Chapter # 03
(Topics in Differentiation)

3.6 L'HÔPITAL'S Rule; Indeterminate Forms: In this section we will discuss a general method for using derivatives to find limits. This method will enable us to establish limits with certainty that earlier in the text we were only able to conjecture using numerical or graphical evidence. The method that we will discuss in this section is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

Indeterminate Forms of Type $\frac{0}{0}$: Recall that a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

in which $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an ***indeterminate form of type 0/0***. Some examples encountered earlier in the text are

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Theorem (L' Hopital's Rule for Form 0/0): Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

Applying L'Hôpital's Rule:

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

Example 1: Find the limit

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution: The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\frac{d}{dx}[x^2-4]}{\frac{d}{dx}[x-2]}=\lim_{x\to 2}\frac{2x}{1}=4$$

This agrees with the computation

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Example 2: In each part confirm that the limit is an indeterminate form of type 0/0, and evaluate it using L'Hôpital's rule.

$$\text{(a)}\ \lim_{x\to 0}\frac{\sin 2x}{x}\qquad \text{(b)}\ \lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}\qquad \text{(c)}\ \lim_{x\to 0}\frac{e^x-1}{x^3}$$

$$\text{(d)}\ \lim_{x\to 0^-}\frac{\tan x}{x^2}\qquad \text{(e)}\ \lim_{x\to 0}\frac{1-\cos x}{x^2}\qquad \text{(f)}\ \lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$$

Solution: (*a*) The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

(*b*) The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2}\frac{\frac{d}{dx}[1-\sin x]}{\frac{d}{dx}[\cos x]}=\lim_{x\to \pi/2}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

(*c*) The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\frac{d}{dx}[e^x-1]}{\frac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

(*d*) The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

(*e*) The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

(*f*) The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to +\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to +\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0$$

Indeterminate Form of Type $\frac{\infty}{\infty}$: When we want to indicate that the limit (or a one-sided limit) of a function is $+\infty\ or-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\begin{array}{llllll}\lim_{x\to a^+} f(x)=\infty & \text{means} & \lim_{x\to a^+} f(x)=+\infty & \text{or} & \lim_{x\to a^+} f(x)=-\infty\\ \lim_{x\to +\infty} f(x)=\infty & \text{means} & \lim_{x\to +\infty} f(x)=+\infty & \text{or} & \lim_{x\to +\infty} f(x)=-\infty\\ \lim_{x\to a} f(x)=\infty & \text{means} & \lim_{x\to a^+} f(x)=\pm\infty & \text{and} & \lim_{x\to a^-} f(x)=\pm\infty\end{array}$$

The limit of a ratio, $\frac{f(x)}{g(x)}$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type $\frac{\infty}{\infty}$.***

Theorem (L'Hôpital's Rule for Form $\frac{\infty}{\infty}$): Suppose that ***f*** and ***g*** are differentiable functions on an open interval containing ***x = a***, except possibly at ***x = a***, and that

$$\lim_{x\to a} f(x)=\infty \quad and \quad \lim_{x\to a} g(x)=\infty$$

If $\lim_{x \to a}[f'(x)/g'(x)]$ *exists, or if this limit is* $+\infty$ *or* $-\infty$, *then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, *or as* $x \to +\infty$.

Example 3: In each part confirm that the limit is an indeterminate form of type $\frac{\infty}{\infty}$ and apply L'Hôpital's rule.

$$\text{(a)} \quad \lim_{x \to +\infty} \frac{x}{e^x} \qquad \text{(b)} \quad \lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

Solution: (*a*) The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type $\frac{\infty}{\infty}$. Applying L'Hôpital's rule yields

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

(*b*) Given,

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type $\frac{\infty}{\infty}$. Applying L'Hôpital's rule yields

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x}$$

This last limit is again an indeterminate form of type $\frac{\infty}{\infty}$. Moreover, any additional applications of L'Hôpital's rule will yield powers of $\frac{1}{x}$ in the numerator and expressions involving $\csc x$ *and* $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms.

Now,

$$\lim_{x \to 0^+} \left(-\frac{\sin x}{x} \tan x \right) = -\lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} \tan x = -(1)(0) = 0$$

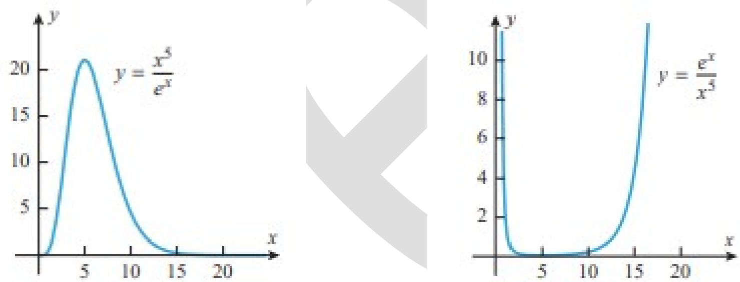
Thus

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = 0$$

Analyzing the Growth of Exponential Functions Using L'Hopital's Rule: If $\boldsymbol{n}$ is any positive integer, then $\boldsymbol{x^n} \to +\infty\ \boldsymbol{as}\ \boldsymbol{x} \to \infty$. Such integer powers of $\boldsymbol{x}$ are sometimes used as "***measuring sticks***" to describe how rapidly other functions grow. For example, we know that $\boldsymbol{e^x} \to +\infty\ \boldsymbol{as}\ \boldsymbol{x} \to \infty$and that the growth of $\boldsymbol{e^x}$ is very rapid; however, the growth of $\boldsymbol{x^n}$ is also rapid when $\boldsymbol{n}$ is a high power, so it is reasonable to ask whether high powers of $\boldsymbol{x}$ grow more or less rapidly than $\boldsymbol{e^x}$. One way to investigate this is to examine the behavior of the ratio $\frac{\boldsymbol{x^n}}{\boldsymbol{e^x}}\ \boldsymbol{as}\ \boldsymbol{x} \to \infty$. For example, Figure **(*a*)** shows the graph of $\boldsymbol{y} = \frac{\boldsymbol{x^5}}{\boldsymbol{e^x}}$. This graph suggests that $\frac{\boldsymbol{x^5}}{\boldsymbol{e^x}} \to \boldsymbol{0}\ \boldsymbol{as}\ \boldsymbol{x} \to \infty$, and this implies that the growth of the function $\boldsymbol{e^x}$ is sufficiently rapid that its values eventually overtake those of $\boldsymbol{x^5}$ and force the ratio toward zero. Stated informally, "$\boldsymbol{e^x}$ eventually grows more rapidly than $\boldsymbol{x^5}$." The same conclusion could have been reached by putting $\boldsymbol{e^x}$ on top and examining the behavior of $\frac{\boldsymbol{e^x}}{\boldsymbol{x^5}}\ \boldsymbol{as}\ \boldsymbol{x} \to \infty$, (Figure ***b***). In this case the values of $\boldsymbol{e^x}$ eventually overtake those of $\boldsymbol{x^5}$ and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that $\boldsymbol{e^x}$ eventually grows more rapidly than any positive integer power of $\boldsymbol{x}$, that is,

$$\lim_{x \to +\infty} \frac{x^n}{e^x} = 0 \quad \text{and} \quad \lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty$$

Both limits are indeterminate forms of type $\frac{\infty}{\infty}$ that can be evaluated using L'Hôpital's rule.



(*a*) (*b*)

Indeterminate Forms of Type $\mathbf{0} \cdot \infty$: Indeterminate forms of type $\mathbf{0} \cdot \infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type **0/0** or $\frac{\infty}{\infty}$:

Example 4: Evaluate

$$\text{(a)} \quad \lim_{x \to 0^+} x \ln x \qquad \text{(b)} \quad \lim_{x \to \pi/4} (1 - \tan x) \sec 2x$$

Solution: (*a*) The factor $\boldsymbol{x}$ has a limit of **0** and the factor ***ln x*** has a limit of $-\infty$, so the stated problem is an indeterminate form of type $\mathbf{0} \cdot \infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} \quad \text{or} \quad \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

the first being an indeterminate form of type $\frac{\infty}{\infty}$ and the second an indeterminate form of type $\frac{\mathbf{0}}{\mathbf{0}}$. However, the first form is the preferred initial choice because the derivative of $\frac{\mathbf{1}}{\boldsymbol{x}}$ is less complicated than the derivative of $\frac{\mathbf{1}}{\boldsymbol{ln\,x}}$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

(***b***) The stated problem is an indeterminate form of type $\mathbf{0}\,.\,\infty$. We will convert it to an indeterminate form of type 0/0:

$$\lim_{x\to \pi/4}(1-\tan x)\sec 2x = \lim_{x\to \pi/4} \frac{1-\tan x}{1/\sec 2x} = \lim_{x\to \pi/4} \frac{1-\tan x}{\cos 2x}$$
$$= \lim_{x\to \pi/4} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1$$

Indeterminate Forms of Type $\infty - \infty$: A limit problem that leads to one of the expressions

$$(+\infty)-(+\infty), \quad (-\infty)-(-\infty),$$
$$(+\infty)+(-\infty), \quad (-\infty)+(+\infty)$$

is called an ***indeterminate form of type*** $\infty - \infty$**.** Indeterminate forms of type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 5: Evaluate

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$$

Solution: Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty - \infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x}$$

which is an indeterminate form of type **0/0**. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$$

Indeterminate Forms of Type $0^0, \infty^0, 1^\infty$: Limits of the form

$$\lim f(x)^{g(x)}$$

can give rise to indeterminate forms of the types $0^0, \infty^0$ ***and*** 1^∞.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then computing the limit of $\ln y$. Since

$$\ln y = \ln[f(x)^{g(x)}] = g(x) \cdot \ln[f(x)]$$

the limit of $\ln y$ will be an indeterminate form of type $0.\infty$, which can be evaluated by methods we have already studied.

Example 6: Find

$$\lim_{x \to 0} (1 + \sin x)^{1/x}$$

Solution: We begin by introducing a dependent variable

$$y = (1 + \sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1 + \sin x)^{1/x} = \frac{1}{x}\ln(1 + \sin x) = \frac{\ln(1 + \sin x)}{x}$$

Thus,

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + \sin x)}{x}$$

which is an indeterminate form of type 0/0, so by L'Hôpital's rule

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + \sin x)}{x} = \lim_{x \to 0} \frac{(\cos x)/(1 + \sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus,

$$\lim_{x \to 0} (1 + \sin x)^{1/x} = e$$

Home Work: Exercise 3.6: Problem No. 7-45