Course Name : Physics – I
Course # PHY 107

Notes-2 : Representation of Vectors and the Product Rules

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Topics to be covered

1. Unit vectors and their properties
2. Addition/Subtraction rules
3. Product rules for vectors:

   a) Dot/Scalar product

   b) Cross/Vector product
4. Polar form of product rules and the geometrical interpretations
5. Examples

Unit Vectors:

- Definition: any vector whose length or magnitude is one is called a unit vector. In Cartesian Coordinate system, the unit vector along the x-axis is denoted by $\hat{i}$ and similarly $\hat{j}, \hat{k}$ are the unit vectors along the y- and z-axes.

- In three dimensions, any vector is expressed as

$$\vec{A} = (A_x, A_y, A_z) = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

- The unit vectors are mutually perpendicular.

- For any two vectors, the addition/subtraction is given by

$$\vec{A} \pm \vec{B} = (A_x, A_y, A_z) \pm (B_x, B_y, B_z) = (A_x \pm B_x)\hat{i} + (A_y \pm B_y)\hat{j} + (A_z \pm B_z)\hat{k}$$

- The polar form of the sum can be obtained by Pythagorean Theorem.

Product Rules for Vectors:

- There are two rules for product between two vectors:

1) Dot or Scalar Product: It is defined for the unit vectors as

$$\underbrace{\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1}_{\text{unit vectors}},\ \underbrace{\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0}_{\text{Orthogonality of vectors}}.$$

2) Cross or Vector Product: It is defined for the unit vectors as

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0,\ (\text{Parallel Vector properties})$$

$$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j},\ (\text{The Right-hand Rule})$$

$$\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j}\ (\text{Note the change in direction})$$

In Cartesian form, these two products between any two vectors are very well known and easy to remember:

- The Dot product between any two vector is now given by

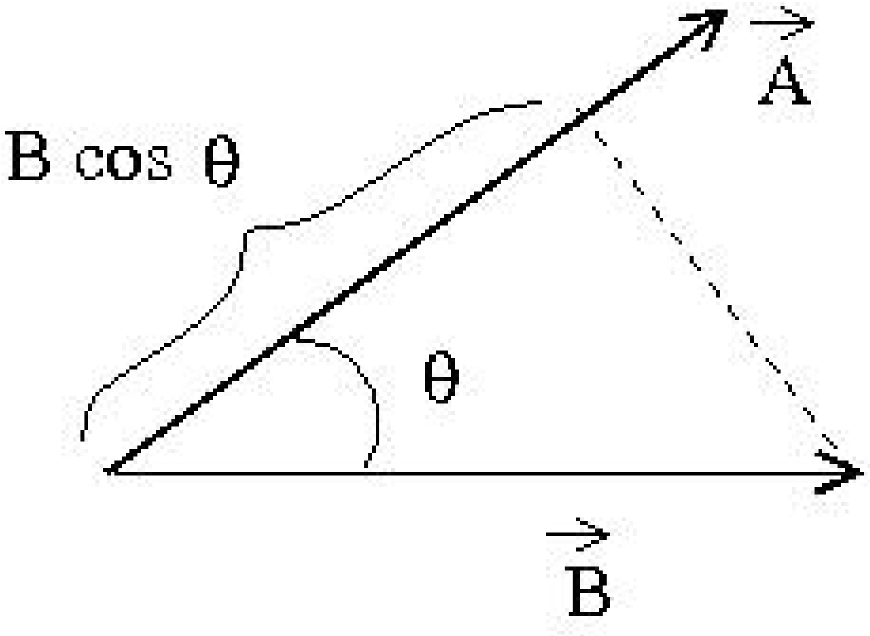
$$\vec{A}\cdot\vec{B}=\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\cdot\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)=A_xB_x+A_yB_y+A_zB_z \rightarrow \text{Scalar}$$

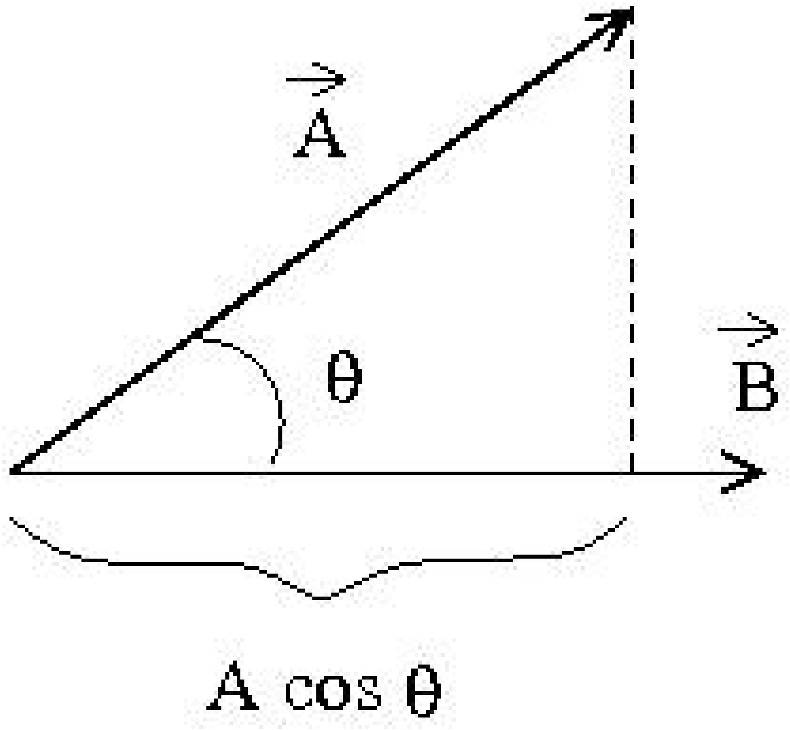
- The Cross product between any two vectors is now given by

$$\vec{A}\times\vec{B}=\hat{i}\left(A_yB_z-A_zB_y\right)-\hat{j}\left(A_xB_z-A_zB_x\right)+\hat{k}\left(A_xB_y-A_yB_x\right)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \rightarrow \text{Vector quantity}$$

Polar form of the Dot product (Geometrical Interpretation)

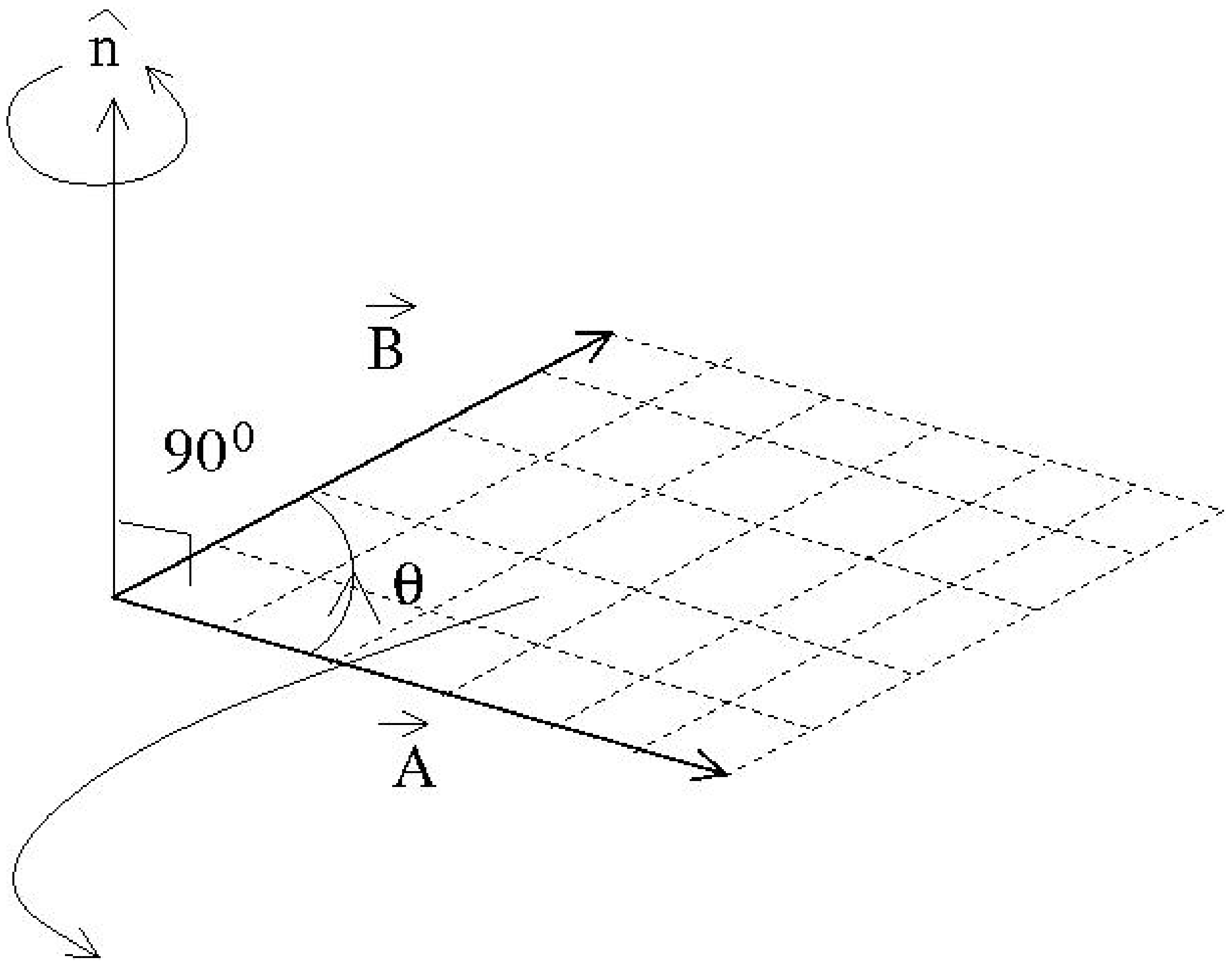




$$\vec{A}\cdot\vec{B}=|A|\left(|B|\cos\theta\right)=\left(|A|\cos\theta\right)|B|=A\,B\cos\theta\,.$$

- So, it is really a scalar multiplication (multiplying a number by another number), also known as scaling. The result is a scalar.

Polar Form of the Cross Product (Geometrical Interpretation)

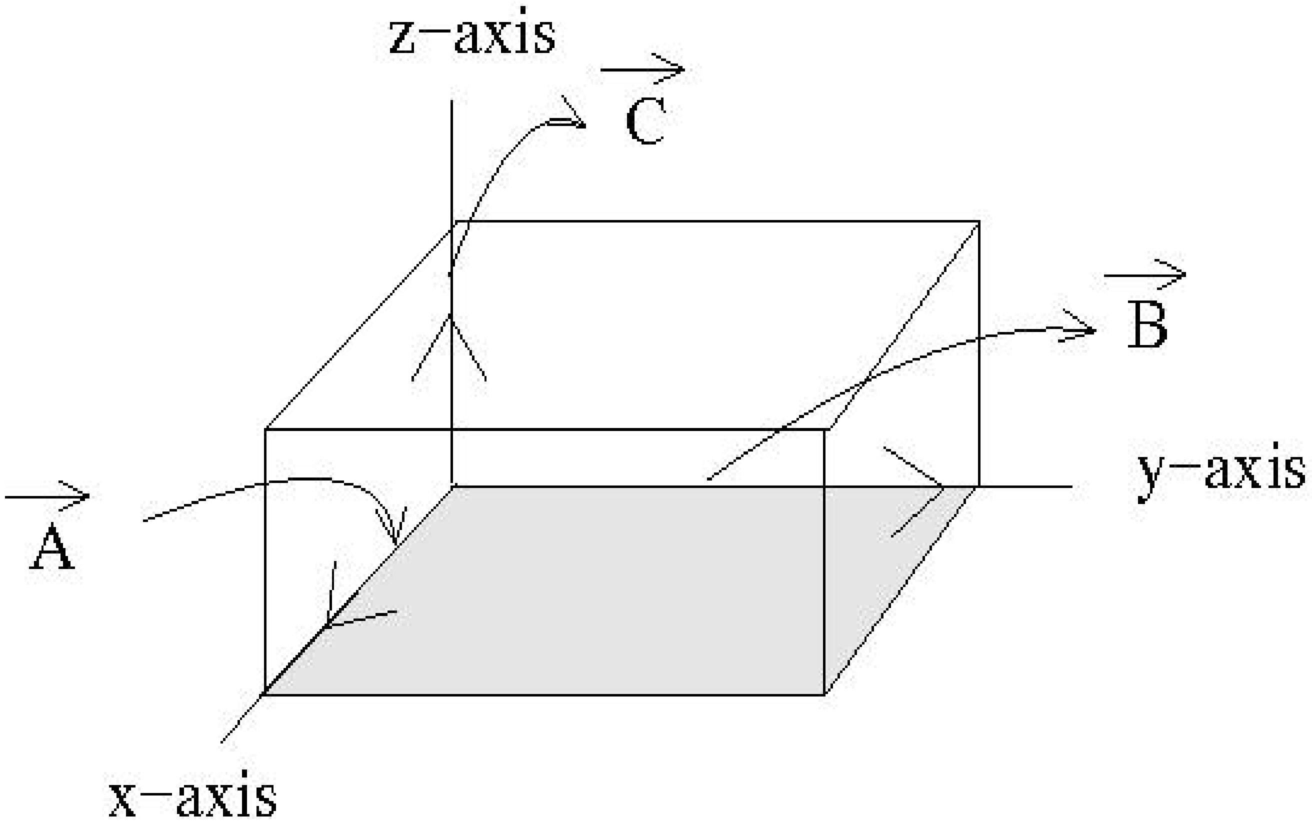


$$\vec{A}\times\vec{B}=|\vec{A}||\vec{B}|\sin\theta\,\hat{n}$$
$$=(AB\sin\theta)\,\hat{n}$$
$$=(\text{Area of the Parallelogran})\,\hat{n}$$

Area of the Parallelogram $= AB\sin\theta$

Note that:
A surface or plane is a vector quantity
The area is the magnitude of the surface and the direction of the surface or plane is given by $\hat{n}$ which is known as the normal unit vector.

Volume or Scalar Triple product:



$$\text{Area of the bottom face} = |\vec{A}\times\vec{B}|$$
$$= AB\sin 90^\circ = AB$$

Volume of the parallelopiped

$$= \vec{C}\cdot(\vec{A}\times\vec{B})$$
$$= \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\rightarrow$ Scalar quantity

Using the properties of Determinant, it is to show that:

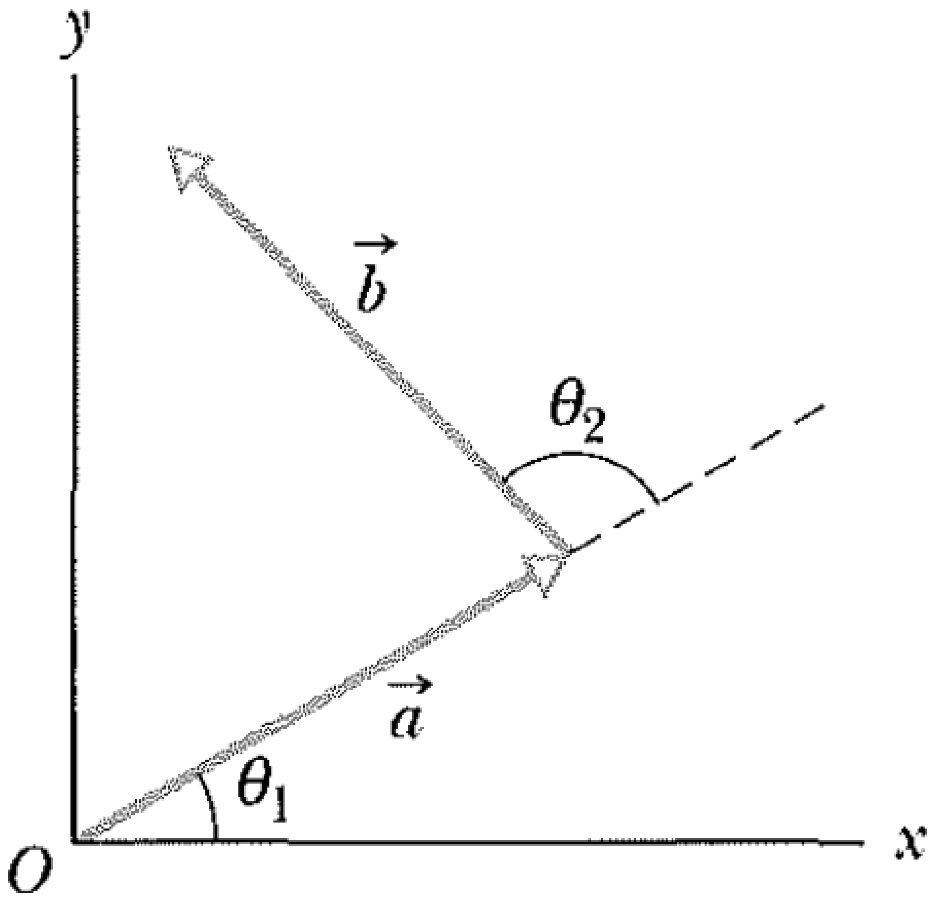
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}).$$

- Example: Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 and 4 units respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if the magnitude of the vector product $\vec{C}\times\vec{D}$ is zero? 12 units?

- By definition of the cross product: $\left|\vec{C}\times\vec{D}\right| = CD\sin\theta$
  Applying the given values, we obtain
  $$0 = (3)(4)\sin\theta \rightarrow \theta = 0, 180^\circ$$
  These are parallel or anti-parallel vectors.

- For the 2nd case, applying the given values, we obtain
  $$12 = (3)(4)\sin\theta \rightarrow \theta = \pi/2 \text{ rad or } 90^\circ$$
  These are orthogonal or perpendicular vectors.

Example: The two vectors **a** and **b** in Figure have equal magnitudes of 10.0 m and the angles are $\theta_1 = 30^O$ and $\theta_2 = 105^O$. Find the (a) *x* and (b) y components of their vector sum ***r***, (c) the magnitude of **r**, and (d) the angle **r** makes with the positive direction of the *x* axis.



- **Solution:**

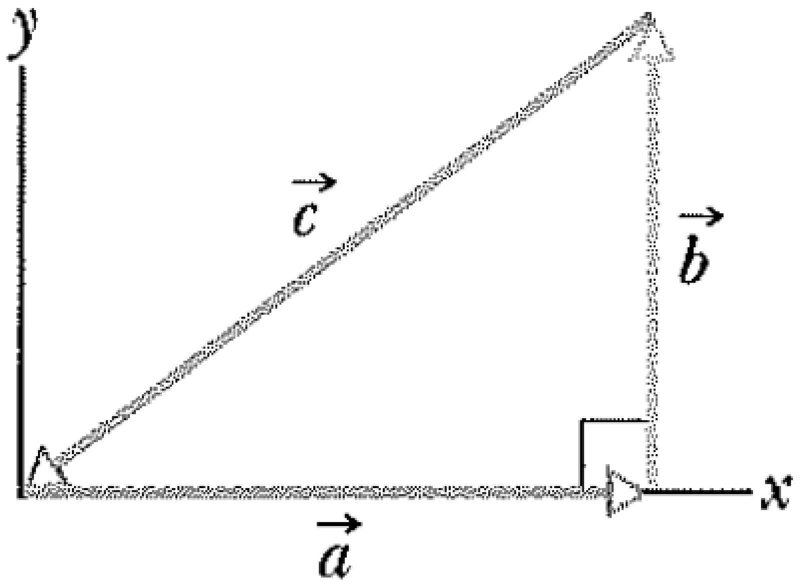
$$r_x = a_x + b_x = a\cos\theta_1 + b\cos(\theta_1 + \theta_2)$$
$$= (10\cos 30^O + 10\cos 135^O)m = 1.59\,\text{m}.$$

$$r_y = a_y + b_y = a\sin\theta_1 + b\sin(\theta_1 + \theta_2)$$
$$= (10\sin 30^O + 10\sin 135^O)m = 12.1\,\text{m}.$$

Therefore, by Pythagorean Theorem,

$$r = \sqrt{r_x^2 + r_y^2} = \sqrt{(1.59)^2 + (12.1)^2}\,\text{m} = 12.2\,\text{m}.$$
$$\theta_r = \tan^{-1}(r_y / r_x) = \tan^{-1}(12.1/1.59) = 82.5^O.$$

Example: For the vectors in the figure below, with a = 4, b = 3, and c = 5, what are (a) the magnitude and (b) the direction of ***a*** x ***b***, (c) the magnitude and (d) the direction of ***a*** x **c**, and (e) the magnitude and (f) the direction of ***b*** x **c**? (The z axis is not shown, but it is perpendicular to the page and outward).



- **Solution:**

From the diagram, the given vectors are:

$$\vec{a}=4\hat{i}=(4,0)=(4,0^{O}),$$
$$\vec{b}=3\hat{j}=(0,3)=(3,90^{O}),$$
$$\vec{c}=-4\hat{i}-3\hat{j}=(4,3)=(5,217^{O}).$$

Firstly, $\vec{a}\times\vec{b}=(4\hat{i})\times(3\hat{j})=12(\hat{i}\times\hat{j})=12\hat{k}$.

Therefore, (a) magnitude of $\vec{a}\times\vec{b}=12$ and

(b) Direction of $\vec{a}\times\vec{b}=\hat{k}$.

Similarly, $\vec{b}\times\vec{c}=(3\hat{j})\times(-4\hat{i}-3\hat{j})$

$$=(-12\hat{j}\times\hat{i})+(-9\hat{j}\times j)$$
$$=-12(-\hat{k})+(-9)\times 0=12\hat{k}.$$

Therefore, (e) magnitude of $\vec{b}\times\vec{c}=12$ and

(f) Direction of $\vec{b}\times\vec{c}=\hat{k}$.