Chapter 5.1

Composite Functions

5.1.1 Formation of a Composite Function

DEFINITION **Composite Function**

Figure 2

EXAMPLE 1 Evaluating a Composite Function

5.1.2 Find the Domain of a Composite Function

EXAMPLE 2 Finding a Composite Function and Its Domain

EXAMPLE 3 Finding the Domain of $f \circ g$

EXAMPLE 4 Finding a Composite Function and Its Domain

EXAMPLE 5 Showing That Two Composite Functions Are Equal

5.1 Assess Your Understanding

Skill Building

In determining the domain of a composite function $(f \circ g)(x) = f(g(x))$ keep the following two thoughts in mind about the input *x*.

1. Any *x* not in the domain of *g* must be excluded.
2. Any *x* for which $g(x)$ is not in the domain of *f* must be excluded.

In Problems 21–28, find the domain of the composite function $f \circ g$ **:**

21. $f(x) = \dfrac{3}{x-1}; \; g(x) = \dfrac{2}{x}$

Solution: We can write $Dom(f) = \{x \mid x \neq 1\}$ and $Dom(g) = \{x \mid x \neq 0\}$.

Now compute $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{2}{x}\right) = \dfrac{3}{\frac{2}{x} - 1} = \dfrac{3x}{2-x}$

Hence, $Dom(f \circ g) = \{x \mid x \neq 0, \; x \neq 2\}$.

23. $f(x)=\dfrac{x}{x-1}$; $g(x)=-\dfrac{4}{x}$

Solution: We can write $Dom(f)=\{x\,|\,x\neq 1\}$ and $Dom(g)=\{x\,|\,x\neq 0\}$.

Now compute $(f\circ g)(x)=f(g(x))=f\left(-\dfrac{4}{x}\right)=\dfrac{-\dfrac{4}{x}}{-\dfrac{4}{x}-1}=\dfrac{4}{x+4}$

Hence, $Dom(f\circ g)=\{x\,|\,x\neq 0,\ x\neq -4\}$.

25. $f(x)=\sqrt{x}$; $g(x)=2x+3$

Solution: We can write $Dom(f)=\{x\,|\,x\geq 0\}$ and $Dom(g)=$ set of all real numbers.

Now compute $(f\circ g)(x)=f(g(x))=f(2x+3)=\sqrt{2x+3}$

We know $\sqrt{2x+3}$ is defined if $2x+3\geq 0 \Rightarrow x\geq -\dfrac{3}{2}$

Hence, $Dom(f\circ g)=\{x\,|\,x\geq -3/2\}$.

27. $f(x)=x^2+1$; $g(x)=\sqrt{x-1}$

Solution: We can write $Dom(f)=$ set of all real numbers and $Dom(g)=\{x\,|\,x\geq 1\}$.

Now compute $(f\circ g)(x)=f(g(x))=f(\sqrt{x-1})=\left(\sqrt{x-1}\right)^2+1=x$

Hence, $Dom(f\circ g)=\{x\,|\,x\geq 1\}$.

In Problems 29-44, for the given functions f and g, find:

(a) $f\circ g$ ***(b)*** $g\circ f$ ***(c)*** $f\circ f$ ***(d)*** $g\circ g$

State the domain of each composite function.

29. $f(x)=2x+3$; $g(x)=3x$

Solution

(a) We can write $Dom(f)=$ set of all real numbers and $Dom(g)=$ set of all real numbers.
Now compute $(f\circ g)(x)=f(g(x))=f(3x)=2(3x)+3=6x+3$
Hence, $Dom(f\circ g)=$ set of all real numbers.

(b) We can write $Dom(f)=$ set of all real numbers and $Dom(g)=$ set of all real numbers.
Now compute $(g\circ f)(x)=g(f(x))=g(2x+3)=3(2x+3)=6x+9$
Hence, $Dom(g\circ f)=$ set of all real numbers.

(c) We can write $Dom(f)=$ set of all real numbers
Now compute $(f\circ f)(x)=f(f(x))=f(2x+3)=2(2x+3)+3=4x+9$
Hence, $Dom(f\circ f)=$ set of all real numbers.

(d) We can write $Dom(g)=$ set of all real numbers
Now compute $(g\circ g)(x)=g(g(x))=g(3x)=3(3x)=9x$.
Hence, $Dom(g\circ g)=$ set of all real numbers.

35. $f(x)=\dfrac{3}{x-1};\ g(x)=\dfrac{2}{x}$

Solution

(a) We can write $Dom(f)=\{x\mid x\neq 1\}$ and $Dom(g)=\{x\mid x\neq 0\}$.

Now compute $(f\circ g)(x)=f(g(x))=f\left(\dfrac{2}{x}\right)=\dfrac{3}{\frac{2}{x}-1}=\dfrac{3x}{2-x}$

Hence, $Dom(f\circ g)=\{x\mid x\neq 0,\ x\neq 2\}$.

(b) We can write $Dom(f)=\{x\mid x\neq 1\}$ and $Dom(g)=\{x\mid x\neq 0\}$.

Now compute $(g\circ f)(x)=g(f(x))=g\left(\dfrac{3}{x-1}\right)=\dfrac{2}{\frac{3}{x-1}}=\dfrac{2(x-1)}{3}$

Hence, $Dom(g\circ f)=\{x\mid x\neq 1\}$.

(c) We can write $Dom(f)=\{x\mid x\neq 1\}$

Now compute $(f\circ f)(x)=f\left(\dfrac{3}{x-1}\right)=\dfrac{3}{\frac{3}{x-1}-1}=\dfrac{3(x-1)}{4-x}$

Hence, $Dom(f\circ f)=\{x\mid x\neq 1, x\neq 4\}$.

(d) We can write $Dom(g)=\{x\mid x\neq 0\}$.

Now compute $(g\circ g)(x)=g\left(\dfrac{2}{x}\right)=\dfrac{2}{\frac{2}{x}}=x.$

Hence, $Dom(g\circ g)=\{x\mid x\neq 0\}$.

39. $f(x)=\sqrt{x};\ g(x)=2x+3$

Solution

(a) We can write $Dom(f)=\{x\mid x\geq 0\}$ and $Dom(g)=$ set of all real numbers.

Now compute $(f\circ g)(x)=f(g(x))=f(2x+3)=\sqrt{2x+3}$

We know $\sqrt{2x+3}$ is defined if $2x+3\geq 0\Rightarrow x\geq -\dfrac{3}{2}$

Hence, $Dom(f\circ g)=\{x\mid x\geq -3/2\}$.

(b) We can write $Dom(f)=\{x\mid x\geq 0\}$ and $Dom(g)=$ set of all real numbers.

Now compute $(g\circ f)(x)=g(f(x))=g(\sqrt{x})=2\sqrt{x}+3$

Hence, $Dom(g\circ f)=\{x\mid x\geq 0\}$.

(c) We can write $Dom(f)=\{x\mid x\geq 0\}$.

Now compute $(f\circ f)(x)=f(\sqrt{x})=\sqrt{\sqrt{x}}=(x^{1/2})^{1/2}=x^{1/4}=\sqrt[4]{x}$

Hence, $Dom(f\circ f)=\{x\mid x\geq 0\}$.

(d) We can write $Dom(g) =$ set of all real numbers.

Now compute $(g \circ g)(x) = g(2x+3) = 2(2x+3)+3 = 4x+9.$

Hence, $Dom(g \circ g) =$ set of all real numbers.

41. $f(x) = x^2 + 1;\ g(x) = \sqrt{x-1}$

Solution

(a) We can write $Dom(f) =$ set of all real numbers and $Dom(g) = \{x \mid x \geq 1\}$.

Now compute $(f \circ g)(x) = f(g(x)) = f(\sqrt{x-1}) = \left(\sqrt{x-1}\right)^2 + 1 = x$

Hence, $Dom(f \circ g) = \{x \mid x \geq 1\}$.

(b) We can write $Dom(f) =$ set of all real numbers and $Dom(g) = \{x \mid x \geq 1\}$.

Now compute $(g \circ f)(x) = g(f(x)) = g(x^2+1) = \sqrt{x^2+1-1} = \sqrt{x^2} = x$

Hence, $Dom(g \circ f) =$ set of all real numbers.

(c) We can write $Dom(f) =$ set of all real numbers.

Now compute $(f \circ f)(x) = f(x^2+1) = (x^2+1)^2 + 1 = x^4 + 2x^2 + 2$

Hence, $Dom(f \circ f) =$ set of all real numbers.

(d) We can write $Dom(g) = \{x \mid x \geq 1\}$.

Now compute $(g \circ g)(x) = g(\sqrt{x-1}) = \sqrt{\sqrt{x-1}-1}$. Now $\sqrt{\sqrt{x-1}-1}$ is defined if $\sqrt{x-1} - 1 \geq 0$ or if $\sqrt{x-1} \geq 1$ or if $x - 1 \geq 1$ or if $x \geq 2$.

Hence, $Dom(g \circ g) = \{x \mid x \geq 2\}$.

Chapter 5.2

One-to-One Functions; Inverse Functions

5.2.1 Determine Whether a Function Is One-to-One

DEFINITION

Figure 8

EXAMPLE 1 Determining Whether a Function Is One-to-One

THEOREM

Horizontal-line Test

If every horizontal line intersects the graph of a function in at most one point, then *f* is one-to-one.

Figure 9

EXAMPLE 2 Using the Horizontal-line Test

Figure 10

THEOREM

A function that is increasing on an interval *I* is a one-to-one function on *I* and a function that is decreasing on an interval *I* is a one-to-one function on *I*.

5.2.2 Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs

DEFINITION

EXAMPLE 4 Finding the Inverse of a Function Defined by a Set of Ordered Pairs

Figure 11

Figure 12

EXAMPLE 5 Verifying Inverse Functions

EXAMPLE 6 Verifying Inverse Functions

5.2.3 Obtain the Graph of the Inverse Function from the Graph of the Function

Figure 13

THEOREM

Figure 14

EXAMPLE 7 Graphing the Inverse Function

Figure 15

5.2.4 Find the Inverse of a Function Defined by an Equation

EXAMPLE 8 How to Find the Inverse Function

Figure 16

Procedure for Finding the Inverse of a One-to-One Function

EXAMPLE 9 Finding the Inverse Function

If a function is not one-to-one, it has no inverse function. Sometimes, though, an appropriate restriction on the domain of such a function will yield a new function that is one-to-one. Then the function defined on the restricted domain has an inverse function. Let's look at an example of this common practice.

EXAMPLE 10 Finding the Inverse of a Domain-restricted Function

Figure 17

5.2 Assess Your Understanding

Concepts and Vocabulary

5. If x_1 and x_2 are two different inputs of a function *f*, then *f* is one-to-one if $\underline{f(x_1) \neq f(x_2)}$.

6. If every horizontal line intersects the graph of a function *f* at no more than one point, then *f* is a *one-to-one* function.

7. If *f* is a one-to-one function and $f(3)=8$, then $\underline{f^{-1}(8)=3}$.

8. If f^{-1} denotes the inverse of a function *f*, then the graph of *f* and f^{-1} are symmetric with respect to the line $\underline{y=x}$.

9. If the domain of a one-to-one function *f* is $[4, \infty)$, the range of its inverse, f^{-1}, is $[4, \infty)$.

Skill Building

In Problems 11–18, determine whether the function is one-to-one.

15. $\{(2,6),(-3,6),(4,9),(1,10)\}$

Solution: Let $f=\{(2,6),(-3,6),(4,9),(1,10)\}$.

Since two different inputs 2 and -3 in *f* have the same output 6, *f* is not one-to-one.

17. $\{(0,0),(1,1),(2,16),(3,81)\}$

Solution: Let $\{(0,0),(1,1),(2,16),(3,81)\}$

The function *f* is one-to-one because no two different inputs have the same output.

In Problems 25–32, find the inverse of each one-to-one function. State the domain and the range of each inverse function.

29. $\{(-3,5),(-2,9),(-1,2),(0,11),(1,-5)\}$

Solution: Let $f=\{(-3,5),(-2,9),(-1,2),(0,11),(1,-5)\}$.

Then $f^{-1}=\{(5,-3),(9,-2),(2,-1),(11,0),(-5,1)\}$.

Therefore, $Dom(f^{-1})=\{5,9,2,11,-5\}$ and $Range(f^{-1})=\{-3,-2,-1,0,1\}$.

In Problems 33–42, verify that the functions f and g are inverses of each other by showing that and Give any values of x that need to be excluded from the domain of f and the domain of g.

35. $f(x)=4x-8;\ g(x)=\dfrac{x}{4}+2$

Solution: Compute $f(g(x))=f\left(\dfrac{x}{4}+2\right)=4\left(\dfrac{x}{4}+2\right)-8=x+8-8=x$

$$g(f(x))=g(4x-8)=\frac{4x-8}{4}+2=x-2+2=x$$

Therefore, by definition, *f* and *g* are inverses of each other.

39. $f(x)=\frac{1}{x};\ g(x)=\frac{1}{x}$

Solution: First find $Dom(f)=\{x\,|\,x\neq 0\}=Dom(g)$.

Now compute $f(g(x))=f\left(\frac{1}{x}\right)=\frac{1}{1/x}=x$ provided $x\neq 0$

$$g(f(x))=g\left(\frac{1}{x}\right)=\frac{1}{1/x}=x \text{ provided } x\neq 0$$

Therefore, by definition, *f* and *g* are inverses of each other.

41. $f(x)=\frac{2x+3}{x+4};\ g(x)=\frac{4x-3}{2-x}$

Solution: First find $Dom(f)=\{x\,|\,x\neq -4\}$ and $Dom(g)=\{x\,|\,x\neq 2\}$.

Now compute $f(g(x))=f\left(\frac{4x-3}{2-x}\right)=\frac{2\times\frac{4x-3}{2-x}+3}{\frac{4x-3}{2-x}+4}=\frac{8x-6+6-3x}{4x-3+8-4x}=\frac{5x}{5}=x$ provided $x\neq 2$

$$g(f(x))=g\left(\frac{2x+3}{x+4}\right)=\frac{4\times\frac{2x+3}{x+4}-3}{2-\frac{2x+3}{x+4}}=\frac{8x+12-3x-12}{2x+8-2x-3}=\frac{5x}{5}=x \text{ provided } x\neq -4$$

Therefore, by definition, *f* and *g* are inverses of each other.

EXAMPLE 9 $f(x)=\frac{2x+1}{x-1},\ x\neq 1$

Solution:

Step I: Replace $f(x)$ with y in $f(x)=\frac{2x+1}{x-1}$ to obtain $y=\frac{2x+1}{x-1}$.

Step II: Interchange the variables *x* and y to obtain $x=\frac{2y+1}{y-1}$ which defines f^{-1} implicitly

Step III: To find the explicit form of f^{-1}solve $x=\frac{2y+1}{y-1}$ for *y* to get

$$x(y-1)=2y+1\Rightarrow xy-x=2y+1\Rightarrow xy-2y=1+x\Rightarrow y=\frac{1+x}{x-2}$$

Hence, $f^{-1}(x)=\frac{1+x}{x-2}$.

Check

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right) = \frac{1+\dfrac{2x+1}{x-1}}{\dfrac{2x+1}{x-1}-2} = \frac{x-1+2x+1}{2x+1-2x+2} = x \quad \text{provided } x \neq 1$$

$$f(f^{-1}(x)) = f\left(\frac{1+x}{x-2}\right) = \frac{2\times\dfrac{1+x}{x-2}+1}{\dfrac{1+x}{x-2}-1} = \frac{2+2x+x-2}{1+x-x+2} = x \quad \text{provided } x \neq 2$$

Exploration

We noticed that if $f(x) = \dfrac{2x+1}{x-1}$ then $f^{-1}(x) = \dfrac{1+x}{x-2}$.

Comparing the vertical and horizontal asymptotes of f and f^{-1}, we get

(i) The vertical asymptote of f is $x=1$ and the horizontal asymptote of f is $y=2$ because as $x \to -\infty$ or as $x \to +\infty$, the improper integral $f(x) = \dfrac{2x+1}{x-1} \to 2$ for

$$f(x) = \frac{2x+1}{x-1} = \frac{2(x-1)+3}{x-1} = 2 + \frac{3}{x-1} \to 2 \text{ since } \frac{3}{x-1} \to 0 \text{ as } x \to -\infty \text{ or as } x \to +\infty$$

(ii) The vertical asymptote of f^{-1} is $x=2$ and the horizontal asymptote of f^{-1} is $y=1$.

In Problems 49-60, the function f is one-to-one. Find its inverse and check your answer. Graph f, f^{-1} and $y = x$ on the same coordinate axes.

49. $f(x) = 3x$

Solution: Replace $f(x)$ with y in $f(x) = 3x$ to obtain $y = 3x$.

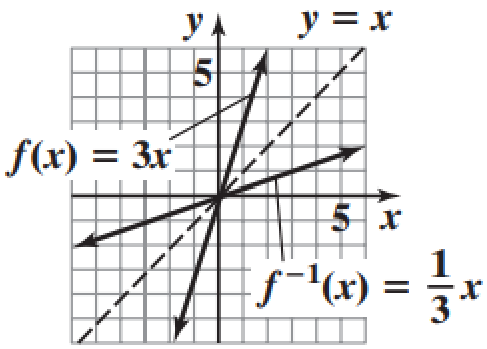
Interchanging the variables x and y to obtain $x = 3y$ and then solve this equation for y to obtain $y = \dfrac{1}{3}x$.

Hence, $f^{-1}(x) = \dfrac{1}{3}x$.

Check

$$f^{-1}(f(x)) = f^{-1}(3x) = \frac{1}{3}(3x) = x$$

and $$f(f^{-1}(x)) = f\left(\frac{1}{3}x\right) = 3\left(\frac{1}{3}x\right) = x$$



51. $f(x) = 4x + 2$

Solution: Replace $f(x)$ with y in $f(x) = 4x + 2$ to obtain $y = 4x + 2.$

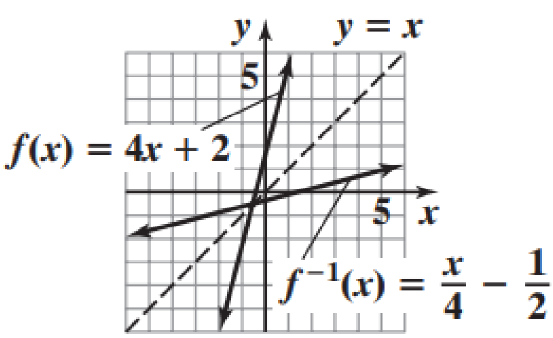
Interchanging the variables *x* and *y* to obtain $x = 4y + 2$ and then solve this equation for *y* to obtain

$$x = 4y + 2 \Rightarrow 4y = x - 2 \Rightarrow y = \frac{1}{4}(x-2)$$

Hence, $f^{-1}(x) = \frac{1}{4}(x-2).$

<u>Check</u>

$f^{-1}(f(x)) = f^{-1}(4x+2) = \frac{1}{4}[(4x+2)-2] = x$ and $f(f^{-1}(x)) = f\left(\frac{1}{4}(x-2)\right) = 4\left(\frac{1}{4}(x-2)\right) + 2 = x$



53. $f(x) = x^3 - 1$

Solution: Replace $f(x)$ with y in $f(x) = x^3 - 1$ to obtain $y = x^3 - 1.$

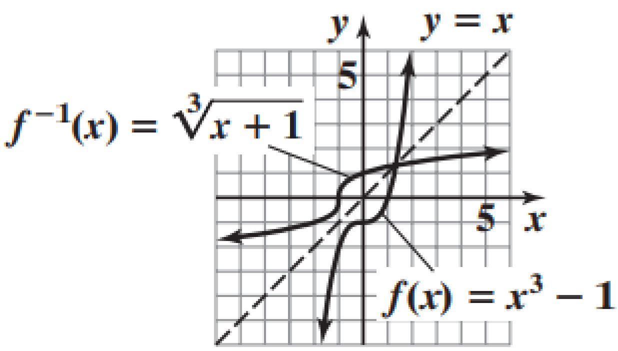
Interchanging the variables *x* and *y* to obtain $x = y^3 - 1$ and then solve this equation for *y* to obtain

$$x = y^3 - 1 \Rightarrow y^3 = x + 1 \Rightarrow y = \sqrt[3]{x+1}$$

Hence, $f^{-1}(x) = \sqrt[3]{x+1}.$

Check

$f^{-1}(f(x)) = f^{-1}(x^3-1) = \sqrt[3]{x^3-1+1} = x$ and $f(f^{-1}(x)) = f\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = x$



55. $f(x) = x^2 + 4,\ x \geq 0$

Solution: Replace $f(x)$ with y in $f(x) = x^2 + 4$ to obtain $y = x^2 + 4$.

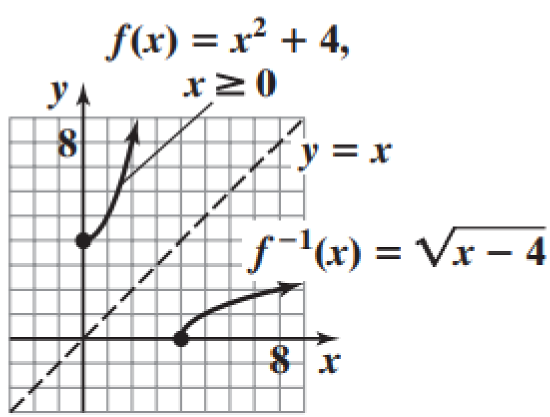
Interchanging the variables x and y to obtain $x = y^2 + 4$ and then solve this equation for y to obtain

$$x = y^2 + 4 \Rightarrow y^2 = x - 4 \ \Rightarrow y = \sqrt{x-4} \text{ provided } x \geq 0$$

Hence, $f^{-1}(x) = \sqrt{x-4}$.

Check

$f^{-1}(f(x)) = f^{-1}(x^2+4) = \sqrt{(x^2+4)-4} = x$ and $f(f^{-1}(x)) = f\left(\sqrt{x-4}\right) = \left(\sqrt{x-4}\right)^2 + 4 = x$



57. $f(x)=\frac{4}{x}$

Solution: Replace $f(x)$ with y in $f(x)=\frac{4}{x}$ to obtain $y=\frac{4}{x}$.

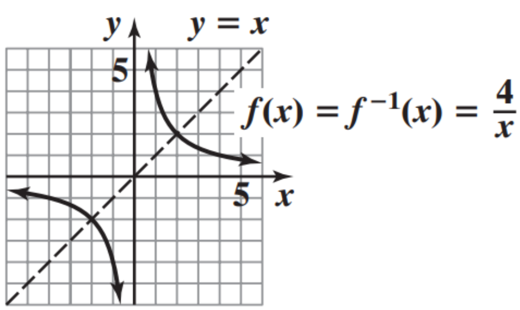
Interchanging the variables x and y to obtain $x=\frac{4}{y}$ and then solve this equation for y to obtain

$$x=\frac{4}{y}\Rightarrow y=\frac{4}{x}$$

Hence, $f^{-1}(x)=\frac{4}{x}$.

Check

$$f^{-1}(f(x))=f^{-1}\left(\frac{4}{x}\right)=\frac{4}{4/x}=x \quad \text{and} \quad f(f^{-1}(x))=f\left(\frac{4}{x}\right)=\left(\frac{4}{4/x}\right)=x$$



59. $f(x)=\frac{1}{x-2}$

Solution: Replace $f(x)$ with y in $f(x)=\frac{1}{x-2}$ to obtain $y=\frac{1}{x-2}$.

Interchanging the variables x and y to obtain $x=\frac{1}{y-2}$ and then solve this equation for y to obtain

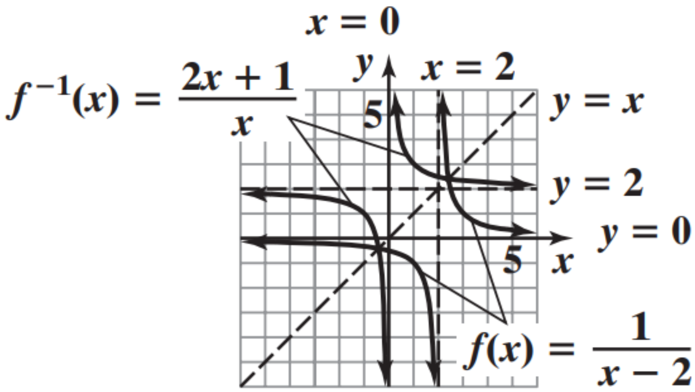
$$x=\frac{1}{y-2}\Rightarrow y-2=\frac{1}{x}\Rightarrow y=\frac{1}{x}+2=\frac{1+2x}{x}$$

Hence, $f^{-1}(x)=\frac{1+2x}{x}$.

Check

$$f^{-1}(f(x))=f^{-1}\left(\frac{1}{x-2}\right)=\frac{1+2\times\frac{1}{x-2}}{\frac{1}{x-2}}=\frac{x-2+2}{1}=x$$

$$f(f^{-1}(x)) = f\left(\frac{1+2x}{x}\right) = \frac{1}{\frac{1+2x}{x}-2} = \frac{x}{1+2x-2x} = x$$



Chapter 5.3

Exponential Functions

5.3.1 Evaluate Exponential Functions

EXAMPLE 1 Using a Calculator to Evaluate Powers of 2

THEOREM Laws of Exponents

If s, t, a and b are real numbers with $a > 0$ and $b > 0$, then

(a) $a^s \cdot a^t = a^{s+t}$ (b) $(a^s)^t = a^{st}$ (c) $(ab)^s = a^s \cdot b^s$

(d) $1^s = 1$ (e) $a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s$ (f) $a^0 = 1$

Introduction to Exponential Growth

Definition

An **exponential function** is a function of the form

$$f(x) = Ca^x$$

where a is a positive real number, $a > 0$, $a \neq 1$ and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor** and the number C is called the **initial value**.

Theorem

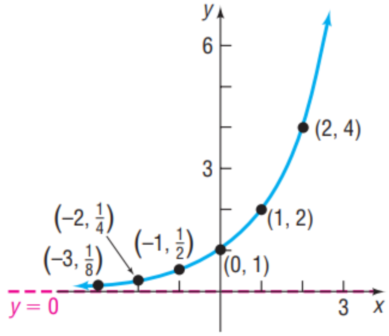
For an exponential function $f(x) = Ca^x$, where $a > 0$ and $a \neq 1$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)$$

5.3.2 Graph Exponential Functions

EXAMPLE 3 Graphing an Exponential Function

Figure 18



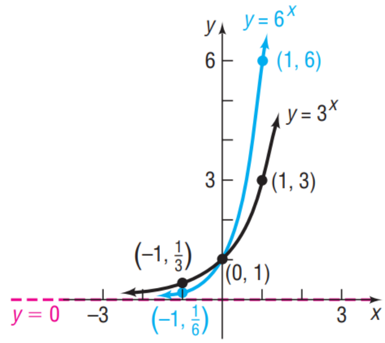
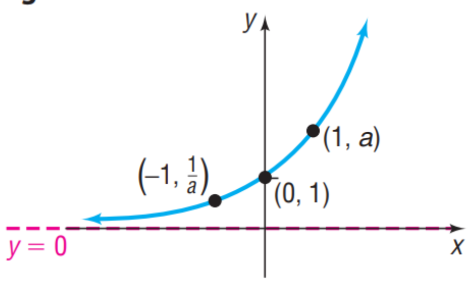


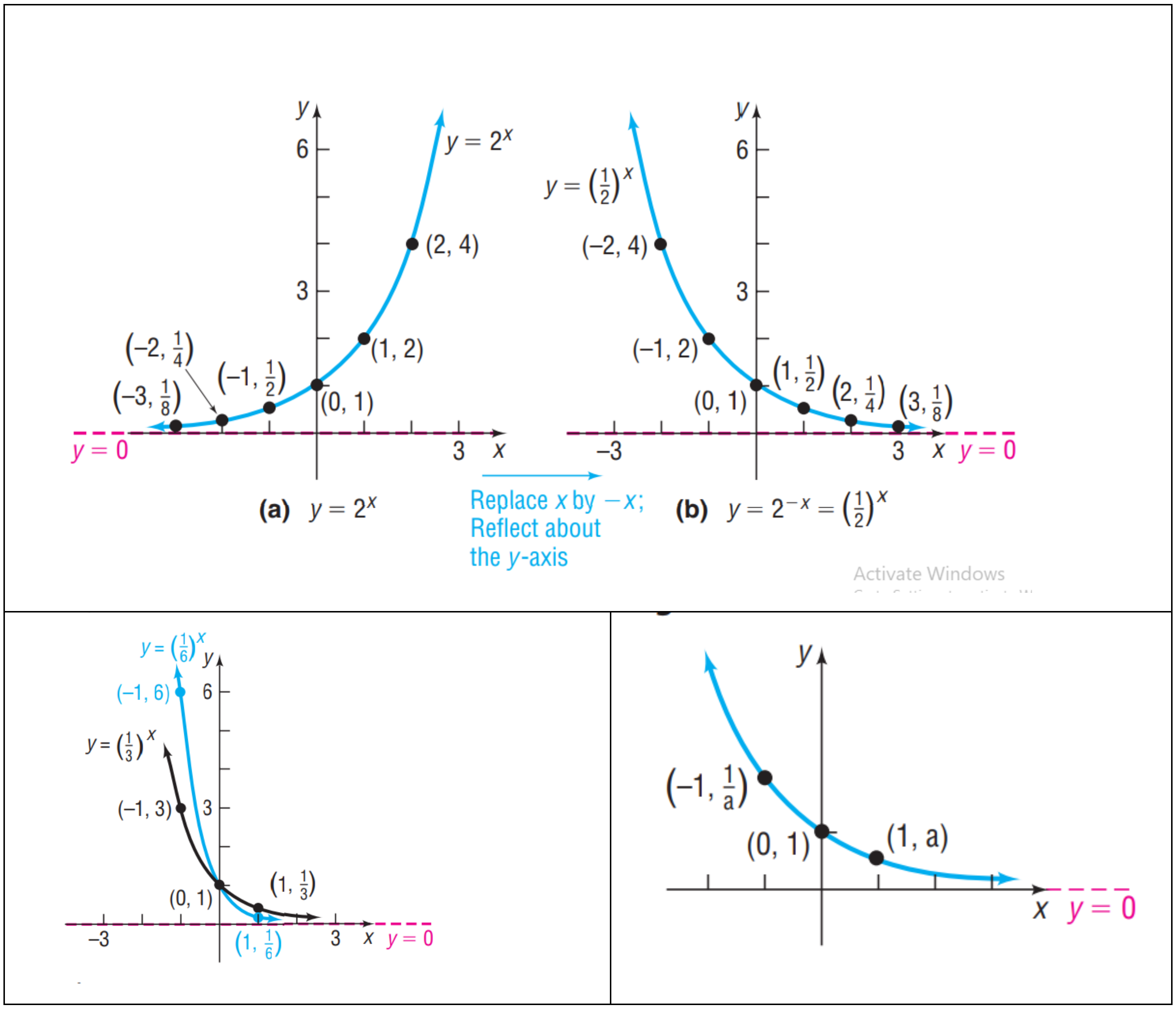
Figure 21



Properties of the Exponential Function $f(x) = a^x,\ a > 1$

1. The domain of f is the set of all real numbers and the range is the set of positive real numbers.
2. There are no x-intercepts and the y-intercept is 1.
3. The x-axis ($y = 0$) is a horizontal asymptote as $x \to -\infty$, i.e. $\lim\limits_{x \to -\infty} a^x = 0$.
4. $f(x) = a^x$, for $a > 1$, is an increasing function and is one-to-one.
5. The graph of f contains the points $\left(-1, \dfrac{1}{a}\right)$, $(0,1)$ and $(1, a)$.
6. The graph of f is smooth and continuous with no corners or gaps.

EXAMPLE 4 Graphing an Exponential Function



Properties of the Exponential Function $f(x) = a^x,\ 0 < a < 1$

1. The domain of *f* is the set of all real numbers and the range is the set of positive real numbers.
2. There are no *x*-intercepts and the *y*-intercept is 1.
3. The *x*-axis ($y = 0$) is a horizontal asymptote as $x \to \infty$, i.e. $\lim_{x\to\infty} a^x = 0$.
4. $f(x) = a^x$, for $0 < a < 1$, is a decreasing function and is one-to-one.
5. The graph of *f* contains the points $\left(-1, \dfrac{1}{a}\right)$, $(0,1)$ and $(1, a)$.
6. The graph of *f* is smooth and continuous with no corners or gaps.

EXAMPLE 5 Graphing Exponential Functions Using Transformations

5.3.3 Define the Number *e*

Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter *e*. The letter *e* was chosen to represent this irrational number in honor of the Swiss mathematician Leonhard Euler (pronounced "oiler") (1707-1783).

Definition

The number *e* is defined as the number that the expression

$$\left(1+\frac{1}{n}\right)^n$$

approaches as $n \to \infty$.

In calculus, this is expressed using limit notation as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

EXAMPLE 6 Graphing Exponential Functions Using Transformations

5.3.4 Solve Exponential Equations

Definition

Equations that involve terms of the form a^x, where $a > 0$ and $a \neq 1$, are referred to as **exponential equations.** Such equations can sometimes be solved by appropriately applying the Laws of Exponents with the property given by

$$\boxed{\text{If } a^u = a^v \text{ then } u = v}$$

EXAMPLE 7 Solving Exponential Equations

Solve each exponential equation

(a) $3^{x+1} = 81$ (b) $4^{2x-1} = 8^{x+3}$

Solution:

(a) $3^{x+1} = 81 \Rightarrow 3^{x+1} = 3^4 \Rightarrow x+1 = 4 \Rightarrow x = 4-1 \Rightarrow x = 3$

Hence the solution is $x = 3$.

(b) $4^{2x-1} = 8^{x+3} \Rightarrow (2^2)^{2x-1} = (2^3)^{x+3} \Rightarrow 2^{4x-2} = 2^{3x+9}$

Using formula, we get

$$4x-2 = 3x+9 \Rightarrow 4x-3x = 9+2 \Rightarrow x = 11$$

Hence the solution is $x = 11$.

EXAMPLE 8 Solving an Exponential Equation

Solve the exponential equation $e^{-x^2} = (e^x)^2 \cdot \dfrac{1}{e^3}$

Solution: Using Laws of Exponents, we can write

$$e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3} = e^{2x} \cdot e^{-3} \Rightarrow e^{-x^2} = e^{2x-3}$$

$$\Rightarrow -x^2 = 2x-3 \Rightarrow x^2 + 2x - 3 = 0$$

$$\Rightarrow (x+3)(x-1) = 0 \Rightarrow \quad x = -3 \text{ or } \quad x = 1$$

Hence the solution is $(x, y) = (-3, 1)$.

SUMMARY Properties of the Exponential Function

5.3 Assess Your Understanding

Skill Building

In Problems 41–52, use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

In Problems 53–60, begin with the graph of [Figure 27] and use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

Solve the following exponential equations:

63. $2^{-x} = 16$ **65.** $\left(\dfrac{1}{5}\right)^x = \dfrac{1}{25}$ **67.** $2^{2x-1} = 4$ **69.** $3^{x^3} = 9^x$

71. $8^{-x+14} = 16^x$ **73.** $3^{x^2-7} = 27^{2x}$ **75.** $4^x \cdot 2^{x^2} = 16^2$ **77.** $e^x = e^{3x+8}$

79. $e^{x^2} = e^{3x} \cdot \dfrac{1}{e^2}$