Course Name : Physics – I
Course # PHY 107

Notes-10 : Rotation: Kinematics and Dynamics (Ch. 10 & 11)

Abu Mohammad Khan
Department of Mathematics and Physics
North South University
https://abukhan.weebly.com

Topics to be studied

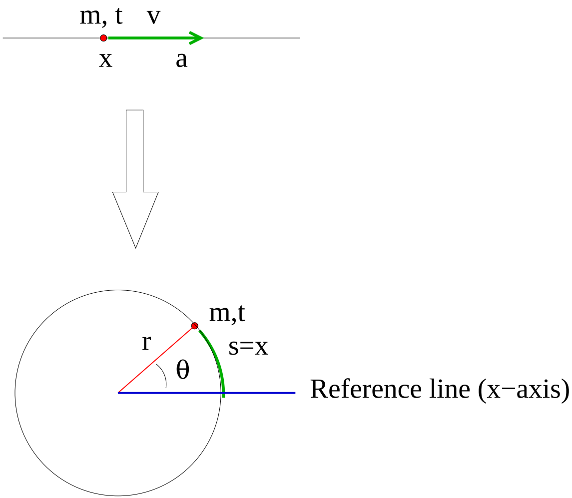
- Definitions: Angular position, velocity and acceleration
- Relation between linear and rotational variables
- Rotational equation of motion (constant acceleration)
- Properties of rotational motion: turning point, speeding. sowing, etc.
- Definition of Torque: Properties
- Moment of Inertia
- Angular Momentum: Definition and Properties
- Work-energy theorem for rotation
- Short Summary
- Examples
- Suggested Problems

Angular position, velocity and acceleration

- The angular position $\theta(t)$ is defined as the angle the radius vector makes with the x-axis (or reference line) at time t, measured clockwise (positive rotation).
- The angular velocity is defined as the rate of change of angular position: $\omega(t) = \dfrac{d\theta}{dt}$.
- Similarly, the angular acceleration is defined as the rate of change of the angular velocity: $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$.
- Geometrically, the slope at a point on the θ vs. t graph is the angular velocity, and the curvature at time t is the angular acceleration.
- Integrating ω gives the angular position (indefinite integral) or angular displacement (definite integral).
- Geometrically, the displacement is the area under the curve on the ω vs. t graph.
- These are the basic kinematical parameters for angular motion.
- Note that, by convention, counter-clockwise (anti-clockwise) rotation is positive and clockwise rotation is negative.

Relation between linear and rotational variables

- Let's consider one dimensional linear motion. At time t, the object of mass m has position x, velocity v and acceleration a.
- Now bend one dimensional motion to form a circular path of radius r. The arc length x subtends an angle θ at the center as shown in the adjacent figure.
- Suppose r=constant. Then $x(t) \propto \theta(t)$. Geometrically, $\boxed{x = r\theta}$. By differentiating this relation, other relations can be derived.



Rotational equations of motion

- Differentiating $x(t) = r\theta(t)$, we find: $\dfrac{dx}{dt} = r\dfrac{d\theta}{dt} \implies \boxed{v = r\omega}$.
- Taking derivative again, we get: $\dfrac{dv}{dt} = r\dfrac{d\omega}{dt} \implies \boxed{a = r\alpha}$.
- Using these equations, all rotational equations motion can be derived. For each linear variable x, v and a, we need to substitute $r\theta$, $r\omega$ and $r\alpha$ respectively, and then simplifying/rearranging will give the corresponding equation of motion for rotation. Hence, we get:

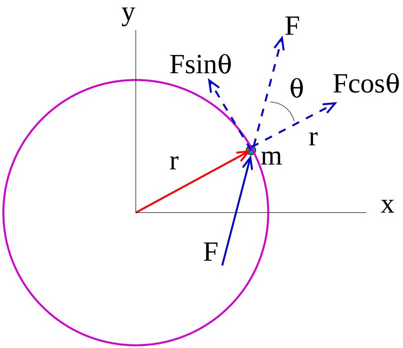
$$
\begin{aligned}
x = x_0 + v_{\mathrm{av}}t \quad &\Rightarrow \quad r\theta = r\theta_0 + r\omega_{\mathrm{av}}t. \qquad \therefore\ \theta = \theta_0 + \omega_{\mathrm{av}}t. \\
v = v_0 + at \quad &\Rightarrow \quad r\omega = r\omega_0 + r\alpha t. \qquad \therefore\ \omega = \omega_0 + \alpha t. \\
x = x_0 + v_0 t + \frac{1}{2}at^2 \quad &\Rightarrow \quad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2. \\
v^2 = v_0^2 + 2a(x - x_0) \quad &\Rightarrow \quad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0).
\end{aligned}
$$

- The above equations are valid only for constant angular acceleration (α=constant), exactly like linear motion.
- Mathematically, the equations for linear and rotational motions are same, except the fact that the symbols have been changed.
- In fact, all the rules and conditions that we used for linear motion, are also valid for rotational motions with the corresponding change in symbols or variables.
- We can write:

$$
\begin{aligned}
\text{Average velocity} \quad &\Rightarrow \quad \overline{\omega} = \frac{\Delta\theta_0}{\Delta t} \quad \text{(Definition)} \\
&\quad\quad\ \overline{\omega} = \frac{\omega + \omega_0}{2} \quad \text{(only if } \alpha\text{=constant)} \\
\text{Rest} \quad &\Rightarrow \quad \omega = 0 \text{ and } \alpha = 0. \\
\text{Turning point} \quad &\Rightarrow \quad \omega = 0 \text{ and } \alpha \neq 0. \\
\text{Slowing down} \quad &\Rightarrow \quad \omega < 0, \alpha > 0 \text{ or } \omega > 0, \alpha < 0. \\
\text{Speeding up} \quad &\Rightarrow \quad \omega < 0, \alpha < 0 \text{ or } \omega > 0, \alpha > 0
\end{aligned}
$$

Torque and Moment of Inertia

- The torque, also called rotational force (responsible for rotation) is defined as $\vec{\tau} = \vec{r} \times \vec{F} = rF \sin\theta \, \hat{n}$.
- Here $\vec{r}$ is the position vector of the object of mass m, and $r = |\vec{r}|$=constant. So the path is circular. a force $\vec{F}$ is applied to th mass. Since r is fixed, instead of moving linearly, the mass will rotate. The component of force responsible for rotation is shown below:



- We define, $F_\perp = F\sin\theta$ which is the cause of rotation. Other component $F\cos\theta$ is balenced by the centripetal force.
- Using 2nd law , we can write:

$$\tau = rF_\perp = r(ma) = r(mr\alpha) = (mr^2)\alpha. \quad \Rightarrow \ \tau = I\alpha \ .$$

  Here we defined: $I = mr^2$, and it is called the 'Moment of Inertia'.
- The moment of Inertia plays role of mass in rotational motion.
- In vector form, we write: $\vec{\tau} = I\alpha\,\hat{n}$. Here $\hat{n}$ is the direction of the torque, and it is also called the axis of rotation.
- $\hat{n}$ is determined by the 'Right-Hand-Rule' which is the direction of the cross product.
- Finally, torque can be computed as:

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta\,\hat{n} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{pmatrix} = I\alpha\hat{n} \ .$$

Angular Momentum:

- The angular momentum is defined as: $\vec{L} = \vec{r} \times \vec{p} = rp\sin\theta\,\hat{n}$.
- Here $p_\perp = p\sin\theta$ is responsible for rotation. Therefore, we can write, using linear relations:

$$L = rp_\perp = r(mv) = r(mr\omega) = (mr^2)\omega. \quad \Rightarrow \; L = I\omega \; .$$

- Since r is constant, I is also constant. Therefore, taking derivative, we get:

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha \quad \Rightarrow \boxed{\sum \vec{\tau} = \frac{d\vec{L}}{dt}} \; .$$

This is known as the Newton's 2nd Law for rotation.

- Finally, angular momentum can be computed as:

$$\vec{L} = \vec{r} \times \vec{p} = rp\sin\theta\,\hat{n} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix} = I\omega\hat{n} \; .$$

Work-energy Theorem

- Starting from the linear motion, the corresponding equation for totation can be derived by the change of variables.
- The work done is

$$W = \int F dx \Longrightarrow W_{\rm rot} = \int madx = \int m(r\alpha)d(r\theta) = \int (mr^2)d\theta = \int \tau d\theta \ .$$

- The kinetic energy is

$$K = \frac{1}{2}mv^2 \Longrightarrow K_{\rm rot} = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}(mr^2)\omega = \frac{1}{2}I\omega^2 \ .$$

- The Work-Energy Theorem for rotation becomes

$$W_{\rm tot} = \Delta K = K_f - K_i \Longrightarrow W_{\rm tot} = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \Delta K_{\rm rot} \ .$$

Short Summary

- From the above analysis, we see that strating from the linear equation, the corresponding angular or rotation equation can easily be derived by the method of substitution.
- All we need to do is to replace the linear variable by the corresponding rotational variables as shown in the following:

$$
\begin{aligned}
\text{Linear variable} &\Longrightarrow \text{Rotational variable} \\
x &\Longrightarrow \theta \\
v &\Longrightarrow \omega \\
a &\Longrightarrow \alpha \\
m &\Longrightarrow I \\
F &\Longrightarrow \tau \\
p &\Longrightarrow L
\end{aligned}
$$

Examples:
Problem # 10.4

The angular position of a point on a rotating wheel is given by $\theta = 2.0 + 4.0t^2 + 2.0t^3$, where θ is in radians and t is in seconds. At $t = 0$, what are (a) the point's angular position and (b) its angular velocity? (c) What is its angular velocity at $t = 4.0$s? (d) Calculate its angular acceleration at $t = 2.0$s. (e) Is its angular acceleration constant?

Solution: The angular velocity and angular acceleration are

$$\omega = 8t + 6t^2 \quad \text{and} \quad \alpha = 8 + 12t \ .$$

Substituting the values of t we can get the answers.

- (a) $\theta_0 = \theta\big|_{t=0} = 2\,\mathrm{rad}.$
- (b) $\omega_0 = \omega\big|_{t=0} = 0.$
- (c) $\omega_4 = \omega\big|_{t=4} = (8(4) + 6(4^2))\mathrm{r/s} = 128\,\mathrm{r/s}.$
- (d) $\alpha_2 = \alpha\big|_{t=2} = (8 + 12(2))\mathrm{r/s^2} = 32\,\mathrm{r/s^2}.$
- (e) α is not constant.

Problem # 10.16:

A merry-go-round rotates from rest with an angular acceleration of $1.50\,\mathrm{rad/s^2}$. How long does it take to rotate through (a) the first $2.00\,\mathrm{rev}$ and (b) the next $2.00\,\mathrm{rev}$?

Solution: Here: $\theta_0 = 0$, $\omega_0 = 0$ and $\alpha = 1.50\,\mathrm{r/s^2}$.

- (a) In this case, $\theta = 2.0\,\mathrm{rev} = 4\pi\,\mathrm{rad}$. Using the 3rd equation of motion, we obtain,

$$\theta = \cancelto{0}{\theta_0} + \cancelto{0}{\omega_0} + \frac{1}{2}\alpha t^2. \quad \Rightarrow t = \sqrt{\frac{2\theta}{\alpha}} = \sqrt{\frac{2 \times 4\pi}{1.5}} = 4.1\,\mathrm{sec}\ .\checkmark$$

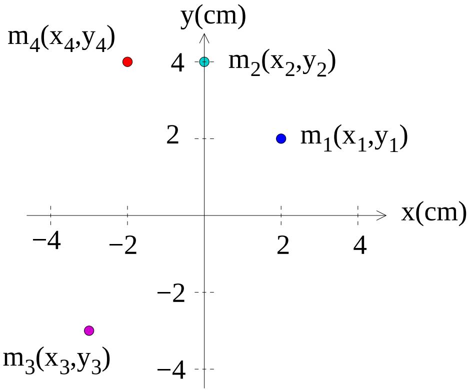
- (b) Here, we compute the time t' for four revolutions, and then subtract the time for the first two revolutions. The rest of the time is for the next two (*i.e.* 3rd and 4th) revolutions. Using $\theta = 4\,\mathrm{rev} = 8\pi\,\mathrm{rad}$ and the 3rd equation of motion, we get,

$$\theta = \cancelto{0}{\theta_0} + \cancelto{0}{\omega_0} + \frac{1}{2}\alpha t'^2. \quad \Rightarrow t' = \sqrt{\frac{2\theta}{\alpha}} = \sqrt{\frac{2 \times 8\pi}{1.5}} = 5.8\,\mathrm{sec}\ .\checkmark$$

Therefore, the time for the next two revolutions $= (5.8 - 4.1)\mathrm{sec} = 1.7\,\mathrm{sec}.\ \checkmark$

Problem # 10.42:

The adjacent figure shows the masses and coordinates of four particles are as follows:
$50\text{g},\ x = 2.0\text{cm},\ \ y = 2.0\text{cm};$
$25\,\text{g},\ x = 0,\ \ y = 4.0\text{cm};$
$25\,\text{g},\ x = -3.0\text{cm}, y = -3.0\text{cm};$
$30\text{g},\ x = -2.0\text{cm}, y = 4.0\text{cm}.$
What are the rotational inertias of this collection about the (a) x, (b) y, and (c) z axes? (d) Suppose the answers to (a) and (b) are A and B, respectively. Then what is the answer to (c) in terms of A and B?



Solution: The masses and their coordinated are as follows (see the diagram):

$$m_1 = 50\text{g},\ (x_1, y_1) = (2.0\text{cm}, 2.0\text{cm});\quad m_2 = 25\text{g},\ (x_2, y_2) = (0, 4.0\text{cm});$$

$$m_3 = 25\text{g},\ (x_3, y_3) = (-3.0\text{cm}, -3.0\text{cm});\quad m_4 = 30\text{g},\ (x_4, y_4) = (-2.0\text{cm}, 4.0\text{cm}).$$

Solution # 10.42:

Now, the moment of inertia is defined as $i = mr^2$, where r is the shortest or radial distance of the mass m from the axis or rotation. In this case, the total moment of inertia, $I = I_1 + I_2 + I_3 + I_4$, where I_1 is the moment inertia for m_1, etc.

- (a) For rotation about x-axis: $r_1 = y_1 = 2.0\mathrm{cm}$, $r_2 = y_2 = 4.0\mathrm{cm}$, $r_3 = |y_3| = 3.0\mathrm{cm}$ and $r_4 = y_4 = 4.0\mathrm{cm}$. Therefore,

$$I = \left[50 \times 2^2 + 25 \times 4^2 + 25 \times 3^2 + 30 \times 4^2\right]\mathrm{g.cm}^2 = 1305\,\mathrm{g.cm}^2\ .\checkmark$$

- (b) For rotation about y-axis: $r_1 = x_1 = 2.0\mathrm{cm}$, $r_2 = x_2 = 0\,\mathrm{cm}$, $r_3 = |x_3| = 3.0\mathrm{cm}$ and $r_4 = |x_4| = 2.0\mathrm{cm}$. Therefore,

$$I = \left[50 \times 2^2 + 25 \times 0 + 25 \times 3^2 + 30 \times 2^2\right]\mathrm{g.cm}^2 = 545\,\mathrm{g.cm}^2\ .\checkmark$$

- (c) For rotation about z-axis: $r_1^2 = x_1^2 + y_1^2$ and so on. Therefore,

$$I = m_1(x_1^2 + y_1^2) + \cdots = I_x + I_y = (1305 + 545)\mathrm{g.cm}^2 = 1850\,\mathrm{g.cm}^2\ .\checkmark$$

- (d) $I = A + B$. ✓

Problem # 11.22:

A particle moves through an xyz coordinate system while a force acts on the particle. When the particle has the position vector $\vec{r} = (2.00\mathrm{m})\hat{i} - (3.00\mathrm{m})\hat{j} + (2.00\mathrm{m})\hat{k}$, the force is given by $\vec{F} = F_x + (7.00\mathrm{N})\hat{j} - (6.00\mathrm{N})\hat{k}$ and the corresponding torque about the origin is $\vec{\tau} = (4.00\mathrm{N.m})\hat{i} + (2.00\mathrm{N.m})\hat{j} - (1.00\mathrm{N.m})\hat{k}$. Determine F_x.

Solution: By definition the torque is

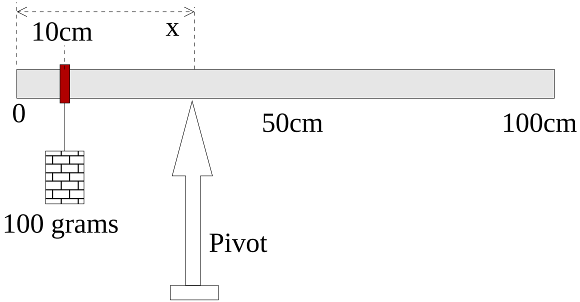
$$\begin{aligned}\vec{\tau} = \vec{r} \times \vec{F} &= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{pmatrix} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 2 \\ F_x & 7 & -6 \end{pmatrix} \mathrm{N.m}\ , \\ &= \left[4\hat{i} + (12 + 2F_x)\hat{j} + (14 + 3F_x)\hat{k}\right] \mathrm{N.m}\ .\end{aligned}$$

This result needs to be compared to the torque,
$\vec{\tau} = (4.00\mathrm{N.m})\hat{i} + (2.00\mathrm{N.m})\hat{j} - (1.00\mathrm{N.m})\hat{k}$. Comparing the y-component gives,

$$12 + 2F_x = 2 \quad \Longrightarrow \quad F_x = -5.0\,\mathrm{N}\ .\checkmark$$

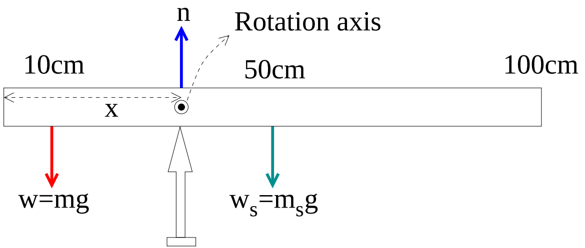
Static Equilibrium:

The static equilibrium means that Newton's 1st law for both linear and rotational motion will be satisfied: $\boxed{\sum \vec{F} = 0 \text{ and } \sum \vec{\tau} = 0}$. Consider the following diagram.



A 100 grams mass is hanging at 10cm mark of a meter stick and the whole set up is in static equilibrium when put on the pivot as shown. The mass of the meter stick is 200 grams. Find: (a) the normal force and (b) the distance x of the pivot.

- For simplicity, we choose the rotation axis at x (at the location of the pivot), and directed out of the page. The figure below shows the forces and the rotation axis.



- (a) $\sum \vec{F} = 0 \implies \sum F_{\text{upward}} = \sum F_{\text{downward}}$. Therefore, we find:

$$n = w + w_s = (m + m_s)g = (0.100 + 0.200)9.80\,\mathrm{N} = 2.94\,\mathrm{N}\ .\checkmark$$

- (b) $\sum \vec{\tau} = 0 \implies \sum \tau_{\text{clockwise}} = \sum \tau_{\text{anti-clockwise}}$. Using the above diagram, we can write:

$$\begin{aligned} \tau_w + \tau_{w_s} &= \tau_n \quad \Rightarrow \quad (0.5 - x)m_s g = (x - 0.1)mg\ , \\ \therefore\ x &= \frac{0.5m_s + 0.1m}{m + m_s} = \frac{0.5 \times 0.2 + 0.1 \times 0.1}{0.2 + 0.1}\,\mathrm{m} = 0.367\,\mathrm{m} = 36.7\,\mathrm{cm}\ .\checkmark \end{aligned}$$

Suggested Problems:

Chapter 10: 4, 6, 9, 13, 15, 16, 22, 33, 38, 41, 45, 52 and 61.
Chapter 11: 19, 22, 24, 26, 27, 29, 30, 33 and 37.