DAY-5 : Section 7.4 Integration by Trogonometric Substitutions

Understanding:

- ➔ Trigonometric Functions [MAT 116]
- ➔ Inverse Trigonometric Functions. [From a different book]

TRIGONOMETRIC SUBSTITUTIONS: $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$

Formulas:

$1 - \sin^2 x = \cos^2 x$

$1 + \tan^2 x = \sec^2 x$

$\sec^2 x - 1 = \tan^2 x$

We need this method to integrate type (b) integrals.

Examples: (1) $\boldsymbol{a})\ \int x\,(4-x^2)^{44}\,dx$ *and* $\boldsymbol{b})\ \int (4-x^2)^{44}\,dx$

$(2)\ a)\ \int 2x\sqrt{1-x^2}\,dx$ *and* $b)\ \int \sqrt{1-x^2}\,dx$

Note: $\sqrt{9} = 3$ and $-\sqrt{9} = 3$ [Solve $x^2 = 9 \rightarrow x = \pm\sqrt{9}$, that is, $x = +\sqrt{9}$, $x = -\sqrt{9}$]

Definition: For any real number x, $\sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

Notes:

(1) $y = \sqrt{x}$: This is called **positive square root**, and we only get non-negative number y from this equation. Here, $y = 0$ if $x = 0$. Example: $\sqrt{4} = 2$.

(2) $y = -\sqrt{x}$: This is called **negative square root,** and we only get non-positive number y from this equation. Here, $y = 0$ if $x = 0$. Example: $-\sqrt{4} = -2$.

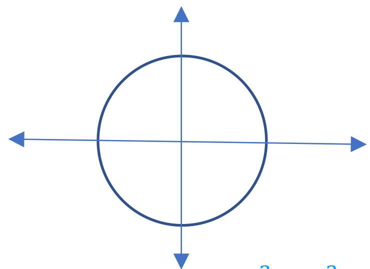
Recall: To solve $x^2 = 9 \Rightarrow x = \pm\sqrt{9} \Rightarrow x = \pm 3$. That is, $x = 3$ or $x = -3$

There are 3-cases

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \leq \theta \leq \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \leq \theta < \pi/2 & (\text{if } x \geq a) \\ \pi/2 < \theta \leq \pi & (\text{if } x \leq -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Recall: Consider the circle of radius r and with center at the origin.



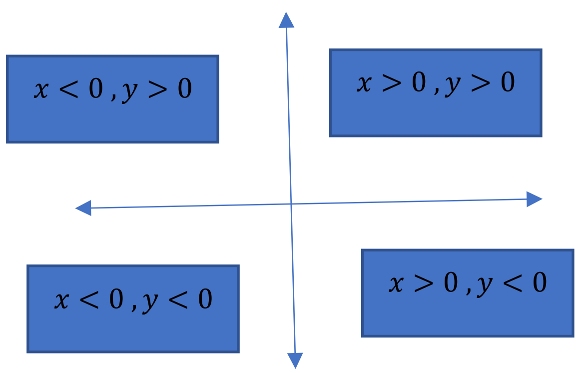
For any point (x, y) on the circle, we get $x^2 + y^2 = r^2$, and also, in polar coordinates,

$$x = r\cos\theta \quad , \quad y = r\sin\theta.$$

In particular, for any point (x, y) on the circle, we get $x^2 + y^2 = 1$, and also, in polar coordinates,

$$x = \cos\theta \quad , \quad y = \sin\theta.$$

Remember that $(x, y) = (\cos\theta\ , \sin\theta\ )$.



Case: 1 $\boldsymbol{\sqrt{a^2 - x^2}}$; $\boldsymbol{a > 0}$

Set $x = a\sin\theta$. Then $\frac{dx}{d\theta} = a\cos\theta \Rightarrow dx = a\cos\theta\, d\theta$

Also, $\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\, sin\,\theta)^2}$

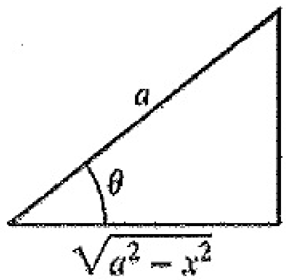
$= \sqrt{a^2 - a^2\, sin^2\,\theta} = \sqrt{a^2(1 - sin^2\theta\,)}$

$= \sqrt{a^2\, cos^2\theta} = |a|\sqrt{cos^2\theta} = a\ |\cos\theta|$

That is, $\sqrt{a^2 - x^2} = a\ \cos\theta$; when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Again, $x = a\sin\theta$ $\Rightarrow \frac{x}{a} = \sin\theta$. Hence $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

Now, $\sin\theta = \frac{x}{a} = \frac{Opposite}{Hypotenuse} = \frac{x \text{ is the opposite}}{a \text{ is the hypotenuse}}$



$x = a\sin\theta$

$cos\theta = \frac{\sqrt{a^2-x^2}}{a}$, $\sec\theta = \frac{a}{\sqrt{a^2-x^2}}$, $\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$, $\cot\theta = \frac{\sqrt{a^2-x^2}}{x}$, $\csc\theta = \frac{a}{x}$

Case: 2 $\sqrt{a^2+x^2}$; $a > 0$

Set $x = a\tan\theta$. Then $\frac{dx}{d\theta} = a\sec^2\theta$ $\Rightarrow dx = a\sec^2\theta \; d\theta$

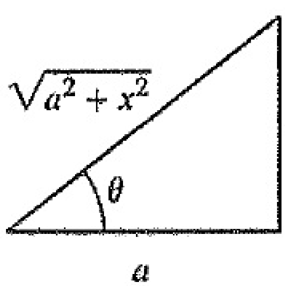
$\sqrt{a^2+x^2} = \sqrt{a^2+(a\,tan\,\theta)^2}$

$= \sqrt{a^2(1+\ tan^2\theta\,)} = \sqrt{a^2sec^2\theta} = a\sqrt{\ sec^2\theta} = a|\sec\theta|$

$\sqrt{a^2+x^2} = a\ \sec\theta$; when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Again, $x = a\tan\theta$ $\Rightarrow \frac{x}{a} = \tan\theta$. Hence $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

Now, $\tan\theta = \frac{x}{a} = \frac{Opposite}{Adjecent} = \frac{x \text{ is the opposite}}{a \text{ is the Adjecent}}$



$x = a\tan\theta$

$cos\theta = \frac{a}{\sqrt{a^2+x^2}}$, $\sec\theta = \frac{\sqrt{a^2+x^2}}{\mathrm{a}}$, $\sin\theta = \frac{x}{\sqrt{a^2+x^2}}$, $\cot\theta = \frac{\mathrm{a}}{x}$, $\csc\theta = \frac{\sqrt{a^2+x^2}}{x}$

Case: 3 $\sqrt{x^2 - a^2}$; $a > 0$

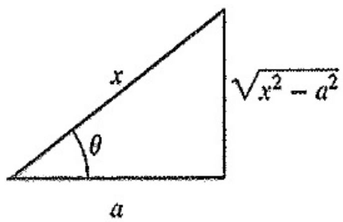
Set $x = a\sec\theta$. Then $\frac{dx}{d\theta} = a\sec\,\theta\tan\theta \Rightarrow dx = a\,\sec\,\theta\tan\theta\,\,d\theta$

$$\sqrt{x^2 - a^2} = \sqrt{(a\,sec\,\theta)^2 - a^2} = \sqrt{a^2(\,sec^2\theta - 1)} = \sqrt{a^2tan^2\theta} = a\sqrt{\,tan^2\theta} = a|\tan\theta|$$

$\sqrt{x^2 - a^2} = a\,\tan\theta$; when $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

Again, $\boldsymbol{x = a\sec\theta} \Rightarrow \boldsymbol{\frac{x}{a} = \sec\theta}$. Hence $\boldsymbol{\theta = \sec^{-1}\left(\frac{x}{a}\right)}$

Now, $\sec\theta = \frac{x}{a} = \frac{Hypotenuse}{Adjecent} = \frac{x\text{ is the }Hypotenuse}{a\text{ is the }Adjecent}$



$x = a\sec\theta$

$cos\theta = \frac{a}{x}$, $\cot\theta = \frac{a}{\sqrt{x^2-a^2}}$, $\sin\theta = \frac{\sqrt{x^2-a^2}}{x}$, $\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$, $\csc\theta = \frac{x}{\sqrt{x^2-a^2}}$

Examples

1. $\int_{\frac{2\pi}{3}}^{\pi} |\tan\theta|\,\,d\theta = \int_{\frac{2\pi}{3}}^{\pi} (-\tan\theta\,)\,\,d\theta\,;$ Here $\frac{2\pi}{3} \le \theta \le \pi$, hence $\theta \le 0.$

$[x = -2\,,\,\,|x| = |-2| = \,-(-2)]$

2. $\int_0^{\pi} |\cos\theta|\,\,d\theta = \int_0^{\frac{\pi}{2}} |\cos\theta|\,\,d\theta + \int_{\frac{\pi}{2}}^{\pi} |\cos\theta|\,\,d\theta = \int_0^{\frac{\pi}{2}} \cos\theta\,\,d\theta + \int_{\frac{\pi}{2}}^{\pi} (-\cos\theta\,)\,\,d\theta$

Section 7.4 Integration by Trogonometric Substitutions

EXERCISES

TRIGONOMETRIC SUBSTITUTIONS: $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$

Exercise: 1 (a) $\int \frac{\sqrt{1+t^2}}{t}\, dt$

Set $t = \tan\theta$. Then $\frac{dt}{d\theta} = \sec^2\theta$, that is, $dt = \sec^2\theta \; d\theta$

Now,

$$\int \frac{\sqrt{1+t^2}}{t}\, dt = \int \frac{\sqrt{1+tan^2\,\theta}}{\tan\theta} \sec^2\theta \; d\theta = \int \frac{\sqrt{\sec^2\theta}}{\tan\theta} \sec^2\theta \; d\theta$$

$$= \int \frac{|\sec\theta|}{\tan\theta} \sec^2\theta \; d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} \sec^2\theta \; d\theta \; ; \quad \text{when} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \int \frac{\sec^3\theta}{\tan\theta}\, d\theta = \int \frac{\frac{1}{\cos^3\theta}}{\frac{(\sin\theta)}{\cos\theta}} d\theta$$

$$= \int \frac{\sin\theta}{\cos^2\theta}\, d\theta \; ; \; Set\;\; u = \cos\theta, \quad then \;\; -du = \sin\theta \; d\theta$$

$$= \int \frac{1}{u^2}\,(-1)\, du = -\int u^{-2}\, du$$

$$= -\frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{1}{u} + C = \frac{1}{\cos\theta} + C = \sec\theta + C = \sqrt{1+t^2} + C \; ; \;\; \text{if } t = \tan\theta, \text{ then } \sec\theta = \sqrt{1+t^2}$$

$$\int \frac{\sqrt{1+t^2}}{t}\, dt = \sqrt{1+t^2} + C$$

$$(b) \int \frac{1}{x^2\sqrt{x^2+5}}\, dx$$

Set $x = \sqrt{5}\,\tan\theta$. $dx = \sqrt{5}\sec^2\theta \; d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+5}}\, dx$$

$$= \int \frac{1}{(\sqrt{5}\,\tan\theta\,)^2\sqrt{(\sqrt{5}\,\tan\theta\,)^2+5}} \sqrt{5}\sec^2\theta \; d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\sqrt{5(\tan^2\theta+1)}} \; d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\ \sqrt{5\sec^2\theta}}\quad d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\ |\sqrt{5}\sec\theta\,|}\quad d\theta$$

$= \int \frac{\sqrt{5}\sec^2\theta}{5\tan^2\theta\ \sqrt{5}\sec\theta}\quad d\theta\quad ;$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \frac{1}{5}\int \frac{\sec\ \theta}{\tan^2\theta}\ d\theta$$

$$= \frac{1}{5}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\ d\theta$$

$$= \frac{1}{5}\int \frac{\cos\ \theta}{\sin^2\theta}\ d\theta\quad ;\quad set\ u = \sin\theta\,,\ \ du = \cos\theta\ \ d\theta$$

$$= \frac{1}{5}\int \frac{1}{u^2}\ du = = \frac{1}{5}\int u^{-2}\ du = \frac{1}{5}\ \frac{u^{-2+1}}{-2+1} = -\frac{1}{5}\frac{1}{u} + C$$

$$= -\frac{1}{5\sin\theta} + C\ = -\frac{\sqrt{x^2+5}}{5x} + C$$

Now, $x = \sqrt{5}\ \tan\theta\ \Rightarrow \tan\theta = \frac{x}{\sqrt{5}}$. Hence, $\sin\theta\ = \frac{x}{\sqrt{x^2+5}}$, that is, $\frac{1}{\sin\theta}\ \ \frac{x}{\sqrt{x^2+5}}$

<u>**Exercise: 2**</u> $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\ dx$.

Solution: Set $x = \sin\theta$. Then, $dx = \cos\theta\ \ d\theta$. Now,

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\ dx$$

$$= \int \frac{1}{(1-(\sin\theta)^2)^{\frac{3}{2}}}\ dx$$

$$= \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}\ dx$$

$$= \int \frac{1}{(cos^2\theta)^{\frac{3}{2}}}\ dx$$

$$= \int \frac{1}{(cos^2\theta)^{\frac{3}{2}}}\ dx$$

$$= \int \frac{1}{\left(\sqrt{cos^2\theta}\right)^3}\ dx$$

$$= \int \frac{1}{(|\cos\theta\,|)^3}\ dx$$

$$= \int \frac{1}{\cos^3\theta}\ dx\ ; \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

$$= \int \sec^3\theta\ \ dx$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\ \ d\theta\ \ ;\int \sec^{\mathrm{n}} x\ \ dx = \frac{1}{n-1}\sec^{n-2}x\ \tan x + \frac{n-2}{n-1}\int \sec^{\mathrm{n}-2}x\ \ dx$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2}\sqrt{1-x^2}\ \ \frac{x}{\sqrt{1-x^2}} + \frac{1}{2}\ln\left|\sqrt{1-x^2} + \frac{x}{\sqrt{1-x^2}}\right| + C$$

$$;[\text{since } \frac{x}{1} = \sin\theta, \sec\theta = \sqrt{1-x^2}\quad,\quad \tan\theta = \frac{x}{\sqrt{1-x^2}}]$$

$$= \frac{1}{2}x\ + \frac{1}{2}\ln\left|\frac{1+x-x^2}{\sqrt{1-x^2}}\right| + C.$$

Exercise: 3 $\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\ dx$

Given $\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\ dx = \int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{2^2-x^2}}\ dx$

Set $x = 2\ \sin y$. Then $dx = 2\cos y\ \ dy$.

Also,

x	1	$\sqrt{2}$
y	$\frac{\pi}{6}$	$\frac{\pi}{4}$

From, $x = 2\sin y$; *If* $x = 1$, *then* $1 = 2\sin y \quad \Rightarrow \sin y = ½$, that is, $y = \frac{\pi}{6}$

If $x = \sqrt{2}$, *then* $y = \frac{\pi}{4}$

Hence, $\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{(2\sin y)^2\ \sqrt{4-4\,sin^2 y}}\ 2\cos y\ dy$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{4\ sin^2 y\ \sqrt{4\cos^2 y}}\ 2\cos y\ dy = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{sin^2 y\ \sqrt{(2\cos y)^2}}\ 2\cos y\ dy$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{sin^2 y\ |2\cos y|}\ 2\cos y\ dy$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{sin^2 y\ 2\cos y}\ 2\cos y\ dy$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{sin^2 y}\ dy$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc^2 y\ dy = \frac{1}{4}\left[-\cot y\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{4}\left[-\cot\frac{\pi}{4} + \cot\frac{\pi}{6}\right] = \frac{-1+\sqrt{3}}{4}$$

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{\sqrt{3}-1}{4}$$

Exercise: 4 $\int \frac{x^2}{(x^2-1)^{\frac{3}{2}}}\,dx$ Homework