Homework 2
Due Date: Sunday, 3 November 2023

Objective: In this homework, given the equations for a robot (a dynamic system) we'll learn how to generate a specific class of trajectories called differentially flat trajectories. These trajectories are not the optimal trajectory (i.e., the best given the constraints) but they are feasible. We'll also learn how to generate the controls to move on these trajectories.

1) The equations for a **nonholonomic integrator** are given by,

$$\dot{x_1} = u_1$$
$$\dot{x_2} = u_2$$
$$\dot{x_3} = x_2 u_1$$

a) Show that this system is differentially flat for the flat output z = (x_1, x_3).

b) Assume the system moves from initial time $t_i = 0$ to final time $t_f = T$. Assume the following initial conditions:

At $t_i = 0$, $x_1(0)$, $x_2(0)$, $x_3(0)$, $\dot{x_1}(0) = 1$
At $t_f = T$, $x_1(T)$, $x_2(T)$, $x_3(T)$, $\dot{x_1}(T) = 1$

Using four basis functions, $\psi_1 = 1$, $\psi_2 = t$, $\psi_3 = t^2$, $\psi_4 = t^3$, write down the matrix-vector equations for this differentially flat system. [**Hint:** Slide 22 in the differential flatness presentation might be helpful]

c) Repeat part (b), but this time use six basis functions, $\psi_1 = 1$, $\psi_2 = t$, $\psi_3 = t^2$, $\psi_4 = t^3$, $\psi_5 = t^4$, $\psi_6 = t^5$

d) Using the four basis functions of 1(b) and $\Delta t = 0.1$, plot the differentially flat trajectory for this system for the following initial conditions. Also plot, u_1 vs t and u_2 vs t. Submit your code and screenshots.

i) At $t_i = 0$, $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = -3$, $\dot{x_1}(0) = 1$
At $t_f = 10$, $x_1(10) = 5$, $x_2(10) = 5$, $x_3(10) = 5$, $\dot{x_1}(10) = 1$

ii) At $t_i = 0$, $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 1$, $\dot{x_1}(0) = 1$
At $t_f = 15$, $x_1(15) = 10$, $x_2(15) = 10$, $x_3(15) = 5$, $\dot{x_1}(15) = 1$

2) Trajectory tracking using an open loop controller

In this problem, first we will generate a trajectory using the differential flatness technique. Then, we will generate the control inputs for the robot to navigate this desired trajectory using open loop controller techniques.

Consider the equations for the dynamically extended unicycle robot,

$$\dot{x}(t) = V(t)\cos\theta(t)$$
$$\dot{y}(t) = V(t)\sin\theta(t)$$
$$\dot{V}(t) = a(t)$$
$$\dot{\theta}(t) = \omega(t)$$

The initial conditions are: $x(0) = 0$, $y(0) = 0$, $V(0) = 0.5$, $\theta(0) = -\pi/2$

$$x(t_f) = 5,\ y(t_f) = 5,\ V(t_f) = 0.5,\ \theta(t_f) = -\pi/2 \text{ where } t_f = 15$$

Here the controls are a(t) and ω(t).

a) Show this system is differentially flat with flat output z = (x, y).

b) Generate a differentially flat trajectory using the four basis functions of problem 1(b). Plot your generated trajectory.

c) We will now navigate our robot along this desired trajectory. To do that we need to compute the controls a(t) and ω(t). Notice that differentiating the velocities $\dot{x}, \dot{y}$ we get the following equations,

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos\theta\,(t) & -V(t)\sin\theta\,(t) \\ \sin\theta\,(t) & V(t)\cos\theta(t) \end{bmatrix} \begin{bmatrix} a(t) \\ \omega(t) \end{bmatrix}$$

From your generated differentially flat trajectory x_d, y_d (the subscript "d" refers to the desired values) you can estimate $\dot{x}_d, \dot{y}_d, \theta_d, V_d, \ddot{x}_d, \ddot{y}_d$. Then a(t) and ω(t) can be calculated by taking the matrix inverse in the equation above.

Once the controls a(t) and ω(t) are calculated, you can navigate the robot by numerically integrating the unicycle equations using Euler's method. See slide 5 of the "Open loop motion control and differential flatness" presentation.

Integrate the unicycle equations using Euler's method, generate the control inputs as described above. Plot the result for time steps Δt = 0.1 and 0.01. You can also try experimenting with other values for the time step. The smaller the time step, the more accurate the result but greater the computation.

3) Open loop controller performance in the presence of noise or disturbance

In problem 2(c), your robot should track the trajectory perfectly. This is because there is no noise or disturbance in the system. If noise and disturbance is present, which is the case for real systems, the robot won't be able to track the trajectory with an open loop controller. We will see this in this exercise.

We can randomly inject noise or disturbances into the system using the Gaussian (Normal) distribution. Inject noise in the V and θ equations in the following way,

$$V(t+\Delta t) = V(t) + \Delta t\, a(t) + \mathcal{N}(0, 0.01)$$
$$\theta(t+\Delta t) = \theta(t) + \Delta t\, \omega(t) + \mathcal{N}(0, 0.001)$$

Here $\mathcal{N}(0, 0.01)$ is a random number generated from a Gaussian (Normal) distribution with mean 0 and standard deviation 0.01. Similarly, $\mathcal{N}(0, 0.001)$ is a random number generated from a Gaussian distribution with mean 0 and standard deviation 0.001.

With this noise injection, integrate the unicycle equations using Euler's method and the open loop controller of problem 2(c). Plot the result for time step Δt = 0.1 and Δt = 0.01. You should be able to see the robot deviate from the desired trajectory. Submit your code and screenshots.