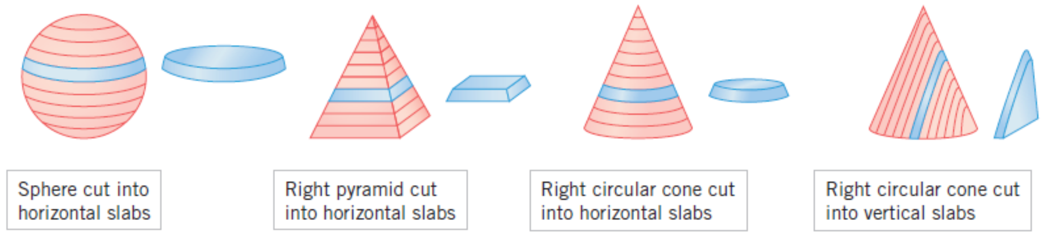
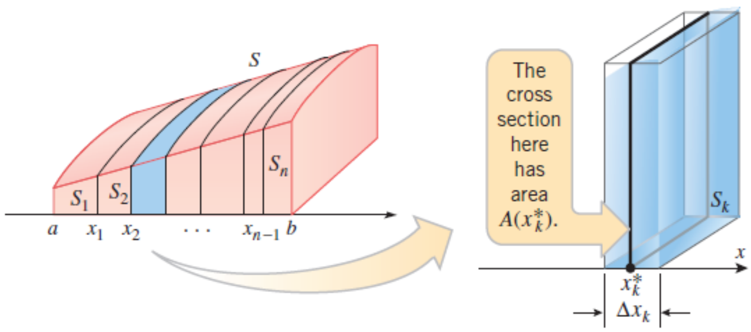
SECTION 6.2: VOLUME BY SLICING: DISC AND WASHER METHOD





Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

In summary, we have the following result.

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\, dy \tag{4}$$

provided $A(y)$ is integrable.

Summary:

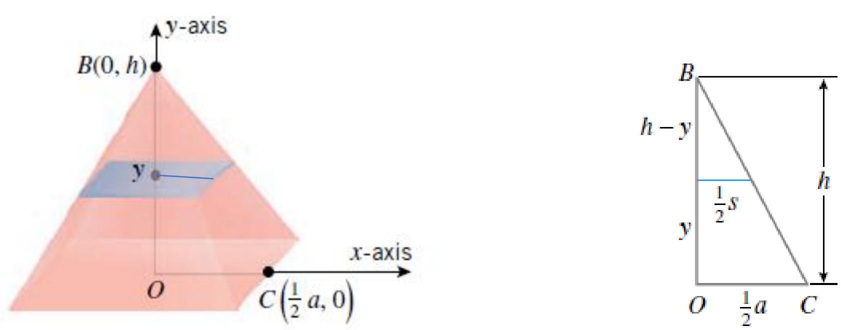
The volume of the solid S, **by slicing,** is given by

$V = \int_a^b (\text{Area of the cross} - \text{section})\ dx$ Or $V = \int_c^d (\text{Area of the cross} - \text{section})\ dy$.

Lecture 11, Date: 18th August, 2020

Example 1

Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length $\boldsymbol{a}$.



(a) (b)

Project the Pyramid along the y-axis, placing the height of the pyramid along the axis with the center of the base at the origin.

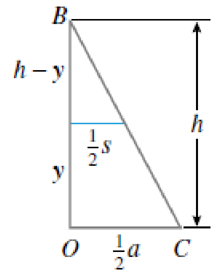
Now, take any cross-section of the pyramid at any $y,\ 0 \le y \le h$. The cross-section is a square that is perpendicular to the y −axis.

Volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy \ldots\ldots\ldots (1)$

Let the length of a side of the cross-section be s. Then the area of the cross-section is

$A(y) = s^2 \ldots\ldots\ldots (2)$

By the similar triangle property on the triangles



(b)

$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \Rightarrow \frac{s}{a} = \frac{h-y}{h}$. Hence $s = \frac{a}{h}(h-y)$

From equation (2): Area of the cross-section is $A(y) = \left[\frac{a}{h}(h-y)\right]^2 = \frac{a^2}{h^2}(h-y)^2$

Then volume $V = \int_0^h (\text{Area of the cross} - \text{section})\ dy$

$$= \int_0^h \frac{a^2}{h^2}(h-y)^2\ dy$$

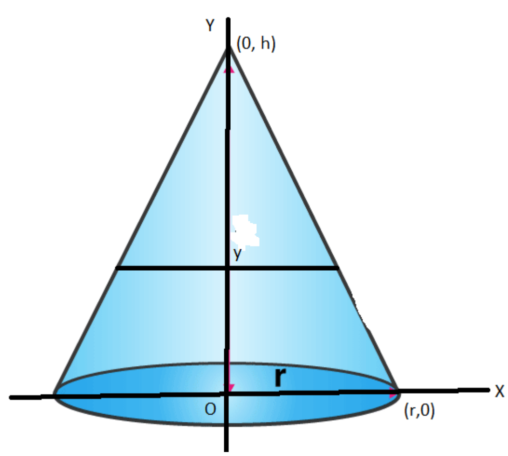
$$= \frac{a^2}{h^2}\int_0^h [h^2 - 2hy + y^2]\ dy$$

$$= \frac{a^2}{h^2}\left[h^2y - hy^2 + \frac{1}{3}y^3\right]_0^h$$

$$V = \frac{1}{3}a^2h \quad unit^3.$$

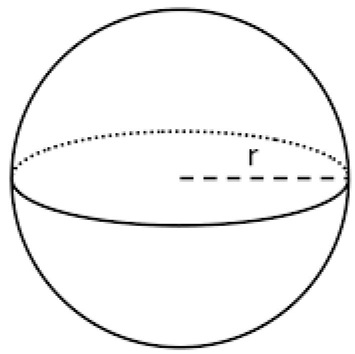
Example 2 [homework]

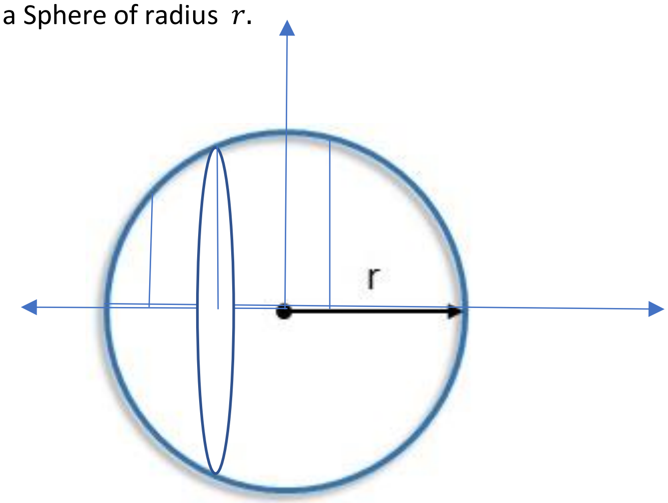
Derive the formula for the volume of a **right circular cone** whose altitude is h and whose base is a circle of radius r.



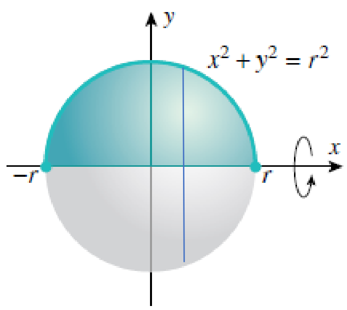
Example 3

Derive the formula for the volume of a Sphere of radius r.





The projection of the sphere on the xy −plane is a disk of radius r, bounded by the circle $x^2 + y^2 = r^2$.
But $x^2 + y^2 = r^2$ is not a function. The upper-half circle represents a function of x given by
$y = \sqrt{r^2 - x^2}$.

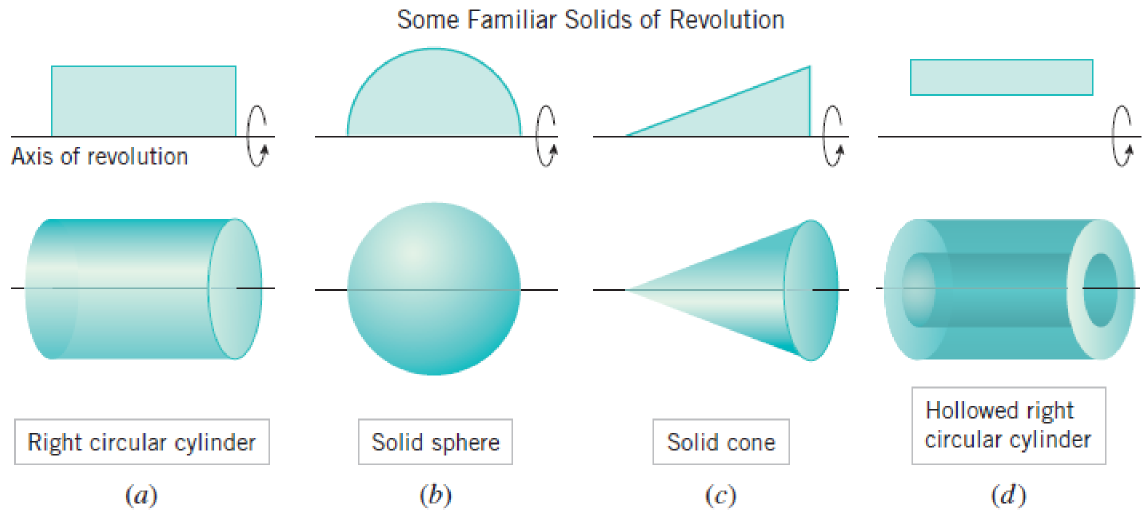


Interval = $[-r, r]$, The cross-section at any x, $-r \le x \le r$, is a disk of radius, **say r_1** .
Here $r_1 = \sqrt{r^2 - x^2} - 0 = \sqrt{r^2 - x^2}$.
Area of the cross-section $A(x) = \pi r_1^2 = \pi(r^2 - x^2)$.
volume $V = \int_a^b (\text{Area of the cross} - \text{section})\ dx$
$V = \int_{-r}^{r} \pi(r^2 - x^2)\ dx = \frac{4}{3}\pi r^3$. [complete !!]

[Note: $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$
Lower-half circle$y = -\sqrt{r^2 - x^2}$,
Upper-half circle $y = \sqrt{r^2 - x^2}$]

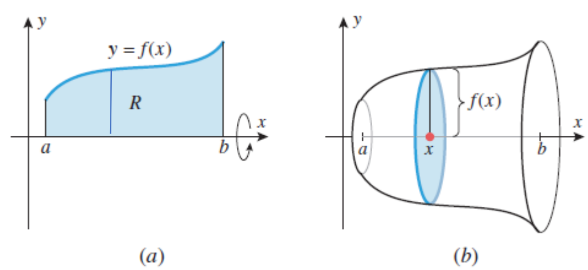
SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type



VOLUMES BY DISKS PERPENDICULAR TO THE x −AXIS

Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x −axis, and on the sides by the vertical lines $x = a$ and $x = b$. Then the volume of the solid of revolution that is generated by revolving the region R about the x −axis is given by

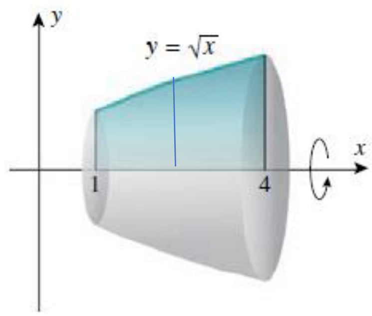


The cross-section is a disk of radius $r = f(x)$ that is perpendicular to the x −axis. Hence, the volume is

$$V = \int_a^b \pi[f(x)]^2 dx$$

Example 4 Find the volume of the solid that is obtained when the region **under the curve** $y = \sqrt{x}$ **over** the interval $[\mathbf{1}, \mathbf{4}]$ is revolved about the x —axis.

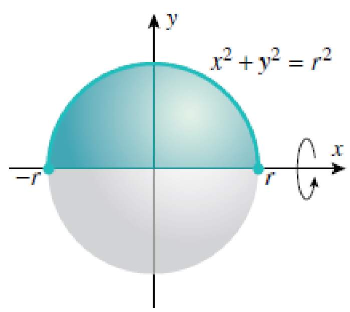
Solution:



The Volume $V = \int_a^b \pi[f(x)]^2 dx = \int_1^4 \pi[\sqrt{x}]^2 dx$ complete!!!

Homework

Example 5 Find the volume of the solid generated by revolving the circle $x^2 + y^2 = r^2$ about the x —axis.

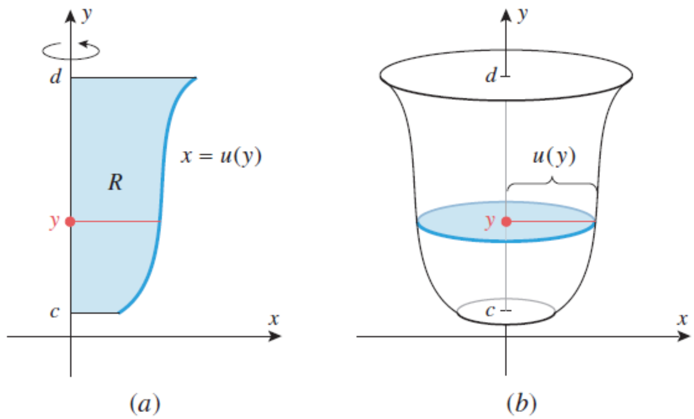


VOLUMES BY DISKS PERPENDICULAR TO THE y —AXIS

Let $x = u(y)$ be continuous and nonnegative on $[c, d]$, and let R be the region that is bounded on the right by $x = u(y)$, on the left by the y —axis, and at the bottom and top by the horizontal lines $y = c$ and $y = d$. Then the volume of the solid of revolution that is generated by revolving the region R about the y —axis is given by

$$V = \int_c^d \pi[u(y)]^2 dy$$

Note that the cross-section is a disk of radius $r = u(y)$ that is perpendicular to the y —axis.



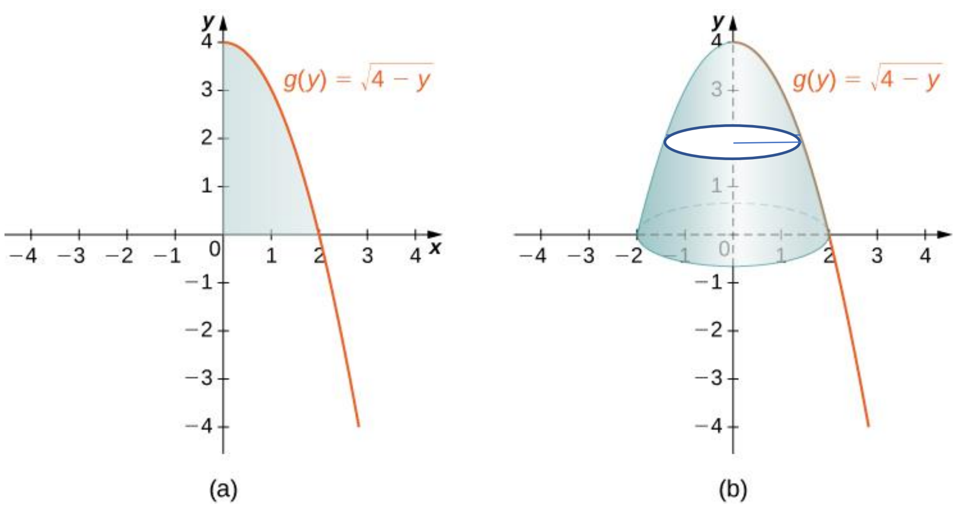
Disks

Information comes from the origin $\boldsymbol{R}$. Interval, radius, height come from the region.

Example 6 [Complete !!!]

Find the volume of the solid that is obtained when the region R is revolved about the $\boldsymbol{y}$ **−axis,** where R is bounded by the curve $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{y}) = \sqrt{4-y}$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]

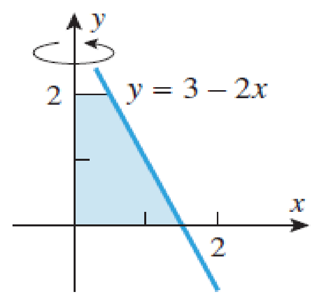


$V = \int_c^d \pi[u(y)]^2 dy = \int_0^4 \pi\left[\sqrt{4-y}\right]^2 dy$ complete!

Example 7

Find the volume of the solid that is obtained when the region R revolved about the y −axis, where R is bounded by the curve $y = 3 - 2x$, $y = 2$, $y = 0$ and $x = 0$.

Solution: [Information comes from the origin]



Interval, Shape of the cross-section, Area of the cross-section, Variable of the function that gives you the area of the cross-section.

$$\boldsymbol{y = 3 - 2x}\ \textbf{. That is,} \qquad \boldsymbol{x = \frac{1}{2}\,(3 - y)}$$

Radius of the cross-section is $= \frac{1}{2}\,(3 - y)$

$$I = [0, 2], \text{Cross} - \text{section is a disk,} \qquad A(y) = \pi \left(\frac{3}{2} - \frac{y}{2}\right)^2$$

Lecture 12, Date: 23rd August, 2020

Midterm

Date: 30th August, 2020
Section:8 at 11:20 AM
Section:9 at 1:00 PM

Syllabus: Chapter 7: sections 7.1 – 7.5
Chapter 6: sections 6.1 – **6.3**

Policy: We have to follow the following policies, otherwise midterm is worth 0

1) Mic and video must be on
2) No headphone / blue-tooth
3) No late submission
4) You must be on the grid
5) **Cam must be on during uploading.**
6) **Copy work is worth 00.**

Discussion Class:

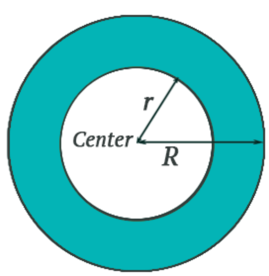
Section: 8

Questions and Answers Session, Weekly.
Tuesday, 8pm.

Section: 9

Questions and Answers Session, Weekly.
Wednesday, 8pm.

Annulus or ring or washer



Inner radius = Radius of the inner circle $= \boldsymbol{r}$

Outer radius = Radius of the outer circle $= \boldsymbol{R}$

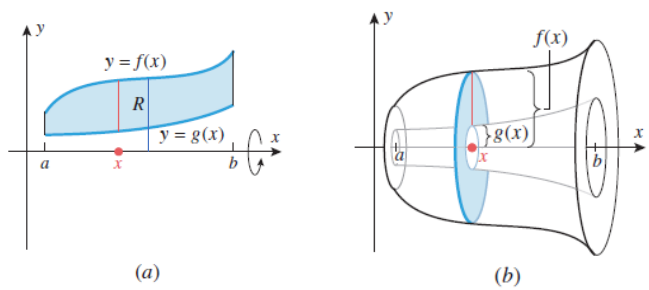
Area of the Annulus $= \boldsymbol{\pi R^2 - \pi r^2}$.

In, Washer method by slicing, the cross-section would be an annulus.

VOLUMES BY WASHERS PERPENDICULAR TO THE x –AXIS

Let f and g be continuous and **non-negative** on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. The volume of the solid of revolution that is generated by revolving the region R about the x –axis is given by

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

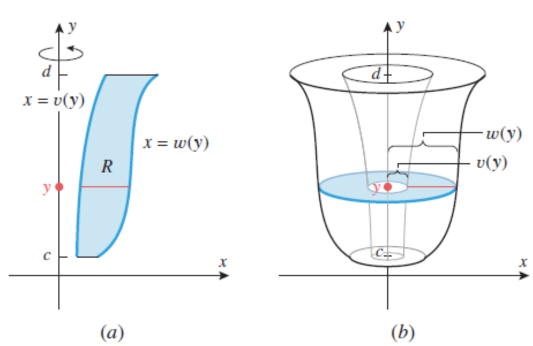


VOLUMES BY WASHERS PERPENDICULAR TO THE y –AXIS

Let w and v be continuous and nonnegative on $[c, d]$, and suppose $w(y) \geq v(y)$ for all y in the interval $[c, d]$. Let R be the region that is bounded on the right by $x = w(y)$, on the left by $x = v(y)$, and at the bottom and top by the lines $y = c$ and $y = d$. The volume of the solid of revolution that is generated by revolving the region R about the y –axis is given by

$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy$$

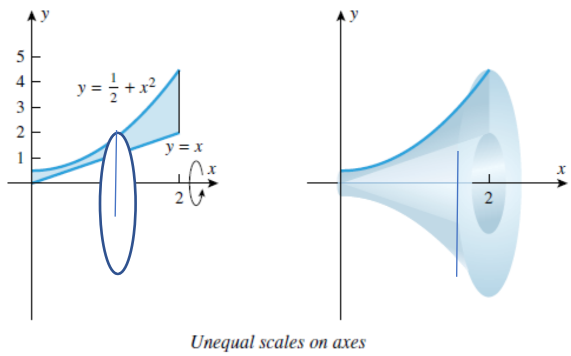
Washers



Washers

Examples

Example 1 Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x —axis.



Region defined by f and g

(a)

The resulting solid of revolution

(b)

The interval $I = [0, 2]$

The inner radius $r = x$

The outer radius $R = \frac{1}{2} + x^2$

The area of the cross section at any x: $A(x) = \pi R^2 - \pi r^2 = \pi\left[\left(\frac{1}{2} + x^2\right)^2 - x^2\right]$

The volume of the solid is

$$V = \int_0^2 A(x)\ dx = \int_0^2 \pi\left[\left(\frac{1}{2} + x^2\right)^2 - x^2\right] dx$$

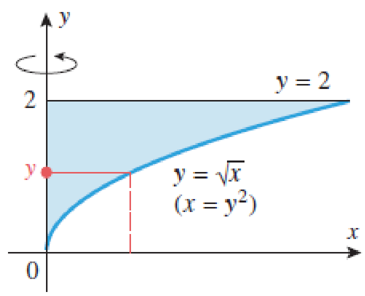
$$= \pi \int_0^2 \left[\frac{1}{4} + x^2 + x^4 - x^2\right] dx$$

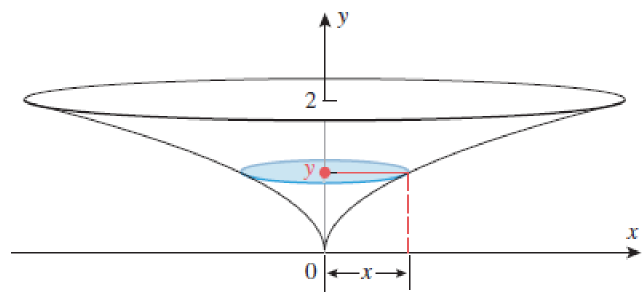
$$= \pi \int_0^2 \left[\frac{1}{4} + x^4\right] dx = \pi\left[\frac{1}{4}(2) + \frac{1}{5}(2)^5\right]$$

$$= \frac{1}{20}[10 + 128]\pi = \frac{69}{10}\pi \quad unit^3$$

Example 2

(a) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is evolved about the y –axis.





The cross-section is a disk (revolving about a boundary).

Radius of the disk= $R = y^2$

Interval $I = [0, 2]$

Area of the cross-section $A(y) = \pi R^2 = \pi (y^2)^2 = \pi y^4$

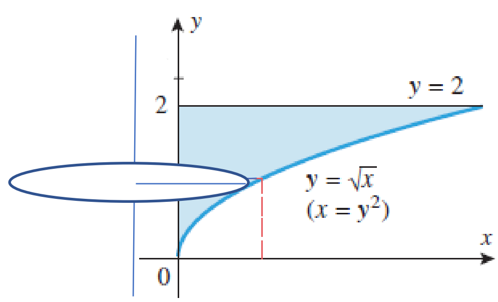
The volume of the solid is

$V = \int_0^2 A(y)\, dy = \int_0^2 \pi y^4\, dy$ Complete!!

Example 2

(b) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is evolved about the line $\boldsymbol{x = -1}$.

Solution: R: $y = \sqrt{x} \Rightarrow x = y^2$, $x = 0$, $y = 2$. Axis of the solid: $\boldsymbol{x = -1}$.



Interval $= [0, 2]$

Inner radius $r = 0 - (-1) = 1$

Outer radius $R = y^2 - (-1) = y^2 + 1$

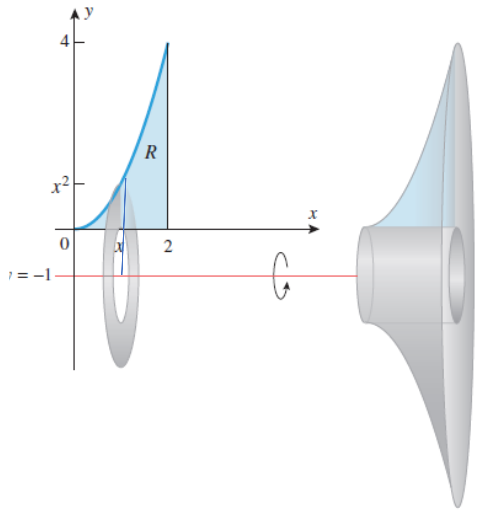
$V = \int_0^2 A(y)\, dy = \int_0^2 \pi[(y^2+1)^2 - 1^2]\, dy$; Please complete!!

Example 3

Find the volume of the solid generated when the region **under** the curve $y = x^2$ **over** the interval $[0, 2]$ is rotated about the line $\boldsymbol{y} = \mathbf{-1}$.

Solution: The region under the curve $y = x^2$ over the interval $[0, 2]$, that is, R is bounded by

$$R: \quad y = x^2, \qquad y = 0, \quad x = 0, \qquad x = 2.$$



Interval $= [0, 2]$

Inner radius $r = 0 - (-1) = 1$

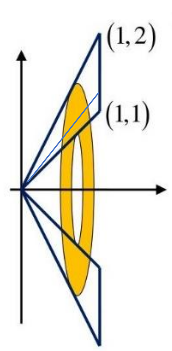
Outer radius $R = x^2 - (-1) = x^2 + 1$

$V = \int_0^2 A(x)\ dx \ = \int_0^2 \pi[(x^2 + 1)^2 - 1^2]\ dx$; Please complete!!

Example 4

Find the volume of the solid generated when the region R revolves about the x −axis, where R is bounded by the lines $y = 2x, \qquad y = x$ **over** the interval $[0, 1]$.

Solution:



$$V = \int_0^1 A(x)\ dx \ = \int_0^1 \pi[(2x)^2 - x^2]\ dx$$

$$= \pi \int_0^1 [4x^2 - x^2]\ dx$$

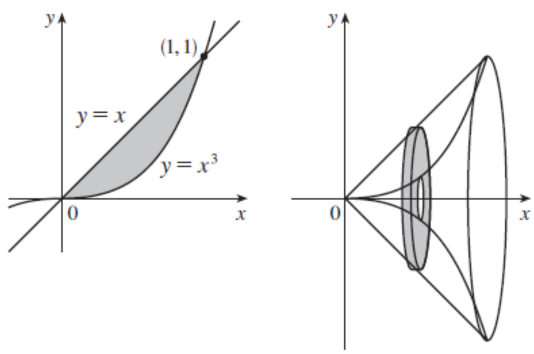
$$= \pi \int_0^1 3x^2\ dx$$

$$= \pi\ unit^3$$

Example 5

Find the volume of the solid generated when the region R revolves about the x −axis, where R is the region in the first quadrant bounded by the lines $y = x^3$ and $y = x$.

Solution: Given region R: $y = x^3$ and $y = x$.



To find the interval, set : $x^3 = x \ \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = 0, 1, -1.$

Interval $I = [0, 1]$.

Inner Radius $r = x^3$

Outer Radius $R = x$

Area of the cross section $A(x) = \pi[x^2 - (x^3)^2] = \pi\ [x^2 - x^6]$

$$V = \int_0^1 A(x)\ dx \ = \int_0^1 \pi[x^2 - x^6]\, dx = \frac{4}{21}\pi\ \ unit^3$$

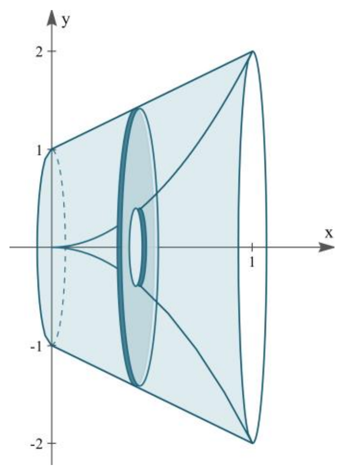
Example 6

Find the volume of the solid generated when the region R revolves about the x −axis, where R is the region **in the first quadrant** bounded by the lines $y = 2x^2$, $x = 0$ and $y = x + 1$.

Solution: Given region R: $y = x^2$, $y = x + 1$ and $x = 0$.

For the interval, find the point of intersection in first quadrant. Set $2x^2 = x + 1 \Rightarrow 2x^2 - x - 1 = 0$.

Hence, $x = -\frac{1}{2}$, 1. Interval $I = [0,\ 1]$



Complete!

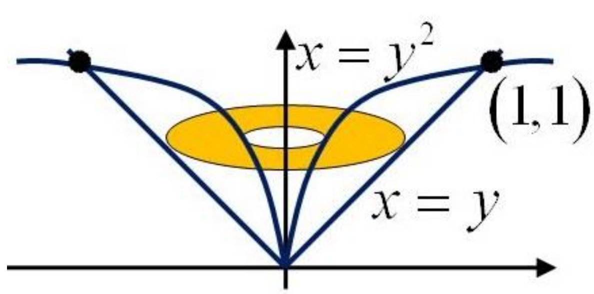
Example 7

Find the volume of the solid generated when the region R revolves about the y −axis, where R is the region bounded by $y = \sqrt{x}$ and $y = x$.

Solution: Given region R: $y = \sqrt{x}$ and $y = x$.

For the interval, set $\sqrt{x} = x \Rightarrow x^2 - x = 0$.

Hence, $x = 0,\ 1$. Interval $I = [0,\ 1]$



DAY-13
DATE: 25th August, 2020

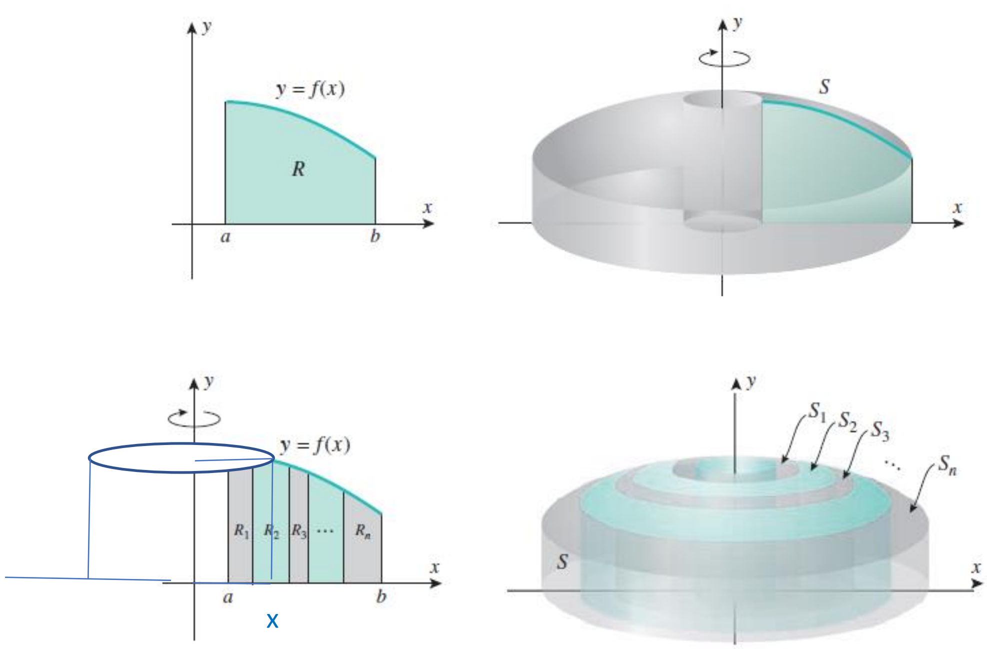
SECTION 6.3: VOLUME BY CYLINDRICAL SHELLS METHOD

1) (a) In disk/washer, when the region R is revolved about the $x-$axis/line parallel to $x-$axis, then the cross-section is perpendicular to the $x-$axis.

(b) In cylindrical shells method, when the region R is revolved about the $x-$axis/line parallel to $x-$axis, then the cross-section is perpendicular to the $y-$axis.

2) (a) In disk/washer, when the region R is revolved about the $y-$axis/line parallel to $y-$axis, then the cross-section is perpendicular to the $y-$axis.

(b) In cylindrical shells method, when the region R is revolved about the $y-$axis/line parallel to $y-$axis, then the cross-section is perpendicular to the $x-$axis.



(a) (b)

The cross-section is a cylinder. Radius of the cylinder $\boldsymbol{r = x - 0 = x}$,

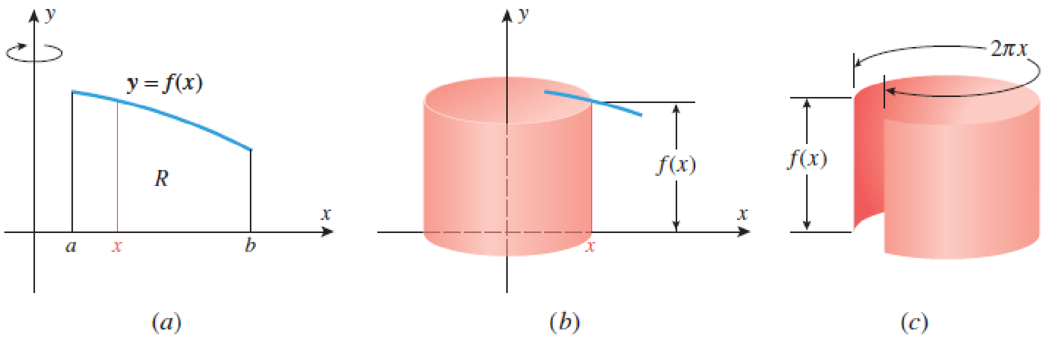
Height of the cylinder $\boldsymbol{H = f(x) - 0 = f(x)}$.

The cross-section is a cylinder, and the area of the cross-section (which is the surface of the cylinder) is $\boldsymbol{A(x) = 2\pi r H = 2\pi x f(x)}$.

VOLUMES BY CYLINDRICAL SHELLS PERPENDICULAR TO THE x –AXIS

Let f be continuous and **non-negative** on $[a, b], 0 \leq a < b$. Let R be the region that is bounded above by $y = f(x)$, below by the $x -$ axis and on the sides by the vertical lines $x = a$ and $x = b$. The volume of the solid of revolution that is generated by revolving the region R about the y –axis is given by, **using the cylindrical shells method**,

$$V = \int_a^b A(x)\ dx = \int_a^b 2\pi\, x\, f(x)\ dx\ .$$



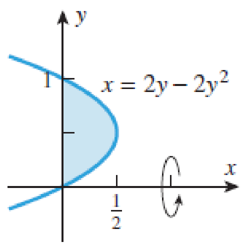
VOLUMES BY CYLINDRICAL SHELS PERPENDICULAR TO THE y –AXIS

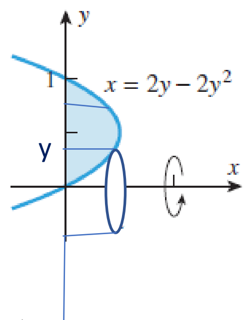
Let $x = g(y)$ be continuous and **non-negative** on $[c, d],\ 0 \leq c < d$. Let R be the region that is bounded on the right by $x = g(y)$, on the left by the $y -$ axis and at the bottom and top by the horizontal lines $y = c$ and $y = d$. The volume of the solid of revolution that is generated by revolving the region R about the x –axis is given by, **using the cylindrical shells method,**

$$V = \int_c^d A(y)\ dy = \int_c^d 2\pi\, y\, g(y)\ dy$$

Example 1

Use cylindrical shells to find the volume of the solid generated when the region enclosed between $x = 2y - 2y^2$ and $x = 0$ by revolving about the x −axis.





To find the interval, set $2y - 2y^2 = 0$. That is, $y = 0, 1$.

Radius $r = y - 0 = y$ and Height $H = g(y) = 2y - 2y^2$, $I = [0, 1]$

Area of the cross-section $A(y) = 2\pi\, rH = 2\pi\, y\, g(y) = 2\pi\, y\, (2y - 2y^2)$

The volume of the solid is,

$$V = \int_c^d A(y)\; dy = \int_0^1 2\pi\, y\, g(y)\; dy$$

$$= \int_0^1 2\pi\, y\, (2y - 2y^2)\; dy$$

$$= 2\pi \int_0^1 [2y^2 - 2y^3]\; dy$$

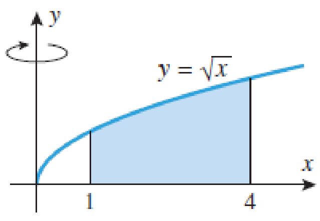
$$= 2\pi \left[\frac{2}{3}y^3 - \frac{2}{4}y^4\right]_0^1$$

$$= 2\pi \left(\frac{2}{3} - \frac{1}{2}\right)$$

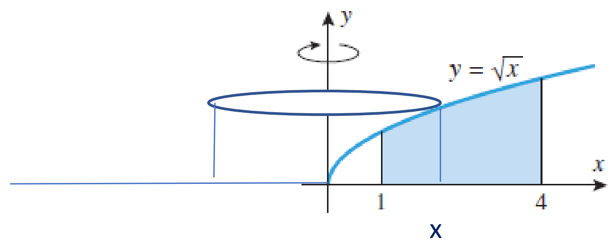
$$= 2\pi \frac{1}{6} = \frac{\pi}{3} \quad unit^3$$

Example 2

Use cylindrical shells to find the volume of the solid generated when the region is bounded by $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x −axis is revolved about the y −axis.



(a)



Radius of the cylinder $r = x - 0 = x$

Height of the cylinder $H = \sqrt{x} - 0 = \sqrt{x}$,

Interval $I = [1,\ 4]$

The volume of the solid is,

$$V = \int_a^b A(x)\ dx = \int_1^4 2\pi\, x\, f(x)\ dx$$

$$= \int_1^4 2\pi\, x\, \sqrt{x}\, dx \quad = 2\pi \int_1^4 x^{\frac{3}{2}}\ dx$$

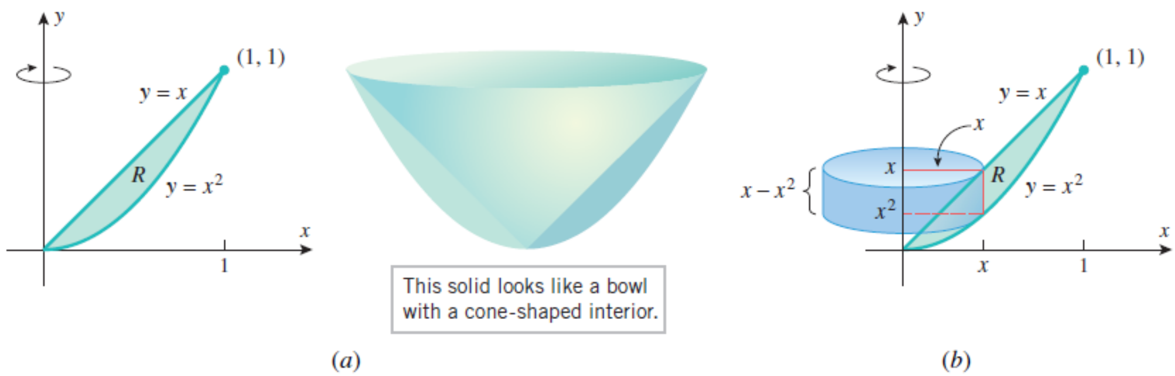
$$= 2\pi \left(\frac{2}{5}\right) \left[4^{\frac{5}{2}} - 1^{\frac{5}{2}}\right]$$

$$= \frac{4\pi}{5}\ [32 - 1]$$

$$= \frac{124}{5}\ \pi \quad unit^3$$

Example 3 **Homework !!**

Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the y −axis



Radius r=?

Height H=?

Interval I=?

Area of the cross-section A =?

V=?

Example 4

Use cylindrical shells to find the volume of the solid generated when the region R **under** $y = x^2$ **over** the interval $[\mathbf{0}, \mathbf{2}]$ is revolved about the line $\boldsymbol{y} = -\mathbf{1}$.

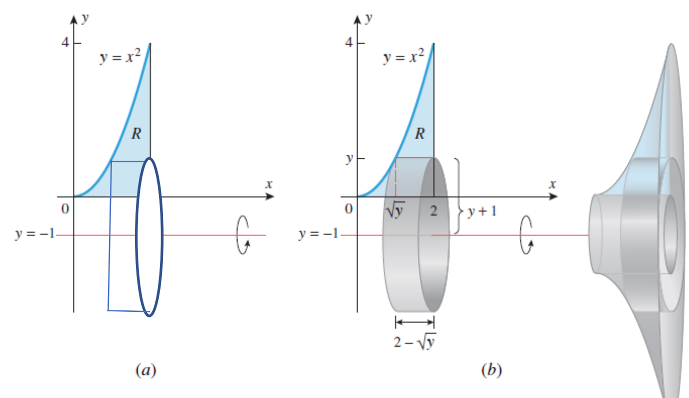


Figure 6.3.9

Solution:

Radius $r = y - (-1) = y + 1$, Height $H = 2 - \sqrt{y}$, Interval $I = [0, 4]$

Area of the cross-section $A = 2\pi r H = 2\pi(y + 1)(2 - \sqrt{y})$

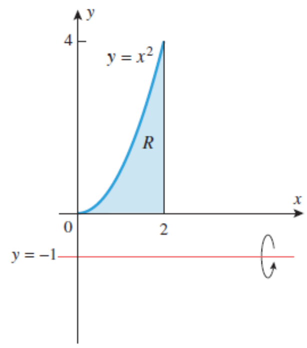
The volume of the solid, using Cylindrical Shells, is

$$V = \int_0^4 [2\pi(y+1)(2-\sqrt{y})]\,dy = 2\pi \int_0^4 [2y - y^{\frac{3}{2}} - y^{\frac{1}{2}} + 2]\,dy$$

Complete !!

Example 5

Let R be the region that is **under** $y = x^2$ **over** the interval $[\mathbf{0}, \mathbf{2}]$ **.**



Use cylindrical shells to find the volume of the solid generated when the region R is revolved

(a) about the line $\boldsymbol{y} = -\mathbf{2}$.

(b) about the line $\boldsymbol{y} = \mathbf{0}$.

(c) about the line $\boldsymbol{y} = \mathbf{4}$.

(d) about the line $\boldsymbol{y} = \mathbf{5}$.

(e) about the line $\boldsymbol{x} = \mathbf{0}$.

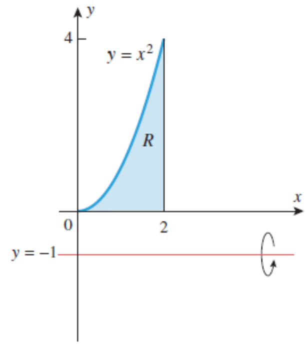
(f) about the line $\boldsymbol{x} = \mathbf{2}$.

(g) about the line $\boldsymbol{x} = -\mathbf{1}$.

(h) about the line $\boldsymbol{x} = \mathbf{3}$.

Example 6

Let R be the region that is **under** $y = x^2$ **over** the interval $[\mathbf{0}, \mathbf{2}]$ **.**



Use disk/washer to find the volume of the solid generated when the region R is revolved

(a) about the line $\boldsymbol{y} = -\mathbf{2}$.

(b) about the line $\boldsymbol{y} = \mathbf{0}$.

(c) about the line $\boldsymbol{y} = \mathbf{4}$.

(d) about the line $\boldsymbol{y} = \mathbf{5}$.

(e) about the line $\boldsymbol{x} = \mathbf{0}$.

(f) about the line $\boldsymbol{x} = \mathbf{2}$.

(g) about the line $\boldsymbol{x} = -\mathbf{1}$.

(h) about the line $\boldsymbol{x} = \mathbf{3}$.