Complex zeros

A variable in the complex number system is refferred to as a complex variable.

A complex polynomial function f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1} \ldots a_0$ are complex number, $a_n \neq 0$, n is a nonnegative integer and x is complex variable.

a_n is called the leading coefficient of f. A complex number r is called a complex zero of f if $f(r)=0$

Fundamental theorem of Algebra:

Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

Conjugate pairs theorem:

Let $f(x)$ be a polynomial function whose coefficients are real numbers. If $r = a+bi$ is a zero of f, the complex conjugate $\bar{r} = a-bi$ is also a zero of f.

Find the complex zeros of a Polynomial Function

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

write f in factored form.

Solution

Step-1 : The degree of f is 4. So f will have four complex zeros.

Step-2 : The potential rational zeros are $\quad | a_0 = -18$, $a_4 = 3$

$\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18.$

Test 1 first : $f(1) = 3 + 5 + 25 + 45 - 18 = 60 \neq 0$

Test -1 : $f(-1) = 3 - 5 + 25 - 45 - 18 = -40 \neq 0$

Test 2 : $f(2) = 3.16 + 40 + 100 + 90 - 18 = 260 \neq 0$

Test -2 : $f(-2) = 48 - 40 + 100 - 90 - 18 = 0$

Since $f(-2) = 0$, then -2 is a zero and $x+2$ is a factor of f.

$\therefore f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$

$= 3x^4 + 6x^3 - x^3 - 2x^2 + 27x^2 + 54x - 9x - 18$

$= 3x^3(x+2) - x^2(x+2) + 27x(x+2) - 9(x+2)$

$= (x+2)(3x^3 - x^2 + 27x - 9)$

Here the depressed eqn can be factored by grouping.

$3x^3 + 27x - x^2 - 9 = 3x(x^2+9) - 1(x^2+9)$

$= (3x-1)(x^2+9) = 0$

$\therefore 3x - 1 = 0 \quad$ or $\quad x^2 + 9 = 0$

$x = \frac{1}{3}$ or $x^2 = -9$

$\Rightarrow x = -3i, \; x = 3i$

The four complex zeros are $\{-3i, 3i, -2, \frac{1}{3}\}$

The factored form of f is

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

$$= 3(x+3i)(x-3i)(x+2)\left(x-\frac{1}{3}\right)$$

Finding a polynomial function f of degree 4 whose coefficients are real numbers that has zeros 1, 1 and $-4+i$.

Soln: Since $-4+i$ is a zero, by the conjugate pairs $-4-i$ must also be a zero of f. Because of the factor theorem if $f(c)=0$, then $x-c$ is a factor of $f(x)$. So we can now write f as

$$f(x) = a(x-1)(x-1)[x-(-4+i)][x-(-4-i)]$$

where a is any real number. Then

$$f(x) = a(x-1)^2[x-(-4+i)][x-(-4-i)]$$

$$= a(x^2-2x+1)[x^2-(-4+i)x-(-4-i)x+(-4)^2-i^2]$$

$$= a(x^2-2x+1)[x^2+4x-ix+4x+ix+16+1]$$

$$= a(x^2-2x+1)(x^2+8x+17)$$

$$= a(x^4+6x^3+2x^2-26x+17)$$

Exercise set: 4.6 $\Rightarrow$ 17–40

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Composite Functions

Given two functions f and g, the composite function, denoted by $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

Example: 1

$f(x) = 2x^2 - 3 \quad g(x) = 4x$. Find

a. $(f \circ g)(1)$ b. $(g \circ f)(1)$ c. $(f \circ f)(-2)$ d. $(g \circ g)(-1)$

Sol^n:

$$\begin{aligned} f \circ g &= f(g(x)) \\ &= f(4x) \\ &= 2(4x)^2 - 3 \\ &= 32x^2 - 3 \end{aligned}$$

$\therefore (f \circ g)(1) = 32 - 3 = 29$

$$\begin{aligned} (f \circ g)(1) &= f(g(1)) \\ &= f(4) \\ &= 2 \cdot 4^2 - 3 \\ &= 32 - 3 \\ &= 29 \end{aligned}$$

Example: 2

If $f(x) = x^2 + 3x - 1$ and $g(x) = 2x + 3$. Then find $f \circ g$ and $g \circ f$. Then find the domain of each composite function.

Sol^n: The domain of f and g are the set of all real numbers.

1. $f \circ g = f(g(x)) = f(2x+3)$

$$\begin{aligned} &= (2x+3)^2 + 3(2x+3) - 1 \\ &= 4x^2 + 12x + 9 + 6x + 9 - 1 \\ &= 4x^2 + 18x + 17 \end{aligned}$$

The domain of $f \circ g$ is also set of all real numbers.

$g \circ f = ?$

Example: 3

Find the domain of $f \circ g$ if $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{4}{x-1}$

Soln: The domain of $f(x) = \frac{1}{x+2}$ is $\{x \mid x \neq -2\}$

The domain of $g(x) = \frac{4}{x-1}$ is $\{x \mid x \neq 1\}$

$$(f \circ g)(x) = f(g(x))$$

$$= f\left(\frac{4}{x-1}\right)$$

$$= \frac{1}{\frac{4}{x-1} + 2} = \frac{1}{\frac{4+2(x-1)}{x-1}} = \frac{x-1}{4+2x-2}$$

$$= \frac{x-1}{2x+2} = \frac{x-1}{2(x+1)}$$

The domain of $f \circ g$ is $\{x \mid x \neq -1, x \neq 1\}$

Note: To determine the domain of composite function, keep the following thoughts in mind:

1. Any x not in the domain of g must be excluded.
2. Any x for which $g(x)$ is not in the domain of f must be excluded.

b. $(f \circ f)(x) = f(f(x))$

$= f\left(\frac{1}{x+2}\right)$

$= \dfrac{1}{\frac{1}{x+2}+2} = \dfrac{x+2}{1+2(x+2)} = \dfrac{x+2}{2x+5}$

Domain $= \{x \mid x = -5/2,\ x \neq -2\}$

Ex: $f(x) = \frac{3}{x-1}$, $g(x) = \frac{2}{x}$ what is the domain of $f \circ g$?

Example:

If $f(x) = 3x-4$ and $g(x) = \frac{1}{3}(x+4)$, show that

$(f \circ g)(x) = (g \circ f)(x) = x$

soln: $(f \circ g)(x) = f(g(x))$

$= f\left(\frac{x+4}{3}\right)$

$= 3\left(\frac{x+4}{3}\right) - 4 = x+4-4 = x$

$(g \circ f)(x) = g\,f(x))$

$= g(3x-4)$

$= \frac{1}{3}(3x-4+4)$

$= \frac{1}{3} \cdot 3x = x$

$\therefore f \circ g)(x) = (g \circ f)(x) = x$ (Showed)

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Example:

Find functions f and g such that $f \circ g = (x^2+1)^{50}$

Soln:

Inside function is $g(x)$ and outside function is $f(x)$

since $f \circ g(x) = f(g(x))$

So here $g(x) = x^2+1$ and $f(x) = x^{50}$

Example:

If $f \circ g = \sqrt{x+1}$, ~~that~~ then $g(x) = x+1$ and $f(x) = \sqrt{x}$

Example:

If $f \circ g = \frac{1}{x+1}$ then $g(x) = x+1$ and $f(x) = \frac{1}{x}$.