MAT 350
Engineering mathematics

Lecture -3

- First order Linear ODE
- Homogeneous ODE, Bernoulli's Equations

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First-order differential equation: (Chapter 2.3)

Linear differential equations:

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

is said to be a **linear equation in the dependent variable *y*.**

When g(x) = 0, the linear equation (1) is said to be **homogeneous; otherwise, it** is **nonhomogeneous.** The **standard form, of the above linear equation is**

$$\boxed{\frac{dy}{dx} + P(x)y = f(x),} \qquad (2)$$

where $P(\mathrm{x}) = \dfrac{a_0(x)}{a_1(x)}$, and $f(x) = \dfrac{g(x)}{a_1(x)}$.

We seek a solution of (2) on an interval I for which both coefficient functions P and f are continuous.
If f(x)=0, one can solve (2) by variable separable method.

Linear differential equations:

$$\frac{dy}{dx} + P(x)y = f(x), \qquad (2)$$

When (2) is multiplied by a function μ (x), it becomes an exact differential equation. This function, μ (x) is known as **integrating factor**.

For a linear ODE (2) ; the integrating factor is given by

$$\mu(x) = e^{\int P(x)dx} \qquad (3)$$

Multiply both sides of (2)

$$\frac{dy}{dx} e^{\int P(x)dx} + P(x)y\, e^{\int P(x)dx} = f(x)\, e^{\int P(x)dx}$$

$$\frac{d}{dx}(ye^{\int P(x)dx}) = f(x)\, e^{\int P(x)dx}$$

Integrating both sides w.r.t. x

$$ye^{\int P(x)dx} = \int f(x)\, e^{\int P(x)dx} dx + \mathbf{constant(C)}$$

$$y = e^{-\int P(x)dx} \int f(x)\, e^{\int P(x)dx} dx + Ce^{-\int P(x)dx}$$

Steps:

(i) Identify P(x) from the standard form (2)

(ii) Compute I.F. μ(x)

(iii) Multiply equation (2) with I.F and reduce to exact differential.

(iv) Solve for y

Linear differential equations

Solve $\dfrac{dy}{dx} - 3y = 6.$

Solution: Comparing to the standard form (2), we see that P(x)= -3, and the integrating factor is

$$e^{\int(-3)dx} = e^{-3x}$$

Then multiplying the given equation by this factor gives

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x},$$

$$\frac{d}{dx}[e^{-3x}y] = 6e^{-3x}$$

Integrating both sides,

$$e^{-3x}y = -2e^{-3x} + c$$

$$\boxed{y = -2 + ce^{3x}, \quad -\infty < x < \infty.}$$

Linear differential equations:

$y = y_c + y_p$, where
$y_c = ce^{3x}$ is the solution of the corresponding homogeneous eqn.,

$$y' - 3y = 0$$

And, $y_p = -2$ is a particular solution of the nonhomogeneous equation

$$y' - 3y = 6.$$

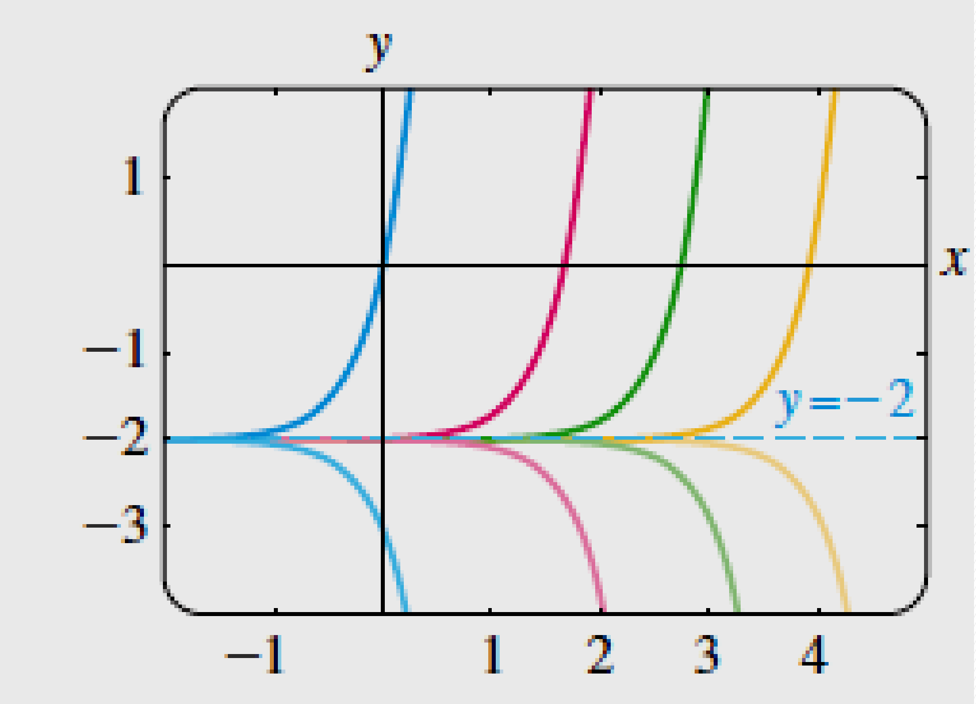


FIGURE 2.3.1 Some solutions of $y' - 3y = 6$

Linear differential equations:

Find the general solution of $(x^2 - 9)\dfrac{dy}{dx} + xy = 0.$

SOLUTION We write the differential equation in standard form

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0$$

and identify $P(x) = x/(x^2 - 9)$.

Although P is continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$, we shall solve the equation on the first and third intervals.

$$e^{\int x\,dx/(x^2-9)} = e^{\frac{1}{2}\int 2x\,dx/(x^2-9)} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2 - 9}.$$

Multiplying by I.F gives $$\frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = 0.$$

Integrating both sides of the last equation gives $\sqrt{x^2 - 9}\,y = c.$

$$\boxed{y = \frac{c}{\sqrt{x^2 - 9}}.}$$

?
(-3,3)
ignored

Series Circuits For a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor $(L(di/dt))$ and the voltage drop across the resistor (iR) is the same as the impressed voltage $(E(t))$ on the circuit. See Figure 3.1.7.

Thus we obtain the linear differential equation for the current $i(t)$,

$$\boxed{L\frac{di}{dt} + Ri = E(t),} \tag{7}$$

where L and R are constants known as the inductance and the resistance, respectively. The current $i(t)$ is also called the **response** of the system.

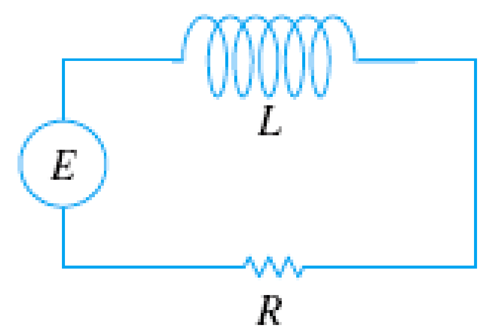


FIGURE 3.1.7 *LR*-series circuit

$$L\frac{di}{dt} + Ri = E, \quad i(0) = i_0, \quad L, R, E, i_0 \text{ constants}$$

Solution: Hints

$$\text{For } \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

an integrating factor is $e^{\int (R/L)\, dt} = e^{Rt/L}$

$$\text{so that } \quad \frac{d}{dt}\left[e^{Rt/L} i\right] = \frac{E}{L} e^{Rt/L}$$

or,

$$i = \frac{E}{R} + ce^{-Rt/L} \text{ for } -\infty < t < \infty.$$

If $i(0) = i_0$ then $c = i_0 - E/R$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{Rt/L}$$

EXAMPLE 7 Series Circuit

A 12-volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

Solution:

we see that we must solve

$$\frac{1}{2}\frac{di}{dt} + 10i = 12,$$

subject to $i(0) = 0$. First, we multiply the differential equation by 2 and read off the integrating factor e^{20t}. We then obtain

$$\frac{d}{dt}[e^{20t}i] = 24e^{20t}.$$

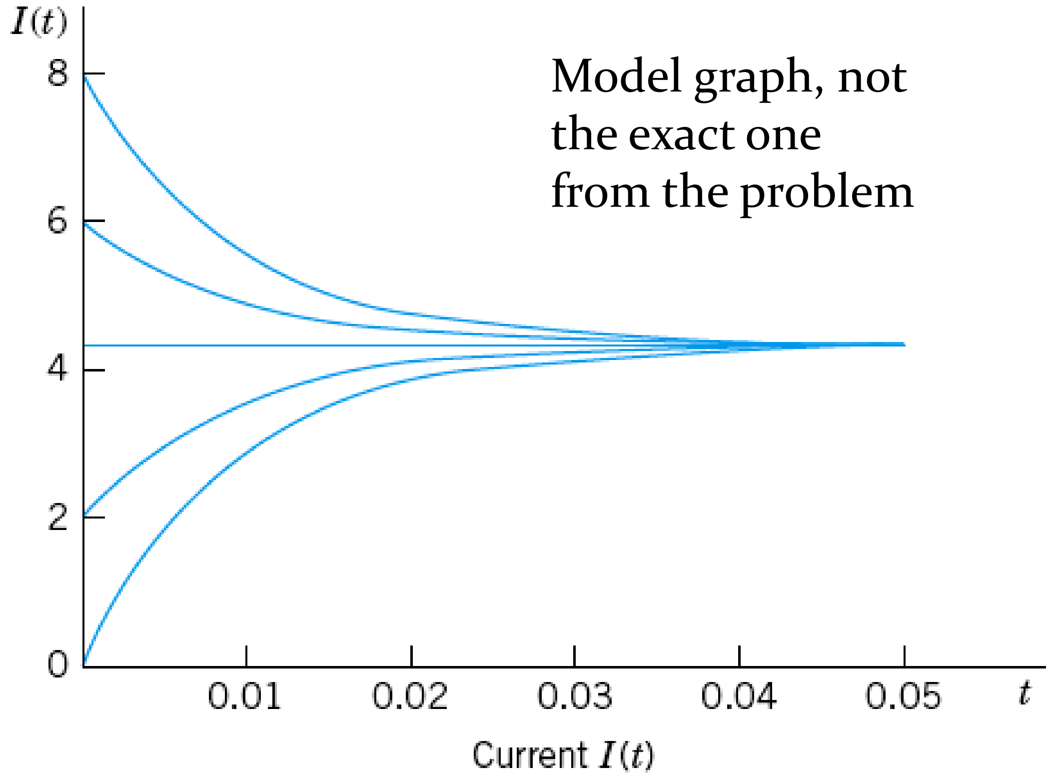
Integrating each side of the last equation and solving for i gives

$$i(t) = \tfrac{6}{5} + ce^{-20t}.$$

Now $i(0) = 0$ implies that $0 = \frac{6}{5} + c$ or $c = -\frac{6}{5}$.

Therefore the response is

$$i(t) = \frac{6}{5} - \frac{6}{5} e^{-20t}.$$



Exercises: 2.3

11. $x\dfrac{dy}{dx} + 4y = x^3 - x$

12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$

16. $y\,dx = (ye^y - 2x)\,dy$

23. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

24. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$

28. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

29. $L\dfrac{di}{dt} + Ri = E, \quad i(0) = i_0, \quad L, R, E, i_0$ constants

30. $\dfrac{dT}{dt} = k(T - T_m), \quad T(0) = T_0, \quad k, T_m, T_0$ constants

Solving Homogeneous First Order ODEs (Chapter 2.5)

- Homogeneous first order ODE
- Bernoulli's Equations

If a function *f possesses the property*

$$f(tx, ty) = t^{\alpha} f(x, y)$$

for some real number α , then *f is said to be a* ***homogeneous function of*** degree α .

For example, $f(x, y) = x^3 + y^3$

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

is a homogeneous function of degree 3.

whereas $f(x, y) = x^3 + y^3 + 1$ is not homogeneous.

A first-order DE in differential equation of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \qquad \text{(H1)}$$

is said to be **homogeneous if both coefficient functions *M* *and N are homogeneous*** functions of the *same degree.*

In other words, (H1) is homogeneous if

$$M(tx, ty) = t^{\alpha}M(x, y) \qquad \text{and} \qquad N(tx, ty) = t^{\alpha}N(x, y).$$

How to Solve: Either of the substitutions **y= ux** or **x= vy**, where u and v are new dependent variables, will reduce a homogeneous equation to a **separable first-order differential equation**.

Solve $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$.

SOLUTION

$$M(x, y) = x^2 + y^2 \text{ and } N(x, y) = x^2 - xy$$

Inspection of $M(x, y) = x^2 + y^2$ and $N(x, y) = x^2 - xy$ shows that these coefficients are homogeneous functions of degree 2.

If we let $y = ux$,

$$dy = u\,dx + x\,du,$$

After substituting, the given equation becomes

$$(x^2 + u^2x^2)\,dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$$

$$x^2(1 + u)\,dx + x^3(1 - u)\,du = 0$$

$$\frac{1-u}{1+u}\,du + \frac{dx}{x} = 0$$

$$\left[-1 + \frac{2}{1+u}\right] du + \frac{dx}{x} = 0.$$

After integration the last line gives

$$-u + 2\ln|1+u| + \ln|x| = \ln|c|$$

resubstituting $u = y/x$

$$-\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|c|.$$

$$\ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x}$$

or $\qquad (x+y)^2 = cxe^{y/x}.$

Bernoulli's Equation

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \qquad \text{(H2)}$$

where n is any real number, is called **Bernoulli's equation.**

Note that for *n= 0 and n=1, the above equation is a linear ODE*

For $n \neq 0$ and $n \neq 1$

the substitution $u = y^{1-n}$ reduces

any equation of form (H2) to a linear equation.

Solving a Bernoulli DE

Solve $x\dfrac{dy}{dx} + y = x^2y^2$.

SOLUTION We first rewrite the equation a

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$ by dividing by x.

With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$.

We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \qquad \leftarrow \text{Chain Rule}$$

into the given equation and simplify.

The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor for this linear equation

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

Integrating

$$\frac{d}{dx}[x^{-1}u] = -1$$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$.

Since $u = y^{-1}$, we have $y = 1/u$,

So a solution of the given equation is

$$y = 1/(-x^2 + cx).$$

Exercise 2.5

12. $(x^2 + 2y^2)\dfrac{dx}{dy} = xy, \quad y(-1) = 1$

Letting $y = ux$ we have

$$(x^2 + 2u^2x^2)dx - ux^2(u\,dx + x\,du) = 0$$

$$x^2(1 + u^2)dx - ux^3\,du = 0$$

$$\frac{dx}{x} - \frac{u\,du}{1 + u^2} = 0$$

$$\ln|x| - \frac{1}{2}\ln(1 + u^2) = c$$

$$\frac{x^2}{1 + u^2} = c_1$$

$$x^4 = c_1(x^2 + y^2).$$

Using $y(-1) = 1$ we find $c_1 = 1/2$.

The solution of the initial-value problem is $2x^4 = y^2 + x^2$.

21. $x^2 \dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

From $y' - \dfrac{2}{x}y = \dfrac{3}{x^2}y^4$ and $w = y^{-3}$ we obtain

$$\frac{dw}{dx} + \frac{6}{x}w = -\frac{9}{x^2}$$

An integrating factor is x^6

$$x^6 w = -\tfrac{9}{5}x^5 + c$$

or $y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$.

If $y(1) = \frac{1}{2}$ then $c = \frac{49}{5}$

and $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$.

End