MAT 350

ENGINEERING MATHEMATICS

Higher Order ODEs with Variable Coefficient:
Cauchy-Euler Equation

Lecture: 8

Dr. M. Sahadet Hossain (Mth)
Associate Professor
Department of Mathematics and Physics, NSU.

Cauchy-Euler Equation

A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x), \qquad (1)$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants,
is known as a **Cauchy-Euler equation.**

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

Higher Order ODE

The differential equation is named in honor of two of the most prolife mathematicians of all time. **Augustin-Louis Cauchy (French, 1789–1857) and Leonhard Euler (Swiss, 1707–1783).**

The above equation (1) is also know as **differential equation of variable coefficients.**

same same

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots .$$

Consider the 2nd order form for simplicity,

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0. \qquad (2)$$

Method of Solution: We try a solution of the form

$$y = x^m,$$

$$a_k x^k \frac{d^k y}{dx^k} = a_k x^k m(m-1)(m-2) \cdots (m-k+1) x^{m-k}$$

$$= a_k m(m-1)(m-2) \cdots (m-k+1) x^m.$$

For example, when we substitute $y = x^m$, the second-order equation becomes

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = am(m-1)x^m + bmx^m + cx^m = (am(m-1) + bm + c)x^m.$$

auxiliary equation

$$am(m-1)+bm+c=0$$

$$\boxed{am^2+(b-a)m+c=0.}$$

Auxiliary equation

There are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex.

Case I: Distinct Real Roots

Let m_1 *and* m_2 *denote the real roots of (2) such* that $m_1 \neq m_2$.

Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions.

The general solution is: $$y = c_1x^{m_1} + c_2x^{m_2}.$$

Case II: Repeated Real Roots

If the roots of (2) are repeated (that is, $m_1 = m_2$), *then we obtain only one solution —namely*

$$y = x^{m_1}.$$

The general solution is:

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x.$$

Case III: Conjugate Complex Roots

$m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real,

$$y = C_1 x^{\alpha + i\beta} + C_2 x^{\alpha - i\beta}.$$

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)].$$

Nonhomogeneous Equations

Solution methods:
(a) Variation of parameters
(b) Changing to constant coefficients

Variation of Parameters

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

Solution: First solve

$x^2y'' - 3xy' + 3y = 0$ for the Auxiliary equation.

We substitute $y = x^m$

Hence, we have the auxiliary equation $(m-1)(m-3) = 0$

$$y_c = c_1x + c_2x^3.$$

Now before using variation of parameters to find a particular solution

$$y_p = u_1y_1 + u_2y_2,$$

recall that the formulas $u_1' = W_1/W$ and $u_2' = W_2/W$,

$y'' + P(x)y' + Q(x)y = f(x).$

Therefore we divide the given equation by x^2, and from

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

we make the identification $f(x) = 2x^2e^x$. Now with $y_1 = x$, $y_2 = x^3$, and

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x,$$

$$u_1' = -\frac{2x^5e^x}{2x^3} = -x^2e^x \qquad \text{and} \qquad u_2' = \frac{2x^3e^x}{2x^3} = e^x.$$

The results are $u_1 = -x^2e^x + 2xe^x - 2e^x$ and $u_2 = e^x$.

$y_p = u_1y_1 + u_2y_2$ is

$$y_p = (-x^2e^x + 2xe^x - 2e^x)x + e^xx^3 = 2x^2e^x - 2xe^x.$$

Finally, $\boxed{y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x.}$

Exercise 4.7

22. $x^2y'' - 2xy' + 2y = x^4e^x$

The auxiliary equation is

$$m^2 - 3m + 2 = (m-1)(m-2) = 0$$

so that $y_c = c_1x + c_2x^2$ and

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2.$$

Identifying $f(x) = x^2e^x$

we obtain $u_1' = -x^2e^x$ and $u_2' = xe^x$.

Then $u_1 = -x^2e^x + 2xe^x$ $\quad$ $u_2 = xe^x - e^x$,

$$y = c_1x + c_2x^2 - x^3e^x + 2x^2e^x - 2xe^x + x^3e^x - x^2e^x$$

$$= c_1x + c_2x^2 + x^2e^x - 2xe^x.$$

Changing to Constant Coefficient

$\boxed{\text{Solve } x^2y'' - xy' + y = \ln x.}$

SOLUTION With the substitution $x = e^t$ or $t = \ln x$, it follows that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \qquad \leftarrow \text{Chain Rule}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) \qquad \leftarrow \text{Product Rule and Chain Rule}$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Substituting in the given differential equation and simplifying yields

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t.$$

Since this last equation has constant coefficients, its auxiliary equation is $m^2 - 2m + 1 = 0$, or $(m - 1)^2 = 0$. Thus we obtain $y_c = c_1 e^t + c_2 t e^t$.

By undetermined coefficients we try a particular solution of the form $y_p = A + Bt$. This assumption leads to $-2B + A + Bt = t$, so $A = 2$ and $B = 1$. Using $y = y_c + y_p$, we get

inf

$$y = c_1 e^t + c_2 t e^t + 2 + t.$$

By resubstituting $e^t = x$ and $t = \ln x$ we see that the general solution of the original differential equation on the interval (0,) is $y = c_1 x + c_2 x \ln x + 2 + \ln x$.

34. $x^2y'' - 4xy' + 6y = \ln x^2$

Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2t.$$

The auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$

so that $y_c = c_1e^{2t} + c_2e^{3t}$.

Using Superposition principle

we try $y_p = At + B$. This leads to $(-5A + 6B) + 6At = 2t$, so that

$A = 1/3$, $B = 5/18$, and

$$y = c_1e^{2t} + c_2e^{3t} + \frac{1}{3}t + \frac{5}{18} = c_1x^2 + c_2x^3 + \frac{1}{3}\ln x + \frac{5}{18}.$$