Chapter # 05
(Integration)

5.5 The Definite Integral: In this section we will introduce the concept of a ***"definite integral,"*** which will link the concept of area to other important concepts such as length, volume, density, probability, and work.

Riemann Sums and the Definite Integral: In our definition of net signed area, we assumed that for each positive number ***n***, the interval **[*a, b*]** was subdivided into ***n*** subintervals of equal length to create bases for the approximating rectangles.

A partition of the interval **[*a, b*]** is a collection of points

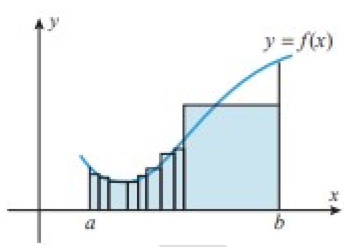
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

that divides **[*a, b*]** into ***n*** subintervals of lengths

$$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1, \quad \Delta x_3 = x_3 - x_2, \ldots, \quad \Delta x_n = x_n - x_{n-1}$$

The partition is said to be ***regular*** provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$



For a regular partition, the widths of the approximating rectangles approach zero as n is made large. Since this need not be the case for a general partition, we need some way to measure the "size" of these widths. One approach is to let $\max \Delta x_k$ denote the largest of the subinterval widths. The magnitude $\max \Delta x_k$ is called the ***mesh size*** of the partition.

Definition: A function ***f*** is said to be integrable on a finite closed interval **[*a, b*]** if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

which is called the ***definite integral*** of ***f*** from ***a*** to ***b***. The numbers ***a*** and ***b*** are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and $\boldsymbol{f(x)}$ is called the ***integrand***.

The above sum is called a ***Riemann sum***, and the definite integral is sometimes called the ***Riemann integral*** in honor of the German mathematician Bernhard Riemann who formulated many of the basic concepts of integral calculus.

Theorem: If a function ***f*** is continuous on an interval **[*a, b*]**, then ***f*** is integrable on **[*a, b*]**, and the net signed area ***A*** between the graph of ***f*** and the interval **[*a, b*]** is

$$A = \int_a^b f(x)\,dx$$

Example 1: Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\int_1^4 2\,dx$ (b) $\int_{-1}^2 (x+2)\,dx$ (c) $\int_0^1 \sqrt{1-x^2}\,dx$

Solution: **(*a*)** The graph of the integrand is the horizontal line, $\boldsymbol{y = 2}$, so the region is a rectangle of height **2** extending over the interval from 1 to 4 (figure - ***a***). Thus,

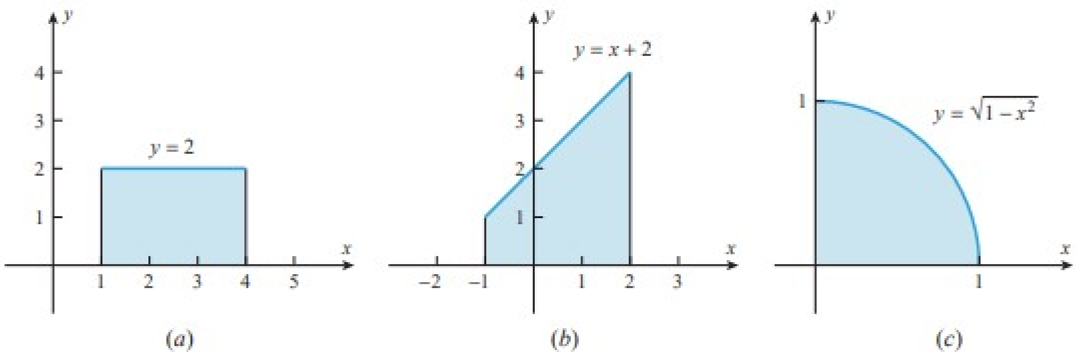
$$\int_1^4 2\,dx = (\text{area of rectangle}) = 2(3) = 6$$

(*b*) The graph of the integrand is the line $\boldsymbol{y = x + 2}$, so the region is a trapezoid whose base extends from ***x* = −1** to ***x* = 2** (figure - ***b***). Thus,

$$\int_{-1}^2 (x+2)\,dx = (\text{area of trapezoid}) = \frac{1}{2}(1+4)(3) = \frac{15}{2}$$

(*c*) The graph of $\boldsymbol{y = \sqrt{1-x^2}}$ is the upper semicircle of radius **1**, centered at the origin, so the region is the right quarter-circle extending from ***x* = 0** to ***x* = 1** (figure - ***c***). Thus,

$$\int_0^1 \sqrt{1-x^2}\,dx = (\text{area of quarter-circle}) = \frac{1}{4}\pi(1^2) = \frac{\pi}{4}$$



Properties of the Definite Integral:

Definition: (*a*) If ***a*** is in the domain of ***f***, we define

$$\int_a^a f(x)\,dx = 0$$

(*b*) If ***f*** is integrable on **[*a, b*]**, then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Example 3:

(a) $\int_1^1 x^2\,dx = 0$

(b) $\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}\,dx = -\frac{\pi}{4}$

Theorem: If ***f*** and ***g*** are integrable on **[*a, b*]** and if ***c*** is a constant, then ***cf***, ***f + g***, and ***f − g*** are integrable on **[*a, b*]** and

(*a*) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(*b*) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(*c*) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Theorem: If ***f*** is integrable on a closed interval containing the three points ***a***, ***b***, and ***c***, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the points are ordered.

Theorem: (*a*) If ***f*** is integrable on **[*a, b*]** and $\boldsymbol{f(x) \geq 0}$ for all ***x*** in **[*a, b*]**, then

$$\int_a^b f(x)\,dx \geq 0$$

(*b*) If ***f*** and ***g*** are integrable on **[*a, b*]** and $\boldsymbol{f(x) \geq g(x)}$ for all ***x*** in **[*a, b*]**, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Home Work: Exercise 5.5: Problem No. 13-18, 21-28 and 37-38