Fourier Analysis

[1]JEAN-BAPTISTE JOSEPH FOURIER (1768–1830), French physicist and mathematician, lived and taught in Paris, accompanied Napoléon in the Egyptian War, and was later made prefect of Grenoble. The beginnings on Fourier series can be found in works by Euler and by Daniel Bernoulli, but it was Fourier who employed them in a systematic and general manner in his main work, *Théorie analytique de la chaleur* (*Analytic Theory of Heat,* Paris, 1822), in which he developed the theory of heat conduction (heat equation; see Sec. 12.5), making these series a most important tool in applied mathematics.

11.1 Fourier Series

Fourier series are infinite series that represent periodic functions in terms of cosines and sines. As such, Fourier series are of greatest importance to the engineer and applied mathematician. To define Fourier series, we first need some background material.

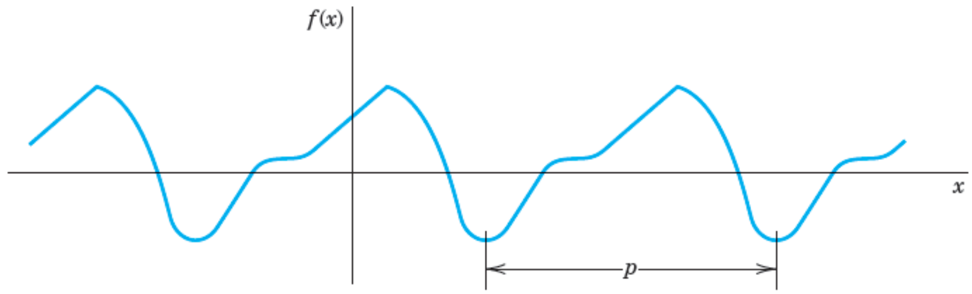


Fig. 258. Periodic function of period p

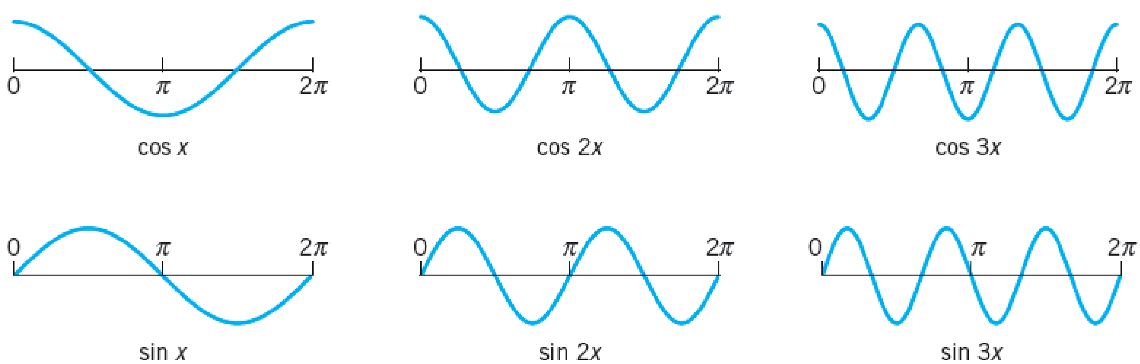


Fig. 259. Cosine and sine functions having the period 2π (the first few members of the trigonometric system (3), except for the constant 1)

The series to be obtained will be a **trigonometric series**, that is, a series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

(4)
$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$a_0, a_1, b_1, a_2, b_2, \cdots$ are constants, called the **coefficients** of the series. We see that each term has the period 2π. Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π.

and we integrate over x from $-L$ to L. Consequently, we obtain for a function $f(x)$ of period $2L$ the Fourier series

(5)
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas** (π/L in dx cancels $1/\pi$ in (3))

(6)

(0)
$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

(a)
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots$$

(b)
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots.$$

EXAMPLE 1 **Periodic Rectangular Wave (Fig. 260)**

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 260. The formula is

(7)
$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

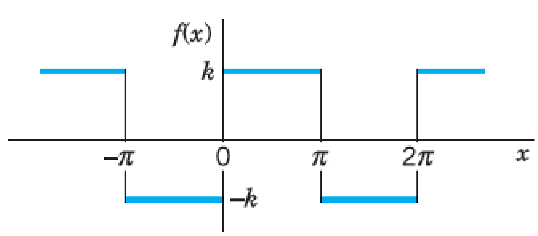


Fig. 260. Given function $f(x)$ (Periodic reactangular wave)

Since the a_n are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots\right). \tag{8}$$

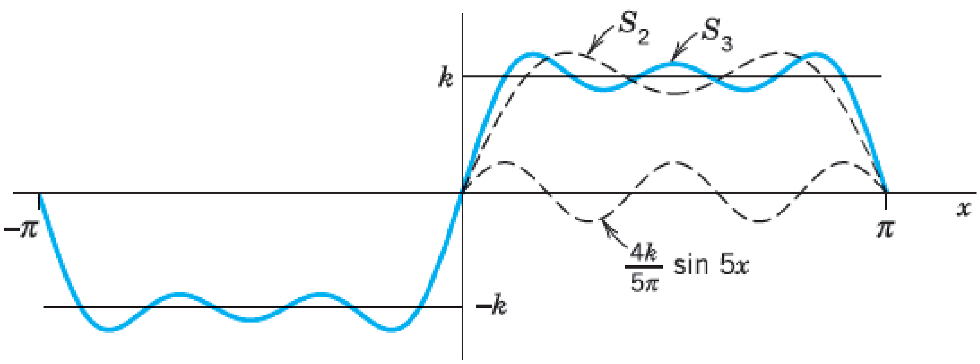


Fig. 261. First three partial sums of the corresponding Fourier series

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

Thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result obtained by Leibniz in 1673 from geometric considerations. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points. ■

11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

20. Numeric Values. Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.

EXAMPLE 1 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 263)

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

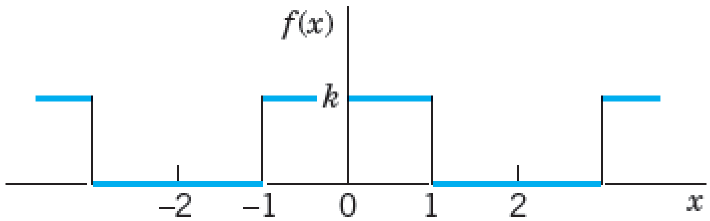


Fig. 263. Example 1

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

[Fourier Series Animation (Square Wave)]

[Fourier Series: Modeling Nature]

EXAMPLE 2 **Periodic Rectangular Wave. Change of Scale**

Find the Fourier series of the function (Fig. 264)

$$f(x) = \begin{cases} -k & \text{if } \; -2 < x < 0 \\ k & \text{if } \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

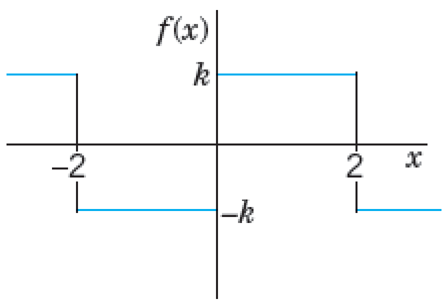


Fig. 264. Example 2

$$f(x) = \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right).$$

EXAMPLE 3 **Half-Wave Rectifier**

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Fig. 265). Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } \; -L < t < 0, \\ E \sin \omega t & \text{if } \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}.$$

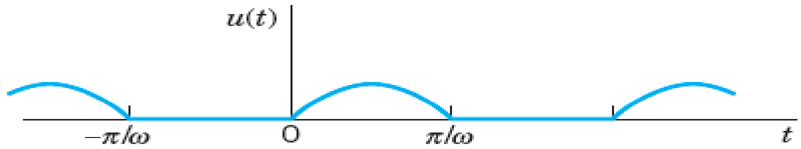


Fig. 265. Half-wave rectifier

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$

2. Simplifications: Even and Odd Functions

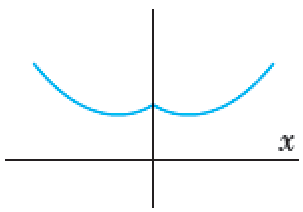


Fig. 266. Even function

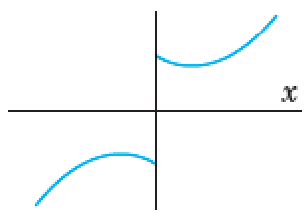


Fig. 267. Odd function

If $f(x)$ is an **even function**, that is, $f(-x) = f(x)$ (see Fig. 266), its Fourier series (5) reduces to a **Fourier cosine series**

(5*) $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad (f \text{ even})$$

with coefficients (note: integration from 0 to L only!)

(6*) $$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

If $f(x)$ is an **odd function**, that is, $f(-x) = -f(x)$ (see Fig. 267), its Fourier series (5) reduces to a **Fourier sine series**

(5)** $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad (f \text{ odd})$$

with coefficients

(6**) $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

EXAMPLE 5 **Sawtooth Wave**

Find the Fourier series of the function (Fig. 268)

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

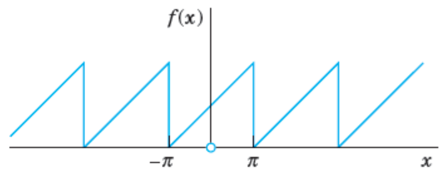


Fig. 268. The function $f(x)$. Sawtooth wave

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right). \qquad \text{(Fig. 269)}$$

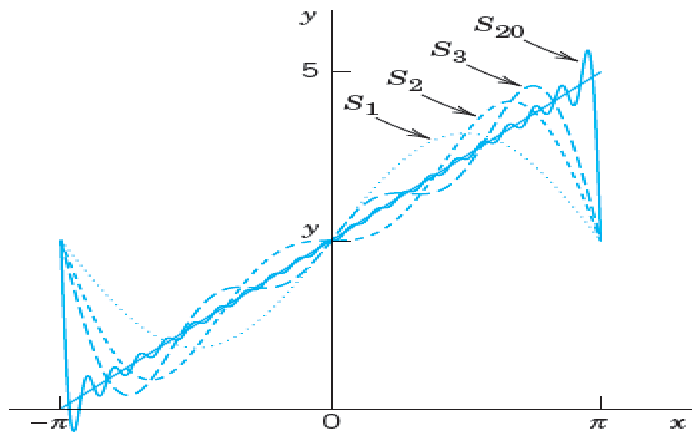
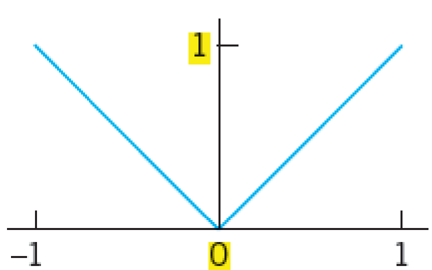


Fig. 269. Partial sums S_1, S_2, S_3, S_{20} in Example 5

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

8.



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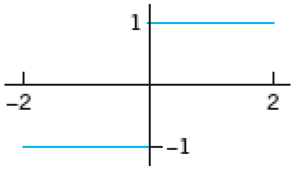

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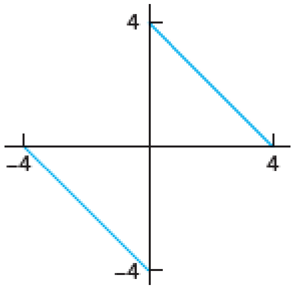


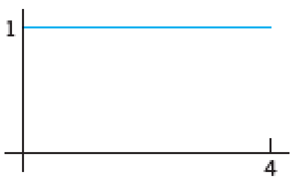
11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$

23–29 HALF-RANGE EXPANSIONS

Find (**a**) the Fourier cosine series, (**b**) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

23.



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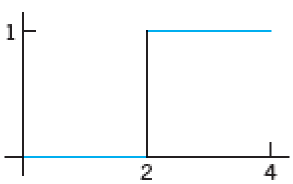

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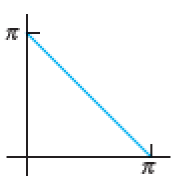
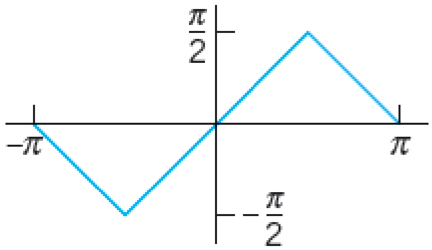


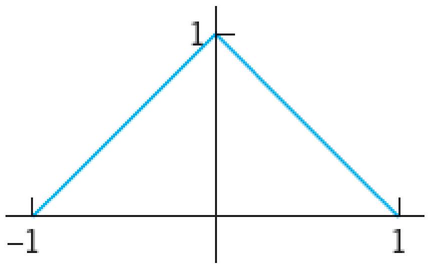
14. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$

15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$

17.



18. Rectifier. Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.

19. Trigonometric Identities. Show that the familiar identities $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$ and $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.

11.7 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.2 and 11.3 first illustrated this, and various further applications follow in Chap. 12. Since, of course, many problems involve functions that are ***nonperiodic*** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

From Fourier Series to Fourier Integral

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will suggest that we should expect an integral (instead of a series) involving $\cos wx$ and $\sin wx$ with w no longer restricted to integer multiples $w = w_n = n\pi/L$ of π/L but taking *all* values. We shall also see what form such an integral might have.

If we insert a_n and b_n from the Euler formulas (6), Sec. 11.2, and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right]. \tag{1}$$

This representation is valid for any fixed L, arbitrarily large, but finite.

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following (finite!) limits exist:

$$(2) \qquad \lim_{a\to-\infty} \int_a^0 |f(x)|\,dx + \lim_{b\to\infty} \int_0^b |f(x)|\,dx \quad \left(\text{written } \int_{-\infty}^{\infty} |f(x)|\,dx\right).$$

Then $1/L \to 0$, and the value of the first term on the right side of (1) approaches zero. Also $\Delta w = \pi/L \to 0$ and it seems ***plausible*** that the infinite series in (1) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

$$(3) \qquad f(x) = \frac{1}{\pi} \int_0^\infty \left[\cos wx \int_{-\infty}^{\infty} f(v) \cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v) \sin wv\,dv\right] dw.$$

If we introduce the notations

$$\textbf{(4)} \qquad A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\,dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\,dv$$

we can write this in the form

$$\textbf{(5)} \qquad f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx]\,dw.$$

This is called a representation of $f(x)$ by a **Fourier integral**.

THEOREM 1

Fourier Integral

If $f(x)$ is piecewise continuous (see Sec. 6.1) *in every finite interval and has a right-hand derivative and a left-hand derivative at every point* (see Sec 11.1) *and if the integral* (2) *exists, then $f(x)$ can be represented by a Fourier integral* (5) *with A and B given by* (4). *At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point* (see Sec. 11.1). (Proof in Ref. [C12]; see App. 1.)

Applications of Fourier Integrals

The main application of Fourier integrals is in solving ODEs and PDEs,

EXAMPLE 2 Single Pulse, Sine Integral. Dirichlet's Discontinuous Factor. Gibbs Phenomenon

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases} \qquad \text{(Fig. 281)}$$

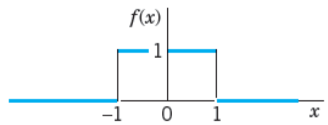


Fig. 281. Example 2

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos wx \sin w}{w}\, dw.$$

The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $(1 + 0)/2$, that is, $\frac{1}{2}$.
Furthermore, from (6) and Theorem 1 we obtain (multiply by $\pi/2$)

$$\int_0^\infty \frac{\cos wx \sin w}{w}\, dw = \begin{cases} \pi/2 & \text{if} \quad 0 \leqq x < 1, \\ \pi/4 & \text{if} \quad x = 1, \\ 0 & \text{if} \quad x > 1. \end{cases} \tag{7}$$

We mention that this integral is called **Dirichlet's discontinous factor**. (For P. L. Dirichlet see Sec. 10.8.)
The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

$$\int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}. \tag{8*}$$

We see that this integral is the limit of the so-called **sine integral**

$$\mathrm{Si}(u) = \int_0^u \frac{\sin w}{w}\, dw \tag{8}$$

as $u \to \infty$. The graphs of $\mathrm{Si}(u)$ and of the integrand are shown in Fig. 282.

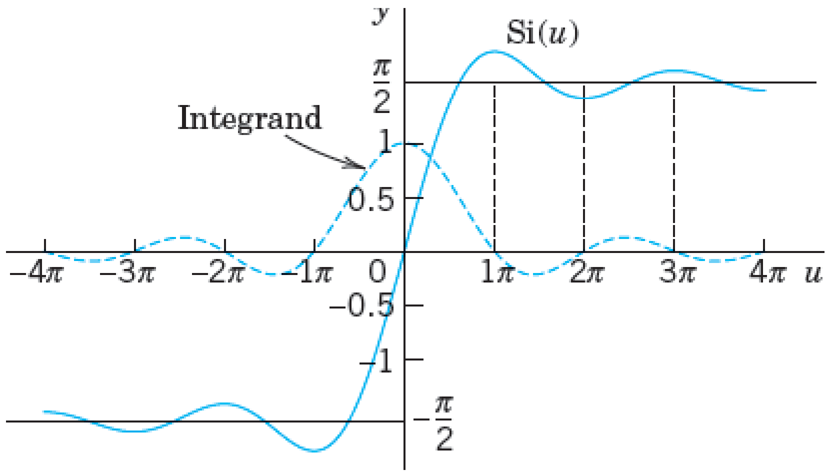


Fig. 282. Sine integral Si(u) and integrand

In the case of a Fourier series the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series. Similarly, in the case of the Fourier integral (5), approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw \tag{9}$$

approximates the right side in (6) and therefore $f(x)$.

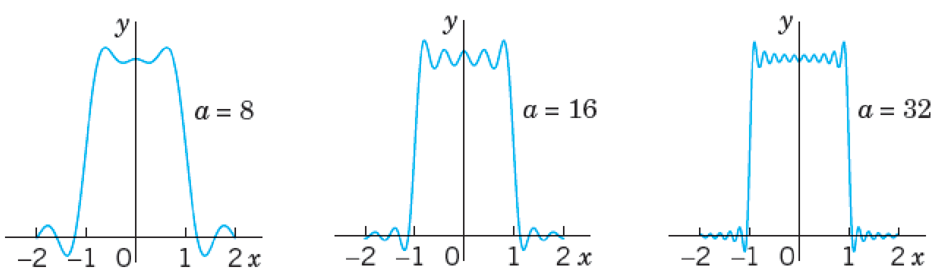


Fig. 283. The integral (9) for $a = 8, 16$, and 32, illustrating the development of the Gibbs phenomenon

Figure 283 shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as a approaches infinity. But this is not true; with increasing a, they are shifted closer to the points $x = \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series (see Sec. 11.2), is known as the **Gibbs phenomenon**. We can explain it by representing (9) in terms of sine integrals as follows. Using (11) in App. A3.1, we have

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^a \frac{\sin (w + wx)}{w}\,dw + \frac{1}{\pi}\int_0^a \frac{\sin (w - wx)}{w}\,dw.$$

In the first integral on the right we set $w + wx = t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x + 1)a$. In the last integral we set $w - wx = -t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x - 1)a$. Since $\sin(-t) = -\sin t$, we thus obtain

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t}\,dt.$$

From this and (8) we see that our integral (9) equals

$$\frac{1}{\pi}\,\mathrm{Si}(a[x+1]) - \frac{1}{\pi}\,\mathrm{Si}(a[x-1])$$

and the oscillations in Fig. 283 result from those in Fig. 282. The increase of a amounts to a transformation of the scale on the axis and causes the shift of the oscillations (the waves) toward the points of discontinuity -1 and 1.

Fourier Cosine Integral and Fourier Sine Integral

reduces to a **Fourier cosine integral**

(10) $$f(x) = \int_0^\infty A(w)\cos wx\,dw \qquad \text{where} \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv.$$

is true because the integrand of $A(w)$ is odd. Then (5) becomes a **Fourier sine integral**

(11) $$f(x) = \int_0^\infty B(w)\sin wx\,dw \qquad \text{where} \qquad B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv.$$

EXAMPLE 3 **Laplace Integrals**

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ (Fig. 284). The result will be used to evaluate the so-called Laplace integrals.

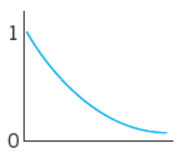


Fig. 284. $f(x)$ in Example 3

Solution. (a) From (10) we have $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\,dv$. Now, by integration by parts,

$$\int e^{-kv}\cos wv\,dv = -\frac{k}{k^2 + w^2}\,e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right).$$

If $v = 0$, the expression on the right equals $-k/(k^2 + w^2)$. If v approaches infinity, that expression approaches zero because of the exponential factor. Thus $2/\pi$ times the integral from 0 to ∞ gives

$$A(w) = \frac{2k/\pi}{k^2 + w^2}. \tag{12}$$

By substituting this into the first integral in (10) we thus obtain the Fourier cosine integral representation

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2 + w^2}\,dw \qquad (x > 0, \quad k > 0).$$

From this representation we see that

$$\int_0^\infty \frac{\cos wx}{k^2 + w^2}\,dw = \frac{\pi}{2k}e^{-kx} \qquad (x > 0, \quad k > 0). \tag{13}$$

(b) Similarly, from (11) we have $B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\,dv$. By integration by parts,

$$\int e^{-kv}\sin wv\,dv = -\frac{w}{k^2 + w^2}e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

$$B(w) = \frac{2w/\pi}{k^2 + w^2}. \tag{14}$$

From (14) we thus obtain the Fourier sine integral representation

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\,dw.$$

From this we see that

$$\int_0^\infty \frac{w\sin wx}{k^2 + w^2}\,dw = \frac{\pi}{2}e^{-kx} \qquad (x > 0, \quad k > 0). \tag{15}$$

The integrals (13) and (15) are called the **Laplace integrals**.

PROBLEM SET 11.7

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

11. $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

16–20 FOURIER SINE INTEGRAL REPRESENTATIONS

18. $f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

19. $f(x) = \begin{cases} e^{x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

11.8 Fourier Cosine and Sine Transforms

An **integral transform** is a transformation in the form of an integral that produces from given functions new functions depending on a different variable. One is mainly interested in these transforms because they can be used as tools in solving ODEs, PDEs, and integral equations and can often be of help in handling and applying special functions. The Laplace

(1a)
$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

and

(1b)
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw.$$

Formula (1a) gives from $f(x)$ a new function $\hat{f}_c(w)$, called the **Fourier cosine transform** of $f(x)$. Formula (1b) gives us back $f(x)$ from $\hat{f}_c(w)$, and we therefore call $f(x)$ the **inverse Fourier cosine transform of** $\hat{f}_c(w)$.

(2a)
$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx,$$

and the **inverse Fourier sine transform** of $\hat{f}_s(w)$, given by

(2b)
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw.$$

The process of obtaining $f_s(w)$ from $f(x)$ is also called the **Fourier sine transform** or the *Fourier sine transform method.*

EXAMPLE 1 **Fourier Cosine and Fourier Sine Transforms**

Find the Fourier cosine and Fourier sine transforms of the function

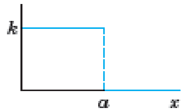


Fig. 285. $f(x)$ in Example 1

$$f(x) = \begin{cases} k & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases} \qquad \text{(Fig. 285).}$$

Solution. From the definitions (1a) and (2a) we obtain by integration

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\,k\int_0^a \cos wx\,dx = \sqrt{\frac{2}{\pi}}\,k\left(\frac{\sin aw}{w}\right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\,k\int_0^a \sin wx\,dx = \sqrt{\frac{2}{\pi}}\,k\left(\frac{1-\cos aw}{w}\right).$$

This agrees with formulas 1 in the first two tables in Sec. 11.10 (where $k = 1$).

Note that for $f(x) = k = \text{const}$ $(0 < x < \infty)$, these transforms do not exist. (Why?) ■

EXAMPLE 2 **Fourier Cosine Transform of the Exponential Function**

Find $\mathcal{F}_c(e^{-x})$.

Solution. By integration by parts and recursion,

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos wx\,dx = \sqrt{\frac{2}{\pi}}\,\frac{e^{-x}}{1+w^2}(-\cos wx + w\sin wx)\bigg|_0^\infty = \frac{\sqrt{2/\pi}}{1+w^2}.$$

This agrees with formula 3 in Table I, Sec. 11.10, with $a = 1$. See also the next example. ■

What have we gained? We show next that these transforms have operational properties that permit them to convert differentiations into algebraic operations (just as the Laplace transform does). This is the key to their application in solving differential equations.

Linearity, Transforms of Derivatives

$$\text{(3)} \qquad \begin{aligned} &\text{(a)} \quad \mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \\ &\text{(b)} \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g). \end{aligned}$$

THEOREM 1 **Cosine and Sine Transforms of Derivatives**

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then

$$\textbf{(4)} \qquad \begin{aligned} &\textbf{(a)} \quad \mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\,f(0), \\ &\textbf{(b)} \quad \mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}. \end{aligned}$$

(5a) $$\mathscr{F}_c\{f''(x)\} = -w^2\mathscr{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0).$$

(5b) $$\mathscr{F}_s\{f''(x)\} = -w^2\mathscr{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\,wf(0).$$

EXAMPLE 3 **An Application of the Operational Formula (5)**

Find the Fourier cosine transform $\mathscr{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a > 0$.

Solution. By differentiation, $(e^{-ax})'' = a^2e^{-ax}$; thus

$$a^2f(x) = f''(x).$$

From this, (5a), and the linearity (3a),

$$\begin{aligned} a^2\mathscr{F}_c(f) &= \mathscr{F}_c(f'') \\ &= -w^2\mathscr{F}_c(f) - \sqrt{\frac{2}{\pi}}f'(0) \\ &= -w^2\mathscr{F}_c(f) + a\sqrt{\frac{2}{\pi}}. \end{aligned}$$

Hence

$$(a^2 + w^2)\mathscr{F}_c(f) = a\sqrt{2/\pi}.$$

The *answer* is (see Table I, Sec. 11.10)

$$\mathscr{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right) \qquad (a > 0).$$

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 11.7]

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv.$$

Substituting A and B into the integral for f, we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)[\cos wv\cos wx + \sin wv\sin wx]\,dv\,dw.$$

(1*) $$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right]dw.$$

$$(1) \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\,dv\right]dw.$$

We claim that the integral of the form (1) with sin instead of cos is zero:

$$(2) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\,dv\right]dw = 0.$$

This is true since $\sin(wx - wv)$ is an odd function of w, which makes the integral in brackets an odd function of w, call it $G(w)$. Hence the integral of $G(w)$ from $-\infty$ to ∞ is zero, as claimed.

We now take the integrand of (1) plus $i\,(=\sqrt{-1})$ times the integrand of (2) and use the **Euler formula** [(11) in Sec. 2.2]

$$(3) \qquad e^{ix} = \cos x + i\sin x.$$

Taking $wx - wv$ instead of x in (3) and multiplying by $f(v)$ gives

$$f(v)\cos(wx - wv) + if(v)\sin(wx - wv) = f(v)e^{i(wx-wv)}.$$

Hence the result of adding (1) plus i times (2), called the **complex Fourier integral**, is

$$\textbf{(4)} \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{iw(x-v)}\,dv\,dw \qquad (i = \sqrt{-1}).$$

To obtain the desired Fourier transform will take only a very short step from here.

Fourier Transform and Its Inverse

Writing the exponential function in (4) as a product of exponential functions, we have

$$(5) \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv\right]e^{iwx}\,dw.$$

The expression in brackets is a function of w, is denoted by $\hat{f}(w)$, and is called the **Fourier transform** of f; writing $v = x$, we have

$$\textbf{(6)} \qquad \hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx.$$

With this, (5) becomes

(7)
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw$$

and is called the **inverse Fourier transform** of $\hat{f}(w)$.

THEOREM 1 **Existence of the Fourier Transform**

If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ given by (6) exists.

EXAMPLE 1 **Fourier Transform**

Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

Solution. Using (6) and integrating, we obtain

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-iwx}}{-iw}\Big|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}}(e^{-iw} - e^{iw}).$$

As in (3) we have $e^{iw} = \cos w + i \sin w$, $e^{-iw} = \cos w - i \sin w$, and by subtraction

$$e^{iw} - e^{-iw} = 2i \sin w.$$

Substituting this in the previous formula on the right, we see that i drops out and we obtain the answer

$$\hat{f}(w) = \sqrt{\frac{\pi}{2}}\, \frac{\sin w}{w}.$$

EXAMPLE 2 **Fourier Transform**

Find the Fourier transform $\mathcal{F}(e^{-ax})$ of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0$; here $a > 0$.

Solution. From the definition (6) we obtain by integration

$$\mathcal{F}(e^{-ax}) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, \frac{e^{-(a+iw)x}}{-(a + iw)}\Big|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}(a + iw)}.$$

This proves formula 5 of Table III in Sec. 11.10.

[Time vs Frequency Domains]

[The intuition behind Fourier and Laplace transforms]

But what is the Fourier Transform? A visual introduction.

Linearity. Fourier Transform of Derivatives

THEOREM 2 **Linearity of the Fourier Transform**

The Fourier transform is a ***linear operation****; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b, the Fourier transform of $af + bg$ exists, and*

(8) $$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

THEOREM 3 **Fourier Transform of the Derivative of $f(x)$**

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

(9) $$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}.$$

(10) $$\mathcal{F}\{f''(x)\} = -w^2\mathcal{F}\{f(x)\}.$$

EXAMPLE 3 **Application of the Operational Formula (9)**

Find the Fourier transform of xe^{-x^2} from Table III, Sec 11.10.

Solution. We use (9). By formula 9 in Table III

$$\begin{aligned}\mathcal{F}(xe^{-x^2}) &= \mathcal{F}\{-\tfrac{1}{2}(e^{-x^2})'\} \\ &= -\tfrac{1}{2}\mathcal{F}\{(e^{-x^2})'\} \\ &= -\tfrac{1}{2}iw\mathcal{F}(e^{-x^2}) \\ &= -\frac{1}{2}iw\frac{1}{\sqrt{2}}e^{-w^2/4} \\ &= -\frac{iw}{2\sqrt{2}}e^{-w^2/4}.\end{aligned}$$

■

THEOREM 4 **Convolution Theorem**

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g). \tag{12}$$