Chapter # 02
(The Derivative)

2.3 Introduction to Techniques of Differentiation: In the last section we defined the derivative of a function f as a limit, and we used that limit to calculate a few simple derivatives. In this section we will develop some important theorems that will enable us to calculate derivatives more efficiently.

Theorem: The derivative of a constant function is ***0***; that is, if ***c*** is any real number, then

$$\frac{d}{dx}[c] = 0$$

Theorem (The Power Rule): If ***n*** is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Proof: Let $\boldsymbol{f(x) = x^n}$. Thus, from the definition of a derivative and the binomial formula for expanding the expression $\boldsymbol{(x+h)^n}$, we obtain

$$\begin{aligned}
\frac{d}{dx}[x^n] = f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\
&= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \\
&= nx^{n-1} + 0 + \cdots + 0 + 0 \\
&= nx^{n-1} \quad \blacksquare
\end{aligned}$$

Theorem (Extended Power Rule): If ***r*** is any real number, then

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

Example 3:

$$\frac{d}{dx}[x^{\pi}] = \pi x^{\pi-1}$$
$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$
$$\frac{d}{dw}\left[\frac{1}{w^{100}}\right] = \frac{d}{dw}[w^{-100}] = -100w^{-101} = -\frac{100}{w^{101}}$$
$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$
$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} \blacktriangleleft$$

Theorem (Constant Multiple Rule): If ***f*** is differentiable at ***x*** and ***c*** is any real number, then ***cf*** is also differentiable at ***x*** and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Theorem (Sum and Difference Rules): If ***f*** and ***g*** are differentiable at ***x***, then so are ***f*** + ***g*** and ***f*** − ***g*** and

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$
$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Example 5:

$$\frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}] = 12x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{\sqrt{x} - 2x}{\sqrt{x}}\right] &= \frac{d}{dx}[1 - 2\sqrt{x}\,] \\ &= \frac{d}{dx}[1] - \frac{d}{dx}[2\sqrt{x}\,] = 0 - 2\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{\sqrt{x}}\end{aligned}$$

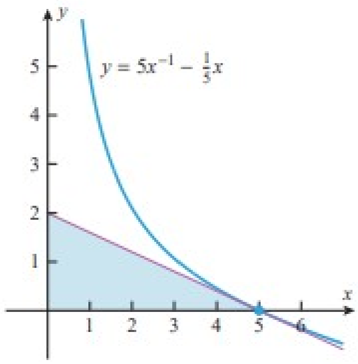
Example 8: Find the area of the triangle formed from the coordinate axes and the tangent line to the curve $\boldsymbol{y = 5x^{-1} - \frac{1}{5}x}$ at the point (5, 0).

Solution: Since the derivative of ***y*** with respect to ***x*** is

$$y'(x) = \frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right] = \frac{d}{dx}[5x^{-1}] - \frac{d}{dx}\left[\frac{1}{5}x\right] = -5x^{-2} - \frac{1}{5}$$

the slope of the tangent line at the point (5, 0) is $\boldsymbol{y'(5) = -\frac{2}{5}}$. Thus, the equation of the tangent line at this point is

$$y - 0 = -\frac{2}{5}(x - 5) \quad \text{or equivalently} \quad y = -\frac{2}{5}x + 2$$



Since the $\boldsymbol{y}$-intercept of this line is 2, the right triangle formed from the coordinate axes and the tangent line has legs of length 5 and 2, so its area is $\boldsymbol{\frac{1}{2}(5)(2) = 5.}$

Higher Derivatives: The derivative $\boldsymbol{f'}$ of a function $\boldsymbol{f}$ is itself a function and hence may have a derivative of its own. If $\boldsymbol{f'}$ is differentiable, then its derivative is denoted by $\boldsymbol{f''}$ and is called the second derivative of $\boldsymbol{f}$. As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of $\boldsymbol{f}$. These successive derivatives are denoted by

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

If $\boldsymbol{y = f(x)}$, then successive derivatives can also be denoted by

$$y', \quad y'', \quad y''', \quad y^{(4)}, \quad y^{(5)}, \ldots$$

Other common notations are

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{d}{dx}[f(x)] \\
y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)] \\
y''' &= \frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)] \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

A general nth order derivative can be denoted by

$$\frac{d^ny}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

and the value of a general ***n***th order derivative at a specific point $x = x_0$ can be denoted by

$$\left.\frac{d^n y}{dx^n}\right|_{x=x_0} = f^{(n)}(x_0) = \left.\frac{d^n}{dx^n}[f(x)]\right|_{x=x_0}$$

Home Work: Exercise 2.3: Problem No. 9-24, 37-48 and 65-68

2.4 The Product and Quotient Rules: In this section we will develop techniques for differentiating products and quotients of functions whose derivatives are known.

Theorem (The Product Rule): If ***f*** and ***g*** are differentiable at ***x***, then so is the product ***f · g***, and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Proof:

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\cdot\frac{f(x+h)-f(x)}{h}\right] \\
&= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \left[\lim_{h\to 0} f(x+h)\right]\frac{d}{dx}[g(x)] + \left[\lim_{h\to 0} g(x)\right]\frac{d}{dx}[f(x)] \\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]
\end{aligned}$$

Example 1: Find $\frac{dy}{dx}$ $if\;\; y = (4x^2 - 1)(7x^3 + x)$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[(4x^2-1)(7x^3+x)] \\
&= (4x^2-1)\frac{d}{dx}[7x^3+x] + (7x^3+x)\frac{d}{dx}[4x^2-1] \\
&= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) = 140x^4 - 9x^2 - 1
\end{aligned}$$

Theorem (The Quotient Rule): If $\boldsymbol{f}$ and $\boldsymbol{g}$ are both differentiable at $\boldsymbol{x}$ and if $\boldsymbol{g(x) \neq 0}$, then $\frac{\boldsymbol{f}}{\boldsymbol{g}}$ is differentiable at $\boldsymbol{x}$ and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example 3: Find $\boldsymbol{y'(x)\ for\ \ y = \frac{x^3+2x^2-1}{x+5}}$

Solution: Applying the quotient rule yields

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3+2x^2-1}{x+5}\right] &= \frac{(x+5)\frac{d}{dx}[x^3+2x^2-1] - (x^3+2x^2-1)\frac{d}{dx}[x+5]}{(x+5)^2} \\ &= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)(1)}{(x+5)^2} \\ &= \frac{(3x^3+19x^2+20x) - (x^3+2x^2-1)}{(x+5)^2} \\ &= \frac{2x^3+17x^2+20x+1}{(x+5)^2}\end{aligned}$$

Home Work: Exercise 2.4: Problem No. 5-20, 29-34