DAY-7 : Section 7.5 Integration by Partial Fractions

How to integrate following functions?

$$\int \frac{x^2-2x-1}{x^2-1}\,dx, \qquad \int \frac{x^3-2x-1}{x^2-1}\,dx, \qquad \int \frac{x^2-2x-1}{x^3-x}\,dx.$$

Partial Fraction is based on Logarithmic Derivative.

Definition: Logarithmic Derivative

$$\frac{d}{dx}\left[\ln\big(f(x)\big)\right] = \frac{1}{f(x)}\,f'(x) = \frac{f'(x)}{f(x)} = \frac{\text{Derivative of the Denominator}}{\text{Denomintor}}.$$

The fraction of the type $\frac{f'(x)}{f(x)}$ is called the logarithmic derivative.

Here, $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$; $\int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$

$$\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\int \frac{2x}{x^2+5}\,dx = ln\,(x^2+5) + C$$

Example :1

(a) Evaluate $\int \frac{x^2-2x-1}{x^3-x}\,dx.$

(b) Evaluate $\int \frac{2x-1}{x^3-x}\,dx.$

(c) Evaluate $\int \frac{x^2-1}{x^3-x}\,dx.$

Solution [Example: 1, Part (a)]

Let's start with the fraction.

$$\frac{x^2-2x-1}{x^3-x} = \frac{x^2-2x-1}{x(x-1)(x+1)}$$

Set $\frac{x^2-2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$ … … … (1) ; [Rational Function with D= all reals, except -1, 0, 1]

Multiply the both sides of the equation above by $x(x-1)(x+1)$:

$x^2-2x-1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$ … … … (2) ; [Polynomial, D= all reals]

[(2) gives us a polynomial, and the domain of a polynomial is all real numbers. So, we can plug in any real value for x in (2)]

If $x=0$: $-1 = A(-1)(1) \Rightarrow -A = -1 \quad \therefore \quad A = 1.$

If $x=1$: $1^2 - 2(1) - 1 = B(1)(2) \Rightarrow 2B = -2 \quad \therefore \quad B = -1.$

If $x = -1$: $(-1)^2 - 2(-1) - 1 = C(-1)(-2) \quad \Rightarrow 2C = \ 2 \ \therefore \quad C = 1.$

Now,

$$\frac{x^2-2x-1}{x^3-x} = \frac{x^2-2x-1}{x(x-1)(x+1)} = \frac{1}{x} + \frac{-1}{x-1} + \frac{1}{x+1}$$

$$\int \frac{x^2-2x-1}{x^3-x}\,dx = \int \left[\frac{1}{x} + \frac{-1}{x-1} + \frac{1}{x+1}\right] dx$$

$$= \int \left[\frac{1}{x} - \frac{1}{x-1} + \frac{1}{x+1}\right] dx$$

$$= \ln|x| - \ln|x-1| + \ln|x+1| + C \ = \ln\frac{|x(x+1)|}{|x-1|} + C$$

Example: 2 Evaluate $\int \frac{x^3-2x-1}{x^2-1}\,dx$

[Dividend / divisor = Quotient + Remainder/Divisor]

Given Fraction $\frac{x^3-2x-1}{x^2-1} = x + \frac{-x-1}{x^2-1} = x - \frac{(x+1)}{(x-1)(x+1)} = \ x - \frac{1}{x-1}$

Here $\int \frac{x^3-2x-1}{x^2-1} = \int \left[x - \frac{1}{x-1}\right] dx = \frac{x^2}{2} - \ln|x-1| + C$

__**Understanding:**__ Since $\frac{d}{dx}(x^4) = 4x^3$, the derivative of x^4 is a polynomial of degree 3.

Also, general form of a polynomial of degree 3 is given by $ax^3 + bx^2 + cx + d$

$$\frac{Ax^3+Bx^2+Cx+D}{[\,x\,]^4} = \frac{A}{x^1} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4}$$

$$\frac{2x+1}{(\ \boldsymbol{x^2+4}\ )^4} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3} + \frac{Gx+H}{(x^2+4)^4}$$

Example: 3 Evaluate $\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

Solution: The given fraction is $\frac{3x^2-x+1}{x^3-x^2} = \frac{3x^2-x+1}{x^2(x-1)}$

Set $\frac{3x^2-x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \quad \dots\dots\dots (1)$

Multiply the both sides of the equation above by $x^2(x-1)$:

$3x^2 - x + 1 = Ax(x-1) + B(x-1) + Cx^2 \quad \dots\dots\dots \quad (2)$

[(2) gives us a polynomial, and the domain of a polynomial is all real numbers. So, we can plug in any real value for x in (2)]

If $x = 0$: $1 = 0 + B(-1) + 0 \ \Rightarrow \ -B = 1 \quad \therefore \boldsymbol{B = -1}$

If $x = 1$: $\quad 3 = C(1) \quad \therefore \boldsymbol{C = 3}$

If $x = -1$: $\quad 5 = A(-1)(-2) + B(-2) + C$

$\Rightarrow 5 = 2A - 2B + C$

$\Rightarrow 5 = 2A - 2(-1) + 3$

$\Rightarrow 2A = 5 - 5$

$\Rightarrow \boldsymbol{A = 0}$

$$\frac{3x^2 - x + 1}{x^2(x-1)} = \frac{0}{x} + \frac{-1}{x^2} + \frac{3}{x-1}$$

$$\int \frac{3x^2 - x + 1}{x^3 - x^2}\, dx = \int [\frac{-1}{x^2} + 3\ \frac{1}{x-1}]\, dx$$

$$= \int [-x^{-2} + 3\ \frac{1}{x-1}]\, dx \quad = \frac{1}{x} + 3\ \ln|x-1| + C$$

$$\int \frac{3x^2 - x + 1}{x^3 - x^2}\, dx = \frac{1}{x} + 3\ \ln|x-1| + C.$$

Example: 4 $\quad \int \frac{3x^2-x+1}{x^3+x}\, dx$ Homework

Solution: $\quad \frac{3x^2-x+1}{x^3+x} = \frac{3x^2-x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

Example: 5 $\quad \int \frac{5+2\ln x}{x(1+\ln x)^2}\, dx$

Given $\int \frac{5+2\ln x}{x(1+\ln x)^2}\, dx = \int \frac{2(1+\ln x)+3}{x(1+\ln x)^2}\, dx = \int \frac{2(1+\ln x)+3}{(1+\ln x)^2}\ \frac{1}{x} dx$

$$= \int \frac{2u+3}{u^2}\, du \quad ; \text{ Set } u = 1 + \ln x, \text{ then } du = \frac{1}{x}\, dx.$$

That is, $\int \frac{5+2\ln x}{x(1+\ln x)^2}\, dx = \int \frac{2u+3}{u^2}\, du = \int \left[\frac{2u}{u^2} + \frac{3}{u^2}\right] du$

$$= \int \left[2\frac{1}{u} + 3u^{-2}\right] du = 2\ln|1 + \ln x| - \frac{3}{1 + \ln x} + C$$

Example: 6 $\int \frac{7\cos\theta}{\sin^2\theta+5\cos\theta-6}\,d\theta$

Solution: Given $\int \frac{7\cos\theta}{\sin^2\theta+5\sin\theta-6}\,d\theta$

Set $u = \sin\theta$, $du = \cos\theta\ d\theta$

Then $\int \frac{7\cos\theta}{\sin^2\theta+5\sin\theta-6}\,d\theta = \int \frac{7}{\sin^2\theta+5\sin\theta-6}\cos\theta\,d\theta = 7\int \frac{1}{u^2+5u-6}\,du$ Complete!!

Example: 7 $\int \frac{3x^4-x^2+1}{x^3+x}\,dx$ Homework.

Example: 4 $\int \frac{3x^2-x+1}{x^3+x}\,dx$ Homework

Solution: $\frac{3x^2-x+1}{x^3+x} = \frac{3x^2-x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \ldots\ldots\ldots (1)$

$3x^2 - x + 1 = A(x^2+1) + (Bx+C)x \ \ldots\ldots\ldots\ (2)$

Set $x = 0,\ 1,\ -1$ in (2)

Please complete!

Example: 5 $\int \frac{5+2\ln x}{x(1+\ln x)^2}\,dx$

Given $\int \frac{5+2\ln x}{x(1+\ln x)^2}\,dx = \int \frac{2(1+\ln x)+3}{x(1+\ln x)^2}\,dx = \int \frac{2(1+\ln x)+3}{(1+\ln x)^2}\,\frac{1}{x}dx$

$= \int \frac{2u+3}{u^2}\,du$; Set $u = 1+\ln x$, then $du = \frac{1}{x}\,dx$.

That is, $\int \frac{5+2\ln x}{x(1+\ln x)^2}\,dx = \int \frac{2u+3}{u^2}\,du = \int \left[\frac{2u}{u^2} + \frac{3}{u^2}\right] du$

$= \int \left[2\frac{1}{u} + 3u^{-2}\right] du = 2\ln|1+\ln x| - \frac{3}{1+\ln x} + C$

Example: 6 $\int \frac{7\cos\theta}{\sin^2\theta+5\sin\theta-6}\,d\theta$

Solution: Given $\int \frac{7\cos\theta}{\sin^2\theta+5\sin\theta-6}\,d\theta$

Set $u=\sin\theta$, $du=\cos\theta\ d\theta$

Then $\int \frac{7\cos\theta}{\sin^2\theta+5\sin\theta-6}\,d\theta = \int \frac{7}{\sin^2\theta+5\sin\theta-6}\cos\theta\,d\theta = 7\int \frac{1}{u^2+5u-6}\,du$ Complete !!

Example: 7 $\int \frac{3x^4-x^2+1}{x^3+x}\,dx$ **Homework.**

$$\begin{array}{r|c|l} x^3+x & 3x^4-x^2+1 & 3x \\ & 3x^4+3x^2 & \\ \hline & -4x^2+1 & \end{array}$$

Note that $(3x^4-x^2+1)$**-**$(3x^4+3x^2)$**=** $-4x^2+1$

$$\frac{3x^4-x^2+1}{x^3+x}=3x+\frac{-4x^2+1}{x^3+x}$$

Find the partial fraction of $\frac{-4x^2+1}{x^3+x}$.

$\int \frac{3x^4-x^2+1}{x^3+x}\,dx = \int \left[3x+\frac{-4x^2+1}{x^3+x}\right]dx$

$$\frac{-4x^2+1}{x^3+x}=\frac{-4x^2+1}{x(x^2+1)}$$

Set $\frac{-4x^2+1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$ ……… (1)

Then $-4x^2+1=A(x^2+1)+(Bx+C)x$ ……… (2)

$Set\ x=0,\quad 1,-1$

Finally, $\int \frac{3x^4-x^2+1}{x^3+x}\,dx = \int \left[3x+\frac{A}{x}+\frac{Bx+C}{x^2+1}\right]dx$

Please Complete!