MAT350: Engineering Mathematics

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Office Room : SAC 1063A

Office Hour : 02:00 PM - 03:00 PM (T)

03:00 PM - 04:00 PM (W)

09:00 AM - 11:00 AM (R)

Course Assessment Process: Summer 2022

Category	Weight
Attendance	10%
Assignments (Minimum 3)	10%
Quizzes (Best 3 of 4)	15%
Mid-Term	30%
Final Exam	35%

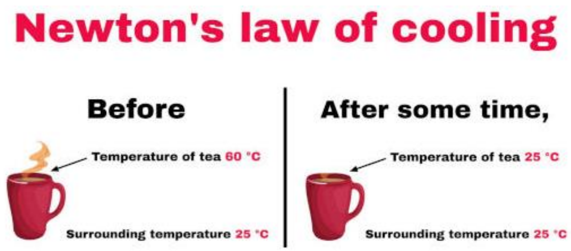
Differential Equations

❑ Modeling Approach

- **Newton's Law of cooling/heating:**

$$\frac{dT(t)}{dt} \propto T - T_m \Rightarrow \frac{dT(t)}{dt} = k(T - T_m)$$

where T: body temperature, T_m: surrounding temperature, k: proportional constant

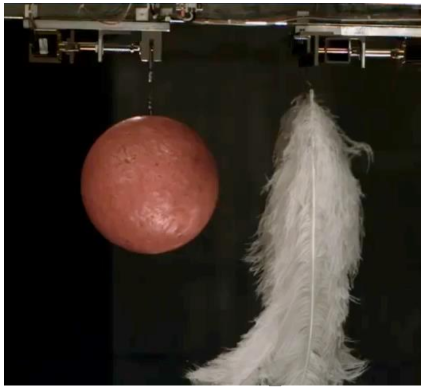


- **Bacterial Population Model:** $\frac{dP(t)}{dt} \propto P(t) \Rightarrow \frac{dP(t)}{dt} = r\,P(t)$

where r is proportional constant which can be called as the intrinsic rate of natural increase and $P(0) = 2, P(10) = 4$.

https://www.youtube.com/watch?v=gEwzDydciWc

- **Free Falling Objects:** $\frac{d^2y(t)}{dt^2} = -g$, where g is called the gravitational constant and $y(0) = h,\ \left.\frac{dy}{dt}\right|_{t=0} = 0.$

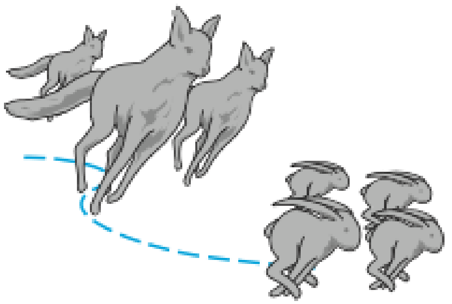


https://www.youtube.com/watch?v=74MUjUj7bp8

Differential Equations

❑ Modeling Approach

- **Predator-Prey Population Model:** $$\frac{dP_1}{dt} = aP_1 - bP_1P_2$$ $$\frac{dP_2}{dt} = -mP_2 + nP_1P_2$$



where P_1: population of rabbits, P_2: population of foxes

a: growth rate of rabbits, b: killing rate of rabbits,

m: death rate of foxes, and n: growth rate of foxes.

- **Spread of a Disease:**

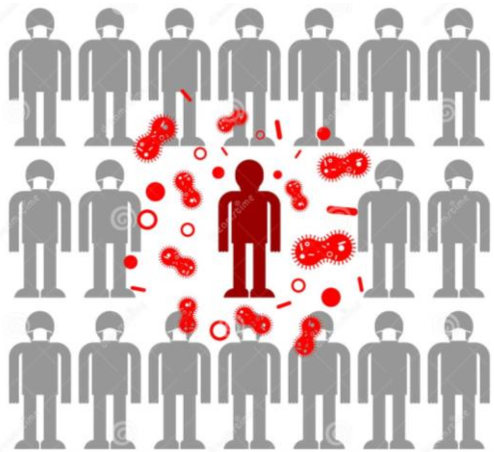
$$\frac{dP(t)}{dt} \propto P(t)Q(t) \Rightarrow \frac{dP(t)}{dt} = kP(t)Q(t) = kP(n+1-P)$$

where $P(t)$: infected population,

$Q(t)$: non-infected population, k: proportional constant,

$n+1$: total population with one infected person so that

$P+Q=n+1$ and $P(0)=1$.



https://thumbs.dreamstime.com/z/infected-person-coronavirus-infection-epidemic-disease-spread-sickness-171599936.jpg

Differential Equations

Definitions and Terminology

Differential Equations: An equation containing the derivative of one or more unknown function (or dependent variables), with respect to one or more independent variables. E.g.,

$$\frac{dy}{dx} + 5y = e^x$$

$$y'' - y' + 6y = 0$$

$$y'y''' - \frac{3}{2}y'^2 = 0$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Types of Differential equations

Ordinary Differential Equation (ODE): Differential equation containing only ordinary derivatives of one or more unknown functions with respect to a ***single*** independent variable. Here prime ($'$) denotes differentiation with respect to x.

Partial Differential Equation (PDE): Differential equation involving partial derivatives of one or more unknown functions of two or more independent variables.

Ordinary Differential Equations

Classifications of ODE

- **By Order:** The **order of an ODE** is the order of the highest derivative in the equation. For example,

$$\frac{dy}{dx} + 2y = 5 \qquad \text{(First order)}$$

$$(y - x)\,dx + 4x\,dy = 0 \qquad \text{(First order)}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \qquad \text{(Second order)}$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0 \qquad \text{(Second order)}$$

$$y'y''' - \frac{3}{2}y'^2 = 0 \qquad \text{(Third order)}$$

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0 \qquad \text{(}n\text{-th order)}$$

Ordinary Differential Equations

Classifications of ODE

- **By Linearity**: An *n*th-order ordinary differential equation $F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$ is said to be **linear** if the dependent variable y and its various derivatives $y', y'', \dots, y^{(n)}$ appear linearly in the equation. Otherwise, the ODE is said to be nonlinear. For example,

$\frac{dy}{dx} + 2y = 5$	(First order, Linear)
$(y - x)\, dx + 4x\, dy = 0$	(First order, Linear)
$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$	(Second order, Linear)
$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 6y = 0$	(second order, Non-linear)
$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y^2 = 0$	(second order, Non-linear)
$\frac{d^2y}{dx^2} + \sin y = 0$	(Second order, Non-linear)

Ordinary Differential Equations

Solution of an ODE:

Any function $y = \phi(x)$, defined on an interval $I = (a, b)$, is said to be a **solution** of an n-th order ODE

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$$

if $\phi(x)$ is defined and differentiable throughout the interval and is such that

$$F\left(x, \phi(x), \phi'(x), \phi''(x), \dots, \phi^{(n)}(x)\right) = 0 \quad \forall\, x \in (a, b).$$

Example. $y = \frac{1}{16}x^4$ is a solution of the ODE: $\frac{dy}{dx} - x\sqrt{y} = 0$ on the interval $(-\infty, \infty)$.

Example. $y = c/x$ is a solution of the ODE: $x\frac{dy}{dx} = -y$ on the interval $(-\infty, 0)$ or $(0, \infty)$.

Example. $y = xe^x$ is a solution of the ODE: $y'' - 2y' + y = 0$ on the interval $(-\infty, \infty)$.

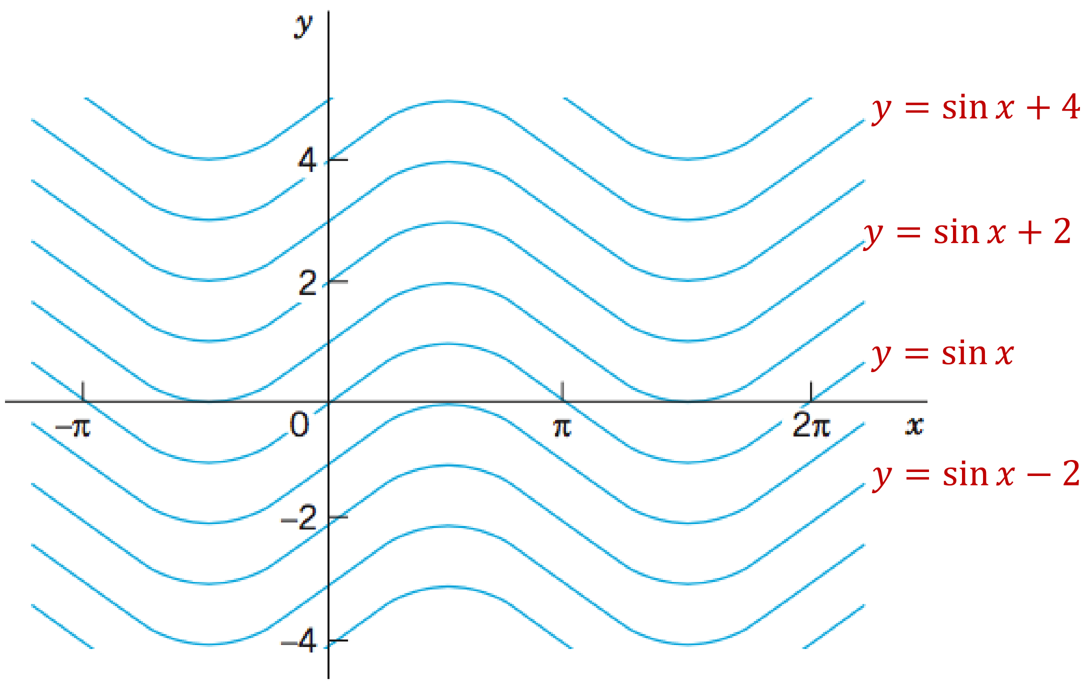
Example. $x^2 + y^2 = 4$ is a solution of the ODE: $y' = -\frac{x}{y}$ on the interval $(-2, 2)$.

Ordinary Differential Equations

Family of Solutions and Solution Curve:

Example. $y = \sin x + c$, where c is an arbitrary constant, is a family of solutions on the interval $(-\infty, \infty)$ for the ODE:

$$\frac{dy}{dx} = \cos x \;\; \textbf{since } dy = \cos x \;\, dx \Rightarrow \int dy = \int \cos x \, dx \Rightarrow y = \sin x + c$$



Here, $y = \sin x + 2$ is a **particular solution** for $c = 2$.

Here, $y = \sin x - 2$ is a **particular solution** for $c = -2$.

Ordinary Differential Equations

Initial Value Problem (IVP):

An n-th order initial value problem on an open interval $I = (a, b)$ can be defined as

$Solve$: $\quad y^{(n)}(x) = f(x, y, y', \dots, y^{n-1})$

$Subject\ to$: $y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \dots, y^{(n-1)}(x_0) = y_{n-1}$, where $x_0 \in I$.

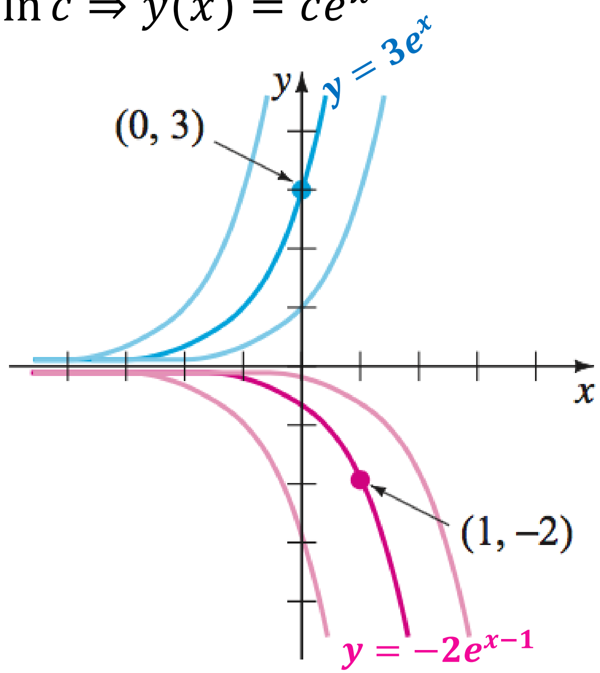
Example. Solve $\frac{dy}{dx} = y$ subject to the condition $y(0) = 3$.

Solution. The ODE is given by,

$$\frac{dy}{dx} = y \Rightarrow \frac{1}{y} dy = dx \Rightarrow \int \frac{1}{y} dy = \int dx \Rightarrow \ln y = x + \ln c \Rightarrow y(x) = ce^x$$

Since, $y(0) = 3 \Rightarrow ce^0 = 3 \Rightarrow \mathrm{c} = 3$

Thus, the solution of the given IVP becomes, $y = 3e^x$.



Example. Solve $\frac{dy}{dx} = y$ subject to the condition $y(1) = -2$.

Solution. From the previous solution, $y(x) = ce^x$

Since, $y(1) = -2 \Rightarrow ce^1 = -2 \Rightarrow \mathrm{c} = -2\mathrm{e}^{-1}$

Thus, the solution of the given IVP becomes, $y = -2e^{x-1}$.

Ordinary Differential Equations

H.W. from the text book

Exercises 1.1

Determine the order and the linearity of the following ODE.

1. $(1-x)y'' - 4xy' + 5y = \cos x$

2. $x\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^4 + y = 0$

3. $t^5y^{(4)} - t^3y'' + 6y = 0$

4. $\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + u = \cos(r+u)$

5. $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

6. $\dfrac{d^2R}{dt^2} = -\dfrac{k}{R^2}$

7. $(\sin\theta)y''' - (\cos\theta)y' = 2$

8. $\ddot{x} - \left(1 - \dfrac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

9. $(y^2-1)\,dx + x\,dy = 0$; in y; in x

10. $u\,dv + (v + uv - ue^u)\,du = 0$; in v; in u

Verify that the indicated function is an explicit solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

11. $2y' + y = 0;\quad y = e^{-x/2}$

12. $\dfrac{dy}{dt} + 20y = 24;\quad y = \dfrac{6}{5} - \dfrac{6}{5}e^{-20t}$

13. $y'' - 6y' + 13y = 0;\quad y = e^{3x}\cos 2x$

14. $y'' + y = \tan x;\quad y = -(\cos x)\ln(\sec x + \tan x)$

15. $(y-x)y' = y - x + 8;\quad y = x + 4\sqrt{x+2}$

16. $y' = 25 + y^2;\quad y = 5\tan 5x$

17. $y' = 2xy^2;\quad y = 1/(4-x^2)$

18. $2y' = y^3\cos x;\quad y = (1-\sin x)^{-1/2}$

Ordinary Differential Equations

H.W. from the text book

Exercises 1.1

Verify that the indicated family of functions is a solution of the given differential equation.

21. $\dfrac{dP}{dt} = P(1 - P); \quad P = \dfrac{c_1e^t}{1 + c_1e^t}$

22. $\dfrac{dy}{dx} + 2xy = 1; \quad y = e^{-x^2}\displaystyle\int_0^x e^{t^2}\,dt + c_1e^{-x^2}$

23. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0; \quad y = c_1e^{2x} + c_2xe^{2x}$

24. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 12x^2;$

$y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$

Verify that the indicated pair of functions is a solution of the given system of differential equations

37. $\dfrac{dx}{dt} = x + 3y$

$\dfrac{dy}{dt} = 5x + 3y;$

$x = e^{-2t} + 3e^{6t},$

$y = -e^{-2t} + 5e^{6t}$

38. $\dfrac{d^2x}{dt^2} = 4y + e^t$

$\dfrac{d^2y}{dt^2} = 4x - e^t;$

$x = \cos 2t + \sin 2t + \frac{1}{5}e^t,$

$y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$