Day-2, 17th February, 2021

Section 7.2 Integration by Parts

Recall Last Lecture: 7.1 Integration by Substitution

Example: Integration by substitutions works $\int x^2(2+4x^3)^{12}\,dx = \frac{1}{12}\int 12\,x^2(2+4x^3)^{12}\,dx$,

Integration by Substitution does not work for $\int x\,(2+4x^3)^{12}\,dx$.

Please practice following exercises from Chapter 7:

Section 7.1: All odd numbered exercises

Section 7.2: 11, 15, 19, 23, 31, 37, 60, 64

Section 7.3: 7, 9, 15, 19, 21, 23, 25, 31, 33, 43, 45 – 51 (odd), 57

Section 7.4: 3, 5, 7, 11, 15, 19, 21, 23, 33, 35*, 37, 41, 47

Section 7.5: 9, 13, 17, 21, 23, 25, 39*, 41*

Section 7.8: 3, 4, 5, 10, 15, 17, 23

Section 7.2: Integration by Parts [Backward Product Rule]

Works to integrate a product of two functions where one is a differential function. For example,

$$\int f(x)\,\cdot g'(x)\,dx$$

Recall Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

We get,

$$f(x)g'(x) = \frac{d}{dx}[f(x)g(x)] - f'(x)g(x)$$

$$\int f(x)\,\cdot g'(x)\,dx = \int \left[\frac{d}{dx}[f(x)g(x)] - f'(x)g(x)\right]\,dx$$

$$= \int \frac{d}{dx}[f(x)g(x)]\,\,dx - \int f'(x)g(x)\,\,dx = f(x)g(x) - \int f'(x)g(x)\,\,dx$$

$$\int f(x) \cdot g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$$

If we set $u = f(x)$, and $dv = g'(x)dx$, then

$\frac{du}{dx} = f'(x) \Rightarrow du = f'(x)dx$, and $\int 1\, dv = \int g'(x)dx \Rightarrow v = g(x) + 0$;

[Note: C must be 0 for this method]

Then from

$\int f(x) \cdot g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$ we get $\int u\, dv = uv - \int v\, du$

Formulas:

Indefinite Integral: $\int u\, dv = uv - \int v\, du \; \ldots\ldots\ldots (1)$

Definite Integral: $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du \; \ldots\ldots\ldots (2)$

Fundamental Theorem of Calculus PARTY-II

If $\int f(x)\, dx = \boldsymbol{F(x)} + C$ over the closed interval $[a, b]$, then

$$\int_a^b f(x)\, dx = [F(x) + C]_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a).$$

LIATE

L → Logarithmic Function

I → Inverse Trigonometric Function

A → Algebraic Function, means radicals, polynomials, etc.

T → Trigonometric Function

E → Exponential Function

This order helps you to choose u and dv. **The function which comes first in this order is set up as $\boldsymbol{u}$.**

Exercises:

1) **(a/i)** $\int \boldsymbol{x^2 \ln x\, dx}$

Set $u = \ln x$, $dv = x^2 dx$

$\int x^2 \ln x\, dx = \int \boldsymbol{u\, dv} = \boldsymbol{uv - \int v\, du} \; \ldots\ldots\ldots \boldsymbol{(1)}$

Now, $u = \ln x \;\Rightarrow \frac{du}{dx} = \frac{1}{x}$, that is, $du = \frac{1}{x} dx$

and $dv = x^2 dx \;\Rightarrow \int 1\, dv = \int x^2\, dx$, that is, $v = \frac{x^3}{3}$; [For this method, the constant $C = 0$]

So, $\int x^2 \ln x \; dx = \int \boldsymbol{u\, dv} = \boldsymbol{uv} - \int \boldsymbol{v\, du}$

$$= \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \frac{1}{x}\, dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 \; dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C$$

$$\int x^2 \ln x \; dx = \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C.$$

(a/ii) Evaluate

$$\int x^4 \ln x \; dx.$$

Solution: Set $u = \ln x$ and $dv = x^4 dx$. Then we get,

$$\int x^4 \ln x \; dx = \int u\, dv = uv - \int v \; du \ldots\ldots\ldots (1)$$

$\frac{du}{dx} = \frac{1}{x}$, that is, $du = \frac{1}{x} dx$ and $dv = x^4 dx \Rightarrow \int 1 \; dv = \int x^4 \; dx$, that is, $v = \frac{x^5}{5} + 0$

From equation (1):

$$\int x^4 \ln x \; dx$$

$$= \ln x \cdot \frac{x^5}{5} - \int \frac{x^5}{5} \cdot \frac{1}{x} \; dx$$

$$= \frac{1}{5} x^5 \ln x - \frac{1}{5} \int x^4\, dx$$

$$= \frac{1}{5} x^5 \ln x - \frac{1}{5} \frac{x^5}{5} + C$$

$$= \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + C$$

(b) **Evaluate**

$$\int x\, e^{-3x} \; dx.$$

Set $u = x$ and $dv = e^{-3x}\,dx.$

$$\int x\,e^{-3x}\,dx$$
$$= \int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{v\,du} \quad \ldots\ldots\ldots(\mathbf{1})$$

Then $du = dx$, $\quad v = -\frac{1}{3}e^{-3x}$ $\quad$; $\quad$ [Formula $\int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + C$]

From (1):

$$\int x\,e^{-3x}\,dx = \boldsymbol{x} \cdot \frac{\boldsymbol{e^{-3x}}}{\boldsymbol{-3}} - \int \frac{\boldsymbol{e^{-3x}}}{\boldsymbol{-3}}\,\boldsymbol{dx}$$

$$= \boldsymbol{x} \cdot \frac{\boldsymbol{e^{-3x}}}{\boldsymbol{-3}} + \frac{\mathbf{1}}{\mathbf{3}}\int \boldsymbol{e^{-3x}\,dx}$$

$$= \boldsymbol{x} \cdot \frac{\boldsymbol{e^{-3x}}}{\boldsymbol{-3}} + \frac{\mathbf{1}}{\mathbf{3}}\,\frac{\boldsymbol{e^{-3x}}}{\boldsymbol{-3}} + \boldsymbol{C}$$

$$= -\frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{x\,e^{-3x}} - \frac{\mathbf{1}}{\mathbf{9}}\,\boldsymbol{e^{-3x}} + \boldsymbol{C}$$

(c) $\quad \int sin^{-1} x\,dx$

<u>Solution:</u> Set $u = sin^{-1} x$, $\quad dv = dx$. Then we get, $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$, $i.e.,$ $\quad \boldsymbol{du = \frac{1}{\sqrt{1-x^2}}\,dx}$

And $dv = dx \Rightarrow \int 1\,dv = \int 1\,dx,\ i.e.,\ \boldsymbol{v = x.}$

$$\int sin^{-1} x\,dx = \int u\,dv = uv - \int v\,du$$

$$= sin^{-1} x \cdot x - \int x\,\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x\,sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}}\,xdx\ ;$$ Set $z = 1 - x^2 \Rightarrow \frac{dz}{dx} = -2x$, then $-\frac{1}{2}dz = xdx.$

$$= x\,sin^{-1} x - \int \frac{1}{\sqrt{z}}\left(-\frac{1}{2}\right)dz$$

$$= x\,sin^{-1} x + \frac{1}{2}\int z^{-\frac{1}{2}}\,dz$$

$$= x\, sin^{-1} x \quad + \frac{1}{2} \frac{z^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= x\, sin^{-1} x \quad + \sqrt{1-x^2} \quad + C$$

(d) Evaluate

$$\int_0^1 sin^{-1} x \;\; dx \,.$$

Solution:

Consider

$$\int sin^{-1} x \;\; dx \,.$$

Then we get

$$\int sin^{-1} x \;\; dx \; = \; x\, sin^{-1} x \quad + \sqrt{1-x^2} \quad + C \,. \quad \text{Taking help from (c).}$$

Finally,

$$\int_0^1 sin^{-1} x \;\; dx = \left[x\, sin^{-1} x \quad + \sqrt{1-x^2} \right]_0^1$$

$$= \; 1(\, sin^{-1} 1\,) \; + \sqrt{1-1} \;\; - [0 + \sqrt{1-0}\,]$$

$$= \frac{\pi}{2} - 1 \,.$$

2) $\int x \tan^{-1} x \; dx$. Here $u = \tan^{-1} x$, $dv = x\, dx$. Then $du = \frac{1}{1+x^2}\, dx$, $v = \frac{x^2}{2}$

$$\int x \tan^{-1} x \; dx$$

$$= \int \boldsymbol{u\, dv} \; = \; \boldsymbol{uv} - \int \boldsymbol{v\, du}$$

$$= \; \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \frac{1}{1+x^2} \, dx$$

$= \frac{1}{2}x^2\tan^{-1}x \; - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$

$= \frac{1}{2}x^2\tan^{-1}x \; - \frac{1}{2}\int \frac{(1+x^2)-1}{1+x^2}\, dx$

$= \frac{1}{2}x^2\tan^{-1}x \; - \frac{1}{2}\int \left[1 - \frac{1}{1+x^2}\right] dx$

$\int x\tan^{-1}x \; dx \; = \frac{1}{2}x^2\tan^{-1}x \; - \frac{1}{2}\, x + \frac{1}{2}\tan^{-1}x \; + C$

<u>Theorem: Fundamental Theorem of Calculus - part-2</u>

If $F(x)$ is the anti-derivative of the function $f(x)$ over the interval $[a, b]$, that is,

$$\int f(x)\; dx = F(x) + C\,, \;\; \text{for} \;\; a \le x \le b,$$

then $\int_a^b f(x)\; dx = [F(x) + C]_a^b = F(b) - F(a).$

<u>Note:</u> By the help of this Theorem, to integrate $\int_0^1 x\tan^{-1}x\; dx$, we can start with $\int x\tan^{-1}x\; dx$.

Since $\int x\tan^{-1}x\; dx \; = \frac{1}{2}x^2\tan^{-1}x \; - \frac{1}{2}\, x + \frac{1}{2}\tan^{-1}x \; + C$, so

$\int_0^1 x\tan^{-1}x\; dx \; = \left[\frac{1}{2}x^2\tan^{-1}x \; - \frac{1}{2}\, x + \frac{1}{2}\tan^{-1}x \; + C\right]_0^1$

$= \frac{1}{2}\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\;\frac{\pi}{4} + C - 0 + 0 - 0 - C = \frac{\pi}{4}$

Note: There will remain no C in the answer of a definite integral.

3) $\int_0^1 x\cot^{-1}x\; dx$ Homework

4) $\int x^2\,(\ln x\,)^2\; dx$

$u = (\ln x)^2 \;\; , \quad dv = x^2 dx \quad \Rightarrow \quad du = \frac{2\ln x}{x}\, dx, \qquad v = \frac{x^3}{3}$

Now,

$\int x^2\,(\ln x\,)^2\; dx$

$= \int \boldsymbol{u\,dv} \; = \boldsymbol{uv} - \int \boldsymbol{v\,du}$

$$= (\ln \boldsymbol{x})^2 \frac{x^3}{3} - \int \frac{x^3}{3} \; (2\ln x)\frac{1}{x}\, dx$$

$= (\ln \boldsymbol{x})^2 \frac{x^3}{3} - \frac{2}{3}\int x^2 \, \ln x \; dx$; [Set $u = \ln x$, $dv = x^2\, dx \;\Rightarrow du = \frac{1}{x}\, dx$, $\quad v = \frac{x^3}{3}$

$$= (\ln \boldsymbol{x})^2 \frac{x^3}{3} - \frac{2}{3}\left[\ln x \;\cdot \frac{x^3}{3} - \int \frac{x^3}{3}\frac{1}{x}\, dx\right]$$

$$= (\ln \boldsymbol{x})^2 \frac{x^3}{3} - \frac{2}{9}x^3 \ln x + \frac{2}{9}\int x^2 \; dx$$

$$= (\ln \boldsymbol{x})^2 \frac{x^3}{3} - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C$$

Exercise: Homework

A) $\int \boldsymbol{x^3 \, e^{-x} \, dx}$ **; Use integration by parts method 3-times OR Tabular method**

B) $\int \boldsymbol{x^2 \sin x \, dx}$ **; Use integration by parts method 2-times OR Tabular method**

Solution : (A) $\int x^3\, e^{-x}\, dx$; Set $\quad u = x^3$, $dv = e^{-x}dx \quad \Rightarrow du = 3x^2 dx$, $\; v = -e^{-x}$

$$\int x^3\, e^{-x}\; dx \;= \int u\, dv \; = uv - \int v\, du$$

$$= x^3\, (-e^{-x}) - \int (-e^{-x}) 3x^2 dx$$

$= -x^3 e^{-x} + 3\int e^{-x} x^2 dx$; [Set $\quad u = x^2$, $dv = e^{-x}dx \quad \Rightarrow du = 2xdx$, $\; v = -e^{-x}$]

$$= -x^3 e^{-x} + 3\left[x^2(-e^{-x}) - \int (-e^{-x})\; 2x\, dx\right]$$

$= -x^3 e^{-x} - 3x^2\, e^{-x} + 6\int x\; e^{-x} \; dx$; [Set $u = x$, $dv = e^{-x}dx \quad \Rightarrow du = dx$, $\; v = -e^{-x}$]

$$= -x^3 e^{-x} - 3x^2\, e^{-x} + 6\left[x(-e^{-x}) - \int (-e^{-x})\; dx\right]$$

$$= -x^3 e^{-x} - 3x^2\, e^{-x} - 6x\, e^{-x} + 6\int e^{-x}\; dx$$

$$\int x^3\, e^{-x}\; dx \quad = -x^3 e^{-x} - 3x^2\, e^{-x} - 6x\, e^{-x} - 6\, e^{-x} + C.$$

(B) Evaluate

$$\int \boldsymbol{x^2 \sin x \, dx}.$$

Solution: $u = x^2, \quad dv = \sin x \ dx \ \Rightarrow \ du = 2x \, dx \ , \quad v = -\cos x$

$$\int x^2 \sin x \, dx = \int u \, dv = uv - \int v \, du$$

$$= \ x^2 \,(-cos\, x) - \int -cos\, x \ \ 2xdx$$

$$= -x^2 \cos x \ + 2 \int xcos\, x \ dx \quad ;$$

$u = x$ and $dv = \cos x \ dx$, that is, $du = dx$ and $v = \sin x$

$$= -x^2 \cos x \ + 2 \left[x \sin x - \int \sin x \ \ dx \right]$$

$$= -x^2 \cos x \ + 2x \sin x \ - 2 \int \sin x \ \ dx$$

$$= -x^2 \cos x \ + 2x \sin x + 2 \cos x + C$$

5) (a) Evaluate

$$\int_0^{\frac{\pi}{2}} e^{2x} \ \sin(3x) \ dx$$

Type of the exercise: If you apply the method integration by parts twice, then you will get back the given integral. We need to set the integral as I.

Solution: To evaluate $\int_0^{\frac{\pi}{2}} e^{2x} \ \sin(3x) \ dx$, we will start with $\int e^{2x} \ \sin(3x) \ dx$.

Set $I = \int e^{2x} \ \sin(3x) \ dx$;

[(i) Set $u = \sin(3x)$, $dv = e^{2x} \, dx \ \Rightarrow du = 3\cos(3x) \, dx \ , \ v = \frac{1}{2} e^{2x}$]

$$= \int u \, dv \ = uv - \int v \, du$$

$$= \ \sin(3x) \ \frac{1}{2} e^{2x} \ - \int \frac{1}{2} e^{2x} \ 3 \cos(3x) \ dx$$

$$= \frac{1}{2} e^{2x} \sin(3x) \ - \frac{3}{2} \int e^{2x} \ \cos(3x) \ dx \, ;$$

[(ii) Set $u = \cos(3x)$, $dv = e^{2x} \, dx \ \Rightarrow du = -3\sin(3x) \, dx \ , \ v = \frac{1}{2} e^{2x}$]

$$= \frac{1}{2} e^{2x} \sin(3x) \; - \frac{3}{2} \left[\cos(3x) \, \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \left(-3 \, \sin(3x)\right) \, dx \right]$$

$$= \frac{1}{2} e^{2x} \sin(3x) \; - \frac{3}{4} \cos(3x) \, e^{2x} - \frac{9}{4} \int e^{2x} \; \sin(3x) \, dx$$

$$\Rightarrow \; I \; = \frac{1}{2} e^{2x} \sin(3x) \; - \frac{3}{4} \cos(3x) \, e^{2x} - \frac{9}{4} \; I$$

$$\Rightarrow I + \frac{9}{4} \; I = \frac{e^{2x}}{4} [2 \sin(3x) \; - 3 \cos(3x)]$$

$$\Rightarrow \; \frac{13}{4} \; I = \frac{e^{2x}}{4} [2 \sin(3x) \; - 3 \cos(3x)]$$

$$\Rightarrow \; I \; = \frac{4}{13} \cdot \frac{e^{2x}}{4} [2 \sin(3x) \; - 3 \cos(3x)] + C$$

$$\therefore \; \int e^{2x} \; \sin(3x) \; dx \; = \frac{e^{2x}}{13} \; [2 \sin(3x) \; - 3 \cos(3x)] + C$$

Now, by the Fundamental Theorem of Calculus,

$$\int_0^{\frac{\pi}{2}} \; e^{2x} \; \sin(3x) \; dx = \; \left[\frac{e^{2x}}{13} [2 \sin(3x) \; - 3 \cos(3x)] + C \right]_0^{\frac{\pi}{2}}$$

$$= \frac{e^{\pi}}{13} \left[2 \sin\left(3 \frac{\pi}{2}\right) \; - 3 \cos\left(3 \frac{\pi}{2}\right) \right] + C - \; \frac{e^0}{13} \; [\, 2 \sin 0 - 3 \cos 0 \,] - C$$

$$= \frac{e^{\pi}}{13} \; [2 \, (-1) - 3(0)] - \frac{1}{13} \; [2 \; (0) - 3(1)]$$

$$= \; \frac{e^{\pi}}{13} \; (-2) + \frac{1}{13} \; (3)$$

$$\boldsymbol{\int_0^{\frac{\pi}{2}} e^{2x} \; \sin(3x) \; dx = \frac{1}{13} [3 - 2e^{\pi}].}$$

[Tabular Method: Self-Study or ask me during the Q and A session]

5) (b) Evaluate

$$\int_0^{\frac{\pi}{2}} \; e^{-x} \; \cos(2x) \; dx.$$

Solution: To evaluate $\int_0^{\frac{\pi}{2}} \; e^{-x} \; \cos(2x) \; dx$, consider first $\int \; e^{-x} \; \cos(2x) \; dx$. Now set,

$$I = \int e^{-x} \cos(2x)\ dx\ ;$$

$$\left[\text{(i) Set } u = \cos(2x) \text{ and } dv = e^{-x}dx \Rightarrow du = -2\sin(2x)dx \text{ and } v = \frac{e^{-x}}{-1} = -e^{-x}\right]$$

$$= \int u\,dv = uv - \int v\,du = \cos(2x)\ (-e^{-x}) - \int -e^{-x}\ (-2\sin(2x))\ dx$$

$$= -e^{-x}\cos(2x) - 2\int e^{-x}\ \sin(2x)\ dx$$

$$\left[\text{(ii) Set } u = \sin(2x) \text{ and } dv = e^{-x}dx \Rightarrow du = 2\cos(2x)dx \text{ and } v = \frac{e^{-x}}{-1} = -e^{-x}\right]$$

$$= -e^{-x}\cos(2x) - 2\left[\sin(2x)\ (-e^{-x}) - \int -e^{-x}\, 2\cos(2x)\ dx\right]$$

$$= -e^{-x}\cos(2x) + 2\,e^{-x}\sin(2x) - 4\int e^{-x}\ \cos(2x)\ dx$$

$$\Rightarrow I = -e^{-x}\cos(2x) + 2\,e^{-x}\sin(2x) - 4I$$

$$\Rightarrow I + 4I = e^{-x}[2\ \sin(2x) - \cos(2x)]$$

$$\Rightarrow 5I = e^{-x}[2\ \sin(2x) - \cos(2x)]$$

$$\Rightarrow I = \frac{e^{-x}}{5}[2\ \sin(2x) - \cos(2x)] + C$$

$$\therefore \int e^{-x}\ \cos(2x)\ dx = \frac{e^{-x}}{5}[2\ \sin(2x) - \cos(2x)] + C$$

Now, for the definite integral, we get

$$\int_0^{\frac{\pi}{2}} e^{-x}\ \cos(2x)\ dx = \left[\frac{e^{-x}}{5}[2\ \sin(2x) - \cos(2x)]\right]_0^{\frac{\pi}{2}}$$

$$= \frac{e^{-\frac{\pi}{2}}}{5}\left[2\ \sin\left(2\frac{\pi}{2}\right) - \cos\left(2\frac{\pi}{2}\right)\right] - \frac{e^0}{5}[2\ \sin(0) - \cos(0)]$$

$$= \frac{e^{-\frac{\pi}{2}}}{5}[2\ \sin(\pi) - \cos(\pi)] - \frac{e^0}{5}[2\ \sin(0) - \cos(0)]$$

$$= \frac{e^{-\frac{\pi}{2}}}{5}[2\,(0) - (-1)] - \frac{1}{5}[2\,(0) - 1]$$

$$= \frac{e^{-\frac{\pi}{2}}}{5} + \frac{1}{5}$$

$$= \frac{1}{5}\left(e^{-\frac{\pi}{2}} + 1\right)$$