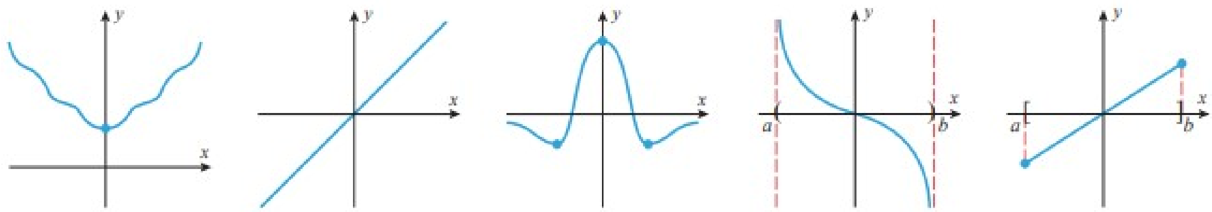
Chapter # 04
(The Derivative in Graphing and Applications)

4.4 Absolute Maxima and Minima:

Absolute Extrema: Consider an interval in the domain of a function f and a point x_0 in that interval. We say that f has an absolute maximum at x_0 if $f(x) \le f(x_0)$ for all x in the interval, and we say that f has an absolute minimum at x_0 if $f(x_0) \le f(x)$ for all x in the interval. We say that f has an absolute extremum at x_0 if it has either an absolute maximum or an absolute minimum at that point.



f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.

f has no absolute extrema on $(-\infty, +\infty)$.

f has an absolute maximum and minimum on $(-\infty, +\infty)$.

f has no absolute extrema on (a, b).

f has an absolute maximum and minimum on $[a, b]$.

Theorem (Extreme-Value Theorem): If a function f is continuous on a finite closed interval [***a,b***], then f has both an absolute maximum and an absolute minimum on [***a,b***].

Theorem: If f has an absolute extremum on an open interval (***a, b***), then it must occur at a critical point of f.

A Procedure for Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval $[a, b]$:

Step 1. Find the critical points of f in **(*a, b*)**.

Step 2. Evaluate f at all the critical points and at the endpoints ***a*** and ***b***.

Step 3. The largest of the values in ***Step 2*** is the absolute maximum value of f on **[*a,b*]** and the smallest value is the absolute minimum

Example 1: Find the absolute maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 36x$ on the interval **[1, 5],** and determine where these values occur

Solution: Since f is continuous and differentiable everywhere, the absolute extrema must occur either at endpoints of the interval or at solutions to the equation $f'(x) = 0$ in the open interval (1, 5).

For stationary points, $f'(x) = 0$

$$\Rightarrow 6x^2 - 30x + 36 = 0 \;\Rightarrow x^2 - 5x + 6 = 0 \;\Rightarrow (x-3)(x-2) = 0 \quad \therefore x = 2 \;\;\&\;\; x = 3$$

Evaluating ***f*** at the endpoints, at ***x = 2***, and at ***x = 3*** yields

$$\begin{aligned} f(1) &= 2(1)^3 - 15(1)^2 + 36(1) = 23 \\ f(2) &= 2(2)^3 - 15(2)^2 + 36(2) = 28 \\ f(3) &= 2(3)^3 - 15(3)^2 + 36(3) = 27 \\ f(5) &= 2(5)^3 - 15(5)^2 + 36(5) = 55 \end{aligned}$$

from which we conclude that the absolute minimum of ***f*** on **[1, 5]** is 23, occurring at ***x = 1***, and the absolute maximum of f on **[1, 5]** is 55, occurring at ***x = 5***.

Example 2: Find the absolute extrema of $\boldsymbol{f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}}$ on the interval **[−1, 1]**, and determine where these values occur.

Solution: Note that ***f*** is continuous everywhere and therefore the Extreme-Value Theorem guarantees that ***f*** has a maximum and a minimum value in the interval **[−1, 1]**. Differentiating, we obtain

$$f'(x) = 8x^{1/3} - x^{-2/3} = x^{-2/3}(8x - 1) = \frac{8x - 1}{x^{2/3}}$$

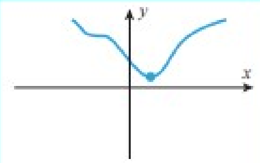
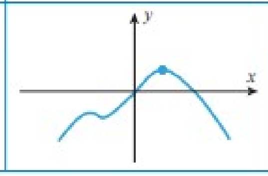
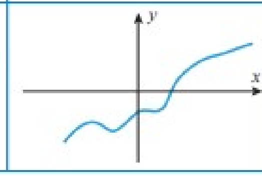
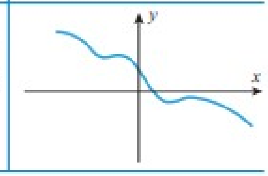
Thus, $\boldsymbol{f'(x) = 0}$ at $\boldsymbol{x = \frac{1}{8}}$, and $\boldsymbol{f'(x)}$ is undefined at ***x* = 0**. Evaluating ***f*** at these critical points and endpoints yields

x	-1	0	$\frac{1}{8}$	1
$f(x)$	9	0	$-\frac{9}{8}$	3

from which we conclude that an absolute minimum value of $-\boldsymbol{\frac{9}{8}}$ occurs at $\boldsymbol{x = \frac{1}{8}}$, and an absolute maximum value of 9 occurs at $\boldsymbol{x = -1}$.

Absolute Extrema on Infinity Intervals: We observed earlier that a continuous function may or may not have absolute extrema on an infinite interval. However, certain conclusions about the existence of absolute extrema of a continuous function ***f*** on $(-\infty, \infty)$ can be drawn from the behavior of $\boldsymbol{f(x)}$ as $x \to -\infty \;\; and\ as \;\; x \to +\infty$.

ABSOLUTE EXTREMA ON INFINITE INTERVALS

LIMITS	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = -\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
CONCLUSION IF f IS CONTINUOUS EVERYWHERE	f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.	f has an absolute maximum but no absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.
GRAPH				

Example 4: Determine by inspection whether $p(x) = 3x^4 + 4x^3$ has any absolute extrema. If so, find them and state where they occur.

Solution: Since $p(x)$ has even degree and the leading coefficient is positive, $p(x) \to +\infty \;\; as \;\; x \to \pm\infty$**.** Thus, there is an absolute minimum but no absolute maximum. The absolute minimum must occur at a critical point of ***p***. Since ***p*** is differentiable everywhere, we can find all critical points by solving the equation $p'(x) = 0$. This equation is

$$12x^3 + 12x^2 = 12x^2(x+1) = 0$$

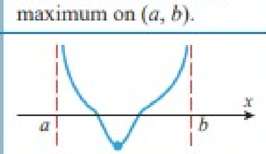
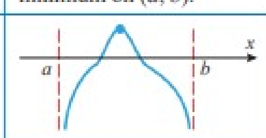
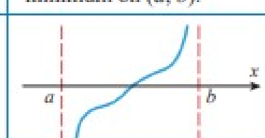
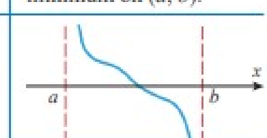
from which we conclude that the critical points are ***x*** **= 0** and ***x*** **= −1**. Evaluating ***p*** at these critical points yields

$$p(0) = 0 \quad \text{and} \quad p(-1) = -1$$

Therefore, ***p*** has an absolute minimum of **−1** at ***x*** **= −1**.

Absolute Extrema on open Intervals: We know that a continuous function may or may not have absolute extrema on an open interval. However, certain conclusions about the existence of absolute extrema of a continuous function ***f*** on a finite open interval **(*a, b*)**.

ABSOLUTE EXTREMA ON OPEN INTERVALS

LIMITS	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
CONCLUSION IF f IS CONTINUOUS ON (a, b)	f has an absolute minimum but no absolute maximum on (a, b).	f has an absolute maximum but no absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).
GRAPH				

Example 5: Determine whether the function

$$f(x) = \frac{1}{x^2 - x}$$

has any absolute extrema on the interval **(0, 1)**. If so, find them and state where they occur.

Solution: Since ***f*** is continuous on the interval **(0, 1)** and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x^2 - x} = \lim_{x \to 0^+} \frac{1}{x(x-1)} = -\infty$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x^2 - x} = \lim_{x \to 1^-} \frac{1}{x(x-1)} = -\infty$$

the function ***f*** has an absolute maximum but no absolute minimum on the interval **(0, 1)**.

The absolute maximum must occur at a critical point of ***f*** in the interval **(0, 1)**. We have

$$f'(x) = -\frac{2x-1}{(x^2-x)^2}$$

For $f'(x) = 0 \quad \Rightarrow \; -\frac{2x-1}{(x^2-x)^2} = 0 \quad \Rightarrow x = \frac{1}{2}.$

Although ***f*** is not differentiable at **x = 0** or at **x = 1**, these values are doubly disqualified since they are neither in the domain of ***f*** nor in the interval **(0, 1)**.

Thus, the absolute maximum occurs at $x = \frac{1}{2}$, and this absolute maximum value is

$$f\left(\tfrac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2 - \frac{1}{2}} = -4$$

Home Work: Exercise 4.4: Problem No. 7-13 and 21-27