# MAT 350
## ENGINEERING MATHEMATICS

**Higher Order ODEs**
**UNDETERMINED COEFFICIENTS**
**Variation of parameters**

**Lecture: 7**

Dr. M. Sahadet Hossain (Mth)
Associate Professor
Department of Mathematics and Physics, NSU.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \qquad (1)$$

By dividing by the leading coefficient $a_2(x)$

$$y'' + P(x)y' + Q(x)y = f(x) \qquad (2)$$

In (2) we suppose that coefficient functions P(x), Q(x), and f(x) are continuous on some common interval I.

The complementary solution is.

$$y_c = c_1y_1(x) + c_2y_2(x), \qquad (3)$$

We assume that the particular solution is of the same type-

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Using the Product Rule to differentiate $y_p$ twice, we get

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2'$$

$$y_p'' = u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2'.$$

Substituting (3) and the foregoing derivatives into (2) and grouping terms yields

$$y_p'' + P(x)y_p' + Q(x)y_p = u_1[\overbrace{y_1'' + Py_1' + Qy_1}^{\text{zero}}] + u_2[\overbrace{y_2'' + Py_2' + Qy_2}^{\text{zero}}]$$

$$+ y_1u_1'' + u_1'y_1' + y_2u_2'' + u_2'y_2' +$$

$$P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2'$$

$$= \frac{d}{dx}[y_1u_1'] + \frac{d}{dx}[y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2'$$

$$= \frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x).$$

We also assume,

that the functions $u_1$ and $u_2$ satisfy $y_1u_1' + y_2u_2' = 0$.

We now have our desired two equations for $u_1'$, and $u_2'$

$$y_1 u_1' + y_2 u_2' = 0 \qquad (4)$$
$$y_1' u_1' + y_2' u_2' = f(x)$$

By Cramer's Rule, the solution of the system can be expressed in terms of determinants:

$$\boxed{u_1' = \frac{W_1}{W}} \qquad \boxed{u_2' = \frac{W_2}{W}} \qquad (5)$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$ (6)

The functions $u_1$ and $u_2$ are found by integrating the results in (5).

The determinant $W$ is recognized as the Wronskian of $y_1$ and $y_2$.

## Example-1:

$$\boxed{\text{Solve } y'' - 4y' + 4y = (x+1)e^{2x}.}$$

Solution: The Auxiliary equation is

$$m^2 - 4m + 4 = (m-2)^2 = 0$$

The complementary solution is.

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

With the identifications $y_1 = e^{2x}$ and $y_2 = xe^{2x}$,

we compute the Wornskian as

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

We identify,

$$f(x) = (x+1)e^{2x}.$$

Then,

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x},$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x},$$

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x,$$

$$u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1.$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$.

Hence, the particular solution is:

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

And, the general solution is

$$\boxed{y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.}$$

Exercise 4.6 ref. Zill

**10.** $y'' - 9y = \dfrac{9x}{e^{3x}}$

The auxiliary equation is $m^2 - 9 = 0$, so $y_c = c_1 e^{3x} + c_2 e^{-3x}$ and

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6.$$

Identifying $f(x) = 9x/e^{3x}$ we obtain $u_1' = \frac{3}{2}xe^{-6x}$ and $u_2' = -\frac{3}{2}x$. Then

$$u_1 = -\frac{1}{24}e^{-6x} - \frac{1}{4}xe^{-6x},$$

$$u_2 = -\frac{3}{4}x^2$$

and

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{24}e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2 e^{-3x}$$

$$= c_1 e^{3x} + c_3 e^{-3x} - \frac{1}{4}xe^{-3x}(1 - 3x).$$

**12.** $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$, so $y_c = c_1 e^x + c_2 x e^x$ and

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}.$$

Identifying $f(x) = e^x / \left(1 + x^2\right)$ we obtain

$$u_1' = -\frac{xe^x e^x}{e^{2x}\left(1+x^2\right)} = -\frac{x}{1+x^2}$$

$$u_2' = \frac{e^x e^x}{e^{2x}\left(1+x^2\right)} = \frac{1}{1+x^2}.$$

Then $u_1 = -\frac{1}{2}\ln\left(1+x^2\right)$, $u_2 = \tan^{-1} x$, and

$$y = c_1 e^x + c_2 x e^x - \frac{1}{2} e^x \ln\left(1+x^2\right) + xe^x \tan^{-1} x.$$

**19.** $4y'' - y = xe^{x/2}$ $y(0) = 1, y'(0) = 0.$

The auxiliary equation is $4m^2 - 1 = (2m-1)(2m+1) = 0$,

so $y_c = c_1 e^{x/2} + c_2 e^{-x/2}$ and

$$W = \begin{vmatrix} e^{x/2} & e^{-x/2} \\ \frac{1}{2}e^{x/2} & -\frac{1}{2}e^{-x/2} \end{vmatrix} = -1.$$

Identifying $f(x) = xe^{x/2}/4$ we obtain $u_1' = x/4$ and $u_2' = -xe^x/4$.

Then $u_1 = x^2/8$ and $u_2 = -xe^x/4 + e^x/4$. Thus

$$y = c_1 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}$$

$$= c_3 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2}$$

$$y' = \frac{1}{2}c_3e^{x/2} - \frac{1}{2}c_2e^{-x/2} + \frac{1}{16}x^2e^{x/2} + \frac{1}{8}xe^{x/2} - \frac{1}{4}e^{x/2}.$$

The initial conditions imply

$$\begin{aligned} c_3 + c_2 &= 1 \\ \frac{1}{2}c_3 - \frac{1}{2}c_2 - \frac{1}{4} &= 0. \end{aligned}$$

Thus $c_3 = 3/4$ and $c_2 = 1/4$, and

$$y = \frac{3}{4}e^{x/2} + \frac{1}{4}e^{-x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2}.$$

Solve each differential equation by variation of parameters.

$$y'' + y = \tan x$$

$$y'' - 4y = \frac{e^{2x}}{x}$$

$$y'' + 3y' + 2y = \frac{1}{1 + e^x}$$

$$y'' + 2y' + y = e^{-t} \ln t$$

$$2y'' + 2y' + y = 4\sqrt{x}$$

solve each differential equation by variation of parameters, subject to the initial conditions $y(0) = 1, y'(0) = 0.$

$$4y'' - y = xe^{x/2}$$

$$2y'' + y' - y = x + 1$$

$$y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$$