Day-1 , 15^{th} February, 2021

Assignment:1

A list of formulas for differentiation and integration from Chapters 2, 3 and 5.

→Due: 22^{nd} February, 2021

MAT-130: Calculus II → Chapters 7, 6 and 10

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = \int \frac{d}{dx}(f(x))\ \ dx = f(x) + C$$

Note: $\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1}x + C$

Section 7.1: Overview

7.1 → 5.3 : Integration by Substitution

Method: Integration by Substitution [u-substitution]

Recall Chain Rule: $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x)$; $[\frac{d}{dx}(f(u) = f'(u) \cdot \frac{du}{dx}]$

For example, $\frac{d}{dx}[\sin^2\left(e^{\sqrt{2x}}\right)] = \frac{d}{dx}[\sin\left(e^{\sqrt{2x}}\right)]^2 = 2\ \sin\left(e^{\sqrt{2x}}\right) \cdot \cos\left(e^{\sqrt{2x}}\right) \cdot e^{\sqrt{2x}} \cdot \frac{1}{2\sqrt{2x}} \cdot 2$

Backward Chain Rule: $\int f'\big(g(x)\big) \cdot g'(x)\ \ dx.$

- Integral function must be a product of two functions, where one of them is a composite function
- The other one must be the derivative or a constant multiple of the derivative of the input function in the composition.

Understanding: Identify the composite function. Set the input function as u, **whose derivatives is given** (somehow, scalar multiple is given).

Now,

$$\int f'\big(g(x)\big) \cdot g'(x)\ \ dx\ \ ;$$

Set $u = g(x)$. Then $\frac{du}{dx} = g'(x)$, than is, $du = g'(x)dx$.

Then,

$$\int f'(g(x)) \cdot g'(x) \quad dx$$

$$= \int f'(u)\, du \ = f(u) + C \ = f(g(x)) + C, \text{ where } C \text{ is a constant}$$

Examples

1) (a) $\displaystyle\int \frac{\mathbf{1}}{\sqrt{\boldsymbol{x}}} \ \mathbf{sec^2}(\sqrt{\boldsymbol{x}}\,) \quad \boldsymbol{dx} \qquad = \ 2 \int \frac{1}{2\sqrt{x}} \ \sec^2(\sqrt{x}\,) \ dx$

$= \ 2 \displaystyle\int \ \sec^2(\sqrt{x}\,) \ \frac{1}{2\sqrt{x}} dx$

$Set \quad u = \sqrt{x}$ implies $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$.

That is, $du \ = \ \frac{1}{2\sqrt{x}}\, dx$

Now,

$$\int \frac{1}{\sqrt{x}} \ \sec^2(\sqrt{x}\,) \ dx = 2 \int \frac{1}{2\sqrt{x}} \ \sec^2(\sqrt{x}\,) \ dx = \ 2 \int \ \sec^2(\sqrt{x}\,) \ \frac{1}{2\sqrt{x}} dx$$

$= \ 2 \ \int \ \sec^2(u\,)\, du \quad = 2\tan u \ + C \quad = 2 \ \tan(\sqrt{x}) \ + C$; [Formula : $\int \sec^2 x \ dx = \ \tan x \ + C$]

(b) $\displaystyle\int \frac{\mathbf{1}}{\sqrt{\boldsymbol{x}}} \ \mathbf{csc^2}(\sqrt{\boldsymbol{x}}) \ \boldsymbol{dx}$

Solution:

$$\int \frac{1}{\sqrt{x}} \ \csc^2(\sqrt{x}) \ dx \ = \ 2 \int \frac{1}{2\sqrt{x}} \ \csc^2(\sqrt{x}) \ dx$$

$= \ 2 \displaystyle\int \csc^2(\sqrt{x}) \ \frac{1}{2\sqrt{x}} \ dx$; Set $u = \sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, that is, $du = \frac{1}{2\sqrt{x}}\, dx$

$$= \ 2 \int \csc^2(u) \ du \ = -2\cot u + C \ = -2\cot(\sqrt{x}) \ + C$$

OR

$\displaystyle\int \frac{1}{\sqrt{x}} \ \csc^2(\sqrt{x}) \ dx$; Set $u = \sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, that is, $2\, du = \frac{1}{\sqrt{x}}\, dx$

$$\int \ \csc^2(\sqrt{x}) \ \frac{1}{\sqrt{x}}\, dx = \int \csc^2(u) \ 2 \ du = \ 2 \int \csc^2(u) \ du \ = -2\cot u + C \ = -2\cot(\sqrt{x}) \ + C$$

2) $(a) \int \frac{x}{\sqrt{1-x^4}}\, dx \quad = \int \frac{x}{\sqrt{1-(x^2)^2}}\, dx$

Set $u = x^2, \quad du = 2x\, dx$. So, $x\, dx = \frac{1}{2} du$

Now, $\int \frac{x}{\sqrt{1-x^4}}\, dx \quad = \int \frac{x}{\sqrt{1-(x^2)^2}}\, dx$

$= \int \frac{1}{\sqrt{1-(x^2)^2}} \quad x\, dx \quad = \int \frac{1}{\sqrt{1-u^2}} \quad \frac{1}{2}\, du$

$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}}\, du$

$= \frac{1}{2} \sin^{-1} u + C$

$= \frac{1}{2} \sin^{-1}(x^2) + C$

2) $(b) \int \frac{x^2}{\sqrt{1-x^3}}\, dx \; ; \quad Set \quad u = 1 - x^3 \, , then \; -\frac{1}{3}\, du = x^2 dx$

$\int \frac{x^2}{\sqrt{1-x^3}}\, dx = \int \frac{1}{\sqrt{u}} \left(-\frac{1}{3}\right) du \; = -\frac{1}{3} \int u^{-\frac{1}{2}}\, du$

$= -\frac{1}{3} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= -\frac{2}{3} \sqrt{1-x^3} + C$

Please practice few exercises from section 7.1

3) $(a) \int \frac{1}{x(\ln x)^3}\, dx$

Set $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$, that is, $du = \frac{1}{x}\, dx$

$\int \frac{1}{x(\ln x)^3}\, dx = \int \frac{1}{(\ln x)^3} \cdot \frac{1}{x}\, dx$

$$= \int \frac{1}{u^3}\, du \;=\; \int u^{-3}\, du = \frac{u^{-3+1}}{-3+1} + C$$

$$= -\frac{1}{2}\, u^{-2} + C \;=\; -\frac{1}{2u^2} + C$$

$$= -\frac{1}{2(\ln x)^2} + C$$

3) $(b) \int x\sqrt{1-x^2}\; dx$; $Set \;\; u = 1 - x^2$

3) $(c) \int \sqrt{1-x^2}\; dx$

3) $(d) \int \frac{1}{\sqrt{1+x^2}}\; dx$

3) $(e) \int \frac{x}{\sqrt{1+x^2}}\; dx$; $Set\; u = 1 + x^2$

Note: parts (c) and (d) need the concept of trigonometric substitution.

4) $\int \frac{x^2}{\sqrt{1-x^6}}\, dx \;=\int \frac{x^2}{\sqrt{1-(x^3)^2}}\, dx \;=\; \int \frac{1}{\sqrt{1-(x^3)^2}} \cdot x^2\, dx$

Set $u = x^3$ Then $x^2 dx = \frac{1}{3}\, du$

$$\int \frac{x^2}{\sqrt{1-x^6}}\, dx \;=\; \int \frac{1}{\sqrt{1-(x^3)^2}} \cdot x^2\, dx$$

$$= \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{3}\, du$$

$= \frac{1}{3}\sin^{-1} u \; + C = \frac{1}{3}\sin^{-1}(x^3) \; + C$; Formula: $\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u \; + C$