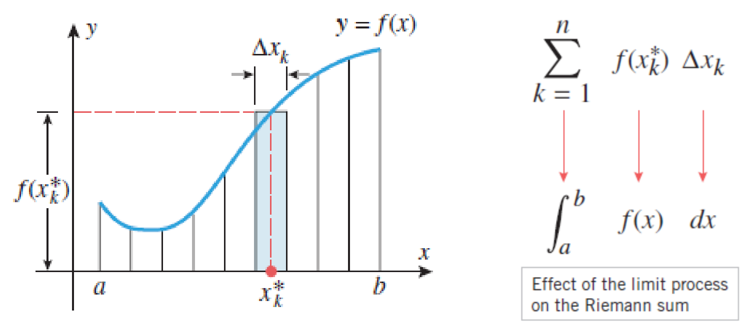
DAY-9

MAT-130, Lecture-9

Chapter 6: Applications of Integration

Section 6.1 Area Between Two Curves

MAT-120: A REVIEW OF RIEMANN SUMS (AREA UNDER THE CURVE $y = f(x)$ on $[a, b]$)



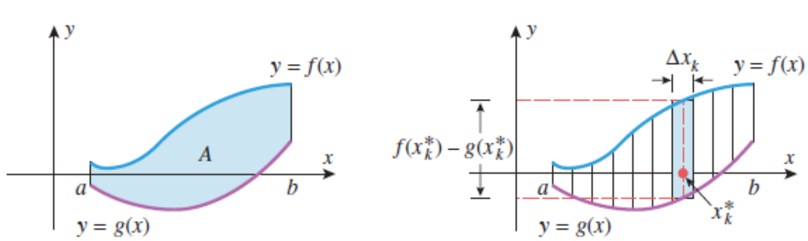
$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

Here $n \to \infty$.

Note that $g(x) = 0$ is the lower boundary. Hence, $A = \int_a^b [f(x) - 0]dx = \int_a^b f(x)\,dx$

AREA BETWEEN $y = f(x)$ AND $y = g(x)$

> **6.1.1 FIRST AREA PROBLEM** Suppose that f and g are continuous functions on an interval $[a, b]$ and
>
> $$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$
>
> [This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$

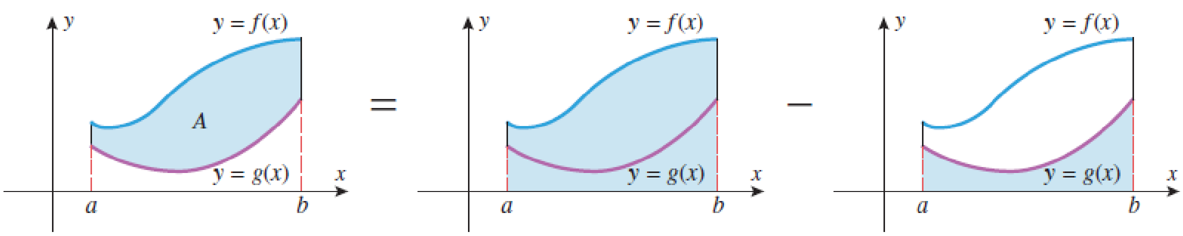


(a) (b)

6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

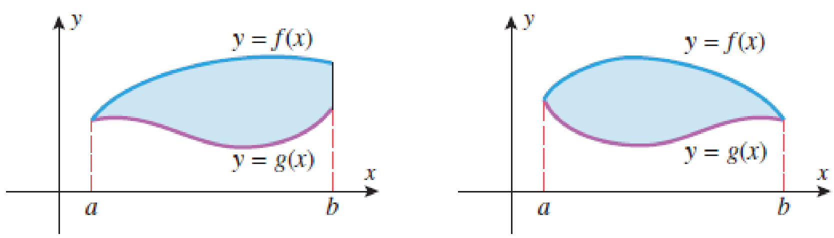
$$A = \int_a^b [f(x) - g(x)]\, dx \qquad (1)$$

GEOMERTICAL INTERPRETATION



Area between f and g | Area below f | Area below g

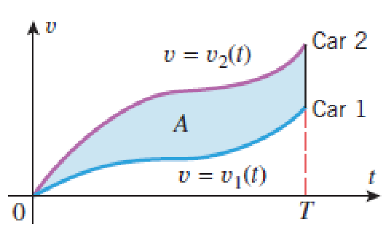
POINT AS A BOUNDARY



The left-hand boundary reduces to a point. | Both side boundaries reduce to points.

Example 1 [Worked out example from book]

The figure below shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area A between the curves over the interval $0 \leq t \leq T$.



Note: Graphing will help you a lot to know the boundaries.

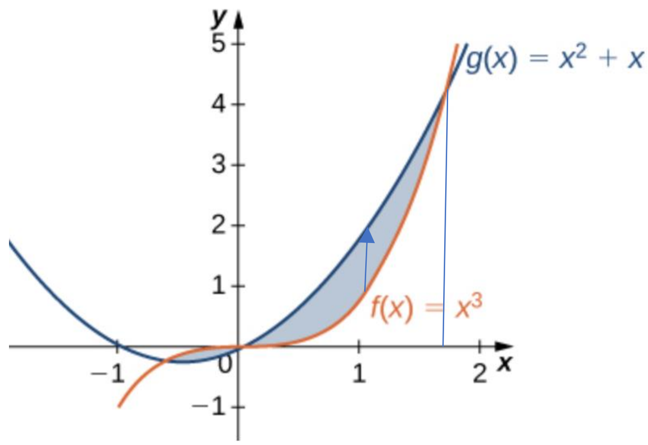
Formula: **Area** $=\int_a^b [\textbf{Upper boundary} - \textbf{Lower boundary}]dx$

Example 2

Find the area of the region bounded by the curves $f(x) = x^3$ and $g(x) = x^2 + x$ in the **first quadrant.**

Solution: Given $f(x) = x^3$ and $g(x) = x^2 + x$.

Note that $g(x) = x^2 + x$ is a parabola opening upward, with $x-$intercepts $x = 0, -1$ since $g(x) = x(x+1)$.



To find the points of intersection, set $f(x) = g(x) \Rightarrow x^3 - x^2 - x = 0$

$$\Rightarrow x(x^2 - x - 1) = 0$$

$$\Rightarrow x = 0, \quad x^2 - x - 1 = 0$$

$$x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

That is, $x = 0, \ \frac{1+\sqrt{5}}{2}, \ \frac{1-\sqrt{5}}{2}$

Interval: $I = [a, b] = [0, \frac{1+\sqrt{5}}{2}]$

The area of the region is $A = \int_0^{\frac{1+\sqrt{5}}{2}} [g(x) - f(x)]dx = \int_0^{\frac{1+\sqrt{5}}{2}} [x^2 + x - x^3]dx$

Please complete at home.

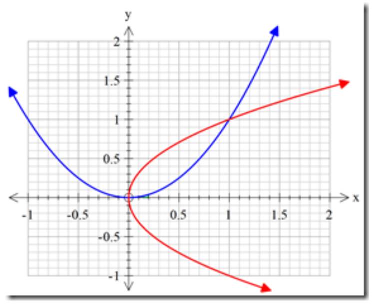
Example 3

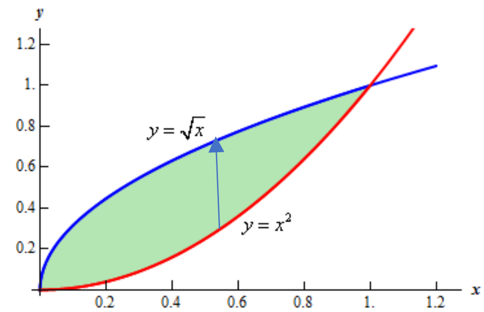
Find the area of the region **bounded** by the curves $y = x^2$ and $y^2 = x$.

Solution: Given $y = x^2$ and $y^2 = x$

From $y^2 = x$, we get $y = \pm\sqrt{x}$.

We the boundary $y = \sqrt{x}$.





To find the interval which is given by the points of intersection,

set $x^2 = \sqrt{x} \quad \Rightarrow x^4 = x$

$\Rightarrow x^4 - x = 0 \quad \Rightarrow x(x^3 - 1) = 0$

We get $x = 0, 1$. Interval $= [0, 1]$

The area of the region is

$$A = \int_0^1 \left[\sqrt{x} - x^2\right] dx = \int_0^1 \left[x^{\frac{1}{2}} - x^2\right] dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} - 0 = \frac{1}{3}\ unit^2$$

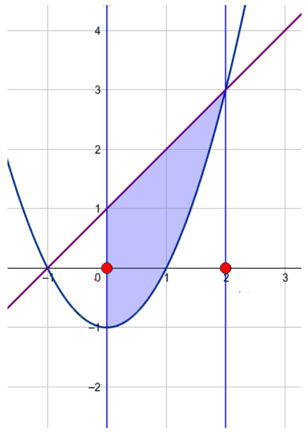
Example 4

Find the area of the region **bounded** by the curves $y = x^2 - 1$ and $y = x + 1$ on the interval $[0,2]$.

Solution: Given $y = x^2 - 1$ and $y = x + 1$ on the interval $[0,2]$.

To find the points of intersection:

Set $x^2 - 1 = x + 1 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow \quad x = -1, 2$.



The area of the region is

$A = \int_0^2 [(x+1) - (x^2 - 1)]dx$

$= \int_0^2 [-x^2 + x + 2]dx$

$= -\frac{8}{3} + 2 + 4$

$= -\frac{8}{3} + 6$

$= \frac{10}{3}\ unit^2$

Example 5

Find the area of the region **bounded** by the curves $y = \sin x$ and $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution:

Given $y = \sin x$ and $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

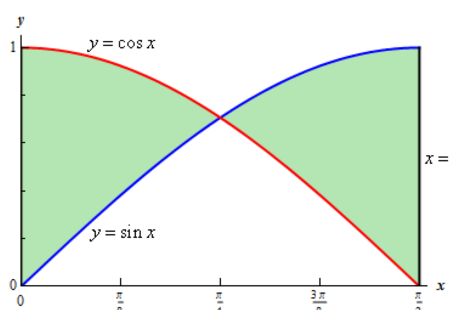
To find the point of intersection, set $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$,

which belongs to the given interval.

Intervals are $I_1 = \left[0, \frac{\pi}{4}\right]$ and $I_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

On $I_1 = \left[0, \frac{\pi}{4}\right]$, $y = \cos x$ is the upper boundary

On $I_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $y = \sin x$ is the upper boundary.



$$\text{Area} = \int_0^{\frac{\pi}{4}} [\cos x - \sin x]\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x]\, dx = ??$$

Lecture 10

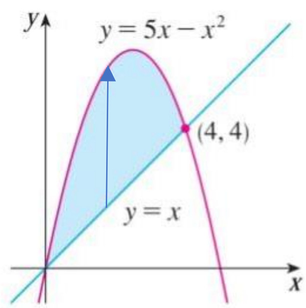
Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x −axis, connecting the top and bottom boundaries.

Step 2. The y −coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$.

Example 6 Find the area of the shaded region given below.



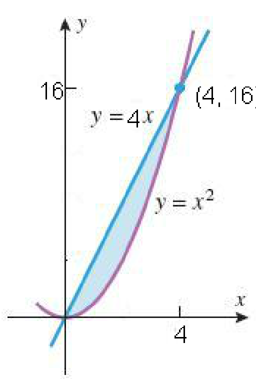
Solution:

$$A = \int_0^4 [(5x - x^2) - x] \; dx$$

Example 7

Find the area of the given graphs in the shaded regions.

A)



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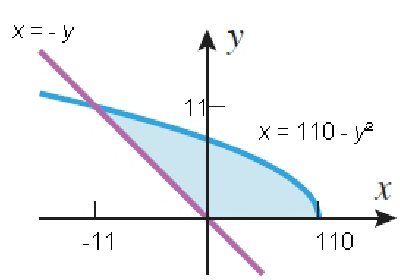

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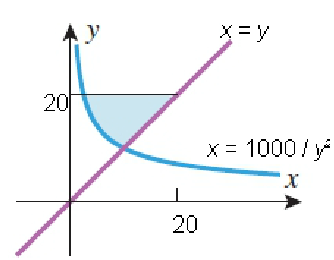
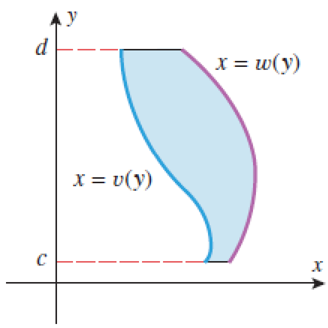
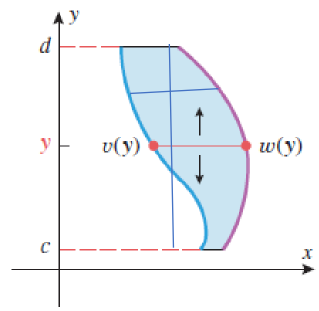


REVERSING THE ROLES OF x AND y : $x = w(y),\ x = v(y)$

> **6.1.4 AREA FORMULA** If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is
>
> $$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$





Formula: If the boundaries are given by functions of y, then Area =$\int_c^d$[Right boundary $-$ Left boundary]dy

Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a horizontal line segment through the region at an arbitrary point y on the y —axis, connecting the left and right boundaries.

Step 2. The x —coordinate of the right endpoint of the line segment sketched in Step 1 will be $w(y)$, the left one $v(y)$, and the length of the line segment will be $w(y) - v(y)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment top and then bottom. The bottommost position at which the line segment intersects the region is $y = c$ and the topmost is $y = d$.

Example 8

Find the area of the region **bounded** by the curves $y^2 = x$ and $y = x - 2$.

[NOTE: If we consider these curves as functions of x, then we get three curves given by

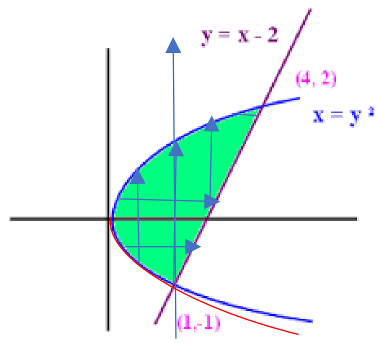
From the equation $y^2 = x \quad \Rightarrow \quad y = \pm\sqrt{x}$

$y = x - 2,\ \ y = \sqrt{x}\,,\ \ y = -\sqrt{x},\ \ INTERVAL = [0, 4]$.

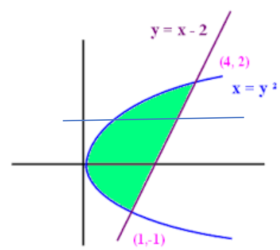
$y = \sqrt{x}$ → Upper part of the parabola which is above the x-axis

$y = -\sqrt{x}$ → Lower part of the parabola which is below the x-axis

$$A = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\, dx + \int_1^4 [\sqrt{x} - (x - 2)] dx \ = ???$$



Alternative method:
Find the area of the region **bounded** by the curves $y^2 = x$ and $y = x - 2$.



Note that $y^2 = x$ is not a function of x, but $x = y^2$ and $x = y + 2$ are functions of y.
The area of the region is

$$A = \int_{-1}^{2} [(y+2) - y^2]\, dy$$

Example 9

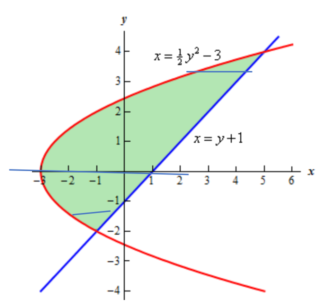
Find the area of the region **bounded** by the curves $x = \frac{1}{2}y^2 - 3$ and $x = y + 1$.

Solution: The region is bounded on the left by $x = v(y) = \frac{1}{2}y^2 - 3$ and on the right by $x = w(y) = y + 1$.

To find the interval, left find the points of intersection.

Set $\frac{1}{2}y^2 - 3 \ = \mathrm{y} + 1 \ \Rightarrow \mathrm{y}^2 - 2y - 8 = 0 \ \Rightarrow (y-4)(y+2) = 0,\ i.e.,\ y = -2, 4.$

Interval $= [c, d] = [-2, 4]$.

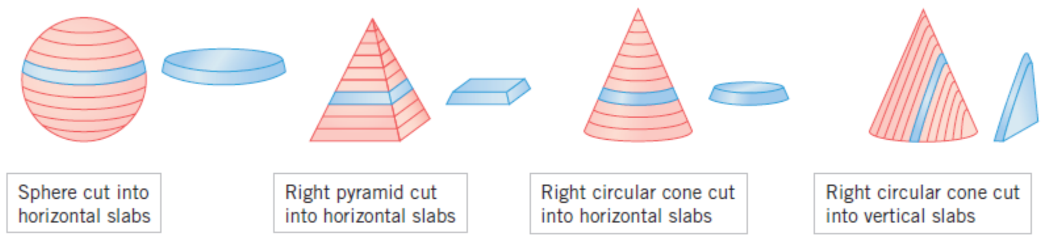


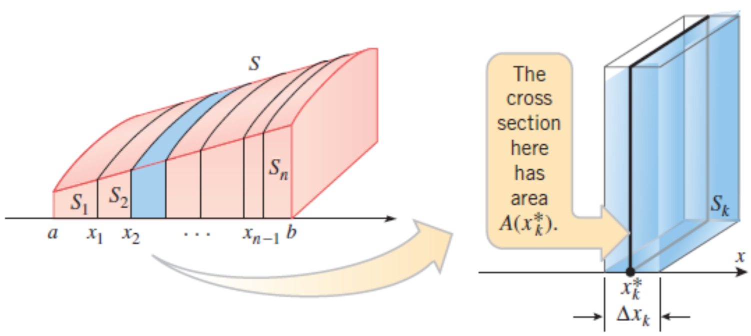
Area $=\int_c^d$**[Right boundary − Left boundary]**dy

$$= \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy = \int_{-2}^{4} \left[-\frac{1}{2}y^2 + y + 4\right] dy$$

$$= \left[-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right]_{-2}^{4} = \frac{1}{6}\left[-y^3 + 3y^2 + 24y\right]_{-2}^{4} = \frac{1}{6}[-72 + 36 + 144] = -12 + 6 + 24 = 18\, unit^2.$$

SECTION 6.2: VOLUME BY SLICING: DISC AND WASHER METHOD





Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

In summary, we have the following result.

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

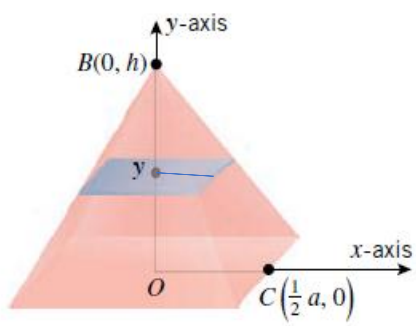
Summary:

The volume of the solid S is given by

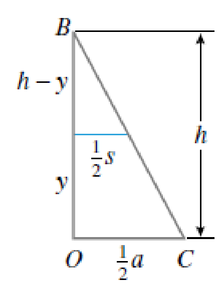
$V = \int_a^b (\text{Area of the cross} - \text{section})\ dx$ Or $V = \int_c^d (\text{Area of the cross} - \text{section})\ dy$

Example 1

Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length $\boldsymbol{a}$.



(a)



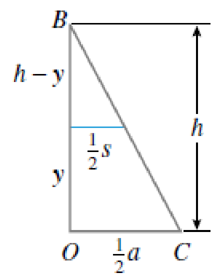
(b)

Project the Pyramid along the y-axis, placing the height of the pyramid along the axis with the center of the base at the origin.

Now, take any cross-section of the pyramid at any y, $0 \le y \le h$. The cross-section is a square that is perpendicular to the y −axis.

Volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$ … … … (1)

Let the length of a side of the cross-section be s. Then the area of the cross-section is $A(y) = s^2$ … … … (2)
By the similar triangle property on the triangles



(b)

$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \Rightarrow \frac{s}{a} = \frac{h-y}{h}$. Hence $s = \frac{a}{h}(h-y)$

From equation (2): Area of the cross-section is $A(y) = \left[\frac{a}{h}(h-y)\right]^2 = \frac{a^2}{h^2}(h-y)^2$
Then volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$

$$= \int_0^h \frac{a^2}{h^2}(h-y)^2\ dy$$

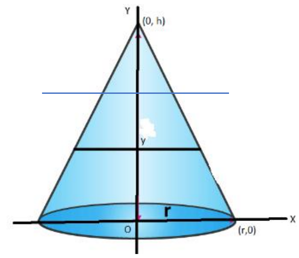
$$= \frac{a^2}{h^2}\int_0^h [h^2 - 2hy + y^2]\ dy$$

$$= \frac{a^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$

$$V = \frac{1}{3}a^2 h \quad unit^3.$$

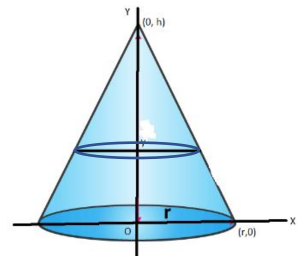
Example 2

Derive the formula for the volume of a **right circular cone** whose altitude is h and whose base is a circle of radius r.



Solution: Project the right circular cone placing the height of the cone along the y −axis with the center of the base at the origin.

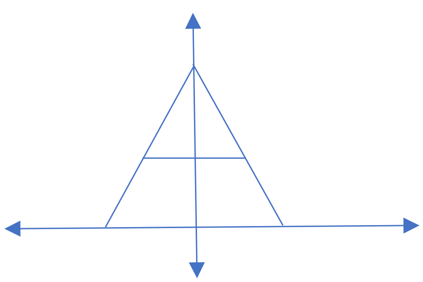
Now, take any cross-section of the cone at any $y,\ 0 \le y \le h.$ The cross-section is a disk that is perpendicular to the y −axis.



$$\text{Volume } V = \int_0^h (\text{Area of the crosssection})\ dy \ \ldots\ldots\ \ldots\ (1)$$

Let the radius of the cross-section be r_1. Then the area of the cross-section is

$$A(y) = \pi r_1^2 \ldots\ldots\ldots (2)$$



By the similar triangle property on the triangles

$$\frac{2r_1}{2r} = \frac{h-y}{h} \ \Rightarrow \frac{r_1}{r} = \frac{h-y}{h}. \quad \text{Hence} \quad r_1 = \frac{r}{h}\,(h-y)$$

From equation (2): Area of the cross-section is $A(y) = \pi\left[\frac{r}{h}\,(h-y)\right]^2 = \pi\frac{r^2}{h^2}(h-y)^2$

Then volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$

$$= \pi\int_0^h \frac{r^2}{h^2}(h-y)^2\ dy$$

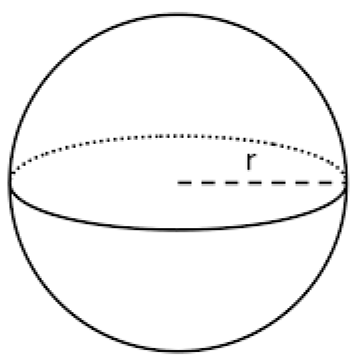
$$= \frac{\pi r^2}{h^2}\int_0^h [h^2 - 2hy + y^2]\ dy$$

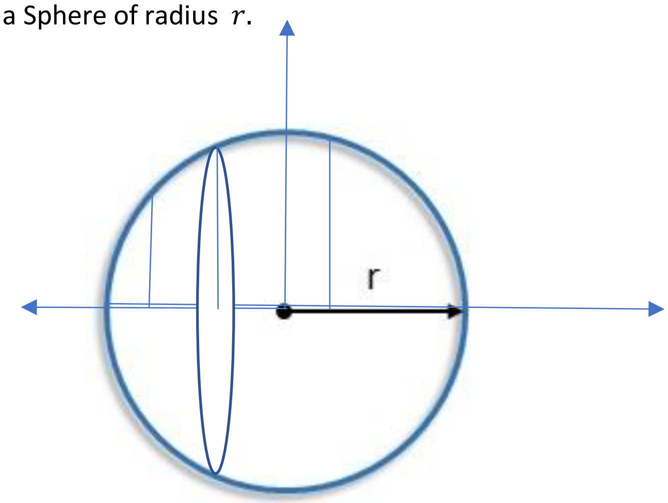
$$= \frac{\pi r^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$

$$V = \frac{\pi}{3}r^2 h \quad unit^3.$$

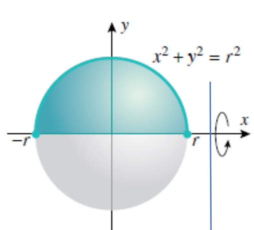
Example 3

Derive the formula for the volume of a Sphere of radius r.





The projection of the sphere on the xy –plane is a disk of radius r, bounded by the circle $x^2 + y^2 = r^2$.
But $x^2 + y^2 = r^2$ is not a function. The upper-half circle represents a function of x given by
$y = \sqrt{r^2 - x^2}$.



Interval $= [-r, r]$, The cross-section at any x, $-r \le x \le r$, is a disk of radius, **say r_1** .
Here $r_1 = \sqrt{r^2 - x^2} - 0 = \sqrt{r^2 - x^2}$.
Area of the cross-section $A(x) = \pi r_1^2 = \pi(r^2 - x^2)$.

volume $V = \int_a^b (\text{Area of the cross} - \text{section})\ dx$

$V = \int_{-r}^{r} \pi\left(r^2 - x^2\right)\ dx = \frac{4}{3}\pi r^3$. [complete !!]

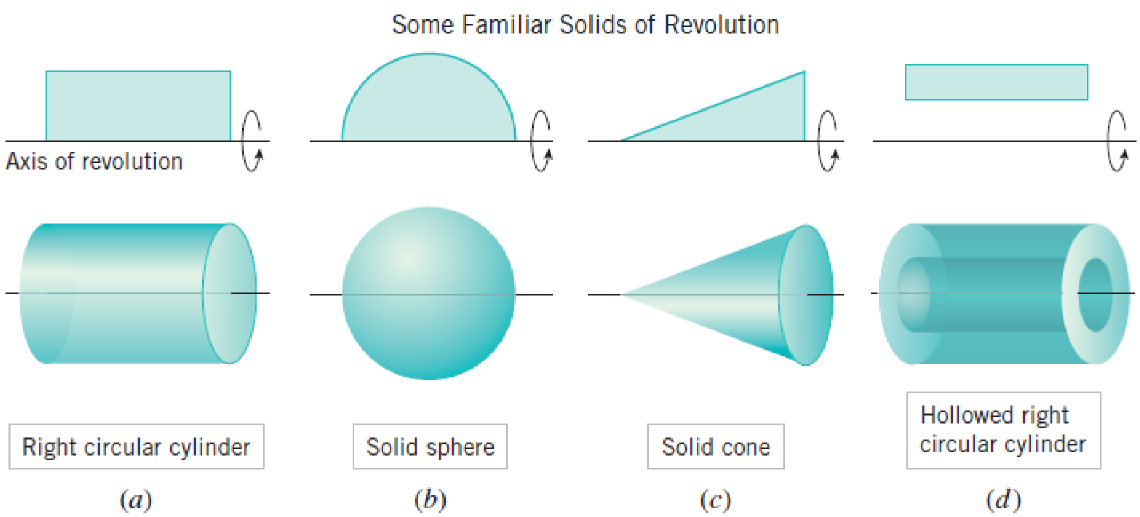
[Note: $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$

Lower-half circle$y = -\sqrt{r^2 - x^2}$,

Upper-half circle $y = \sqrt{r^2 - x^2}$]

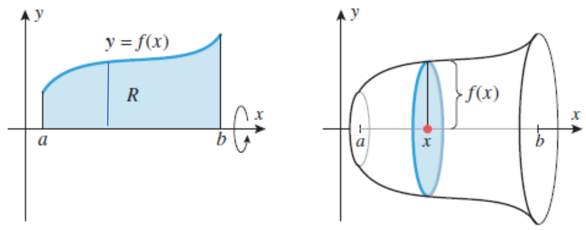
SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type



VOLUMES BY DISKS PERPENDICULAR TO THE x –AXIS

Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x –axis, and on the sides by the vertical lines *x* = *a* and *x* = *b*. Then the volume of the solid of revolution that is generated by revolving the region *R* about the x –axis is given by



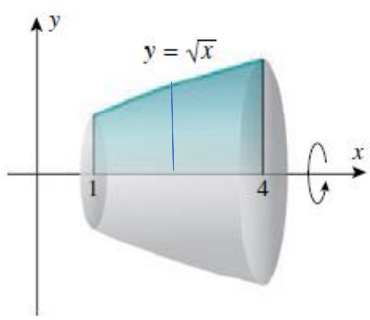
(a) (b)

The cross-section is a disk of radius $r = f(x)$ that is perpendicular to the x −axis. Hence, the volume is

$$V = \int_a^b \pi[f(x)]^2 dx$$

Example 4 Find the volume of the solid that is obtained when the region **under the curve** $y = \sqrt{x}$ **over** the interval $[\mathbf{1}, \mathbf{4}]$ is revolved about the x −axis.

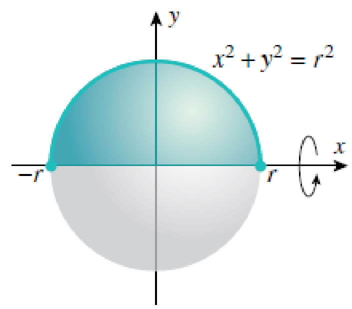
<u>Solution:</u>



The Volume $V = \int_a^b \pi[f(x)]^2 dx = \int_1^4 \pi[\sqrt{x}]^2 dx$ complete!!!

Homework

Example 5 Find the volume of the solid generated by revolving the circle $x^2 + y^2 = r^2$ about the x −axis.

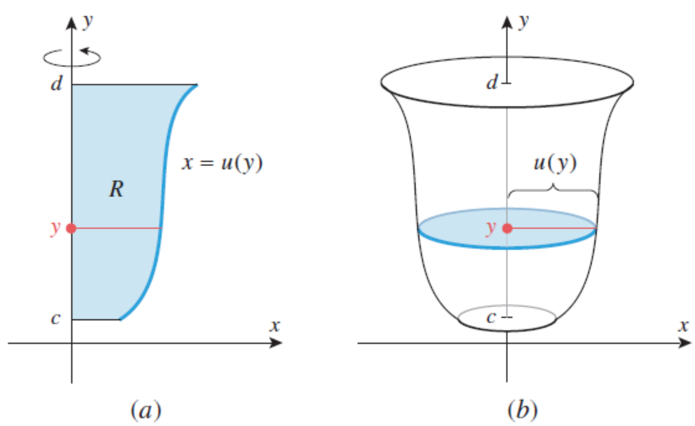


VOLUMES BY DISKS PERPENDICULAR TO THE y −AXIS

Let $x = u(y)$ be continuous and nonnegative on $[c, d]$, and let R be the region that is bounded on the right by $x = u(y)$, on the left by the y −axis, and at the bottom and top by the horizontal lines $y = c$ and $y = d$. Then the volume of the solid of revolution that is generated by revolving the region R about the y −axis is given by

$$V = \int_c^d \pi[u(y)]^2 dy$$

Note that the cross-section is a disk of radius $r = u(y)$ that is perpendicular to the y −axis.



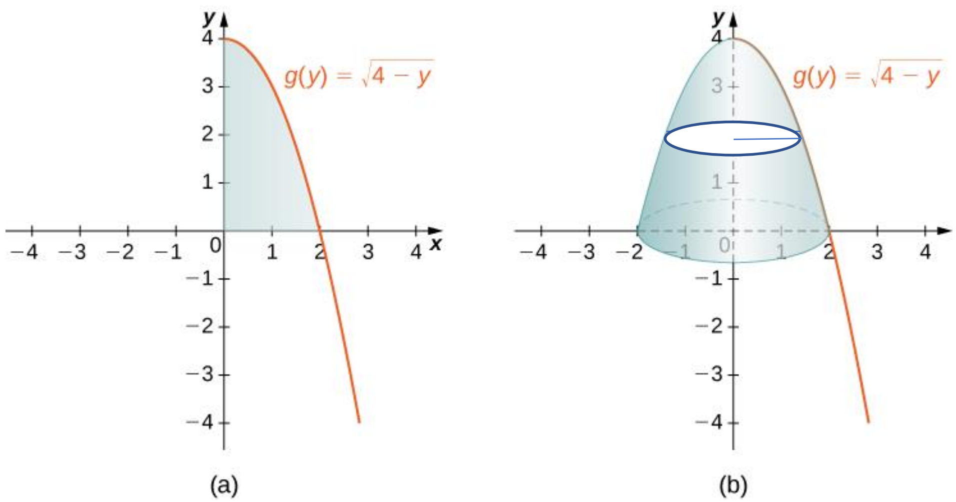
Disks

Information comes from the origin R . Interval, radius, height come from the region.

Example 6 [Complete !!!]

Find the volume of the solid that is obtained when the region R is revolved about the $\boldsymbol{y}$ **−axis,** where R is bounded by the curve $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{y}) = \sqrt{4 - y}$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]

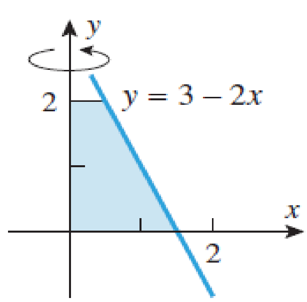


$V = \int_c^d \pi[u(y)]^2 dy = \int_0^4 \pi\left[\sqrt{4-y}\right]^2 dy$ complete!

Example 7

Find the volume of the solid that is obtained when the region R revolved about the y —axis, where R is bounded by the curve $y = 3 - 2x$, $y = 2$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]



Interval, Shape of the cross-section, Area of the cross-section, Variable of the function that gives you the area of the cross-section.

$$\boldsymbol{y = 3 - 2x \text{ . That is,} \qquad x = \frac{1}{2}(3 - y)}$$

Radius of the cross-section is $= \frac{1}{2}(3 - y)$

$$I = [0, 2], \text{Cross} - \text{section is a disk,} \qquad A(y) = \pi\left(\frac{3}{2} - \frac{y}{2}\right)^2$$