Series Solutions of ODEs

Basic Concepts

Consider a second-order homogeneous linear differential equation,

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$
$$\Rightarrow y'' + \frac{a_1(x)}{a_0(x)}y' + \frac{a_2(x)}{a_0(x)}y = 0$$
$$\Rightarrow y'' + P_1(x)y' + P_2(x)y = 0 \qquad (1)$$

If the differential equation has no solution that can be expressible as a finite linear combination of known elementary functions, then we can assume that the solution can be expressible in the form of a power series in $(x - x_0)$,

$$y = f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots\cdots\cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

where $a_0, a_1, a_2, \ldots$ are constants. In eq. (1), if both $P_1(x)$ and $P_2(x)$ are are analytic or defined at the point $x = x_0$, i.e. $a_0(x_0) \neq 0$, then the x_0 is called an ***ordinary point*** of the differential equation (1). Otherwise $x = x_0$ is called a ***singular point*** , then $a_0(x_0) = 0$.

Example:

ODE	Singular points
$(1 - x^2)y'' - 6xy' - 4y = 0$	$x = -1, 1$
$y'' + 2xy' + y = 0$	No singular points
$xy'' + y' + xy = 0$	$x = 0$

Series Solutions of ODEs

Familiar Power Series of $f(x)$ at $x = 0$, known as Maclaurin series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \textbf{geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots .$$

Power series method

-The **power series method** is the standard method for solving linear ODEs with ***scalar*** or ***variable*** coefficients.

Power Series Solutions of ODEs

Example. Solve the ODE using power series: $y' - y = 0$

Solution. Since the equation has no singular points in the finite plane, hence we may expect to find a solution of the form,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots\cdots = \sum_{n=0}^{\infty} a_n x^n \quad (\text{where } x_0 = 0)$$

which is valid for all $x = x_0$ with a_0 is an arbitrary constant. Now, substituting

$$y' = \frac{dy}{dx} = 1.\, a_1 + 2a_2 x + 3a_3 x^2 + \cdots\cdots\cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

into the given ODE we obtain,

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots\cdots\cdots) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots\cdots) = 0$$

$$\Rightarrow (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots\cdots\cdots = 0$$

Now equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0, \qquad \cdots\cdots\cdots\cdots$$

$$\Rightarrow a_1 = a_0, \qquad a_2 = \frac{1}{2} a_1 = \frac{a_0}{2}, \qquad a_3 = \frac{1}{3} a_2 = \frac{a_0}{2 \cdot 3}, \qquad \cdots\cdots\cdots\cdots$$

Thus, the series solution becomes,

$$y = a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{2 \cdot 3} x^3 + \cdots\cdots\cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots \right) = a_0 e^x$$

Power Series Solutions of ODEs

Example. Solve the ODE using power series: $y'' + y = 0$

Solution. Since the equation has no singular points in the finite plane, hence we may expect to find a solution of the form,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots\cdots = \sum_{n=0}^{\infty} a_n x^n \qquad (\text{where } x_0 = 0)$$

which is valid for all $x = x_0$ with a_0 and a_1 arbitrary. Here,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = (2\cdot 1)a_2 + (3\cdot 2)a_3 x + (4\cdot 3)a_4 x^2 + \cdots$$

Now substituting y'' and y into the given ODE we obtain,

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \cdots\ ) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots\cdots) = 0$$

$$\Rightarrow (2a_2 + a_0) + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + \cdots\cdots\cdots = 0$$

Now equating the coefficient of each power of x to zero, we have

$$2a_2 + a_0 = 0, \qquad 6a_3 + a_1 = 0, \qquad 12a_4 + a_2 = 0, \qquad \cdots\cdots\cdots\cdots$$

$$\Rightarrow a_2 = -\frac{1}{2!}a_0, \qquad a_3 = -\frac{1}{6}a_1 = -\frac{a_1}{3!}, \qquad a_4 = -\frac{1}{12}a_2 = \frac{1}{12\cdot 2}a_0 = \frac{1}{4!}a_0, \cdots\cdots\cdots$$

Power Series Solutions of ODEs

Example. Solve the ODE using power series: $y'' + y = 0$

Solution. Thus, the series solution becomes,

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 - \cdots\cdots\cdots$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots\cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\cdots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

Power Series Solutions of ODEs

Example. Solve the following special Legendre near the ordinary point $x = 0$:

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

Solution. Since the equation has singular points at $x_0 = \pm 1$ in the finite plane, hence we may expect to find a solution at the point $x_0 = 0$,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots\cdots = \sum_{n=0}^{\infty} a_n x^n \quad \text{(where } x_0 = 0\text{)}$$

which is valid for $|x| < 1$ and with a_0 and a_1 arbitrary. Here,

$$2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots\cdots\cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow 2xy' = 2\sum_{n=1}^{\infty} n a_n x^n = 2(a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 \ldots)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \Rightarrow (1 - x^2)y'' = (1 - x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow (1 - x^2)y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$= (2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots) - (2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4 + \cdots)$$

$$= 2a_2 + 6a_3 x + (12a_4 - 2a_2)x^2 + (20a_5 - 6a_3)x^3 + \cdots$$

Power Series Solutions of ODEs

Example. Solve the following special Legendre near the ordinary point $x = 0$:

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

Solution. Now substituting $(1 - x^2)y''$ and $2xy'$ into the given ODE we obtain,

$$(2a_2 + 6a_3x + (12a_4 - 2a_2)x^2 + (20a_5 - 6a_3)x^3 + \cdots)$$

$$- (2a_1x + 4a_2x^2 + 6a_3x^3 + 8a_4x^4 \ldots) + (2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + \cdots) = 0$$

$$\Rightarrow (2a_2 + 2a_0) + (6a_3 - 2a_1 + 2a_1)x + (12a_4 - 2a_2 - 4a_2 + 2a_2)x^2$$

$$+(20a_5 - 6a_3 - 6a_3 + 2a_3)x^3 + \cdots\cdots\cdots = 0$$

Now equating the coefficient of each power of x to zero, we have

$$2a_2 + 2a_0 = 0, \qquad 6a_3 = 0, \qquad 12a_4 - 4a_2 = 0, \qquad 20a_5 - 10a_3 = 0, \qquad \cdots\cdots\cdots\cdots$$

$$\Rightarrow a_2 = -a_0, \qquad a_3 = -0, \qquad a_4 = \frac{1}{3}a_2 = -\frac{1}{3}a_0, \qquad a_5 = \frac{1}{2}a_3 = 0, \qquad \cdots\cdots\cdots$$

Thus, the series solution becomes,

$$y = a_0 + a_1x - a_0x^2 - \frac{1}{3}a_0x^4 - \cdots = a_0\left(1 - x^2 - \frac{1}{3}x^4 - \cdots\cdots\cdots\right) + a_1x$$

Power Series Solutions of ODEs

Exercise Problems:

Find a power series solution in powers of x.

10. $y'' - y' + xy = 0$

11. $y'' - y' + x^2y = 0$

12. $(1 - x^2)y'' - 2xy' + 2y = 0$

13. $y'' + (1 + x^2)y = 0$

14. $y'' - 4xy' + (4x^2 - 2)y = 0$