Mat-116

Lec-12

Rational Functions

A rational function is a function of the form

$$R(x) = \frac{P(x)}{q(x)}$$

where p and q are polynomials and q is not the zero polynomials.

The domain of the rational function is the set of all real numbers except those for which the denominator q is zero.

Example:

Find the domain of the following functions:

i. $f(x) = \dfrac{2x^2-4}{x+5}$ Domain $= \{x \mid x \neq 5\}$

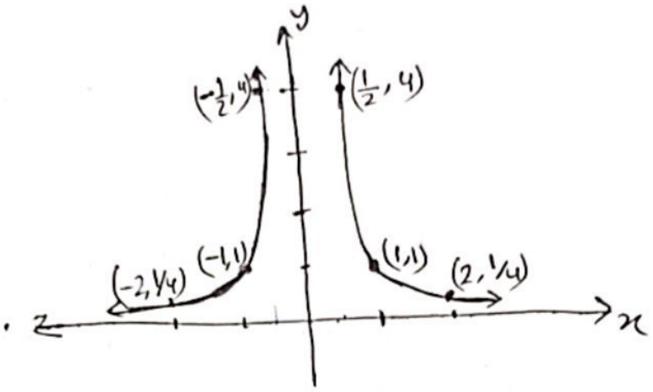
ii) $f(x) = \dfrac{1}{x^2-4}$ Domain $= \{x \mid x \neq -2, \text{ and } x \neq 2\}$

iii) $f(x) = \dfrac{x^3}{x^2+1}$ iv) $f(x) = \dfrac{x^2-1}{x-1}$

** If $R(x) = \dfrac{P(x)}{q(x)}$ is a rational function and if p and q have no common factors, then the rational function R is said to be in lowest terms.

Example:

Graph $y = \frac{1}{x^2}$



Sol^n: let $H(x) = 1/x^2$

Domain $= \{x \mid x \neq 0\}$

y-intercept = NO

x-intercept = NO

$H(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = H(x)$

so $H(x) = \frac{1}{x^2}$ is an even function. So the graph will be symmetric about y-axis

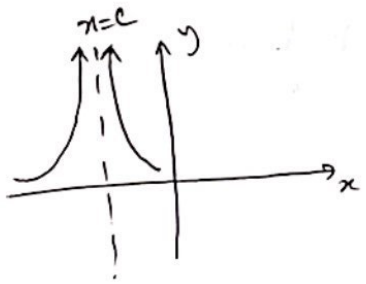
$\lim_{x \to 0} H(x) = \infty$ $\quad\quad$ $\lim_{x \to \infty} H(x) = 0$

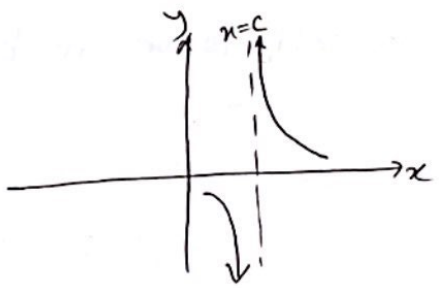
Exercise: $f(x) = \frac{1}{(x-4)^2} + 1$

Asymptotes:

Vertical asymptotes:

If, as x approaches some number c, the values of $|R(x)| \to \infty$ [$R(x) \to -\infty$ or $R(x) \to \infty$], then the line $x=c$ is a vertical asymptote of the graph R.

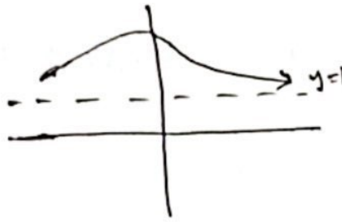


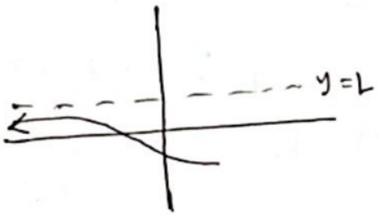


As x approaches c, the values of $|R(x)| \to \infty$, i.e the points on the graph of R getting closer to the line $x=c$; $x=c$ is a vertical asymptote.

Horizontal asymptote:

If, as $x \to -\infty$ or $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y=L$ is a horizontal asymptote of the graph of R.

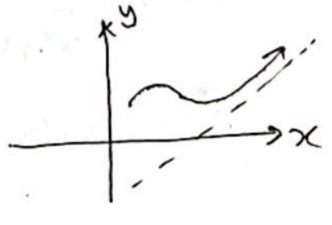




End behavior: As $x \to \infty$, the values of $R(x)$ approach L. that is the points on the graph of R are getting closer to the line $y=L$, $y=L$ line is a horizontal asymptote.

Oblique asymptote:

If, as $x \to \infty$ or $x \to -\infty$, the value of rational function $R(x)$ approaches a linear expression $ax+b$, $a \neq 0$, then the line $y = ax+b$, $a \neq 0$ is an oblique asymptote.



Example: Find the vertical asymptote of the following functions:

i) $F(x) = \frac{x+3}{x-1}$

Soln: A rational function $R(x) = \frac{P(x)}{q(x)}$, in lowest terms, will have a vertical asymptote $x=r$ if r is a real zero of the denominator q.

Given $F(x) = \frac{x+3}{x-1}$

Here F is in lowest terms and the only zero of the denominator is 1. So the line $x=1$ is the vertical asymptote.

ii) $R(x) = \frac{x}{x^2-4}$

Solⁿ: $R(x)$ is in lowest terms.

$x^2 - 4 = 0$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm 2$ so the real zero of the denominator is -2 and 2. Thus $x=-2$ and $x=2$ are the vertical asymptotes of R.

iii) $H(x) = \frac{x^2}{x^2+1}$

Solⁿ: $H(x)$ is in lowest terms.

$x^2+1=0$

$\Rightarrow x^2 = -1$

$\Rightarrow x = \pm\sqrt{-1}$; the given function doesn't have any real solutions. Thus the graph of H has no vertical asymptotes.

iv) $G(x) = \frac{x^2-9}{x^2+4x-21}$

Solⁿ: $G(x) = \frac{x^2-9}{x^2+4x-21} = \frac{(x+3)(x-3)}{(x+7)(x-3)} = \frac{x+3}{x+7}$; $x \neq 3$

The only zero of the denominator of $G(x)$ in lowest term is $x+7=0 \Rightarrow x=-7$.

Thus the line $x=-7$ is the only vertical asymptote of G.

* Note: rational functions can have no vertical asymptotes, one vertical asymptotes or more than one vertical asymptotes.

Finding Horizontal Asymptotes:

1. If a rational function is proper, i.e if the degree of the numerator is less than the degree of denominator, then line $y=0$ is a horizontal asymptote of its graph.
2. If the numerator and denominator are of the same degree, then the ratio of the leading coefficients gives the horizontal asymptote.
3. If the degree of the numerator is greater than the degree of the denominator, then the graph will not have any horizontal asymptote.

Example: 1

$$f(x) = \frac{x-12}{4x^2+x+1}$$

The degree of numerator is $1 <$ the degree of the denominator which is 4.

So $y=0$ line is the horizontal asymptote of $f(x)$.

Example: 2

$$f(x) = \frac{8x^2-x+2}{4x^2-1}$$

The degree of numerator and denominator are the same.

So the horizontal asymptote is $\frac{8}{4} = 2$.

$\therefore$ the line $y=2$ is the horizontal asymptote.

Example: 3

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Here the degree of numerator 4 is greater than the degree of the denominator which is 3. so the graph of the function doesn't have any horizontal asymptotes.

(Either oblique or neither oblique nor ~~horizontal~~ asymptotes)

Oblique asymptotes:

Example: $H(x) = \dfrac{3x^4 - x^2}{x^3 - x^2 + 1}$

Soln:

$$
\begin{array}{r|l}
x^3 - x^2 + 1 & 3x^4 - x^2 \quad (3x + 3 \\
 & 3x^4 - 3x^3 + 3x \\
 & \overline{ 3x^3 - x^2 - 3x} \\
 & \, 3x^3 - 3x^2 \quad + 3 \\
 & \overline{ 2x^2 - 3x - 3}
\end{array}
$$

As a result,

$$H(x) = 3x + 3 + \frac{2x^2 - 3x - 3}{x^3 - x^2 + 1}$$

As $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 - 3x - 3}{x^3 - x^2 + 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

So As $x \to -\infty$ or as $x \to \infty$, we have $H(x) = 3x + 3$.

Thus the graph of $H(x)$ has an oblique asymptote $y = 3x + 3$.

Example:

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

Soln:

$$\begin{array}{r|l}
x^3-1 & \;2x^5 - x^3 + 2 \qquad \big(\,2x^2-1 \\
 & \;2x^5 \qquad\quad \mp 2x^2 \\
 & \;\underline{- \qquad\quad \pm } \\
 & \quad -x^3 + 2x^2 + 2 \\
 & \quad \mp x^3 \qquad\;\; \pm 1 \\
 & \quad\underline{} \\
 & \qquad\quad 2x^2 + 1
\end{array}$$

As a result

$$G(x) = \cancel{2x^2-1} \;\; 2x^2 - 1 + \frac{2x^2+1}{x^3-1}$$

Then as $x \to -\infty$ or $x \to +\infty$

$$\frac{2x^2+1}{x^3-1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

Thus as $x \to -\infty$ or $x \to +\infty$, $G(x) = 2x^2 - 1$. Since $y = 2x^2 - 1$ is not a linear function, G has no horizontal or oblique asymptote.

Notes:- If $R(x) = \dfrac{P(x)}{q(x)}$ is improper, then we can write

$$R(x) = \frac{P(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

Here $r(x)$ is the remainder.
$f(x)$ is the quotient.

Where $f(x)$ is a polynomial and $\dfrac{r(x)}{q(x)}$ is a proper rational function. Since $\dfrac{r(x)}{q(x)} \to 0$ as $x \to -\infty$ or $x \to +\infty$, As a result

$$R(x) = \frac{P(x)}{q(x)} \to f(x) \quad \text{as } x \to -\infty \text{ or } +\infty$$

we have the following possibilities:

1. If $f(x)=b$, a constant, the line $y=b$ is a horizontal asymptote of the graph of R.

2. If $f(x)=ax+b$, $a\neq 0$, the line $y=ax+b$ is an oblique asymptote of the graph of R.

3. In all other cases the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

Exercise section 4.2

13–24, 43–54.

Polynomial and Rational Inequalities

Problem: 1

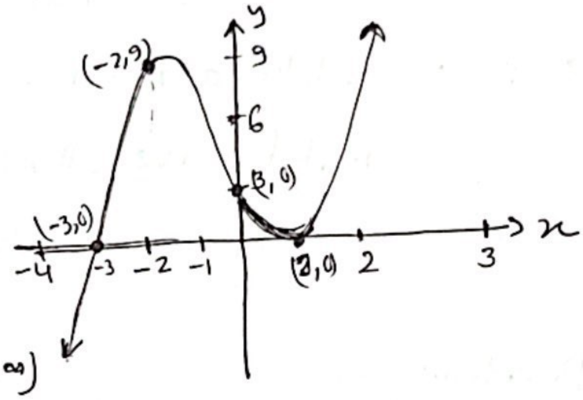
Solve $(x+3)(x-1)^2>0$ using the following graph

Soln: $f(x) = (x+3)(x-1)^2$

From the graph $f(x)>0$ for $-3<x<1$ or $x>1$.

Thus the solution set is $\{x \mid -3<x<1 \text{ or } x>1\}$

or interval notation $(-3,1)\cup(1,\infty)$



Problem: 2

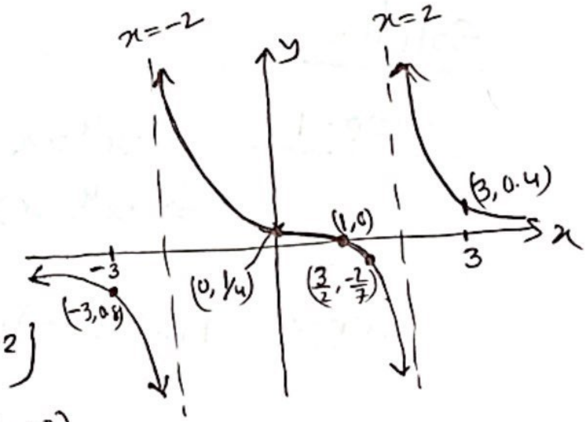
Solve $\frac{x-1}{x^2-4} \geq 0$ using the following graph

Solution: let $f(x) = \frac{x-1}{x^2-4}$

from the graph, we can see that $f(x) \geq 0$ for $-2 \leq x \leq 1$ or $x>2$.

The solution set is $\{x \mid -2<x\leq 1 \text{ or } x>2\}$

or interval notation $(-2,1]\cup(2,\infty)$



Solve inequality algebrically:

Step-1: Write of the form $f(x)>0$, $f(x)\geq 0$, $f(x)<0$, $f(x)\leq 0$

Step-2: Find real zeros or x-intercepts

Step-3: Use zeros to divide real number line into intervals.

Step-4: Select a number in each interval and evaluate f there.

i. If f is +ve, all values of f in that interval are positive.

ii. If f is -ve, all values of f in that interval are negative.

Problem:

Solve the inequality $x^4 > x$ algebrically and graph the solution set.

Solution

Step-1: $x^4 > x$

$\Rightarrow x^4 - x > 0$

Step-2 Find x-intercepts by letting $x^4 - x = 0$

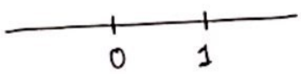
$\Rightarrow x(x^3-1) = 0$

$\Rightarrow x(x-1)(x^2+x+1) = 0$

$\therefore x=0$ or $x-1=0$ or $x^2+x+1=0$

Thus $x=0$ or $x=1$ because $x^2+x+1=0$ does not have any real solution.

Step-3 Divide real number line into intervals.

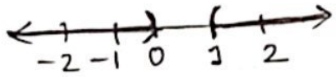


$(-\infty, 0)$, $(0, 1)$, $(1, \infty)$

Step: 4 Select number in each interval and evaluate $f(x) = x^4 - x$

interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
number chosen	-1	$\frac{1}{2}$	2
$f(x)$	2	$-\frac{7}{16}$	14
Conclusion/ sign of $f(x)$	+ve	−ve	+ve

$\Rightarrow f(x) = x^4 - x > 0$ for $x < 0$ or $x > 1$. Interval notation $(-\infty, 0) \cup (1, \infty)$.



Problem: 2

Solve the inequality $\dfrac{4x+5}{x+2} \geq 3$

Soln,

Step-1 $f(x) = \dfrac{4x+5}{x+2} \geq 3$

$$\Rightarrow \frac{4x+5}{x+2} - 3 \geq 0$$

$$\Rightarrow \frac{4x+5-3(x+2)}{x+2} \geq 0$$

$$\Rightarrow \frac{4x+5-3x-6}{x+2} \geq 0 \Rightarrow \frac{x-1}{x+2} \geq 0 \; ; x \neq -2$$

Step-2 Let $f(x) = 0$

$$\therefore \frac{x-1}{x+2} = 0 \Rightarrow x-1=0 \Rightarrow x=1$$

$\Rightarrow x = 1$ is the x-intercept.

Also f is undefined at $x = -2$

Step-3 Divide real number line into intervals.

$(-\infty, -2)$ $(-2, 1)$ $(1, \infty)$

Step-4 Select number in each interval and evaluate $f(x) = \frac{x-1}{x+2}$ there.

interval	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
number chosen	-3	0	2
$f(x)$	4	$-\frac{1}{2}$	$\frac{1}{4}$
sign of $f(x)$	+ve	-ve	+ve

We can see that $f(x) \geq 0$ for $x < -2$ and $x \geq 1$.

Using interval notation $(-\infty, -2) \cup [1, \infty)$.

Note: We didn't include -2 because the function is not defined at $x = -2$.

Exercise 4.4 → $\boxed{19-42}$