DAY-19

MAT-130< Section-8: Viva Session (Only 1 session):

On 1st May, Saturday, from 3pm to 5pm.

MAT-130< Section-9: Viva Session (Only 1 session):

On Tuesday, 27th April, from 7:15pm.

Chapter 10

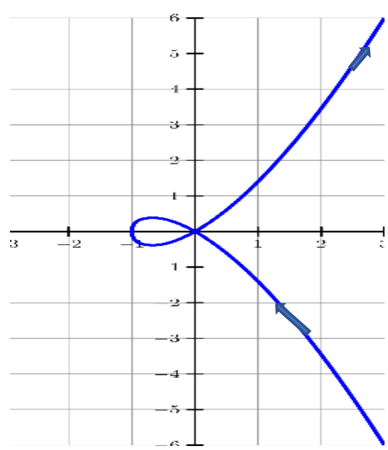
Section 10.1: Parametric Equations

Summery:

1. Graphing Parametric Equations
2. Finding slopes of tangent lines
3. Finding points where the tangent line is either vertical or horizontal
4. **Finding length of a parametric curve (6.4)**
5. **Finding the area of a surface of revolution (6.5)**

PARAMETRIC EQUATIONS:

Suppose that a particle moves along a curve C in the xy-plane in such a way that its $x-$ and $y-$coordinates, as functions of time, are $x = f(t)$, $y = g(t)$. We call these the parametric equations of motion for the particle and refer to C as the trajectory of the particle or the graph of the equations. The variable t is called the parameter of the equations.



$\boldsymbol{x = t^2 - 1}$, $\boldsymbol{y = t^3 - t}$, $\boldsymbol{-\infty < t < \infty}$. **The loop comes for t –values from -1 to 1.**

Functions in parametric form: input $=$ parameter, output $= f(\text{parameter})$

$\boldsymbol{y = f(x)}$: $x = t$, $y = f(t)$, The domain of the function is the set of parameters.

$\boldsymbol{x = f(y)}$: $y = \theta$, $x = f(\theta)$, The domain of the function is the set of parameters.

Parametric Equations for Few well-known Graphs

Linear Line:

1. Equations: $x = at + b$, $y = mt + n$, t any real number.

2. For example, a line segment that starts from (x_1, y_1) and ends to (x_2, y_2) is parametrized by

$$[x = (initial\ \ x - value) + (run)t\,, \qquad y = (initial\ y - value) + (rise)t, \qquad 0 \le t \le 1]$$

$$x = x_1 + (x_2 - x_1)t\,, \qquad y = y_1 + (y_2 - y_1)t, \qquad 0 \le t \le 1.$$

Parametric Equations of few other curves:

Curve	Parametric Equations
Circle	$x = r\cos t$ $y = r\sin t$ (r is radius)
Ellipse	$x = a\cos(ct)$ $y = b\sin(ct)$ (a, b, c are constants ≠ 0)
Parabola	vary

Equation of a parabola	Parametric equations
$y^2 = 4ax$	$x = at^2, y = 2at$
$y^2 = -4ax$	$x = -at^2, y = 2at$
$x^2 = 4ay$	$x = 2at, y = at^2$
$x^2 = -4ay$	$x = 2at, y = -at^2$
$(y-h)^2 = 4a(x-k)$	$x = k + at^2, y = h + 2at$

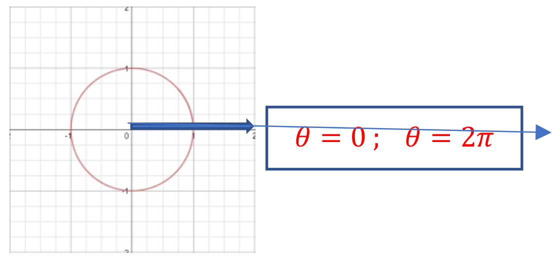
The parametric equations of the circle $x^2 + y^2 = 4^2 \rightarrow x = 4\cos\theta\,,\ y = 4\sin\theta\,,\ \ 0 \le \theta \le 2\pi.$

Equation of a hyperbola	Parametric equations
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with centered at origin	$x = a\sec t,\ y = b\tan t$
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, with centered at (h,k)	$x = h + a\sec t,\ y = k + b\tan t$

Graphing Parametric Curves:

Exercise:1

(a) Find the graph of the parametric equations $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$



Solution:

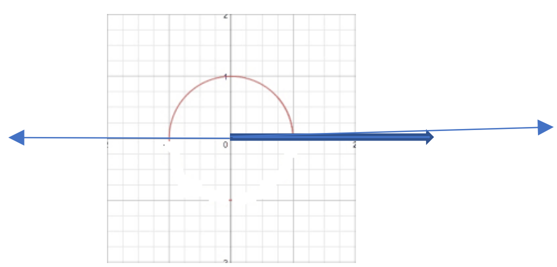
Given, $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

Here we get,

$$x^2 = \cos^2 t, \quad y^2 = \sin^2 t, \quad 0 \le t \le 2\pi.$$

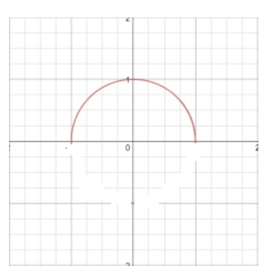
That is,
$x^2 + y^2 = 1$, $0 \le t \le 2\pi$, which gives us the whole circle with center at the $(0,0)$ and radius $r = 1$ in anticlockwise direction.

(b) Find the graph of the parametric equations $x = cost$, $y = sint$, $0 \le t \le \pi$.



Orientation is in anti-clockwise.

(c) Find the graph of the parametric equations $x = cost$, $y = sint$, where t moves from $t = \pi$ to $t = 0$.

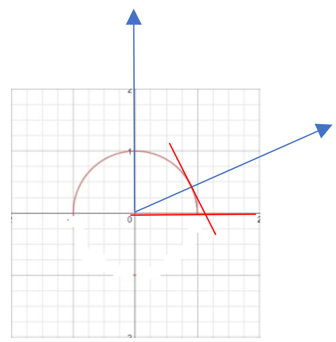


Orientation is clockwise.

(d) Find the graph of the parametric equations $x = \cos t$, $y = \sin t$, $\frac{\pi}{6} \leq t \leq \frac{\pi}{2}$.

From $x = cost$, $y = sint$, we get $x^2 = \cos^2 t$, $y^2 = \sin^2 t$.

Hence, $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. That is, $\boldsymbol{x^2 + y^2 = 1}$.
The graph of the parametric equations is a piece of the circle $x^2 + y^2 = 1$ with center at the origin and radius 1 from $\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ to $(\cos 0, \sin 0) = (0, 1)$.



Orientation is in anti-clockwise.

(e) Find the graph of the parametric equations $x = 3\cos t$, $y = 3\sin t$, $\frac{\pi}{6} < t \leq \frac{\pi}{2}$

➔ Homework [Write down the solution]

Exercise:2

(a) Graph the parametric curve $x = 2t - 3$, $y = 6t - 7$ by eliminating the parameter, and indicate the orientation on the graph.

Solution. To eliminate the parameter, we will solve the first equation for t as a function of x, and then substitute this expression for t into the second equation:

From $x = 2t - 3$, we get $t = \frac{1}{2}(x + 3)$ where t is any real number.

Substitute it in $y = 6t - 7$, then we have

$$y = 6 \cdot \frac{1}{2}(x + 3) - 7 = 3x + 9 - 7$$

$$y = 3x + 2.$$

Picture????

Which is an increasing straight line (slope$= 3 > 0$). The orientation is upward, and goes through the point (0,2).

(b) Graph the parametric curve $x = 2t - 3, \quad y = 6t - 7$ where t is from $\boldsymbol{t = 2}$ **to** $\boldsymbol{t = -3}$. by eliminating the parameter, and indicate the orientation on the graph.

From $x = 2t - 3, \quad y = 6t - 7$, we get

$y = 3x + 2$, which gives you the graph. [Graph]

Orientation: If $t = 2$, then $(x, y) = (1, 5)$ and if $t = -3$, then $(x, y) = (-9, -25)$. It is a line segment from $(1, 5)$ to $(-9, -25)$, in the orientation from right to left, downward.

Exercise:3

Graph the parametric curve $x = 4t^2, \quad y = 8t$ by eliminating the parameter, and indicate the orientation on the graph.

Solution: From $y = 8t$, we get $t = \frac{y}{8}$. Substitute it is $x = 4t^2$: $x = 4\left(\frac{y}{8}\right)^2$

Hence, the graph in rectangular coordinates is: $y^2 = 16x = 4(4)x, \quad i.e., \quad x = \frac{1}{16}y^2$.

Orientation: ???? [The very first parametric curve]

TANGENT LINES TO PARAMETRIC CURVES

[Recall Calculus-1: For the function $y = f(x)$, the slope of the tangent line to the curve of the function $y = f(x)$ at any x is given by $f'(x) = \frac{dy}{dx}$. It is same here.]

We will be concerned with curves that are given by parametric equations $x = f(t), \; y = g(t)$ in which $f(t)$ and $g(t)$ have continuous first derivatives with respect to t. It can be proved that if $\frac{dx}{dt} \neq 0$, then y is a differentiable function of x, in which case the chain rule implies that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \dots\dots\dots (1)$$

This formula makes it possible to find $\frac{dy}{dx}$ directly from the parametric equations without eliminating the parameter.

Exercise: 4

Find the slope of the tangent line to the unit circle $x = cost$, $y = sint$, $0 \le t \le 2\pi$, at the point where $t = \frac{\pi}{6}$.

Solution: The slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \dots\dots\dots\dots (1)$$

Given, $x = cost$, $y = sint$

We have, $\frac{dy}{dt} = \cos t$, $\frac{dx}{dt} = -\sin t$.

So, from equation (1): $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{cos\ t}{-\sin t}$

At $t = \frac{\pi}{6}$, $\left[\frac{dy}{dx}\right]_{t=\frac{\pi}{6}} = \frac{cos\frac{\pi}{6}}{-\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$.

Hence, the slope of the tangent line is $\frac{dy}{dx} = -\sqrt{3}$.

Notes: We know that the slope tangent line to the parametric curve $x = x(t), y = y(t)$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \dots\dots\dots\dots (1)$$

[Note: $\frac{0}{non-zero} = 0$, $\frac{non-zero}{0} = \text{undefined}$, $\frac{0}{0} = \text{indeterminant}$]

Note: A horizontal line has slope = 0, and a vertical line has slope = undefined.

(1) The tangent line to the parametric curve $x = x(t), y = y(t)$ is vertical if and only if $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ =undefined, that is, if and only if $\left[\frac{dy}{dt} \neq 0 \text{ and } \frac{dx}{dt} = 0\right]$.

(2) The tangent line to the parametric curve $x = x(t), y = y(t)$ is horizontal if and only if $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 0$, that is, if and only if $\left[\frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0\right]$.

Exercise: 5[Please solve this later]

Consider the parametric curve

$$x = (1 - \cos t)\cos t\,,\; y = (1 - \cos t)\sin t\;,\quad 0 \le t \le 2\pi.$$

- Find t −values where the tangent line is horizontal.
- Find t −values where the tangent line is vertical.

Exercise: 6

Consider the parametric curve

$$x = t - 3sint,\; y = 4 - 3cost\,,\quad 0 \le t \le 10.$$

1) Find t −values where the tangent line is horizontal.
2) Find t −values where the tangent line is vertical.

Solution: Given,

$$x = t - 3sint,\; y = 4 - 3cost\,,\quad 0 \le t \le 10.$$

$$\frac{dx}{dt} = 1 - 3cost \;\text{ and }\; \frac{dy}{dt} = 3\sin t$$

1) Find t −values where the tangent line is horizontal.

Solution:

The tangent line to the parametric curve $x = x(t), y = y(t)$ is horizontal if and only if

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 0, \text{ that is, if and only if } \left[\boldsymbol{\frac{dy}{dt} = 0} \;\; \textbf{and} \;\; \boldsymbol{\frac{dx}{dt} \ne 0}\right].$$

Setting $\boldsymbol{\frac{dy}{dt} = 0}$ yields the equation $3\sin t = 0$, or, more simply, $\sin t = 0.$

This equation has four solutions in the time interval $0 \le t \le 10$:

$$t = 0,\quad t = \pi,\quad t = 2\pi,\quad t = 3\pi$$

Since $\frac{dx}{dt} = 1 - 3cost \ne 0$ for these values of t, hence the t −values are $t = 0,\; t = \pi,\; t = 2\pi,$ $t = 3\pi$ where the tangent line is horizontal.

2) Solution:

$$\frac{dx}{dt} = 1 - 3cost \;\text{ and }\; \frac{dy}{dt} = 3\sin t$$

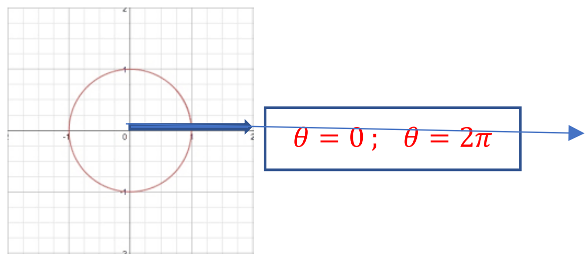
The tangent line to the parametric curve $x = x(t), y = y(t)$ is vertical if and only if

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \text{undefined, that is, if and only if } \left[\frac{dy}{dt} \neq 0 \ \text{ and } \ \frac{dx}{dt} = 0\right]$$

Setting $\frac{dx}{dt} = 0$ yields the equation $1 - 3cost = 0 \ \ or \ \ \cos t = \frac{1}{3}$.

This equation has three solutions in the time interval $0 \leq t \leq 10$:

$$t = \cos^{-1}\left(\frac{1}{3}\right), \quad t = 2\pi - \cos^{-1}\left(\frac{1}{3}\right), \quad t = 2\pi + \cos^{-1}(\frac{1}{3})$$



Periodic Function: $f(x) = f(p + x) = \cdots = f(np + x);$ for integer p.

DONE

Note: Note that the value of $\cos^{-1}(\frac{1}{3})$ is approximately 1.23

Length of a parametric curve:

Definition: If no segment of the parametric curve

$$x = x(t), \qquad y = y(t), \qquad a \leq t \leq b$$

traced more than once as t increases from a to b, then the length of the curve is defined by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

Step: Given a parametric curve $x = x(t), \ y = y(t), \qquad a \leq t \leq b$

Step 1: Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

tep 2: Find $\left(\frac{dx}{dt}\right)^2$ **and** $\left(\frac{dy}{dt}\right)^2$

Step 3: Find $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2$

Step 4: Find $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ **. Here we should try, if possible, to write** $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ **as a perfect square so that we can cancel the square root.**

Step 5: Evaluate the integral to find the length L:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Exercise : 7

Find the length of the curve $x = 3\cos(2\theta)$, $y = 3\sin(2\theta)$ for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-6\sin t\,)^2 + (6\cos t\,)^2} = 6$$

<u>Area of the Surface of Revolution by Revolving a Parametric Curve:</u>

Definition:

A. **Let** $x = x(t),\ y = y(t),\quad a \le t \le b$ **be a smooth parametric curve. Then** the area of the surface of revolution formed by revolving the curve about the $\boldsymbol{x}$ −axis is defined by

$$Surface\ Area = \int_a^b 2\pi\ y(t)\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

B. Let $x = x(t),\ y = y(t),\quad a \le t \le b$ **be a smooth parametric curve. Then the area of the surface of revolution formed by revolving the curve about the** $\boldsymbol{y}$ **−axis is defined by**

$$Area = \int_a^b 2\pi\, x(t)\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Steps: Given a parametric curve $x = x(t),\ y = y(t), \qquad a \le t \le b$

Step 1: Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

Step 2: Find $\left(\frac{dx}{dt}\right)^2$ and $\left(\frac{dy}{dt}\right)^2$

Step 3: Find $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2$

Step 4: Find $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Here we should try, if possible, to write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ as a perfect square so that we can cancel the square root.

Step 5: Evaluate the integral to find the area of the surface of revolution.

Exercise: 8

Find the area of the surface generated by revolving the curve $x = e^t cost, y = e^t sint,\ (0 \le t \le \pi/2)$ about the x −axis.

Solution:

Surface $Area = \int_a^b 2\pi\, y(t)\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$ ... ... (1)

Given curve $x = e^t cost,\ y = e^t sint,\ (0 \le t \le \pi/2)$

$$\frac{dx}{dt} = e^t\,(-\sin t\,) + \cos t\ \ e^t = e^t\,(\cos t - \sin t\,)$$

$$\frac{dy}{dt} = e^t\,(\cos t\,) + \sin t\ \ e^t = e^t\,(\cos t + \sin t\,)$$

So, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}[(\cos t - \sin t\,)^2 + (\cos t + \sin t\,)^2]$

$$= e^{2t}[cos^2 t - 2\cos t \sin t + \sin^2 t + cos^2 t + 2\cos t \sin t + \sin^2 t]$$

$$= e^{2t}[2(cos^2 t + \sin^2 t)] = 2e^{2t}$$

Therefore, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \;=\; \sqrt{2e^{2t}} \;=\; \sqrt{2}\, e^t$

The area of the surface of revolution formed by revolving the curve about the x −axis is defined by

$$Area = \int_a^b 2\pi\, y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt = \int_0^{\frac{\pi}{2}} 2\pi \cdot e^t \sin t \cdot \sqrt{2}\, e^t \; dt$$

$$Area = 2\sqrt{2}\pi \int_0^{\frac{\pi}{2}} e^{2t} \sin t \;\; dt \;\; \ldots\ldots\ldots \;\; (2)$$

Here, Set $I = \int e^{2t} \sin t\, dt$; $\quad u = e^{2t}, \;\; dv = \sin t \; dt.$ So, $du = 2e^{2t}dt, \quad v = -\cos t$

$$= \int udv = uv - \int vdu$$

$$= e^{2t}(-\cos t) - \int (-\cos t)\, 2e^{2t}\, dt$$

$$= -e^{2t} \cos t + 2\int e^{2t} \cos t\, dt \quad ; \quad u = e^{2t}, \;\; dv = \cos t \; dt. \;\; \text{So}, du = 2e^{2t}dt, \quad v = \sin t$$

$$= -e^{2t} \cos t + 2[e^{2t} \sin t - \int \sin t \;\; 2\, e^{2t} dt\,]$$

$$= -e^{2t} \cos t + 2e^{2t} \sin t - 4 \int e^{2t} \sin t\, dt$$

$$\Rightarrow I = e^{2t}[\sin t - \cos t] - 4I$$

$$\Rightarrow 5I = e^{2t}[\sin t - \cos t]$$

$$\Rightarrow I = \frac{\mathrm{e}^{2t}(\sin t - \cos t)}{5} + C$$

$$\int e^{2t} \sin t\, dt \;= \frac{e^{2t}}{5}\; [\sin t \; - \cos t\,] + C$$

From (2):

$$Area = 2\sqrt{2}\pi \int_0^{\frac{\pi}{2}} e^{2t} \sin t \;\; dt \;= 2\sqrt{2}\pi[\tfrac{e^{2t}}{5}(\sin t - \cos t)]_0^{\frac{\pi}{2}} = \frac{2\sqrt{2}}{5}\; [e^{2t}(\sin t - \cos t)]_0^{\frac{\pi}{2}}$$

$$= \frac{2\sqrt{2}}{5}[e^{\frac{2\pi}{2}}(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}) - e^0(\sin 0 \; - \cos 0)] \; = \frac{2\sqrt{2}}{5}[e^{\frac{2\pi}{2}}(1-0) - e^0(0 \; - 1)]$$

$$= \frac{2\sqrt{2}\pi}{5}\left[e^{\frac{2\pi}{2}} + e^0\right]$$

$$Area = \frac{2\sqrt{2}\pi}{5}[e^{\pi} + 1] \; unit^2$$

DAY-20

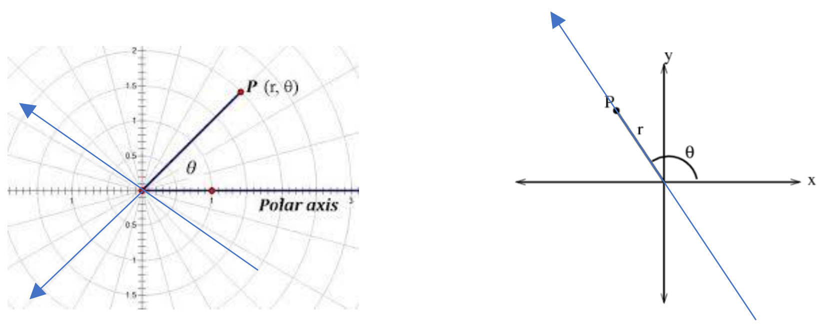
MAT-130< Section-8: Viva Session (Only 1 session):

On 1st May, Saturday, from 3pm to 5pm.

10.2: Polar Coordinates $(\boldsymbol{r}, \boldsymbol{\theta})$

Polar Coordinates is a 2-dimensional coordinate space that is equivalent to Rectangular. The coordinates denoted by $(\boldsymbol{r}, \boldsymbol{\theta})$**.**

There is only one axis in polar coordinates given by the positive x —axis. **The point origin is called the pole.**



The above figures illustrate the polar coordinates (r, θ) of the point P.

The polar coordinates of the point P is denoted by (r, θ) , where

r : the distance from the origin to the point P , and

θ : the angle between the positive x —axis and the line segment from the origin to P.

Relation Between Rectangular $(\boldsymbol{x}, \boldsymbol{y})$ and Polar $(\boldsymbol{r}, \boldsymbol{\theta})$ coordinates:

(1) Given polar coordinates (r, θ), to find the rectangular coordinates (x, y),

Set $\quad x = r\cos\theta\,, \qquad y = r\sin\theta$

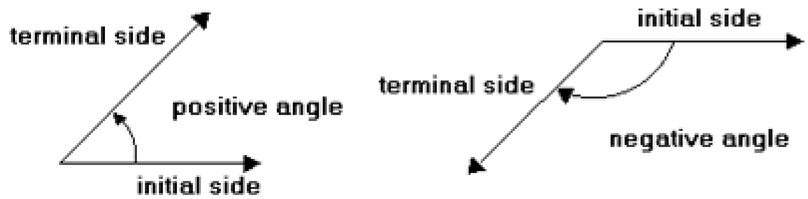
(2) Given the rectangular coordinates (x, y), to find the polar coordinates (r, θ), set

$$\boldsymbol{r} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2} \quad \text{or } r^2 = x^2 + y^2, \quad \text{and} \quad \theta = \begin{cases} \tan^{-1}(\frac{y}{x}) & \text{if } (x, y) \text{ is in Quadrant } I \text{ or } IV \\ \pi + \tan^{-1}(\frac{y}{x}) & \text{if } (x, y) \text{ is in Quadrant } II \text{ or } III \end{cases}$$

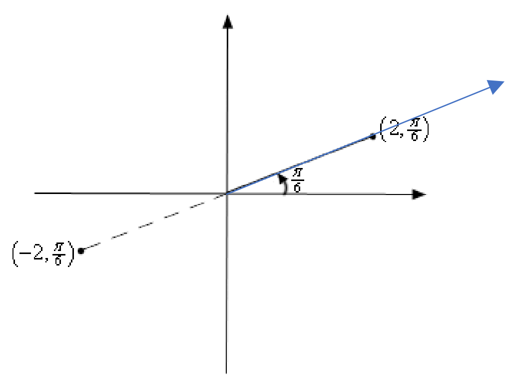
Note that the range of the function $\theta = tan^{-1}\left(\frac{y}{x}\right)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Note:

(1) **The angle $\boldsymbol{\theta}$ must be in radian**. θ is positive in anti-clockwise direction, and θ is negative in clockwise direction. The initial side of the angle θ is the polar axis.



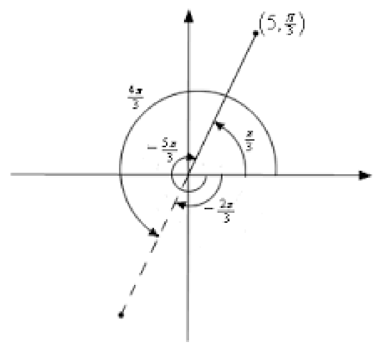
(2) r is always non-negative, that is, $r \geq 0$. $r = 0$ only at the pole (origin). So, $-r \leq 0$. To find the coordinates $(-r,\ \theta)$, we must go to the backward direction. Note that the polar coordinate is a vector coordinate, and $\boldsymbol{\theta}$ **gives the direction**. Backward direction means, beyond the point pole (origin) in the direction of θ. That is



Understanding Angles:

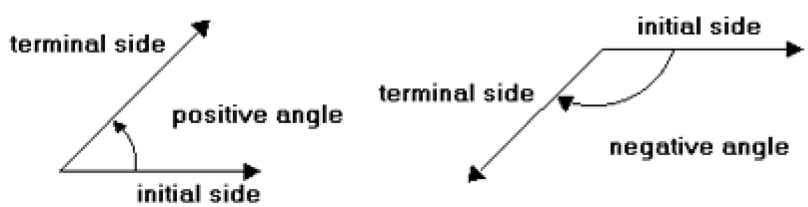
Notes:

(1) Presentation of a point in polar coordinates is not unique.

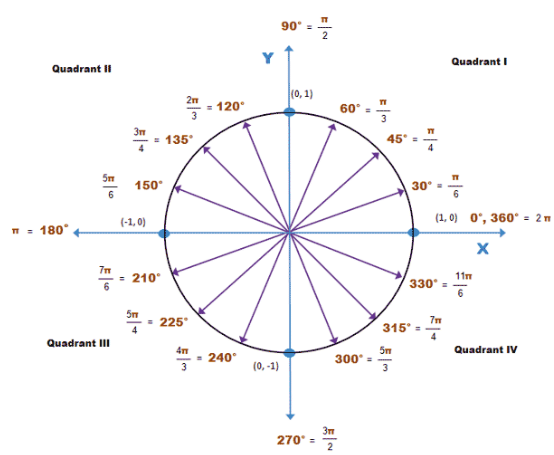


$$\left(5, \frac{\pi}{3}\right) = \left(5, 2n\pi + \frac{\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right)$$; where n is a positive integer.

(2) The angle θ must be in radian. The measurement of θ always starts from the polar axis (positive x –axis), and this line is called the initial side of the angle. And this line is fixed. Other side of the angle, that gives the distance of the point from the origin, is called the terminal side of the angle. Terminal side stars from the pole in outward direction. If you go beyond the pole, then the coordinate of points on the extended part of the terminal side is of the form $(-r, \theta)$, where $r > 0$. Note that $r = 0$ only at the pole.



(3) Relation between degree and radian:



$$1^\circ = \frac{\pi}{180} \text{ rad}$$

Exercise:1

(c) Find the rectangular form of the point $\left(-2, \frac{\pi}{3}\right)$.

Solution: Given polar form is $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\boldsymbol{-2}, \frac{\boldsymbol{\pi}}{\boldsymbol{3}}\right)$. To find (x, y), set

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$

Now, $x = r\cos\theta = (-2)\cos\frac{\pi}{3} = -2\,\left(\frac{1}{2}\right) = -1$

$$y = r\sin\theta = (-2)\sin\frac{\pi}{3} = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

Hence the rectangular form of the given point is $(x, y) = \left(-1, -\sqrt{3}\right)$.

Note: Do not write the coordinates in decimal form.

[Given the rectangular coordinates (x, y), to find the polar coordinates (r, θ), set

$$r = \sqrt{x^2 + y^2} \quad \text{or} \quad r^2 = x^2 + y^2, \quad \text{and} \quad \theta = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } (x, y) \text{ is in Quadrant } I \text{ or } IV \\ \pi + \tan^{-1}\frac{y}{x} & \text{if } (x, y) \text{ is in Quadrant } II \text{ or } III \end{cases}$$]

(d) Find the polar form of the point $\left(-2, 2\sqrt{3}\right)$.

Solution: Given the rectangular point $(x, y) = \left(-2, 2\sqrt{3}\right)$. This point is in Quadrant II.

So, set $r = \sqrt{x^2 + y^2}$ and $\theta = \pi + \tan^{-1}\frac{y}{x}$

Now, $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + \left(2\sqrt{3}\,\right)^2} = \sqrt{4 + 12} = 4$.

And $\theta = \pi + \tan^{-1}\frac{y}{x} = \pi + \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = \pi + \tan^{-1}\left(-\sqrt{3}\right) = \pi - \tan^{-1}\left(\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Hence the polar form of the point is $\left(4, \frac{2\pi}{3}\right)$.

Note: $y = tan^{-1}x$ is an odd function, and hence we get $tan^{-1}(-x) = -tan^{-1}x$

(e) Find the polar form of the point $\left(3, -\sqrt{3}\right)$. [Please do it!!]

Solution: Given the rectangular point $(x, y) = \left(3, -\sqrt{3}\right)$. This point is in Quadrant IV.

So, set $r = \sqrt{x^2 + y^2}$ and $\theta = \pi + \tan^{-1}\frac{y}{x}$

Ans: $(r, \theta) = \left(2\sqrt{3}\ ,\ \frac{5\pi}{6}\right)$

Summery of Lectutre-1:

(1) Definition of polar coordinates
(2) Relation between the polar coordinates and the rectangular coordinates
(3) Type of the coordinates of a point which is on the extended part of the terminal side of an angle θ.
(4) We did one exercise. **[Exercise-1]**

Exercise: 2

Find the equations of the following curves in polar coordinates:

(1) $x^2 + y^2 = 4$
(2) $x^2 + 2x + y^2 = 8$ $(Complete!)$
(3) $\boldsymbol{y = x}$ **(please do it!)**
(4) $y = \sqrt{3}\,x$

Solution: (1) We know that $x^2 + y^2 = r^2$

$x^2 + y^2 = 4 \Rightarrow r^2 = 4 \quad \therefore \quad r = 2$

Hence, $r = 2$ gives us the circle with center at the origin $(0, 0)$ and with radius 2.

Note: $\boldsymbol{r = k,\ \ k}$ **is a positive contant : the circle with center at the origin** $\boldsymbol{(0, 0)}$ **and with radius** $\boldsymbol{k}$**.**

Solution: (2) $x^2 + 2x + y^2 = 8$

$\Rightarrow x^2 + y^2 + 2x = 8$

$\Rightarrow r^2 + 2\, r\cos\theta = 8$

$\Rightarrow \ r^2 + 2r\cos\theta - 8 = 0$

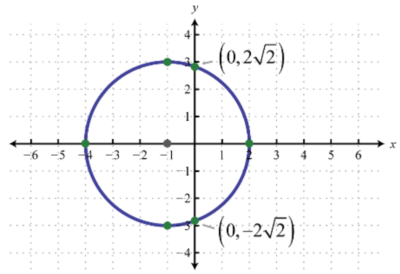
Complete : Solve for r.

Note: A Polar curve is usually denoted by $r = f(\theta)$

[Home task:

Consider $\boldsymbol{r^2 + 2r\cos\theta - 8 = 0}$ **as** $\boldsymbol{x^2 + x\,2\cos\theta - 8 = 0}$ **and use the formula for the Quadratic equation to solve for** $\boldsymbol{r}$**.**

Formula: For $\boldsymbol{ax^2 + bx + c = 0}$**, we get** $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$**.]**



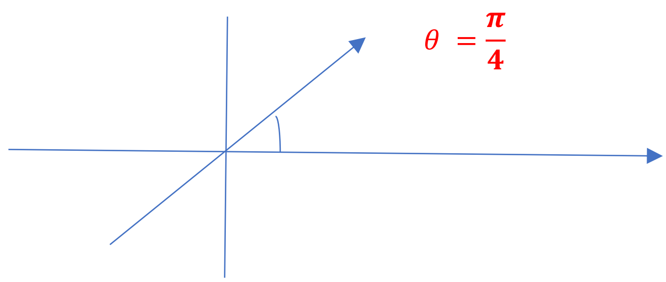
$x^2 + 2x + y^2 = 8 \;\; Or \;\; (x+1)^2 + y^2 = 3^2$

Solution: (3) $y = x \;\Rightarrow r\cos\theta \; = \; r\sin\theta$

$\Rightarrow \dfrac{\mathrm{r}\sin\theta}{\mathrm{r}\cos\theta} = 1 \Rightarrow \tan\theta \; = \mathbf{1}$

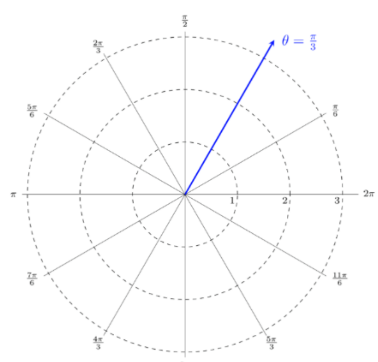
$\Rightarrow \;\; \theta \; = \frac{\boldsymbol{\pi}}{\mathbf{4}}$

Here, $\theta \; = \frac{\boldsymbol{\pi}}{\mathbf{4}}$ is the equation of the line $y = x$. In polar form $\theta \; = \frac{\boldsymbol{\pi}}{\mathbf{4}}$ is a ray, a line with one direction. The direction is given by the terminal side of θ .



Solution: (4) $y = \sqrt{3}\,x \Rightarrow r\sin\theta \; = \; \sqrt{3}\; r\cos\theta \;\Rightarrow \frac{r\sin\theta}{r\cos\theta} = \sqrt{3} \;\Rightarrow \tan\theta = \sqrt{3} \quad \therefore \theta = \frac{\pi}{3}$

So, the polar form of the equation $y = \sqrt{3}\,x$ is $\theta = \frac{\pi}{3}$; which is a Ray.



Note: If $\boldsymbol{\theta = \theta_0}$, $\boldsymbol{\theta_0}$ is a constant: A ray from the pole to infinity in outward direction.

Exercise: 3

Find the equations of the following curves in rectangular coordinates:

A) $r = 3$
B) $\theta = \frac{\pi}{4}$
C) $\boldsymbol{r = \sin\theta}$ [please do it]
D) $r = 3\cos\theta$
E) $\theta = \frac{5\pi}{4}$ (homework)
F) $r = -4\cos\theta$
G) $\boldsymbol{r = -5\sin\theta}$
H) $\theta = \frac{3\pi}{4}$ (homework)

Solutions:

Part (A): Given $r = 3 \Rightarrow \sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 3^2$, a circle with center at the origin and with radius 3.

Part (B) Given $\theta = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} \Rightarrow \frac{y}{x} = \tan\left(\frac{\pi}{4}\right) \Rightarrow \frac{y}{x} = 1 \quad \therefore \; y = x.$

Part (E) : Given $\theta = \frac{5\pi}{4} \Rightarrow \pi + \tan^{-1}\left(\frac{y}{x}\right) = \frac{5\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{5\pi}{4} - \pi$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{y}{x} = 1$ that is $y = x.$

Part (D) Given $r = 3\cos\theta \ldots\ldots\ldots (1)$

Since $x = r\cos\theta$, then $\cos\theta = \frac{x}{r}$. From (1):

$\Rightarrow r = 3\frac{x}{r} \Rightarrow r^2 = 3x$

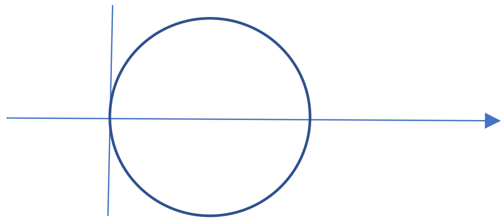
$\Rightarrow x^2 + y^2 = 3x$

$\Rightarrow (x^2 - 3x) + y^2 = 0$

$\Rightarrow \left(x^2 - 2\cdot x\cdot\frac{3}{2} + \left(\frac{3}{2}\right)^2\right) + y^2 = \left(\frac{3}{2}\right)^2$

$$\therefore \left(x - \frac{3}{2}\right)^2 + (y - 0)^2 = \left(\frac{3}{2}\right)^2.$$

$\left(x-\frac{3}{2}\right)^2+(y-0)^2 = \left(\frac{3}{2}\right)^2$ is the circle with center at the point $\left(\frac{3}{2},0\right)$ and radius $\frac{3}{2}$.



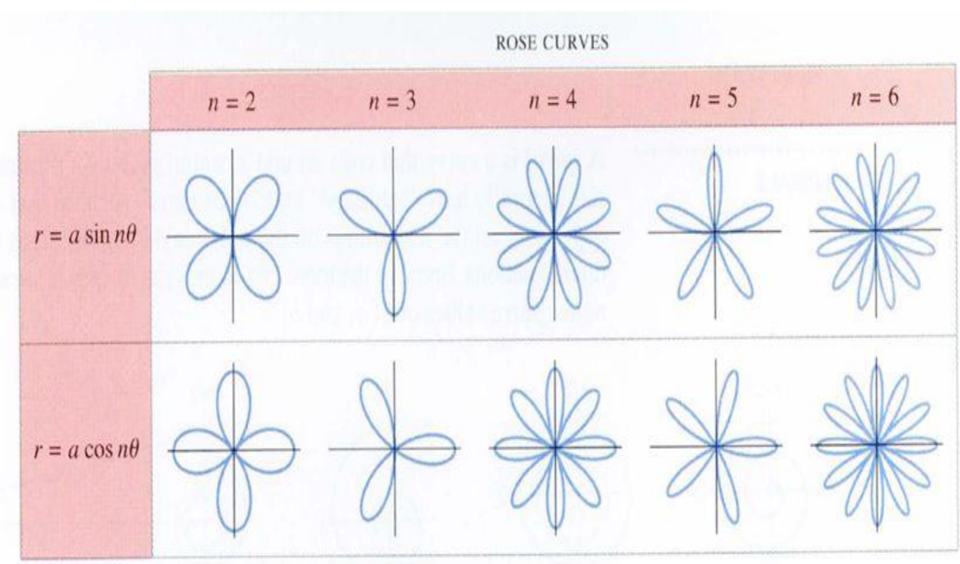
Note: $(\boldsymbol{x}-\boldsymbol{h})^2+(\boldsymbol{y}-\boldsymbol{k})^2 = \boldsymbol{r}^2$ **is a circle with center at the point** $(\boldsymbol{h},\boldsymbol{k})$ **and radius** $\boldsymbol{r}$.

Note: $r = constant$, $r = \text{a}\sin\theta$, $r = b\cos\theta$ are equations of circles, where a, b are constants.

$r = \text{a}\sin\theta$ is a circle of radius $\frac{|a|}{2}$ with center on the y −axis which is $\left(0,\frac{a}{2}\right)$.

Families of Rose Curves

- *Equations of the following form are called rose curves. Notice when n is odd it is the number of rose petals. When n is even, there are 2n rose petals.*



Equations: $\boldsymbol{r = a\sin(n\theta)}$, $\boldsymbol{r = a\cos(n\theta)}$; $\boldsymbol{n > 1}$, $\boldsymbol{a > 0}$.

<u>Notes:</u>

(1) If n is odd, then there are n number of petals in the rose. If n is even, then there are $2n$ number of petals in the rose.

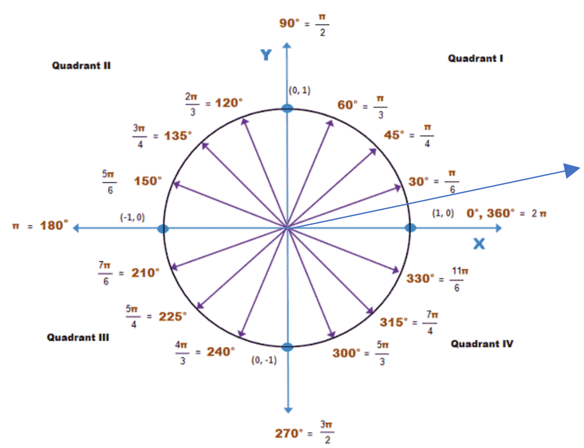
(2) Here $r = a\ \sin(n\theta)$, $r = a\cos(n\theta)$; $n > 1$, $a > 0$; a is the length of each petal.

(3) We divide the xy −plane in $2n$ number of sectors.

(4) If n is odd, then there are n number of petals in the rose where petals will be placed in the sectors alternately.

(5) If n is even, then there are $2n$ number of petals in the rose where petals will be placed in each sector.

(6) Each petal is symmetric with respect to it's axis which is the bisector of the sector.



<u>Exercise: 4</u>

Graph the rose $\boldsymbol{r = 2\sin(3\theta)}$.

Solution: Given $r = 2\ \sin(3\theta)\,,\ a = 2,\ n = 3.$

There are 3 petals.

Divide xy −plane in 2(3)=**6 sectors with rays.**

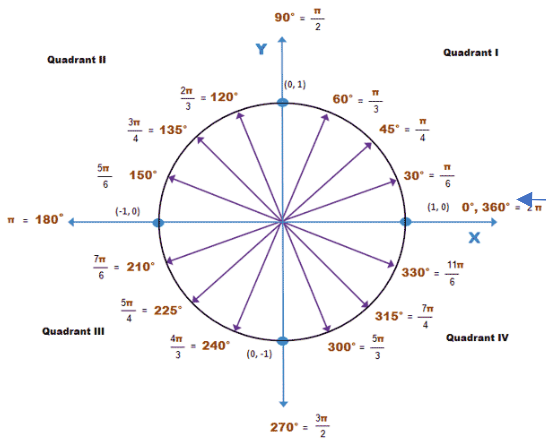
[Note: To find Rays, set $\boldsymbol{r = 0}$**.]**

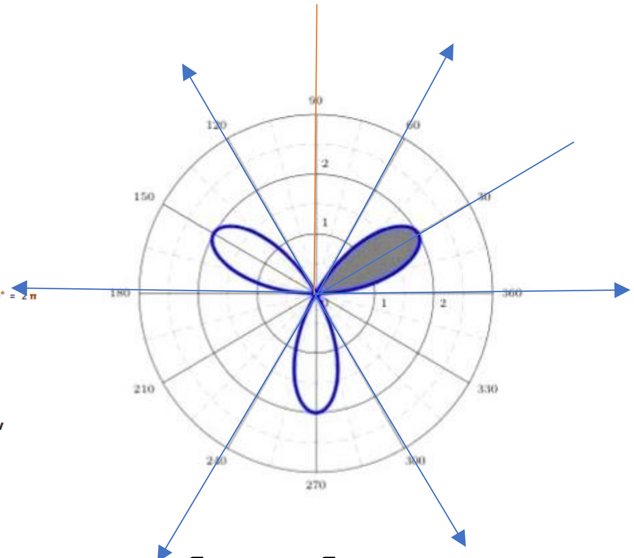
To find equations of rays, set $r = 0 \Rightarrow\ 2\sin(3\theta) = 0 \qquad \Rightarrow \sin(3\theta) = 0.$

$\Rightarrow \sin(3\theta) = 0$; Note that $\sin(k\pi) = 0$ for some integer k.

$\Rightarrow 3\theta = 0,\ \pi,\ \ 2\pi,\ \ 3\pi,\ 4\pi,\ 5\pi$

$\therefore \ \ \theta = 0, \ \frac{\pi}{3}, \ \ \frac{2\pi}{3}, \ \ \pi, \ \frac{4\pi}{3}, \ \frac{5\pi}{3}$ in $[0, 2\pi]$.





Bisector of the first sector (between the rays $\boldsymbol{\theta = 0}$ and $\boldsymbol{\theta = \frac{\pi}{3}}$) is $\boldsymbol{\theta = \frac{\pi}{6}}$: $r = 2\sin(3\theta)$

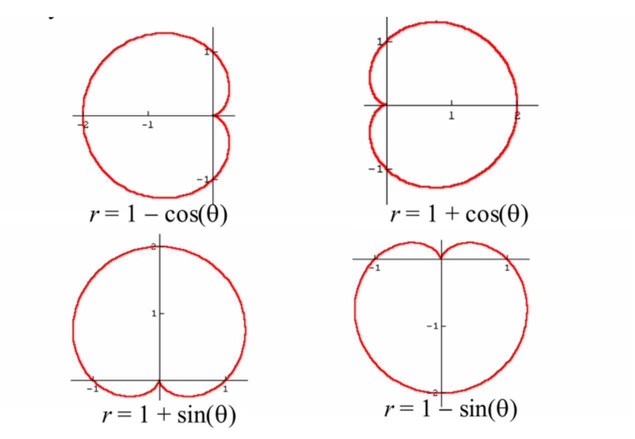
Here $r = 2\sin(3 \cdot \frac{\pi}{6}) = 2\sin\frac{\pi}{2} = 2$. Since $r = 2 > 0$, so this sector contains a petal. Point on the petal is $\left(2, \frac{\pi}{6}\right)$

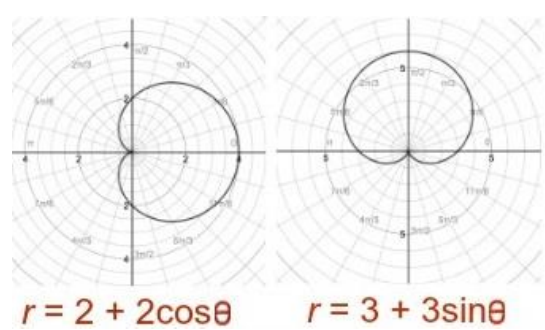
Bisector of the second sector (between the rays $\boldsymbol{\theta = \frac{\pi}{3}}$ and $\boldsymbol{\theta = \frac{2\pi}{3}}$) is $\boldsymbol{\theta = \frac{\pi}{2}}$: $r = 2\sin(3\theta)$

Here $r = 2\sin\left(3 \cdot \frac{\pi}{2}\right) = 2\sin\left(\frac{3\pi}{2}\right) = 2\,(-1) = -2$.

Since $r = -2 < 0$, so this sector contains no petal. Point on the petal is $\left(-2, \frac{\pi}{2}\right)$ which is in the sector opposite to it.

<u>Families of Cardioid Curves</u> $r = a \pm a\sin\theta$, $r = a \pm a\cos\theta$





$r = a \pm a\sin\theta$ symmtric with respect to $y - axis$.

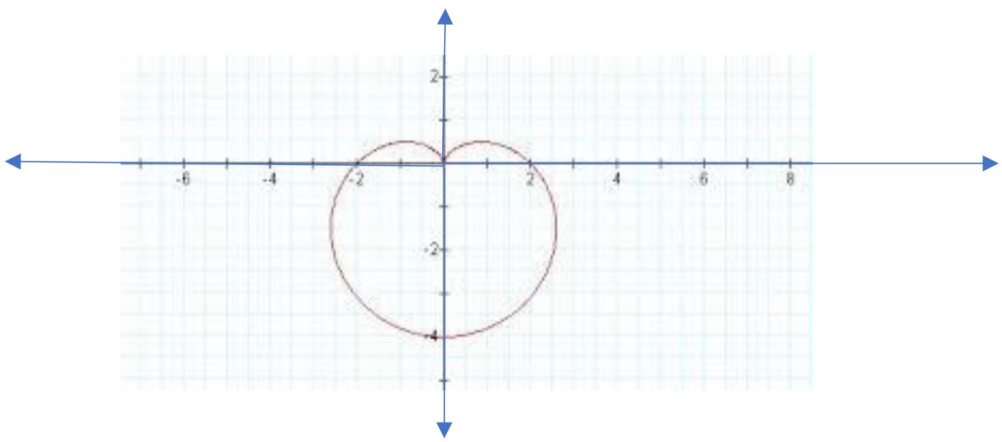
$r = a \pm a\cos\theta$ symmtric with respect to $x - axis$.

<u>Exercise: 5</u>

Graph the Cardioid $r = 2 - 2\sin\theta$.

Solution: Given $r = 2 - 2\sin\theta$.

θ	r	(r, θ)
0	2	$(2, 0)$
$\frac{\pi}{2}$	0	$\left(0, \frac{\pi}{2}\right)$
π	2	$(2, \pi)$
$\frac{3\pi}{2}$	4	$\left(4, \frac{3\pi}{2}\right)$



10.3: Polar Functions: Tangent to the Curve, Length, Area

Summery:

6. Finding slopes of tangent lines
7. Finding points where the tangent line is either vertical or horizontal
8. Finding length of a polar curve
9. Finding the area of a simple polar region

<u>Polar Functions:</u>

A polar function (polar curve) is given by (r, θ): $\boldsymbol{r = f(\theta)}$ in which r is a function of θ.

Parametric Form:

Given a polar curve $r = f(\theta)$.

This can be expressed parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\, cos\theta$ and $y = r\, sin\theta$.

That is, parametric equations are $x = f(\theta)cos\theta, \quad y = f(\theta)\, sin\theta;$ where θ is the parameter.

<u>TANGENT LINES TO POLAR CURVES</u>

[<u>Recall Calculus-1:</u> For the function $y = f(x)$, the slope of the tangent line to the curve of the function $y = f(x)$ at any x is given by $f'(x) = \frac{dy}{dx}$.]

Polar curves are given in parametric form by the equations $x = f(\theta)cos\theta, \quad y = f(\theta)sin\theta;$ where θ is the parameter and $f(\theta)$ has continuous first derivatives with respect to θ.

It can be proved that if $\frac{dx}{d\theta} \neq 0$, then y is a differentiable function of x, in which case the chain rule implies that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \dots\dots\dots (1)$$

This formula makes it possible to find $\frac{dy}{dx}$ directly from the parametric equations without eliminating the parameter.

<u>Exercise: 1</u>

Find the slope of the tangent line to the circle $r = cos\theta,\ 0 \leq \theta \leq 2\pi$, at the point where $\theta = \frac{\pi}{4}$.

<u>Solution:</u> The slope of the tangent line is given by

$$\boldsymbol{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}} \dots\dots\dots\dots (\mathbf{1})$$

Given Polar Curve: $r = cos\theta$

The parametric equations of the circle is given by

$x = r\, cos\theta = \cos\theta\, \cos\theta = \cos^2\theta$

$y = r\, sin\theta = \cos\theta \sin\theta = \frac{1}{2}\sin(2\theta)$

[Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$. $\quad$ ***For*** $\frac{d}{d\theta}(\cos\theta)^2, f(\theta) = \theta^2, \; g(\theta) = \cos\theta$ **]**

Since,
$x = \cos^2\theta$

We have, $\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos^2\theta) = \frac{d}{d\theta}(\cos\theta)^2 = 2\cos\theta\,(-\sin\theta) = -\sin(2\theta)$,

And $y = \frac{1}{2}\sin(2\theta)$, so we get

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\sin(2\theta)\right) = \frac{1}{2}\cos(2\theta) \cdot 2 = \cos(2\theta)$$

So, from equation (1): $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos(2\theta)}{-\sin(2\theta)}$

At $\theta = \frac{\pi}{4}$, $\quad [\frac{dy}{dx}]_{\theta=\frac{\pi}{4}} = \frac{\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = \frac{0}{-1} = 0$.

Hence, the slope of the tangent line is $\frac{dy}{dx} = 0$.

That is, the tangent line to the polar curve $r = \cos\theta$ at $\theta = \frac{\pi}{4}$ is horizontal.

Done!!!

<u>**Note:**</u> $\frac{0}{non-zero} = 0$, $\quad \frac{non-zero}{0} = undefined$, $\quad \frac{0}{0} = indeterminant$

<u>**Notes:**</u>

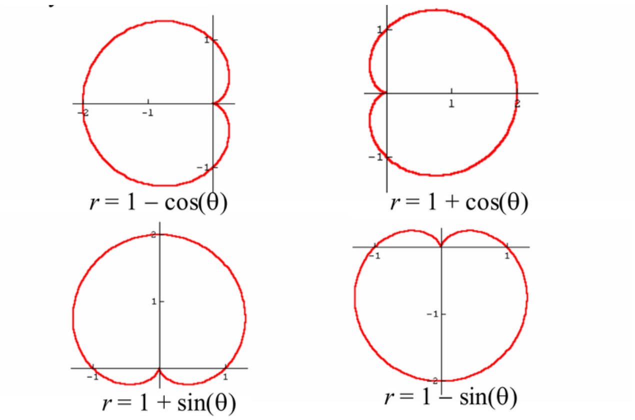
The slope of the tangent line is given by

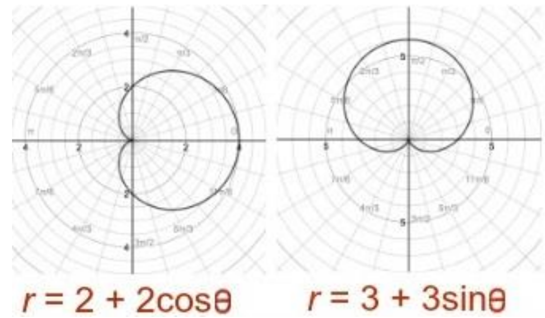
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad \dots\dots\dots\dots(1)$$

(3) The tangent line to the parametric curve $x = f(\theta)cos\theta$, $y = f(\theta)sin\theta$ is vertical if and only if $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ =undefined, that is, if and only if $\left[\frac{dy}{d\theta} \neq 0 \text{ and } \frac{dx}{d\theta} = 0\right]$.

(4) The tangent line to the parametric curve $x = f(\theta)cos\theta$, $y = f(\theta)sin\theta$ is **horizontal** if and only if
$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = 0$, that is, if and only if $\left[\frac{dy}{d\theta} = 0 \text{ and } \frac{dx}{d\theta} \neq 0\right]$.

Families of Cardioid Curves: $\quad r = a(1 \pm \cos\theta), \qquad r = a(1 \pm \sin\theta)$





Observation:

(1) $r = a(1 \pm \cos\theta)$ **is symmetric with respect to** x **−axis.**

(2) $r = a(1 \pm \sin\theta)$ **is symmetric with respect to** y **−axis.**

(3) **Domain of a Cardioid is** $[0,\ 2\pi]$

Exercise: 2

Find the values of θ where the tangent line to the Cardioid $r = 1 - \cos\theta$ is either horizontal or vertical.

Solution:

We know that

(i) vertical tangent iff $\left[\frac{dy}{d\theta} \neq 0 \quad \text{and} \quad \frac{dx}{d\theta} = 0\right]$

(ii) Horizontal tangent iff $\left[\frac{dy}{d\theta} = 0 \quad \text{and} \quad \frac{dx}{d\theta} \neq 0\right]$

Consider the parametric of the polar curve $r = 1 - \cos\theta$

$$x = (1 - \cos\theta)\cos\theta, \quad y = (1 - \cos\theta)\sin\theta, \quad \mathbf{0 \leq \theta \leq 2\pi.}$$

$$\frac{dx}{d\theta} = (1 - \cos\theta)\ \frac{d}{d\theta}(\cos\theta) + \cos\theta\ \frac{d}{d\theta}(1 - \cos\theta\ )$$

$$= (1 - \cos\theta\ )\ (-\sin\theta\ ) + \cos\theta\sin\theta$$

$$= \ -\sin\theta + 2\sin\theta\cos\theta$$

$$\therefore \quad \frac{dx}{d\theta} = \sin\theta\,(2cos\theta\ - 1)$$

$$y = (1 - \cos\theta)\sin\theta$$

$\frac{dy}{d\theta} = (1 - \cos\theta)\ \frac{d}{d\theta}(\sin\theta) + \sin\theta\ \frac{d}{d\theta}(1 - \cos\theta\ )$

$= (1 - \cos\theta)\cos\theta + \sin\theta\sin\theta\ = \cos\theta - \cos^2\theta + \sin^2\theta$

$= \cos\theta - \cos^2\theta + 1 - \cos^2\theta$

$= 1 + \cos\theta - 2\cos^2\theta$

$= 1 - \cos\theta + 2\cos\theta - 2\cos^2\theta$

$= 1(1 - \cos\theta) + 2\cos\theta\,(1 - \cos\theta)$

$$\therefore\ \frac{dy}{d\theta} = (1 - cos\theta)(1 + 2cos\theta)$$

Now, Set, $\frac{dx}{d\theta} = \sin\theta\,(2cos\theta\ - 1) = 0$, then $\sin\theta = 0$, $2cos\theta\ - 1 = 0$

$\Rightarrow\ \sin\theta = 0$, $\cos\theta = \frac{1}{2}$; $0 \le \theta \le 2\pi.$

We know that, $\sin\theta = 0\ \Rightarrow\ \theta = 0,\ \pi,\ 2\pi,$ and $\cos\theta = \frac{1}{2}\ \Rightarrow\ \theta = \frac{\pi}{3}\ ,\ 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$$\frac{dx}{d\theta} = 0 :\ \theta\ = 0,\ \frac{\pi}{3},\ \ \pi,\ \ \frac{5\pi}{3}\ \ , 2\pi$$

If $\frac{dy}{d\theta} = (1 - cos\theta)(1 + 2cos\theta) = 0,$

then $\cos\theta = 1,\ \ \cos\theta = -\frac{1}{2}$ for $0 \le \theta \le 2\pi.$

Here, $\cos\theta = 1\ \Rightarrow\ \theta = 0,\ 2\pi$

and $\cos\theta = -\frac{1}{2} = -\cos\frac{\pi}{3}$

$\Rightarrow \cos\theta = \cos(\pi \pm \frac{\pi}{3})$; $[-\cos\varphi = \cos(\pi \pm \varphi)$ where φ is an acute angle.$]$

$\Rightarrow\ \theta = \pi \pm \frac{\pi}{3}$, that is, $\theta = \frac{2\pi}{3}\ ,\ \frac{4\pi}{3}$

$$\frac{dy}{d\theta} = 0:\ \theta\ = 0,\ \ \frac{2\pi}{3},\ \ \frac{4\pi}{3},\ \ 2\pi$$

Now:

$\boldsymbol{\frac{dy}{d\theta} = 0:\ \theta\ = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi}$ **,** **and**

$\boldsymbol{\frac{dx}{d\theta} = 0 :\ \theta\ = 0,\ \frac{\pi}{3},\ \ \pi,\ \ \frac{5\pi}{3}\ \ ,2\pi}$

(a) The tangent line if vertical iff $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = undefined$, that is, iff $\theta = \frac{\pi}{3}$, π, $\frac{5\pi}{3}$

(b) The tangent line if horizontal iff $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = 0$, that is, iff $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Exercise: 3 (homework)

Consider the parametric curve $x = 3\sin\theta$, $y = 4\cos\theta$, $0 \le \theta \le \pi$.

3) Find θ —values where the tangent line is horizontal.
4) Find θ —values where the tangent line is vertical.

Solution:

Arc Length of a polar curve:

Definition: If no segment of the polar curve $\boldsymbol{r = f(\theta)}$ is traced more than once as θ increases from α to β, and if $\frac{dr}{d\theta} = f'(\theta)$ is continuous for $\alpha \le \theta \le \beta$, then the arc length L from $\theta = \alpha$ to $\theta = \beta$ is

$$\boldsymbol{L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\; d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta} \text{ ... ... (2)}$$

.

Exercise: 4 (homework)

Find the arc length of the spiral $r = e^{3\theta}$ between $\theta = 0$ and $\theta = \pi$.

$Note: \sqrt{e^{6\theta}} = e^{\frac{6\theta}{2}} = e^{3\theta}$

Solution:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta \text{ ... ... } (1)$$

Given $r = e^{3\theta}$, $\frac{dr}{d\theta} = 3\, e^{3\theta}$. Then, $\mathrm{r}^2 = \left(e^{3\theta}\right)^2 = e^{6\theta}$

and $\left(\frac{dr}{d\theta}\right)^2 = \left(3\, e^{3\theta}\right)^2 = 9\, e^{6\theta}$, $0 \le \theta \le \pi$

$$L = \int_{0}^{\pi} \sqrt{e^{6\theta} + 9\, e^{6\theta}}\; d\theta$$

$$= \int_{0}^{\pi} \sqrt{10\, e^{6\theta}}\; d\theta = \int_{0}^{\pi} \sqrt{10}\, e^{3\theta}\; d\theta$$

$$= \frac{\sqrt{10}}{3}\left[e^{3\theta}\right]_0^{\pi} = = \frac{\sqrt{10}}{3}\left(e^{3\pi} - 1\right) unit$$

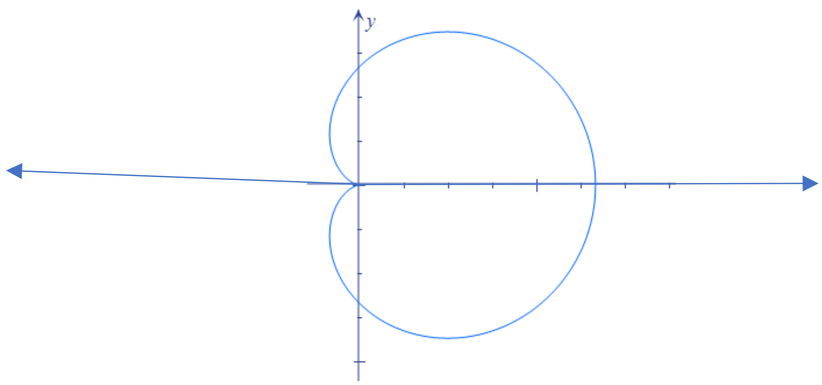
Exercise: 5

Find the total arc length of the cardioid $r = 1 + cos\theta$.

Solution:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \dots\dots\dots (1)$$

Given Cardioid $r = 1 + cos\theta$. Hence $0 \leq \theta \leq 2\pi$.



$\frac{dr}{d\theta} = -\sin\theta.$

Now, from equation (1), Length of the Cardioid,

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{2(1+\cos\theta\,)}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2\left(1+\cos\left(2\left(\frac{\theta}{2}\right)\right)\right)}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 \cdot 2\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \quad ;$$

$[\cos^2 x = \frac{1}{2}[1+\cos(2x)] \Rightarrow \ 1+\cos(2x) = 2\cos^2 x\,.$ If $x = \frac{\theta}{2}$, then $2x = \theta.]$

$$= \int_0^{2\pi} \sqrt{\left(2\cos\frac{\theta}{2}\right)^2}\, d\theta \quad ; \qquad \sqrt{x^2} = |x| \ \text{ for any real number } x.$$

$$= \int_0^{2\pi} 2\left|\cos\frac{\theta}{2}\right|\, d\theta$$

$$= 2\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right|\, d\theta$$

$$= 2\cdot 2\int_0^{\pi} \left|\cos\frac{\theta}{2}\right|\, d\theta\ ; \qquad 0 \le \theta \le \pi \quad ; \quad |x| = \begin{cases} -x, & x < 0 \\ x, & x \ge 0 \end{cases}$$

; [$\boldsymbol{r = a(1 \pm \cos\theta)}$ **is symmetric with respect to** $\boldsymbol{x}$ **−axis.]**

$$= 2\cdot 2\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta \quad ; \qquad \cos\frac{\theta}{2} \ \ge 0 \ \text{ for } \ 0 \le \frac{\theta}{2} \le \frac{\pi}{2}.$$

[The graph $r = 1+\cos\theta$ is symetric with respect to $x -$ axis . $If \ \ 0 \le \theta \le \pi,\ then\ 0 \le \frac{\theta}{2} \le \frac{\pi}{2}.$]

$$= 4\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta \ = 4\left[\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= 8\left[\sin\frac{\theta}{2}\right]_0^{\pi} \ = 8(1-0) = 8\, unit$$

[TANGENT LINES TO POLAR CURVES AT THE ORIGIN

Formula (1) reveals some useful information about the behavior of a polar curve $r = f(\theta)$ that passes through the origin. If we assume that $\boldsymbol{r = 0}$ **and** $\boldsymbol{f'(\theta) = dr/d\theta \neq 0}$ when $\theta = \theta_0$, then it follows from Formula (1) that the slope of the tangent line to the parametric curve

$x = f(\theta)cos\theta,\ y = f(\theta)sin\theta$ at $\theta = \theta_0$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}]_{\theta=\theta_0}}{\frac{dx}{d\theta}]_{\theta=\theta_0}} = \frac{0 + sin\theta_0 \frac{dr}{d\theta}}{0 + cos\theta_0 \frac{dr}{d\theta}} = \frac{sin\theta_0}{cos\theta_0} = tan\theta_0$$

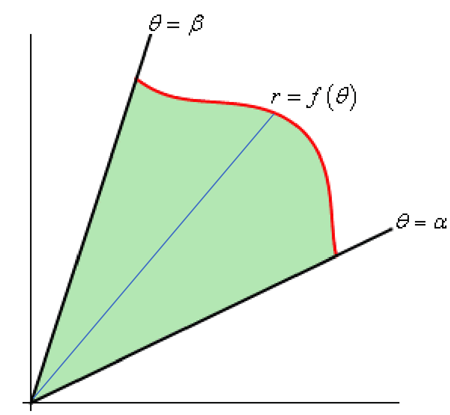
However, $tan\theta_0$ is also the slope of the line $\theta = \theta_0$, so we can conclude that this line is tangent to the curve at the origin. Thus, we have established the following result]

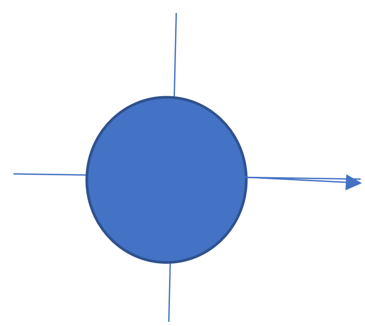
Theorem: 1 If the polar curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, and if $\frac{dr}{d\theta} \neq 0$ at $\theta = \theta_0$, then the line $\theta = \theta_0$ is tangent to the curve at the origin.

Area in polar coordinates:

If α and β are angles that satisfy the condition $\boldsymbol{\alpha < \beta \leq \alpha + 2\pi}$ and if $f(\theta)$ is continuous and either non-negative or non-positive for $\alpha \leq \theta \leq \beta$, then the area A of the region R enclosed by the polar curve $\boldsymbol{r = f(\theta)}$ $(\alpha \leq \theta \leq \beta)$ and the lines $\boldsymbol{\theta = \alpha}$ and $\boldsymbol{\theta = \beta}$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \quad \ldots\ldots\ldots (3)$$





Area in Polar Coordinates: Limits of Integration

Step 1. **Sketch** the region R whose area is to be determined.

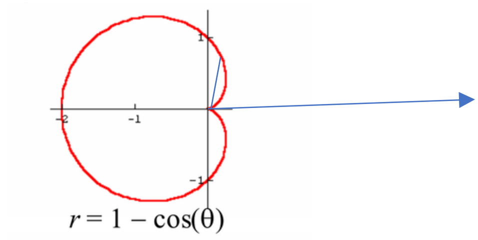
Step 2. Draw an arbitrary "radial line" from the pole to the boundary curve $\boldsymbol{r = f(\theta)}$.

Step 3. Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"

Step 4. Your answer in Step 3 will determine the lower and upper limits of integration.

Exercise: 6

(a) Find the area of the region in the **first quadrant** that is within the cardioid $r = 1 - cos\theta$.



Given $r = 1 - cos\theta$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{0}^{\frac{\pi}{2}} \frac{1}{2} [1 - \cos\theta]^2 \, d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2} [1 - 2\cos\theta + \cos^2\theta] \, d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2} [1 - 2\cos\theta + \frac{1}{2}(1 + \cos(2\theta))] \, d\theta$$

$$= \frac{1}{4} \int_{0}^{\frac{\pi}{2}} [2 - 4\cos\theta + 1 + \cos(2\theta)] \, d\theta$$

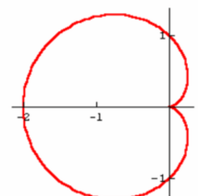
$$= \frac{1}{4} \int_{0}^{\frac{\pi}{2}} [3 - 4\cos\theta + \cos(2\theta)] \, d\theta$$

$$= \frac{1}{4} \left[3\theta - 4\sin\theta + \frac{1}{2}\sin(2\theta) \right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{4} \left(3\,\frac{\pi}{2} - 4 \right)$$

$$= \frac{1}{8} \; (3\pi - 8) \;\; unit^2$$

(b) Find the area of the region that is within the cardioid $r = 1 - cos\theta$.

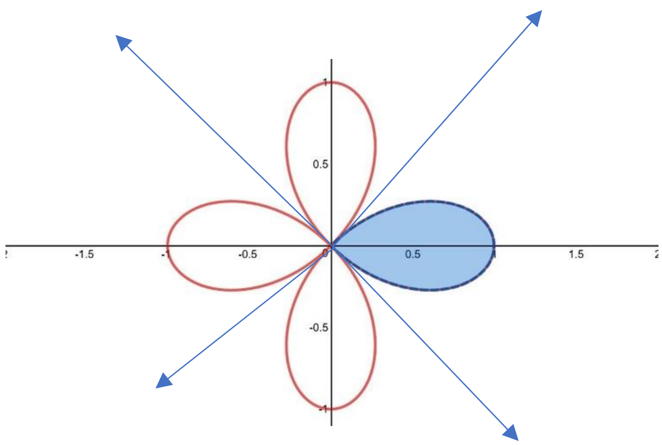


$r = 1 - \cos(\theta)$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{0}^{2\pi} \frac{1}{2} [1 - \cos\theta]^2 \, d\theta = \frac{1}{4}\left[3\theta - 4\sin\theta + \frac{1}{2}\sin(2\theta)\right]_0^{2\pi} = \frac{3\pi}{2} \; unit^2$$

(c) Find the area of the region enclosed by the rose curve $r = \cos 2\theta$.

Solution:



To find rays, set $r = 0$.

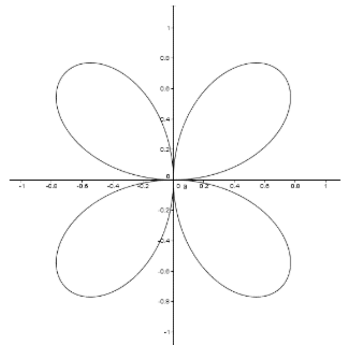
If $r = \cos(2\theta) = 0$; $[\cos(2n+1)\frac{\pi}{2} = 0]$

$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$. Then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. If we want to take the area of the first petal , then the interval must be $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, **since interval starts with smallest number and ends with the largest number.**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = 4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} [\cos(2\theta)]^2 \, d\theta = 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \, d\theta$$

$$= 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} [1 + \cos(4\theta)] \, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [1 + \cos(4\theta)] \, d\theta = \left[\theta + \frac{1}{4}\sin(4\theta)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{2} \; unit^2$$

(d) Find the area of the region enclosed by the rose curve $r = \sin 2\theta$.



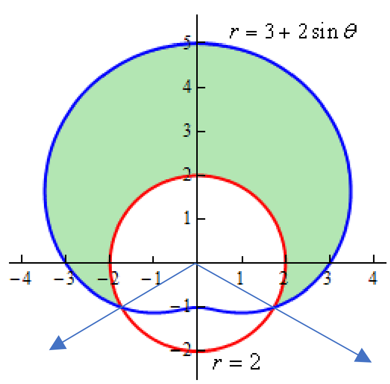
If $r = \sin 2\theta = 0$, then $\theta = 0\,,\ \frac{\pi}{2}\,,\ \pi\,,\ \frac{3\pi}{2}$. If we want to take the area of the petal in the first quadrant, then the interval must be $\left[0\,,\ \frac{\pi}{2}\right]$.

$$A = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}[\sin(2\theta)]^2\ d\theta = 2\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}[1-\cos(4\theta)]\ d\theta = \int_0^{\frac{\pi}{2}} [1-\cos(4\theta)]\ d\theta = \left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\ \ unit^2$$

Exercise: 7

Find the area of the region that is inside the Dimpled limaçon $r = 3 + 2\sin\theta$ outside the circle $r = 2$.



Given $r = \mathbf{3 + 2\sin\theta}$ **and** $\boldsymbol{r = 2}$.

To find the intersection:

Set $\mathbf{2 = 3 + 2\sin\theta}$

$\Rightarrow \sin\theta = -\dfrac{1}{2}$

$\therefore\ \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$, and $\theta = -\frac{\pi}{6}$

[To move anti-clockwise **across the region**, $\theta = -\frac{\pi}{6}$]

So**, interval** $[-\boldsymbol{\frac{\pi}{6}}, \boldsymbol{\frac{7\pi}{6}}]$

Area of the region is on the interval $[-\frac{\pi}{6}, \frac{7\pi}{6}]$ is inside the Dimpled limaçon $r = 3 + 2\sin\theta$ and outside the circle $r = 2$.

$\boldsymbol{A}$ = **Area of the Dimpled limaçon on** $[-\boldsymbol{\frac{\pi}{6}}, \boldsymbol{\frac{7\pi}{6}}]$ − **Area of the circle on** $[-\boldsymbol{\frac{\pi}{6}}, \boldsymbol{\frac{7\pi}{6}}]$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}[3 + 2\sin\theta]^2\ d\theta \quad - \quad \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}[2]^2\ d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} [9 + 12\sin\theta + 4\sin^2\theta]\ d\theta - \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2\ d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left[9 + 12\sin\theta + 4\cdot\frac{1}{2}(1 - \cos(2\theta))\right] d\theta - \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2\ d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} [9 + 12\sin\theta + 2 - 2\cos(2\theta)]\ d\theta - \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2\ d\theta$$

$$= \frac{1}{2}\left[9\theta + 12\,(-\cos\theta) + 2\theta - 2\frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} - \quad [2\theta\,]_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{1}{2}[11\theta - 12\cos\theta - \sin(2\theta)\,]_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} - \quad [2\theta\,]_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{1}{2}\left[11\left(\frac{7\pi}{6}\right) - 12\cos\frac{7\pi}{6} - \sin\left(\frac{7\pi}{3}\right) - 11\left(-\frac{\pi}{6}\right) + 12\cos\left(-\frac{\pi}{6}\right) + \sin\left(-\frac{\pi}{3}\right)\right]$$

$$-\left[2\left(\frac{7\pi}{6}\right) - 2\left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\frac{77\pi}{6} - 12\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + \frac{11\pi}{6} + 12\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right] - \frac{14\pi}{6} - \frac{2\pi}{6}$$

$$= \frac{77\pi}{12} + \frac{11\pi}{12} - \frac{\sqrt{3}}{2} - \frac{7\pi}{3} - \frac{\pi}{3}$$

$$= \frac{88-32}{12}\pi - \frac{\sqrt{3}}{2}$$

$$= \frac{56}{12}\pi - \frac{\sqrt{3}}{2}$$

$$= \frac{1}{12}\left[56\pi - 6\sqrt{3}\right] \quad unit^2$$

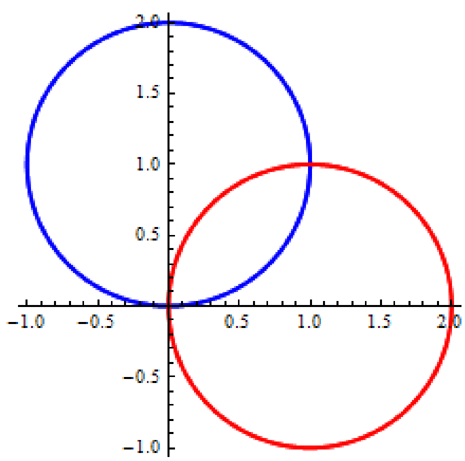
Exercise: 8

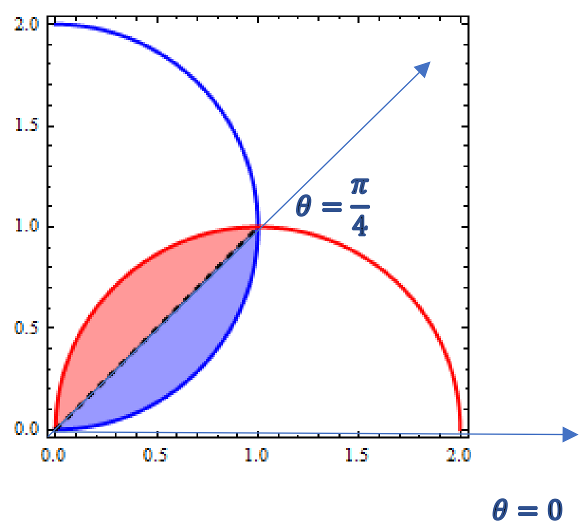
Find the Area of the polar region bounded by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$.

Solution:

$r = 2\cos\theta \;\Rightarrow (x-1)^2 + y^2 = 1^2 \;:\; Center = (1,0), Radius = 1$

and $r = 2\sin\theta \Rightarrow x^2 + (y-1)^2 = 1^2 \;:\; Center = (0,1)\; Radius = 1$





$AREA = 2\; \times$ area of the bluish region

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}[2\sin\theta]^2\; d\theta$$

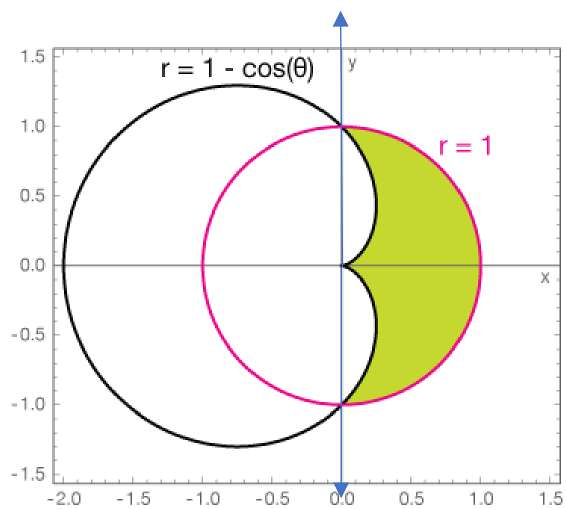
$$= 4\int_{0}^{\frac{\pi}{4}} \frac{1}{2}[1-\cos(2\theta)]\,d\theta$$

$$= 2\left[\theta - \frac{\sin(2\theta)}{2}\right]_{0}^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{2} - 1\right)\,unit^2$$

Exercise: 9

Find the area of the polar region that is inside the circle $r = 1$ and ourside the cardioid $r = 1 - \cos\theta$.



Interval = $\left[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}\right]$

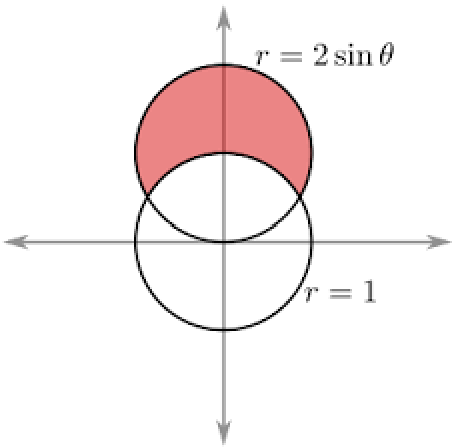
Area of the region = On the interval $\left[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}\right]$, Area of the Circle - area of the cardioid

$$= \frac{1}{2}\pi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}[1-\cos\theta]^2\ d\theta$$

Complete !!

Exercise: 10

Find the area of the polar region that is outside the circle $r = 1$ and ourside the cardioid $r = 2\sin\theta$.



Set $2\sin\theta = 1$. So, $\sin\theta = \frac{1}{2}$.

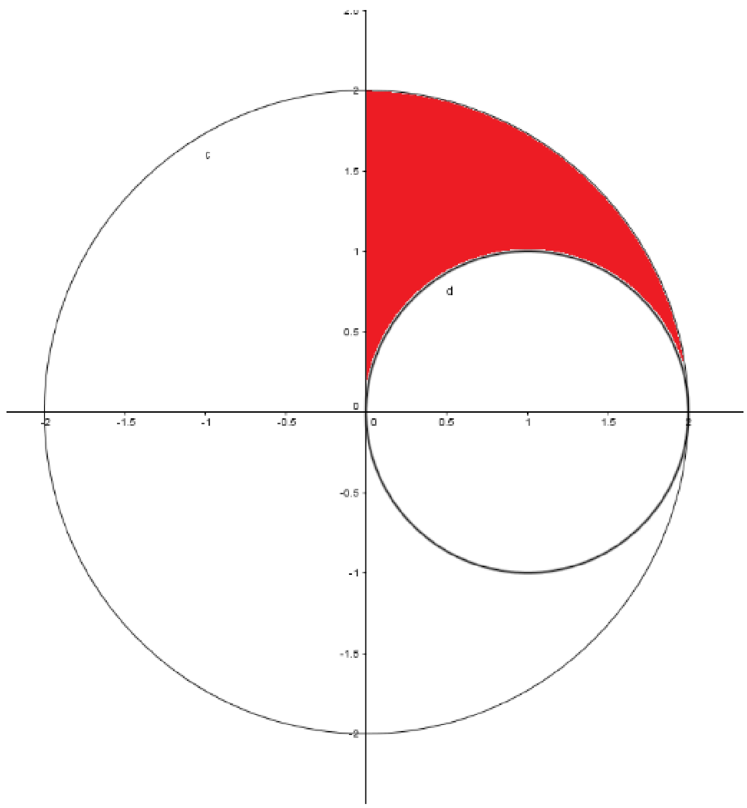
Interval = $\left[\frac{\pi}{6}\ ,\ \frac{5\pi}{6}\right]$

Area = On the interval $\left[\frac{\pi}{6}\ ,\ \frac{5\pi}{6}\right]$, area of the Circle $r = 2\sin\theta -$ area of the circle $r = 1$

Exercise: 11

Find the area of the polar region in the first quadrant that is inside the circle $r = 2$ and outside the circle $r = 2\cos\theta$.

$r = 2\cos\theta \ \Rightarrow (x-1)^2 + y^2 = 1^2 \ : \ Center = (1,0), Radius = 1$



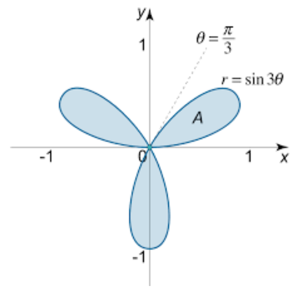
Interval = $\left[0\ ,\ \frac{\pi}{2}\right]$

Area = On the interval $\left[0\ ,\ \frac{\pi}{2}\right]$, area of the Circle $r = 2 -$ area of the Circle $r = 2\cos\theta$

Exercise: 12

Find the area of the rose $r = \sin 3\theta$.

All yours!!!



y
1
$\theta = \frac{\pi}{3}$
$r = \sin 3\theta$
A
-1
0
1
x
-1