(Continuous probability distribution)

- **Central limit theorem:** When sample size is large, the average of a set of independent identically distributed random variables ($\bar{X}$) is always approximately normally distributed with mean population mean and variance population variance divided by sample size. i.e When n is large, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ then $X_1 + X_2 + ... + X_n \sim$ N(nμ, nσ^2)

  Example: The number of flaws in a glass sheet has a Poisson distribution with a parameter $\lambda =$ 0.5. a) What is the distribution of the total number of flaws X in 100 sheets of glass? b) Calculate the probability that there are fewer than 40 flaws in 100 sheets of glass? c) What is the distribution of the average number of flaws per sheet in 100 sheets of glass? d) Calculate the probability that this average is between 0.45 and 0.55.

  Solution:

  The expected number of flaws in 100 sheets of glass is $100 \times \mu = 50$, and the variance of the number of flaws in 100 sheets of glass is $100 \times \sigma^2 = 50$. The central limit theorem then implies that the distribution of X can be approximated by a

  $$N(50, 50)$$

  distribution. The probability that there are fewer than 40 flaws in 100 sheets of glass can therefore be approximated as

  $$P(X \leq 40) \simeq \Phi\left(\frac{40 - 50}{\sqrt{50}}\right) = \Phi(-1.41) = 0.0793$$

  or about 8%.

  The central limit theorem indicates that the average number of flaws per sheet in 100 sheets of glass, $\bar{X}$, has a distribution that can be approximated by a N(μ, $\frac{\sigma^2}{100}$) $=$ N(0.5, 0.005) The probability that this average is between 0.45 and 0.55 is therefore

  $$P(0.45 \leq \bar{X} \leq 0.55) \simeq \Phi\left(\frac{0.55 - 0.50}{\sqrt{0.005}}\right) - \Phi\left(\frac{0.45 - 0.50}{\sqrt{0.005}}\right)$$
  $$= \Phi(0.71) - \Phi(-0.71) = 0.7611 - 0.2389 = 0.5222$$

  Homework
  The number of cracks in a ceramic tile has a Poisson distribution with parameter $\lambda = 2.4$. (a) How would you approximate the distribution of the total number of cracks in 500 ceramic tiles? (b) Estimate the probability that there are more than 1250 cracks in 500 ceramic tiles.

- **Chi-square distribution:**

If the random variable X has a standard normal distribution, then the random variable

$$Y = X^2$$

is said to have a **chi-square distribution** with *one degree of freedom*. More generally, if the random variables $X_i \sim N(0, 1)$, $1 \le i \le n$, are independent, then the random variable

$$Y = X_1^2 + \cdots + X_n^2$$

is said to have a chi-square distribution with *n degrees of freedom*.

The critical points of chi-square distributions are denoted by $\chi^2_{\alpha,\nu}$ and are defined by

$$P\left(X \ge \chi^2_{\alpha,\nu}\right) = \alpha$$

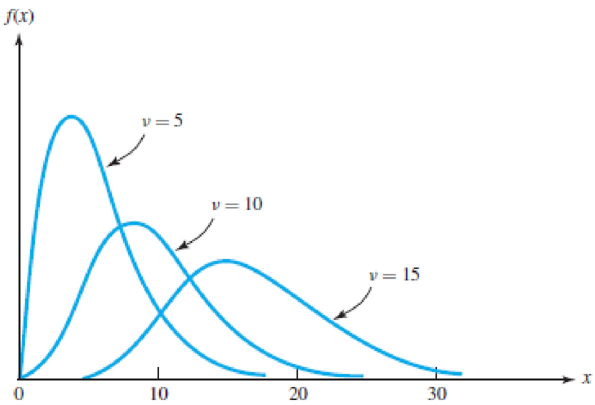


FIGURE 5.36

Probability density functions of the chi-square distribution

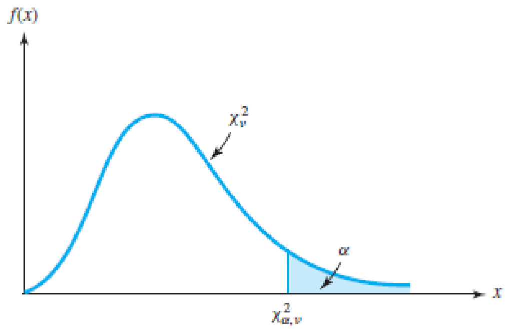


FIGURE 5.37

The critical points $\chi^2_{\alpha,\nu}$ of the chi-square distribution

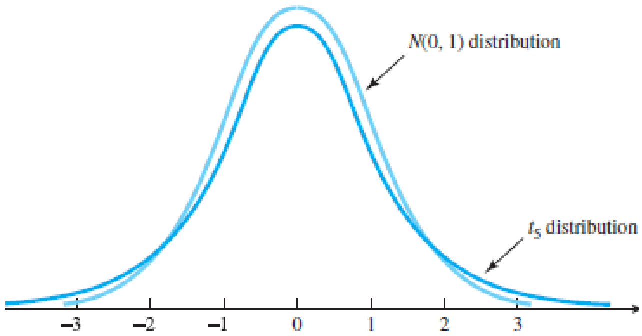
- **t-distribution:**

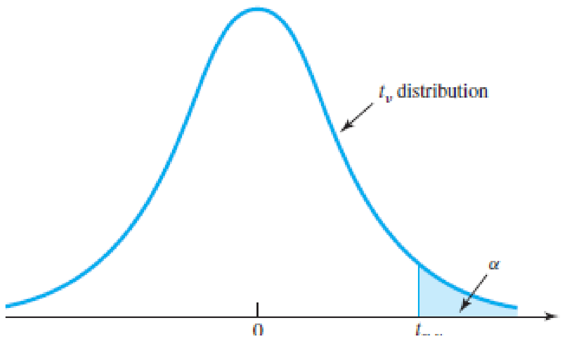
If a standard normal random variable is divided by the square root of an independent χ^2_ν/ν random variable, then the resulting random variable is said to have a ***t*-distribution** with ν degrees of freedom. This can be written

$$t_\nu \sim \frac{N(0, 1)}{\sqrt{\chi^2_\nu/\nu}}$$

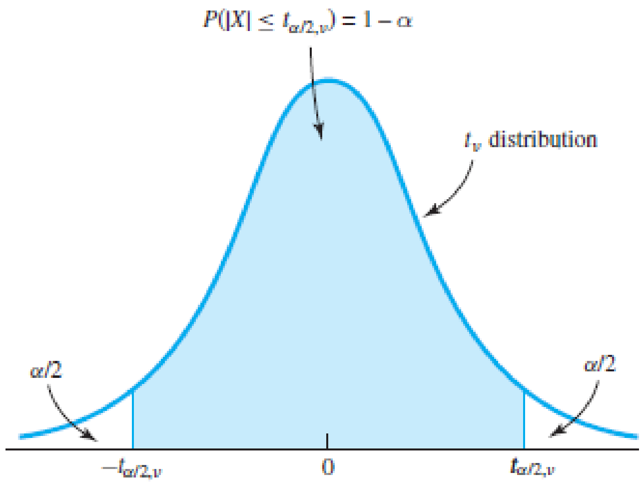
The critical points of a t-distribution are denoted by $t_{\alpha,\nu}$ and are defined by

$$P(X \ge t_{\alpha,\nu}) = \alpha$$





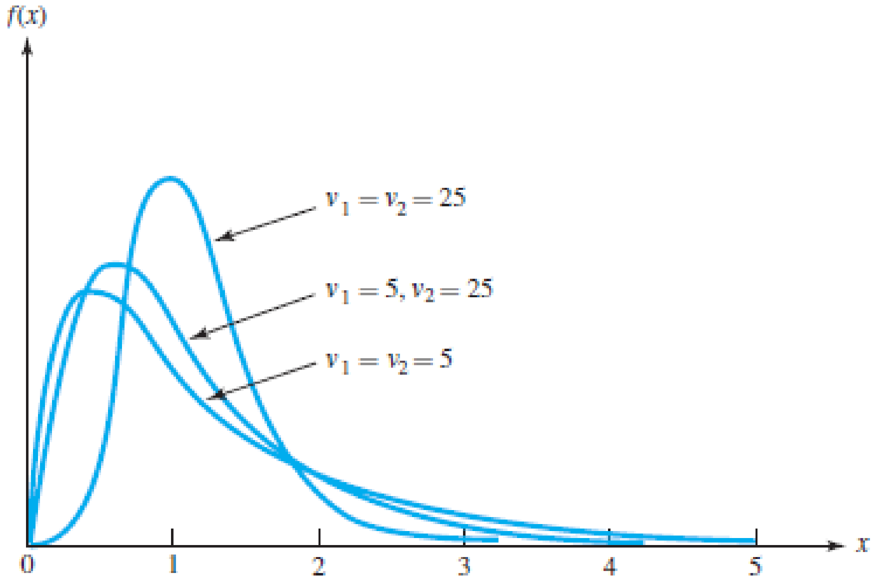
$$P(|X| \le t_{\alpha/2,\nu}) = P(-t_{\alpha/2,\nu} \le X \le t_{\alpha/2,\nu}) = 1 - \alpha$$



- **F distribution:**

  The ratio of two independent chi-square random variables that have been divided by their respective degrees of freedom is defined to be an ***F*-distribution**. This ratio can be written

$$F_{\nu_1,\nu_2} \sim \frac{\chi^2_{\nu_1}/\nu_1}{\chi^2_{\nu_2}/\nu_2}$$



$f(x)$
$v_1 = v_2 = 25$
$v_1 = 5, v_2 = 25$
$v_1 = v_2 = 5$
0
1
2
3
4
5
x