Hyperbolic Functions and Hanging Cables ①

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

even function — odd function

hyperbolic cosine of x

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

hyperbolic Sine of x

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

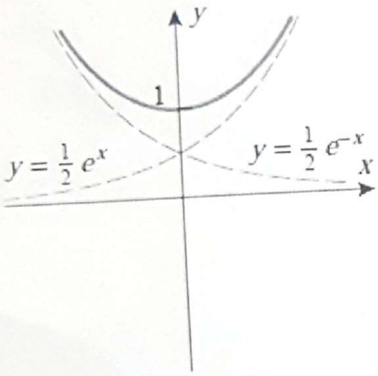
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

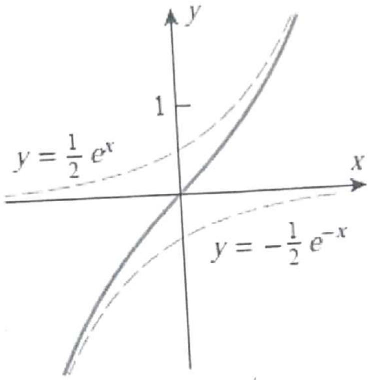
$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = \frac{0}{2} = 0$$

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$$



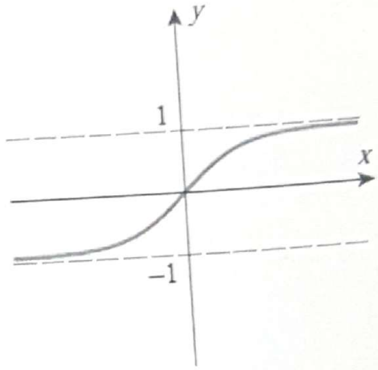
$y = \cosh x$

(*a*)



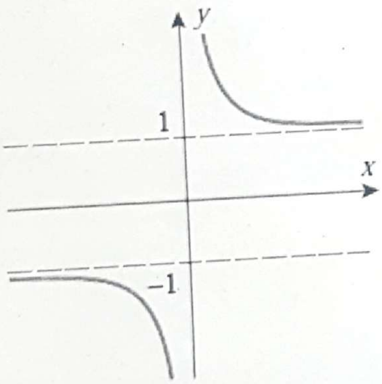
$y = \sinh x$

(*b*)



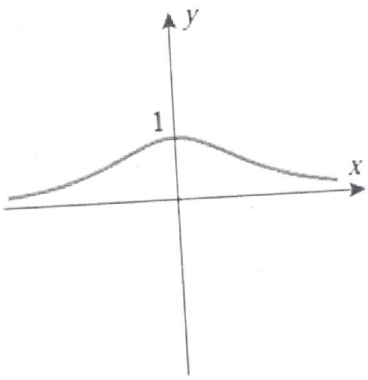
$y = \tanh x$

(*c*)



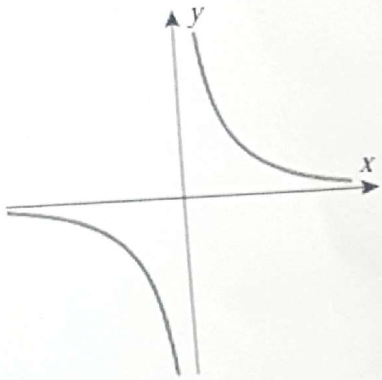
$y = \coth x$

(*d*)



$y = \operatorname{sech} x$

(*e*)



$y = \operatorname{csch} x$

(*f*)

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$$
$$= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$
$$= e^x \cdot e^{-x}$$
$$= e^0$$

$$\boxed{\therefore \cosh^2 x - \sinh^2 x = 1}$$

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$\boxed{\Rightarrow \quad 1 - \tanh^2 x = \operatorname{sech}^2 x}$$

Also $\dfrac{\cosh^2 x}{\sinh^2 x} - \dfrac{\sinh^2 x}{\sinh^2 x} = \dfrac{1}{\sinh^2 x}$

$$\boxed{\Rightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x}$$

$\cosh x + \sinh x = e^x$

$\cosh x - \sinh x = e^{-x}$

$\cosh(-x) = \cosh x$

$\sinh(-x) = -\sinh x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

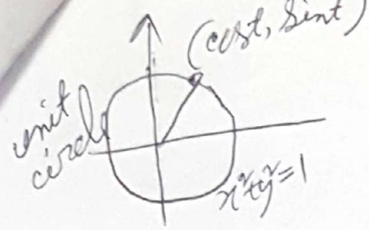
$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh x$

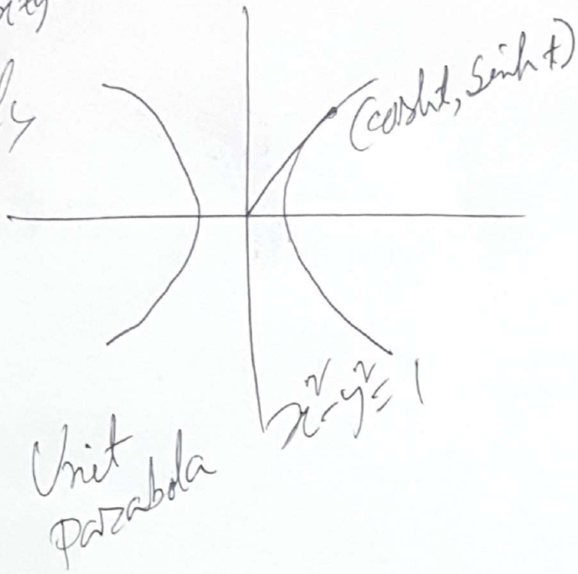
$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$



$$x^2+y^2=\cos^2 t+\sin^2 t=1$$

Similarly



$$x^2-y^2=\cosh^2 t-\sinh^2 t=1$$

Derivative & Integral formulas

$$\frac{d}{dx}(\sinh x)=\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)=\frac{e^x+e^{-x}}{2}=\cosh x$$

$$\frac{d}{dx}(\cosh x)=\frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)=\frac{e^x-e^{-x}}{2}=\sinh x$$

Similarly $\frac{d}{dx}(\tanh x)=\operatorname{sech}^2 x$

$$\frac{d}{dx}(\coth x)=-\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x)=-\operatorname{sech} x\tanh x$$

$$\frac{d}{dx}(\operatorname{csch} x)=-\operatorname{csch} x\coth x$$

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$\int \cosh x\,dx = \sinh x + c$

$\int \sinh x\,dx = \cosh x + c$

$\int \operatorname{sech}^2 x\,dx = \tanh x + c$

$\int \operatorname{csch}^2 x\,dx = -\coth x + c$

$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$

$\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + c$

① $\frac{d}{dx}(\cosh(x^3)) = \sinh(x^3) \cdot \frac{d}{dx}(x^3)$

$= 3x^2 \sinh(x^3)$

② $\frac{d}{dx}\left(\ln(\tanh x)\right) = \frac{1}{\tanh x} \cdot \frac{d}{dx}(\tanh x)$

$= \frac{\operatorname{sech}^2 x}{\tanh x}$

③ $\int \sinh^5 \cosh x\,dx$ put $\sinh x = u$

$\cosh x\,dx = du$

$= \int u^5 . du = \frac{u^6}{6} + c$

$= \frac{1}{6} \sinh^6 x + c$

④ $\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \ln|\cosh x| + c$

$= \ln(\cosh x) + c$

$e^x = y \Rightarrow x = \ln y$

logarithmic function and exponential functions are inverse to each other.

$$y = \sinh x$$

$$\Rightarrow y = \frac{e^x - e^{-x}}{2}$$

$x \longleftrightarrow y$ (interchange x and y)

$$x = \frac{e^y - e^{-y}}{2} = \sinh y$$

$$\Rightarrow e^y - e^{-y} = 2x$$

$$\Rightarrow e^y - 2x - e^{-y} = 0$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0 \quad \text{(Multiply by } e^y\text{)}$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4 \cdot 1(-1)}}{2 \cdot 1} = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2} = \frac{\cancel{2}(x \pm \sqrt{x^2+1})}{\cancel{2}}$$

$$\Rightarrow e^y = x + \sqrt{x^2+1} \quad (\text{Since } e^y > 0)$$

$$\Rightarrow y = \ln(x + \sqrt{x^2+1})$$

$$\Rightarrow \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

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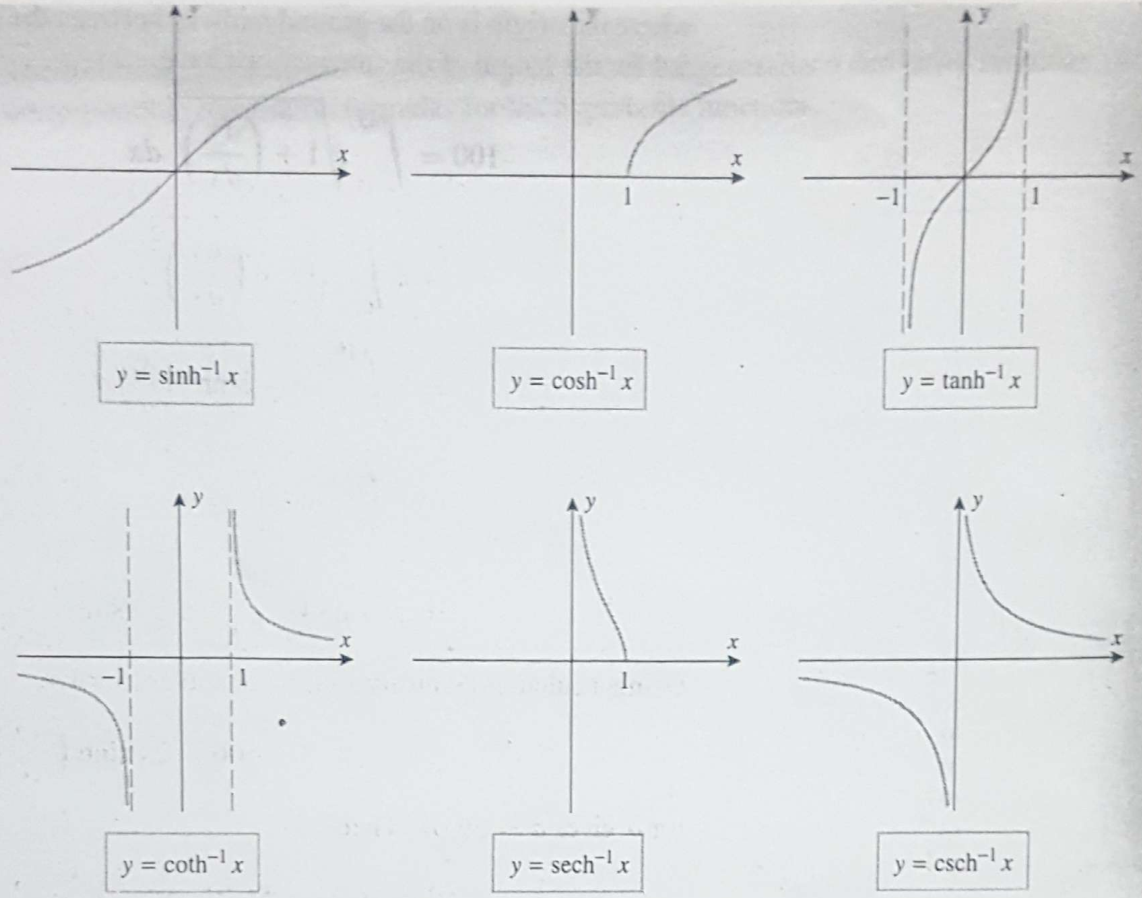

► **Figure 6.8.6**

Table 6.8.1

PROPERTIES OF INVERSE HYPERBOLIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sinh^{-1} x$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < +\infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < +\infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, +\infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \geq 0$ $\cosh(\cosh^{-1} x) = x$ if $x \geq 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, +\infty)$	$\tanh^{-1}(\tanh x) = x$ if $-\infty < x < +\infty$ $\tanh(\tanh^{-1} x) = x$ if $-1 < x < 1$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$ $\coth(\coth^{-1} x) = x$ if $x < -1$ or $x > 1$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = x$ if $x \geq 0$ $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$ if $0 < x \leq 1$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = x$ if $x < 0$ or $x > 0$ $\operatorname{csch}(\operatorname{csch}^{-1} x) = x$ if $x < 0$ or $x > 0$

We will show how to derive the first formula in this theorem and leave the rest as exercises. The basic idea is to write the equation $x = \sinh y$ in terms of exponential functions and solve this equation for y as a function of x. This will produce the equation $y = \sinh^{-1} x$ with $\sinh^{-1} x$ expressed in terms of natural logarithms. Expressing $x = \sinh y$ in terms of exponentials yields

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \quad x>1$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \quad |x|<1$$

$$\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2} \quad |x|>1$$

$$\frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}} \quad 0<x<1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{1+x^2}} \quad \cancel{x} \neq 0$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)+c \ \underline{\text{or}} \ \ln\left(x+\sqrt{x^2+a^2}\right)+c$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)+c \ \underline{\text{or}} \ \ln\left(x+\sqrt{x^2-a^2}\right)+c$$

$$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+c, & |x|<a \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)+c, & |x|>a \end{cases} \ \underline{\text{OR}} \ \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c, \ |x|\neq a$$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{x}{a}\right|+c \ \underline{\text{OR}} \ -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{|x|}\right)+c \quad 0<|x|<a$$

$$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right|+c \ \underline{\text{OR}} \ -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+x^2}}{|x|}\right)+c \quad x\neq 0$$

Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}, \quad x > \frac{3}{2}$

put $u = 2x \Rightarrow du = 2dx \Rightarrow \frac{du}{2} = dx$

$$\therefore \int \frac{dx}{\sqrt{4x^2-9}} = \int \frac{\cancel{dx}\ \frac{du}{2}}{\sqrt{u^2-3^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-3^2}}$$

$$= \frac{1}{2}\cosh^{-1}\left(\frac{u}{3}\right) + c$$

$$= \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + c$$

OR $\frac{1}{2}\ln\left(2x + \sqrt{4x^2-9}\right) + C$

Catenary curve: $y = a\cosh\left(\frac{x}{a}\right) + c$

Arc length of a curve: $L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$

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p-481 $y = \cosh x$

$$\frac{dy}{dx} = \sinh x$$

$$L = \int_0^{\ln 2} \sqrt{1+\sinh^2 x}\, dx = \int_0^{\ln 2} \sqrt{\cosh^2 x}\, dx$$

$$= \int_0^{\ln 2} \cosh x\, dx$$

$$= \sinh x \Big]_0^{\ln 2}$$

$e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$

$$= \sinh(\ln 2)$$

$$= \frac{e^{\ln 2} - e^{\ln 2^{-1}}}{2} = \frac{\frac{2}{1} - \frac{1}{2}}{\frac{2}{1}} = \frac{4-1}{\cancel{2}} \cdot \frac{\cancel{2}\,1}{\cancel{4}\,2}$$

$$= \frac{3}{4}$$

69 $\int_{-a}^{a} e^{tx}\, dx = \int_{-a}^{a} e^{tx}\, dx$

$$= \frac{1}{t}\left[e^{tx}\right]_{-a}^{a} = \frac{1}{t}\left[e^{at} - e^{-at}\right]$$

$$= \frac{2}{t}\left[\frac{e^{at} - e^{-at}}{2}\right]$$

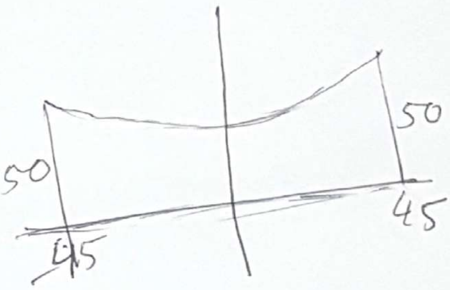
$$= \frac{2}{t}\sinh at$$

Equation of a catenary curve:

$$y = a\cosh\left(\frac{x}{a}\right) + c$$

where the parameters a and c are determine by the distance between the poles and the composition of the cable.

A 100ft wire is attached at its ends to the top of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?



The wire forms a catenary curve: $y = a\cosh\left(\frac{x}{a}\right) + c$

Length of the catenary curve:

$$L = \int_{-45}^{45} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\int_0^{45} \sqrt{1 + \left\{\cancel{a}\sinh\left(\frac{x}{a}\right)\cdot \cancel{a}\right\}^2}\, dx$$

$$= 2\int_0^{45} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\, dx$$

$$= 2\int_0^{45} \sqrt{\cosh^2\left(\frac{x}{a}\right)}\, dx$$

$$= 2\int_0^{45} \cosh\left(\frac{x}{a}\right) dx$$

$$= 2\sinh\left(\frac{x}{a}\right)\cdot a\Big]_0^{45}$$

$$\therefore 100 = 2a\sinh\left(\frac{45}{a}\right) \qquad (L = 100 \text{ ft})$$

$$\Rightarrow a \simeq 56.01$$ (by calculating utility) (numeric solver)

Then putting ~~x=45, we have~~ $a = 56.01$, we have

$$\cancel{100 = 2\cdot(56.01)}$$

$$50 = y(45) = 56.01\cosh\left(\frac{45}{56.01}\right) + c$$

$$\Rightarrow 50 \approx 75.08 + c$$

$$\Rightarrow c = -25.08$$

Thus the middle of the wire is

$$y(0) = (56.01)(1) - 25.08 = 30.93 \text{ ft}$$ above the ground