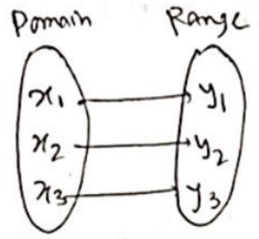
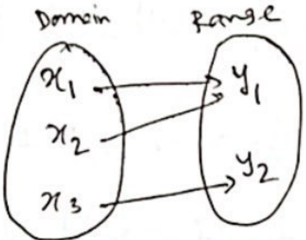
Inverse functions

A function is one-to-one if any two different ~~domain~~ inputs in the domain correspond to two different outputs in the range.
i.e, if x_1 and x_2 are two different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$



one-to-one



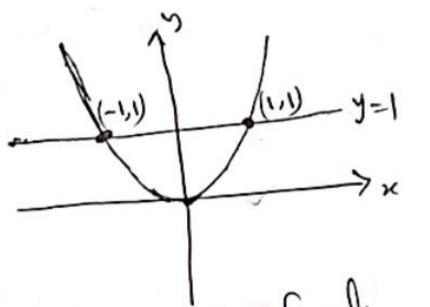
not one-to-one.

Ex $\{(-2,6), (-1,3), (0,2), (1,5), (2,8)\}$ is one-to-one because there are no two distinct inputs that corresponds to same output.

Horizontal line test:

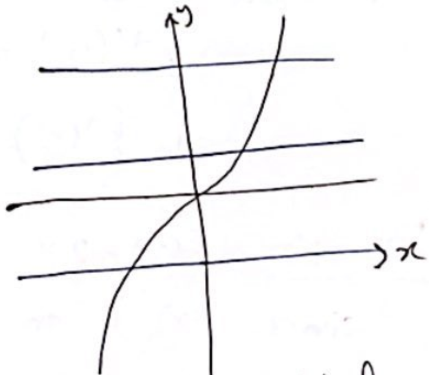
If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

1. $f(x) = x^2$



Not one-to-one function because the horizontal line $y=1$, intersects the graph of f twice, at $(1,1)$ and $(-1,1)$

ii) $f(x) = x^3$



one-to-one function since every horizontal line intersects the graph exactly once.

Ex: Find the inverse of the function

$$\{(-3,-27),(-2,-8),(-1,-1),(0,0),(1,1)\ (2,8),\ (3,27)\}$$

Soln: The given function is one-to-one function. The inverse of the given function can be found by interchanging the entries of each orderedthein.

$$\{(-27,-3),(-8,-2),(-1,-1),(0,0),(1,1),\ (8,2),(27,3)\}$$

Domain of $f = \{-3,-2,-1,0,1,2,3\}$

Range of $f = \{-27,-8,-1,0,1,8,27\}$

Domain of inverse of $f = \{-27,-8,-1,0,1,8,27\}$

Range of " " $f = \{-3,-2,-1,0,1,2,3\}$

Note: If f is one-to-one function then it has an inverse function f^{-1}.

Domain of f = Range of f^{-1}

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In other words,

$f^{-1}(f(x)) = x$ where x is in the domain of f

$f(f^{-1}(x)) = x$ where x in inth domain of f^{-1}.

Example: $f(x) = 2x$

since $f(x)$ is an increasing function so f is one-to-one.

The inverse of $f(x)$ is $f^{-1}(x) = \frac{x}{2}$.

Verify:

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}.2x = x$$

$$f(f^{-1}(x)) = f\left(\frac{x}{2}\right) = 2.\frac{x}{2} = x$$

Example:

Verify that the inverse of $f(x) = \frac{1}{x-1}$ is $f^{-1}(x) = \frac{1}{x}+1$

Soln: The domain of $f = \{x \mid x \neq 1\}$ and

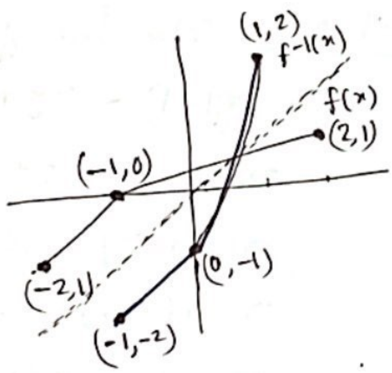
the domain of $f^{-1} = \{x \mid x \neq 0\}$

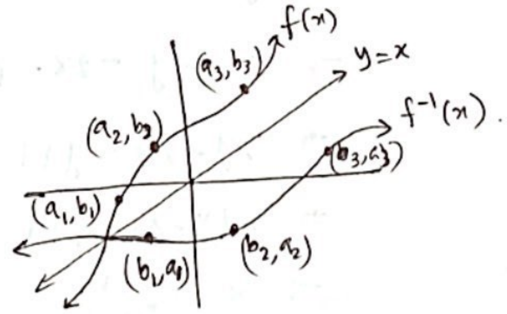
Now. $\cancel{f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}} + 1 = \frac{x-1}{x-1}}$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}} + 1 = x-1+1 = x \quad \text{Provided } x \neq 1$$

$$f(f^{-1}(x)) = f\left(\frac{1}{x}+1\right) = \frac{1}{\frac{1}{x}+1-1} = \frac{1}{\frac{1}{x}} = x \quad \text{Provided } x \neq 0$$

The graph of one-to-one function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y=x$





Find the inverse of a function defined by an equation:

Step 1: In $y=f(x)$, interchange the variable x and y to obtain $x=f(y)$

Step 2: If possible, solve the implicit eq^n for y in terms of x to obtain $y=f^{-1}(x)$

Step 3: Check the result by showing that

$$f^{-1}(f(x))=x \quad \text{and} \quad f(f^{-1}(x))=x$$

Example: Find the inverse of the function $f(x)=\frac{2x+1}{x-1}$, $x\neq 1$.

Solution:

one way to do it

$$y=\frac{2x+1}{x-1}$$

$$\Rightarrow y(x-1)=2x+1$$

$$\Rightarrow xy-y-2x=1$$

$$\Rightarrow xy-2x=y+1$$

$$\Rightarrow x(y-2)=y+1$$

$$\Rightarrow x=\frac{y+1}{y-2}$$

Interchang x and y

$$y=\frac{x+1}{x-2}=f^{-1}(x)$$

another way

Step 1: Interchange x and y.

$$x=\frac{2y+1}{y-1}$$

Step 2: Solve for y

$$x(y-1)=2y+1$$

$$\Rightarrow xy-x=2y+1$$

$$\Rightarrow xy-2y=x+1$$

$$\Rightarrow y(x-2)=x+1$$

$$\Rightarrow y=\frac{x+1}{x-2}=f^{-1}(x)$$

$\therefore$ The inverse is $f^{-1}(x)=\frac{x+1}{x-2}$; $x\neq 2$

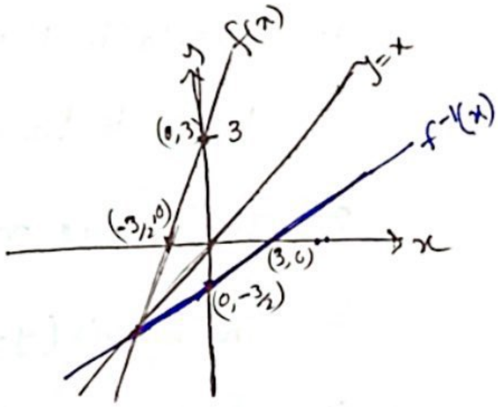
Step 3 Check:

$$f^{-1}(f(x))=f^{-1}\left(\frac{2x+1}{x-1}\right)=\frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2}=\frac{2x+1+x-1}{2x+1-2x+2}=\frac{3x}{3}=x$$

$$f(f^{-1}(x))=f\left(\frac{x+1}{x-2}\right)=\frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1}=\frac{2x+2+x-2}{x+1-x+2}=\frac{3x}{3}=x$$

(checked)

Example: Find inverse of $f(x) = 2x+3$. Graph f and f^{-1} on the same coordinate axes.



Sol^n:

$$y = 2x+3$$

Interchange x and y.

$$x = 2y+3$$

Solve for y. $x = 2y+3$

$$\Rightarrow 2y = x-3$$

$$\Rightarrow y = \frac{1}{2}(x-3) = f^{-1}(x)$$

$\therefore$ The inverse function is $f^{-1}(x) = \frac{1}{2}(x-3)$

Check:

$$f^{-1}(f(x)) = f^{-1}(2x+3) = \frac{1}{2}(2x+3-3) = \frac{1}{2}\cdot 2x = x$$

$$f(f^{-1}(x)) = f\left(\frac{x-3}{2}\right) = 2\frac{(x-3)}{2} + 3 = x-3+3 = x$$

Exponential functions:

An exponential function is a function of the form

$$f(x) = ca^x$$

where a is a positive real number $(a>0)$, $a \neq 1$ and $c \neq 0$ is a real number.

The base a is the growth factor.

Properties of $f(x) = a^x$; $a > 1$, $0 < a < 1$

1. Domain is the set of all real numbers or $(-\infty, +\infty)$

   Range is the set of positive real numbers or $(0, \infty)$

2. There are no x-intercepts, the y-intercepts is 1.

3. The x-axis $(y = 0)$ is the horizontal asymptote as $x \to \infty$.

4. $f(x) = a^x$, where $a > 1$, is an increasing function and is one-to-one.

5. $f(x) = a^x$, $0 < a < 1$, is a decreasing function and is one to one.

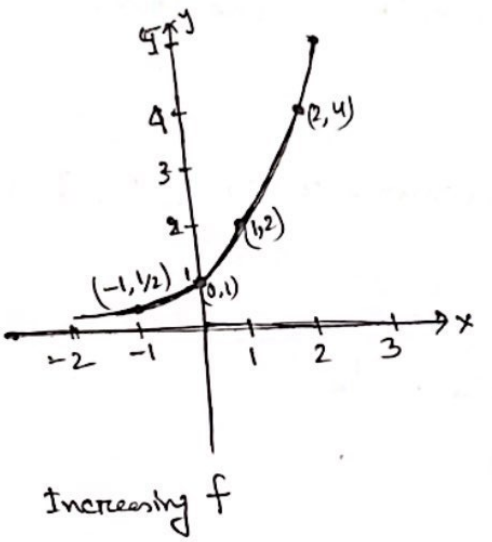
6. The graph of f contains the points $(0, 1)$, $(1, a)$ and $(-1, 1/a)$ when $a > 1$

7. The graph of f contains the points $(-1, \frac{1}{a})$, $(0, 1)$ and $(1, a)$. when $0 < a < 1$

8. The graph of f is smooth and continuous, with no corners or gaps.

Graph of $y = 2^x$

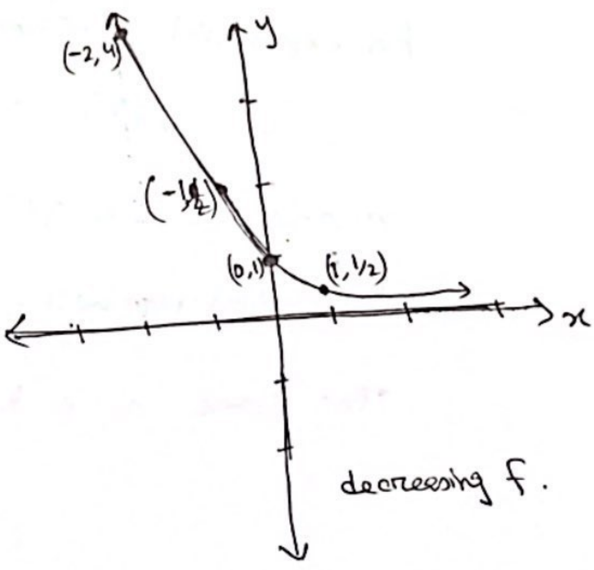
Here $a = 2$ which is > 1



Increasing f

Graph of $y = \left(\frac{1}{2}\right)^x$

Here $a = 1/2$; $0 < \frac{1}{2} < 1$

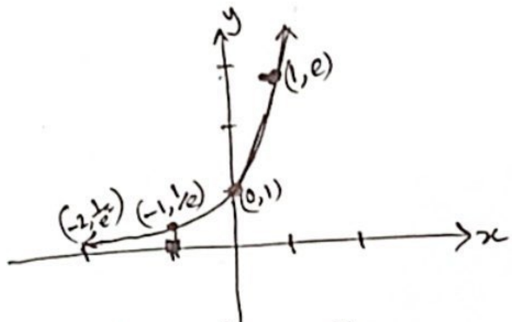


decreasing f.

Ex: Graph $f(x) = 2^{-x} - 5$

The number e is defined as the number that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$

i.e $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$



Graph of $y=e^x$

Ex: Graph of $f(x) = -e^{x-3}$

Ex: Solve (a) $3^{x+1} = 81$

Soln: Given $3^{x+1} = 81$

$\Rightarrow 3^{x+1} = 3^4$

$\Rightarrow x+1 = 4 \Rightarrow x = 4-1 \Rightarrow x = 3$

(b) $4^{2x+1} = 8^{2x+3}$

$\Rightarrow 2^{2(2x+1)} = 2^{3(2x+3)}$

$\Rightarrow 4x+2 = 6x+9$ [If $a^u = a^v$ then $u=v$]

$\Rightarrow 6x - 4x = 2-9$

$\Rightarrow 2x = -7$

$\Rightarrow x = -\frac{7}{2}$

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Ex: Solve $e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3}$

Soln: Given

$$e^{-x^2} = (e^x)^2 \frac{1}{e^3}$$

$$\Rightarrow e^{-x^2} = e^{2x} \cdot e^{-3}$$

$$\Rightarrow e^{-x^2} = e^{2x-3}$$

$$\Rightarrow -x^2 = 2x - 3$$

$$\Rightarrow x^2 + 2x - 3 = 0$$

$$\Rightarrow x^2 + 3x - x - 3 = 0 \quad \cancel{\Rightarrow x}$$

$$\Rightarrow x(x+3) - 1(x+3) = 0$$

$$\Rightarrow (x+3)(x-1) = 0$$

$$\therefore x = -3 \text{ or } x = 1$$

$\therefore$ The solution set is $\{-3, 1\}$