Chapter # 05
(Integration)

5.6 The Fundamental Theorem of Calculus: In this section we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the "***Fundamental Theorem of Calculus***." One part of this theorem will relate the rectangle and antiderivative methods for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.

Theorem (The Fundamental Theorem of Calculus, Part 1): If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example 1: Evaluate

$$\int_1^2 x\,dx.$$

Solution: The function $F(x) = \frac{x^2}{2}$ is an antiderivative of $f(x) = x$, therefore

$$\int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

Note: The requirements in the Fundamental Theorem of Calculus that f be continuous on $[a, b]$ and that F be an antiderivative for f over the entire interval $[a, b]$ are important to keep in mind. Disregarding these assumptions will likely lead to incorrect results. For example, the function $f(x) = \frac{1}{x^2}$ fails on two counts to be continuous at $x = 0$: $f(x)$ is not defined at $x = 0$ and $\lim_{x\to 0} f(x)$ does not exist. Thus, the Fundamental Theorem of Calculus should not be used to integrate f on any interval that contains $x = 0$. However, if we ignore this and mistakenly apply Formula over the interval $[-1, 1]$, we might incorrectly compute $\int_{-1}^{1}\left(\frac{1}{x^2}\right)dx$ by evaluating an antiderivative, $-\frac{1}{x}$, at the endpoints, arriving at the answer

$$-\frac{1}{x}\Big]_{-1}^{1} = -[1 - (-1)] = -2$$

But $f(x) = \frac{1}{x^2}$ is a nonnegative function, so clearly a negative value for the definite integral is impossible.

The Fundamental Theorem of Calculus can be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

Example 2:

$$\int_0^3 (9 - x^2)\, dx = \left[9x - \frac{x^3}{3}\right]_0^3 = \left(27 - \frac{27}{3}\right) - 0 = 18$$

Example 4:

$$\int_1^9 \sqrt{x}\, dx = \int x^{1/2}\, dx\Big]_1^9 = \frac{2}{3}x^{3/2}\Big]_1^9 = \frac{2}{3}(27 - 1) = \frac{52}{3}$$

Example 5:

$$\int_4^9 x^2\sqrt{x}\, dx = \int_4^9 x^{5/2}\, dx = \frac{2}{7}x^{7/2}\Big]_4^9 = \frac{2}{7}(2187 - 128) = \frac{4118}{7} = 588\frac{2}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\, dx = -\frac{1}{5}\cos x\Big]_0^{\pi/2} = -\frac{1}{5}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{1}{5}[0 - 1] = \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\, dx = \tan x\Big]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$\int_0^{\ln 3} 5e^x\, dx = 5e^x\Big]_0^{\ln 3} = 5[e^{\ln 3} - e^0] = 5[3 - 1] = 10$$

$$\int_{-e}^{-1} \frac{1}{x}\, dx = \ln|x|\Big]_{-e}^{-1} = \ln|-1| - \ln|-e| = 0 - 1 = -1$$

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dx = \sin^{-1} x\Big]_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$

Example 6:

$$\int_1^1 x^2\, dx = \frac{x^3}{3}\Big]_1^1 = \frac{1}{3} - \frac{1}{3} = 0$$

$$\int_4^0 x\, dx = \frac{x^2}{2}\Big]_4^0 = \frac{0}{2} - \frac{16}{2} = -8$$

Example 7: Evaluate

$$\int_0^3 f(x)\, dx \text{ if}$$

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Solution:

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$$
$$= \frac{x^3}{3}\Big]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}$$

Total Area: If ***f*** is a continuous function on the interval **[*a, b*]**, then we define the total area between the curve $\boldsymbol{y = f(x)}$ and the interval **[*a, b*]** to be

$$\text{total area} = \int_a^b |f(x)|\,dx$$

Example 8: Find the total area between the curve $\boldsymbol{y = 1 - x^2}$ and the ***x***-axis over the interval **[*0, 2*].**

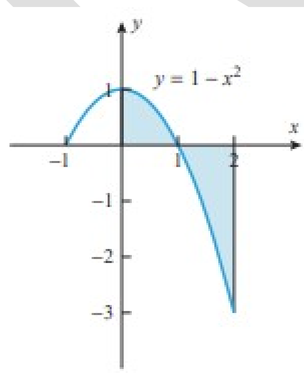
Solution: $\boldsymbol{x}$-intercept for, $\boldsymbol{y = 0} \Rightarrow \boldsymbol{1 - x^2 = 0} \ \ \therefore \boldsymbol{x = \pm 1}$

On $[\mathbf{0}, \mathbf{2}]$, $\boldsymbol{x}$-intercept is $+\mathbf{1}.$

For $(\mathbf{0}, \mathbf{1})$: $\boldsymbol{y} > 0$ $and\ for$ $(\mathbf{1}, \mathbf{2})$: $\boldsymbol{y} < 0.$

The area ***A*** is given by,

$$A = \int_0^2 |1 - x^2|\,dx = \int_0^1 (1 - x^2)\,dx + \int_1^2 -(1 - x^2)\,dx$$
$$= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2$$
$$= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2$$



Theorem (The Mean-Value Theorem for Integrals): If ***f*** is continuous on a closed interval **[*a, b*]**, then there is at least one point $\boldsymbol{x}^*$ in **[*a, b*]** such that

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

Example 9: Since $f(x) = x^2$ is continuous on the interval **[1, 4]**, the Mean-Value Theorem for Integrals guarantees that there is a point x^* in **[1, 4]** such that

$$\int_1^4 x^2\,dx = f(x^*)(4-1) = (x^*)^2(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\bigg]_1^4 = 21$$

so that

$$3(x^*)^2 = 21 \quad \text{or} \quad (x^*)^2 = 7 \quad \text{or} \quad x^* = \pm\sqrt{7}$$

Thus, $x^* = \sqrt{7} \approx 2.65$ is the point in the interval **[1, 4]** whose existence is guaranteed by the Mean-Value Theorem for Integrals.

Theorem (The Fundamental Theorem of Calculus, Part 2): If ***f*** is continuous on an interval, then ***f*** has an antiderivative on that interval. In particular, if a is any point in the interval, then the function ***F*** defined by

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of ***f***; that is, $F'(x) = f(x)$ for each ***x*** in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Example 10: Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

by applying ***Part 2 of the Fundamental Theorem of Calculus***, and then confirm the result by performing the integration and then differentiating.

Solution: The integrand is a continuous function, so from the theorem

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$$

Alternatively, evaluating the integral and then differentiating yields

$$\int_1^x t^3\,dt = \frac{t^4}{4}\Bigg]_{t=1}^{x} = \frac{x^4}{4} - \frac{1}{4}, \quad \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$$

so the two methods for differentiating the integral agree.

Home Work: Exercise 5.6: Problem No. 5-11, 13-26, 31 and 59-63