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MAT 350
Engineering mathematics

Lecture -2
Variable Separable, Exact ODE with modellings

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First Order Ordinary Differential Equations:

Separable Equations:

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be separable or to have separable variables, and the equation is called separable equation.

Examples:

(1) $$\frac{dy}{1+y^2} = dx.$$

(2) $$(1+x)\,dy - y\,dx = 0.$$

(3) $$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3.$$

First Order Ordinary Differential Equations:

Separable Equations:

Solution Technique: When a first order ODE is separable variables, the solution can be found by integration, such that

$$p(y)dy = g(x)dx,$$

where p(y)=1/ h(y)

$$\int p(y)dy = \int g(x)dx + c$$

$$\boxed{H(y) = G(x) + c,}$$

First Order Ordinary Differential Equations:

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION: Dividing by (1+x)y, we can write

$$\frac{dy}{y} = \frac{dx}{(1-x)}$$

from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$\begin{aligned} y &= e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \\ &= |1 + x|\, e^{c_1} \\ &= \pm e^{c_1}(1 + x). \end{aligned}$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

First Order Ordinary Differential Equations:

Separable Equations:

Solve $y' = (x+1)e^{-x}y^2$

Solution: We obtain by side-changing

$$y^{-2}\,dy = (x+1)e^{-x}\,dx.$$

This means the ODE is separable. Then by integration,

$$-y^{-1} = -(x+2)e^{-x} + c,$$

$$y = \frac{1}{(x+2)e^{-x} - c}.$$

If an initial condition y(0)=1 in introduced with the ODE, then we find the value of the constant c, which gives c=1.

$$\mathbf{y} = \frac{\mathbf{1}}{\mathbf{(x+2)e^{-x} - 1}}$$

Exercise 2.2, Zill

25. $x^2 \dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$

Solution: Hints

$$\frac{1}{y}\,dy = \frac{1-x}{x^2}\,dx = \left(\frac{1}{x^2} - \frac{1}{x}\right)dx$$

Integrating both sides,

we obtain $\ln|y| = -\dfrac{1}{x} - \ln|x| = c$

or $xy = c_1 e^{-1/x}$.

Substituting the initial condition

$y(-1) = -1$ we find $c_1 = e^{-1}$.

The solution of the initial-value problem is

$$xy = e^{-1-1/x}$$

$$y = e^{-(1+1/x)}/x.$$

Solve:

$$(e^{y}+1)^{2}e^{-y}dx+(e^{x}+1)^{3}e^{-x}dy=0$$

Solve $(e^{2y} - y)\cos x \dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$

Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y}\, dy = \frac{\sin 2x}{\cos x}\, dx.$$

Using identity $\sin 2x = 2 \sin x \cos x$, and integrating by parts

$$\int (e^y - ye^{-y})\, dy = 2\int \sin x\, dx$$

$$e^y + ye^{-y} + e^{-y} = -2\cos x + c.$$

The initial condition $y = 0$ when $x = 0$ implies $c = 4$.

The solution is: $e^y + ye^{-y} + e^{-y} = 4 - 2\cos x.$

First Order Ordinary Differential Equations:

Reduction to Separable Form:

Certain non-separable ODEs can be made separable by transformations that introduce for y a new unknown function.

Suppose, $y' = f\left(\frac{y}{x}\right).$

Here, f is any (differentiable) function of y/x. The form of such an ODE suggests that we substitute

u=y/x

Thus, $y = ux$, and $y' = u'x + u.$

Substituting into the above ODE gives,

$$u'x + u = f(u) \text{ or } u'x = f(u) - u.$$

if $f(u) - u \neq 0$, this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

Use method of Sep. Vari., and back substitute

$$2xyy' = y^2 - x^2.$$

Solution: We divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substituting, u=y/x, or $y = ux$ and $y' = u'x + u.$

$$u'x + u = \frac{u}{2} - \frac{1}{2u},$$

$$u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}.$$

Separable

Integrating,

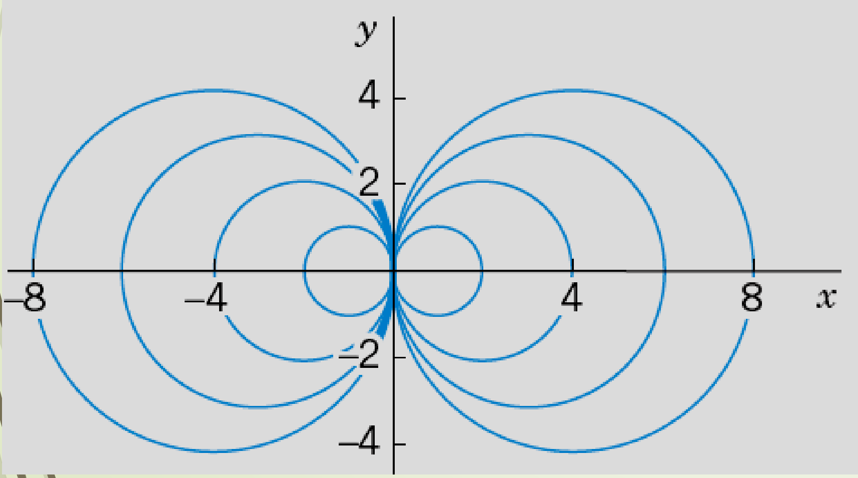
$$\ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Taking exponents on both sides,

$$1 + u^2 = c/x$$

$$\boxed{\text{or } 1 + (y/x)^2 = c/x.}$$

Note: Multiply both sides of the solution by x^2 ,



$$x^2 + y^2 = cx.$$

Thus,

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

Separable Equations: Population Dynamics

In Malthusian model , if P(t) denotes the total population at time t, then this assumption can be expressed as

$$\frac{dP}{dt} \propto P \qquad \text{or} \qquad \frac{dP}{dt} = kP,$$

where k is a constant of proportionality.

$$\frac{1}{P}\,dP = k\,dt \text{ (separable variables)}$$

Integrating, $$\ln|P| = kt + c$$

$$P = P_0\, e^{kt}, \quad \text{where } P_0 = e^{c}$$

In a primary experiment, let the population of insects be N_0. At time t=4 hours, the population is seen to be $2N_0$.
If the growth rate of the insects is proportional to the population of insects at certain time, how many hours later the population of the insects will be 8 times of the current population?

Solution: $$N = N_0 e^{kt},$$

Here, N_0 is the population at time t=0.

At time t=4 hours, the population is seen to be $2N_0$.

Hence, $$2N_0 = N_0 e^{k.4}$$

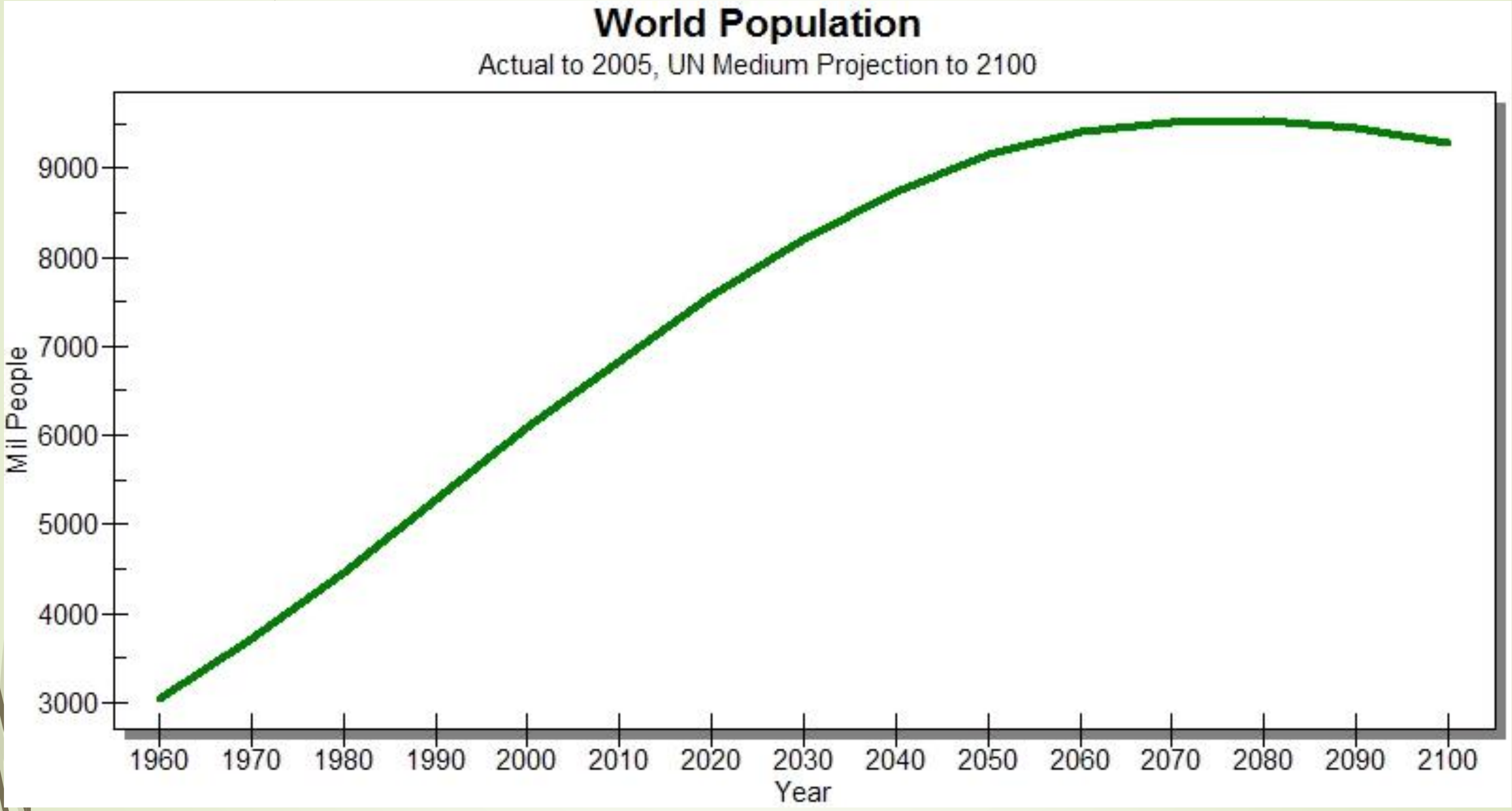
$$or,\ e^{4k} = 2 \qquad or,\ 4k = \ln 2,\ k = \frac{\ln 2}{4} = 0.173$$

Hence, required time (T) for the population of the insects will be 8 times of the current population is,

$$8N_0 = N_0 e^{0.173T}$$

$$e^{0.173T} = 8,\ \ T = \frac{\ln 8}{0.173} = 12.01 \approx 12H$$

Separable Equations: Population Dynamics



Separable Equations: Heating Office Building (Newton's Law of Cooling)

Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M.

On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. Consider that the outside temperature varies between 50°F to 40°F by this time period. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Solution: Let T(t) be the temperature inside the building and T_m be the outside temperature (we consider here T_{out} is the average of 50°F to 40°F, that is T_m = 45°F

$$\frac{dT}{dt} \propto T - T_m \qquad \text{or} \qquad \frac{dT}{dt} = k(T - T_m),$$

$$\frac{dT}{T-45} = k\,dt,$$

$$\ln|T - 45| = kt + c^*,$$

$$T(t) = 45 + ce^{kt} \qquad (c = e^{c^*}).$$

General solution

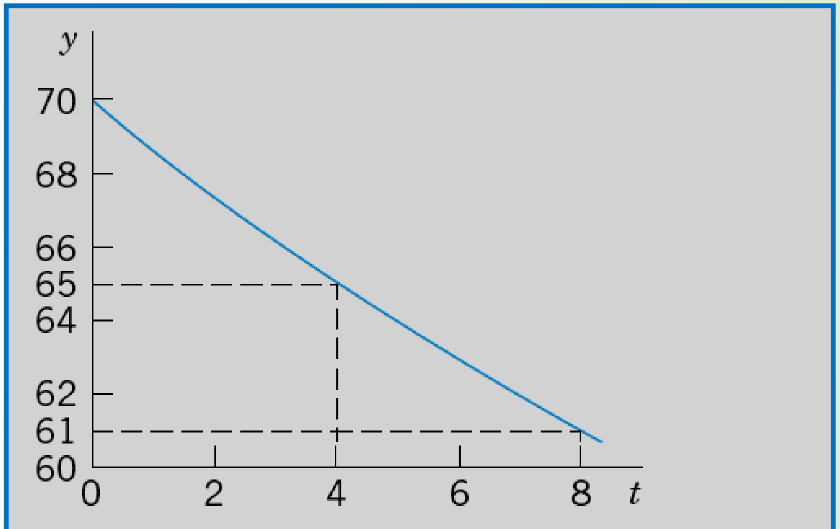
Separable Equations: Heating Office Building (Newton's Law of Cooling)

Particular solution: We choose 10 P.M. to be t=0. Then the given initial condition is T(0)=70 and yields a particular solution, call it T_p . By substitution

$$T(0) = 45 + ce^{0} = 70,$$

$$c = 70 - 45 = 25,$$

$$T_p(t) = 45 + 25e^{kt}.$$



Particular solution (temperature) in Example

We use T(4)=65, where t=4 is 2 A.M

$$T_p(4) = 45 + 25e^{4k} = 65,$$

$$e^{4k} = 0.8, \qquad k = \tfrac{1}{4} \ln 0.8 = -0.056,$$

$$\boxed{T_p(t) = 45 + 25e^{-0.056t}.}$$

Particular solution

First-order differential equation:

EXACT EQUATIONS

First-order differential equation: **EXACT EQUATIONS**

Definition: A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation if the expression on the left-hand side is an** exact differential.

A differential expression M(x, y) dx + N(x, y) dy is an **exact differential in a** region R of the xy-plane if it corresponds to the differential of some function f (x, y) defined in R.

Criterion for an Exact Differential:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

First-order differential equation: **EXACT EQUATIONS**

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$$

Solution Technique:

Consider *M(x, y) dx* + *N(x, y) dy= 0, and* $\dfrac{dM}{dy} = \dfrac{dN}{dx}$ holds true.

Let a function f be the solution of (1), such that

$$\frac{\partial f}{\partial x} = M(x, y). \qquad (2)$$

We can find *f by integrating M(x, y) with respect to x while holding y constant:*

$$f(x, y) = \int M(x, y)\,dx + g(y), \qquad (3)$$

where the arbitrary function *g(y) is the "constant" of integration.*

First-order differential equation: **EXACT EQUATIONS**

Now differentiate (3) with respect to *y* *and assume* $\partial f / \partial y = N(x, y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y) = N(x, y).$$

Hence,

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx. \qquad (4)$$

Finally, integrate (4) with respect to *y and substitute the result in (3). The implicit* solution of the equation is *f (x, y) = c.*

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact.

The solution f(x,y) is given by

$$f(x, y) = \int M(x, y)\,dx + g(y),$$

First-order differential equation: **EXACT EQUATIONS**

$$f(x, y) = x^2y + g(y).$$

The partial derivative of the above f(x,y) with respect to *y* equal to *N(x, y).*

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$.

Hence $f(x, y) = x^2y - y$

So, the solution is $\boxed{x^2y - y = c.}$

Note: The explicit form of the solution is $y = c/(1 - x^2)$ easily seen to be *y c(1 x2) and is defined on any interval not containing either x =1 or x= -1.*

Exercise 2.4 (Zill)

21. $(x + y)^2\, dx + (2xy + x^2 - 1)\, dy = 0, \quad y(1) = 1$

Solution: Hints

Let $M = x^2 + 2xy + y^2$ and $N = 2xy + x^2 - 1$

$$M_y = 2(x+y) = N_x.$$

Therefore,

$$f = \tfrac{1}{3}x^3 + x^2y + xy^2 + h(y),$$

$$h'(y) = -1, \text{ and } h(y) = -y.$$

The solution is $\frac{1}{3}x^3 + x^2y + xy^2 - y = c$

If $y(1) = 1$ then $c = 4/3$

Hence a solution of the initial-value problem is

,

$$\tfrac{1}{3}x^3 + x^2y + xy^2 - y = \tfrac{4}{3}.$$

Examples:

(1) $2xy\,dx + x^2\,dy = 0$

(2) Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0.$

(3) Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$

(4) $\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0.$

(5) $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$