Chapter # 05
(Integration)

5.4 The Definition of Area as a Limit; Sigma Notation: Our main goal in this section is to use the rectangle method to give a precise mathematical definition of the ***"area under a curve"***.

Sigma Notation: To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called ***sigma notation or summation notation*** because it uses the uppercase Greek letter **Σ** (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

in which each term is of the form $\boldsymbol{k^2}$, where $\boldsymbol{k}$ is one of the integers from 1 to 5. In sigma notation this sum can be written as

$$\sum_{k=1}^{5} k^2$$

Example 1:

$$\sum_{k=4}^{8} k^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3$$

$$\sum_{k=1}^{5} 2k = 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5 = 2 + 4 + 6 + 8 + 10$$

$$\sum_{k=0}^{5} (2k+1) = 1 + 3 + 5 + 7 + 9 + 11$$

$$\sum_{k=0}^{5} (-1)^k (2k+1) = 1 - 3 + 5 - 7 + 9 - 11$$

$$\sum_{k=-3}^{1} k^3 = (-3)^3 + (-2)^3 + (-1)^3 + 0^3 + 1^3 = -27 - 8 - 1 + 0 + 1$$

$$\sum_{k=1}^{3} k \sin\left(\frac{k\pi}{5}\right) = \sin\frac{\pi}{5} + 2\sin\frac{2\pi}{5} + 3\sin\frac{3\pi}{5}$$

Properties of Sums:

$$\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = \sum_{j=1}^{5} a_j = \sum_{k=-1}^{3} a_{k+2}$$
$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j = \sum_{k=-1}^{n-2} a_{k+2}$$

Theorem:

$$(a) \quad \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k \qquad (\textit{if } c \textit{ does not depend on } k)$$
$$(b) \quad \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$
$$(c) \quad \sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

Theorem:

$$(a) \quad \sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
$$(b) \quad \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$(c) \quad \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 2: Evaluate

$$\sum_{k=1}^{30} k(k+1).$$

Solution:

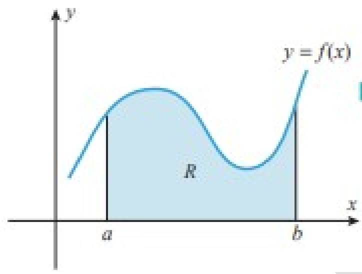
$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30} (k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$
$$= \frac{30(31)(61)}{6} + \frac{30(31)}{2} = 9920$$

In formulas such as

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{or} \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

the left side of the equality is said to express the sum in ***open form*** and the right side is said to express it in ***closed form***. The open form indicates the summands and the closed form is an explicit formula for the sum.

A Definition of Area: We now turn to the problem of giving a precise definition of what is meant by the ***"area under a curve."*** Specifically, suppose that the function $\boldsymbol{f}$ is continuous and nonnegative on the interval **[*a*, *b*]**, and let $\boldsymbol{R}$ denote the region bounded below by the $\boldsymbol{x}$-axis, bounded on the sides by the vertical lines $\boldsymbol{x = a}$ and $\boldsymbol{x = b}$, and bounded above by the curve $\boldsymbol{y = f(x)}$.



Definition (Area Under a Curve): If the function $\boldsymbol{f}$ is continuous on **[*a*, *b*]** and if $\boldsymbol{f(x) \geq 0}$ for all $\boldsymbol{x}$ in **[*a*, *b*]**, then the area $\boldsymbol{A}$ under the curve $\boldsymbol{y = f(x)}$ over the interval **[*a*, *b*]** is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

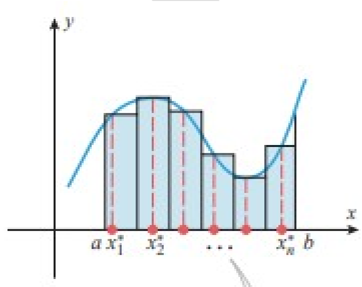
Here, $\Delta x = \frac{b-a}{n}$, $x_1^*, x_2^*, \dots \dots \dots \; x_n^*$ denote the point selected in the subintervals, that is

$$x_k = a + k \,.\, \Delta x \quad for \quad k = 0, 1, \dots \dots \dots, n$$

If x_k^* denotes left endpoint: $\quad x_k^* = x_{k-1} = a + (k-1).\,\Delta x$

If x_k^* denotes right endpoint: $\quad x_k^* = x_k = a + k.\,\Delta x$

If x_k^* denotes midpoint: $\quad x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + \left(k - \frac{1}{2}\right).\,\Delta x$



Example 4: Find the area between the graph of $\boldsymbol{f(x) = x^2}$ and the interval **[0, 1]**.

Solution: Here, $[a, b] = [0, 1]$

$$\therefore \quad \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

If x_k^* denotes right endpoint: $\quad x_k^* = a + k.\Delta x = 0 + k.\frac{1}{n} = \frac{k}{n}$

Thus,

$$\sum_{k=1}^{n} f(x_k^*).\Delta x = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 . \frac{1}{n} = \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{1}{6}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$\therefore$ The area, $A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,.\Delta x = \lim_{x\to\infty}\left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{6}(1)(2) = \frac{1}{3}$ (Ans.)

Theorem:

$$(a)\ \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} 1 = 1 \qquad (b)\ \lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{2}$$

$$(c)\ \lim_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{3} \qquad (d)\ \lim_{n\to+\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{4}$$

Example 5: With $\boldsymbol{x_k^*}$ as the midpoint of each subinterval to find the area under the parabola $\boldsymbol{y = f(x) = 9 - x^2}$ and over the interval [0, 3].

Solution: Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

So if $\boldsymbol{x_k^*}$ denotes as midpoint then,

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x = \left[9 - \left(k-\frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right)$$

$$= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}$$

from which it follows that

$$\begin{aligned}
A &= \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{27}{n}-\frac{27}{n^3}k^2+\frac{27}{n^3}k-\frac{27}{4n^3}\right) \\
&= \lim_{n\to+\infty} 27\left[\frac{1}{n}\sum_{k=1}^{n}1-\frac{1}{n^3}\sum_{k=1}^{n}k^2+\frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n}k\right)-\frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n}1\right)\right] \\
&= 27\left[1-\frac{1}{3}+0\cdot\frac{1}{2}-0\cdot 1\right]=18 \qquad \boxed{\text{Theorem 5.4.4}}
\end{aligned}$$

Definition (Net Signed Area): If the function $\boldsymbol{f}$ is continuous on **[*a*, *b*]**, then the net signed area ***A*** between $\boldsymbol{y = f(x)}$ and the interval **[*a*, *b*]** is defined by

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Example 7: Find the net signed area between the graph of $\boldsymbol{f(x) = x - 1}$ and the interval **[0, 2]** with $\boldsymbol{x_k^*}$ chosen to be the left endpoint of each subinterval.

Solution: Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

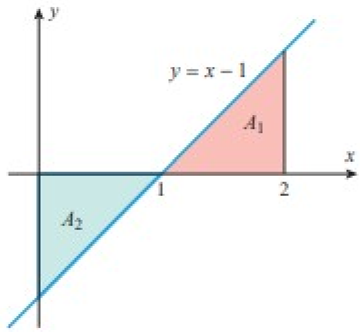
$$x_k^* = a + (k-1)\Delta x = (k-1)\left(\frac{2}{n}\right)$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}(x_k^*-1)\Delta x &= \sum_{k=1}^{n}\left[(k-1)\left(\frac{2}{n}\right)-1\right]\left(\frac{2}{n}\right) \\
&= \sum_{k=1}^{n}\left[\left(\frac{4}{n^2}\right)k-\frac{4}{n^2}-\frac{2}{n}\right]
\end{aligned}$$

from which it follows that

$$\begin{aligned}
A &= \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left[4\left(\frac{1}{n^2}\sum_{k=1}^{n}k\right)-\frac{4}{n}\left(\frac{1}{n}\sum_{k=1}^{n}1\right)-2\left(\frac{1}{n}\sum_{k=1}^{n}1\right)\right] \\
&= 4\left(\frac{1}{2}\right)-0\cdot 1-2\cdot 1 = 0 \qquad \boxed{\text{Theorem 5.4.4}}
\end{aligned}$$



Home Work: Exercise 5.4: Problem No. 11-20, 31, 32, 35-39 and 45-48