Chapter # 02
(The Derivative)

Many real-world phenomena involve changing quantities—the speed of a rocket, the inflation of currency, the number of bacteria in a culture, the shock intensity of an earthquake, the voltage of an electrical signal, and so forth. In this chapter we will develop the concept of a "*derivative*," which is the mathematical tool for studying the rate at which one quantity changes relative to another. The study of rates of change is closely related to the geometric concept of a tangent line to a curve, so we will also be discussing the general definition of a tangent line and methods for finding its slope and equation

2.1 Tangent Lines and Rates of Change: In this section we will discuss three ideas: tangent lines to curves, the velocity of an object moving along a line, and the rate at which one variable changes relative to another. Our goal is to show how these seemingly unrelated ideas are, in actuality, closely linked.

Definition: Suppose that x_0 is in the domain of the function f. The tangent line to the curve $y = f(x)$ at the point $P(x_0,\ f(x_0))$ is the line with equation

$$y - f(x_0) = m_{\tan}(x - x_0)$$

where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided the limit exists. For simplicity, we will also call this the tangent line to $y = f(x)$ *at* x_0.

Example 1: Find an equation for the tangent line to the parabola $y = x^2$ at the point *P*(1, 1).

Solution: Given, $f(x) = x^2$ *and* $x_0 = 1$, we have

$$\begin{aligned} m_{\tan} &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1}(x + 1) = 2 \end{aligned}$$

Thus, the tangent line to $y = x^2$ at (1, 1) has equation

y − 1 = 2(x − 1) or equivalently y = 2x − 1.

An alternative way:

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Let, $\boldsymbol{h = x - x_0} \quad \therefore \quad \boldsymbol{x = x_0 + h}$

If $\boldsymbol{x \to x_0}$ $\quad \boldsymbol{then\ h \to 0}$

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example 3: Find an equation for the tangent line to the parabola $\boldsymbol{y = \frac{2}{x}}$ at the point *P*(2, 1).

Solution: Here $f(x) = \frac{2}{x},\ \ x_0 = 2\ \ and\ \ f(x_0) = 1$

$$m_{tan} = \lim_{h\to0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to0}\frac{\frac{2}{2+h}-1}{h} = \lim_{h\to0}\frac{2-2-h}{h(2+h)}$$

$$= \lim_{h\to0}\frac{-h}{h(2+h)} = \lim_{h\to0}\frac{-1}{2+h} = -\frac{1}{2}$$

Therefore the tangent line at (2, 1) is

$$\boldsymbol{y - f(x_0) = m_{tan}(x - x_0)}$$

$$\Rightarrow \boldsymbol{y - 1 = -\frac{1}{2}(x - 2)} \qquad \therefore \boldsymbol{y = -\frac{1}{2}x + 2} \qquad \textbf{(Ans.)}$$

Example 4: Find the slopes of the tangent lines to the curve $\boldsymbol{y = \sqrt{x}\ \ at\ \ x_0 = 1,\ x_0 = 4\ \ and\ \ x_0 = 9}$.

Solution: Here, $\boldsymbol{f(x) = \sqrt{x}}$

Therefore the slope,

$$\begin{aligned}
m_{\tan} &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \cdot \frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}} \\
&= \lim_{h \to 0} \frac{x_0 + h - x_0}{h(\sqrt{x_0 + h} + \sqrt{x_0})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{x_0 + h} + \sqrt{x_0})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}
\end{aligned}$$

Thus

$$\text{slope at } x_0 = 1: \ \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$\text{slope at } x_0 = 4: \ \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\text{slope at } x_0 = 9: \ \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Velocity: If a particle in rectilinear motion moves along an ***s***-axis so that its position coordinate function of the elapsed time ***t*** is

$$s = f(t)$$

then ***f*** is called the position function of the particle; the graph of the above function is the position versus time curve. The average velocity of the particle over a time interval $[\boldsymbol{t_0},\ \boldsymbol{t_0+h}]$, $\boldsymbol{h} > 0$, is defined to be

$$v_{\text{ave}} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0+h) - f(t_0)}{h}$$

Example 5: Suppose that $\boldsymbol{s = f(t) = 1 + 5t - 2t^2}$ is the position function of a particle, where ***s*** is in meters and ***t*** is in seconds. Find the average velocities of the particle over the time intervals **(*a*)** [0, 2] and **(*b*)** [2, 3].

Solution: **(*a*)** Given,

$$\boldsymbol{s = f(t) = 1 + 5t - 2t^2} \qquad \textbf{interval } [\boldsymbol{t_0},\ \boldsymbol{t_0+h}] = [\boldsymbol{0}, \boldsymbol{2}] \ \ \boldsymbol{i.e.}\ \boldsymbol{t_0 = 0},\ \ \boldsymbol{h = 2}$$

Therefore average velocity is

$$v_{\text{ave}} = \frac{f(t_0+h) - f(t_0)}{h} = \frac{f(2) - f(0)}{2} = \frac{3-1}{2} = \frac{2}{2} = 1 \text{ m/s}$$

(*b*) Given,

$$\boldsymbol{s = f(t) = 1 + 5t - 2t^2} \qquad \textbf{interval } [\boldsymbol{t_0},\ \boldsymbol{t_0+h}] = [\boldsymbol{2}, \boldsymbol{3}] \ \ \boldsymbol{i.e.}\ \boldsymbol{t_0 = 2},\ \ \boldsymbol{t_0 + h = 3}$$

Therefore average velocity is

$$v_{\text{ave}} = \frac{f(t_0+h) - f(t_0)}{h} = \frac{f(3) - f(2)}{1} = \frac{-2-3}{1} = \frac{-5}{1} = -5 \text{ m/s}$$

Instantaneous Velocity: If these average velocities have a limit as ***h*** approaches zero ($\boldsymbol{h \to 0}$), then we can take that limit to be the *instantaneous velocity* of the particle at time $\boldsymbol{t_0}$.

$$v_{\text{inst}} = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h}$$

Average Rate of Change (ARC): If $\boldsymbol{y = f(x)}$, then we define the average rate of change of ***y*** with respect to ***x*** over the interval $[x_0, x_1]$ to be

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let $\boldsymbol{h = x_1 - x_0}$ $\quad \therefore\ \boldsymbol{x_1 = x_0 + h}$ and rewrite the above as

$$r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Instantaneous Rate of Change (IRC): If $\boldsymbol{y = f(x)}$, then we define the instantaneous rate of change of ***y*** with respect to ***x*** over the interval $[x_0, x_1]$ to be

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let $\boldsymbol{h = x_1 - x_0}$ $\quad \therefore\ \boldsymbol{x_1 = x_0 + h}$ and rewrite the above as

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Home Work: Exercise 2.1: Problem No. 11-18

2.2 The Derivative: In this section we will discuss the concept of a "*derivative*," which is the primary mathematical tool that is used to calculate and study rates of change.

Definition: The function $\boldsymbol{f'}$ defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

is called the derivative of ***f*** with respect to ***x***. The domain of $\boldsymbol{f'}$ consists of all ***x*** in the domain of ***f*** for which the limit exists.

Example 1: Find the derivative with respect to ***x*** of $\boldsymbol{f(x) = x^2}$, and use it to find the equation of the tangent line to $\boldsymbol{y = x^2 \ \ at\ \ x = 2}$.

Solution: From the definition,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x \end{aligned}$$

Thus, the slope of the tangent line to $\boldsymbol{y = x^2 \ at \ x = 2 \ \ is \ f'(2) = 4.}$

If $\boldsymbol{x = 2}$, $\boldsymbol{y = f(2) = 2^2 = 4}$

Therefore the point-slope form of the tangent line is

$$y - y_1 = m(x - x_1) \qquad here \quad (x_1, y_1) = (2, 4) \ \& \ m = 4$$

$$\therefore \quad y - 4 = 4(x - 2) \ \Rightarrow \ \ y = 4x - 4 \qquad \text{(Ans.)}$$

Finding an Equation for the Tangent Line to $\boldsymbol{y = f(x) \ at \ x = x_0}$**:**

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Example 2: Find the derivative with respect to ***x*** of $\boldsymbol{f(x) = x^3 - x}$.

Solution: Given, $\boldsymbol{f(x) = x^3 - x}$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h \to 0} [3x^2 + 3xh + h^2 - 1] = 3x^2 - 1 \end{aligned}$$

Example 4: **(*a*)** Find the derivative with respect to ***x*** of $\boldsymbol{f(x) = \sqrt{x}}$.

(*b*) Find the slope of the tangent line to $\boldsymbol{y = \sqrt{x} \ at \ x = 9}$.

(*c*) Find the limits of $f'(x) \ as \ x \to 0^+ \ \ and \ as \ x \to +\infty$.

Solution: **(*a*)** Given, $\boldsymbol{f(x) = \sqrt{x}}$

$$\therefore \ \ \boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}}$$

$$\boldsymbol{= \lim_{h \to 0} \left(\frac{1}{\sqrt{x+h} + \sqrt{x}} \right) = \frac{1}{2\sqrt{x}}} \qquad \textbf{(Ans.)}$$

(*b*) Now $\boldsymbol{f'(x) = \frac{1}{2\sqrt{x}}}$

$\therefore$ The slope of the tangent line to $\boldsymbol{y = \sqrt{x}\ at\ x = 9\ is}$

$$\boldsymbol{f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}} \qquad \textbf{(Ans.)}$$

(*c*) $\quad \boldsymbol{\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{1}{2\sqrt{x}} = +\infty}$ **(Ans.)**

And $\boldsymbol{\lim_{x\to +\infty} f'(x) = \lim_{x\to +\infty} \frac{1}{2\sqrt{x}} = 0}$ **(Ans.)**

Differentiability: It is possible that the limit that defines the derivative of a function $\boldsymbol{f}$ may not exist at certain points in the domain of $\boldsymbol{f}$. At such points the derivative is undefined. To account for this possibility we make the following definition.

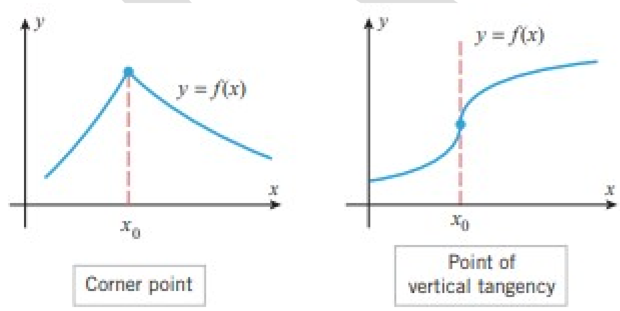
Definition: A function *f* is said to be ***differentiable*** at $\boldsymbol{x_0}$ if the limit

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If ***f*** is ***differentiable*** at each point of the open interval **(*a*, *b*)**, then we say that it is differentiable on **(*a*, *b*)**, and similarly for open intervals of the form $(\boldsymbol{a}, +\infty)$, $(-\infty, \boldsymbol{b})$ $\boldsymbol{and}$ $(-\infty, +\infty)$. In the last case we say that ***f*** is ***differentiable everywhere***.

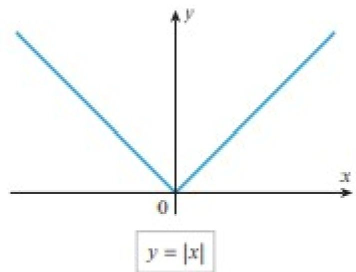
Note: Geometrically, a function $\boldsymbol{f}$ is differentiable at $\boldsymbol{x_0}$ if the graph of $\boldsymbol{f}$ has a tangent line at $\boldsymbol{x_0}$. Thus, $\boldsymbol{f}$ is not differentiable at any point $\boldsymbol{x_0}$ where the secant lines from $\boldsymbol{P(x_0,\ f(x_0))}$ to points $\boldsymbol{Q(x, f(x))}$ distinct from ***P*** do not approach a unique nonvertical limiting position as $\boldsymbol{x \to x_0}$. The following figure illustrates two common ways in which a function that is continuous at $\boldsymbol{x_0}$ can fail to be differentiable at $\boldsymbol{x_0}$. These can be described informally as

- corner points
- points of vertical tangency



Example 6: Prove that $\boldsymbol{f(x) = |x|}$ is not differentiable at $x = 0$.

Solution: Given, $\boldsymbol{f(x) = |x|}$



$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

But

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases}$$

so that

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1 \quad \text{and} \quad \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

So $\lim_{x \to 0} \frac{|h|}{h}$ does not exist and hence $\boldsymbol{f}$ is not differentiable at $x = 0$.

Theorem: If a function $\boldsymbol{f}$ is differentiable at $\boldsymbol{x_0}$, then $\boldsymbol{f}$ is continuous at $\boldsymbol{x_0}$.

Other Derivative Notations: The process of finding a derivative is called differentiation. You can think of differentiation as an operation on functions that associates a function $\boldsymbol{f'}$ with a function $\boldsymbol{f}$. When the independent variable is ***x***, the differentiation operation is also commonly denoted by

$$f'(x) = \frac{d}{dx}[f(x)] \quad \text{or} \quad f'(x) = D_x[f(x)]$$

In the case where there is a dependent variable $\boldsymbol{y = f(x)}$, the derivative is also commonly denoted by

$$f'(x) = y'(x) \quad \text{or} \quad f'(x) = \frac{dy}{dx}$$

Home Work: Exercise 2.2: Problem No. 9-20, 46-49