

From Table I $\Phi(1.75) \approx 0.96$

Solving $\frac{x-10}{\sqrt{2}} = 1.75$, we get, $x = 12.47487$

(h) $P(|X-10| \ge x) = 0.63 \Rightarrow P\left(-\frac{x}{\sqrt{2}} < \frac{X-10}{\sigma} < \frac{x}{\sqrt{2}}\right) = 0.63$

$\Rightarrow \Phi\left(\frac{x}{\sqrt{2}}\right) - \Phi\left(-\frac{x}{\sqrt{2}}\right) = 0.63$

$\Phi\left(\frac{x}{\sqrt{2}}\right) - \left[1 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right] = 0.63$

$\therefore 2\Phi\left(\frac{x}{\sqrt{2}}\right) - 1 = 0.63$, $\Phi\left(\frac{x}{\sqrt{2}}\right) = 0.815$

From Table I, $\Phi(0.895) \approx 0.815$, $\frac{x}{\sqrt{2}} = 0.895$

5.1.9 The thicknesses of glass sheets produced by a certain process are normally distributed with a mean of μ =3.00 mm and a standard deviation of σ=0.12 mm.
(a) What is the probability that a glass sheet is thicker than 3.2 mm?
(b) What is the probability that a glass sheet is thinner than 2.7 mm?

Solution (a) $P(X \geq 3.2) = P\left(\frac{X-\mu}{\sigma} \geq \frac{3.2-3.0}{0.12}\right)$

$= 1 - \Phi\left(\frac{3.2-3.0}{0.12}\right) = 1 - \Phi(1.67)$

$= 1 - 0.9525 = .0475$

(b) $P(X \leq 2.7) = P\left(Z \leq \frac{2.7-3.0}{0.12}\right)$

$= \Phi\left(-\frac{0.3}{0.12}\right) = \Phi(-2.5) = 1 - \Phi(2.5)$

$= 1 - 0.9938$

$= 0.0062$

5.1.9 c) What is the value of *c* for which there is a 99% probability that a glass sheet has a thickness within the interval [3.00 − *c*, 3.00 + *c*]?

(c) $P(3.0 - c < X < 3.0 + c) = 0.99$

$\Rightarrow P\left(\frac{3.0 - c - 3.0}{0.12} < \frac{X - \mu}{\sigma} < \frac{3.0 + c - 3.0}{0.12}\right) = 0.99$

$\Rightarrow \Phi\left(\frac{c}{0.12}\right) - \Phi\left(\frac{-c}{0.12}\right) = 0.99$

$\Rightarrow \Phi\left(\frac{c}{0.12}\right) - 1 + \Phi\left(\frac{c}{0.12}\right) = 0.99$

$\Rightarrow 2\Phi\left(\frac{c}{0.12}\right) = 1.99$

$\Rightarrow \Phi\left(\frac{c}{0.12}\right) = 0.995$

From Table I $\Phi(2.575) = 0.995$

Solving, $\frac{c}{0.12} = 2.575$,

$c = 2.575 \times 0.12 = 0.309$

5.1.1 Suppose that $Z \sim N(0, 1)$. Find:
(a) $P(Z \le 1.34)$
(b) $P(Z \ge -0.22)$
(c) $P(-2.19 \le Z \le 0.43)$
(d) $P(0.09 \le Z \le 1.76)$
(e) $P(|Z| \le 0.38)$
(f) The value of x for which $P(Z \le x) = 0.55$
(g) The value of x for which $P(Z \ge x) = 0.72$
(h) The value of x for which $P(|Z| \le x) = 0.31$

5.1.2 Suppose that $Z \sim N(0, 1)$. Find:
(a) $P(Z \le -0.77)$
(b) $P(Z \ge 0.32)$
(c) $P(-3.09 \le Z \le -1.59)$
(d) $P(-0.82 \le Z \le 1.80)$
(e) $P(|Z| \ge 0.91)$
(f) The value of x for which $P(Z \le x) = 0.23$
(g) The value of x for which $P(Z \ge x) = 0.51$
(h) The value of x for which $P(|Z| \ge x) = 0.42$

5.1.3 Suppose that $X \sim N(10, 2)$. Find:
(a) $P(X \le 10.34)$
(b) $P(X \ge 11.98)$
(c) $P(7.67 \le X \le 9.90)$
(d) $P(10.88 \le X \le 13.22)$
(e) $P(|X - 10| \le 3)$
(f) The value of x for which $P(X \le x) = 0.81$
(g) The value of x for which $P(X \ge x) = 0.04$
(h) The value of x for which $P(|X - 10| \ge x) = 0.63$

5.1.4 Suppose that $X \sim N(-7, 14)$. Find:
(a) $P(X \leq 0)$
(b) $P(X \geq -10)$
(c) $P(-15 \leq X \leq -1)$
(d) $P(-5 \leq X \leq 2)$
(e) $P(|X + 7| \geq 8)$
(f) The value of x for which $P(X \leq x) = 0.75$
(g) The value of x for which $P(X \geq x) = 0.27$
(h) The value of x for which $P(|X + 7| \leq x) = 0.44$

5.1.5 Suppose that $X \sim N(\mu, \sigma 2)$ and that
$P(X \leq 5) = 0.8$ and $P(X \geq 0) = 0.6$
What are the values of μ and $\sigma 2$?

5.1.6 Suppose that $X \sim N(\mu, \sigma 2)$ and that
$P(X \leq 10) = 0.55$ and $P(X \leq 0) = 0.40$
What are the values of μ and $\sigma 2$?

5.1.7 Suppose that $X \sim N(\mu, \sigma 2)$. Show that
$P(X \leq \mu + \sigma z\alpha) = 1 - \alpha$
and that
$P(\mu - \sigma z\alpha/2 \leq X \leq \mu + \sigma z\alpha/2) = 1 - \alpha$

5.1.8 What are the upper and lower quartiles of a N(0, 1) distribution? What is the interquartile range? What is the interquartile range of a $N(\mu, \sigma 2)$ distribution?

5.1.9 The thicknesses of glass sheets produced by a certain process are normally distributed with a mean of μ =3.00 mm and a standard deviation of σ = 0.12 mm.

(a) What is the probability that a glass sheet is thicker than 3.2 mm?
(b) What is the probability that a glass sheet is thinner than 2.7 mm?
(c) What is the value of c for which there is a 99% probability that a glass sheet has a thickness within the interval [3.00 − c, 3.00 + c]?
(This problem is continued in Problem 5.2.8.)

5.1.10 The amount of sugar contained in 1-kg packets is actually normally distributed with a mean of $\mu = 1.03$ kg and a standard deviation of $\sigma = 0.014$ kg.
(a) What proportion of sugar packets are underweight?
(b) If an alternative package-filling machine is used for which the weights of the packets are normally distributed with a mean of $\mu = 1.05$ kg and a standard deviation of $\sigma = 0.016$ kg, does this result in an increase or a decrease in the proportion of underweight packets?
(c) In each case, what is the expected value of the excess package weight above the advertised level of 1 kg?
(This problem is continued in Problem 5.2.9.)

5.1.11 The thicknesses of metal plates made by a particular machine are normally distributed with a mean of 4.3 mm and a standard deviation of 0.12 mm.
(a) What are the upper and lower quartiles of the metal plate thicknesses?
(b) What is the value of c for which there is 80% probability that a metal plate has a thickness within the interval [4.3 − c, 4.3 + c]?
(This problem is continued in Problem 5.2.4.)

5.1.12 The density of a chemical solution is normally distributed with mean 0.0046 and variance $9.6 \times 10{-}8$.

(a) What is the probability that the density is less than 0.005?
(b) What is the probability that the density is between 0.004 and 0.005?
(c) What is the 10th percentile of the density level?
(d) What is the 99th percentile of the density level?

5.1.13 The resistance in milliohms of 1 meter of copper cable at a certain temperature is normally distributed with mean $\mu = 23.8$ and variance $\sigma 2 = 1.28$.
(a) What is the probability that a 1-meter segment of copper cable has a resistance less than 23.0?
(b) What is the probability that a 1-meter segment of copper cable has a resistance greater than 24.0?
(c) What is the probability that a 1-meter segment of copper cable has a resistance between 24.2 and 24.5?
(d) What is the upper quartile of the resistance level?
(e) What is the 95th percentile of the resistance level?

5.1.14 The weights of bags filled by a machine are normally distributed with a standard deviation of 0.05 kg and a mean that can be set by the operator. At what level should the mean be set if it is required that only 1% of the bags weigh less than 10 kg?

5.1.15 Suppose a certain mechanical component produced by a company has a width that is normally distributed with a mean $\mu = 2600$ and a standard deviation $\sigma = 0.6$.
(a) What proportion of the components have a width outside the range 2599 to 2601?
(b) If the company needs to be able to guarantee to its purchaser that no more than 1 in 1000 of the components have a width outside the

range 2599 to 2601, by how much does the value of σ need to be reduced?
(This problem is continued in Problem 5.2.10.)

5.1.16 Bricks have weights that are independently distributed with a normal distribution that has a mean 1320 and a standard deviation of 15. A set of ten bricks is chosen at random. What is the probability that exactly three bricks will weigh less than 1300, exactly four bricks will weigh between 1300 and 1330, and exactly three bricks will weigh more than 1330?

5.1.17 Manufactured items have a strength that has a normal distribution with a standard deviation of 4.2. The mean strength can be altered by the operator. At what value should the mean strength be set so that exactly 95% of the items have a strength less than 100?

5.1.18 An investment in company A has an expected return of $30,000 with a standard deviation of $4000. What is the probability that the return will be at least $25,000 if it has a normal distribution?