1.6 Continuity of Trigonometric, Exponential & Inverse Function:

$\boxed{x = c}$ $\quad y = f(x)$

(i) $f(c)$ —

(ii) $\lim_{x \to c} f(x)$ exist

(iii) $\boxed{\lim_{x \to c} f(x) = f(c)}$

Ex $f(x) = \sin x$

$\lim_{x \to c} \sin x = \sin c$

$\lim_{x \to c} \cos x = \cos c$

$\lim_{x \to c} \tan x = \tan c$

$\lim_{x \to c} \cot x = \cot c$

$\lim_{x \to c} \operatorname{cosec} x = \operatorname{cosec} c$

$\lim_{x \to c} \sec x = \sec c$

* $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(g(c))$

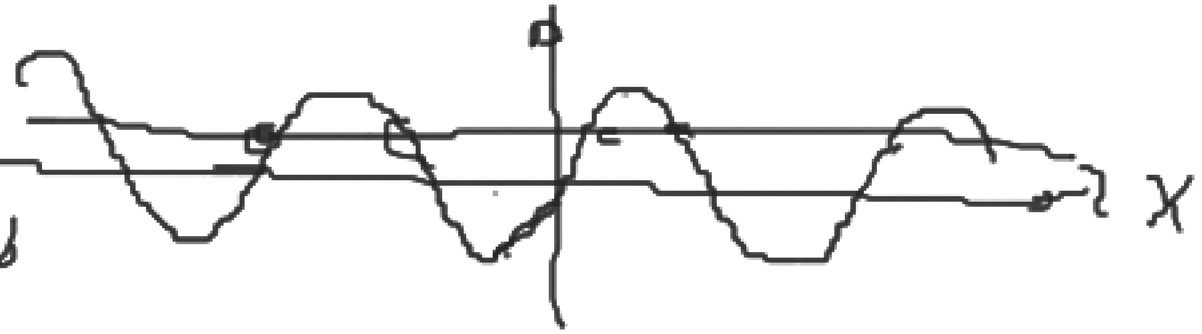
Ex1: $\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right) = ?$

Sol: $\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right) = \cos\left(\lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)}\right)$

$= \cos\left(\lim_{x\to 1} (x+1)\right) = \cos 2$

Inverse Trigonometric Function:

$f(x) = \sin x$



$\boxed{f(x) = \sin x}$; $-\pi/2 \le x \le \pi/2$,
$-1 \le y \le 1$

i.e. $f^{-1}(x) = \sin^{-1} x$. D: $[-1, 1]$
R: $[-\pi/2, \pi/2]$

$\checkmark$ $\boxed{f(x) = \cos x}$; $\boxed{0 \le x \le \pi}$

$\checkmark$ $f^{-1}(x) = \boxed{\cos^{-1} x}$; D : $[-1, 1]$

R : $[0, \pi]$

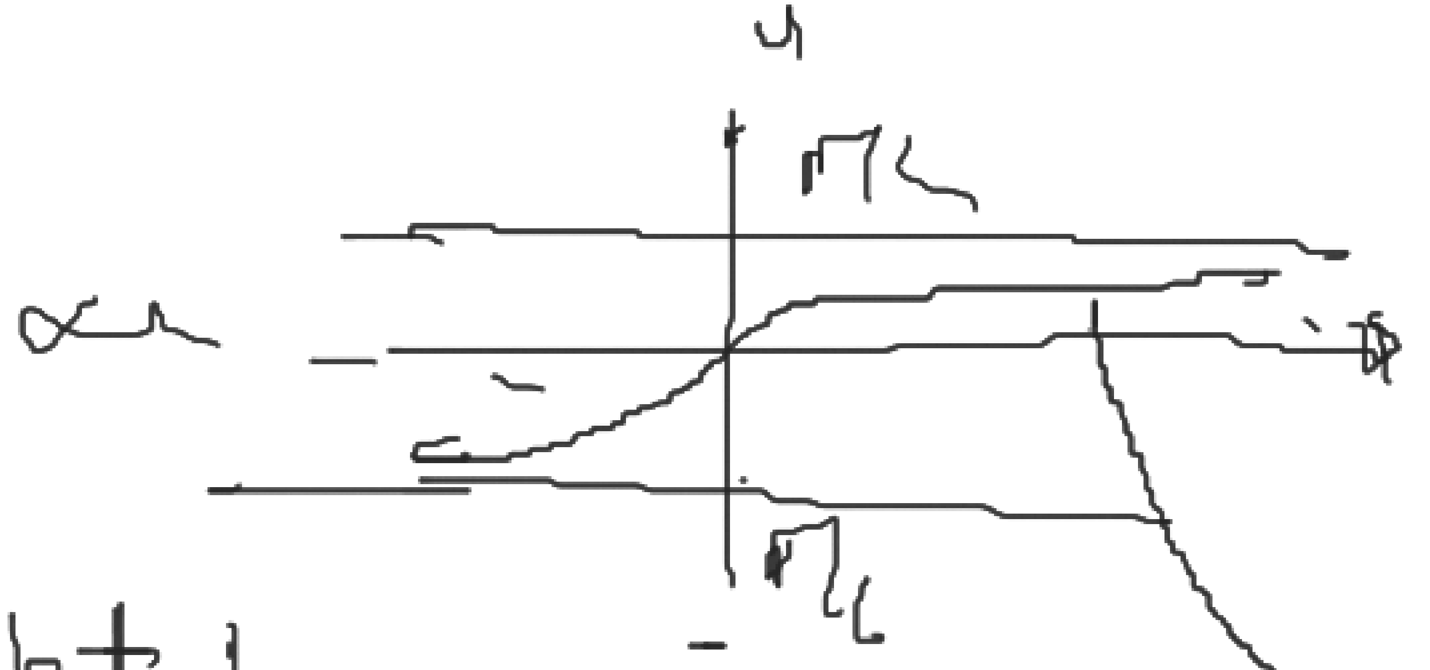
$\checkmark$ $f(x) = \tan x$; $-\pi/2 \le x \le \pi/2$

R : $(-\infty, +\infty)$

$\checkmark$ $\boxed{f(x) = \tan^{-1}}$

$\checkmark$ $\boxed{D : (-\infty, +\infty)}$

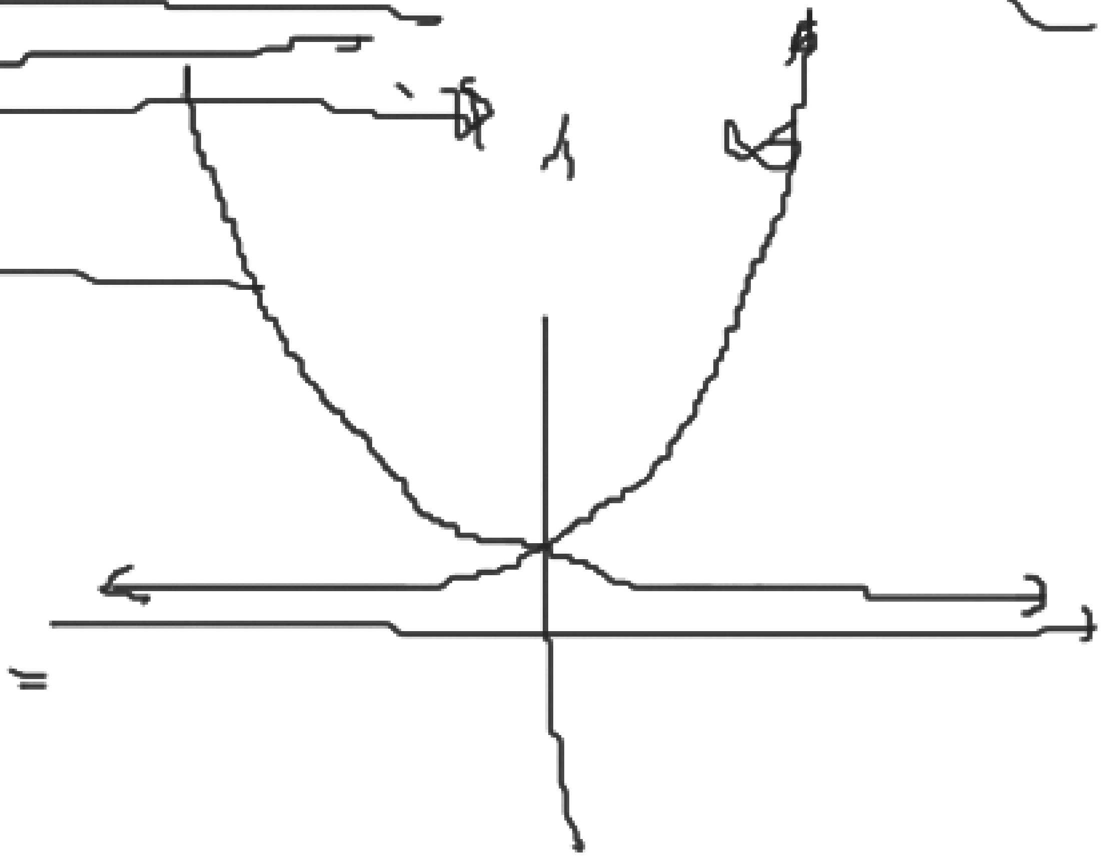
R : $(-\pi/2, \pi/2)$



<u>Theorem</u> : $b > 0$, $b \ne 1$

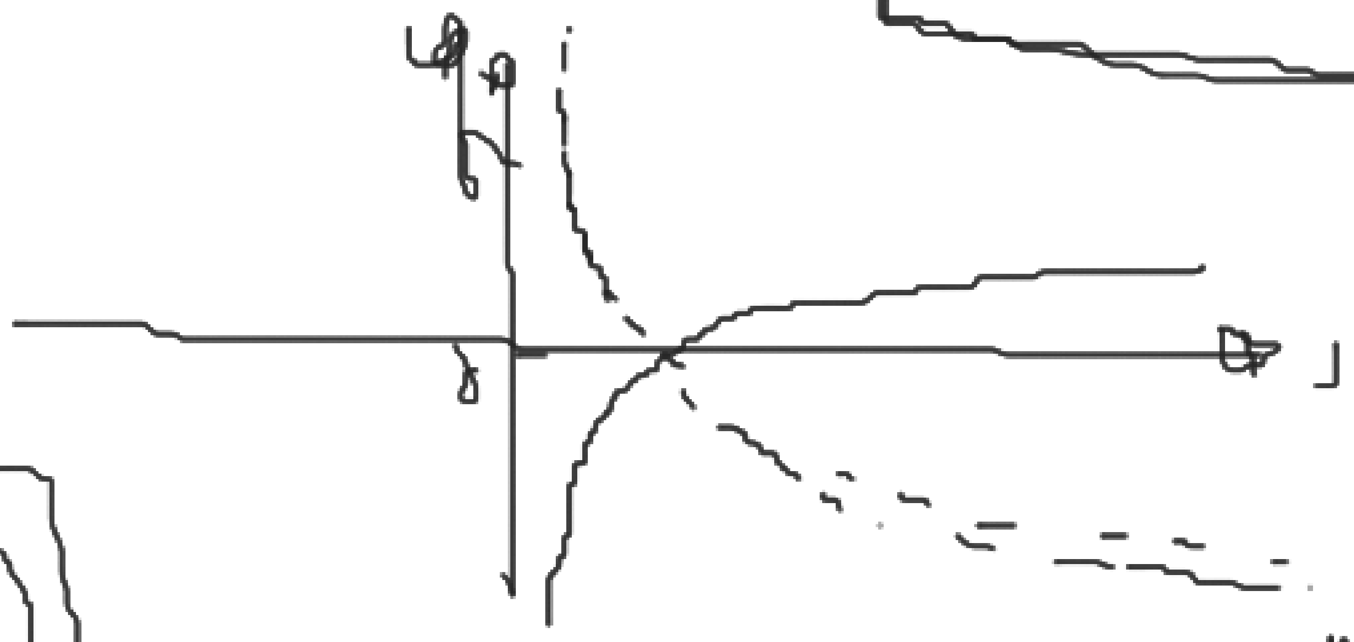
$\checkmark$ $\boxed{f(x) = b^x}$

$(-\infty, +\infty)$



* $b > 0$, $b \neq 1$, $x > 0$

$\boxed{f(x) = \log_b x}$



$\boxed{D: (0, \infty)}$

$R: (-\infty, +\infty)$

$\boxed{f(x) = b^x}$ $\boxed{D: (-\infty, \infty)}$

$R: (0, \infty)$

(I) $\uparrow\downarrow$

(II) $\int n$

(III) $\underline{Ex} \rightarrow$

$\underline{Log}$ $(0, \infty)$.

(IV)

Ex3 : $f(x) = \dfrac{\boxed{\tan^{-1}x + \ln x}}{x^2-4}$ where the function $f(x)$ continuous?

$f(x) = \dfrac{h(x)}{g(x)}$

$D : \underline{\text{Dom of } h(x)} \cap \underline{\text{Dom of } g(x)}$

$f(x) = \boxed{g(x) + h(x)}$

$D : \boxed{\text{Dom of } g(x)} \cap \boxed{D \text{ of } h(x)}$

Sol

$\tan^{-1}x$: $D : (-\infty, +\infty)$

$\ln x$: $D : (0, \infty)$

$\tan^{-1}x + \ln x$: $\boxed{D : (0, \infty)}$

$x^2 - 4$: $\boxed{D : (-\infty, +\infty)}$

$\therefore$ Domain of $f(x)$: $(0, \infty) \cap \{\boxed{x : x^2 - 4 \neq 0}\}$

$x^2 - 4 \neq 0$

$\therefore x \neq \pm 2$

$\therefore D : (0, 2) \cup (2, \infty)$

Theorem : If $g(x) \le \boxed{f(x)} \le h(x)$ and $\lim_{x \to c} g(x) = L$

$\lim_{x \to c} h(x) = L$

then $\boxed{\lim_{x \to c} f(x) = L}$

Theorem : (i) $\lim_{x \to 0} \boxed{\frac{\sin x}{x} = 1}$ (ii) $\lim_{x \to 0} \boxed{\frac{1 - \cos x}{x} = 0}$

Ex 4 : (a) $\lim_{x \to 0} \frac{\tan x}{x} = ?$

Sol: $\lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right)$

$= \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \cdot \left(\lim_{x \to 0} \frac{1}{\cos x} \right)$

$= 1 \cdot \frac{1}{1} = 1$ Ans

(b) $\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} = ?$

Sol: $\lim_{\theta \to 0} 2 \cdot \frac{\sin 2\theta}{2\theta}$

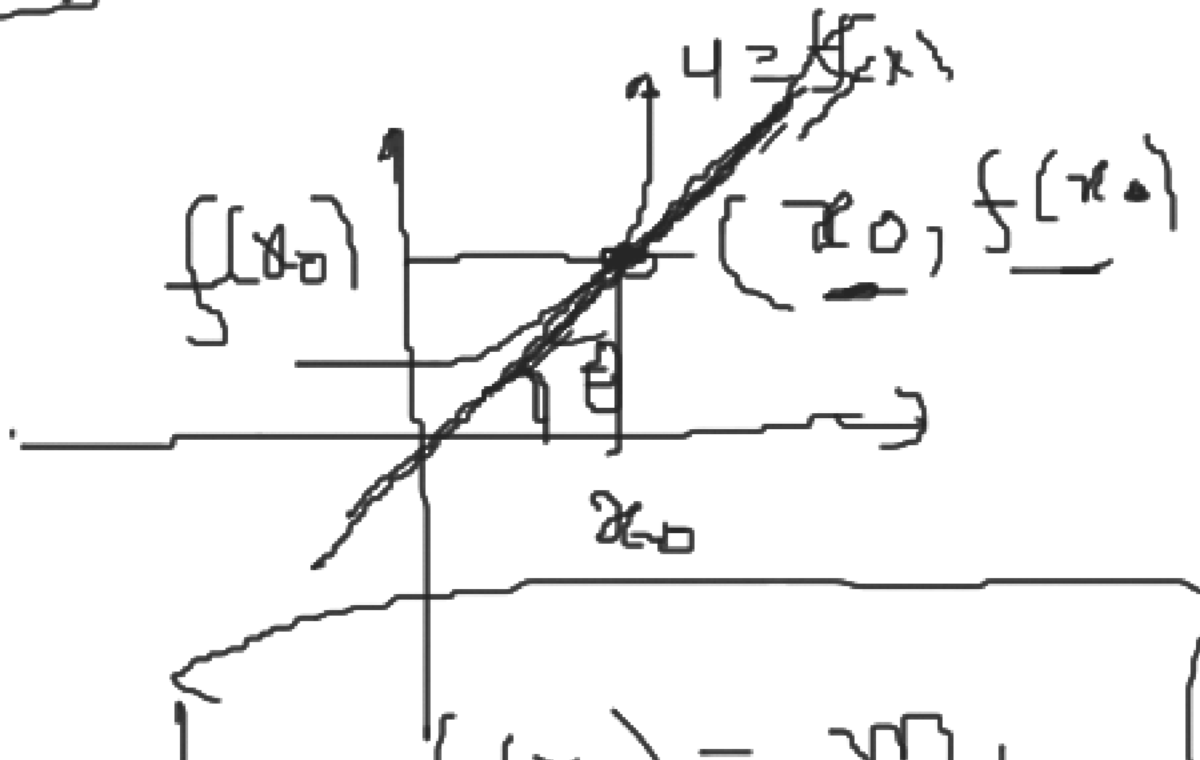
$= 2 \cdot \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta}$ | If $\theta \to 0$ then $2\theta \to 0$

$= 2 \cdot \left(\lim_{2\theta \to 0} \frac{\sin 2\theta}{2\theta} \right)$

$= 2 \cdot 1 = 2$ Ans

H.W. Ex 1.6 : $\boxed{17\text{-}40 \ \& \ 49\text{-}55}$

Chapter # 01 (Derivative)



(x_1, y_1), m

$$y - y_1 = \boxed{m}(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

2.1 Tangent Line:

$$\boxed{y - f(x_0) = m_{tan}(x - x_0)}$$

$$m_{tan} = \boxed{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

$\boxed{y - f(x_0) = m_{tan}(x - x_0)}$

$$m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Alternatively: Let, $h = \boxed{x - x_0}$ $\therefore$ $\boxed{x = x_0 + h}$

If $x \to x_0$ then $h \to 0$

$$\boxed{m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}}$$

Ex 3 : Tangent line = ? for $\boxed{y = \frac{2}{x}}$ at $P(2, 1)$ $(x_0, f(x_0))$

Sol: Let $\boxed{f(x) = \frac{2}{x}}$, $\boxed{x_0 = 2}$ $\boxed{f(x_0) = 1}$

We know that, $y - \boxed{f(x_0)} = m_{tan}(x - x_0)$ ---- (1)

Here $m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{2}{2+h} - 1}{h}$

$= \lim_{h \to 0} \frac{\frac{2 - 2 - h}{(2+h)}}{h} = \lim_{h \to 0} \frac{-h}{h(2+h)} = -\lim_{h \to 0} \left(\frac{1}{2+h}\right)$

$= -\frac{1}{2}$

(1) becomes:

$y - 1 = -\frac{1}{2}(x - 2)$

$\Rightarrow y = -\frac{1}{2}x + 1 + 1$

$\therefore y = -\frac{1}{2}x + 2$ Ans

Ex4: Find the slope of the tangent line to the curve

$\boxed{y = \sqrt{x}}$ at $\boxed{x_0 = 1}$, $\boxed{x_0 = 4}$, $\boxed{x_0 = 9}$

Sol: $\boxed{f(x) = \sqrt{x}}$

$\therefore$ The slope, $m_{tan} = \lim\limits_{h \to 0} \dfrac{\boxed{f(x_0 + h)} - f(x_0)}{h}$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x_0 + h} - \sqrt{x_0}\right)}{h} \times \frac{\left(\sqrt{x_0 + h} + \sqrt{x_0}\right)}{\sqrt{x_0 + h} + \sqrt{x_0}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x_0 + h}\right)^2 - \left(\sqrt{x_0}\right)^2}{h\left(\sqrt{x_0 + h} + \sqrt{x_0}\right)} = \lim_{h \to 0} \frac{\cancel{x_0} + h - \cancel{x_0}}{h\left(\sqrt{x_0 + h} + \sqrt{x_0}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x_0+h} + \sqrt{x_0}\right)} = \lim_{h \to 0} \frac{1}{\left(\sqrt{x_0+h} + \sqrt{x_0}\right)}$$

$$= \frac{1}{\sqrt{x_0} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}$$

$$\therefore \boxed{m_{tan} = \frac{1}{2\sqrt{x_0}}}$$

At $\boxed{x_0 = 1}$: $m_{tan} = \frac{1}{2\sqrt{1}} = \boxed{1/2}$

" $\boxed{x_0 = 4}$: $m_{tan} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

" $x_0 = 9$: $m_{tan} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

Ans

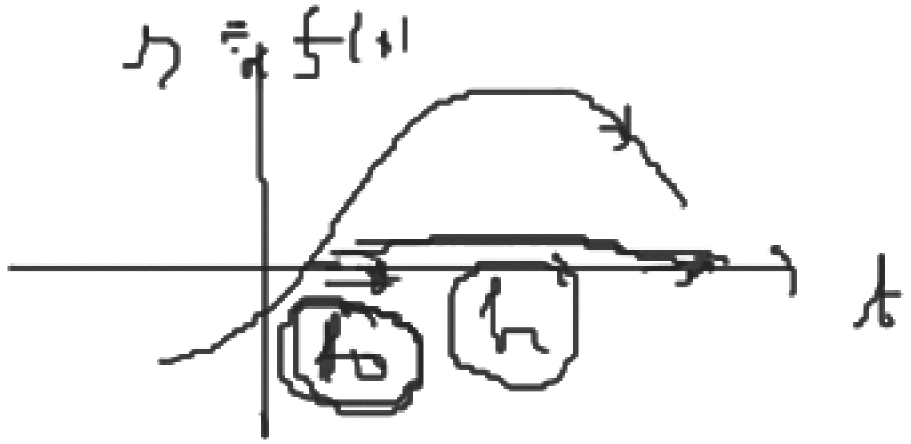
$$y - f(x_0) = m_{tan}(x - x_0)$$

$f(x) = \sqrt{x}$

$f(x_0) = f(1) = \sqrt{1} = 1$, $x_0 = 1$, $m_{tan} = \frac{1}{2}$

$\therefore y - 1 = \frac{1}{2}(x - 1) \Rightarrow$ Ans

Velocity :



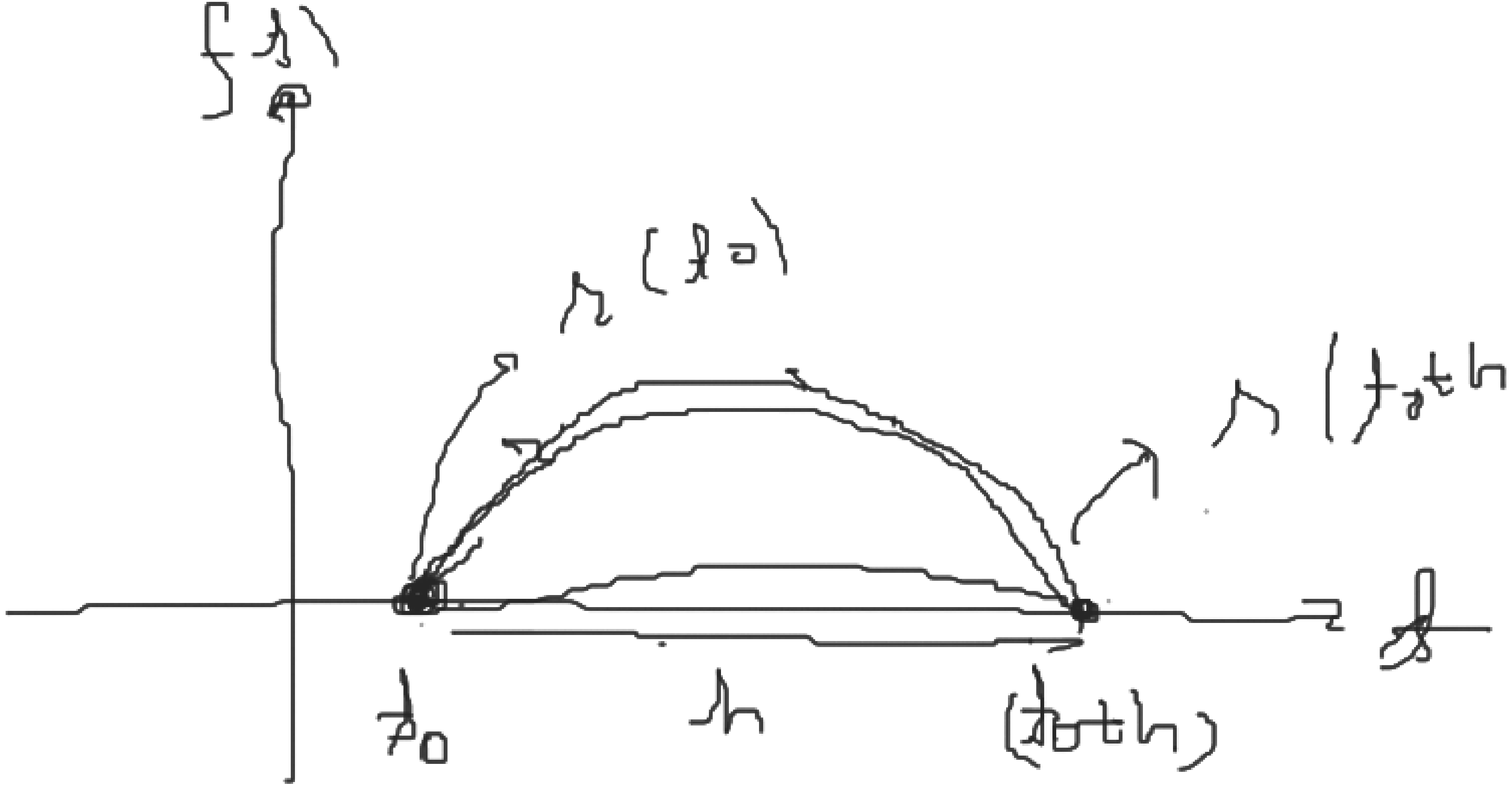
$$\boxed{s = f(t)}$$

Average Velocity, $v_{avg} = \dfrac{\text{Change in Position}}{\text{Time elapsed}}$

$$= \frac{f(t_0+h) - f(t_0)}{h}$$

Instantaneous Velocity :

$$v_{inst} = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h}$$



$$\frac{s(t_0+h) - s(t_0)}{\cancel{t_0}+h-\cancel{t_0}} = \boxed{\frac{s(t_0+h) - s(t_0)}{h}}$$

Ex 5 $s = f(t) = 1 + 5t - 2t^2$ (a) Average Velocity on $[0, 2] = ?$

(b) " " " $[2, 3] = ?$

Sol: (a) $f(t) = 1 + 5t - 2t^2$, $[t_0, t_0 + h] = [0, 2]$

$\therefore t_0 = 0$,

$\boxed{t_0 + h = 2} \Rightarrow 0 + h = 2$

$\therefore \boxed{h = 2}$

$$\therefore v_{ave} = \frac{f(t_0 + h) - f(t_0)}{h}$$

$$= \frac{f(2) - f(0)}{2}$$

$$= \frac{3 - 1}{2} = 1 \quad \text{Ans}$$

$f(t) = 1 + 5t - 2t^2$

$\therefore f(2) = 1 + 5.2 - 2(2)^2$

$= 3$

(b) $f(t) = 1 + 5t - 2t^2$, $\quad [t_0, t_0+h] = [2, 3]$

$\therefore t_0 = 2, \quad t_0 + h = 3$

$\Rightarrow 2 + h = 3$

$\boxed{\therefore h = 1}$

$$\therefore V_{ave} = \frac{f(t_0+h) - f(t_0)}{h}$$

$$= \frac{f(3) - f(2)}{1}$$

$$= \frac{[1 + 5 \cdot 3 - 2 \cdot 3^2] - [1 + 5 \cdot 2 - 2 \cdot 2^2]}{1}$$

$$= \frac{-2-3}{1} = -5 \text{ m/sec Ans}$$

Average Rate of Change (ARC):

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Alternatively:

$x_1 - x_0 = h$

$\therefore x_1 = x_0 + h.$

$$r_{ave} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Instantaneous Rate of Change (IRC)

$$r_{inst} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

HW - Ex 2.1 : $\boxed{11-18}$