Chapter # 03
(Topics in Differentiation)

3.2 Derivative of Logarithmic Functions: In this section we will obtain derivative formulas for logarithmic functions, and we will explain why the natural logarithm function is preferred over logarithms with other bases in calculus.

Derivative of Logarithmic Functions: We will establish that $f(x) = \ln x$ is differentiable for $x > 0$ by applying the derivative definition to $f(x)$. To evaluate the resulting limit, we will need the fact that $\ln x$ is continuous for $x > 0$, and we will need the limit

$$\lim_{v \to 0} (1+v)^{1/v} = e$$

Now,

$$\begin{aligned}
\frac{d}{dx}[\ln x] &= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right) \\
&= \lim_{v \to 0} \frac{1}{vx} \ln(1+v) \\
&= \frac{1}{x} \lim_{v \to 0} \frac{1}{v} \ln(1+v) \\
&= \frac{1}{x} \lim_{v \to 0} \ln(1+v)^{1/v} \\
&= \frac{1}{x} \ln\left[\lim_{v \to 0} (1+v)^{1/v}\right] \\
&= \frac{1}{x} \ln e \\
&= \frac{1}{x}
\end{aligned}$$

Thus

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

Similarly,

$$\frac{d}{dx}[\log_b x] = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b}\frac{d}{dx}[\ln x]$$

It follows from this that

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0$$

And

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx}$$

Example 2: Find,

$$\frac{d}{dx}[\ln(x^2 + 1)].$$

Solution: Let, $u = x^2 + 1$. We obtain

$$\frac{d}{dx}[\ln(x^2 + 1)] = \frac{1}{x^2 + 1} \cdot \frac{d}{dx}[x^2 + 1] = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}$$

Example 3:

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)\right] &= \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right] \\ &= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)} \\ &= \frac{2}{x} + \cot x - \frac{1}{2+2x}\end{aligned}$$

Example 5: Find $\frac{dy}{dx}$ for

$$y = \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4}$$

Solution: Take the natural logarithm of both sides and then use its properties, we can write

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x - 14) - 4\ln(1 + x^2)$$

Differentiating both sides with respect to $\boldsymbol{x}$ yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x - 14} - \frac{8x}{1 + x^2}$$

Thus,

$$\frac{dy}{dx} = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right]$$

Derivatives of Real Powers of x:

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

Home Work: Exercise 3.2: Problem No. 1-30, 35 - 44