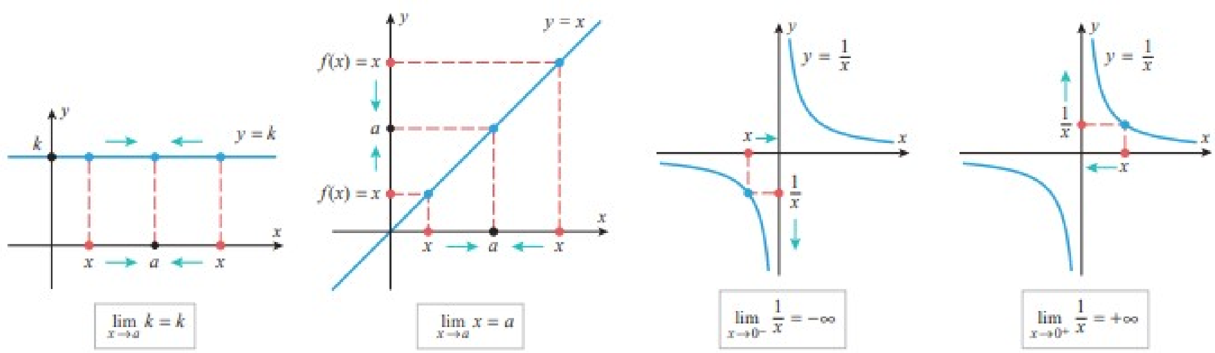
Chapter # 01
(Limits and Continuity)

1.2 Computing Limits: In this section we will discuss techniques for computing limits of many functions. We base these results on the informal development of the limit concept discussed in the preceding section.

Theorem: Let ***a*** and ***k*** be real numbers.

$$(a)\ \lim_{x \to a} k = k \qquad (b)\ \lim_{x \to a} x = a \qquad (c)\ \lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad (d)\ \lim_{x \to 0^+} \frac{1}{x} = +\infty$$



Example:

$$\lim_{x \to -25} 3 = 3, \qquad \lim_{x \to 0} 3 = 3, \qquad \lim_{x \to \pi} 3 = 3$$

$$\lim_{x \to 0} x = 0, \qquad \lim_{x \to -2} x = -2, \qquad \lim_{x \to \pi} x = \pi$$

Theorem: Let ***a*** be a real number, and suppose that

$$\lim_{x \to a} f(x) = L_1 \quad \text{and} \quad \lim_{x \to a} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then:

$$(a)\ \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$$

$$(b)\ \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$$

$$(c)\ \lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$$

$$(d)\ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}, \quad \text{provided } L_2 \neq 0$$

(e) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, *provided* $L_1 > 0$ *if* n *is even.*

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ *or as* $x \to a^+$.

Example 5: Find

$$\lim_{x \to 5} (x^2 - 4x + 3).$$

Solution:

$$\begin{aligned}\lim_{x \to 5} (x^2 - 4x + 3) &= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3\\ &= \lim_{x \to 5} x^2 - 4 \lim_{x \to 5} x + \lim_{x \to 5} 3\\ &= 5^2 - 4(5) + 3\\ &= 8 \end{aligned}$$

Theorem: For any polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number ***a***,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

Example 7: Find

$$\lim_{x \to 2} \frac{5x^3 + 4}{x - 3}.$$

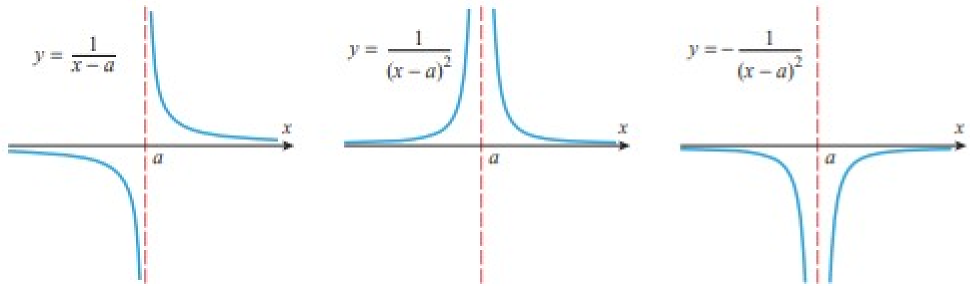
Solution:

$$\begin{aligned}\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} &= \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)}\\ &= \frac{5 \cdot 2^3 + 4}{2 - 3} = -44\end{aligned}$$

Note:

- The limit may be $-\infty$ from one side and $+\infty$ from the other.
- The limit may be $+\infty$.
- The limit may be $-\infty$.

The following figure illustrates these three possibilities graphically for rational functions of the form $\frac{1}{(x-a)}$, $\frac{1}{(x-a)^2}$ $\boldsymbol{and}$ $-\frac{1}{(x-a)^2}$.



$$\lim_{x\to a^+}\frac{1}{x-a}=+\infty \qquad \lim_{x\to a^-}\frac{1}{x-a}=-\infty$$

$$\lim_{x\to a}\frac{1}{(x-a)^2}=+\infty$$

$$\lim_{x\to a}-\frac{1}{(x-a)^2}=-\infty$$

Example 9: Find

$$\text{(a)}\ \lim_{x\to 3}\frac{x^2-6x+9}{x-3} \qquad \text{(b)}\ \lim_{x\to -4}\frac{2x+8}{x^2+x-12} \qquad \text{(c)}\ \lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}$$

Solution: (*a*) The numerator and the denominator both have a zero at $\boldsymbol{x=3}$, so there is a common factor of $(\boldsymbol{x-3})$. Then

$$\lim_{x\to 3}\frac{x^2-6x+9}{x-3}=\lim_{x\to 3}\frac{(x-3)^2}{x-3}=\lim_{x\to 3}(x-3)=0$$

(*b*) The numerator and the denominator both have a zero at $\boldsymbol{x=-4}$, so there is a common factor of $(\boldsymbol{x+4})$. Then

$$\lim_{x\to -4}\frac{2x+8}{x^2+x-12}=\lim_{x\to -4}\frac{2(x+4)}{(x+4)(x-3)}=\lim_{x\to -4}\frac{2}{x-3}=-\frac{2}{7}$$

(*c*) The numerator and the denominator both have a zero at $\boldsymbol{x=5}$, so there is a common factor of $(\boldsymbol{x-5})$. Then

$$\lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}=\lim_{x\to 5}\frac{(x-5)(x+2)}{(x-5)(x-5)}=\lim_{x\to 5}\frac{x+2}{x-5}$$

However,

$$\lim_{x\to 5}(x+2)=7\neq 0 \quad \text{and} \quad \lim_{x\to 5}(x-5)=0$$

so

$$\lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}=\lim_{x\to 5}\frac{x+2}{x-5}$$

does not exist. More precisely,

$$\lim_{x\to 5^+} \frac{x^2-3x-10}{x^2-10x+25} = \lim_{x\to 5^+} \frac{x+2}{x-5} = +\infty$$

And

$$\lim_{x\to 5^-} \frac{x^2-3x-10}{x^2-10x+25} = \lim_{x\to 5^-} \frac{x+2}{x-5} = -\infty$$

Theorem: Let

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let ***a*** be any real number.

(*a*) *If* $q(a) \neq 0$, *then* $\lim_{x\to a} f(x) = f(a)$.

(*b*) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x\to a} f(x)$ *does not exist.*

Limits of Piecewise-Defined Functions: For functions that are defined piecewise, a two-sided limit at a point where the formula changes is best obtained by first finding the one-sided limits at that point.

Example 11: Let

$$f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2-5, & -2 < x \leq 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

Find,

(a) $\lim_{x\to -2} f(x)$ (b) $\lim_{x\to 0} f(x)$ (c) $\lim_{x\to 3} f(x)$

Solution: (*a*) We will determine the stated two-sided limit by first considering the corresponding one-sided limits. For example, as ***x*** approaches ***−2*** from the left, the applicable part of the formula is

$$f(x) = \frac{1}{x+2}$$

and as ***x*** approaches ***−2*** from the right, the applicable part of the formula near ***−2*** is

$$f(x) = x^2 - 5$$

Thus,

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} = -\infty$$
$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 5) = (-2)^2 - 5 = -1$$

from which it follows that $\lim_{x\to -2} f(x)$ does not exist.

(*b*) The applicable part of the formula is $f(x) = x^2 - 5$ on both sides of ***0***, so there is no need to consider one-sided limits here. We see directly that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 5) = 0^2 - 5 = -5$$

(*c*) Using the applicable parts of the formula for $f(x)$, we obtain

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 5) = 3^2 - 5 = 4$$
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x+13} = \sqrt{\lim_{x \to 3^+} (x+13)} = \sqrt{3+13} = 4$$

Since the one-sided limits are equal, we have

$$\lim_{x \to 3} f(x) = 4$$

Home Work: Exercise 1.2: Problem No. 3-32 and 37-40

1.3 Limits at Infinity; End Behavior of a Function: Up to now we have been concerned with limits that describe the behavior of a function $f(x)$ as x approaches some real number a. In this section we will be concerned with the behavior of $f(x)$ as x increases or decreases without bound.

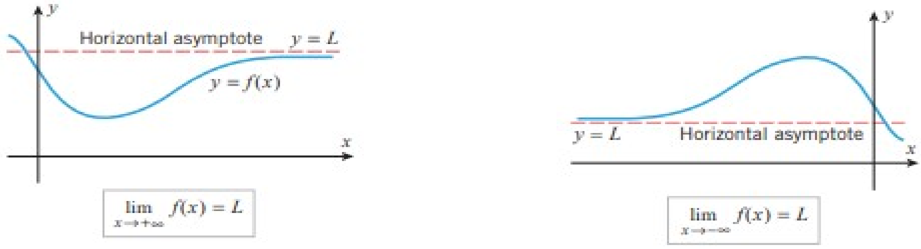
Limits at infinity: If the values of $f(x)$ eventually get as close as we like to a number L as x increases without bound, then we write

$$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to +\infty$$

Similarly, if the values of $f(x)$ eventually get as close as we like to a number L as x decreases without bound, then we write

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to -\infty$$

Example:



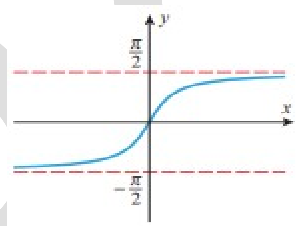
Above figure illustrates the end behavior of a function $\boldsymbol{f}$ when

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

In the first case the graph of $\boldsymbol{f}$ eventually comes as close as we like to the line $\boldsymbol{y = L}$ as $\boldsymbol{x}$ increases without bound, and in the second case it eventually comes as close as we like to the line $\boldsymbol{y = L}$ as $\boldsymbol{x}$ decreases without bound. If either limit holds, we call the line $\boldsymbol{y = L}$ a ***horizontal asymptote*** for the graph of $\boldsymbol{f}$.

Example 2: Consider, $\boldsymbol{f(x) = tan^{-1}x}$. Determine $\boldsymbol{\lim_{x\to\infty} f(x)}$ $\boldsymbol{and}$ $\boldsymbol{\lim_{x\to-\infty} f(x)}$ using graph.

Solution: Given, $\boldsymbol{f(x) = tan^{-1}x}$



Therefore,

$$\lim_{x\to+\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Here the line ***y* = *π*/2** is a horizontal asymptote for $\boldsymbol{f}$ in the positive direction and the line ***y* = −*π*/2** is a horizontal asymptote in the negative direction.

Limit Laws for Limits at Infinity: If ***n*** is a positive integer, then

$$\lim_{x\to+\infty} (f(x))^n = \left(\lim_{x\to+\infty} f(x)\right)^n \qquad \lim_{x\to-\infty} (f(x))^n = \left(\lim_{x\to-\infty} f(x)\right)^n$$

provided the indicated limit of $\boldsymbol{f(x)}$ exists.

$$\lim_{x\to+\infty} kf(x) = k \lim_{x\to+\infty} f(x) \qquad \lim_{x\to-\infty} kf(x) = k \lim_{x\to-\infty} f(x)$$

provided the indicated limit of $\boldsymbol{f(x)}$ exists.

Finally, if $\boldsymbol{f(x) = k}$ is a constant function, then the values of $\boldsymbol{f}$ do not change as $\boldsymbol{x} \to +\infty$ $\boldsymbol{or\ as\ x} \to -\infty$, so

$$\lim_{x \to +\infty} k = k \qquad \lim_{x \to -\infty} k = k$$

Infinite limits at infinity: If the values of $\boldsymbol{f(x)}$ increase without bound as $\boldsymbol{x} \to +\infty$ $\boldsymbol{or\ as\ x} \to -\infty$, then we write

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = +\infty$$

as appropriate; and if the values of $\boldsymbol{f(x)}$ decrease without bound as $\boldsymbol{x} \to +\infty$ $\boldsymbol{or\ as\ x} \to -\infty$, then we write

$$\lim_{x \to +\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

as appropriate.

Example:

$$\lim_{x \to +\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots \qquad \lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases}$$

Limits of Polynomials as $\boldsymbol{x} \to \pm\infty$: There is a useful principle about polynomials which, expressed informally, states: ***The end behavior of a polynomial matches the end behavior of its highest degree term***.

More precisely, if $\boldsymbol{c_n \neq 0}$, then

$$\lim_{x \to -\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to -\infty} c_n x^n$$

$$\lim_{x \to +\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to +\infty} c_n x^n$$

Example 6:

$$\lim_{x \to -\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x \to -\infty} 7x^5 = -\infty$$

$$\lim_{x \to -\infty} (-4x^8 + 17x^3 - 5x + 1) = \lim_{x \to -\infty} -4x^8 = -\infty$$

Limits of Rational Functions as $x \to \pm\infty$: One technique for determining the end behavior of a rational function is to divide each term in the numerator and denominator by the highest power of ***x*** that occurs in the denominator, after which the limiting behavior can be determined using results we have already established.

Example 8: Find

$$\text{(a)} \lim_{x\to-\infty} \frac{4x^2 - x}{2x^3 - 5} \qquad \text{(b)} \lim_{x\to+\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x}$$

Solution: (*a*) Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, namely, $\boldsymbol{x^3}$. We obtain

$$\lim_{x\to-\infty} \frac{4x^2 - x}{2x^3 - 5} = \lim_{x\to-\infty} \frac{\frac{4}{x} - \frac{1}{x^2}}{2 - \frac{5}{x^3}}$$

$$= \frac{\lim\limits_{x\to-\infty}\left(\frac{4}{x} - \frac{1}{x^2}\right)}{\lim\limits_{x\to-\infty}\left(2 - \frac{5}{x^3}\right)}$$

$$= \frac{\lim\limits_{x\to-\infty}\frac{4}{x} - \lim\limits_{x\to-\infty}\frac{1}{x^2}}{\lim\limits_{x\to-\infty} 2 - \lim\limits_{x\to-\infty}\frac{5}{x^3}}$$

$$= \frac{4\lim\limits_{x\to-\infty}\frac{1}{x} - \lim\limits_{x\to-\infty}\frac{1}{x^2}}{2 - 5\lim\limits_{x\to-\infty}\frac{1}{x^3}}$$

$$= \frac{0-0}{2-0} = 0 \qquad \text{(Ans.)}$$

(*b*) Divide each term in the numerator and denominator by the highest power of ***x*** that occurs in the denominator, namely, $\boldsymbol{x^1 = x}$. We obtain

$$\lim_{x\to+\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x} = \lim_{x\to+\infty} \frac{5x^2 - 2x + \frac{1}{x}}{\frac{1}{x} - 3}$$

We have,

$$\lim_{x\to+\infty} 5x^2 - 2x = +\infty, \quad \lim_{x\to+\infty} \frac{1}{x} = 0, \quad \lim_{x\to+\infty}\left(\frac{1}{x} - 3\right) = -3$$

Thus,

$$\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^2-2x+\dfrac{1}{x}}{\dfrac{1}{x}-3}=-\infty$$

A Quick Method for Finding Limits of Rational Functions as $x \to +\infty$ or $x \to -\infty$: Since the end behavior of a polynomial matches the end behavior of its highest degree term, one can reasonably conclude: ***The end behavior of a rational function matches the end behavior of the quotient of the highest degree term in the numerator divided by the highest degree term in the denominator***.

Example 9:

$$\lim_{x\to+\infty}\frac{3x+5}{6x-8}=\lim_{x\to+\infty}\frac{3x}{6x}=\lim_{x\to+\infty}\frac{1}{2}=\frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4x^2}{2x^3}=\lim_{x\to-\infty}\frac{2}{x}=0$$

$$\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^3}{(-3x)}=\lim_{x\to+\infty}\left(-\frac{5}{3}x^2\right)=-\infty$$

Limits Involving Radicals:

Example 10: Find

$$\text{(a)}\ \lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}\qquad\text{(b)}\ \lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$$

Solution: (*a*) As $x \to \infty$, the values of x under consideration are positive, so we can replace $|x|$ by x where helpful. We obtain

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to+\infty}\frac{\dfrac{\sqrt{x^2+2}}{|x|}}{\dfrac{3x-6}{|x|}}=\lim_{x\to+\infty}\frac{\dfrac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\dfrac{3x-6}{x}}$$

$$=\lim_{x\to+\infty}\frac{\sqrt{1+\dfrac{2}{x^2}}}{3-\dfrac{6}{x}}=\frac{\lim\limits_{x\to+\infty}\sqrt{1+\dfrac{2}{x^2}}}{\lim\limits_{x\to+\infty}\left(3-\dfrac{6}{x}\right)}$$

$$=\frac{\sqrt{\lim\limits_{x\to+\infty}\left(1+\dfrac{2}{x^2}\right)}}{\lim\limits_{x\to+\infty}\left(3-\dfrac{6}{x}\right)}=\frac{\sqrt{\left(\lim\limits_{x\to+\infty}1\right)+\left(2\lim\limits_{x\to+\infty}\dfrac{1}{x^2}\right)}}{\left(\lim\limits_{x\to+\infty}3\right)-\left(6\lim\limits_{x\to+\infty}\dfrac{1}{x}\right)}$$

$$=\frac{\sqrt{1+(2\cdot 0)}}{3-(6\cdot 0)}=\frac{1}{3}$$

End Behavior of Trigonometric, Exponential, and Logarithmic Functions: In general, the trigonometric functions fail to have limits as $x \to +\infty$ ***and*** $x \to -\infty$ because of periodicity. There is no specific notation to denote this kind of behavior. But for an exponential and logarithmic functions-

$$\lim_{x \to +\infty} \ln x = +\infty \qquad \lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to 0^+} \ln x = -\infty$$

Home Work: Exercise 1.3: Problem No. 9-40, 47, 48 and 52