Question: Suppose that two continuous random variables X and Y have a joint probability density function

$$f(x, y) = A(x - 3)y \quad -2 \le x \le 3 \quad 4 \le y \le 6$$

a) What is the value of A? /if A is known, show that total area under the probability density function = 1.

b) What is $P(0 \le x \le 1 \text{ and } 4 \le y \le 5)$?

c) Construct the marginal probability density functions. g(x) and h(y)

d) Are the random variables X and Y independent?

e) If Y = 5, what is the conditional probability density function of X?

f) What are the expectation and variance of the random variable X and Y.

g) What is the covariance of X and Y.

h) What is the correlation between X and Y.

Solution:

a) $\int_4^6 \int_{-2}^3 f(x,y)\,dxdy = 1$

=> $\int_4^6 \int_{-2}^3 A(x-3)y\,dxdy = 1$

=> $A \int_4^6 \left[\frac{x^2}{2} - 3x\right]_{-2}^3 y\,dy = 1$

=> $A \int_4^6 \left[\frac{9}{2} - 9 - \left(\frac{4}{2} + 6\right)\right] y\,dy = 1$

=> $A \cdot \left(-\frac{25}{2}\right) \left[\frac{y^2}{2}\right]_4^6 = 1$

=> $A \cdot \left(-\frac{25}{2}\right) \cdot 10 = 1$

=> $A \cdot (-125) = 1$

$\therefore A = -\frac{1}{125}$

b) $P\,(0 \le x \le 1,\ 4 \le y \le 5)$

$$= \int_4^5 \int_0^1 -\frac{1}{125}\,(x-3)\,y\;dx\;dy$$

$$= -\frac{1}{125} \int_4^5 y \int_0^1 (x-3)\;dx\;dy$$

$$= -\frac{1}{125} \int_4^5 y \left[\frac{x^2}{2} - 3x\right]_0^1 dy$$

$$= -\frac{1}{125} \int_4^5 y \left[\frac{1}{2} - 3\right] dy$$

$$= -\frac{1}{125} \cdot -\frac{5}{2} \cdot \left[\frac{y^2}{2}\right]_4^5$$

$$= \frac{1}{125} \cdot \frac{5}{2} \cdot \frac{9}{2}$$

$$= .09$$

c) $g\,(x) = \int_4^6 -\frac{1}{125}\,(x-3)\,y\;dy$

$$= -\frac{1}{125}\,(x-3)\left[\frac{y^2}{2}\right]_4^6$$

$$= -\frac{1}{125}\,(x-3)\cdot 10$$

$$= -\frac{2}{25}\,(x-3)$$

$h\,(y) = \int_{-2}^{3} -\frac{1}{125}\,(x-3)\,y\;dx$

$$= -\frac{1}{125}\,y \int_{-2}^{3} (x-3)\,dx$$

$$= -\frac{y}{125}\left[\frac{x^2}{2} - 3x\right]_{-2}^{3}$$

$$= -\frac{y}{125}\left[\frac{9}{2} - 9 - 2 - 6\right]$$

$$= \frac{y}{10}$$

d) Two random variables are said to be independent if $g\,(x)\cdot h\,(y) = f\,(x,y)$

$g\,(x)\cdot h\,(y) = -\frac{2}{25}\,(x-3)\cdot\frac{y}{10}$

$$= -\frac{1}{125}\,(x-3)\,y$$

$$= f(x, y)$$

So, X and Y are independent.

e) $f\,(y = 5) = \frac{f(x,\ \ y=5)}{h(y=5)}$

$$= \frac{-\frac{1}{125}(x-3)\cdot 5}{\frac{5}{10}}$$

$$=- \tfrac{2}{25}\,(x - 3)$$

f) $E\,(x) = \int_{-2}^{3} x\,g\,(x)\ \ dx$

$$= \int_{-2}^{3} x \cdot - \tfrac{2}{25}\,(x - 3)\,dx$$

$$=- \tfrac{2}{25} \int_{-2}^{3} (x^2 - 3x)\ dx$$

$$=- \tfrac{2}{25} \left[\tfrac{x^3}{3} - 3\tfrac{x^2}{2} \right]_{-2}^{3}$$

$$=- \tfrac{2}{25}\left[9 - \tfrac{27}{2} + \tfrac{8}{3} + \tfrac{12}{2}\right]$$

$$=- \tfrac{2}{25} \cdot \tfrac{25}{6}$$

$$=- \tfrac{1}{3}$$

$E\,(y) = \int_{4}^{6} y\ h\,(y)\ \ dy$

$$= \int_{4}^{6} y \cdot \tfrac{y}{10} dy$$

$$= \tfrac{1}{10} \int_{4}^{6} y^2\ dy$$

$$= \tfrac{1}{10} \left[\tfrac{y^3}{3}\right]_{4}^{6}$$

$$= \tfrac{1}{10}\left[\tfrac{6^3}{3} - \tfrac{4^3}{3}\right]$$

$$= \tfrac{1}{10} \bullet \tfrac{152}{3}$$

$$= \tfrac{76}{15}$$

$$E(x^2) = \int_{-2}^{3} x^2\, g(x)\; dx$$

$$= \int_{-2}^{3} x^2 \cdot -\tfrac{2}{25}(x-3)\, dx$$

$$= -\tfrac{2}{25}\int_{-2}^{3} (x^3 - 3x^2)\; dx$$

$$= -\tfrac{2}{25}\left[\tfrac{x^4}{4} - 3\tfrac{x^3}{3}\right]_{-2}^{3}$$

$$= -\tfrac{2}{25}\left[\tfrac{3^4}{4} - 3^3 - \tfrac{(-2)^4}{4} + (-2)^3\right]$$

$$= -\tfrac{2}{25}\cdot\left(-\tfrac{75}{4}\right)$$

$$= \tfrac{3}{2}$$

$$E(y^2) = \int_{4}^{6} y^2\, h(y)\; dy$$

$$= \int_{4}^{6} y^2 \cdot \tfrac{y}{10} dy$$

$$= \tfrac{1}{10}\int_{4}^{6} y^3\; dy$$

$$= \tfrac{1}{10}\left[\tfrac{y^4}{4}\right]_{4}^{6}$$

$$= \tfrac{1}{10}\left[\tfrac{6^4}{4} - \tfrac{4^4}{4}\right]$$

$$= \tfrac{1}{10}\cdot 260$$

$$= 26$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \tfrac{3}{2} - \left(-\tfrac{1}{3}\right)^2$$

$$= \tfrac{25}{18}$$

$$V(y) = E(y^2) - (E(y))^2$$

$$= 26 - \left(\tfrac{76}{15}\right)^2$$

$$= \tfrac{5774}{225}$$

g) $Cov\,(x,y) = E\,(xy) - E\,(x)\,E(y)$

Here, $E\,(xy) = \int_{4}^{6}\int_{-2}^{3} xy\cdot -\frac{1}{125}\,(x-3)\,y dx\,dy$

$$= -\frac{1}{125}\int_{4}^{6}\int_{-2}^{3} (x^2-3x)\;y^2\;dx\,dy$$

$$= -\frac{1}{125}\int_{4}^{6} y^2 \int_{-2}^{3} (x^2-3x)\quad dx\,dy$$

$$= -\frac{1}{125}\int_{4}^{6} y^2\left[\frac{x^3}{3} - 3\frac{x^2}{2}\right]_{-2}^{3} dy$$

$$= -\frac{1}{125}\left[\frac{3^3}{3} - 3\cdot\frac{3^2}{2} + \frac{8}{3} + 6\right]\int_{4}^{6} y^2\;dy$$

$$= -\frac{1}{125}\left[\frac{3^3}{3} - 3\cdot\frac{3^2}{2} + \frac{8}{3} + 6\right]\left[\frac{y^3}{3}\right]_4^6$$

$$= -\frac{1}{125}\cdot\frac{25}{6}\cdot\frac{152}{3}$$

$$= -\frac{76}{45}$$

$Cov\,(x,y) = E\,(xy) - E\,(x)\,E\,(y)$

$$= -\frac{76}{45} - \left(-\frac{1}{3}\right)\bullet\frac{76}{15}$$

$$= 0$$

h) $Corr\,(x,y) = \frac{Cov(x,y)}{\sqrt{v(x)v(y)}}$

$$= \frac{0}{\sqrt{v(x)v(y)}}$$

$$= 0$$