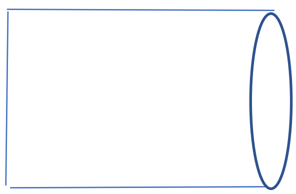
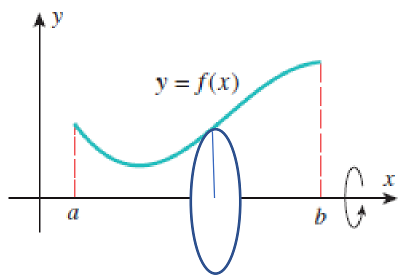
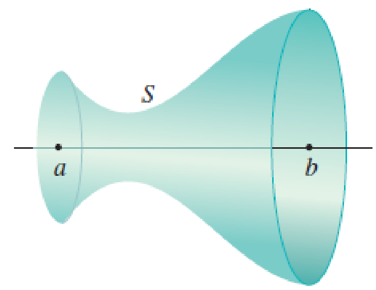
Lecture 15

Recall 6.4: The length of a curve $y = f(x)$, $[a, b]$, $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

6.5: Area of the Surface of Revolution



Surface Area of a cylindre $= 2\pi r h$, $r =$ radius, $h =$ height/length

Definitions:

Definition 1: Let $y = f(x)$ be a non-negative smooth curve on the interval $[a, b]$. Then the area of the surface of revolution formed by revolving the portion of the graph of $f(x)$ about the x —axis is defined by

$$\text{Surface Area} = \int_a^b 2\pi\, f(x)\, \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \mathbf{2\pi\, f(x)} \sqrt{\mathbf{1} + \left[\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right]^{\mathbf{2}}}\, dx$$

Steps: Given a smooth function $\boldsymbol{y = f(x)}$; $\boldsymbol{x = a}$ **to** $\boldsymbol{x = b}$

Step 1: Find $\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \boldsymbol{f'(x)}$

Step 2: Find $\left(\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right)^{\mathbf{2}} = [\boldsymbol{f'(x)}]^{\mathbf{2}}$

Step 3: Find $\mathbf{1} + \left(\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right)^{\mathbf{2}} = \mathbf{1} + [\boldsymbol{f'(x)}]^{\mathbf{2}}$

Step 4: Find $\sqrt{\mathbf{1} + \left(\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right)^{\mathbf{2}}} = \sqrt{\mathbf{1} + [\boldsymbol{f'(x)}]^{\mathbf{2}}}$ **. Here we should try, if possible, to write** $\mathbf{1} + [\boldsymbol{f'(x)}]^{\mathbf{2}}$ **as a perfect square so that we can cancel the square root.**

Step 5: Evaluate the integral to find the area of the surface of revolution: Surface area is given by

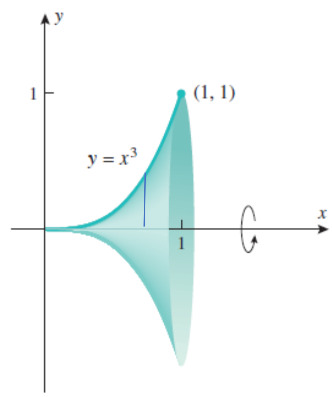
$$S_A = \int_a^b 2\pi f(x)\, \sqrt{1 + [f'(x)]^2}\, dx$$

Exercise :1

Find the area of the surface of revolution that is generated by revolving $\boldsymbol{y = x^3}$ from $x = 0$ to $x = 1$ about the $\boldsymbol{x}$ −axis.

Solution: We know that the area of the surface of revolution is:

$$S_A = \int_a^b 2\pi f(x) \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\ dx \ \ldots\ldots\ldots (1)$$



Here given $y = x^3, \quad 0 \le x \le 1.$

Then we get,

$$\frac{dy}{dx} = 3x^2 \ \Rightarrow \left(\frac{dy}{dx}\right)^2 = (3x^2)^2 = 9x^4$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + 9x^4$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 9x^4}$$

From equation (1), the area of the surface of revolution:

$$S_A = \int_a^b 2\pi f(x) \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\ dx$$

$$= \int_0^1 2\pi\, x^3 \ \sqrt{1 + 9x^4}\, dx$$

$$= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx$$

set $\boldsymbol{u = 1 + 9x^4}$. then $\dfrac{du}{dx} = 36\, x^3$, that is, $\dfrac{1}{36}\, du = x^3 dx$.

Now, if $x = 0$, then $\boldsymbol{u = 1 + 9(0)^4 = 1 + 0 = 1}$

if $x = 1$, then $\boldsymbol{u = 1 + 9(1)^4 = 1 + 9 = 10}$

$$= 2\pi \int_1^{10} \sqrt{u}\, \frac{1}{36}\, du$$

$$= \frac{2\pi}{36} \int_1^{10} u^{\frac{1}{2}}\, du$$

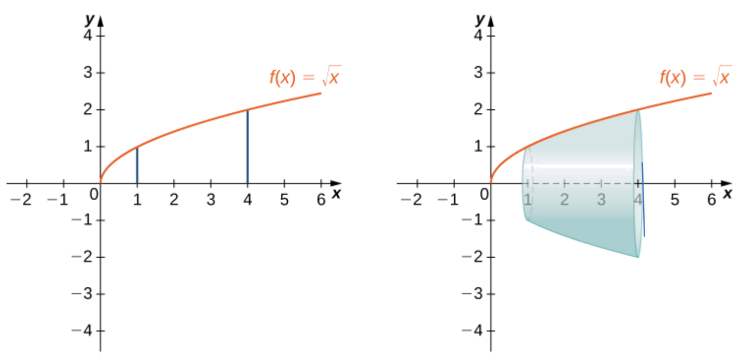
$$= \frac{\pi}{18}\, \frac{2}{3} \left[u^{\frac{3}{2}} \right]_1^{10} = \frac{\pi}{27}\, \left[10\sqrt{10} - 1 \right]\ unit^2$$

Exercise :2

Find the area of **the surface of revolution** generated by revolving $y = f(x) = \sqrt{x}$ from $x = 1$ to $x = 4$ about the x –axis.

Solution: We know that the area of the surface of revolution:

$$S_A = \int_a^b 2\pi f(x)\, \sqrt{1 + [f'(x)]^2}\, dx$$



(a) (b)

Solution: Given

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$\Rightarrow f'(x) = \frac{1}{2}\, x^{-\frac{1}{2}}$

$$\Rightarrow \left(f'(x)\right)^2 = \left(\frac{1}{2}\, x^{-\frac{1}{2}}\right)^2 = \frac{1}{4}\; x^{-1} = \frac{1}{4}\;\frac{1}{x} = \frac{1}{4x}$$

$$\Rightarrow 1 + \left(f'(x)\right)^2 = 1 + \frac{1}{4x} = \frac{4x+1}{4x}$$

$$\Rightarrow \sqrt{1 + \left(f'(x)\right)^2} = \sqrt{\frac{4x+1}{4x}} = \frac{\sqrt{4x+1}}{\sqrt{4x}} = \frac{\sqrt{4x+1}}{2\sqrt{x}} \qquad \text{for } 1 \le x \le 4.$$

So, Surface Area,

$$S_A = \int_a^b 2\pi f(x)\,\sqrt{1+[f'(x)]^2}\;dx$$

$$= \int_1^4 2\pi\sqrt{x}\;\frac{\sqrt{4x+1}}{2\sqrt{x}}\;dx$$

$$= \pi\int_1^4 \sqrt{4x+1}\;dx$$

Set $u = 4x + 1$. Then $\frac{du}{dx} = 4 \Rightarrow du = 4dx \Rightarrow \frac{1}{4}du = dx.$

If $x = 1$, then $u = 5$ and if $x = 4$, then $u = 17$.

$$= \pi\int_5^{17} \sqrt{u}\;\frac{1}{4}du$$

$$= \frac{\pi}{4}\,\frac{2}{3}\,\left[17\sqrt{17} - 5\sqrt{5}\,\right]$$

Surface Area = $\frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]\; unit^2$.

[Note that $a^{\frac{3}{2}} = \left(\sqrt{a}\right)^3 = \sqrt{a}\sqrt{a}\sqrt{a} = a\sqrt{a}$.]

Exercise: 3 **[Similar to Exercises 1-2] Homework**

Let $f(x) = \sqrt{1-x}$ over the interval $[1, \frac{1}{2}]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the $x-$axis. Round the answer to three decimal places.

Definition 2:

Let $x = g(y)$ be a non-negative smooth curve on the interval $[c, d]$. Then the area of the surface of revolution formed by revolving the portion of the curve of $x = g(y)$ about the $y-$axis is defined by

$$\text{Surface Area} = \int_c^d 2\pi\, g(y)\sqrt{1+[g'(y)]^2}\; dy \quad \int_c^d 2\pi\, g(y)\sqrt{1+\left[\frac{dx}{dy}\right]^2}\; dy$$

Steps: Given a smooth function $\boldsymbol{x = g(y)}$; $\boldsymbol{y = c}$ **to** $\boldsymbol{y = d}$

Step 1: Find $\frac{dx}{dy} = g'(y)$

Step 2: Find $\left(\frac{dx}{dy}\right)^2$

Step 3: Find $1 + \left(\frac{dx}{dy}\right)^2$

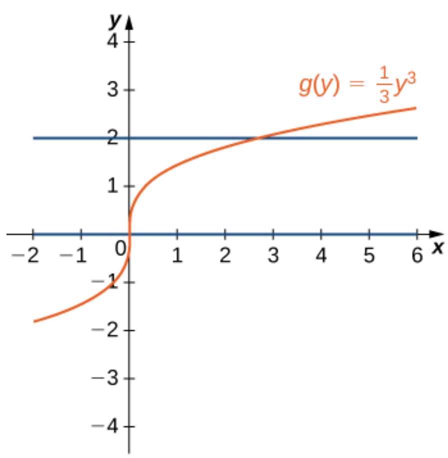
Step 4: Find $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$ **. Here we should try, if possible, to write** $1 + \left[\frac{dx}{dy}\right]^2$ **as a perfect square so that we can cancel the square root.**

Step 5: Evaluate the integral to find the area of the surface of revolution:

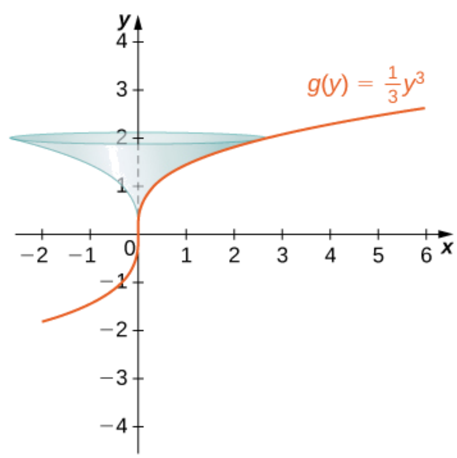
$$\text{Surface Area} = \int_c^d 2\pi g(y)\ \sqrt{1+[g'(y)]^2}\; dy$$

Exercise : 4 Find the surface area of revolution generated by revolving the curve $x = \frac{1}{3}y^3$ over $[0,\ 2]$ about the y –axis.

[Same: Find the surface area of revolution generated by revolving the curve $y = (3x)^{\frac{1}{3}}$ over $\left[0, \frac{8}{3}\right]$ about the y –axis. Hint: revolving the curve $x = \frac{1}{3}y^3$ over $[0,\ 2]$.]



(a)



(b)

Solution:

$$\text{Surface Area} = \int_c^d 2\pi g(y) \sqrt{1+\left[\frac{dx}{dy}\right]^2}\, dy \quad \ldots\ldots\ldots (1)$$

Given $x = g(y) = \frac{1}{3}y^3$

$$\Rightarrow \frac{dx}{dy} = g'(y) = \frac{1}{3}(3y^2) = y^2$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = [g'(y)]^2 = y^4$$

$$\Rightarrow 1+\left(\frac{dx}{dy}\right)^2 = 1+[g'(y)]^2 = 1+y^4$$

$$\therefore \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+[g'(y)]^2} = \sqrt{1+y^4} \qquad \text{for } 0 \le y \le 2.$$

So, Surface Area:

$$S_A = \int_c^d 2\pi\ g(y) \sqrt{1+\left[\frac{dx}{dy}\right]^2}\, dy$$

$$= \int_0^2 2\pi\ \frac{1}{3}\, y^3 \sqrt{1+y^4}\, dy$$

$$= \frac{2\pi}{3} \int_0^2 \sqrt{1+y^4}\ \ y^3 dy$$

Now, set $u = 1+y^4 \Rightarrow du = 4y^3 dy$. Then $\frac{1}{4}\, du = y^3 dy$ and $1 \le u \le 17$.

Hence, the surface area

$$= \frac{2\pi}{3} \int_1^{17} u^{\frac{1}{2}}\ \frac{1}{4}\, du$$

$$= \frac{2\pi}{3}\ \frac{1}{4}\ \frac{2}{3}\ \left[17\sqrt{17} - 1\right]$$

$$= \frac{\pi}{9}\ \left[17\sqrt{17} - 1\right] unit^2$$

Exercise: 5 [Similar to exercise 4] Homework

Find the surface area of revolution generated by revolving the curve $g(y) = \sqrt{9-y^2}$ over $[0, 2]$ about the y −axis.

Solution: We know that,

$$\text{Surface Area} = \int_c^d 2\pi g(y)\ \sqrt{1+[g'(y)]^2}\ dy \ \ldots\ldots\ldots (1)$$

Here given, $g(y) = \sqrt{9-y^2}$,

Then

$$g'(y) = \frac{1}{2\sqrt{9-y^2}}(-2y) = -\frac{y}{\sqrt{9-y^2}}$$

$$\Rightarrow \left(g'(y)\right)^2 = \frac{y^2}{9-y^2}$$

$$\Rightarrow 1+\left(g'(y)\right)^2 = 1+\frac{y^2}{9-y^2}$$

$$\Rightarrow 1+\left(g'(y)\right)^2 = \frac{9}{9-y^2}$$

$$\therefore \sqrt{1+\left(g'(y)\right)^2} = \frac{3}{\sqrt{9-y^2}} \qquad \text{for } 0 \le y \le 2.$$

Hence the surface area is:

$$\int_c^d 2\pi g(y)\ \sqrt{1+[g'(y)]^2}\ dy$$

$$= \int_0^2 2\pi\ \sqrt{9-y^2}\ \frac{3}{\sqrt{9-y^2}}\ dy$$

$$= 2\pi \int_0^2 3\ dy = 12\pi\ unit^2$$

Exercise: 6

Find the surface area of revolution generated by revolving the curve $y = \sqrt{1-x^2}$ over $[1, 2]$ about the x −axis.

Solution: We know that the area of the surface of revolution is:

$$S_A = \int_a^b 2\pi f(x) \sqrt{1 + \left[\frac{dy}{dx}\right]^2} \; dx \quad \dots \dots \dots (1)$$

Here given, $y = \sqrt{1 - x^2}$, $1 \le x \le 2$

Then,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1 - x^2}} (-2x) = -\frac{x}{\sqrt{1 - x^2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1 - x^2}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2}$$

$$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{1}{1 - x^2}}$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{\sqrt{1 - x^2}}$$

Hence, from equation (1), the area of the surface of revolution is:

$$S_A = \int_a^b 2\pi f(x) \sqrt{1 + \left[\frac{dy}{dx}\right]^2} \; dx$$

$$= \int_1^2 2\pi \sqrt{1 - x^2} \; \frac{1}{\sqrt{1 - x^2}} \, dx$$

$$= \int_1^2 2\pi \; dx$$

$$= 2\pi \; unit^2$$

Definition 3:

A. Let $x = x(t)$, $y = y(t)$, $a \le t \le b$ be a smooth parametric curve. Then the area of the surface of revolution formed by revolving the curve about the x −axis is defined by

$$S_A = \int_a^b 2\pi \ y(t) \ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

B. Let $x = x(t),\ y = y(t), \quad a \le t \le b$ be a smooth parametric curve. Then the area of the surface of revolution formed by revolving the curve about the y −axis is defined by

$$S_A = \int_a^b 2\pi \ x(t) \ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

Steps: Given a parametric curve $x = x(t), \quad y = y(t), \quad a \le t \le b$

Step 1: Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

Step 2: Find $\left(\frac{dx}{dt}\right)^2$ and $\left(\frac{dy}{dt}\right)^2$

Step 3: Find $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2$

Step 4: Find $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Here we should try, if possible, to write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ as a perfect square so that we can cancel the square root.

Step 5: Evaluate the integral to find the area of the surface of revolution.

Exercise: 7

Find the area of the surface of revolution generated by revolving the parametric curve

$$x = t^2, \quad y = 2t, \quad 0 \le t \le 4$$

about the x −axis.

Solution: We know that, the surface area is given by

$$S_A = \int_a^b 2\pi \ y(t) \ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt \ldots\ldots\ldots (1)$$

Here given $x = t^2, \quad y = \ 2t, \quad 0 \le t \le 4.$

$$S_A = \int_0^4 2\pi \ 2t \ \sqrt{4t^2 + 4} \ dt$$

$$= \int_0^4 8\pi t \ \sqrt{t^2 + 1} \ dt$$

$$= 4\pi \int_0^4 \ 2t \ \sqrt{t^2 + 1} \ dt \ \ ; \quad u = t^2 + 1, \qquad du = 2t \, dt$$

$$= 4\pi \int_1^{17} \ \sqrt{u} \ du$$

$$= \ 4\pi \ \frac{2}{3} \ [17\sqrt{17} - 1]$$

$$= \frac{8\pi}{3} \left[17\sqrt{17} - 1\right] \ unit^2$$

Exercise: 8

Find the area of the surface of revolution generated by revolving the parametric curve

$$x = 3\cos\theta\,, \quad y = 3\sin\theta\,, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

about the y –axis.

Solution:

$x = x(\theta) = 3\cos\theta\,, \quad y = y(\theta) = 3\sin\theta\,, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

We know that the area of the surface of revolution is

$$S_A = \int_a^b 2\pi \ x(\theta) \ \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \ d\theta \ \ldots\ldots\ldots (1)$$

$$\frac{dx}{d\theta} = -3\sin\theta \quad , \frac{dy}{d\theta} = 3\cos\theta$$

That is, $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 3^2 = 9$

From equation (1), the area of the surface of revolution is

$$S_A = \int_a^b 2\pi \ x(\theta) \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \ d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi \ (3\ )\cos\theta \ \ (3) \ \ d\theta$$

$$= 18\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \ \ d\theta$$

$$= 18\pi \ [\ \sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \ 36\pi$$

Exercise: 9 Homework

Find the area of the surface of revolution generated by revolving the parametric curve

$$x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad 0 \leq \theta \leq \frac{2\pi}{3}$$

about the x −axis.

*****Exercise:10** Homework

Find the area of the surface of revolution generated by revolving the curve $x = 9y + 1, \quad 0 \leq y \leq 3$ about the line $x = -1$