Last lecture:

- Completed discussion of quadrotor dynamics. Obtained nonlinear dynamics: $\dot{\bar{x}} = f(\bar{x}, \bar{u})$.
- Discussed how to linearize dynamics about a "nominal" (i.e., reference) state $\bar{x}_0$ and control input $\bar{u}_0$ to obtain:
$$\dot{\bar{x}} = A(\bar{x} - \bar{x}_0) + B(\bar{u} - \bar{u}_0). \tag{1}$$
- Started discussion of feedback control:
  - Said that we want to find a function that maps states to control inputs in order to achieve some desired behavior (e.g., hovering). Such a function is called a **controller** (or **"control law"**). One option is:
$$\bar{u}(\bar{x}) \triangleq \bar{u}_0 + K(\bar{x} - \bar{x}_0). \tag{2}$$
  - Ended lecture with discussion of ways in which feedback control allows us to deal with different kinds of uncertainty.

Plan for today: Continue discussion of feedback control. We will introduce some important definitions and then discuss particular ways to design feedback controllers.

1. Fixed point (equilibrium point)

A **fixed point (a.k.a. equilibrium point)** for a system $\dot{\bar{x}} = f(\bar{x}, \bar{u})$ is a state $\bar{x}_0$ such that:
$$\dot{\bar{x}} = f(\bar{x}_0, \bar{u}_0) = 0 \text{ for some choice of } u_0. \tag{3}$$

Recall that the hovering configuration for the quadrotor satisfied this. Intuitively, a fixed point $\bar{x}_0$ is a state that you remain at when you apply $\bar{u}_0$.

For the planar quadrotor, the fixed point associated with hover was given by:
$$\bar{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \bar{u}_0 = \begin{bmatrix} mg \\ 0 \end{bmatrix}. \tag{4}$$

Question: Is it possible to have a fixed point of the form:
$$\bar{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ 0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{\theta}_0 \end{bmatrix} \text{ with } \dot{x}_0, \dot{y}_0, \text{or } \dot{\theta}_0 \text{ being non-zero?} \tag{5}$$

[Answer: **No!**. Convince yourself of this by writing down the equations of motion for the planar quadrotor.]

2. Stability

What are we trying to achieve with feedback? Suppose we come up with some feedback controller $\bar{u}(\bar{x})$. The **closed-loop system** is the system we obtain when we run this controller. The dynamics are then given by:

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}(\bar{x})) \triangleq f_{\text{cl}}(\bar{x}). \tag{6}$$

These are known as the **closed-loop dynamics**. What properties would we like f_{cl} to have?

→ We would like the closed-loop system to be **"stable"**. But what does this mean?

There are actually many different definitions/flavors of stability. Here is a reasonable one.

Stability in the sense of Lyapunov: A system is said to be stable in the sense of Lyapunov (with respect to a fixed point $\bar{x}_0$) if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that:

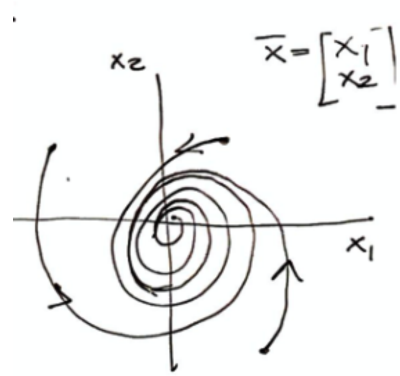
$$\|\bar{x}(0) - \bar{x}_0\| < \delta \implies \|\bar{x}(t) - \bar{x}_0\| < \epsilon, \ \forall t \geq 0. \tag{7}$$

Asymptotic stability: A system is asymptotically stable if (i) it is stable in the sense of Lyapunov and (ii) for every initial condition (i.e., initial state) $\bar{x}(0)$, the following holds:

$$\lim_{t \to \infty} \|\bar{x}(t) - \bar{x}_0\| = 0, \tag{8}$$

where $\bar{x}(t)$ is the state at time t when the system starts off at $\bar{x}(0)$ and $\bar{x}_0$ is the reference state (which is a fixed point). If this condition holds, we say that the system is asymptotically stable at $\bar{x}_0$.

The picture below illustrates this when the state is two-dimensional: $\bar{x} = [x_1, x_2]^T$.

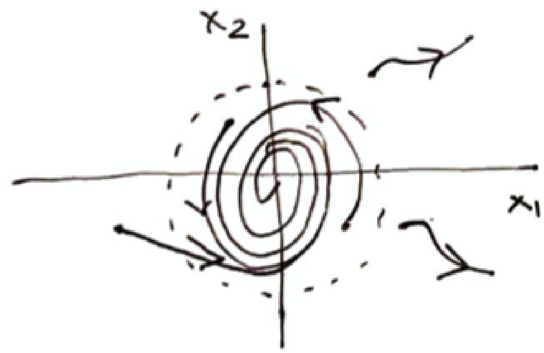


What we have describe above is really **global** asymptotic stability. This is a very strong condition (and may be impossible to achieve in practice). For a quadrotor, this is asking for it to go to $\bar{x}_0$ no matter what the initial state is, e.g., 0.99 × the speed of light!

Here is another (more reasonable) flavor of stability.

Local asymptotic stability: A system is locally asymptotically stable if (i) it is stable in the sense of Lyapunov, and (ii) for every initial condition $\bar{x}(0)$ with $\|\bar{x}(0) - \bar{x}_0\| \leq R$ for some R (i.e., every initial condition in some ball of radius R around $\bar{x}_0$), we have:

$$\lim_{t \to \infty} \|\bar{x}(t) - \bar{x}_0\| = 0. \tag{9}$$



Note: Global asymptotic stability $\implies$ local asymptotic stability.

There are other reasonable flavors of stability too. For example, you could ask that trajectories remain bounded in some region (this is known as *invariance*).

We won't prove this here, but it turns out that for **linear** systems $\dot{\bar{x}} = A(\bar{x} - \bar{x}_0) + B(\bar{u} - \bar{u}_0)$, local asymptotic stability implies global asymptotic stability! So these two notions are the same for linear systems. This is *not* true for nonlinear systems.

3. Stabilizability

We have described what we want to achieve with feedback (i.e., stability). But can we actually achieve it?

Consider the following system:

$$\dot{\bar{x}} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + u \end{bmatrix}. \tag{10}$$

Here, the control input u can only affect x_2. Moreover, the dynamics of x_1 and x_2 are completely decoupled. So the control input has no effect on x_1. The dynamics of x_1 are unstable; hence, no matter what controller you choose, this system cannot be stabilized.

Stabilizability (w.r.t. a fixed point $\bar{x}_0$): Any initial state can be asymptotically driven to $\bar{x}_0$ by choosing the control inputs $\bar{u}$ appropriately, i.e., for any initial state $\bar{x}(0)$,

$$\lim_{t \to \infty} \|\bar{x}(t) - \bar{x}_0\| = 0. \tag{11}$$

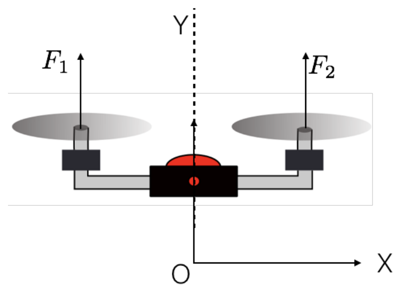
by choosing appropriate control inputs.

As the example above shows, not every system is stabilizable (not even all linear systems). But, it turns out that we can *check* stabilizability of linear systems $\dot{\bar{x}} = A(\bar{x} - \bar{x}_0) + B(\bar{u} - \bar{u}_0)$ by checking eigenvalue conditions on A, B. [We won't describe these conditions here, but I'll say a little bit more about this in the next lecture.; see MAE 433/434 for a thorough treatment.]

The question now is how to actually design a controller that achieves (asymptotic) stability (for linear systems), assuming the system is stabilizable.

4. Proportional-Derivative (PD) Control

Consider the planar quadrotor constrained to move only in the y direction (as we did in Lecture 2).



The dynamics are given by:

$$\ddot{y} = \frac{u_1}{m} - g. \tag{12}$$

The state is:

$$\bar{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{13}$$

and recall that the control input u_1 is the total thrust produced by the propellers.

Suppose we want to stabilize the system to:

$$\bar{x}_0 = \begin{bmatrix} y_0 \\ 0 \end{bmatrix}. \tag{14}$$

With $u_0 = mg$, this is a fixed point (since $\ddot{y} = mg/m - g = 0$).

How should we choose $u(\bar{x})$ to stabilize the system to $\bar{x}_0$? Here is one possibility: choose

$$u(\bar{x}) \triangleq u_0 + k_p(y - y_0). \tag{15}$$

The closed-loop system is then:

$$\ddot{y} = \frac{mg}{m} - g + \frac{k_p}{m}(y - y_0) = \frac{k_p(y - y_0)}{m}, \tag{16}$$

where $k_p < 0$ is a constant that we choose.

Now define $\tilde{y} \triangleq y - y_0$. Then $\dot{\tilde{y}} = \dot{y}$ and $\ddot{\tilde{y}} = \ddot{y}$. Thus:

$$\ddot{\tilde{y}} = \frac{k_p}{m}(y - y_0) = \frac{k_p}{m}\tilde{y}. \tag{17}$$

What does this system remind you of?

$\rightarrow$ Spring-mass system!

Would this controller work (i.e., stabilize the system)? (Not quite)

Here is another option:

$$u(\bar{x}) = u_0 + \frac{k_p}{m}(y - y_0) + k_d\dot{y}, \tag{18}$$

where $k_d < 0$ is another constant that we can choose.

We then have:

$$\ddot{\tilde{y}} = \frac{k_p}{m}\tilde{y} + \frac{k_d}{m}\dot{\tilde{y}}. \tag{19}$$

Now what does this system remind you of?

→ Spring-mass-damper!

This is an example of **proportional-derivative (PD) control**. Such controllers are extremely popular and effective in practice.

Some terminology:

$$k_p : \text{“proportional gain”},$$
$$k_d : \text{“derivative gain”}.$$

Note that the controller is of the form:

$$\bar{u}(\bar{x}) = \bar{u}_0 + K(\bar{x} - \bar{x}_0) \tag{20}$$

$$= \bar{u}_0 + [k_p, k_d]\begin{bmatrix} y - y_0 \\ \dot{y} \end{bmatrix}. \tag{21}$$

The matrix K is referred to as the **gain matrix**.

Also notice that for this example, any choice of $k_p, k_d < 0$ leads to (global) asymptotic stability. In general, things will not be so easy!

How can we choose K in general? Maybe we want to ensure asymptotic stability while also optimizing some performance metric?

→ We will do this in the next lecture!