Q: Let $x_1, x_2 \cdots x_m$ be a random sample from binomial dist^n with parameter θ. Find $100(1-\alpha)\%$ confidence interval of θ.

Solution: Test statistic is $\dfrac{T_m - \theta}{\sqrt{m\,i\,(T_m)}} \sim N(0,1)$

where $i(\theta) = -E\left(\dfrac{\partial^2}{\partial\theta^2} \ln f(x;\theta)\right)$

$$f(x;\theta) = \binom{n}{x}\theta^{x}(1-\theta)^{n-x} \qquad x = 0, 1, 2 \cdots n$$

$$\ln f(x;\theta) = \ln\left[\binom{n}{x}\theta^{x}(1-\theta)^{n-x}\right]$$

$$= \ln\binom{n}{x} + \ln\theta^{x} + \ln(1-\theta)^{n-x}$$

$$= \ln\binom{n}{x} + x\ln\theta + (n-x)\ln(1-\theta)$$

$$\frac{\partial}{\partial\theta}\ln f(x;\theta) = \frac{x}{\theta} - \frac{n-x}{1-\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = -\frac{x}{\theta^2} - \frac{n-x}{(1-\theta)^2}$$

$$-E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right) = \frac{E(x)}{\theta^2} + \frac{n-E(x)}{(1-\theta)^2}$$

$$= \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2}$$

$$= \frac{n}{\theta} + \frac{n(1-\theta)}{(1-\theta)^2}$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta}$$

$$\therefore i(\theta) = \frac{n}{\theta} + \frac{n}{1-\theta}$$

i.e $$i(T_m) = \frac{n}{T_m} + \frac{n}{1-T_m}$$

Now,

$$P\left(-Z_{\frac{\alpha}{2}} < \frac{T_m - \theta}{\sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(-Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)} < T_m - \theta < Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)}\right) = 1-\alpha$$

$$\Rightarrow P\left(-Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)} - T_m < -\theta < Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)} - T_m\right) = 1-\alpha$$

$$\Rightarrow P\left(Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)} + T_m > \theta > -Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)} + T_m\right) = 1-\alpha$$

$$\therefore P\left(T_m - Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)} < \theta < T_m + Z_{\frac{\alpha}{2}} \sqrt{m\left(\frac{n}{T_m} + \frac{n}{1-T_m}\right)}\right) = 1-\alpha$$

Q: Let $x_1, x_2 \cdots x_n$ be a random sample from bernoulli dist^n with parameter θ.

Find $100(1-\alpha)\%$ confidence interval of θ.

Solution: $f(x;\theta) = \theta^x (1-\theta)^{1-x} \qquad x = 0, 1$

$$\ln f(x;\theta) = x \ln \theta + (1-x) \ln (1-\theta)$$

$$\frac{\partial}{\partial \theta} \ln f(x;\theta) = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2}{\partial \theta^2} \ln f(x;\theta) = -\frac{x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}$$

$$-E\left(\frac{\partial^2}{\partial \theta^2} \ln f(x;\theta)\right) = \frac{E(x)}{\theta^2} + \frac{1-E(x)}{(1-\theta)^2}$$

$$= \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2}$$

$$= \frac{1}{\theta} + \frac{1}{1-\theta}$$

$$\therefore i(\theta) = \frac{1}{\theta} + \frac{1}{1-\theta}$$

ie $i(T_n) = \dfrac{1}{T_n} + \dfrac{1}{1-T_n}$

Now, $P\left(-Z_{\frac{\alpha}{2}} < \dfrac{T_n - \theta}{\sqrt{n\left(\frac{1}{T_n} + \frac{1}{1-T_n}\right)}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$

$$\Rightarrow P\left(-Z_{\frac{\alpha}{2}} \sqrt{n\left(\frac{1}{T_n} + \frac{1}{1-T_n}\right)} - T_n < -\theta < Z_{\frac{\alpha}{2}} \sqrt{n\left(\frac{1}{T_n} + \frac{1}{1-T_n}\right)} - T_n\right) = 1-\alpha$$

$$\therefore P\left(T_n - Z_{\frac{\alpha}{2}} \sqrt{n\left(\frac{1}{T_n} + \frac{1}{1-T_n}\right)} < \theta < T_n + Z_{\frac{\alpha}{2}} \sqrt{n\left(\frac{1}{T_n} + \frac{1}{1-T_n}\right)}\right) = 1-\alpha$$

Q: Let $x_1, x_2 \cdots x_n$ be a r.s. from poisson dist^n with parameter θ. Find $100(1-\alpha)\%$ C.I. of θ.

Sol^n: $f(x;\theta) = \dfrac{e^{-\theta}\theta^{x}}{x!}$ $\quad x = 0, 1, 2 \cdots$

$$\ln f(x;\theta) = \ln\left(\frac{e^{-\theta}\theta^{x}}{x!}\right) = \ln\left(e^{-\theta}\theta^{x}\right) - \ln(x!)$$

$$= -\theta \ln e + x \ln\theta - \ln(x!) \quad [\text{Here } \ln e = 1]$$

$$\frac{\partial}{\partial\theta}\ln f(x,\theta) = -1 + \frac{x}{\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = -\frac{x}{\theta^2}$$

$$-E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right) = \frac{E(x)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$\therefore i(\theta) = \frac{1}{\theta} \quad \text{ie} \quad i(T_n) = \frac{1}{T_n}$$

$$P\left(-Z_{\frac{\alpha}{2}} < \frac{T_n - \theta}{\sqrt{n.\frac{1}{T_n}}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(-Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n}} < T_n - \theta < Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n}}\right) = 1-\alpha$$

$$\Rightarrow P\left(-Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n}} - T_n < -\theta < Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n}} - T_n\right) = 1-\alpha$$

$$\Rightarrow P\left(T_n - Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n}} < \theta < T_n + Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n}}\right) = 1-\alpha$$

Q: Let $x_1, \dots x_n$ be a r.s. from exponential distⁿ with parameter θ. Find $100(1-\alpha)\%$ C.I. of θ.

Solⁿ: $f(x;\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x} \quad , x > 0$

$$\ln f(x;\theta) = \ln\left(\frac{1}{\theta}\right) + \ln\left(e^{-\frac{1}{\theta}x}\right)$$

$$= \ln 1 - \ln\theta - \frac{1}{\theta}x = -\ln\theta - \frac{1}{\theta}x$$

$$\frac{\partial}{\partial\theta}\ln f(x;\theta) = -\frac{1}{\theta} + \frac{1}{\theta^2}x$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = \frac{1}{\theta^2} - \frac{2}{\theta^3}x$$

$$-E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right) = -\frac{1}{\theta^2} + \frac{2}{\theta^3}E(x)$$

$$= -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = -\frac{1}{\theta^2} + \frac{2}{\theta^2} = \frac{1}{\theta^2}$$

$\therefore i(\theta) = \frac{1}{\theta^2}$ ie $i(T_n) = \frac{1}{T_n^2}$

$$P\left(-Z_{\frac{\alpha}{2}} < \frac{T_n - \theta}{\sqrt{n\frac{1}{T_n^2}}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(-Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n^2}} < T_n - \theta < Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n^2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(-Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n^2}} - T_n < -\theta < Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n^2}} - T_n\right) = 1-\alpha$$

$$\therefore P\left(T_n - Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n^2}} < \theta < T_n + Z_{\frac{\alpha}{2}}\sqrt{\frac{n}{T_n^2}}\right) = 1-\alpha$$