2.1 : $r_{inst} = \lim_{h \to 0} \frac{f(\boxed{x_0}+h) - f(x_0)}{h}$. At $x = \boxed{x_0}$ u

$r_{ave} = \frac{f(x_0+h) - f(x_0)}{h}$

$v_{inst} = \lim_{h \to 0} \frac{h(x_0+h) - h(x_0)}{h}$

2.2 The Derivative :

$y = f(x)$ $\frac{dy}{dx}$, $f'(x)$, $\frac{df}{dx}$

$y' = \frac{dy}{dx}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex 1 : $f(x) = x^2$, $f'(x) = ?$, tangent line (at $x = 2$) = ?

Sol:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = x^2$

$\therefore f(x+h) = (x+h)^2$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} (2x+h) = \boxed{2x}$$ Ans

At $x = 2$, $y = f(2) = 2^2 = 4$ $\therefore$ point $\boxed{(2, 4)}$

Slope at $x = 2$, $m = f'(2) = 2(2) = 4$ $\therefore$ $\boxed{m = 4}$

$\therefore$ The tangent line : $y - y_1 = m(x - x_1)$ $\Rightarrow$ $y - 4 = 4(x - 2)$

$\therefore$ $y = 4x - 4$ Ans

Point $(2, 4)$. $\boxed{m = 4}$

Point slope form : $\boxed{y - y_1 = m(x - x_1)}$

Here. $(x_1, y_1) = (2, 4)$

& $\boxed{m = 4}$

Finding an Equation for the $\boxed{\text{Tangent Line}}$ to $\boxed{y = f(x)}$ at $\boxed{x = x_0}$

Step 1 : $y = f(x_0) = ?$ point $(x_0, f(x_0))$

Step 2 : Find $\boxed{f'(x)}$ then $\boxed{m = f'(x_0)}$

Step 3 : $\boxed{y - f(x_0) = m(x - x_0)}$

Ex 2: $f(x) = x^3 - x$, $f'(x) = ?$

Sol: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = x^3 - x$

$\therefore f(x+h) = (x+h)^3 - (x+h)$

$= \lim_{h \to 0} \frac{\left[(x+h)^3 - (x+h)\right] - \left[x^3 - x\right]}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}}$ $= \lim_{h \to 0} (3x^2 + \cancel{3xh} + \cancel{h^2} - 1)$

$= 3x^2 - 1$ Ans

Ex4 : (a) $f(x)=\sqrt{x}$, $f'(x)=?$

(b) $m=?$ at $x=9$

(c) $\lim_{x\to 0^+} f'(x)=?$ $\lim_{x\to +\infty} f'(x)=?$

Sol: (a) $f(x)=\sqrt{x}$

$$\therefore f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\therefore f'(x) = \frac{1}{2\sqrt{x}} \quad \text{Ans}$$

(b) $\because f'(x) = \frac{1}{2\sqrt{x}}$

$\therefore$ The slope at $x=4$ is: $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ Ans

(c) $$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty \quad \text{Ans}$$

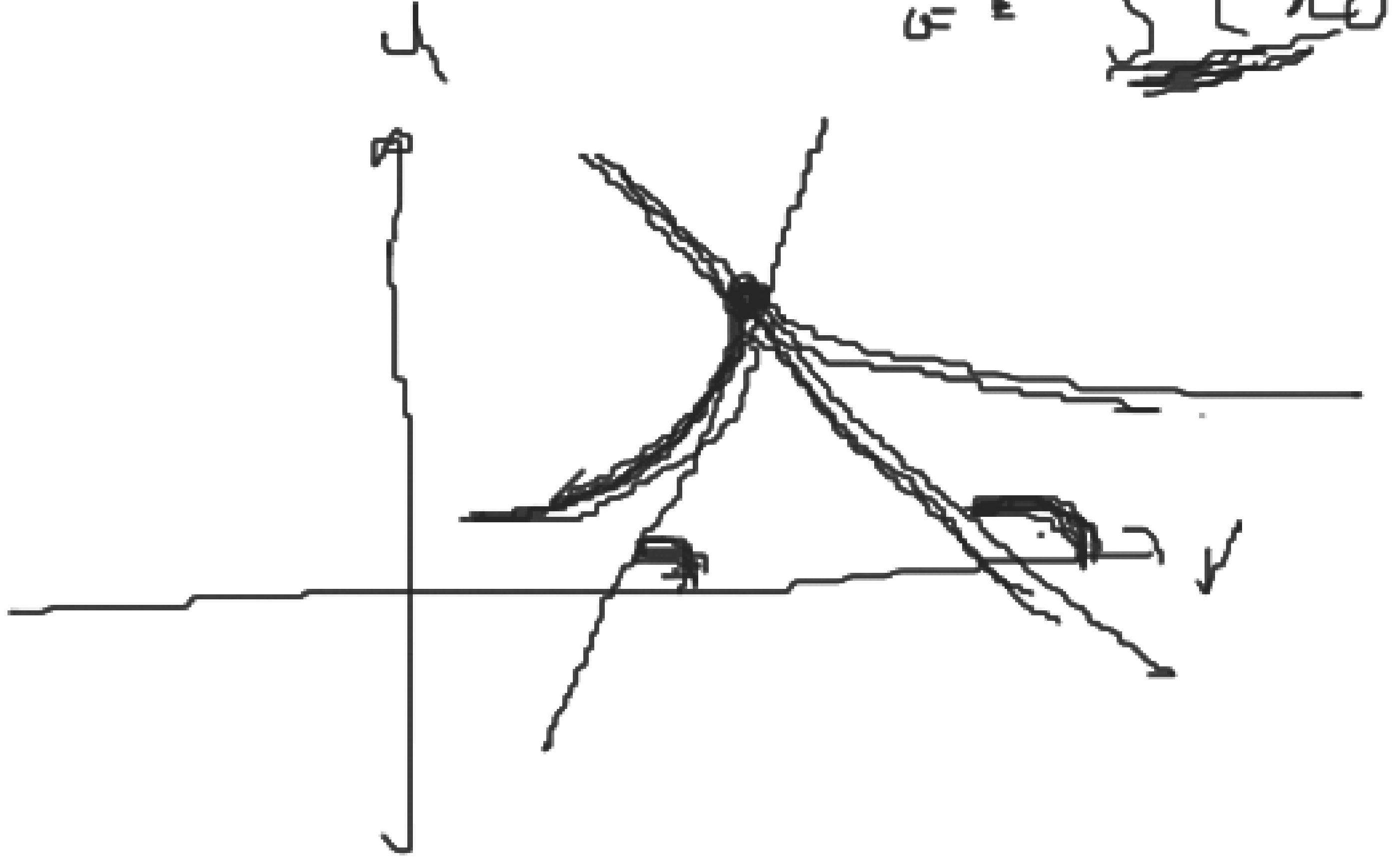
$$\lim_{x \to +\infty} f'(x) = \lim_{x \to +\infty} \frac{1}{2\sqrt{x}} = 0 \quad \text{Ans}$$

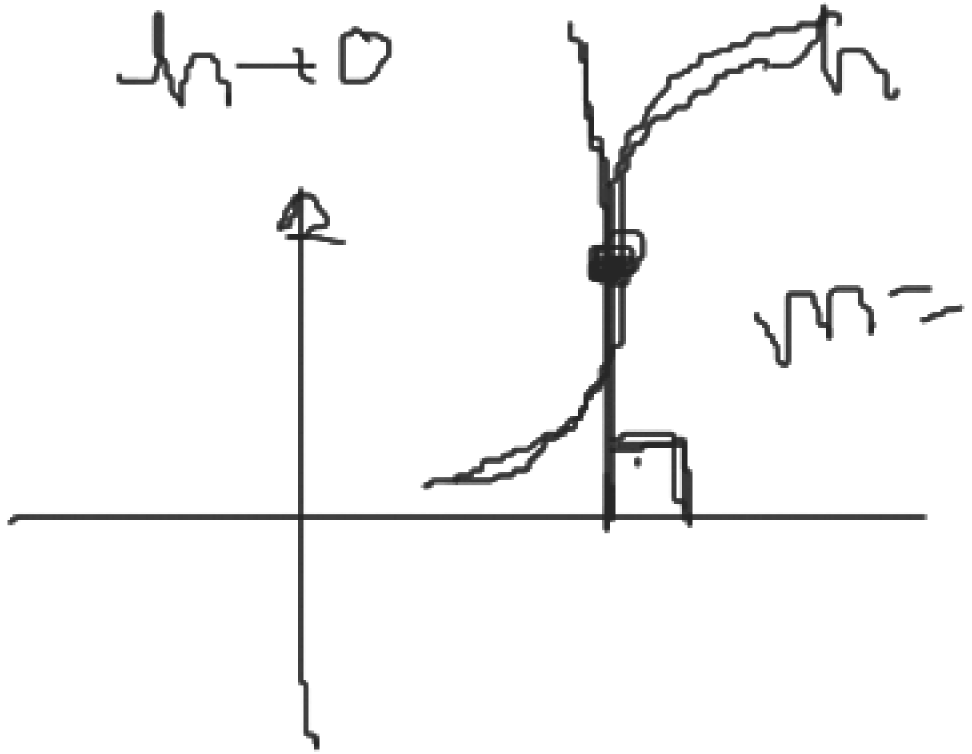
Differentiability:

$\boxed{x = x_0}$ $\boxed{f(x)}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

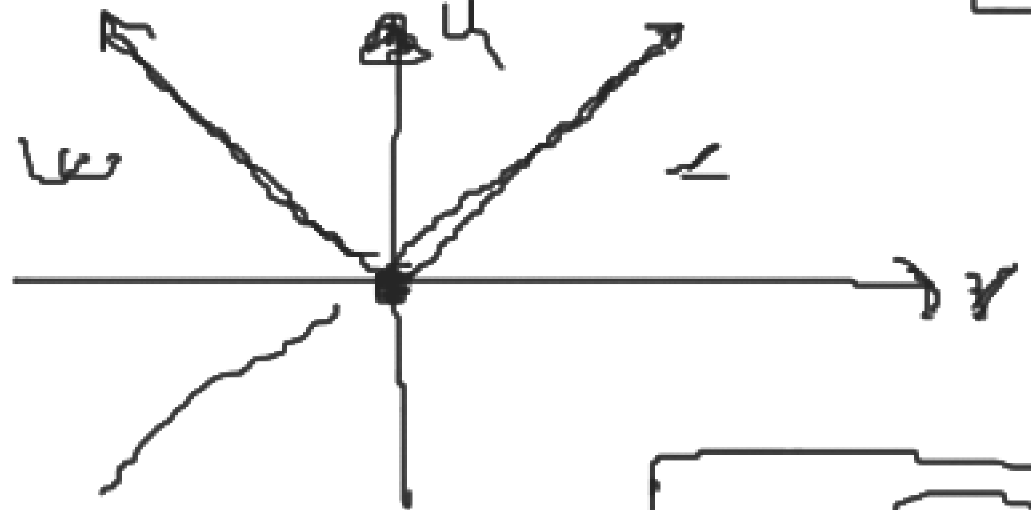
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$





$m = \tan 90^{\circ}$

Ex 6: Prove that $\boxed{f(x)=|x|}$ is not differentiable at $\boxed{x=0}$



Proof:

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-|0|}{h}=\boxed{\lim_{h\to 0}\frac{|h|}{h}}$$

Here. $\frac{|h|}{h}=\begin{cases}\boxed{1}, & h>0\\ -1, & h<0\end{cases}$

Now. $\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}(-1)=\boxed{-1}$

and $\lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}(1)=\boxed{1}$

$\therefore \lim_{h\to 0}\frac{|h|}{h}$ does not exist

$\therefore f(x)=|x|$ is not differentiable at $\boxed{x=0}$ proved

Other Derivative Notation

$y = f(x)$

$\frac{dy}{dx}$, $\frac{df}{dx}$, $f'(x)$, $y'(x)$

$y'(x) = \frac{dy}{dx}$

H.W Ex 2.2 : $\boxed{9-20 \ \& \ 46-49}$

$\boxed{\text{Quiz \# 02}}$

2.3 Differentiation:

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Theorem: $f(x) = c$ $\quad f'(x) = 0$

$$\frac{d}{dx}[c] = 0$$

Ex $f(x) = 10$

$\therefore f'(x) = 0$

Theorem: $f(x) = x^n$, $\quad \frac{d}{dx}[f(x)] = \boxed{f'(x) = n x^{n-1}}$

Proof: $\because f(x) = x^n \quad \therefore f(x+h) = (x+h)^n$ $\because$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left[\cancel{x^n} + n x^{n-1}.h + \frac{n(n-1)}{2!}.x^{n-2}.h^2 + \cdots + h^n\right] - \cancel{x^n}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}\left[nx^{n-1} + \frac{n(n-1)}{2} \cdot x^{n-2}h + \cdots + h^{n-1}\right]}{\cancel{h}}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} \cdot x^{n-2} \cdot h + \cdots + h^{n-1}\right]$$

$$= nx^{n-1} + 0 + \cdots + 0$$

$$= nx^{n-1}$$

<u>Theorem</u>: n is any real number

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

Ex 3 : $\frac{d}{dx}\left[x^{n}\right] = n\,x^{n-1}$.

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = -1\,x^{-1-1} = -x^{-2} = -\frac{1}{x^{2}} \quad \text{Ans}$$

$$\frac{d}{dx}\left[\sqrt[3]{x}\right] = \frac{d}{dx}\left[x^{1/3}\right] = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$$

$$= \frac{1}{3x^{2/3}} \quad \text{Ans}$$

Theorem : $\frac{d}{dx}\left[c\,f(x)\right] = c\,\frac{d}{dx}\left[f(x)\right]$

Theorem : (i) $\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$

(ii) $\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right]$

Ex 5 : $f(x) = 2x^6 + x^{-9}$, $f'(x) = ?$

Soln $f'(x) = \frac{d}{dx}\left[2x^6 + x^{-9}\right]$

$$= \frac{d}{dx}\left[2x^6\right] + \frac{d}{dx}\left[x^{-9}\right]$$

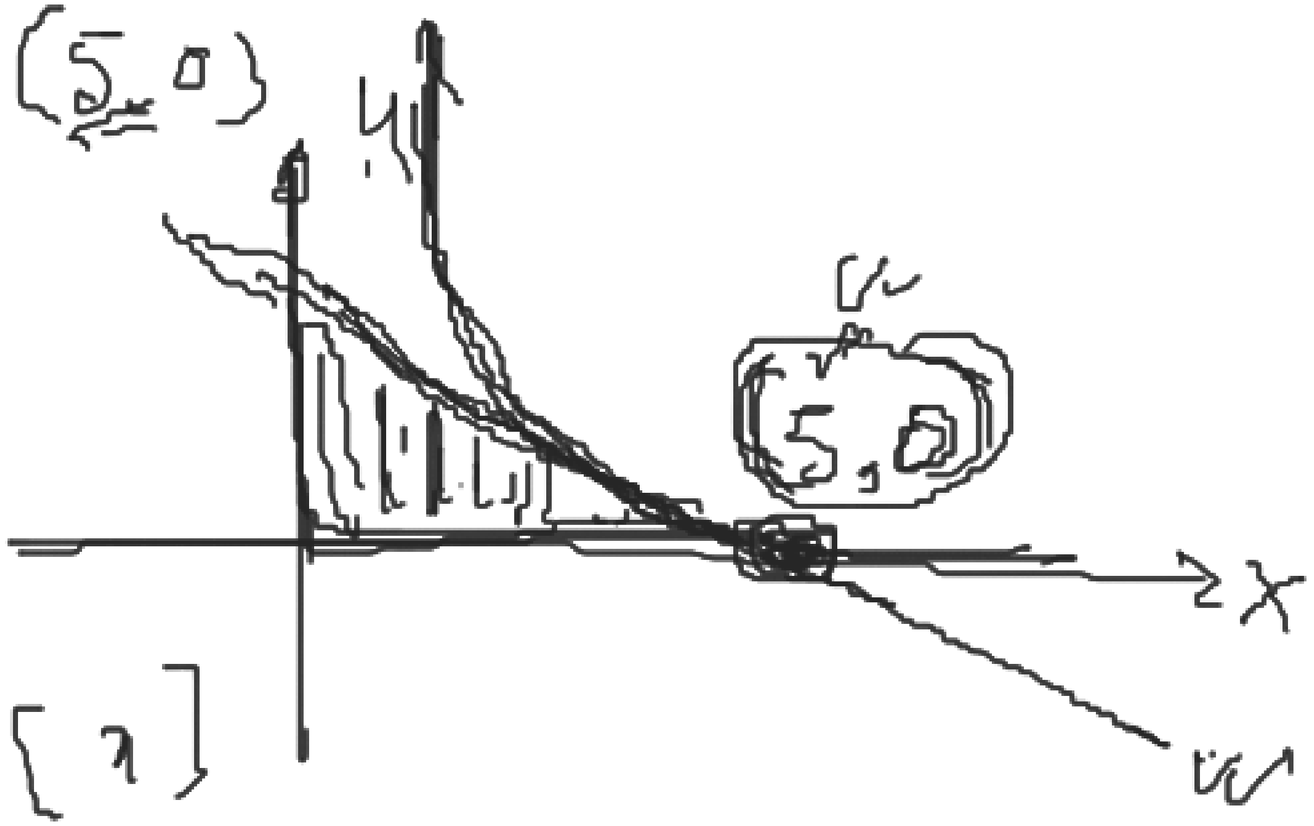
$$= 2\frac{d}{dx}\left[x^6\right] + \frac{d}{dx}\left[x^{-9}\right]$$

$$= 2 \cdot 6x^5 + \left(-9x^{-10}\right)$$

$$= 12x^5 - \frac{9}{x^{10}}$$

Ans

Ex8: Find the area of the triangle formed from the coordinate axes and the tangent line to the curve $\boxed{y = 5x^{-1} - \frac{1}{5}x}$ at $(5, 0)$



Sol: $y = 5x^{-1} - \frac{1}{5}x$

$\therefore\ y'(x) = \frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right]$

$= 5\frac{d}{dx}\left[x^{-1}\right] - \frac{1}{5}\frac{d}{dx}[x]$

$= 5\left(-1x^{-2}\right) - \frac{1}{5}\cdot 1 = -5x^{-2} - \frac{1}{5}$

$\therefore\ \boxed{y'(x) = -\frac{5}{x^2} - \frac{1}{5}}$

$\therefore$ The slope at $(5, 0)$ is

$m = -\frac{5}{5^2} - \frac{1}{5} = -\frac{1}{5} - \frac{1}{5} = -\frac{2}{5}$

$(5,0)$, $m = -\frac{2}{5}$

∴ The equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

Here, $(x_1, y_1) = (5,0)$

$m = -\frac{2}{5}$

$$\Rightarrow y - 0 = -\frac{2}{5}(x - 5)$$

$$\Rightarrow \boxed{y = -\frac{2}{5}x + 2}$$

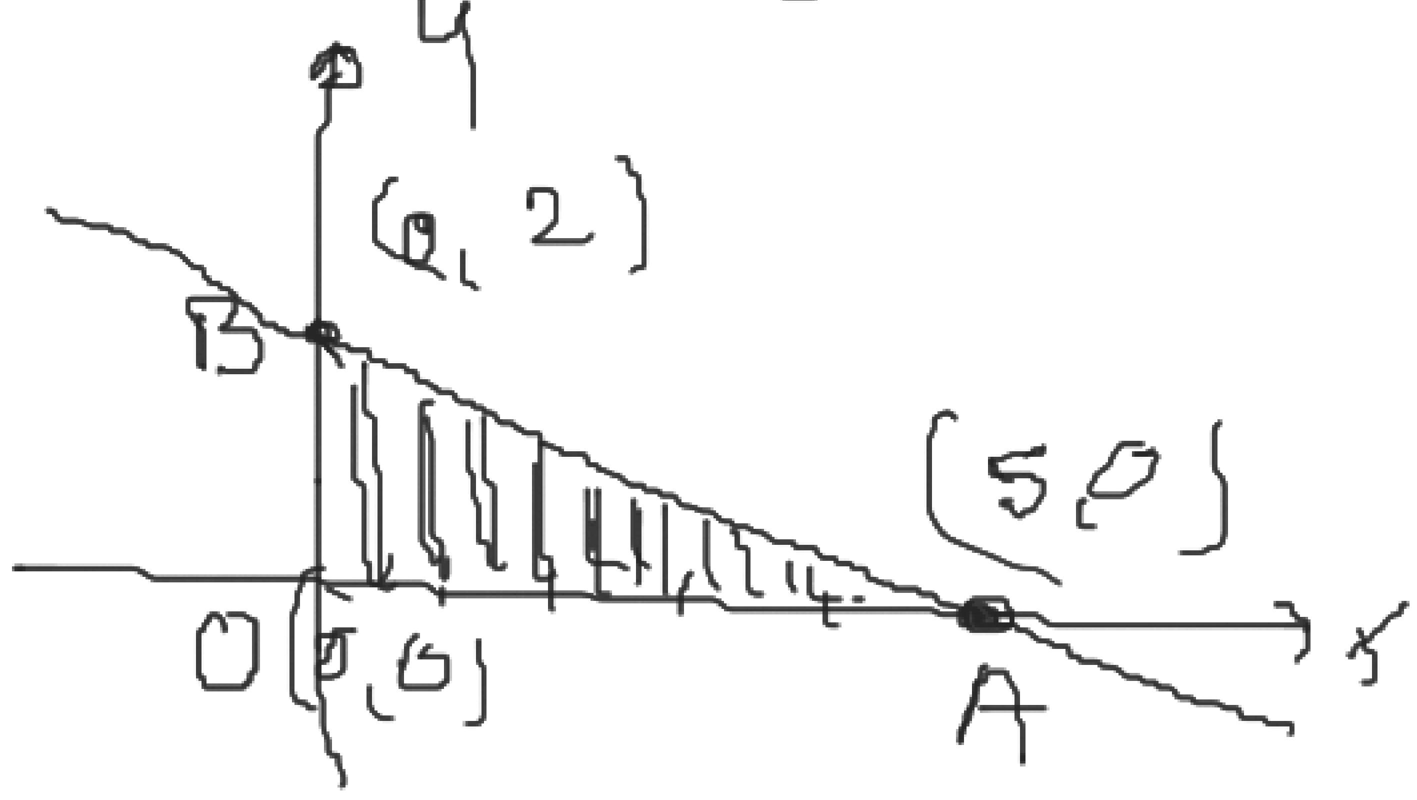
x	5	0
y	0	2

∴ The area of $\triangle OAB$

$= \frac{1}{2} \times OA \times OB$

$= \frac{1}{2} \times 5 \times 2$

$= 5$ sq-unit



Higher Derivative : $y = f(x)$.

1st order

$\frac{dy}{dx}$ or $f'(x)$

2nd order

$\frac{d^2y}{dx^2}$ or $f''(x)$

3rd order

$\frac{d^3y}{dx^3}$ or $f'''(x)$

. . .

nth order

$\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$

$$\frac{d^ny}{dx^n} = f^n(x) = \frac{d^n}{dx^n}\left[f(x)\right]$$

H.W. Ex 2.3 : $\boxed{9-24,\ 37-48\ \&\ 65-69}$

41. (d) $y=(5x^2-3)(7x^3+x)$, $\frac{d^2y}{dx^2}=?$

Soln: $y=35x^5+5x^3-21x^3-3x$

$\Rightarrow y=35x^5-16x^3-3x$

Diff. with respect to x

$$\frac{dy}{dx}=35\frac{d}{dx}[x^5]-16\frac{d}{dx}[x^3]-3\frac{d}{dx}[x]$$

$$=35\cdot 5x^4-16\cdot 3x^2-3\cdot 1$$

$$=175x^4-48x^2-3$$

$$\therefore \frac{d^2y}{dx^2}=175\frac{d}{dx}[x^4]-48\frac{d}{dx}[x^2]-\frac{d}{dx}[3]$$

$$= 700x^3 - 96x - 0$$

$$\therefore \frac{d^2y}{dx^2} = 700x^3 - 96x$$