MAT 350

Engineering mathematics

Modeling with 2^{nd} order ODE:
Mass–Spring System.

Lecture: 5

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Modeling of Free Oscillations of a Mass–Spring System

Linear ODEs with constant coefficients have important applications in mechanics, and in electrical circuits.
In this section we model and solve a basic mechanical system consisting of a mass on an elastic spring (a so-called "mass–spring system," Fig. below), which moves up and down.

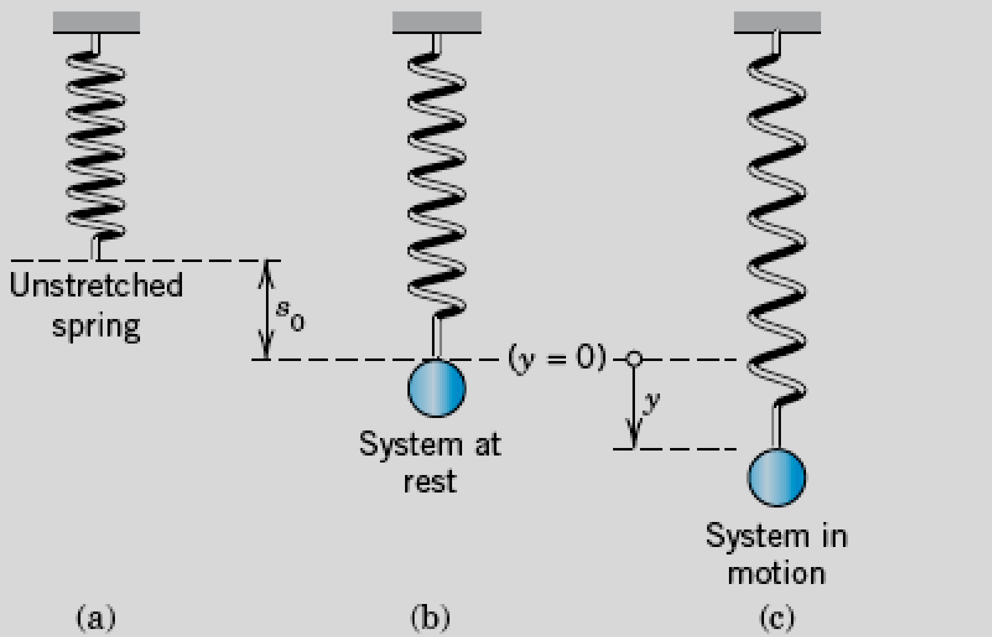


Fig. 33. Mechanical mass–spring system

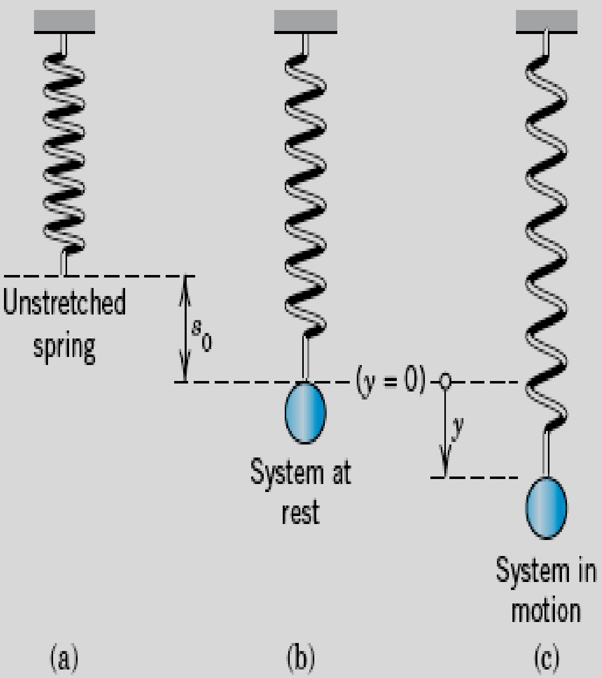


Fig. 33. Mechanical mass–spring system

$$F = ky \quad (1)$$

$$\text{Mass} \times \text{Acceleration} = my'' = \text{Force} \quad (2)$$

If the damping is small and the motion of the system is considered over a relatively short time, we may disregard damping.

Then Newton's law with F=-F_1 gives the model

$$my'' = -F_1 = -ky;$$

$$\boxed{my'' + ky = 0.} \quad (3)$$

This is a homogeneous linear ODE with constant coefficients. A general solution is obtained as

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}. \qquad (4)$$

This motion is called a **harmonic oscillation.** Its *frequency is* $f = \omega_0/2\pi$ Hertz

($=$ cycles/sec) because cos and sin in (4) have the period $2\pi/\omega_0$.

The frequency *f is called* the **natural frequency of the system.**

Please note, $y'(t) = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t$

$$y'(0) = B\omega_0$$

An alternative representation of (4), which shows the physical characteristics of amplitude and phase shift of (4), is

$$y(t) = C \cos(\omega_0 t - \delta)$$

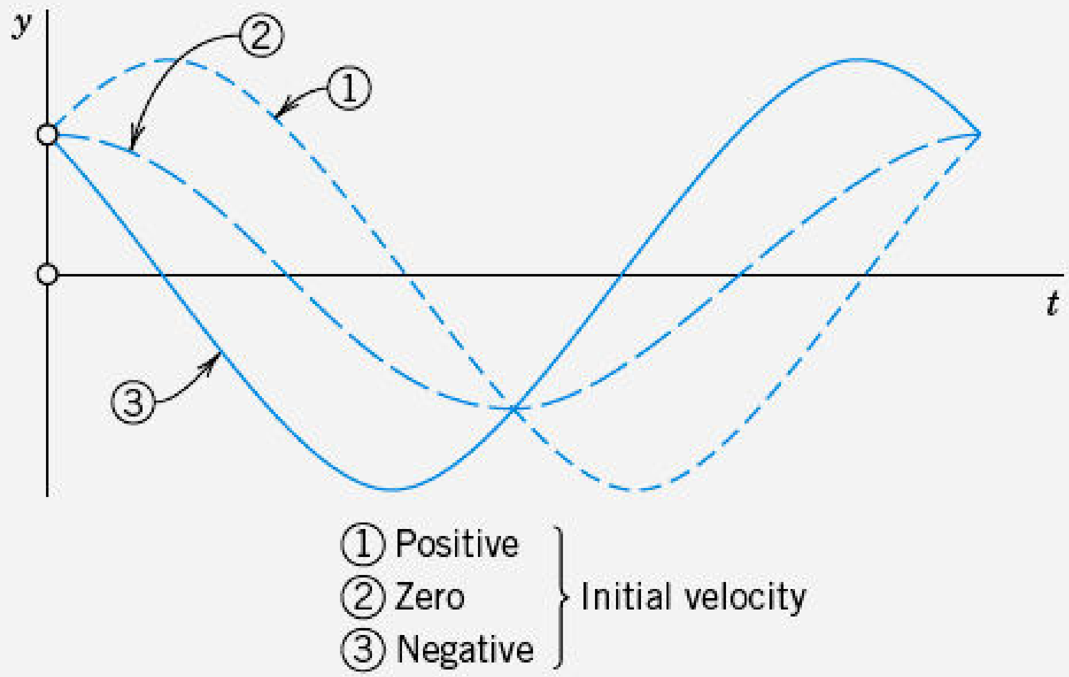


Fig. 34. Typical harmonic oscillations (4) and (4*) with the same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$, positive ①, zero ②, negative ③

Modeling of Free Oscillations of a Mass–Spring System

Example 1: If a mass–spring system with an iron ball of weight W=98 nt (about 22 lb) can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m (about 43 in.), how many cycles per minute will the system execute?

What will its motion be if we pull the ball down from rest by 16 cm (about 6 in.) and let it start with zero initial velocity?

Solution. Hooke's law with W as the force and 1.09 meter as the stretch gives

$$W = 1.09\,k \quad \text{(since } W=ky\text{, } y \text{ displacement)}$$

$$k = W/1.09 = 98/1.09 = 90\ [\text{kg/sec}^2] = 90\ [\text{nt/meter}].$$

The mass is $m = W/g = 98/9.8 = 10$ [kg].

$$\boxed{my'' + ky = 0.}$$

Generate mathematical model

Frequencey, $f = \omega_0/(2\pi) = \sqrt{k/m}/(2\pi)$

$$= 3/(2\pi) = 0.48\ [\text{Hz}] = 29\ [\text{cycles/min}].$$

Initial conditions $y(0) = A = 0.16$ [meter] and $y'(0) = \omega_0 B = 0.$

Note that we have, $y'(t) = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t$. Hence $y'(0) = B\omega_0$

Hence B=0, and $\delta = \tan^{-1}(B/A) = 0$,

Therefore, the general solution is:

$$y(t) = C\cos(\omega_0 t - \delta)$$

$$y(t) = 0.16\cos 3t \text{ [meter]}$$

Here, 16 cm=0.16 m

C=sqrt(A²+B²)

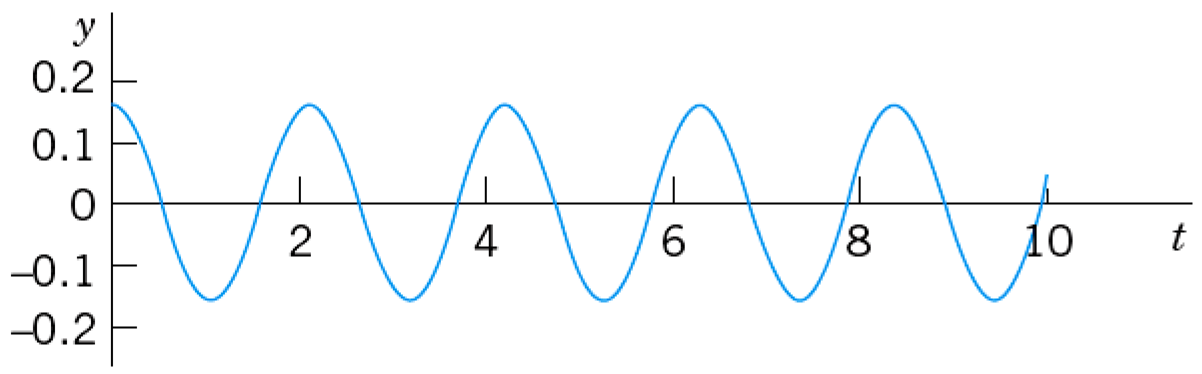


Fig. 35. Harmonic oscillation in Example 1

ODE of the Damped System

$$my'' = -ky$$

$$F_2 = -cy',$$

$$my'' + cy' + ky = 0.$$

The characteristic equation is

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

$$\lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta,$$

where $\alpha = \dfrac{c}{2m}$ and $\beta = \dfrac{1}{2m}\sqrt{c^2 - 4mk}.$

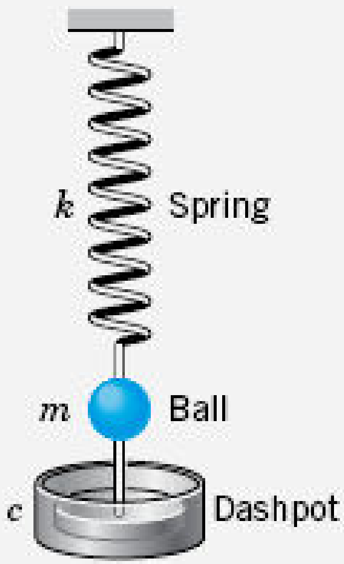


Fig. 36.
Damped system

It is now interesting that depending on the amount of damping present—whether a lot of damping, a medium amount of damping or little damping—three types of motions occur, respectively:

Case I.	$c^2 > 4mk$.	*Distinct real roots* λ_1, λ_2.	**(Overdamping)**
Case II.	$c^2 = 4mk$.	*A real double root.*	**(Critical damping)**
Case III.	$c^2 < 4mk$.	*Complex conjugate roots.*	**(Underdamping)**

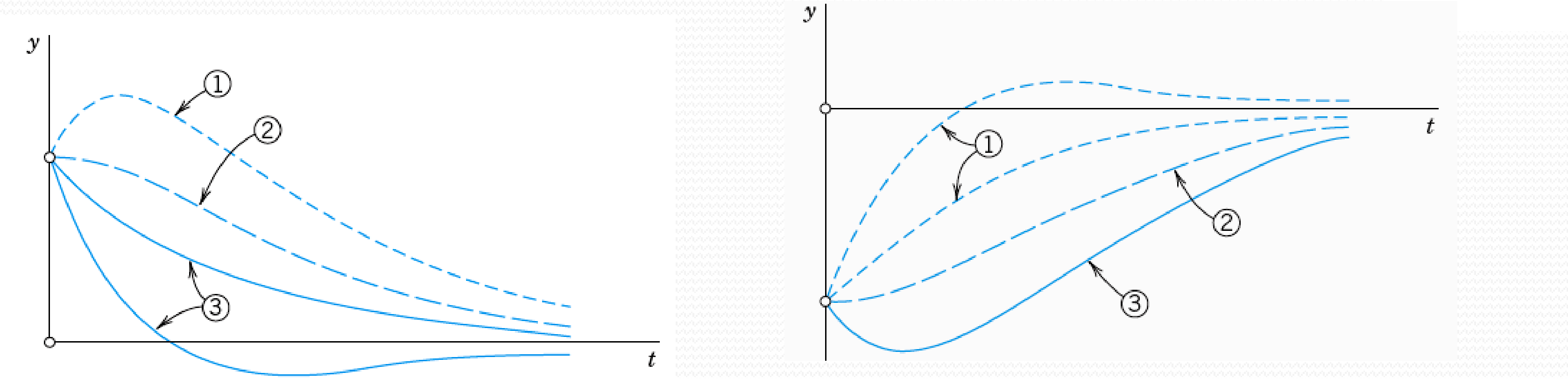
Case I. Overdamping

If the damping constant c is so large that $c^2 > 4mk$,
then λ_1 and λ_2 are distinct real roots.

In this case the corresponding general solution of (5) is

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0, \beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$. Practically



Typical motions (7) in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

Case II. Critical Damping

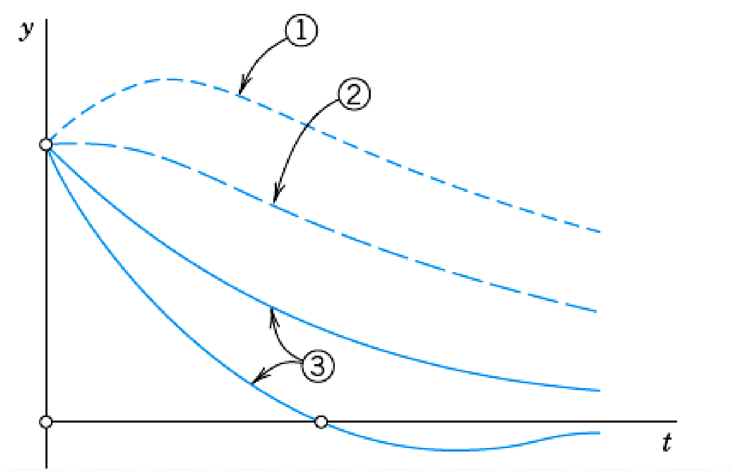
It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$.

Then the corresponding general solution

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}.$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{-\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.

Note that they look almost like those in the previous figure.



(1) Positive
(2) Zero
(3) Negative

} Initial velocity

Fig. 38. Critical damping [see (8)]

Case III. Underdamping

It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (>0).$$

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^* \quad \text{with } \alpha = c/(2m),$$

Hence the corresponding general solution is

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos (\omega^* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$.

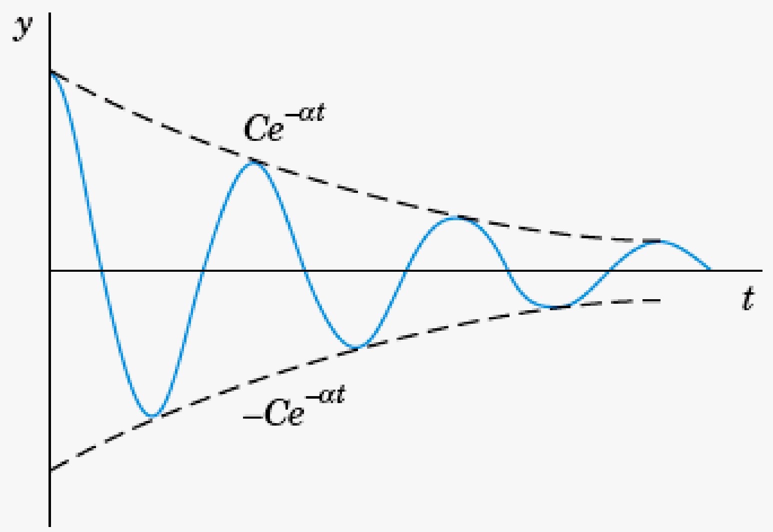
This represents **damped oscillations**. Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 39, touching them when $\omega^* t - \delta$ is an integer multiple of π because these are the points at which $\cos(\omega^* t - \delta)$ equals 1 or -1.

The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec).

If c approaches 0,

then ω^* approaches $\omega_0 = \sqrt{k/m}$, giving the harmonic oscillation

whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.



. 39. Damped oscillation in Case III [see (10)]

Examples: from Zill 10th edition.

Example 2: A mass weighing 2 pounds stretches a spring 6 inches. At *t=0 the mass is released* from a point 8 inches below the equilibrium position with an upward velocity of 4/3 ft/s.

Determine the equation of motion.

SOLUTION:
Because we are using the engineering system of units, the measurements given in terms of inches must be converted into feet

$$6 \text{ in.} = \tfrac{1}{2} \text{ ft}; \; 8 \text{ in.} = \tfrac{2}{3} \text{ ft}.$$

In addition, we must convert the units of weight given in pounds into units of mass.

From $m = W/g$ we have $m = \frac{2}{32} = \frac{1}{16}$ slug.

Also, from Hooke's law, $2 = k\left(\frac{1}{2}\right)$ $\quad k = 4$ lb/ft.

$$\frac{1}{16}\frac{d^2x}{dt^2} = -4x \qquad \text{or} \qquad \frac{d^2x}{dt^2} + 64x = 0.$$

$$\frac{d^2x}{dt^2} + 64x = 0.$$

The initial displacement and initial velocity are $x(0) = \frac{2}{3}$, $x'(0) = -\frac{4}{3}$,

Here the negative sign in the last condition is a consequence of the fact that the mass is given an initial velocity in the negative, or upward, direction.

Now $\omega^2 = 64$ or $\omega = 8$, so the general solution

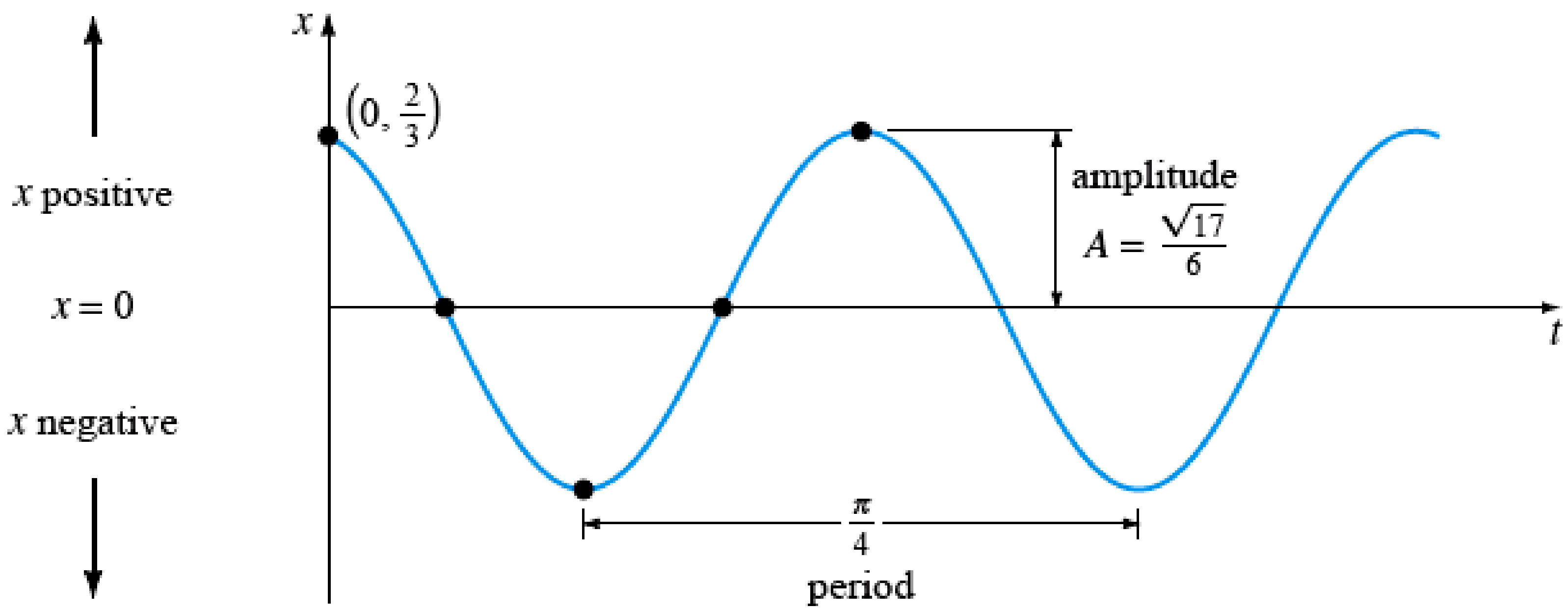
$$x(t) = c_1 \cos 8t + c_2 \sin 8t.$$

Applying the initial conditions to $x(t)$ and $x'(t)$ gives

$$c_1 = \frac{2}{3} \text{ and } c_2 = -\frac{1}{6}.$$

Thus, the equation of motion is

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t.$$



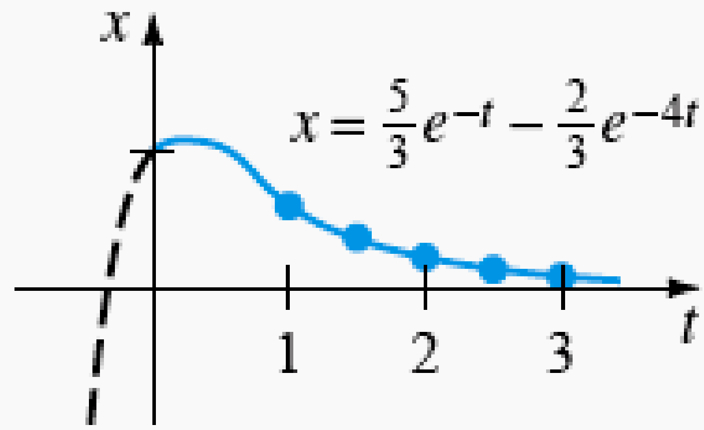
Simple harmonic motion

Example 3: **Overdamped motion**

Consider the overdamped motion described by the equation below. The mass is initially released from a position 1 unit *below the equilibrium* position with a *downward velocity of 1 ft/s.*

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0, \quad x(0) = 1, \quad x'(0) = 1$$

$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}.$$



Example 4: Critically damped

A mass weighing 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of 3 ft/s.

SOLUTION From Hooke's law we see that $8 = k(2)$ gives $k = 4$ lb/ft and $W = mg$ gives $m = \frac{8}{32} = \frac{1}{4}$ slug. The differential equation of motion is then

$$\frac{1}{4}\frac{d^2x}{dt^2} = -4x - 2\frac{dx}{dt} \qquad \text{or} \qquad \frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0.$$

Auxiliary equation is:

$$m^2 + 8m + 16 = (m + 4)^2 = 0, \text{ so } m_1 = m_2 = -4.$$

Hence the system is critically damped, and

$$x(t) = c_1e^{-4t} + c_2te^{-4t}.$$

Applying the initial conditions $x(0) = 0$ and $x'(0) = -3$,

we find, in turn, that $c_1 = 0$ and $c_2 = -3$. Thus the equation of motion is

$$x(t) = -3te^{-4t}.$$

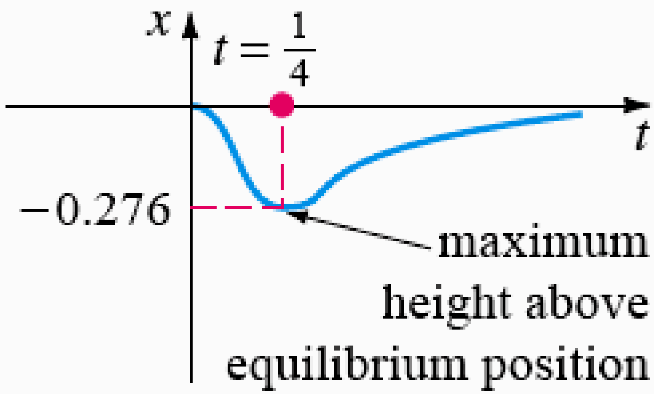


FIGURE 5.1.10 Critically damped system in Example 4

Example: (Underdamped condition)

A mass weighing 16 pounds is attached to a 5-foot-long spring. At equilibrium the spring measures 8.2 feet. If the mass is initially released from rest at a point 2 feet above the equilibrium position, find the displacements x(t) if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

The Three Cases of Damped Motion

Consider the motion in Example 1 (of free oscillation) change if we change the damping constant c *from one to another of the* following three values

(I) $c = 100$ kg/sec, (II) $c = 60$ kg/sec, (III) $c = 10$ kg/sec.

(I) With $m = 10$ and $k = 90$, as in Example 1, the model is the initial value problem

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16 \text{ [meter]}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0$.

It has the roots -9 and -1.

$y = c_1e^{-9t} + c_2e^{-t}$. We also need $y' = -9c_1e^{-9t} - c_2e^{-t}$.

$c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$.

$$y = -0.02e^{-9t} + 0.18e^{-t}.$$

(II) The model is as before, with $c = 60$ instead of 100.

$$10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0.$$

It has the double root -3.

$$y = (c_1 + c_2 t)e^{-3t}.$$

We also need $\quad y' = (c_2 - 3c_1 - 3c_2 t)e^{-3t}.$

$$y(0) = c_1 = 0.16,\ y'(0) = c_2 - 3c_1 = 0,\ c_2 = 0.48.$$

Hence in the critical case the solution is

$$y = (0.16 + 0.48t)e^{-3t}.$$

It is always positive and decreases to 0 in a monotone fashion.

(III) The model now is $10y'' + 10y' + 90y = 0$.

Since c =10 is smaller than the critical *c, we shall get* oscillations.

$$10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \tfrac{1}{2})^2 + 9 - \tfrac{1}{4}] = 0.$$

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i.$$

$$y = e^{-0.5t}(A \cos 2.96t + B \sin 2.96t).$$

Thus $y(0) = A = 0.16$. We also need the derivative

$$y' = e^{-0.5t}(-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t).$$

Hence $y'(0) = -0.5A + 2.96B = 0$,

$B = 0.5A/2.96 = 0.027$. This gives the solution

$$y = e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t)$$

$$= 0.162e^{-0.5t} \cos (2.96t - 0.17).$$

We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3.00 by about 1%). Their amplitude goes to zero.

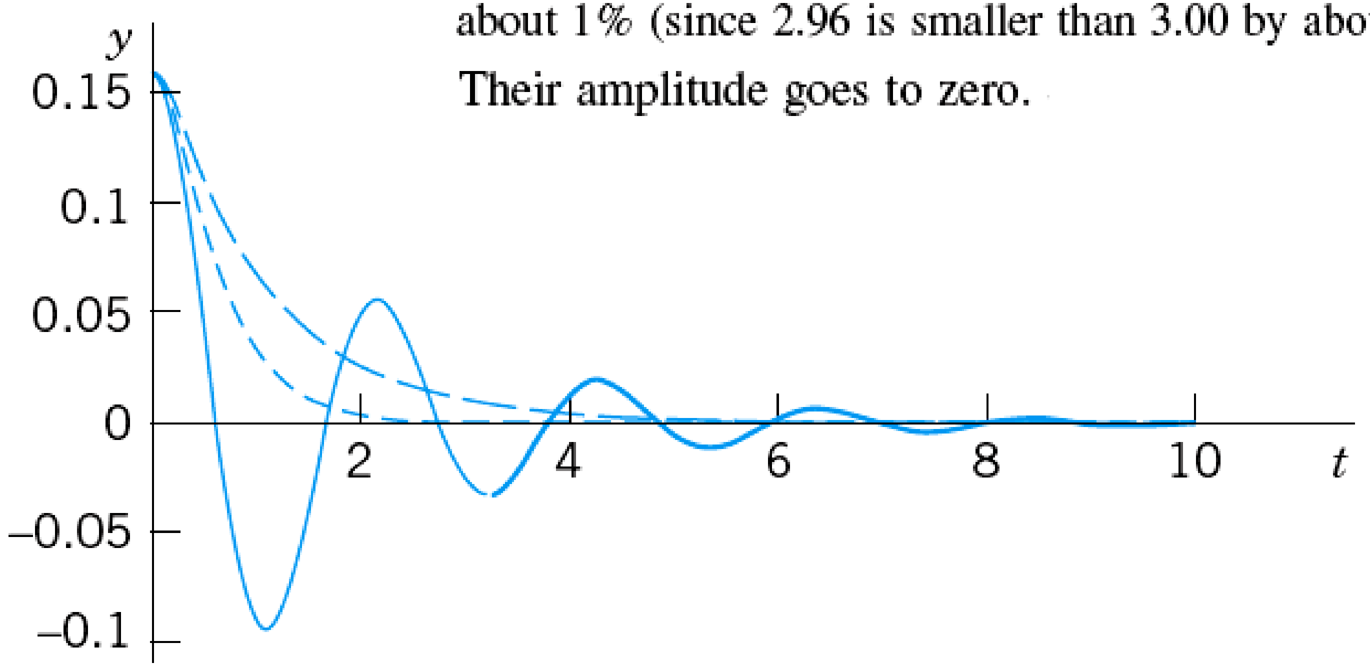


Fig. 40. The three solutions in Example 2