Linear Functions and their Properties

A linear function is a function of the form

$$y = f(x) = mx + b$$

The graph of a linear function is a line with slope m and y-intercept b.

Domain is the set of all real numbers.

Functions that are not linear are said to be non-linear.

Example: $y = -3x + 5$

This is a linear function with slope -3 and y-intercept 5.

Average rate of change of linear function:

The average rate of change of a linear function $f(x) = mx + b$ is $m = \frac{\Delta y}{\Delta x}$

Example: The average rate of change of $g(x) = -\frac{2}{5}x + 5$ is $-\frac{2}{5}$

Increasing, decreasing or constant linear function:

A linear function $f(x) = mx + b$ is increasing over its domain if its slope m is positive.

It is decreasing if the slope m is negative.

It is constant over its domain if its slope m is zero.

Example:

(a) $f(x) = 5x - 2$. Linear function with positive slope 5.

So the function is increasing on the interval $(-\infty, \infty)$

(b) $f(x) = -2x + 8$

(c) $s(t) = \frac{3}{4}t - 9$

(d) $h(z) = 5$

Supply: The quantity supplied of a good is the amount of a product that a company is willing to make available for sale at a given price.

Demand: The quantity demanded of a good is the amount of product that consumers are willing to purchase at a given price.

Equilibrium price:

The equilibrium price of a product is defined as the price at which quantity supplies equals quantity demanded.

Example:

Suppose that the quantity supplied S and quantity demanded D of a cellular telephones each months are given by the following functions:

$$S(P) = 60P - 900$$

$$D(P) = -15P + 2850$$

(a) Find the equilibrium price of the cellular telephones. What is the equilibrium quantity at the equilibrium price?

Soln:

To find the equilibrium price, solve the eqn $S(P) = D(P)$

$$60P - 900 = -15P + 2850$$

$$\Rightarrow 60P + 15P = 2850 + 90$$

$$\Rightarrow 75P = 3750$$

$$\Rightarrow P = 50$$

The equilibrium price is \$50 per cellular phone.

To find the equilibrium quantity, evaluate either $S(P)$ or $D(P)$ at $P = 50$.

$$S(50) = 60 \times 50 - 900 = 2100$$

Therefore, at a price \$50, the company will produce and cell 2100 phones each month and have no shortage or excess inventory.

(b) Determine the price for which quantity supplied is greater than quantity demanded?

Soln: The inequality $S(P) > D(P)$ is

$$60P - 900 > -15P + 2850$$

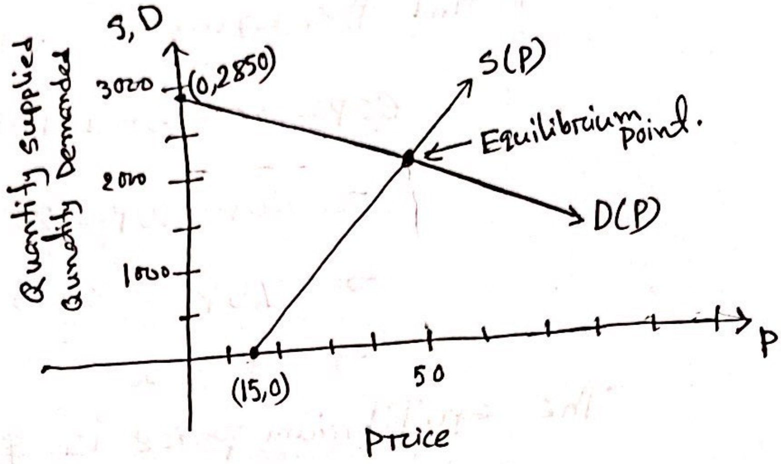
$$\Rightarrow 60P + 15P > 2850 + 900$$

$$\Rightarrow 75P > 3750$$

$$\Rightarrow P > 50$$

So if the company charges more than \$50 per phone then quantity supplied will exceed quantity demanded. In this case the company will have excess phones in inventory.

(C) Graph $S = S(P)$ and $D = D(P)$ and Label the equilibrium Price.



Quadratic Functions

A quadratic function is of the form

$$y = f(x) = ax^2 + bx + c \quad ; \; a \neq 0$$

Domain is the set of all real numbers.

Given $f(x) = ax^2 + bx + c$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\frac{b^2}{4a^2}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

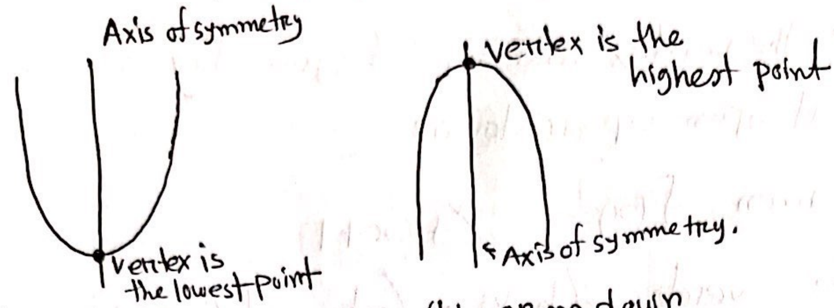
$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

If $h = -\frac{b}{2a}$ and $k = \frac{4ac - b^2}{4a}$ then

$f(x) = ax^2 + bx + c = a(x-h)^2 + k^2$ is the Parabola $y = ax^2$ shifted horizontally h units and vertically k units.

Here (h, k) is called vertex.

Identifying vertex, axis of symmetry and intercepts:



a) opens up $a>0$ (b) opens down $a<0$

In ax^2+bx+c ; $a\neq 0$, if $a>0$ then the parabola is opens up and has a lowest point which is called vertex.

If $a<0$ then the parabola opens down and has a highest point, that point is called vertex.

* The vertical line passing through the vertex point is called the axis of symmetry.

Properties of the graph of a quadratic function:

$$f(x) = ax^2+bx+c \qquad a\neq 0$$

vertex: $(h,k) = \left(\frac{-b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Axis of symmetry = the line $x = -\frac{b}{2a}$

Parabola opens up if $a>0$; the vertex is a minimum point

Parabola opens down if $a<0$; the vertex is a maximum point.

Intercepts:

If $b^2-4ac>0$, $f(x)=ax^2+bx+c$ has two distinct x-intercepts.

If $b^2-4ac=0$, $f(x)=ax^2+bx+c$ has one x-intercept so it touches the x-axis at its vertex.

If $b^2-4ac<0$, $f(x)=ax^2+bx+c$ has no x-intercepts.

Example: Suppose $f(x) = -3x^2+6x+1$.

(a) Locate the vertex and axis of symmetry of the parabola. Does it open up or down?

Sol^n: Given, $f(x) = -3x^2+6x+1$

We know vertex $=(h,k) = \left(\frac{-b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

$\therefore h = -\frac{b}{2a} = \frac{-6}{-6} = 1$ [$\because a=-3, b=6$

$k = f\left(-\frac{b}{2a}\right) = f(1) = 4$

$\therefore$ vertex $=(1,4)$.

Axis of symmetry is the line $x=1$.

Because $a=-3<0$, the parabola opens down.

(b) Find the intercepts of the function.

Sol^n: For y-intercept, let $x=0$

$\therefore f(0)=1$. So the y-intercept $=1$.

For x-intercept let $y=f(x)=0$

$\therefore -3x^2+6x+1=0$

The discriminant $b^2-4ac = (6)^2-4(-3)1 = 36+12 = 48>0$

so the eq^n has two real solution and the graph has two x-intercepts.

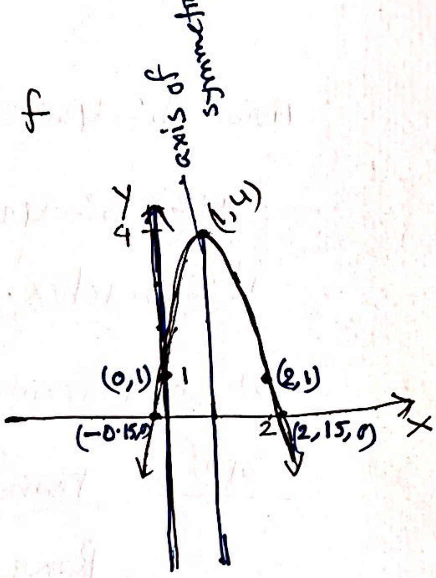
$$\therefore x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-6\pm\sqrt{48}}{-6} = \frac{-6\pm4\sqrt{3}}{-6}$$

i.e $x = \frac{-6+4\sqrt{3}}{-6}$ or $x = \frac{-6-4\sqrt{3}}{-6}$

≈ -0.15 ≈ 2.15

The x-intercepts are approximately -0.15 and 2.15.

(c) Determine the domain and Range of f



Sol^n: Domain, $D = (-\infty, \infty)$

Range, $R = (-\infty, 4]$

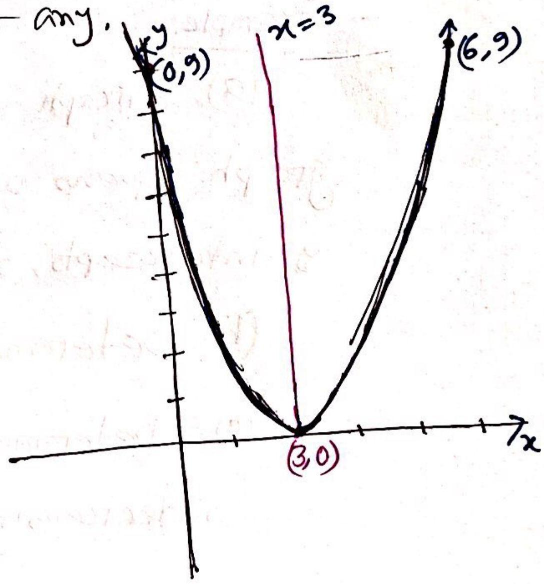
(d) Determine where f is increasing and where its decreasing?

Sol^n: increasing on the interval $(-\infty, 1)$

and decreasing on the interval $(1, \infty)$

Example:

(a) Graph $f(x) = x^2 - 6x + 9$. Determine whether the graph opens up or down and find its vertex, axis of symmetry, y-intercepts and x-intercepts, if any.



Solution:

Given $f(x) = x^2 - 6x + 9$

$a = 1, \; b = -6$

Since $a = 1 > 0$, the parabola opens up.

$$\text{vertex} = (h,k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

$$= \left(\frac{-(-6)}{2}, f\left(\frac{6}{2}\right)\right)$$

$$= (3, f(3))$$

$$= (3, 0)$$; Axis of symmetry is the line $x = 3$.

y-intercept, let $x = 0$ which yields $f(0) = 9$.

So y-intercept $= 9$

For x-intercept let $f(x) = 0$

$\therefore \; x^2 - 6x + 9 = 0$

Now $b^2-4ac = (-6)^2 - 4\cdot1\cdot9 = 36-36=0$

Since $b^2-4ac=0$, so the graph touches the x-axis at its vertex.

(b) determine domain and Range of f.

Soln:

Domain $= (-\infty, \infty)$

Range $= [0, \infty)$

(c) Determine where f is increasing and where it is decreasing?

Soln: The function f is decreasing on the interval $(-\infty, 3)$ and increasing on the interval $(3, \infty)$.

Example:

(a) Graph $f(x) = 2x^2+x+1$, determine whether the graph opens up or down. Find its vertex, axis of symmetry, y-intercepts, x-intercepts if any.

(b) Determine the domain and Range of f.

(c) Determine where f is increasing and where it is decreasing?

Maximum and Minimum value of quadratic function:

$f(x) = ax^2+bx+c;\ a\neq 0$ has vertex $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

* If the vertex is the highest point, then $f\left(-\frac{b}{2a}\right)$ is the maximum value of f.

* If the vertex is the lowest point, then $f(-b/2a)$ is the minimum value of f.

Example:

Determine whether the quadratic function $f(x) = x^2-4x-5$ has a maximum or minimum value. Then find the maximum or minimum value.

Solution: Given $f(x) = x^2-4x-5$

Here $a=1$, $b=-4$; since $a>0$, f opens up, so the vertex is a minimum point.

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

$$= \left(\frac{-(-4)}{2}, \left(f\left(\frac{-(-4)}{2}\right)\right)\right)$$

$$= (2, f(2))$$

$$= (2,-9)$$

The minimum point occurs at $x=2$ and the minimum value is $f(2)=-9$

If we have given the vertex (h, k) and one additional point on the graph of a quadratic function $f(x) = ax^2+bx+c, a\neq 0$ we can use $f(x) = a(x-h)^2+k$ to obtain the quadratic function.

Example:

Determine the quardatic function whose vertex is $(1, -5)$ and whose y-intercept is -3.

Solution: The vertex $(1, -5)$ so $h=1$ and $k=-5$.

$$\therefore \quad f(x) = a(x-h)^2+k$$
$$= a(x-1)^2-5$$

To determine the value of a, we use the fact that $f(0) = -3$.

$$f(x) = a(x-1)^2-5$$
$$\Rightarrow -3 = a(0-1)^2-5$$
$$\Rightarrow -3 = a-5 \Rightarrow a = 2.$$

$\therefore$ The quardatic function is

$$f(x) = 2(x-1)^2-5 = 2x^2-4x-3$$