Calculus and Analytical Geometry-I
(Exponential and Logarithmic Function)

0.5 Exponential and Logarithmic Functions: When logarithms were introduced in the seventeenth century as a computational tool, they provided scientists of that period computing power that was previously unimaginable. Although computers and calculators have replaced logarithm tables for numerical calculations, the logarithmic functions have wide-ranging applications in mathematics and science. In this section we will review some properties of exponents and logarithms and then use our work on inverse functions to develop results about exponential and logarithmic functions.

Exponential Functions: A function of the form $\boldsymbol{f(x) = b^x}$, where $\boldsymbol{b > 0}$, $\boldsymbol{b \neq 1}$ is called an exponential function with base $\boldsymbol{b}$. Some examples are

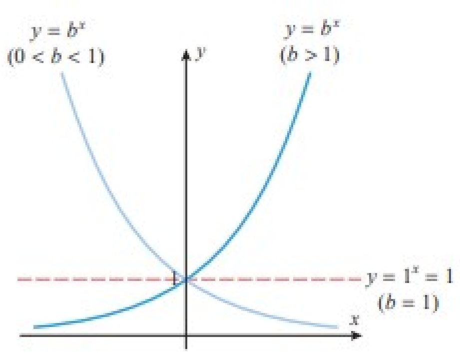
$$f(x) = 2^x, \quad f(x) = \left(\tfrac{1}{2}\right)^x, \quad f(x) = \pi^x$$

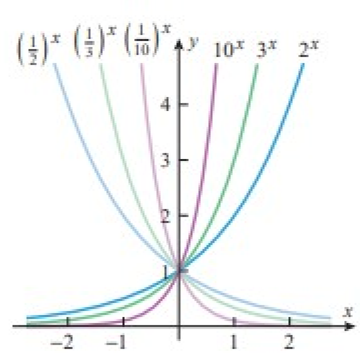
Note that an exponential function has a constant base and variable exponent. Thus, functions such as $f(x) = x^2$ and $f(x) = x^\pi$ would not be classified as exponential functions, since they have a variable base and a constant exponent.

The graph of $y = b^x$ has the following properties:

- The graph passes through (0, 1) because $\boldsymbol{b^0 = 1}$.
- If $\boldsymbol{b > 1}$, the value of $\boldsymbol{b^x}$ increases as $\boldsymbol{x}$ increases. The $\boldsymbol{x}$-axis is a horizontal asymptote of the graph of $\boldsymbol{b^x}$.
- If $\mathbf{0} < b < 1$, the value of $\boldsymbol{b^x}$ decreases as $\boldsymbol{x}$ increases.
- If $\boldsymbol{b = 1}$, then the value of $\boldsymbol{b^x}$ is constant.

Example: The graph of $\boldsymbol{y = b^x}$



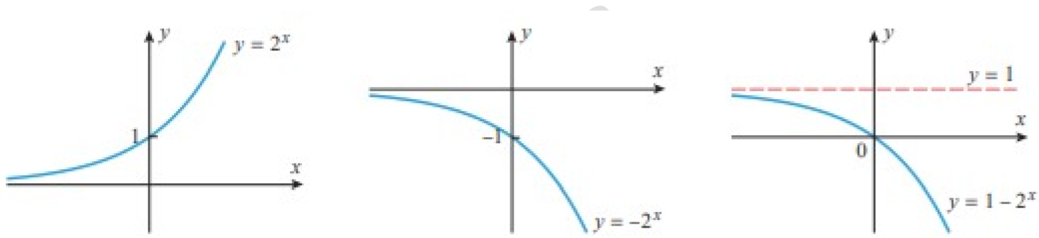


The domain and range of the exponential function $\boldsymbol{f(x) = b^x}$:

- If $\boldsymbol{b > 0}$, then $\boldsymbol{f(x) = b^x}$ is defined and has a real value for every real value of $\boldsymbol{x}$, so the natural domain of every exponential function is $(-\infty, \infty)$.
- If $\boldsymbol{b > 0}$ and $\boldsymbol{b \neq 1}$, then the range of $\boldsymbol{f(x) = b^x}$ is $(\mathbf{0}, \infty)$.

Example 1: Sketch the graph of the function $\boldsymbol{f(x) = 1 - 2^x}$ and find its domain and range.

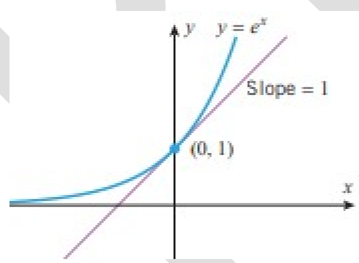
Solution:



Domain: $(-\infty, \infty)$ and **Range:** $(-\infty, 1)$.

The Natural Exponential Function: The function $\boldsymbol{f(x) = e^x}$ is called the natural exponential function. To simplify typography, the natural exponential function is sometimes written as $\mathbf{exp}\,(\boldsymbol{x})$, in which case the relationship $\boldsymbol{e^{x_1+x_2} = e^{x_1}e^{x_2}}$ would be expressed as

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$



Logarithmic Functions: If $\boldsymbol{b} > 0$ *and* $\boldsymbol{b} \neq \mathbf{1}$, then for a positive value of ***x*** the expression

$$\log_b x$$

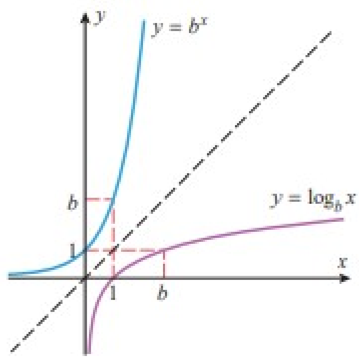
(read "***the logarithm to the base b of x***") denotes that exponent to which ***b*** must be raised to produce ***x***. Thus for example

$$\underset{\boxed{10^2 = 100}}{\log_{10} 100 = 2}, \quad \underset{\boxed{10^{-3} = 1/1000}}{\log_{10}(1/1000) = -3}, \quad \underset{\boxed{2^4 = 16}}{\log_2 16 = 4}, \quad \underset{\boxed{b^0 = 1}}{\log_b 1 = 0}, \quad \underset{\boxed{b^1 = b}}{\log_b b = 1}$$

Definition: The function $\boldsymbol{f(x) = log_b^x}$ is called the logarithmic function with base $\boldsymbol{b}$, where $\boldsymbol{b} > 0 \; and \; b \neq 1,\; x$ is any positive value.

Theorem: If $\boldsymbol{b} > 0 \; and \; b \neq 1$, then $\boldsymbol{b^x}$ and $\boldsymbol{log_b^x}$ are inverse function.

Example: The graphs of $\boldsymbol{y = b^x \; and \; y = log_b^x}$ are reflections of one another about the line $\boldsymbol{y = x}$ (for the case where ***b > 1***).



Correspondence between Properties of Logarithmic and Exponential Functions:

PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

Theorem (Algebraic Properties of Logarithms): If $\boldsymbol{b} > 0, b \neq 1, a > 0, c > 0 \; and \; r$ is any real number, then:

(*a*) $\log_b(ac) = \log_b a + \log_b c$ Product property

(*b*) $\log_b(a/c) = \log_b a - \log_b c$ Quotient property

(*c*) $\log_b(a^r) = r \log_b a$ Power property

(*d*) $\log_b(1/c) = -\log_b c$ Reciprocal property

Example 2: Find $\boldsymbol{x}$ such that

(a) $\log x = \sqrt{2}$ (b) $\ln(x + 1) = 5$ (c) $5^x = 7$

Solution: (*a*) Converting the equation to exponential form yields

$$x = 10^{\sqrt{2}} \approx 25.95$$

(*b*) Converting the equation to exponential form yields

$$x + 1 = e^5 \quad \text{or} \quad x = e^5 - 1 \approx 147.41$$

(*c*) Converting the equation to logarithmic form yields

$$x = \log_5 7 \approx 1.21$$

Change of Base Formula for Logarithms: The following formula expresses a logarithm with base ***b*** in terms of natural logarithms:

$$\log_b x = \frac{\ln x}{\ln b}$$

Home Work: Exercise 0.5: Problem No. 5, 6, 16-23, 24-29, 57 and 58

Chapter # 01
(Limits and Continuity)

The development of calculus in the seventeenth century by ***Newton*** and ***Leibniz*** provided scientists with their first real understanding of what is meant by an "***instantaneous rate of change***" such as velocity and acceleration. Once the idea was understood conceptually, efficient computational methods followed, and science took a quantum leap forward. The fundamental building block on which rates of change rest is the concept of a "***limit***," an idea that is so important that all other calculus concepts are now based on it.

In this chapter we will develop the concept of a ***limit*** in stages, proceeding from an informal, intuitive notion to a precise mathematical definition. We will also develop theorems and procedures for calculating limits, and we will conclude the chapter by using the limits to study "***continuous***" curves.

1.1 Limits: The concept of a "***limit***" is the fundamental building block on which all calculus concepts are based. In this section we will study limits informally, with the goal of developing an intuitive feel for the basic ideas. In the next three sections we will focus on computational methods and precise definitions.

Definition: If the values of $f(x)$ can be made as close as we like to ***L*** by taking values of ***x*** sufficiently close to ***a*** (but not equal to ***a***), then we write

$$\lim_{x \to a} f(x) = L$$

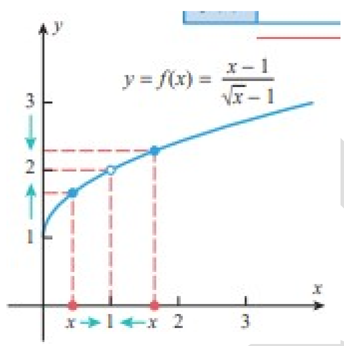
which is read "the limit of $f(x)$ as ***x*** approaches ***a*** is ***L***" or "$f(x)$ approaches ***L*** as ***x*** approaches ***a***." The above expression can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a$$

Example 2: Find the value of

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$$

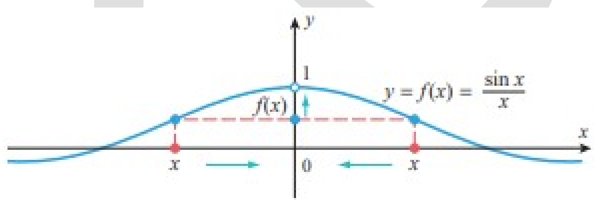
Solution: $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \lim_{x\to 1}(\sqrt{x}+1) = 2$



Example 3: Use numerical evidence to make a conjecture about the value of

$$\lim_{x\to 0} \frac{\sin x}{x}$$

Solution:



Therefore, $\lim_{x\to 0} \frac{\sin x}{x} = 1$

One-Sided Limits: If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

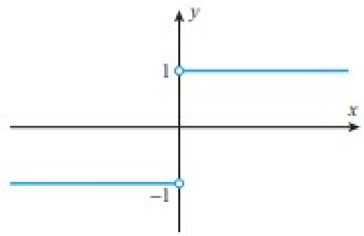
$$\lim_{x\to a^+} f(x) = L$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x\to a^-} f(x) = L$$

Example: Consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



$y = \frac{|x|}{x}$

We denote these limits by writing

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

The relationship between one-sided and two-sided limits: The twosided limit of a function $\boldsymbol{f(x)}$ exists at $\boldsymbol{a}$ if and only if both of the one-sided limits exist at $\boldsymbol{a}$ and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example 4: Explain why

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

Solution: Here,

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Therefore,

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

Since,

$$\lim_{x \to 0^+} \frac{|x|}{x} \neq \lim_{x \to 0^-} \frac{|x|}{x}$$

So, $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Infinite limits: The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty$$

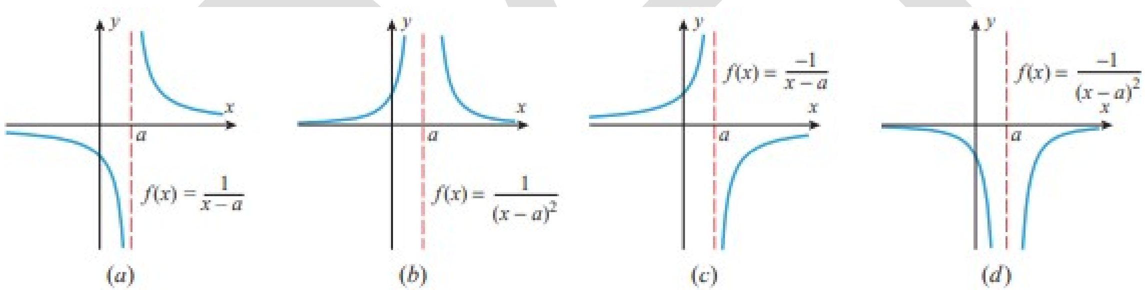
Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$

Example 7: For the functions in following figure, describe the limits at $x = a$ in appropriate limit notation.



Solution: **(*a*)** In Figure **(*a*)**, the function increases without bound as x approaches a from the right and decreases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

(*b*) In Figure **(*b*)**, the function increases without bound as x approaches a from both the left and right. Thus,

$$\lim_{x \to a} \frac{1}{(x-a)^2} = \lim_{x \to a^+} \frac{1}{(x-a)^2} = \lim_{x \to a^-} \frac{1}{(x-a)^2} = +\infty$$

(*c*) In Figure **(*c*)**, the function decreases without bound as x approaches a from the right and increases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{-1}{x-a} = -\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{-1}{x-a} = +\infty$$

(*d*) In Figure **(*d*)**, the function decreases without bound as ***x*** approaches ***a*** from both the left and right. Thus,

$$\lim_{x \to a} \frac{-1}{(x-a)^2} = \lim_{x \to a^+} \frac{-1}{(x-a)^2} = \lim_{x \to a^-} \frac{-1}{(x-a)^2} = -\infty$$

Home Work: Exercise 1.1: Problem No. 3-7 and 13-16