Higher Order Linear ODEs : Non-Homogeneous

- ***Linear differential equation of order TWO : constant coefficients*** (a_0, a_1, a_2)

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x) \Rightarrow a_0 D^2 y + a_1 Dy + a_2 y = R(x) \Rightarrow f(D)y = R(x)$$

$$Where\ f(D) = a_0 D^2 + a_1 D + a_2$$

- ***Linear differential equation of order Three : constant coefficients*** (a_0, a_1, a_2, a_3)

$$a_0 \frac{d^3y}{dx^3} + a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = R(x) \Rightarrow a_0 D^3 y + a_1 D^2 y + a_2 Dy + a_3 y = R(x)$$

$$\Rightarrow f(D)y = R(x)$$

$$Where\ f(D) = a_0 D^3 + a_1 D^2 + a_2 D + a_3$$

Example.

$$D^3 y - D^2 y = 3e^x \qquad [\text{Non−Homogeneous, third order}]$$

$$(D^2+1)y = \sin x \qquad [\text{Non−Homogeneous, second order}]$$

$$(D^3 - D)y = 4e^{-x} + 3e^{2x} \qquad [\text{Non−Homogeneous, third order}]$$

$$D^2 y - 6Dy + 9y = e^x \qquad [\text{Non−Homogeneous, second order}]$$

Higher Order Linear ODEs : Non-Homogeneous

General solutions of a non-homogeneous Linear ODEs

The solution of the non-homogeneous linear differential equation $f(D)y = R(x)$ is of the form

$y = y_c + y_p$ where y_c: general solution of $f(D)y = 0$ and

y_p: particular solution of $f(D)y = R(x)$.

Here, y_c is called the complementary function for $f(D)y = R(x)$.

A number of methods are used to obtain particular integrals for non-homogeneous differential equations. Some of the standard methods are

1. **Variation of Parameters**
2. **Inverse Operator method**
3. The method of Undetermined Coefficients

Higher Order Linear ODEs : Non-Homogeneous

General Solution using Variation of Parameters

Example. Find the general solution of the following non-homogeneous ODE:

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Solution. Here, the auxiliary equation is, $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2.$

Therefore, the complementary solution yields, $y_c = (c_1 + c_2 x)e^{2x} = c_1 e^{2x} + c_2 x e^{2x}$.
Here, the **Wronskian** of the solutions $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ yields,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x} \neq 0$$

Therefore, the solutions y_1 and y_2 are linearly independent. We also compute,

$$W_1(y_1, y_2) = \begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2(y_1, y_2) = \begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Now, let us consider a particular solution of the form, $y_p = A(x)e^{2x} + B(x)xe^{2x}$,

Higher Order Linear ODEs : Non-Homogeneous

General Solution using Variation of Parameters

Example. Find the general solution of the following non-homogeneous ODE:

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Solution. Now, let us consider a particular solution of the form, $y_p = A(x)e^{2x} + B(x)xe^{2x}$, where

$$A(x) = \int \frac{W_1}{W}\,dx = -\int (x^2 + x)\,dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2$$

$$B(x) = \int \frac{W_2}{W}\,dx = \int (x+1)\,dx = \frac{1}{2}x^2 + x$$

$$\boxed{\begin{aligned} \frac{W_1(x)}{W(x)} &= -\frac{(x+1)xe^{4x}}{e^{4x}} \\ \frac{W_2(x)}{W(x)} &= \frac{(x+1)e^{4x}}{e^{4x}} \end{aligned}}$$

Thus the particular solution becomes,

$$\begin{aligned} y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} &= \left(-\frac{1}{3}x^3 + \frac{1}{2}x^3 - \frac{1}{2}x^2 + x^2\right)e^{2x} \\ &= \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x} \end{aligned}$$

Therefore, the general solution becomes,

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x}.$$

Higher Order Linear ODEs : Non-Homogeneous

General Solution using Variation of Parameters

Example. Find the general solution of the following non-homogeneous ODE:

$$y'' + y = \sec x \tan x$$

Solution. Here, the auxiliary equation is, $m^2 + 1 = 0 \Rightarrow m = \pm i$.

Therefore, the complementary solution yields, $y_c = c_1 \cos x + c_2 \sin x$.
Here, the **Wronskian** of the solutions $y_1 = \cos x$ and $y_2 = \sin x$ yields,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

Therefore, the solutions y_1 and y_2 are linearly independent. We also compute,

$$W_1(y_1, y_2) = \begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix} = -\sin x \sec x \tan x = -\tan^2 x$$

$$W_2(y_1, y_2) = \begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix} = \tan x$$

Now, let us consider a particular solution of the form, $y_p = A(x) \cos x + B(x) \sin x$,

Higher Order Linear ODEs : Non-Homogeneous

General Solution using Variation of Parameters

Example. Find the general solution of the following non-homogeneous ODE:

$$y'' + y = \sec x \tan x$$

Solution. Now, let us consider a particular solution of the form, $y_p = A(x)\cos x + B(x)\sin x$, where

$$A(x) = \int \frac{W_1}{W}\,dx = -\int \tan^2 x\,dx = -\int (\sec^2 x - 1)\,dx = -\tan x + x$$

$$B(x) = \int \frac{W_2}{W}\,dx = \int \tan x\,dx = \ln|\sec x|$$

Thus the particular solution becomes,

$$y_p = (x - \tan x)\cos x + (\sin x)\ln|\sec x| = x\cos x - \sin x + \sin x \ln|\sec x|$$

Therefore, the general solution becomes,

$$y = y_c + y_p = c_1\cos x + c_2\sin x + x\cos x + \sin x \ln|\sec x|\,.$$

Higher Order Linear ODEs : Non-Homogeneous

General Solution using Variation of Parameters

Example. Find the general solution of the following non-homogeneous ODE:

$$4y'' + 36y = \csc 3x$$

Solution. The ODE can be rewritten in it's standard form as, $y'' + 9y = \frac{1}{4}\csc 3x$

Here, the auxiliary equation is, $m^2 + 9 = 0 \Rightarrow m = \pm 3i$.

Therefore, the complementary solution yields, $y_c = c_1 \cos 3x + c_2 \sin 3x$.

Now, the **Wronskian** of the solutions $y_1 = \cos 3x$ and $y_2 = \sin 3x$ yields,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3(\cos^2 3x + \sin^2 3x) = 3 \neq 0$$

Therefore, the solutions y_1 and y_2 are linearly independent. We also compute,

$$W_1(y_1, y_2) = \begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}\sin 3x \csc 3x = -\frac{1}{4}$$

$$W_2(y_1, y_2) = \begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix} = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\cos 3x \csc 3x = \frac{1}{4}\cot 3x$$

Higher Order Linear ODEs : Non-Homogeneous

General Solution using Variation of Parameters

Example. Find the general solution of the following non-homogeneous ODE:

$$4y'' + 36y = \csc 3x$$

Solution. Now, let us consider a particular solution of the form,

$$y_p = A(x)\cos 3x + B(x)\sin 3x,$$

where

$$A(x) = \int \frac{W_1}{W}\,dx = -\frac{1}{12}\int 1\,dx = -x/12$$

$$B(x) = \int \frac{W_2}{W}\,dx = \frac{1}{12}\int \cot 3x\,dx = \frac{1}{12\times 3}\ln|\sin 3x| = \frac{1}{36}\ln|\sin 3x|$$

Thus the particular solution becomes,

$$y_p = \left(-\frac{x}{12}\right)\cos 3x + \left(\frac{1}{36}\ln|\sin 3x|\right)\sin 3x = -\frac{1}{12}x\cos 3x + \frac{1}{36}\sin 3x\ln|\sin 3x|$$

Therefore, the general solution becomes,

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}\sin 3x\ln|\sin 3x|.$$

Higher Order Linear ODEs : Non-Homogeneous

Exercise 4.6

1-18. Solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$

2. $y'' + y = \tan x$

3. $y'' + y = \sin x$

4. $y'' + y = \sec\theta \tan\theta$

5. $y'' + y = \cos^2 x$

6. $y'' + y = \sec^2 x$

7. $y'' - y = \cosh x$

8. $y'' - y = \sinh 2x$

9. $y'' - 4y = \dfrac{e^{2x}}{x}$

10. $y'' - 9y = \dfrac{9x}{e^{3x}}$

11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

13. $y'' + 3y' + 2y = \sin e^x$

15. $y'' + 2y' + y = e^{-t} \ln t$

16. $2y'' + 2y' + y = 4\sqrt{x}$

17. $3y'' - 6y' + 6y = e^x \sec x$

18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

19-22. Solve each differential equation by variation of parameters, subject to the initial conditions $y(0) = 1, y'(0) = 0$

19. $4y'' - y = xe^{x/2}$

20. $2y'' + y' - y = x + 1$

21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$