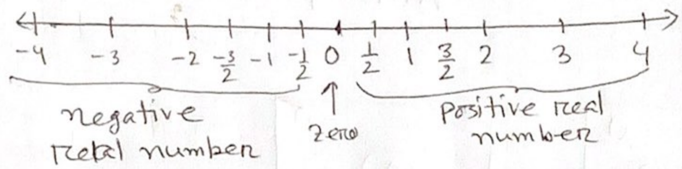
Real number line:

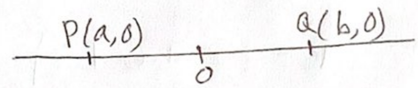
The real number can be represented by points on a line is called the real number line.



Find distance on a real number line:

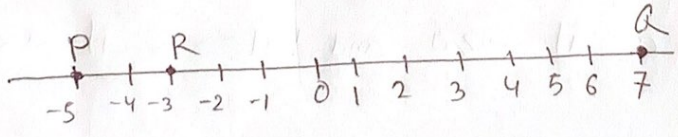
If P and Q are points on a real number line with coordinates a and b respectively, then the distance betn P and Q, denoted by $d(P,Q)$ as

$d(P,Q) = |b-a|$



Example:

Say P, Q, R be points on a real number line with coordinates $-5, 7$ and -3 respectively. Find the distance between (I) P and Q (II) Q and R, (III) P and R.



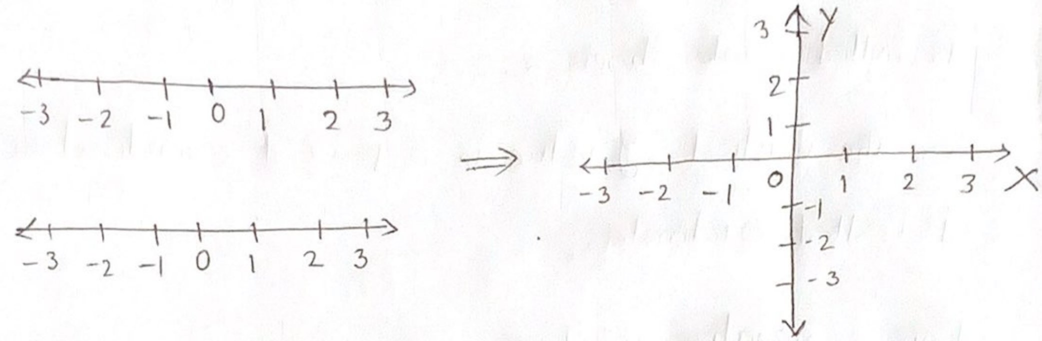
(a) $d(P,Q) = |b-a| = |7-(-5)| = |7+5| = 12$

(b) $d(Q,R) = |b-a| = |7-(-3)| = |7+3| = 10$

(c) $d(P,R) = |b-a| = |-3-(-5)| = |-3+5| = 2$

(d) $d(R,P) = ?$

If we put two real number lines together we get 2D coordinates or cartesian coordinates.



$\xrightarrow{x}$ the left-right (horizontal direction)

$\uparrow y$ the up-down (vertical direction)

Any point P in the xy plane can be located by using an ordered pair (x, y) of real numbers.

Here, x denote the signed distance P from y-axis
and y " " " " P from x-axis

* Signed distance means, if P is to the right of y-axis then $x>0$;
if P is to the left of y-axis, then $x<0$

For example, ~~3 units~~ $(3, 2)$ means 3 units to the right along x direction and 2 units up/vertical direction.

The origin:

The point $(0, 0)$ is called origin.

Abscissa and ordinate:

The horizontal x value in a pair of coordinates is called ~~ab~~ abscissa.

The vertical y value in a pair of coordinates is called ordinate.

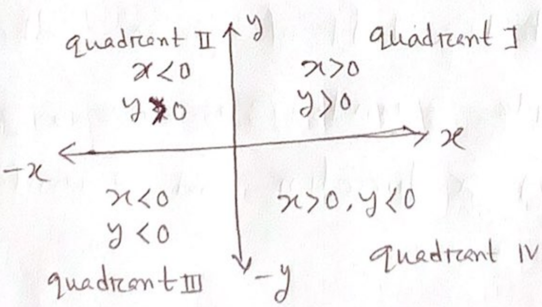
For a negative number

- go backwards for x
- go down for y

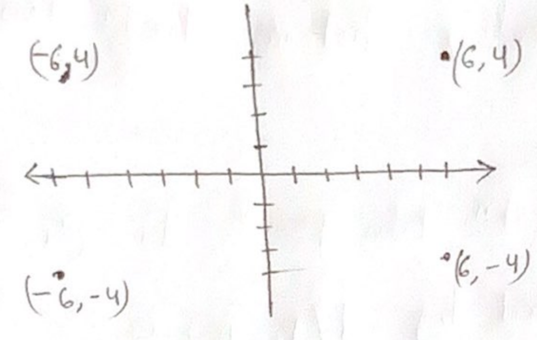
For example $(-6,-5)$ means go back along the x axis 6 units and then go down along y axis 5 units.

Quadrants:

When we include negative values, the x and y axes divides the space into 4 pieces:



Thus we can now locate points;



Exercise:

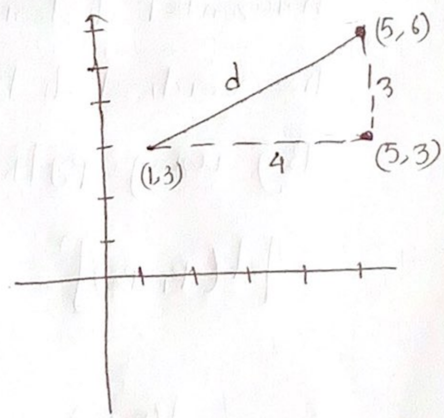
Tell in which quadrant or on what coordinate axis each point lies?

(I) $A = (-3, 2)$ (II) $B = (6, 0)$ (III) $C = (-2, -2)$ (IV) $D = (6, 5)$

(V) $E = (0, -3)$ (VI) $F = (6, -3)$

Finding distance between two points:

Find the distance d between the points $(1, 3)$ and $(5, 6)$



Soln:

We have to find d.

Draw horizontal line from $(1,3)$ to $(5,3)$

Draw vertical line from $(5,3)$ to $(5,6)$ form a right angle triangle.

One lege of the triangle is 4 since $|5-1| = 4$ and another leg of the triangle is 3 since $|6-3| = 3$

By pythagorean theorem

$$d^2 = 4^2 + 3^2 = 25$$

$$\Rightarrow d = \sqrt{25} = 5$$

The distance formula provides a straight forward method for computing distance betn two points.

Distance formula:

The distance betn two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ denoted by $d(P_1, P_2)$ is

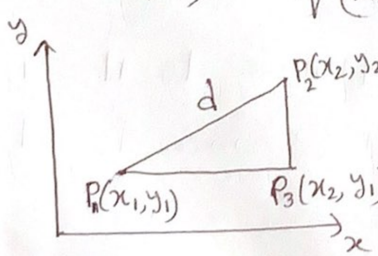
$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

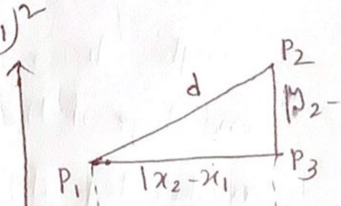
Proof:

Let (x_1, y_1) and (x_2, y_2) denote the coordinates of point P_1 and P_2 respectively. Assume that the line joining P_1 and P_2 are neither horizontal nor vertical. The coordinate of P_3 are (x_2, y_1). Then the horizontal distance from P_1 to P_3 is $|x_2 - x_1|$. The vertical distance from P_3 to P_2 is $|y_2 - y_1|$ By using pythagorean theorem, it follows that

$$[d(P_1, P_2)]^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

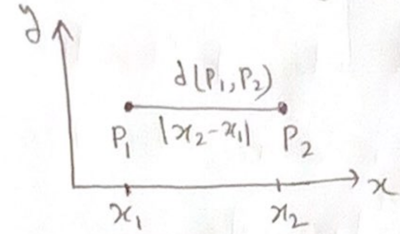




If the line P_1 and P_2 is horizontal then the y coordinate of P_1 is equal to y coordinate of P_2 i.e $y_1 = y_2$

In this case

$$d(P_1, P_2) = \sqrt{(x_2-x_1)^2 + 0^2} = \sqrt{(x_2-x_1)^2} = |x_2 - x_1|$$



Example:

Using distance formula, find the distance d betn the points $(-4, 5)$ and $(3, 2)$.

Soln.

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{[3-(-4)]^2 + (2-5)^2} = \sqrt{7^2+(-3)^2}$$

$$= \sqrt{49+9}$$

$$= \sqrt{58} \simeq 7.62$$

Exercise:

Plot the following points and find distance between P_1 and P_2.

i) $P_1 = (4,-3)$, $P_2 = (6,4)$
ii) $P_1 = (-4,-3)$, $P_2 = (6,2)$
iii) $P_1 = (a,b)$, $P_2 = (0,0)$
iv) $P_1 = (-1,0)$, $P_2 = (9,8)$

Example:

Consider the three points $A=(-2,1)$, $B=(2,3)$ and $C=(3,1)$

(a) Plot each point and form the triangle ABC

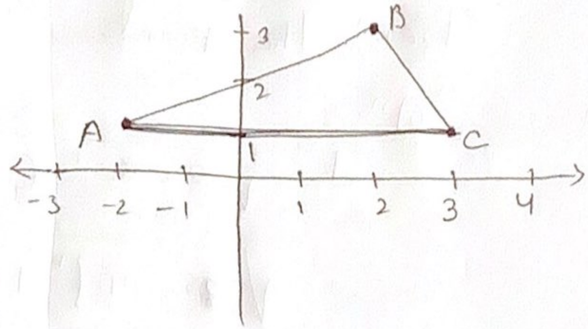
(b) Find the length of each side of the triangle.

(c) Verify that the triangle is a right triangle.

(d) Find the area of the triangle.

Soln:

(a)



(b) $d(A,B) = \sqrt{[2-(-2)]^2+(3-1)^2}$

$= \sqrt{16+4}$

$= \sqrt{20}$

$= \sqrt{4\times 5}$

$= 2\sqrt{5}$

$d(B,C) = \sqrt{(3-2)^2+(1-3)^2} = \sqrt{1+4} = \sqrt{5}$

$d(A,C) = \sqrt{(3-(-2))^2+(1-1)^2} = \sqrt{25+0} = 5$

(c) To show that the triangle is a right angled triangle, we need to show that the sum of the squares of lengths of two sides equals to the square of the length of the third side.

So let's check whether

$$[d(A,B)]^2 + [d(B,C)]^2 = [d(A,C)]^2$$

From part (b)

$$[d(A,B)]^2 + [d(B,C)]^2$$

$$= (2\sqrt{5})^2 + (\sqrt{5})^2 = (4\times 5)+5 = 20+5 = 25$$

$$= [d(A,C)]^2$$

So it follows that the given triangle is a right handed triangle.

(d) Since the right angle is at vertex B, then the sides AB and BC form the base and height of the triangle. It's area is

$$\text{Area} = \frac{1}{2}(\text{Base} \times \text{height}) = \frac{1}{2}(2\sqrt{5})\sqrt{5} = 5 \text{ square units.}$$

Exercise:

Plot each point and form the triangle ABC. Verify that the triangle is a right trianle. Find its area:

1) $A = (-2, 5)$; $B = (1, 3)$; $C = (-1, 0)$

11) $A = (-6, 3)$; $B = (3, -5)$; $C = (-1, 5)$

111) $A = (4, 3)$; $B = (4, 1)$; $C = (2, 1)$

Derive midpoint formula for a line segment.

Let P_1 and P_2 be the endpoints of a line segment and let $M=(x,y)$ be the midpoint on the line segment.

The triangles P_1AM and P_2MB are congurent, so the corresponding sides are equal in length:

$x - x_1 = x_2 - x$

$\Rightarrow x + x = x_2 + x_1$

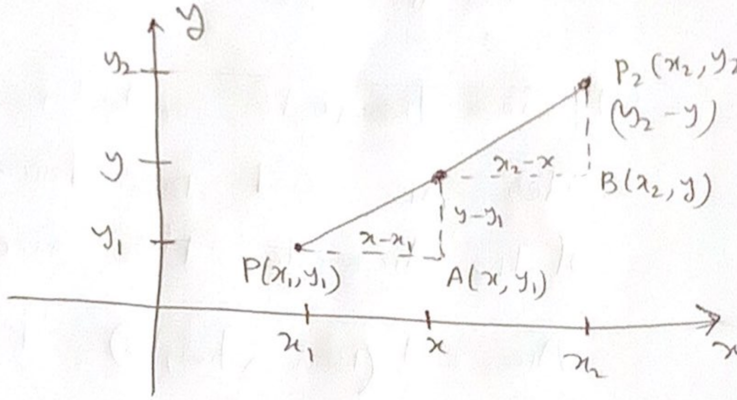
$\Rightarrow 2x = x_1 + x_2$

$\Rightarrow x = \frac{x_1 + x_2}{2}$

$y - y_1 = y_2 - y$

$\Rightarrow 2y = y_2 + y_1$

$\Rightarrow y = \frac{y_1 + y_2}{2}$



Example.

If you have $P_1 = (\cancel{+}3, -4)$ and $P_2 = (5, 4)$, then find the midpoint of the line segment

Soln: $P_1 = (3, -4)$, $P_2 = (5, 4)$

Here $x_1 = 3$, $x_2 = 5$, $y_1 = -4$, $y_2 = 4$

$$\text{midpoint } (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{3+5}{2}, \frac{-4, +4}{2}\right)$$

$$= (4, 0)$$

Exercise:

(i) $P_1 = (-3, 2)$, $P_2 = (6, 0)$

(ii) $P_1 = (-4, -3)$, $P_2 = (2, 2)$

Example:

The diameter of a circle has endpoints $(-1, -4)$ and $(5, -4)$. Find the center of the circle.

Solⁿ: The center of the circle is the center or midpoint of its diameter. Thus the midpoint formula will yield the center point.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-1+5}{2}, \frac{-4-4}{2}\right)$$

$$= \left(\frac{4}{2}, \frac{-8}{2}\right)$$

$$= (2, -4)$$