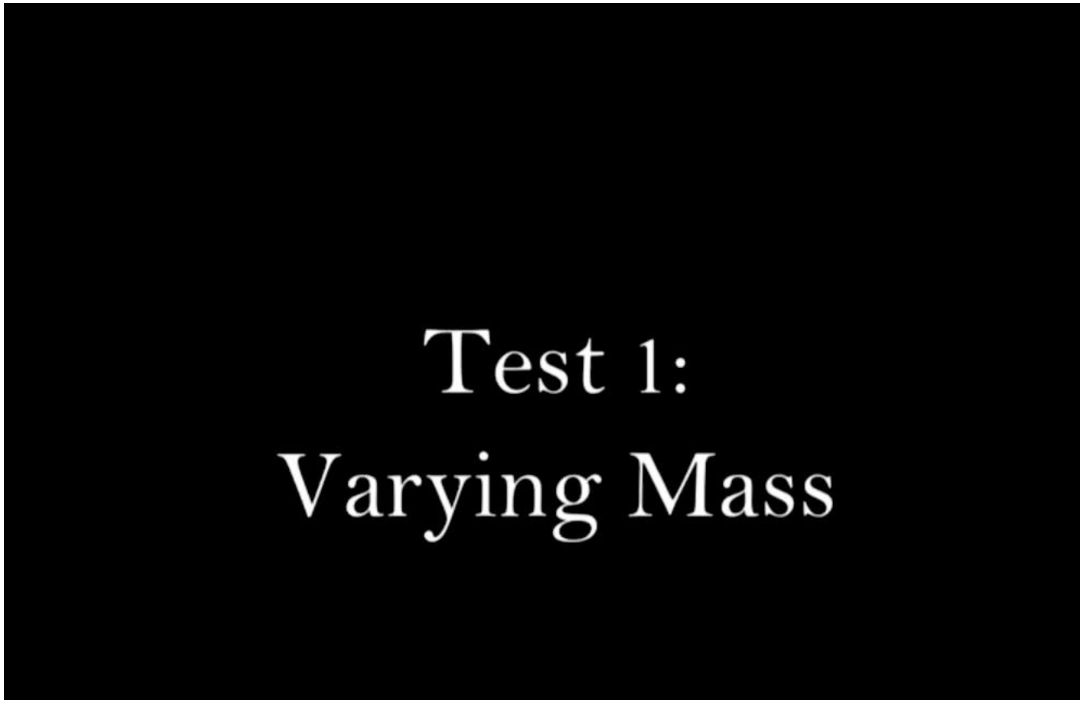
Modelling with Higher Order Linear ODEs

Spring-Mass System:

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0,$$

where m is the mass of the body and k is called the spring constant with the initial conditions $x(0) = x_0, \left.\frac{dx}{dt}\right|_{t=0} = x_1$.



https://www.youtube.com/watch?v=lZPtFDXYQRU

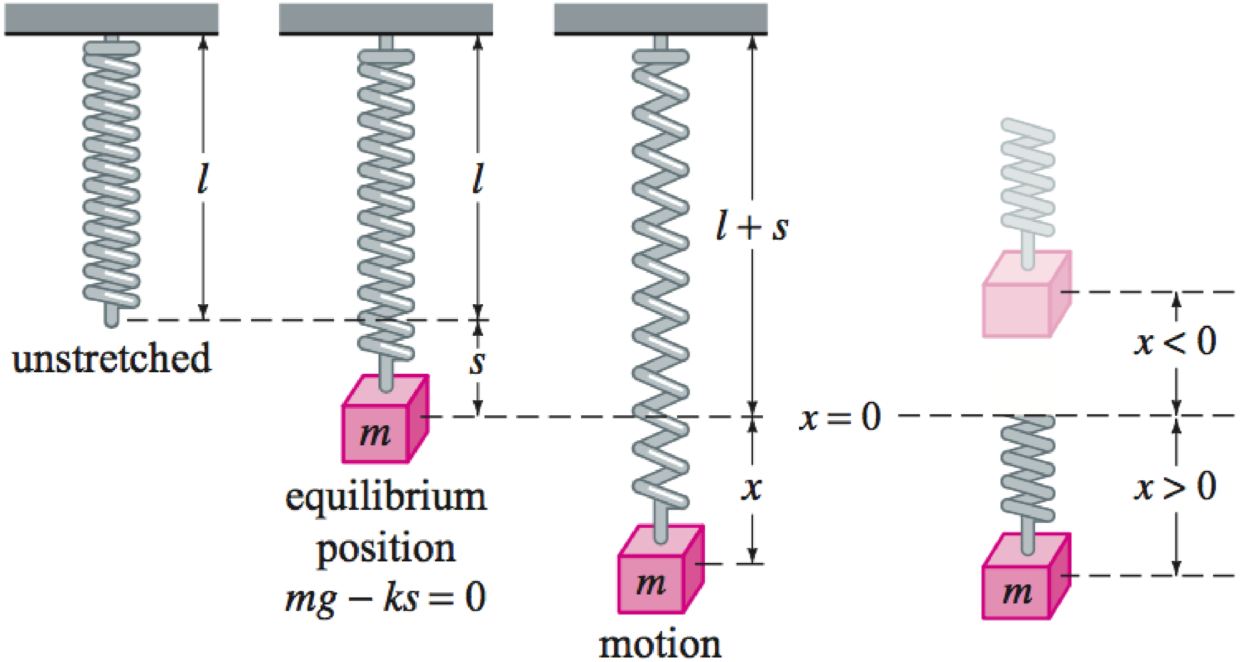
Modelling with Higher Order Linear ODEs

Spring-Mass System:

$$\mathrm{m}\frac{d^2x(t)}{dt^2}+kx(t)=0,$$

where m is the mass of the body and k is called the spring constant with the initial

conditions $x(0)=x_0, \left.\frac{dx}{dt}\right|_{t=0}=x_1$.



Hook's Law: $F=-ks$

$$\Rightarrow |F|=k|s|$$

Where,

F: Restoring force of spring

s: Stretching amount of spring

At equilibrium: $mg=ks$

Newton's second law of motion:

$$m\frac{d^2x}{dt^2}=-k(x+s)+mg$$

$$\Rightarrow m\frac{d^2x}{dt^2}=-kx$$

$$\Rightarrow m\frac{d^2x}{dt^2}+kx=0$$

Modelling with Higher Order Linear ODEs

Spring-Mass System: Simple Harmonic motion/Free Undamped motion

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \left[\omega^2 = \frac{k}{m}\right]$$

With the initial conditions $x(0) = x_0$, $\left.\frac{dx}{dt}\right|_{t=0} = x'(0) = x_1$.

Auxiliary equation: $m^2 + \omega^2 = 0 \Rightarrow m = \pm\omega i$

General solution: $x(t) = c_1 \cos\omega t + c_2 \sin\omega t$

Period of the motion: $T = \frac{2\pi}{\omega}$ which is the length of the time interval between two successive maxima (or minima) of $x(t)$.

Frequency of motion: $f = \frac{1}{T} = \frac{\omega}{2\pi}$ which is the number of cycles completed each second.

Circular frequency: $\omega = \sqrt{\frac{k}{m}}$

Modelling with Higher Order Linear ODEs

Spring-Mass System: Simple Harmonic motion/Free Undamped motion

Example. A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of motion.

Solution. Given that, the body weight, $W = 2$ lbf;

$$\text{stretching amount, } |s| = |-6| \text{ in.} = \frac{6}{12}\text{ ft} = \frac{1}{2}\text{ ft;}$$

$$\text{initial displacement, } x(0) = 8 \text{ in.} = \frac{8}{12}\text{ ft.} = \frac{2}{3}\text{ ft.;}$$

$$\text{initial velocity, } x'(0) = -\frac{4}{3}\text{ ft/s.}$$

Here, the mass of the body, $m = \frac{W}{g} = \frac{2}{32}\text{lbf.}\frac{\text{s}^2}{\text{ft}} = \frac{1}{16}$ slug.

Also, from Hook's law, $|F| = k|s| \Rightarrow 2 = k.\tfrac{1}{2} \Rightarrow k = 4\frac{\text{lb}}{\text{ft}}$.

Therefore, the circular frequency, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{\frac{1}{16}}} = \sqrt{64} = 8$

Thus the general solution of the simple harmonic motion: $x(t) = c_1 \cos 8t + c_2 \sin 8t$

Applying the initial conditions: $c_1 = \frac{2}{3}$ and $c_2 = -\frac{1}{6}$

The equation of simple harmonic motion: $x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t = A\sin(8t + \phi)$.

Modelling with Higher Order Linear ODEs

Spring-Mass System: Simple Harmonic motion/Free Undamped motion

Example. A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of motion.

Solution. The equation of simple harmonic motion: $x = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t = A\sin(8t + \phi)$.

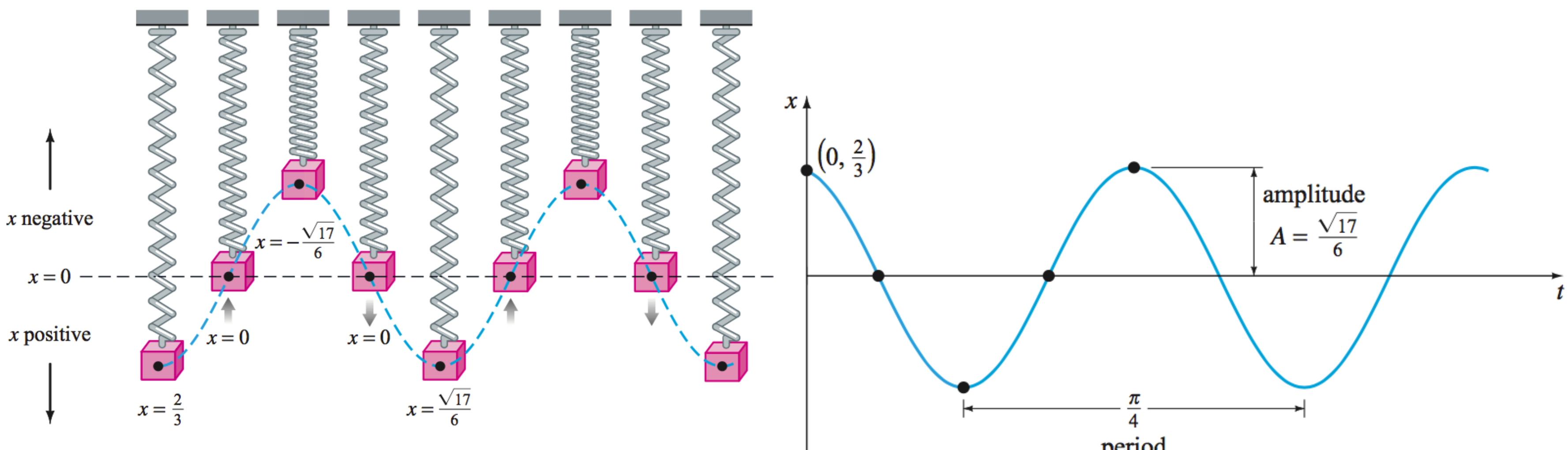
Where, $A = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \frac{\sqrt{17}}{6}$

and $\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right) = \tan^{-1}\left(\frac{2/3}{-1/6}\right) = \pi + \tan^{-1}(-4) = 1.816$ rad.

Thus the solution turns into, $x(t) = \frac{\sqrt{17}}{6}\sin(8t + 1.816)$.

The period of the motion is, $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}\,s$.

Graphical interpretation



Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion

$$m\frac{d^2x}{dt^2} + kx(t) + \beta\frac{dx}{dt} = 0,$$

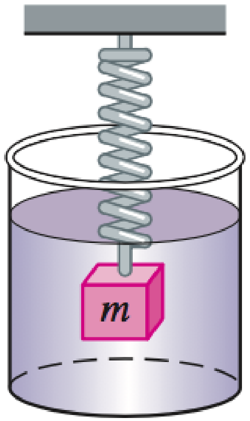
conditions $x(0) = x_0, \left.\frac{dx}{dt}\right|_{t=0} = x_1.$

Hook's Law: $F = -ks$
Where,
F: Restoring force of spring
s: Stretching amount of spring

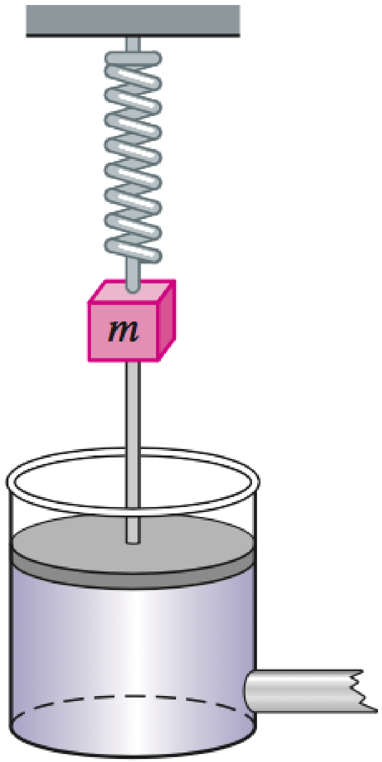
At equilibrium: $mg = ks$
Newton's second law of motion:

$$m\frac{d^2x}{dt^2} = -k(x+s) + mg - \beta\frac{dx}{dt}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$



Mass suspended in a viscous medium



Mass connected to a dashpot damping device.

Where β is a positive *damping constant* and the negative sign is a consequence of the fact that the damping force acts in a direction opposite to the motion.

Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion

$$\frac{d^2x}{dt^2}+\frac{\beta}{m}\frac{dx}{dt}+\frac{k}{m}x=0 \Rightarrow \frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=0 \qquad \left[\omega^2=\frac{k}{m} \text{ and } 2\lambda=\frac{\beta}{m}\right]$$

With the initial conditions $x(0)=x_0$, $\left.\frac{dx}{dt}\right|_{t=0}=x'(0)=x_1$.

Auxiliary equation: $m^2+2\lambda m+\omega^2=0 \Rightarrow m=-\lambda\pm\sqrt{\lambda^2-\omega^2}$

General solution: $x(t)=e^{-\lambda t}\left(c_1e^{\sqrt{\lambda^2-\omega^2}t}+c_2e^{-\sqrt{\lambda^2-\omega^2}t}\right)$

Damping factor: $D=e^{-\lambda t}$ $(\lambda>0)$, i.e. the displacements of the mass become negligible as time t increases.

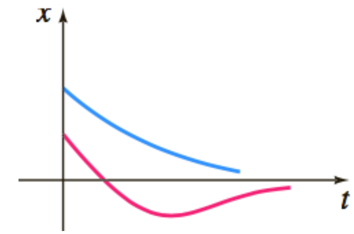
Case – I : Overdamped motion $(\lambda^2-\omega^2>0)$

Case – II : Critically damped motion $(\lambda^2-\omega^2=0)$

Case – III : Underdamped motion $(\lambda^2-\omega^2<0)$

Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion



Case – I : Overdamped motion $(\lambda^2 - \omega^2 > 0)$

Example. A mass weighing 32 pounds stretches a spring 8 feet. Assuming that a damping force numerically equal to 5 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from a position 1 ft *below* the equilibrium position with a *downward* velocity of 1 ft/s.

Solution. Given that, the body weight, $W = 32$ lbf; stretching amount, $|s| = |-8|$ ft $= 8$ ft; initial displacement, $x(0) = 1$ ft.; initial velocity, $x'(0) = 1$ ft/s.

Here, the mass of the body, $m = \frac{W}{g} = \frac{32}{32}\,\text{lbf.}\frac{\text{s}^2}{\text{ft}} = 1$ slug; damping force constant, $\beta = 5$

Also, from Hook's law, $|F| = k|s| \Rightarrow 32 = k \cdot 8 \Rightarrow k = 4\,\frac{\text{lb}}{\text{ft}}$.

Now, the differential equation of motion forms,

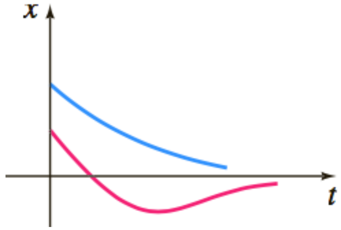
$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} = -4x - 5\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0$$

The auxiliary equation of the ODE becomes,

$$m^2 + 5m + 4 = 0 \Rightarrow (m+4)(m+1) = 0 \Rightarrow m = -4, -1$$

Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion



Case – I : Overdamped motion $(\lambda^2 - \omega^2 > 0)$

Example. A mass weighing 32 pounds stretches a spring 8 feet. Assuming that a damping force numerically equal to 5 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from a position 1 ft *below* the equilibrium position with a *downward* velocity of 1 ft/s.

Solution. Since the roots are real and distinct, the system is overdamped and the general form of the equation of motion can be written as,

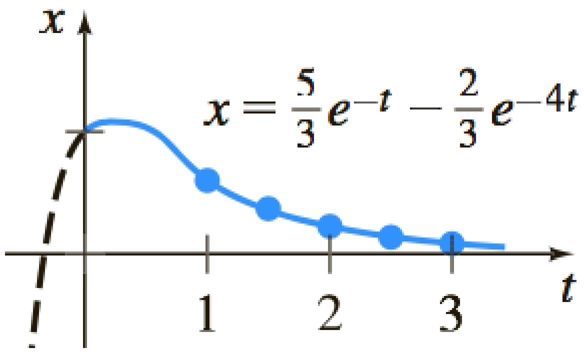
$$x(t) = c_1 e^{-4t} + c_2 e^{-t}$$

From which, $x'(t) = -4c_1 e^{-4t} - c_2 e^{-t}$.

Applying the initial conditions, $x(0) = 1$ and $x'(0) = 1$, we obtain $c_1 = -\frac{2}{3}$ and $c_2 = \frac{5}{3}$.

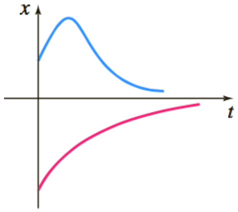
The equation of motion of the overdamped system yields,

$$x(t) = -\frac{2}{3}e^{-4t} + \frac{5}{3}e^{-t}.$$



Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion



Case – II : Critically damped motion $(\lambda^2 - \omega^2 = 0)$

Example. A mass weighing 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an *upward* velocity of 3 ft/s.

Solution. Given that, the body weight, $W = 8$ lbf; stretching amount, $|s| = |-2|$ ft $= 2$ ft; initial displacement, $x(0) = 0$ ft.; initial velocity, $x'(0) = -3$ ft$/$s.

Here, the mass of the body, $m = \frac{W}{g} = \frac{8}{32}\,\text{lbf}.\frac{\text{s}^2}{\text{ft}} = \frac{1}{4}$ slug; damping force constant, $\beta = 2$

Also, from Hook's law, $|F| = k|s| \Rightarrow 8 = k \cdot 2 \Rightarrow k = 4\,\frac{\text{lb}}{\text{ft}}$.

Now, the differential equation of motion forms,

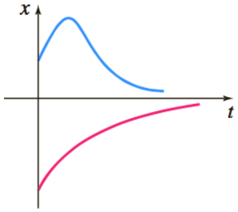
$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \Rightarrow \frac{1}{4}\frac{d^2x}{dt^2} = -4x - 2\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$$

The auxiliary equation of the ODE becomes,

$$m^2 + 8m + 16 = 0 \Rightarrow (m+4)^2 = 0 \Rightarrow m = -4, -4$$

Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion



Case – II : Critically damped motion $(\lambda^2 - \omega^2 = 0)$

Example. A mass weighing 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an *upward* velocity of 3 ft/s.

Solution. Since the roots are real and equal, the system is critically damped and the general form of the equation of motion can be written as,

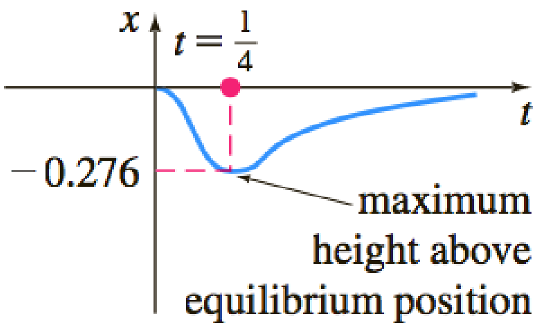
$$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$$

From which, $x'(t) = -4c_1 e^{-4t} - c_2(e^{-4t} - 4te^{-4t})$.

Applying the initial conditions, $x(0) = 0$ and $x'(0) = -3$, we obtain $c_1 = 0$ and $c_2 = -3$.

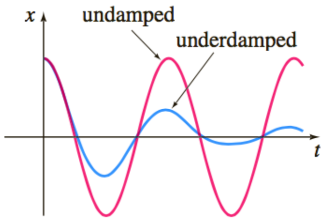
The equation of motion of the critically damped system yields,

$$x(t) = -3te^{-4t}.$$



Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion



Case – III : Underdamped motion $(\lambda^2 - \omega^2 < 0)$

Example. A mass weighing 16 pounds is attached to a 5-foot-long spring. At equilibrium the spring measures 8.2 feet. If the mass is initially released from rest at a point 2 feet above the equilibrium position, find the displacements $x(t)$ if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

Solution. Given that, the body weight, $W = 16$ lbf;
stretching amount, $|s| = |5 - 8.2|$ ft $= 3.2$ ft;
initial displacement, $x(0) = -2$ ft.; initial velocity, $x'(0) = 0$ ft/s.

Here, the mass of the body, $m = \frac{W}{g} = \frac{16}{32}\text{lbf.}\frac{\text{s}^2}{\text{ft}} = \frac{1}{2}$ slug; damping force constant, $\beta = 1$

Also, from Hook's law, $|F| = k|s| \Rightarrow 16 = k \cdot (3.2) \Rightarrow k = 5\,\frac{\text{lb}}{\text{ft}}$.

Now, the differential equation of motion forms,

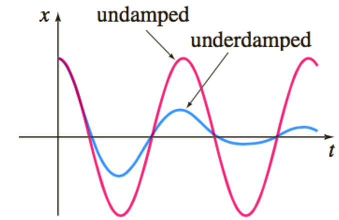
$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \Rightarrow \frac{1}{2}\frac{d^2x}{dt^2} = -5x - 1\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$$

The auxiliary equation of the ODE becomes,

$$m^2 + 2m + 10 = 0 \Rightarrow (m+1)^2 = -9 \Rightarrow m = -1 \pm 3i$$

Modelling with Higher Order Linear ODEs

Spring-Mass System: Free Damped Motion



Case – III : Underdamped motion $(\lambda^2 - \omega^2 < 0)$

Example. A mass weighing 16 pounds is attached to a 5-foot-long spring. At equilibrium the spring measures 8.2 feet. If the mass is initially released from *rest* at a point 2 feet above the equilibrium position, find the displacements $x(t)$ if it is further known that the surrounding medium offers a resistance numerically *equal* to the instantaneous velocity.

Solution. Since the roots are complex, the system is underdamped and the general form of the equation of motion can be written as,

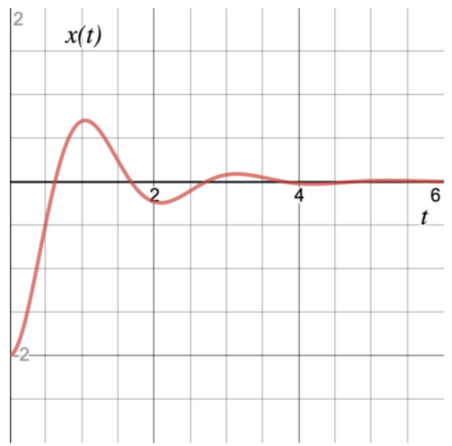
$$x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

From which, $x'(t) = -e^{-t}(c_1 \cos 3t + c_2 \sin 3t) + e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$.

Applying the initial conditions, $x(0) = -2$ and $x'(0) = 0$, we obtain $c_1 = -2$ and $c_2 = -\frac{2}{3}$.

The equation of motion of the underdamped system yields,

$$x(t) = e^{-t}\left(-2\cos 3t - \frac{2}{3}\sin 3t\right).$$



Modelling with Higher Order Linear ODEs

Exercise 5.1 Spring-Mass System: Free Undamped motion

1. A mass weighing 4 pounds is attached to a spring whose spring constant is 16 lb/ft. What is the period of simple harmonic motion?

2. A 20-kilogram mass is attached to a spring. If the frequency of simple harmonic motion is $2/\pi$ cycles/s, what is the spring constant k? What is the frequency of simple harmonic motion if the original mass is replaced with an 80-kilogram mass?

3. A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially, the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

4. Determine the equation of motion if the mass in Problem 3 is initially released from the equilibrium position with a downward velocity of 2 ft/s.

5. A mass weighing 20 pounds stretches a spring 6 inches. The mass is initially released from rest from a point 6 inches below the equilibrium position.

(a) Find the position of the mass at the times $t = \pi/12$, $\pi/8$, $\pi/6$, $\pi/4$, and $9\pi/32$ s.

(b) What is the velocity of the mass when $t = 3\pi/16$ s? In which direction is the mass heading at this instant?

(c) At what times does the mass pass through the equilibrium position?

6. A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/s. Find the equation of motion.

7. Another spring whose constant is 20 N/m is suspended from the same rigid support but parallel to the spring/mass system in Problem 6. A mass of 20 kilograms is attached to the second spring, and both masses are initially released from the equilibrium position with an upward velocity of 10 m/s.

(a) Which mass exhibits the greater amplitude of motion?

(b) Which mass is moving faster at $t = \pi/4$ s? At $\pi/2$ s?

(c) At what times are the two masses in the same position? Where are the masses at these times? In which directions are the masses moving?

8. A mass weighing 32 pounds stretches a spring 2 feet. Determine the amplitude and period of motion if the mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of 2 ft/s. How many complete cycles will the mass have completed at the end of 4π seconds?

Modelling with Higher Order Linear ODEs

Exercise 5.1

9. A mass weighing 8 pounds is attached to a spring. When set in motion, the spring/mass system exhibits simple harmonic motion.

(a) Determine the equation of motion if the spring constant is 1 lb/ft and the mass is initially released from a point 6 inches below the equilibrium position with a downward velocity of $\frac{3}{2}$ ft/s.

(b) Express the equation of motion in the form given in (6).

(c) Express the equation of motion in the form given in (6′).

10. A mass weighing 10 pounds stretches a spring $\frac{1}{4}$ foot. This mass is removed and replaced with a mass of 1.6 slugs, which is initially released from a point $\frac{1}{3}$ foot above the equilibrium position with a downward velocity of $\frac{5}{4}$ ft/s.

(a) Express the equation of motion in the form given in (6).

(b) Express the equation of motion in the form given in (6′)

(c) Use one of the solutions obtained in parts (a) and (b) to determine the times the mass attains a displacement below the equilibrium position numerically equal to $\frac{1}{2}$ the amplitude of motion.

12. A mass of 1 slug is suspended from a spring whose spring constant is 9 lb/ft. The mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of $\sqrt{3}$ ft/s. Find the times at which the mass is heading downward at a velocity of 3 ft/s.

Spring-Mass System: Free Damped motion

21. A mass weighing 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/s. Determine the time at which the mass passes through the equilibrium position. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

Modelling with Higher Order Linear ODEs

Exercise 5.1 Spring-Mass System: Free Damped motion

22. A 4-foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 ft/s. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

23. A 1-kilogram mass is attached to a spring whose constant is 16 N/m, and the entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity. Determine the equations of motion if

(a) the mass is initially released from rest from a point 1 meter below the equilibrium position, and then

(b) the mass is initially released from a point 1 meter below the equilibrium position with an upward velocity of 12 m/s.

24. In parts (a) and (b) of Problem 23 determine whether the mass passes through the equilibrium position. In each case find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

25. A force of 2 pounds stretches a spring 1 foot. A mass weighing 3.2 pounds is attached to the spring, and the system is then immersed in a medium that offers a damping force that is numerically equal to 0.4 times the instantaneous velocity.

(a) Find the equation of motion if the mass is initially released from rest from a point 1 foot above the equilibrium position.

(b) Express the equation of motion in the form given in (23).

(c) Find the first time at which the mass passes through the equilibrium position heading upward.

26. After a mass weighing 10 pounds is attached to a 5-foot spring, the spring measures 7 feet. This mass is removed and replaced with another mass that weighs 8 pounds. The entire system is placed in a medium that offers a damping force that is numerically equal to the instantaneous velocity.

(a) Find the equation of motion if the mass is initially released from a point $\frac{1}{2}$ foot below the equilibrium position with a downward velocity of 1 ft/s.

(b) Express the equation of motion in the form given in (23).

(c) Find the times at which the mass passes through the equilibrium position heading downward.

(d) Graph the equation of motion.