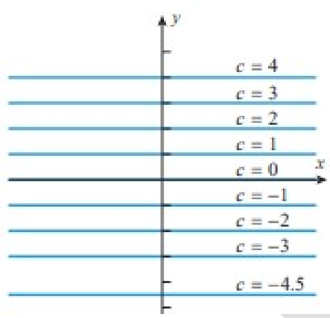
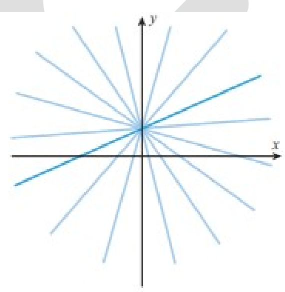
Calculus and Analytical Geometry-I
(Families of Functions and Inverse Functions)

0.3 Families of Functions: Functions are often grouped into families according to the form of their defining formulas or other common characteristics. In this section we will discuss some of the most basic families of functions.

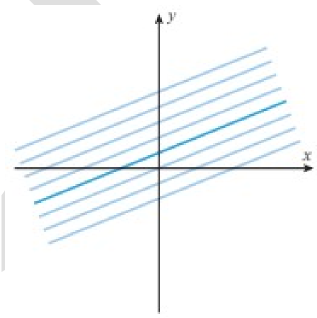
Families of Curves: The graph of a constant function $f(x) = c$ is the graph of the equation $y = c$, which is the horizontal line. If we vary c, then we obtain a set or family of horizontal lines.

Constants that are varied to produce families of curves are called parameters. For example, recall that an equation of the form $y = mx + b$ represents a line of slope m and y-intercept b. If we keep b fixed and treat m as a parameter, then we obtain a family of lines whose members all have y-intercept b, and if we keep m fixed and treat b as a parameter, we obtain a family of parallel lines whose members all have slope m.



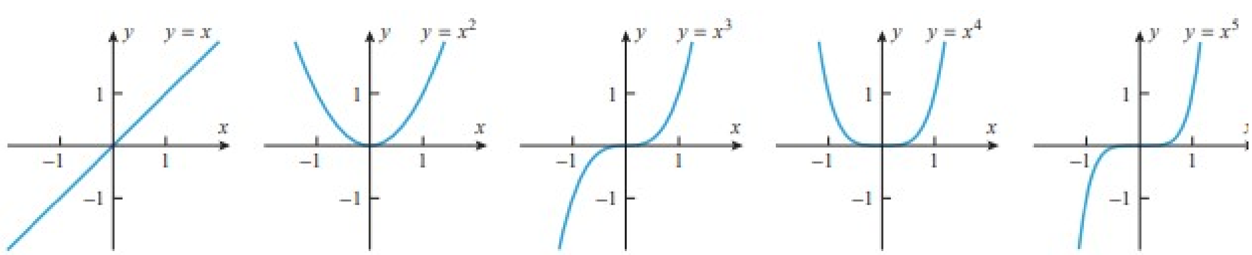


The family $y = mx + b$ (b fixed and m varying)



The family $y = mx + b$ (m fixed and b varying)

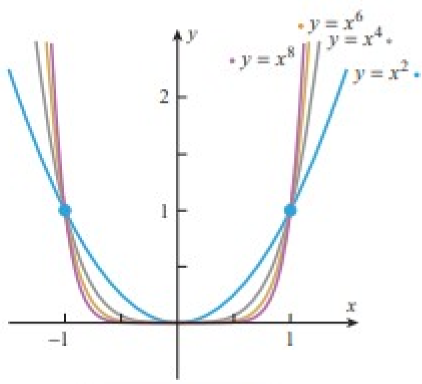
Power Functions; The Family $y = x^n$: A function of the form $f(x) = x^p$, where p is constant, is called a power function. For the moment, let us consider the case where p is a positive integer, say $p = n$. The graphs of the curves $y = x^n$ for n = 1, 2, 3, 4, and 5 are shown in the following figure.



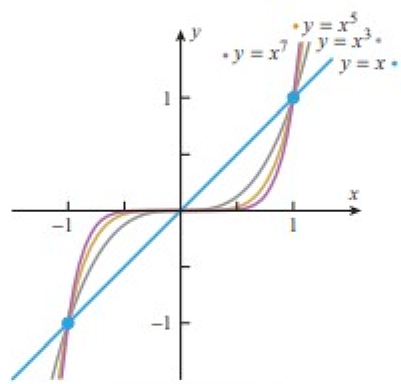
Note: For $\boldsymbol{n \geq 2}$ the shape of the curve $\boldsymbol{y = x^n}$ depends on whether ***n*** is even or odd.

(i) For even values of ***n***, the functions $\boldsymbol{f(x) = x^n}$ are even, so their graphs are symmetric about the ***y***-axis. The graphs all have the general shape of the graph of $\boldsymbol{y = x^2}$, and each graph passes through the points (−1, 1), (0, 0), and (1, 1). As ***n*** increases, the graphs become flatter over the interval $-\mathbf{1} < x < 1$ and steeper over the intervals $\boldsymbol{x} > 1$ *and* $x < -1$**.**

(ii) For odd values of n, the functions $\boldsymbol{f(x) = x^n}$ are odd, so their graphs are symmetric about the origin. The graphs all have the general shape of the curve $\boldsymbol{y = x^3}$, and each graph passes through the points (−1, −1), (0, 0), and (1, 1). As ***n*** increases, the graphs become flatter over the interval $-\mathbf{1} < x < 1$ and steeper over the intervals $\boldsymbol{x} > 1$ *and* $x < -1$**.**



The family $y = x^n$
(n even)



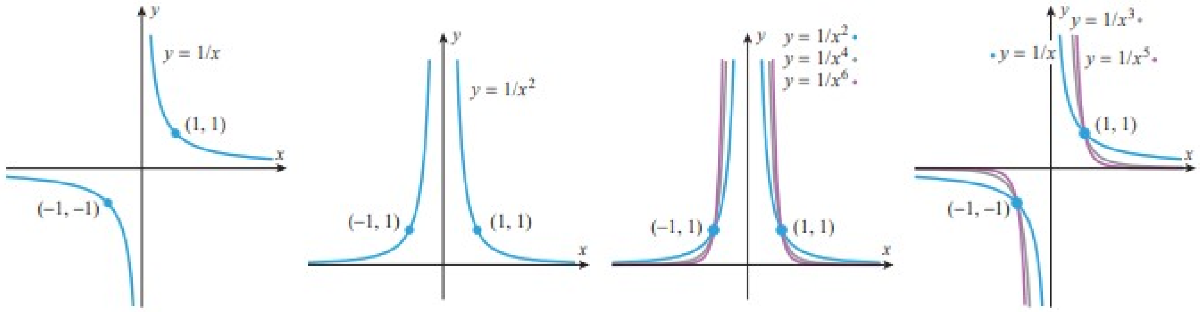
The family $y = x^n$
(n odd)

The Family $\boldsymbol{y = x^{-n}}$**:** If ***p*** is a negative integer, say $\boldsymbol{p = -n}$, then the power functions $\boldsymbol{f(x) = x^p}$ have the form $\boldsymbol{f(x) = x^{-n} = \frac{1}{x^n}}$**.** The graph of $\boldsymbol{y = \frac{1}{x}}$ is called an equilateral hyperbola. The shape of the curve $\boldsymbol{y = \frac{1}{x^n}}$ depends on whether ***n*** is even or odd.

(i) For even values of ***n***, the functions $\boldsymbol{f(x) = \frac{1}{x^n}}$ are even, so their graphs are symmetric about the ***y***-axis. The graphs all have the general shape of the curve $\boldsymbol{y = \frac{1}{x^2}}$, and each graph passes through the points (−1, 1) and (1, 1). As ***n*** increases, the graphs become ***steeper*** over the intervals ***−1 <x< 0*** and ***0 <x< 1*** and become ***flatter*** over the intervals ***x > 1*** and ***x < −1.***

(ii) For odd values of ***n***, the functions $\boldsymbol{f(x) = \frac{1}{x^n}}$ are odd, so their graphs are symmetric about the origin. The graphs all have the general shape of the curve $\boldsymbol{y = \frac{1}{x}}$, and each graph passes through the points (1, 1) and (−1, −1). As ***n*** increases, the graphs become ***steeper*** over the intervals ***−1 <x< 0*** and ***0 <x< 1*** and become ***flatter*** over the intervals ***x > 1*** and ***x < −1***.

(iii) For both even and odd values of n the graph $y = \frac{1}{x^n}$ has a break at the origin (called a discontinuity), which occurs because division by zero is undefined.



The family $y = 1/x^n$ (n even)

The family $y = 1/x^n$ (n odd)

Polynomials: A polynomial in x is a function that is expressible as a sum of finitely many terms of the form cx^n, where c is a constant and n is a nonnegative integer. Some examples of polynomials are

$$2x + 1, \quad 3x^2 + 5x - \sqrt{2}, \quad x^3, \quad 4\,(= 4x^0), \quad 5x^7 - x^4 + 3$$

The function $(x^2 - 4)^3$ is also a polynomial because it can be expanded by the binomial formula and expressed as a sum of terms of the form cx^n:

$$(x^2 - 4)^3 = (x^2)^3 - 3(x^2)^2(4) + 3(x^2)(4^2) - (4^3) = x^6 - 12x^4 + 48x^2 - 64$$

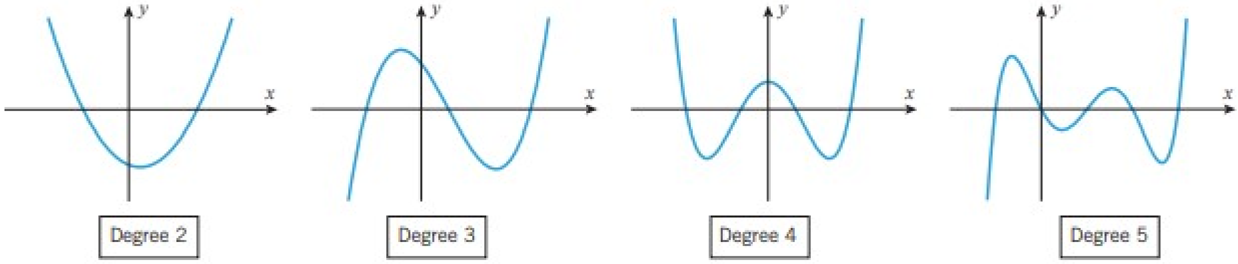
A general polynomial can be written in either of the following forms, depending on whether one wants the powers of x in ascending or descending order:

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$
$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

The constants $c_0, c_1, \ldots\ldots, c_n$ are called the coefficients of the polynomial. When a polynomial is expressed in one of these forms, the highest power of x that occurs with a nonzero coefficient is called the degree of the polynomial. Nonzero constant polynomials are considered to have degree 0, since we can write $c = cx^0$. Polynomials of degree 1, 2, 3, 4, and 5 are described as linear, quadratic, cubic, quartic, and quintic, respectively. For example

$3 + 5x$	$x^2 - 3x + 1$	$2x^3 - 7$
Has degree 1 (linear)	Has degree 2 (quadratic)	Has degree 3 (cubic)
$8x^4 - 9x^3 + 5x - 3$	$\sqrt{3} + x^3 + x^5$	$(x^2 - 4)^3$
Has degree 4 (quartic)	Has degree 5 (quintic)	Has degree 6 [see (3)]

Note: The natural domain of a polynomial in x is $(-\infty, \infty)$, since the only operations involved are multiplication and addition; the range depends on the particular polynomial. We will see later that the number of peaks and valleys is less than the degree of the polynomial.



Rational Functions: A function that can be expressed as a ratio of two polynomials is called a rational function. If P (x) and Q(x) are polynomials, then the domain of the rational function consists of all values of x such that $\boldsymbol{Q(x) \neq 0}$.

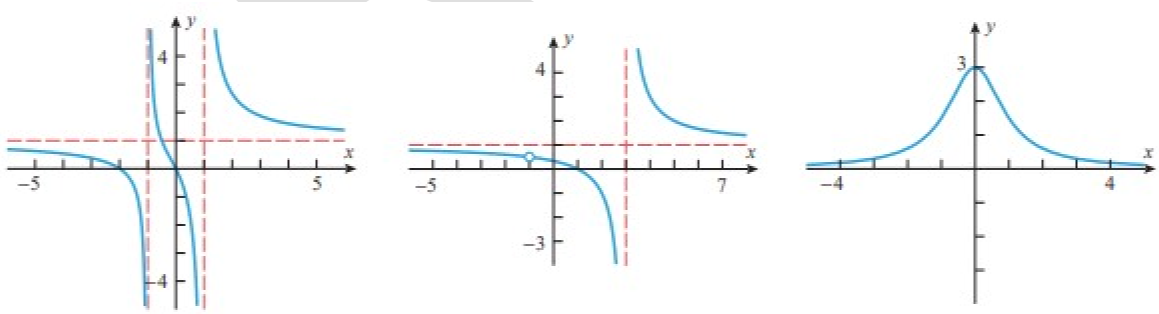
$$f(x) = \frac{P(x)}{Q(x)}$$

The graphs of rational functions with nonconstant denominators differ from the graphs of polynomials in some essential ways:

(i) Unlike polynomials whose graphs are continuous (unbroken) curves, the graphs of rational functions have discontinuities at the points where the denominator is zero.

(ii) Unlike polynomials, rational functions may have numbers at which they are not defined. Near such points, many rational functions have graphs that closely approximate a vertical line, called a vertical asymptote.

(iii) Unlike the graphs of nonconstant polynomials, which eventually rise or fall indefinitely, the graphs of many rational functions eventually get closer and closer to some horizontal line, called a horizontal asymptote.



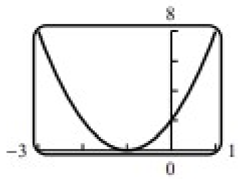
$y = \dfrac{x^2+2x}{x^2-1}$ $\quad y = \dfrac{x^2-1}{x^2-2x-3}$ $\quad y = \dfrac{3}{x^2+1}$

Question 17 (exercise): Sketch the graph of

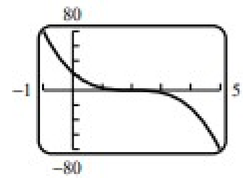
(a) $y = 2(x+1)^2$ (b) $y = -3(x-2)^3$

(c) $y = \dfrac{-3}{(x+1)^2}$ (d) $y = \dfrac{1}{(x-3)^5}$

Solution: (*a*)



(*b*)



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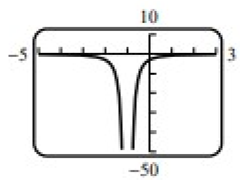
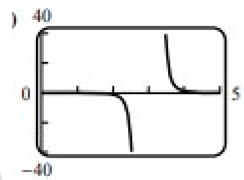


(*d*)



Home Work: Exercise 0.3: Problem No. 13-17 and 35

0.4 Inverse Functions; Inverse Trigonometric Function: In everyday language the term “inversion” conveys the idea of a reversal. For example, in meteorology a temperature inversion is a reversal in the usual temperature properties of air layers, and in music a melodic inversion reverses an ascending interval to the corresponding descending interval. In mathematics the term inverse is used to describe functions that reverse one another in the sense that each undoes the effect of the other. In this section we discuss this fundamental mathematical idea. In particular, we introduce inverse trigonometric functions to address the problem of recovering an angle that could produce a given trigonometric function value.

Inverse Function: If the functions ***f*** and ***g*** satisfy the two conditions

$\boldsymbol{g(f(x)) = x}$ for every ***x*** in the domain of ***f***

$\boldsymbol{f(g(y)) = y}$ for every ***y*** in the domain of ***g***

then we say that ***f*** is an inverse of ***g*** and ***g*** is an inverse of ***f*** or that ***f*** and ***g*** are inverse functions.

Thus, if a function ***f*** has an inverse, then we are entitled to talk about “the” inverse of ***f***, in which case we denote it by the symbol $\boldsymbol{f^{-1}}$.

Example 2: Confirm each of the following

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

Solution: (*a*)

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$
$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

(*b*)

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$
$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x$$

Domain and Range of Inverse Functions: The domains and ranges of $\boldsymbol{f}$ and f^{-1} are as follows

domain of f^{-1} = range of f
range of f^{-1} = domain of f

A Method for Finding Inverse Functions:

Theorem: If an equation $\boldsymbol{y = f(x)}$ can be solved for ***x*** as a function of ***y***, say $\boldsymbol{x = g(y)}$, then ***f*** has an inverse and that inverse is $\boldsymbol{g(y) = f^{-1}(y)}$.

A Procedure for Finding the Inverse of a Function *f*:

Step 1. Write down the equation $\boldsymbol{y = f(x)}$.

Step 2. If possible, solve this equation for ***x*** as a function of ***y***.

Step 3. The resulting equation will be $\boldsymbol{x = f^{-1}(y)}$, which provides a formula for $\boldsymbol{f^{-1}}$ with ***y*** as the independent variable.

Step 4. If ***y*** is acceptable as the independent variable for the inverse function, then you are done, but if you want to have ***x*** as the independent variable, then you need to interchange ***x*** and ***y*** in the equation $\boldsymbol{x = f^{-1}(y)}$ to obtain $\boldsymbol{y = f^{-1}(x)}$.

Example 4: Find a formula for the inverse of $\boldsymbol{f(x) = \sqrt{3x-2}}$ with ***x*** as the independent variable, and state the domain of $\boldsymbol{f^{-1}}$.

Solution: We first write

$$\boldsymbol{y = \sqrt{3x-2}}$$

Then we solve this equation for ***x*** as a function of ***y***:

$$y^2 = 3x - 2$$
$$x = \frac{1}{3}(y^2 + 2)$$

which tells us that

$$f^{-1}(y) = \frac{1}{3}(y^2 + 2)$$

Since we want ***x*** to be the independent variable, we reverse ***x*** and ***y*** in above to produce the formula

$$f^{-1}(x) = \frac{1}{3}(x^2 + 2)$$

Again, $\boldsymbol{f(x) = \sqrt{3x-2} \quad here \quad 3x - 2 \geq 0 \;\Rightarrow\; x \geq \frac{2}{3}}$

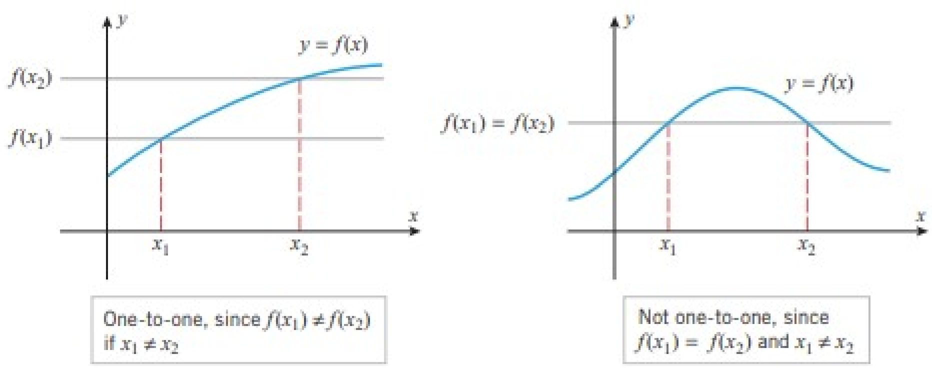
∴ The domain of ***f*** is $[\frac{2}{3},\ \infty)$

And the range of ***f*** is $[0, \infty)$.

Therefore the domain of f^{-1} is $[0, \infty)$.

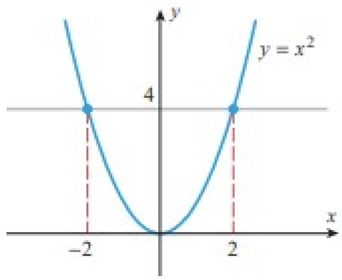
Theorem: A function has an ***inverse*** if and only if it is ***one-to-one***.

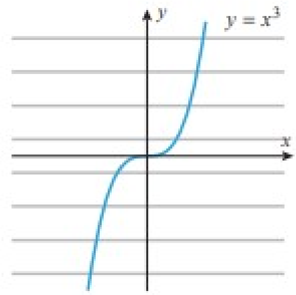
Theorem (The Horizontal Line Test): A function has an inverse function if and only if its graph is cut at most once by any horizontal line.



Example 5: Use the horizontal line test to show that $\boldsymbol{f(x) = x^2}$ has no inverse but that $\boldsymbol{f(x) = x^3}$ does.

Solution: A horizontal line that cuts the graph of $\boldsymbol{y = x^2}$ more than once, so $\boldsymbol{f(x) = x^2}$ is not invertible. The graph of $y = x^3$ is cut at most once by any horizontal line, so $\boldsymbol{f(x) = x^3}$ is invertible.



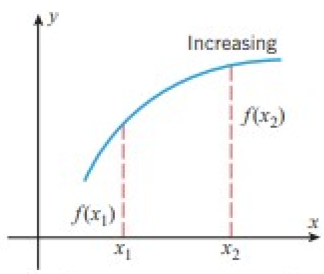


Increasing or Decreasing Functions are Invertible: A function whose graph is always rising as it is traversed from left to right is said to be an ***increasing function***, and a function whose graph is always falling as it is traversed from left to right is said to be a ***decreasing function***. If x_1 and x_2 are points in the domain of a function f, then f is increasing if

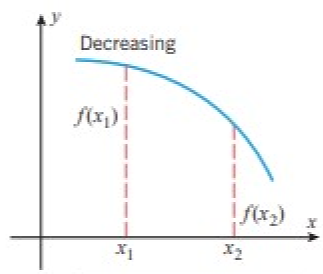
$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2$$

and f is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2$$



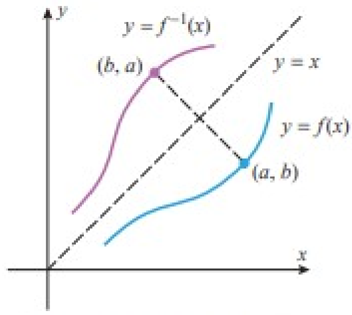
$f(x_1) < f(x_2)$ if $x_1 < x_2$



$f(x_1) > f(x_2)$ if $x_1 < x_2$

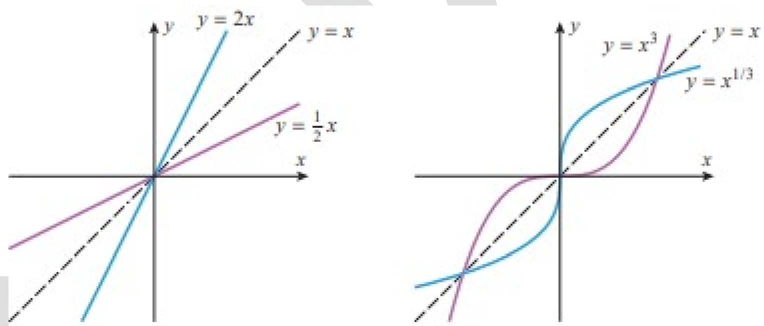
Theorem: If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.

Example: The graph of $f\ and\ f^{-1}$ are reflections about $\boldsymbol{y = x}$**.**



The graphs of f and f^{-1} are reflections about $y = x$.

Example 7: The following figure shows the graphs of the inverse functions.



Restricting Domains for Invertibility: If a function ***g*** is obtained from a function ***f*** by placing restrictions on the domain of ***f***, then ***g*** is called a restriction of ***f***. Thus, for example, the function

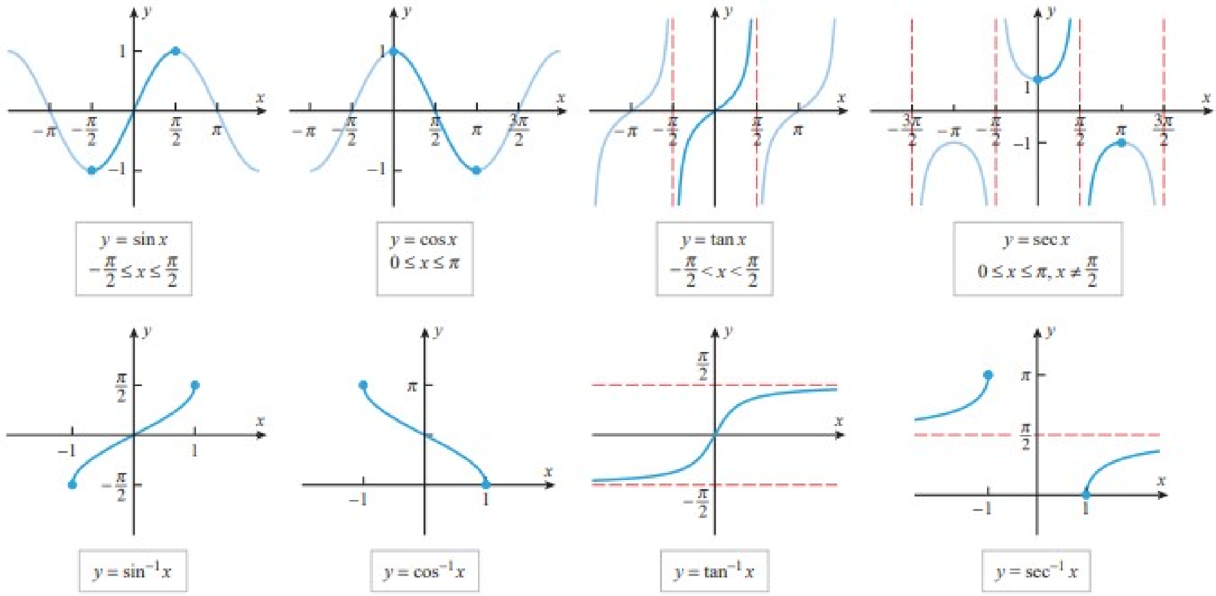
$$g(x) = x^3, \quad x \geq 0$$

is a restriction of the function $\boldsymbol{f(x) = x^3}$. More precisely, it is called the restriction of $\boldsymbol{x^3}$ to the interval $[0, \infty)$.

Inverse Trigonometric Functions: A common problem in trigonometry is to find an angle $\boldsymbol{x}$ using a known value of $\boldsymbol{\sin x}$, $\boldsymbol{cos\ x}$, or some other trigonometric function. Recall that problems of this type involve the computation of "***arc functions***" such as $\boldsymbol{arc \sin x}$, $\boldsymbol{arc \cos x}$, and so forth. We will conclude this section by studying these arc functions from the viewpoint of general inverse functions.

The six basic trigonometric functions do not have inverses because their graphs repeat periodically and hence do not pass the horizontal line test. To circumvent this problem we will restrict the domains of the trigonometric functions to produce one-to-one functions and then define the "***inverse trigonometric functions***" to be the inverses of these restricted functions. The top part of the following figure shows geometrically how these restrictions are made for ***sin x***, ***cos x***, ***tan x***, and ***sec x***, and the bottom part of the figure shows the graphs of the corresponding inverse functions

$$\sin^{-1}x, \quad \cos^{-1}x, \quad \tan^{-1}x, \quad \sec^{-1}x$$



Definition: The ***inverse sine function***, denoted by $\boldsymbol{sin^{-1}}$, is defined to be the inverse of the restricted sine function

$$\sin x, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Definition: The ***inverse cosine function***, denoted by $\boldsymbol{cos^{-1}}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \qquad 0 \le x \le \pi$$

Definition: The ***inverse tangent function***, denoted by $\boldsymbol{tan^{-1}}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Definition: The ***inverse secant function***, denoted by $\boldsymbol{sec^{-1}}$, is defined to be the inverse of the restricted secant function

$$\sec x, \qquad 0 \le x \le \pi \quad with \quad x \neq \frac{\pi}{2}$$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

Identities for Inverse Trigonometric Functions:

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

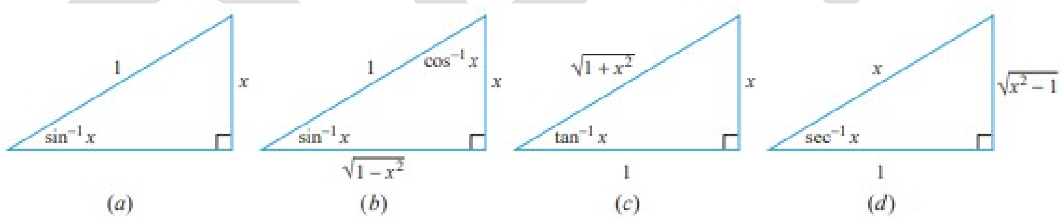
$$\cos(\sin^{-1} x) = \sqrt{1 - x^2}$$

$$\sin(\cos^{-1} x) = \sqrt{1 - x^2}$$

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}$$

$$\sec(\tan^{-1} x) = \sqrt{1 + x^2}$$

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2 - 1}}{x} \quad (x \ge 1)$$



Home Work: Exercise 0.4: Problem No. 1(c, d), 3(c, e), 9-20, 40 and 41