DAY-3:

7.3: Integration of Trigonometric Functions

→ There are 7 groups, where each group has 3 sub-groups.

Group -1: All six trigonometric functions with power 1

1) $\int \sin x \; dx \; = \; -\cos x + C$
2) $\int \cos x \; dx \; = \; \sin x + C$

3) $\int \tan x \; dx \; = \; \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{\cos x} \, \mathbf{\sin x \, dx}$

[Set $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$, that is, $du = -\sin x \; dx$. Hence, $-du = \mathbf{\sin x \; dx}$]

$$= \int \frac{1}{u} (-1) du = -\int \frac{1}{u} \, du \quad ; \quad [\text{Note: When } \frac{1}{u} \text{ is given, we only know that } u \neq 0]$$

$= -\ln |u| + C$

$= -\ln |\cos x| + C$

$= \ln |\cos x|^{-1} + C$; $[n \log_b x = \log_b x^n]$

$= \ln |(\cos x)^{-1}| + C$

$= \ln |\frac{1}{\cos x}| + C$

$\int \tan x \; dx \; = \ln |\sec x| + C$

4) $\int \cot x \; dx \; = \; \int \frac{\cos x}{\sin x} \, dx = \int \frac{1}{\sin x} \cos x \, dx$

$$= \int \frac{1}{u} \; du \, ; \; set \;\; u = \sin x \rightarrow \; du = \cos x \; dx$$

$= \ln|\sin x| + C$

Example: $\int \; 3x^2 \, dx \; = x^3 + C$

5) $\int \sec x \; dx = \int \sec x \; \cdot 1 \, dx$; $1 = \frac{\sec x + \tan x}{\sec x + \tan x}$

$$= \int \frac{\sec x \ (\tan x + \sec x)}{\tan x \ + \ \sec x} \ dx$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} \ dx == \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \ dx \ ;$$

[Set $u = \tan x + \sec x$, then $\frac{du}{dx} = \sec^2 x \ + \sec x \ \tan x$. Hence, $du = (\sec^2 x \ + \sec x \ \tan x\ )dx$]

$$\int \sec x \ dx = \int \frac{1}{u} \ du \ = \ \ln|u| + C$$

$$\int \sec x \ dx \ = \ \ln|\tan x + \sec x| + C$$

Definition: Logarithmic Derivative

Since $\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)} = \frac{\frac{d}{dx}(D)}{D}$, $\frac{f'(x)}{f(x)}$ fraction is called the logarithmic derivative. And hence,

$$\int \frac{f'(x)}{f(x)} \ dx \ = \ln|f(x)| + C.$$

Note: (1) $\int \sec x \tan x \, dx \ = \sec x + C$ and $\int \sec^2 x \, dx \ = \tan x + C$

(2) $\int \csc x \cot x \, dx \ = -\csc x + C$ and $\int \csc^2 x \, dx \ = -\cot x + C$

Also, $\cot^2 x + 1 = \csc^2 x \ \Rightarrow \ \mathbf{1 = \csc^2 x - \cot^2 x}$

6) $\int \csc x \ dx$ Homework

$$\int \csc x \ dx = \int \csc x \ \cdot 1 \ dx = \int \csc x \ \cdot \frac{\csc x + \cot x}{\csc x + \cot x} \ dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \ dx$$

$$= \int \frac{\csc x \cot x + \csc^2 x}{\csc x + \cot x} \ dx \ ;$$

set $u = \csc x + \cot x$, then $\dfrac{du}{dx} = -\csc x \cot x - \csc^2 x \rightarrow -du = (\csc x \cot x + \csc^2 x)dx$

$$= \int \frac{1}{u}(-1)\, du = -\ln|u| + C$$

$$= -\ln|\csc x + \cot x| + C$$

$= \ln|\csc x + \cot x|^{-1} + C$; Note: $x^{-1} = \frac{1}{x}$.

$= \ln\left|\dfrac{1}{\csc x + \cot x}\right| + C$; Formula: $1 + \cot^2 x = \csc^2 x$

$$= \ln\left|\frac{\csc^2 x - \cot^2 x}{\csc x + \cot x}\right| + C$$

$$= \ln|\csc x - \cot x| + C,$$

Alternative Method:

$\int \csc x \; dx = \int \frac{1}{\sin x}\, dx =$

$$= \int \frac{1}{2\sin(\frac{x}{2})\cos\left(\frac{x}{2}\right)}\, dx$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right)}{2\sin(\frac{x}{2})\cos\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2\left(\frac{x}{2}\right)}{\sin(\frac{x}{2})\cos\left(\frac{x}{2}\right)\frac{1}{\cos^2\left(\frac{x}{2}\right)}}\, dx$$

$= \displaystyle\int \frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)}{\tan(\frac{x}{2})}\, dx$; $Set\ u = \tan(\frac{x}{2})$, then $du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$.

$$= \int \frac{1}{u}\, du\ dx = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

Group -2: All six trigonometric functions with power 2

Formulas:

(i) $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$

(ii) $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$

(iii) $\tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

(iv) $cot^2 x + 1 = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1$

1) $(a) \int \sin^2 x \; dx = \int \frac{1}{2}[1 - \cos(2x)] \; dx$

$$= \frac{1}{2} \int [1 - \cos(2x)] \; dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C \quad ;$$

$$\left[\text{Formula:} \quad \int \cos(kx) \; dx = \frac{\sin(kx)}{k} + C \right]$$

$$\int \sin^2 x \; dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$(b) \int \sin^2(3x) \; dx$$

$$= \int \frac{1}{2}[1 - \cos(6x)] \; dx$$

$$= \frac{1}{2} \int [1 - \cos(6x)] \; dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(6x)}{6}\right] + C$$

2) $(a)\ \int \cos^2 x\ dx$

$$= \int \frac{1}{2}[1+\cos(2x)]\ \ dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin(2x)\ + C$$

$(b)\ \displaystyle\int \cos^2(5x)\ dx$

$$= \int \frac{1}{2}[1+\cos(10x)]\ \ dx$$

$$= \frac{1}{2}\int [1+\cos(10x)]\ \ dx$$

$$= \frac{1}{2}\left[x + \frac{\sin(10x)}{10}\right] + C$$

$$= \frac{1}{2}x + \frac{1}{20}\ \sin(10x) + C$$

3) $\int \tan^2 x\ dx\ =\ \int[\ \sec^2 x\ -1]\, dx\ \ =\ \tan x\ - x\ + C$

4) $\int \cot^2 x\ dx\ =\ \int[\ \csc^2 x\ -1]\, dx\ \ =\ -\cot x\ - x\ + C$

5) $\int \sec^2 x\ dx\ = \tan x + C$

6) $\int \csc^2 x\ dx\ =\ -\cot x + C$

Group -3: All six trigonometric functions with power n, any integer $n \geq 2$.

Reduction Formulas for Integration:

(1) * $\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$

(2) $\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$

(3) * $\int \tan^n x \, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \, dx$

(4) $\int \cot^n x \, dx = -\frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x \, dx$

(5) * $\int \sec^n x \, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$

(6) $\int \csc^n x \, dx = -\frac{1}{n-1}\csc^{n-2} x \cot x + \frac{n-2}{n-1}\int \csc^{n-2} x \, dx$

HINT: To derive the formula [Homework]

(1) $\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$

Start with $I = \int \sin^n x \, dx$

$= \int \sin^{n-1} x \ \sin x \ dx$; **Set** $u = \sin^{n-1} x$, $dv = \sin x \, dx$

***Also,* for** $\int \sec^n x \, dx$, **set**

$I = \int \sec^n x \, dx = \int \sec^{n-2} x \cdot \sec^2 x \, dx$;

$u = \sec^{n-2} x$ and $dv = \sec^2 x \, dx$

Definition: Co-functions

(i) Sine and Cosine are co-functions
(ii) Tangent and Cotangent are co-functions
(iii) Secant and Cosecant are co-functions

1) Evaluate $\int \sec^5 x\ dx$

Solution: We know that for $n \geq 2$,

$$\int \sec^{n} x\ dx = \frac{1}{n-1}\sec^{n-2} x\ \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx \ldots\ldots\ldots \quad (1)$$

Here $\int \sec^5 x\ dx$; Given $n=5,\ n-1=4,\quad n-2=3$

$$\int \sec^5 x\ dx = \frac{1}{4}\sec^3 x\ \tan x + \frac{3}{4}\int \sec^3 x\ dx\ ;$$

$; [\text{here } n=3, \quad n-1=2, \quad n-2=1]$

$$= \frac{1}{4}\sec^3 x\ \tan x + \frac{3}{4}\left[\frac{1}{2}\sec\ x\ \tan x + \frac{1}{2}\int \sec\ x\ dx\right]$$

$$= \frac{1}{4}\sec^3 x\ \tan x + \frac{3}{8}\sec\ x\ \tan x + \frac{3}{8}\int \sec\ x\ dx$$

$$= \frac{1}{4}\sec^3 x\ \tan x + \frac{3}{8}\sec\ x\ \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$$

Homework:

2) $\int \sec^7 x\ dx\ +\ \int \sec^5 x\ dx$

$= \frac{1}{6}\sec^5 x\ \tan x + \frac{5}{6}\ \int \sec^5 x\ dx\ +\ \int \sec^5 x\ dx$

$= \frac{1}{6}\sec^5 x\ \tan x + \frac{11}{6}\ \int \sec^5 x\ dx$

Please complete!

3) $\int \sin^6 x\ dx\ +\ \int \sin^4 x\ dx$

4) $\int \tan^6 x\ dx\ +\ \int \tan^5 x\ dx$

Group-4: $\int \boldsymbol{sinA\ cosB\ dx}$; $\int \boldsymbol{sinA\ sinB\ dx}$; $\int \boldsymbol{cosA\ cosB\ dx}$ **here** $\mathbf{A \neq B}$.

Formulas:

1) $\sin A \cos B = \frac{1}{2}\,[\sin(A-B) + \sin(A+B)]$
2) $\sin A \sin B = \frac{1}{2}\,[\cos(A-B) - \cos(A+B)]$
3) $\cos A \cos B = \frac{1}{2}\,[\cos(A-B) + \cos(A+B)]$

Definitions

1) $y = \sin x$ is an odd function, that is, $\sin(-x) = -\sin x$

$$[f(-x) = -f(x)]$$

2) $y = \cos x$ is an even function, that is, $\cos(-x) = \cos x$

$$[f(-x) = f(x)]$$

Here,

$$\cos(5x)\sin(2x) = \sin(2x)\cos(5x)$$

$$= \frac{1}{2}\,[\sin(-3x) + \sin(7x)] = \frac{1}{2}\,[-\sin(3x) + \sin(7x)]$$

Example:1 Evaluate

$$\int_{0}^{\frac{\pi}{2}} \boldsymbol{sin\,(3x)\,sin\,(6x)\ dx}$$

Solution:

$$\int_{0}^{\frac{\pi}{2}} sin\,(3x)\,sin\,(6x)\ dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[\cos(3x-6x) - \cos(3x+6x)]\ dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[\cos(-3x) - \cos(9x)]\ dx\ ;\quad \cos(-3x) = \cos(3x)$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} [\cos(3x) - \cos(9x)]\ dx$$

$$= \frac{1}{2}\left[\frac{\sin(3x)}{3} - \frac{\sin(9x)}{9}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{1}{6}\sin(3x) - \frac{1}{18}\sin(9x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{6}\sin\left(3\frac{\pi}{2}\right) - 0 - \frac{1}{18}\sin\left(9\frac{\pi}{2}\right) - 0\quad ;$$

$$\left[\sin 0 = 0,\quad \sin\left(9\frac{\pi}{2}\right) = \sin\left(2\pi\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1,\quad \sin\left(3\frac{\pi}{2}\right) = -1\right]$$

$$= \frac{1}{6}(-1) - \frac{1}{18}(1) = -\frac{1}{6} - \frac{1}{18}$$

$$= -\frac{4}{18} = -\frac{2}{9}$$

Example: 2 Evaluate

$$\int_{\mathbf{0}}^{\frac{\boldsymbol{\pi}}{\mathbf{2}}} cos\ (3x)cos(6x)\ \boldsymbol{dx}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos(3x - 6x) + \cos(3x + 6x)]\ dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[\cos(-3x) + \cos(9x)]\ dx$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} [\cos(3x) + \cos(9x)]\ dx$$

$$= \frac{1}{2}\left[\frac{\sin(3x)}{3} + \frac{\sin(9x)}{9}\right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{6}\sin\left(3\ \frac{\pi}{2}\right) - 0 + \frac{1}{18}\sin\left(9\ \frac{\pi}{2}\right) - 0$$

$$= \frac{1}{6}(-1) + \frac{1}{18}\ (1) = -\frac{1}{6} + \frac{1}{18}$$

$$= -\frac{2}{18} = -\frac{1}{9}$$