Chapter 4.3
The Graph of a Rational Function

4.3.1 Analyze the Graph of a Rational Function

We commented earlier that calculus provides the tools required to graph a polynomial function accurately. The same holds true for rational functions. However, we can gather together quite a bit of information about their graphs to get an idea of the general shape and position of the graph.

Example 1 How to Analyze the Graph of a Rational Function

Step 1: Factor the numerator and denominator of R. Find the domain of the rational function.

Step 2: Write R in lowest terms.

Step 3: Locate the intercepts of the graph. Determine the behavior of the graph of R near each x-intercept using the same procedure as for polynomial functions. Plot each x-intercept and indicate the behavior of the graph near it.

Step 4: Locate the vertical asymptotes. Graph each vertical asymptote using a dashed line.

Step 5: Locate the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of R intersects this asymptote. Graph the asymptotes using a dashed line. Plot any points at which the graph of R intersects the asymptote.

Step 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals.

Now construct Table 11.

Step 7: Analyze the behavior of the graph of R near each asymptote and indicate this behavior on the graph.

Step 8: Use the results obtained in Steps 1 through 7 to graph R.

SUMMARY Analyzing the Graph of a Rational Function R

STEP 1: Factor the numerator and denominator of R. Find the domain of the rational function.
STEP 2: Write R in lowest terms.
STEP 3: Locate the intercepts of the graph. The x-intercepts are the zeros of the numerator of R that are in the domain of R. Determine the behavior of the graph of R near each x-intercept.
STEP 4: Determine the vertical asymptotes. Graph each vertical asymptote using a dashed line.
STEP 5: Determine the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of R intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.
STEP 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.
STEP 7: Analyze the behavior of the graph of R near each asymptote and indicate this behavior on the graph.
STEP 8: Use the results obtained in Steps 1 through 7 to graph R.

Example 2 Analyzing the Graph of a Rational Function

Example 3 Analyzing the Graph of a Rational Function

Example 4 Analyzing the Graph of a Rational Function

Example 5 Analyzing the Graph of a Rational Function with a Hole

4.3 Assess Your Understanding

Skill Building

7. $R(x)=\dfrac{x+1}{x(x+4)}$

(a) $Dom(R)=\{x: x\neq 0, x\neq -4\}$.

(b) Since there are no common factors between the numerator and denominator, R is in lowest terms.

(c) Since 0 is not in the domain of R, there is no y-intercept. To find x-intercepts, solve $x+1=0$ or $x=-1$. Therefore, the only real zero of the numerator is $x=-1$, i.e. the only x-intercept of the graph of R is -1.

Near -1: $R(x)=\dfrac{x+1}{x(x+4)}\approx\dfrac{x+1}{(-1)(-1+4)}=-\dfrac{1}{3}(x+1)$

Plot the point $(-1,0)$ and draw a line through $(-1,0)$ with a negative slope.

(d) Since R is in lowest terms, the vertical asymptotes are $x=0$ and $x=-4$.

(e) Since R is proper, the horizontal asymptote is $y=0$ intersected at $(-1,0)$ because

$$R(x)=\frac{x+1}{x(x+4)}=0 \Rightarrow x+1=0 \Rightarrow x=-1$$

(f) Now construct a table.

Table 1 for Question No. 7

Interval	Number picked	Value of *R*	Point on graph	Location of graph
$(-\infty,-4)$	-5	$R(-5)=-\dfrac{4}{5}$	$\left(-5,-\dfrac{4}{5}\right)$	Below the *x*-axis
$(-4,-1)$	-2	$R(-2)=\dfrac{1}{4}$	$\left(-2,\dfrac{1}{4}\right)$	Above the *x*-axis
$(-1,0)$	$-\dfrac{1}{2}$	$R\left(-\dfrac{1}{2}\right)=-\dfrac{2}{7}$	$\left(-\dfrac{1}{2},-\dfrac{2}{7}\right)$	Below the *x*-axis
$(0,+\infty)$	1	$R(1)=\dfrac{2}{5}$	$\left(1,\dfrac{2}{5}\right)$	Above the *x*-axis

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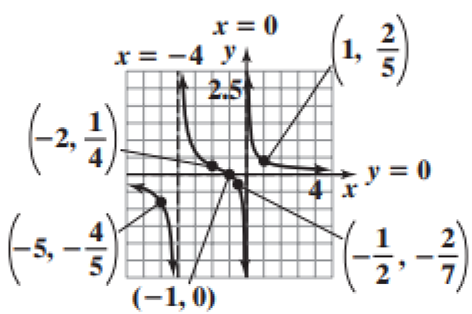

9. $R(x)=\dfrac{3x+3}{2x+4}=\dfrac{3(x+1)}{2(x+2)}=\dfrac{3}{2}\dfrac{(x+2)-1}{x+2}=\dfrac{3}{2}\left(1-\dfrac{1}{x+2}\right)=\dfrac{3}{2}-\dfrac{3}{2(x+2)}$

(a) $Dom(R)=\{x:x\neq -2\}$.

(b) Since there are no common factors between the numerator and denominator, R is in lowest terms.

(c) Since 0 is in the domain of R, the y-intercept is $R(0)=\dfrac{3}{4}$. To find x-intercepts, solve $3(x+1)=0$ or $x=-1$. Therefore, the only real zero of the numerator is -1, i.e. the only x-intercept of the graph of R is -1.

Near -1: $R(x)=\dfrac{3(x+1)}{2(x+2)}\approx\dfrac{3(x+1)}{2(-1+2)}=\dfrac{3}{2}(x+1)$

Plot the point $(-1,0)$ and draw a line through $(-1,0)$ with a positive slope.

(d) Since R is in lowest terms, the only vertical asymptote is $x=-2$.

(e) Since R is proper, the horizontal asymptote is $y=\dfrac{3}{2}$ which does not intersect at $(-1,0)$ because

$$R(x)=\frac{3x+3}{2x+4}=\frac{3}{2}\Rightarrow 6x+12=6x+6\Rightarrow 12=6 \text{ is absurd}$$

(f) Now construct a table.

Table 2 for Question No. 9

Interval	Number picked	Value of *R*	Point on graph	Location of graph
$(-\infty,-2)$	-3	$R(-3)=3$	$(-3,3)$	Above the x-axis
$(-2,-1)$	$-\dfrac{3}{2}$	$R\left(-\dfrac{3}{2}\right)=-\dfrac{3}{2}$	$\left(-\dfrac{3}{2},-\dfrac{3}{2}\right)$	Below the x-axis
$(-1,+\infty)$	0	$R(0)=\dfrac{3}{4}$	$\left(0,\dfrac{3}{4}\right)$	Above the x-axis

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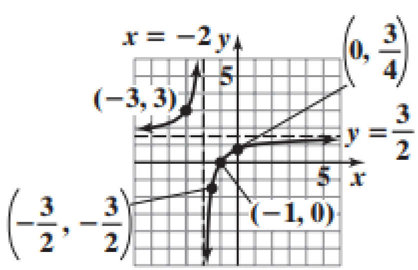

11. $R(x)=\dfrac{3}{x^2-4}=\dfrac{3}{(x+2)(x-2)}$

(a) $Dom(R)=\{x:x\neq -2, x\neq 2\}$.

(b) Since there are no common factors between the numerator and denominator, R is in lowest terms.

(c) Since 0 is in the domain of R, the y-intercept is $R(0)=-\dfrac{3}{4}$ and there is no x-intercept.

(d) Since R is in lowest terms, the vertical asymptotes are $x=-2$ and $x=2$.

(e) Since R is proper, the horizontal asymptote is $y = 0$.
(f) Now construct a table.

Table 3 for Question No. 11

Interval	Number picked	Value of *R*	Point on graph	Location of graph
$(-\infty, -2)$	-3	$R(-3) = \frac{3}{5}$	$R\left(-3, \frac{3}{5}\right)$	Above the *x*-axis
$(-2,\ 2)$	0	$R(0) = -\frac{3}{4}$	$\left(0,\ -\frac{3}{4}\right)$	Below the *x*-axis
$(2, +\infty)$	3	$R(3) = \frac{3}{5}$	$\left(3,\ \frac{3}{5}\right)$	Above the *x*-axis

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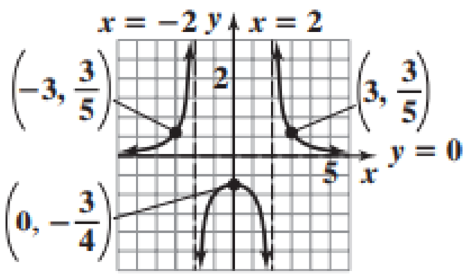

13. $P(x) = \dfrac{x^4 + x^2 + 1}{x^2 - 1}$

6.

	−1		1
Interval	**$(-\infty, -1)$**	**$(-1, 1)$**	**$(1, \infty)$**
Number Chosen	**-2**	**0**	**2**
Value of *P*	**$P(-2) = 7$**	**$P(0) = -1$**	**$P(2) = 7$**
Location of Graph	**Above *x*-axis**	**Below *x*-axis**	**Above *x*-axis**
Point on Graph	**$(-2, 7)$**	**$(0, -1)$**	**$(2, 7)$**

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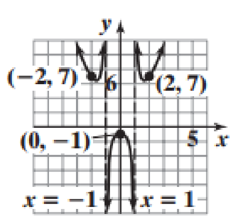

15. $H(x)=\dfrac{x^3-1}{x^2-9}$

6.

Interval	$(-\infty, -3)$	$(-3, 1)$	$(1, 3)$	$(3, \infty)$
Number Chosen	-4	0	2	4
Value of H	$H(-4) \approx -9.3$	$H(0) = \frac{1}{9}$	$H(2) = -1.4$	$H(4) = 9$
Location of Graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on Graph	$(-4, -9.3)$	$\left(0, \frac{1}{9}\right)$	$(2, -1.4)$	$(4, 9)$

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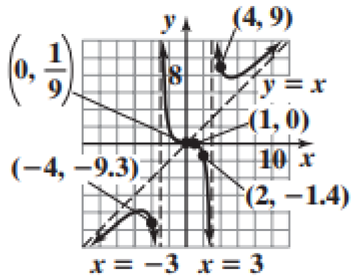

17. $R(x)=\dfrac{x^2}{x^2+x-6}$

6.

Interval	$(-\infty, -3)$	$(-3, 0)$	$(0, 2)$	$(2, \infty)$
Number Chosen	-6	-1	1	3
Value of R	$R(-6) = 1.5$	$R(-1) = -\frac{1}{6}$	$R(1) = -0.25$	$R(3) = 1.5$
Location of Graph	Above x-axis	Below x-axis	Below x-axis	Above x-axis
Point on Graph	$(-6, 1.5)$	$\left(-1, -\frac{1}{6}\right)$	$(1, -0.25)$	$(3, 1.5)$

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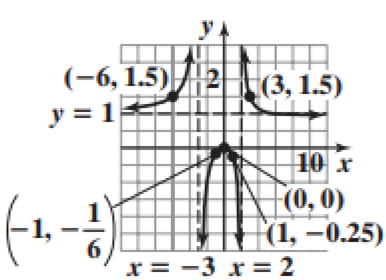

19. $G(x)=\dfrac{x}{x^2-4}$

6.

-2 $\quad$ 0 $\quad$ 2

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,\infty)$
Number Chosen	-3	-1	1	3
Value of G	$G(-3)=-\frac{3}{5}$	$G(-1)=\frac{1}{3}$	$G(1)=-\frac{1}{3}$	$G(3)=\frac{3}{5}$
Location of Graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on Graph	$\left(-3,-\frac{3}{5}\right)$	$\left(-1,\frac{1}{3}\right)$	$\left(1,-\frac{1}{3}\right)$	$\left(3,\frac{3}{5}\right)$

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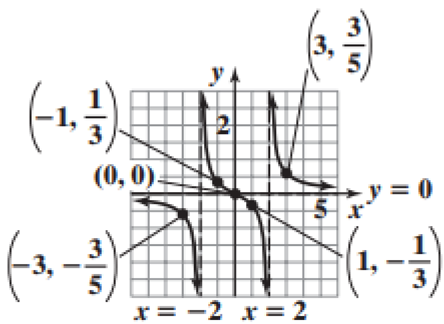

21. $R(x)=\dfrac{3}{(x-1)(x^2-4)}$

23. $H(x)=\dfrac{x^2-1}{x^4-16}$

25. $F(x)=\dfrac{x^2-3x-4}{x+2}$

27. $R(x)=\dfrac{x^2+x-12}{x-4}$