Chapter # 03
(Topics in Differentiation)

We begin this chapter by extending the process of differentiation to functions that are either difficult or impossible to differentiate directly. We will discuss a combination of direct and indirect methods of differentiation that will allow us to develop a number of new derivative formulas that include the derivatives of logarithmic, exponential, and inverse trigonometric functions. Later in the chapter, we will consider some applications of the derivative. These will include ways in which different rates of change can be related as well as the use of linear functions to approximate nonlinear functions. Finally, we will discuss L'Hôpital's rule, a powerful tool for evaluating limits

3.1 Implicit Differentiation: Up to now we have been concerned with differentiating functions that are given by equations of the form $\boldsymbol{y = f(x)}$. In this section we will consider methods for differentiating functions for which it is inconvenient or impossible to express them in this form.

Functions Defined Explicitly and Implicitly: An equation of the form $\boldsymbol{y = f(x)}$ is said to define ***y*** ***explicitly*** as a function of ***x*** because the variable ***y*** appears alone on one side of the equation and does not appear at all on the other side. However, sometimes functions are defined by equations in which ***y*** is not alone on one side; for example, the equation

$$yx + y + 1 = x$$

is not of the form $\boldsymbol{y = f(x)}$, but it still defines ***y*** as a function of ***x*** since it can be rewritten as

$$y = \frac{x-1}{x+1}$$

Thus, we say that first one defines ***y implicitly*** as a function of ***x***, the function being

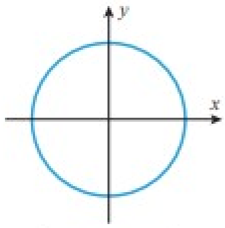
$$f(x) = \frac{x-1}{x+1}$$

Example: An equation in ***x*** and ***y*** can implicitly define more than one function of ***x***. This can occur when the graph of the equation fails the vertical line test, so it is not the graph of a function of ***x***. For example, if we solve the equation of the circle

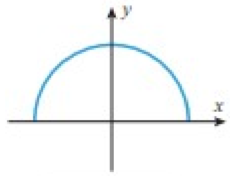
$$x^2 + y^2 = 1 \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (i)$$

for ***y*** in terms of ***x***, we obtain $\boldsymbol{y = \pm\sqrt{1 - x^2}}$, so we have found two functions that are defined implicitly by ***(i)***, namely

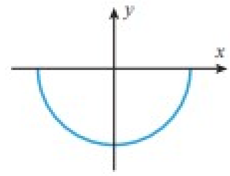
$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2}$$



$x^2 + y^2 = 1$



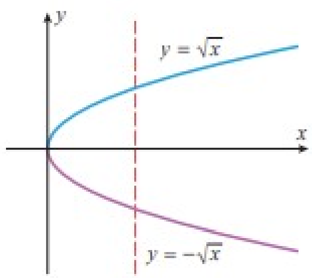
$y = \sqrt{1 - x^2}$



$y = -\sqrt{1 - x^2}$

Example 1: The graph of $\boldsymbol{x = y^2}$ is not the graph of a function of $\boldsymbol{x}$, since it does not pass the vertical line test (following figure). However, if we solve this equation for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$, we obtain the equations $\boldsymbol{y = \sqrt{x}\ and\ y = -\sqrt{x}}$, whose graphs pass the vertical line test and are portions of the graph of $\boldsymbol{x = y^2}$. Thus, the equation $\boldsymbol{x = y^2}$ implicitly defines the functions

$$f_1(x) = \sqrt{x} \quad \text{and} \quad f_2(x) = -\sqrt{x}$$



Example 2: Use implicit differentiation to find $\boldsymbol{\frac{dy}{dx}\ if\ 5y^2 + \sin y = x^2}$**.**

Solution:

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

Solving for $\boldsymbol{\frac{dy}{dx}}$ we obtain

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

Example 3: Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $\boldsymbol{4x^2 - 2y^2 = 9}$.

Solution: Differentiating both sides of $\boldsymbol{4x^2 - 2y^2 = 9}$ with respect to ***x*** yields

$$8x - 4y\frac{dy}{dx} = 0$$

from which we obtain

$$\frac{dy}{dx} = \frac{2x}{y}$$

Differentiating both sides with respect to ***x***

$$\frac{d^2y}{dx^2} = \frac{(y)(2) - (2x)(dy/dx)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y - 2x(2x/y)}{y^2} = \frac{2y^2 - 4x^2}{y^3} = -\frac{9}{y^3}$$ (Ans.)

Example 4: Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points

(2, −1) and (2, 1).

Solution: We could proceed by solving the equation for ***y*** in terms of ***x***, and then evaluating the derivative of $y = \sqrt{x-1}\ at\ (2,1)\ \ and\ \ y = -\sqrt{x-1}\ at\ (2,-1)$. Differentiating implicitly yields

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$

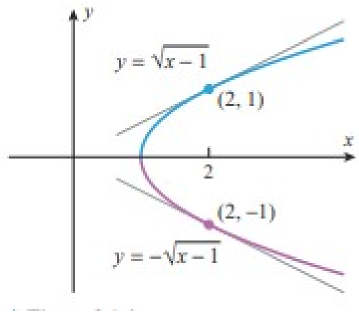
$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the tangent lines to the curve at those points are

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2}$$



Example 5:

(a) Use implicit differentiation to find dy/dx for the Folium of Descartes $x^3 + y^3 = 3xy$.
(b) Find an equation for the tangent line to the Folium of Descartes at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.
(c) At what point(s) in the first quadrant is the tangent line to the Folium of Descartes horizontal?

Solution: (*a*) Differentiating implicitly yields

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \qquad \text{(i)}$$

(*b*) At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, we have $x = \frac{3}{2}\ \ and\ \ y = \frac{3}{2}$, so from **(*i*)** the slope $\boldsymbol{m_{tan}}$ of the tangent line at this point is

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=3/2 \\ y=3/2}} = \frac{(3/2) - (3/2)^2}{(3/2)^2 - (3/2)} = -1$$

Thus, the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$y - \tfrac{3}{2} = -1\left(x - \tfrac{3}{2}\right) \quad \text{or} \quad x + y = 3$$

(*c*) The tangent line is horizontal at the points where $\frac{dy}{dx} = 0$, and from (i) this occurs only where

$$\boldsymbol{y - x^2 = 0 \quad or \quad y = x^2} \qquad \textbf{(ii)}$$

Substituting this expression for ***y*** in the equation $\boldsymbol{x^3 + y^3 = 3xy}$ for the curve yields

$$x^3 + (x^2)^3 = 3x^3$$
$$x^6 - 2x^3 = 0$$
$$x^3(x^3 - 2) = 0$$

whose solutions are $x = 0$ *and* $x = 2^{\frac{1}{3}}$. From (ii), the solutions $x = 0$ *and* $x = 2^{\frac{1}{3}}$ yield the points (0, 0) and $\left(2^{\frac{1}{3}},\ 2^{\frac{2}{3}}\right)$, respectively. Of these two, only $\left(2^{\frac{1}{3}},\ 2^{\frac{2}{3}}\right)$ is in the first quadrant. Substituting $x = 2^{\frac{1}{3}}, y = 2^{\frac{2}{3}}$ into (i) yields

$$\left.\frac{dy}{dx}\right|_{\substack{x=2^{1/3} \\ y=2^{2/3}}} = \frac{0}{2^{4/3} - 2^{2/3}} = 0$$

We conclude that $\left(2^{\frac{1}{3}}, 2^{\frac{2}{3}}\right) \approx (1.26, 1.59)$ is the only point on the Folium of Descartes in the first quadrant at which the tangent line is horizontal.

Home Work: Exercise 3.1: Problem No. 3-20, 25-28, 32