DAY-3:

7.3: Integration of Trigonometric Functions

Group -1: All six trigonometric functions with power 1

1) $\int \sin x \ dx = -\cos x + C$
2) $\int \cos x \ dx = \sin x + C$

3) $\int \tan x \ dx = \int \frac{\sin x}{\cos x} \ dx = \int \frac{1}{\cos x} \ \boldsymbol{\sin x \, dx}$

[Set $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$, that is, $du = -\sin x \ dx$. Hence, $-du = \boldsymbol{\sin x \ dx}$]

$= \int \frac{1}{u} (-1) du = -\int \frac{1}{u} \ du$

$= -\ln |u| + C$

$= -\ln |\cos x| + C$

$= \ln |\cos x|^{-1} + C$; $[n \log_b x = \log_b x^n]$

$= \ln |(\cos x)^{-1}| + C$

$$\int \tan x \ dx = \ln |\sec x| + C$$

4) $\int \cot x \ dx = \int \frac{\cos x}{\sin x} \ dx = \int \frac{1}{\sin x} \cos x \, dx = \ln|\sin x| + C$

5) $\int \sec x \ dx = \int \sec x \cdot 1 \, dx = \int \frac{\sec x \, (\tan x + \sec x)}{\tan x + \sec x} \ dx$

$= \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} \ dx == \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \ dx$;

[Set $u = \tan x + \sec x$, then $\frac{du}{dx} = \sec^2 x + \sec x \ \tan x$

Hence, $du = (\sec^2 x\ + \sec x\ \tan x\ )dx]$

$$\int \sec x\ dx = \int \frac{1}{u}\ du\ \ =\ \ \ln|u| + C$$

$$\int \sec x\ dx\ \ =\ \ \ln|\tan x + \sec x| + C$$

Note: (1) $\int \sec x \tan x\, dx\ = \sec x + C$ and $\int \sec^2 x\, dx\ = \tan x + C$

(2) $\int \csc x \cot x\, dx\ = -\csc x + C$ and $\int \csc^2 x\, dx\ = -\cot x + C$

Also, $\cot^2 x + 1 = \csc^2 x\ \ \Rightarrow\ \ \mathbf{1 = \csc^2 \boldsymbol{x} - \cot^2 \boldsymbol{x}}$

6) $\int \csc x\ dx$ Homework

$\int \csc x\ dx = \ln|\csc x - \cot x| + C,\quad \text{hint:}\quad \text{set}\quad u = \csc x + \cot x$

Group -2: All six trigonometric functions with power 2

Formulas:

(i) $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$

(ii) $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$

(iii) $\tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

(iv) $cot^2 x + 1 = \csc^2 x \Rightarrow \cot^2 x = \csc^2 x - 1$

1) $\int \sin^2 x \; dx = \int \frac{1}{2}[1 - \cos(2x)] \; dx$

$$= \frac{1}{2} \int [1 - \cos(2x)] \; dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C \quad ;$$

$$\left[\text{ Formula:} \quad \int \cos(kx) \; dx = \frac{\sin(kx)}{k} + C \right]$$

$$\int \sin^2 x \; dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

2) $\int \cos^2 x \; dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

3) $\int \tan^2 x \; dx = \int [\sec^2 x - 1] \, dx = \tan x - x + C$

4) $\int \cot^2 x \; dx = \int [\csc^2 x - 1] \, dx = -\cot x - x + C$

5) $\int \sec^2 x \; dx = \tan x + C$

6) $\int \csc^2 x \; dx = -\cot x + C$

Group -3: All six trigonometric functions with power n, any integer $n \geq 2$.

Reduction Formulas for Integration:

(1) * $\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

(2) $\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

(3) * $\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$

(4) $\int \cot^n x \, dx = -\frac{1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x \, dx$

(5) * $\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

(6) $\int \csc^n x \, dx = -\frac{1}{n-1} \csc^{n-2} x \cot x + \frac{n-2}{n-1} \int \csc^{n-2} x \, dx$

HINT: To derive the formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Start with $I = \int \sin^n x \, dx$

$= \int \sin^{n-1} x \ \sin x \ dx$; **Set** $u = \sin^{n-1} x$, $dv = \sin x \, dx$

Definition: Co-functions

(i) Sine and Cosine are co-functions
(ii) Tangent and Cotangent are co-functions
(iii) Secant and Cosecant are co-functions

Evaluate $\int \sec^5 x \; dx$

Solution: we know that for $n \geq 2$,

$$\int \sec^{n} x \; dx = \frac{1}{n-1} \sec^{n-2} x \; \tan x + \frac{(n-2)}{n-1} \int \sec^{n-2} x \; dx \; \ldots \ldots \ldots \; (1)$$

Here $\int \sec^5 x \; dx$; Given $n = 5, \; n-1 = 4, \quad n-2 = 3$

$$\int \sec^5 x \; dx = \frac{1}{4} \sec^3 x \; \tan x + \frac{3}{4} \int \sec^3 x \; dx \; ;$$

here $n = 3, \qquad n-1 = 2, \quad n-2 = 1$

$$= \frac{1}{4} \sec^3 x \; \tan x + \frac{3}{4} \left[\frac{1}{2} \sec x \; \tan x + \frac{1}{2} \int \sec x \; dx \right]$$

$$= \frac{1}{4} \sec^3 x \; \tan x + \frac{3}{8} \sec x \; \tan x + \frac{3}{8} \ln |\sec x + \tan x| + C$$

Homework:

1. $\int \sec^7 x \; dx + \int \sec^5 x \; dx$

$$= \frac{1}{6} \sec^5 x \; \tan x + \frac{5}{6} \int \sec^5 x \; dx + \int \sec^5 x \; dx$$

$$= \frac{1}{6} \sec^5 x \; \tan x + \frac{11}{6} \int \sec^5 x \; dx$$

2. $\displaystyle\int \sin^6 x \; dx + \int \sin^4 x \; dx$

3. $\displaystyle\int \tan^6 x \; dx + \int \tan^5 x \; dx$

Group-4: $\int \boldsymbol{sinA\ \ cosB\ dx}$; $\int \boldsymbol{sinA\ \ sinB\ dx}$; $\int \boldsymbol{cosA\ \ cosB\ dx}$ **here A ≠ B.**

Formulas:

1) $\sin A \cos B = \frac{1}{2}\,[\sin(A-B) + \sin(A+B)]$
2) $\sin A \sin B = \frac{1}{2}\,[\cos(A-B) - \cos(A+B)]$
3) $\cos A \cos B = \frac{1}{2}\,[\cos(A-B) + \cos(A+B)]$

Definitions

1) $y = \sin x$ is an odd function, that is, $\sin(-x) = -\sin x$
2) $y = \cos x$ is an even function, that is, $\cos(-x) = \cos x$

Here,

$$\cos(5x)\sin(2x) = \sin(2x)\cos(5x)$$

$$= \frac{1}{2}\,[\sin(-3x) + \sin(7x)] = \frac{1}{2}\,[-\sin(3x) + \sin(7x)]$$

Example:1 Evaluate $\int_{\boldsymbol{0}}^{\frac{\pi}{2}} \boldsymbol{sin\ (3x)\ sin\ (6x)\ \ dx}$

Solution: $\int_0^{\frac{\pi}{2}} sin\ (3x)\ sin\ (6x)\ \ dx$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos(3x-6x) - \cos(3x+6x)]\ dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos(-3x) - \cos(9x)]\ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [\cos(3x) - \cos(9x)]\ dx = \frac{1}{2}\left[\frac{\sin(3x)}{3} - \frac{\sin(9x)}{9}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{6}\sin\left(3\ \frac{\pi}{2}\right) - 0 - \frac{1}{18}\sin\left(9\ \frac{\pi}{2}\right) - 0\ = -\frac{1}{6} - \frac{1}{18} = -\frac{4}{18} = -\frac{2}{9}$$

Example: 2 Evaluate $\int_{\boldsymbol{0}}^{\frac{\pi}{2}}\ cos\ (3x)cos(6x)\ \ \boldsymbol{dx}$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos(3x-6x) + \cos(3x+6x)]\ dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos(-3x) + \cos(9x)] \ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [\cos(3x) + \cos(9x)] \ dx$$

$$= \frac{1}{2}\left[\frac{\sin(3x)}{3} + \frac{\sin(9x)}{9}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{6}\sin\left(3\ \frac{\pi}{2}\right) - 0 + \frac{1}{18}\sin\left(9\ \frac{\pi}{2}\right) - 0$$

$$= \frac{1}{6}(-1) + \frac{1}{18}\ (1) = \ -\frac{1}{6} + \frac{1}{18}$$

$$= \ -\frac{2}{18} = \ -\frac{1}{9}$$

Group: 5 $\int \boldsymbol{sin^n\ x\ \ cos^m x\ \ \ dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

There are 3 −cases.

Case-1: When $\boldsymbol{n}$ is even and $\boldsymbol{m}$ is odd

Steps:

1) Split off a factor $\cos x\ \ dx$
2) Write $\cos^2 x \ = \ 1 - \sin^2 x$
3) Set $u = \sin x$. Then $\frac{du}{dx} = \ \cos x$, that is, $\cos x\ \ dx = \ du$

Example: 3 $\int sin^4\ x\ \ cos^5 x\ \ \ dx$

Solution: Given, $\int sin^4\ x\ \ cos^5 x\ \ \ dx$

$$= \int sin^4 x \ \ cos^4 x \ \ \cos x \, dx$$

$$= \int sin^4 x \ \ (cos^2 x)^2 \ \ \cos x \, dx$$

$$= \int sin^4 x \ \ (1 - \sin^2 x)^2 \ \cdot \ \cos x \, dx$$

[Set $u = \sin x$. Then $\frac{du}{dx} = \cos x$, that is, $\cos x \ dx = du$]

$$= \int u^4 \ \ (1 - u^2)^2 \ \ du$$

$$= \int u^4 \ \ [1 - 2u^2 + u^4] \ \ du$$

$$= \int \ [u^4 - 2u^6 + u^8] \ \ du$$

$$= \frac{1}{5} u^5 - \frac{2}{7} u^7 + \frac{1}{9} u^9 + C$$

$$= \frac{1}{5} sin^5 x - \frac{2}{7} sin^7 x + \frac{1}{9} sin^9 x + C$$

Case-2: When n is odd and m is even

Steps:

1) Split off a factor $\sin x \ dx$
2) Write $\sin^2 x = 1 - \cos^2 x$
3) Set $u = \cos x$. Then $\frac{du}{dx} = -sin\ x \ \Rightarrow \ \sin x \, dx = -du$

Example:4 Evaluate $\int sin^7 x \ \ cos^8 x \ \ dx$

Solution: $\int sin^7 x \ \ cos^8 x \ \ dx = \int sin^6 x \ \ cos^8 x \ \ \sin x \ \ dx$

$$= \int (sin^2 x)^3 \ \ cos^8 x \ \ \sin x \ dx = = \int (1 - \cos^2 x\ )^3 \ \ cos^8 x \ \ \sin x \ dx$$

Set $u = \cos x$. Then $\frac{du}{dx} = -sin\ x \Rightarrow sin\ x\, dx = -du$

$$\int sin^7 x \ \ cos^8 x \ \ dx \ = \int (1 - u^2\ )^3 \ u^8 \ \ (-1)\ du$$

$$= \ -\int (1 - u^2\ )^3 \ u^8 \quad du$$

$$= \ -\int [1 - 3u^2 + 3u^4 - u^6]\ u^8 \quad du$$

$$= \ -\int [u^8 - 3u^{10} + 3u^{12} - u^{14}]\ du$$

$$= -\left[\frac{1}{9}u^9 - \frac{3}{11}u^{11} + \frac{3}{13}u^{13} - \frac{1}{15}u^{15}\right] + C$$

$$= -\left[\frac{1}{9}cos^9 x - \frac{3}{11}cos^{11} x + \frac{3}{13}cos^{13} x - \frac{1}{15}cos^{15} x\right] + C$$

Case-3: When n and m both are even

Step: Write $\sin^2 x \ = \ \frac{1}{2}[1 - \cos(2x)]$ and $\cos^2 x \ = \ \frac{1}{2}[1 + \cos(2x)]$; and whenever we get $\cos^2 \theta$, we must apply the formula $\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)]$.

Example: Evaluate $\int sin^4 x \ \ cos^4 x \ \ dx$

$$= \int (sin^2 x \ \ cos^2 x)^2 \ \ dx$$

$$= \int \left(\frac{1}{2}[1 - cos\ (2x)]\frac{1}{2}[1 + cos\ (2x)]\right)^2 \ \ dx$$

$$= \frac{1}{4}\frac{1}{4} \int \left(1 - cos^2\ (2x)\right)^2 \ \ dx$$

$$= \frac{1}{16} \int \left(1 - \frac{1}{2}[1 + cos\ (4x)]\right)^2 \ dx$$

$$= \frac{1}{16} \int \left(1 - \frac{1}{2} - \frac{1}{2} cos\ (4x)\right)^2 dx$$

$$= \frac{1}{16} \int \left(\frac{1}{2} - \frac{1}{2} cos\ (4x)\right)^2 dx$$

$$= \frac{1}{16} \ \frac{1}{4} \int (1 - cos\ (4x))^2 \, dx$$

$$= \frac{1}{64} \int [1 - 2\, cos(4x) + cos^2(4x)] \, dx$$

$$= \frac{1}{64} \int [1 - 2\, cos(4x)] dx \quad + \frac{1}{64} \int cos^2(4x) \, dx$$

$$= \frac{1}{64} \ [x - 2\ \frac{\sin(4x)}{4}] + \frac{1}{64} \int \frac{1}{2} [1 + \cos(8x)] \, dx$$

$$= \frac{1}{64}\, x - \frac{1}{128} \sin(4x) + \frac{1}{64}\, \frac{1}{2} \left[x + \frac{\sin(8x)}{8}\right] + C$$

$$= \frac{3}{128}\, x - \frac{1}{128} \sin(4x) + \frac{1}{1024} \ \sin(8x) + C$$

Alternative Method:

Evaluate $\int sin^4\, x\ \ (1 - \sin^2 x)^2 \ \ dx = \int [sin^4\, x - 2 \sin^6 x + \sin^8 x\,] \, dx$

$$= \int \sin^4 x \ dx \ - 2 \int \sin^6 x \ \ dx \ + \int \sin^8 x \ dx$$

<u>Homework</u>

Evaluate $\int sin^{11}\, x\ \ cos^5 x \ \ dx$

*****Evaluate** $\boldsymbol{\int sin^6\, x\ \ cos^4 x \ \ dx \ = \int (sin^4\, x\ \ cos^4 x) sin^2 x \ \ dx}$ **Homework**

Evaluate $\int sin^6\, x\ \ cos^5 x \ \ dx$

Evaluate $\int sin^{99}\, x\ \ cos^7 x \ \ dx$

Group: 6 $\int \boldsymbol{tan^n\,x\;\; sec^m x\,dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

Group: 7 $\int \boldsymbol{cot^n\,x\;\; csc^m x\,dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

Group: 6 $\int \boldsymbol{tan^n\,x\;\; sec^m x\;\; dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

There are 3-cases in Group 6

Case-1: If m is even

Steps:

1) Split off the factor $\mathbf{sec^2}\,\boldsymbol{x\,dx}$; [Do not change this factor. Save it for du]
2) Write $\sec^2 x = 1 + \tan^2 x$
3) Set $u = \tan x$. Then we get $du = \sec^2 x\;\; dx$

Example: 5 $\int tan^{100}\,x\;\; sec^2 x\;\; dx = \int u^{100}\,du = \frac{1}{101}\tan^{101} x + C$

$$Set\;\; u = \tan x\;\; .\;\; Then\; du = \sec^2 x\;\; dx$$

Example: 6 $\int tan^9\,x\;\; sec^6 x\;\; dx$

$$= \int tan^9\,x\;\; sec^4 x\;\; \mathbf{sec^2}\,\boldsymbol{x\,dx}$$

$$= \int tan^9\,x\;\; (sec^2 x)^2\;\; \mathbf{sec^2}\,\boldsymbol{x\,dx}$$

$$= \int tan^9\,x\;\; (1 + \tan^2 x)^2\;\; \mathbf{sec^2}\,\boldsymbol{x\,dx}\;\; ;\quad \text{Set}\;\; u = \tan x\,,\text{then}\;\; du = \sec^2 x\;\; dx$$

$$= \int u^9\;\; (1 + u^2)^2\;\; du$$

$$= \int u^9\;\; [1 + 2u^2 + u^4]\;\; du$$

$$= \int [u^9 + 2u^{11} + u^{13}]\ du$$

$$= \frac{1}{10}u^{10} + \frac{2}{12}u^{12} + \frac{1}{14}u^{14} + C$$

$$= \frac{1}{10}tan^{10}x + \frac{2}{12}tan^{12}x + \frac{1}{14}tan^{14}x + C$$

Group: 6 $\int \boldsymbol{tan^n\ x\ \ sec^m x\ \ \ dx}$; **here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers**

Case-2: If n is odd

Steps:

1) Split off the factor $\boldsymbol{sec\ x\ tan\ x}\ dx$
2) Write $\tan^2 x = \sec^2 x - 1$
3) Set $u = \sec x$. Then we get $du = \sec\ x \tan x\ \ dx$

Example: 7 $\int tan^7\ x\ \ sec^{10}x\ \ \ dx$ **Homework**

Case-3: If n is even **and** m is odd

Step: Write $\tan^2 x = \sec^2 x - 1$, and then we will get sum of integrals of the form $\int \sec^{\mathrm{k}} x\ \ dx$ for $k \geq 2$. So, apply the reduction formula

$$\int \boldsymbol{\sec^{k} x\ \ dx} = \frac{\boldsymbol{1}}{\boldsymbol{k-1}}\boldsymbol{\sec^{k-2} x\ \ \tan x} + \frac{\boldsymbol{k-2}}{\boldsymbol{k-1}}\ \int \boldsymbol{\sec^{k-2} x\ \ dx} \ldots\ldots\ldots\ (\boldsymbol{1})$$

Example: 8 $\int tan^4\ x\ \ sec^3x\ \ \ dx = \int (tan^2\ x)^2\ \ sec^3x\ \ \ dx$

$$= \int (Sec^2x - 1)^2\ \ sec^3x\ \ \ dx$$

$$= \int [\sec^4 x - 2\sec^2 x + 1]\ sec^3x\ \ \ dx$$

$$= \int [\sec^7 x - 2\sec^5 x + sec^3 x] \; dx$$

$$= \int \sec^7 x \; dx \; - 2\int \sec^5 x \; dx + \int \sec^3 x \; dx$$

Group: 7 $\int \boldsymbol{cot^n\, x \;\; csc^m x\, dx}$ **; here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers .**

Homework: Write all the steps for all 3-cases for Group-7 and then solve following exercises

Example: 9 $\int \boldsymbol{cot^9\, x \;\; csc^6 x \;\; dx}$

Example: 10 $\int \boldsymbol{cot^7\, x \;\; csc^{16} x \;\; dx}$

Group: 7 $\int \boldsymbol{cot^n\, x \;\; csc^m x\, dx}$ **; here $\boldsymbol{m}$ and $\boldsymbol{n}$ are positive integers .**

Case-3: If n is even **and** m is odd

Step: Write $\cot^2 x = \csc^2 x - 1$, and then we will get sum of integrals of the form $\int \csc^{k} x \; dx$ for $k \geq 2$. So, apply the reduction formula

(1) … … … $\int \csc^{k} x \; dx = -\frac{1}{k-1}\csc^{k-2} x \; \cot x + \frac{k-2}{k-1} \; \int csc^{k-2} x \; dx$

Example: 11 $\int \cot^4 x \;\; csc^3 x \;\; dx = \int (\cot^2 x)^2 \;\; csc^3 x \;\; dx$

$$= \int (\csc^2 x - 1)^2 \;\; csc^3 x \;\; dx$$

$$= \int [\csc^4 x - 2\csc^2 x + 1] \;\; csc^3 x \;\; dx$$

$$= \int [\csc^7 x - 2\csc^5 x + csc^3 x \,]\, dx$$

$$[\int \csc^{k} x \; dx = -\frac{1}{k-1}\csc^{k-2} x \; \cot x + \frac{k-2}{k-1} \; \int csc^{k-2} x \; dx]$$

$$= \int \csc^7 x \; dx - 2\int \csc^5 x \; dx + \int \csc^3 x \; dx \,;\, k = 7, \qquad k - 1 = 6, k - 2 = 5$$

$$= -\frac{1}{6}\csc^5 x \cot x + \frac{5}{6}\int \csc^5 x \ dx - 2\int \csc^5 x \ dx + \int \csc^3 x \ dx$$

$$= -\frac{1}{6}\csc^5 x \cot x - \frac{7}{6}\int \csc^5 x \ dx + \int \csc^3 x \ dx$$

$$= -\frac{1}{6}\csc^5 x \cot x - \frac{7}{6}\left[-\frac{1}{4}\csc^3 x \cot x + \frac{3}{4}\int \csc^3 x \ dx\right] + \int \csc^3 x \ dx$$

… … Continue !!