Chapter 4.2

Properties of Rational Functions

4.2.1 Domain of a Rational Function

Example 1 Finding the Domain of a Rational Function

The rational function R is in lowest terms if in $R(x)=\dfrac{p(x)}{q(x)}$, the functions p and q have no common factors.

Example 2

Graphing $y=\dfrac{1}{x^2}$

Example 3 Using Transformations to Graph a Rational Function

Graph the rational function: $R(x)=\dfrac{1}{(x-2)^2}+1$

Definition

Let *R* denote a function.
If, as $x\to-\infty$ or as $x\to+\infty$, the values of $R(x)$ approach some fixed number L, then the line $y=L$ is a **horizontal asymptote** of the graph of *R*.
See **Figures 30(a)** and **Figures 30(b)**.
If, as *x* approaches some number *c*, the values of $|R(x)|\to\infty$, i.e. $R(x)\to-\infty$ or $R(x)\to+\infty$, then the line $x=c$ is a **vertical asymptote** of the graph of *R*.
See **Figures 30(c)** and **Figures 30(d)**.
A horizontal asymptote, when it occurs, describes the **end behavior** of the graph as $x\to-\infty$ or as $x\to+\infty$.
The graph of a function may intersect a horizontal asymptote.
A vertical asymptote, when it occurs, describes the behavior of the graph when *x* is close to some number *c*.
The graph of a rational function will never intersect a vertical asymptote.
If, as $x\to-\infty$ or as $x\to+\infty$, the value of a rational function $R(x)$ approach a linear expression $ax+b, a\neq 0,$ then the line $y=ax+b, a\neq 0,$ is an oblique asymptote of *R*. **Figure 31** shows an oblique asymptote. An oblique asymptote, when it occurs, describes the end behavior of the graph. **The graph of a function may intersect an oblique asymptote.**

4.2.2 Vertical Asymptotes of a Rational Function

Theorem Locating Vertical Asymptotes

A rational function $R(x)=\dfrac{p(x)}{q(x)}$, in lowest terms, will have a vertical asymptote $x=r$ if *r* is a real zero of the *denominator q*, i.e. if $x-r$ is a factor of the denominator *q* of a rational function $R(x)=\dfrac{p(x)}{q(x)}$, in lowest terms, R will have the vertical asymptote $x=r$.

Example 4 Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each of rational function.

(a) $F(x)=\dfrac{x+3}{x-1}$ (b) $R(x)=\dfrac{x}{x^2-4}$ (c) $H(x)=\dfrac{x^2}{x^2+1}$

(d) $H(x)=\dfrac{x^2-9}{x^2+4x-21}$

Solution: See the **Textbook** page 192.

4.2.3 The Horizontal or Oblique Asymptote of a Rational Function

To find horizontal and oblique asymptotes, we need to know how the values of a function behave as $x\to-\infty$ or as $x\to+\infty$, i.e. we need to find the end behavior of the rational function.

If a rational function $R(x)$ is proper, i.e. if the degree of the numerator is less than the degree of the denominator, then as $x\to-\infty$ or as $x\to+\infty$, the value of $R(x)$ approaches 0. Consequently, the line $y=0$ (the x-axis) is a horizontal asymptote of the graph.

Theorem

If a rational function is proper, the line $y=0$ is a horizontal asymptote of its graph.

Example 5 Finding a Horizontal Asymptote

Find the horizontal asymptote, if one exists, of the graph of $R(x)=\dfrac{x-12}{4x^2+x+1}$

Solution: Since the degree of the numerator is less than that of the denominator, the rational function R is proper.

We now investigate the behavior of R as $x\to-\infty$ or as $x\to+\infty$. When $|x|$ is very large, the numerator of R can be approximated by the power function $y=x$, while the denominator of R can be approximated by the power function $y=4x^2$. Applying these ideas to $R(x)$, we get

$$R(x)=\frac{x-12}{4x^2+x+1}\approx\frac{x}{4x^2}=\frac{1}{4x}\to 0$$

as $x\to-\infty$ or as $x\to+\infty$. This shows that the line $y=0$ is the horizontal asymptote of the graph of R.

If a rational function $R(x)=\dfrac{p(x)}{q(x)}$ is **improper**, i.e. if the degree of the numerator is greater than or equal to the degree of the denominator, we use long division to write the rational function as the sum of a polynomial $f(x)$ plus a proper rational function $\dfrac{r(x)}{q(x)}$, in which $r(x)$ is the remainder, i.e. we can write

$$R(x)=\frac{p(x)}{q(x)}=f(x)+\frac{r(x)}{q(x)}$$

where $f(x)$ is a polynomial and $\dfrac{r(x)}{q(x)}$ is a proper rational function.

Since $\dfrac{r(x)}{q(x)}$ is proper, $\dfrac{r(x)}{q(x)}\to 0$ as $x\to-\infty$ or as $x\to+\infty$. As a result, we get

$$R(x)=\frac{p(x)}{q(x)}\to f(x) \text{ as } x\to-\infty \text{ or as } x\to+\infty.$$

We can consider the following possibilities:

1. If $f(x)=b$, a constant, the line $y=b$ is a horizontal asymptote of the graph of R.
2. If $f(x)=ax+b$, $a\neq 0$, the line $y=ax+b$ is an oblique asymptote of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and so there are no horizontal or no oblique asymptotes of the graph of R.

Each of the above possibilities are illustrated in Examples 6, 7 and 8.

Example 6 Finding an Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of $H(x)=\dfrac{3x^4-x^2}{x^3-x^2+1}$

Solution: See the **Textbook**.

Example 7 Finding a Horizontal Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of $R(x)=\dfrac{8x^2-x+2}{4x^2-1}$

Solution: See the **Textbook**.

Note

In Example 7, notice that the quotient 2 that is obtained by long division is the quotient of the leading coefficients of the numerator polynomial and the denominator polynomial. This means that we can avoid the long division process for rational functions where the numerator and denominator are of the same degree. Thus, we can conclude that the quotient of the leading coefficients will give us the horizontal asymptote.

Example 8 Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of $G(x)=\dfrac{2x^5-x^3+2}{x^3-1}$

Solution: See the **Textbook**.

4.2 Assess your understanding

Skill Building

Find the domain of each rational function:

15. $H(x) = \dfrac{-4x^2}{(x-2)(x+4)}$

Solution: The domain of a rational function is the set of all real numbers except where the denominator is zero. Therefore, we get

$$Dom(H) = \{x : x \neq 2,\ x \neq -4\}$$

17. $F(x) = \dfrac{3x(x-1)}{2x^2 - 5x - 3}$

Solution: Rearranging the function, we get $F(x) = \dfrac{3x(x-1)}{2(x+1/2)(x-3)}$

Therefore, $Dom(F) = \{x : x \neq -1/2,\ x \neq 3\}$

19. $R(x) = \dfrac{x}{x^3 - 8}$

Solution: Rearranging the function, we get $R(x) = \dfrac{x}{x^3 - 8} = \dfrac{x}{(x-2)(x^2 + 2x + 4)}$

Since the equation $x^2 + 2x + 4 = 0$ has no real solution, the domain of R is the set of all real numbers except 2, i.e. $Dom(R) = \{x : x \neq 2\}$.

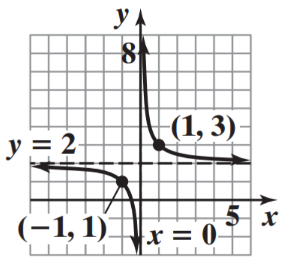
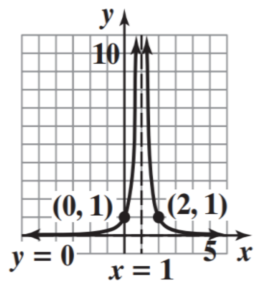
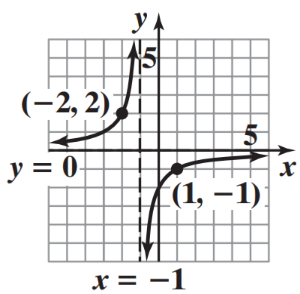
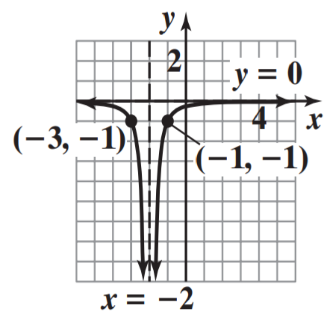
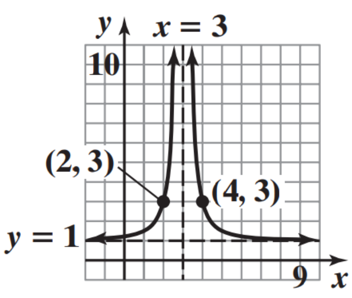
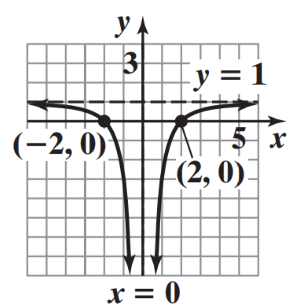
21. $H(x) = \dfrac{3x^2 + x}{x^2 + 4}$

Solution: Since the equation $x^2 + 4 = 0$ has no real solution, the domain of H is the set of all real numbers, i.e. $Dom(H) = \{x : -\infty < x < \infty\}$.

23. $R(x) = \dfrac{3(x^2 - x - 6)}{4(x^2 - 9)}$

Solution: $Dom(R) = \{x : x \neq -3,\ x \neq 3\}$

Graph each rational function using transformations:

31. $F(x)=2+\dfrac{1}{x}$ See Chapter 2.5, Example 12 **31.**  	**33.** $R(x)=\dfrac{1}{(x-1)^2}$ See Chapter 4.2, Example 2, Example 3 **33.** 
35. $H(x)=\dfrac{-2}{x+1}$ See Chapter 2.5, Example 12 **35.**  	**37.** $R(x)=\dfrac{-1}{x^2+4x+4}$ = (-1)(x+2)^(-2) See Chapter 4.2, Example 3 **37.** 
39. $G(x)=1+\dfrac{2}{(x-3)^2}$ See Chapter 4.2, Example 3 **39.**  	**41.** $R(x)=\dfrac{x^2-4}{x^2}$ = 1 – 4x^(-2) See Chapter 4.2, Example 3 **41.** 

Find the vertical, horizontal and oblique asymptotes, if any, of each of the following rational functions:

43. $R(x)=\dfrac{3x}{x+4}$

Solution: We can write $R(x)=\dfrac{3x}{x+4}=\dfrac{3(x+4)-12}{x+4}=3-\dfrac{12}{x+4}$

First Part

Since R is in lowest terms and the only zero of the denominator is -4, the line $x=-4$ is the vertical asymptote of the graph of R.

Second Part

Since the degree of the numerator equals the degree of the denominator, the rational function R is improper.

Therefore, as $x\to-\infty$ or as $x\to+\infty$, $\dfrac{12}{x+4}\approx\dfrac{12}{x}\to 0$ and so $R(x)\to 3$.

This shows that the line $y=3$ is the horizontal asymptote of the graph of R.

45. $H(x)=\dfrac{x^3-8}{x^2-5x+6}$

Solution: We can write $H(x)=\dfrac{x^3-8}{x^2-5x+6}=\dfrac{(x-2)(x^2+2x+4)}{(x-2)(x-3)}=\dfrac{x^2+2x+4}{x-3}$

First Part

Since R is in lowest terms and the only zero of the denominator is 3, the line $x=3$ is the vertical asymptote of the graph of R.

Second Part

Since the degree of the numerator is greater than the degree of the denominator, the rational function R is improper. To find a horizontal or oblique asymptote, we use long division.

$$
\begin{array}{r|l}
 & x+5 \\
x^2-5x+6 & x^3-8 \\
 & \underline{x^3-5x^2+6x} \\
 & \quad 5x^2-6x-8 \\
 & \quad \underline{5x^2-25x+30} \\
 & \qquad 19x-38
\end{array}
$$

As a result, $H(x)=\dfrac{x^3-8}{x^2-5x+6}=(x+5)+\dfrac{19x-38}{x^2-5x+6}$

Then as $x\to-\infty$ or as $x\to+\infty$, $\dfrac{19x-38}{x^2-5x+6}\approx\dfrac{19x}{x^2}=\dfrac{19}{x}\to 0$ and so $H(x)\to x+5$.

Hence we can conclude that the line $y=x+5$ is the oblique asymptote of the graph of R.

47. $T(x)=\dfrac{x^3}{x^4-1}$

Solution: We can write $T(x)=\dfrac{x^3}{x^4-1}=\dfrac{x^3}{(x+1)(x-1)(x^2+1)}$

First Part

Since R is in lowest terms and the zeros of the denominator are -1 and 1, the lines $x=-1$ and $x=1$ are the vertical asymptotes of the graph of *R*.

Second Part

Since R is proper, the line $y=0$ is the horizontal asymptote of the graph of R because as $x\to-\infty$ or as $x\to+\infty$, $\dfrac{x^3}{x^4-1}\approx\dfrac{x^3}{x^4}=\dfrac{1}{x}\to 0.$

49. $Q(x)=\dfrac{2x^2-5x-12}{3x^2-11x-4}$

Solution: We can write $Q(x)=\dfrac{2x^2-5x-12}{3x^2-11x-4}=\dfrac{(2x+3)(x-4)}{(3x+1)(x-4)}=\dfrac{2x+3}{3x+1}$

First Part

Since the only zero of the denominator of $Q(x)$ in lowest terms is $\dfrac{-1}{3}$, the line $x=-\dfrac{1}{3}$ is the only vertical asymptote of the graph of *R*.

Second Part

Since the degree of the numerator equals the degree of the denominator, the rational function R is improper. Therefore, as $x\to-\infty$ or as $x\to+\infty$, $Q(x)=\dfrac{2x^2-5x-12}{3x^2-11x-4}\approx\dfrac{2x^2}{3x^2}=\dfrac{2}{3}$ and so $Q(x)\to\dfrac{2}{3}$.

Hence, we can conclude that the line $y=\dfrac{2}{3}$ is the horizontal asymptote of the graph of *R*.

51. $R(x)=\dfrac{6x^2+7x-5}{3x+5}$

53. $G(x)=\dfrac{x^4-1}{x^2-x}$