First Order ODEs - Exact equations

- ***Differential of Function of two variables:*** $z = f(x, y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

- ***For the special case:*** $z = f(x, y) = c$

$$dz = 0 \Rightarrow \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 0 \Rightarrow M(x, y)dx + N(x, y)\, dy = 0$$

$$where,\;\; M(x, y) = \frac{\partial z}{\partial x} \; and \; N(x, y) = \frac{\partial z}{\partial y}$$

$$Then, \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 z}{\partial y\, \partial x} \; and \; \frac{\partial N}{\partial x} = \frac{\partial^2 z}{\partial x\, \partial y}.$$

[If $M(x, y)$ and $N(x, y)$ are continuous and have first partials]

$$Thus, \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

First Order ODEs - Exact equations

Definition. A first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called an **exact differential equation** if $M(x, y)dx + N(x, y)dy$ is **exactly** the total differential of $f(x, y)$.

Criterion for an Exact ODEs. If $M(x, y)$, $N(x, y)$ are continuous in x and y, and have continuous first partial derivatives, then a necessary and sufficient condition that

$$M(x, y)dx + N(x, y)dy = 0$$

be an exact differential equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

First Order ODEs - Exact equations

Example. Solve the differential equations,

$$(x+y)\,dx + (x-y)dy = 0$$

[The ODE is not separable]

Solution. Here, $M(x,y) = x + y$ and $N(x,y) = x - y$ where

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = 1 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

STEP-01
Exactness
Test

Thus, the given equation is an exact and there exists a solution $f(x,y) = c$ such that

STEP-02
Find $f(x,y)$

$$\frac{\partial f}{\partial x} = M(x,y) = x + y, \qquad and \qquad \frac{\partial f}{\partial y} = N(x,y) = x - y$$

$$\Rightarrow \int \frac{\partial f}{\partial x}\,dx = \int (x+y)\,dx$$

$$\Rightarrow f(x,y) = \frac{x^2}{2} + xy + g(y)$$

STEP-03
Find $g(y)$

$$\Rightarrow x + g'(y) = x - y \Rightarrow g'(y) = -y$$

$$\Rightarrow \int g'(y)\,dy = -\int y\,dy$$

$$\Rightarrow g(y) = -\frac{y^2}{2}$$

STEP-04
Write the Solution

The desired solution becomes, $f(x,y) = c \Rightarrow \frac{x^2}{2} + xy - \frac{y^2}{2} = c \Rightarrow \boldsymbol{x^2 + 2xy - y^2 = c}$

First Order ODEs - Exact equations

Example. Solve the differential equations,

$$2xy\,dx + (x^2 - 1)dy = 0 \qquad \text{[The ODE is separable]}$$

Solution. Here, $M(x,y) = 2xy$ and $N(x,y) = x^2 - 1$ where

$$\frac{\partial M}{\partial y} = 2x, and \ \frac{\partial N}{\partial x} = 2x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, the given equation is an exact and there exists a solution $f(x,y) = c$ such that

$$\frac{\partial f}{\partial x} = M(x,y) = 2xy, \qquad and \qquad \frac{\partial f}{\partial y} = N(x,y) = x^2 - 1$$

$$\Rightarrow \int \frac{\partial f}{\partial x} dx = \int 2xy\,dx \qquad\qquad \Rightarrow x^2 + g'(y) = x^2 - 1 \ \Rightarrow g'(y) = -1$$

$$\Rightarrow f(x,y) = x^2 y + g(y) \qquad\qquad \Rightarrow \int g'(y)\,dy = -\int dy$$

$$\Rightarrow g(y) = -y$$

The desired solution becomes, $f(x,y) = c \Rightarrow \boldsymbol{x^2 y - y = c}$

First Order ODEs - Exact equations

Example. Solve the differential equations,

$$(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$$ [The ODE is not separable]

Solution. Here, $M(x,y) = e^{2y} - y\cos xy$ and $N(x,y) = 2xe^{2y} - x\cos xy + 2y$ where

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy\,, and\ \frac{\partial N}{\partial x} = 2e^{2y} - \cos xy + xy\sin xy \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, the given equation is an exact and there exists a solution $f(x,y) = c$ such that

$$\frac{\partial f}{\partial x} = M(x,y) = e^{2y} - y\cos xy\,, \qquad and \qquad \frac{\partial f}{\partial y} = N(x,y) = 2xe^{2y} - x\cos xy + 2y$$

$$\Rightarrow \int \frac{\partial f}{\partial x}\,dx = \int (e^{2y} - y\cos xy)\,dx$$

$$\Rightarrow f(x,y) = xe^{2y} - \sin xy + g(y)$$

$$\Rightarrow 2xe^{2y} - x\cos xy + g'(y) = 2xe^{2y} - x\cos xy + 2y$$

$$\Rightarrow g'(y) = 2y \Rightarrow \int g'(y)\,dy = \int 2y\,dy$$

$$\Rightarrow g(y) = y^2$$

The desired solution becomes, $f(x,y) = c \Rightarrow \boldsymbol{xe^{2y} - \sin xy + y^2 = c}$

First Order ODEs - Exact equations

Exercises 2.4

H.W. from the text book

Determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$

2. $(2x + y)\,dx - (x + 6y)\,dy = 0$

3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$

4. $(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$

5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$

7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$

8. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)\,dy$

9. $(x - y^3 + y^2\sin x)\,dx = (3xy^2 + 2y\cos x)\,dy$

10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$

11. $(y\ln y - e^{-xy})\,dx + \left(\dfrac{1}{y} + x\ln y\right)dy = 0$

12. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$

13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$

14. $\left(1 - \dfrac{3}{y} + x\right)\dfrac{dy}{dx} + y = \dfrac{3}{x} - 1$

15. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$

16. $(5y - 2x)y' - 2y = 0$

17. $(\tan x - \sin x\sin y)\,dx + \cos x\cos y\,dy = 0$

18. $(2y\sin x\cos x - y + 2y^2e^{xy^2})\,dx$
$= (x - \sin^2 x - 4xye^{xy^2})\,dy$

19. $(4t^3y - 15t^2 - y)\,dt + (t^4 + 3y^2 - t)\,dy = 0$

20. $\left(\dfrac{1}{t} + \dfrac{1}{t^2} - \dfrac{y}{t^2 + y^2}\right)dt + \left(ye^y + \dfrac{t}{t^2 + y^2}\right)dy = 0$

First Order ODEs - Exact equations

Exercises 2.4

H.W. from the text book

Solve the given initial-value problem.

21. $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$

22. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$

23. $(4y + 2t - 5)\,dt + (6y + 4t - 1)\,dy = 0, \quad y(-1) = 2$

24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$

25. $(y^2 \cos x - 3x^2y - 2x)\,dx$

$+ (2y \sin x - x^3 + \ln y)\,dy = 0, \quad y(0) = e$

26. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \quad y(0) = 1$

Verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor $\mu(x, y)$ and verify that the new equation is exact. Solve.

29. $(-xy \sin x + 2y \cos x)\,dx + 2x \cos x\,dy = 0;$
$\mu(x, y) = xy$

30. $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0;$
$\mu(x, y) = (x + y)^{-2}$