chapter - 1

Probability Theory

Introduction:

Jointly with statistic, probability theory is a branch of mathematics that has been developed to deal with uncertainty.

The theory of probability has been developed as a scientific tool to deal with chance.

Sample Space:

The sample space (s) of an experiment is a set consisting of all the possible experimental outcomes.

Example 1: (Machine Breakdowns)

An engineer in charge of the maintenance of a particular machine notices that its breakdowns can be characterized as due to an electrical failure within the machine, a mechanical failure of some component of the machine or operator misuse. When the machine is running, the enginner is uncertain what will be the cause of the next breakdown. The problem can be thought of as an experiment with the sample space.

$$S = \{\text{electrical, mechanical, misuse}\}$$

Example 2: (Defective Computer Chips)

A company sells computer chips in boxes of 500 and each chips can be classified as either satisfactory or defective. The number defective chips in a perticular box is uncertain. What is the sample space of defective chips?

$$S = \{0 \text{ defective}, 1 \text{ defective}, 2 \text{ defectives}, \ldots\ldots 500 \text{ defectives}\}$$ in a box.

Example 3 : (Software Errors)

The control of errors in computer software products is obviously of great importance. The number of separate errors in a perticular piece of software can be viewed in what sample space ? [0 – unlimited errors]

$$S = \{ \cancel{0 \text{ defective}, 1 \text{ defective}}, $$

$$S = \{ 0 \text{ errors}, 1 \text{ errors}, 2 \text{ errors}, \ldots \}$$

Example 4 : (Power Plant Operation)

A manager supervises the operation of three power plants x, y, z. At any given time each of the three plants can be classified as either generating electricity or being idle. What is the sample space of all three plant at a perticular time ?

[0 = being idle ; 1 = generating electricity]

$$S = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

$x, y, z \; [0,1]$

$$\boxed{\begin{array}{ll} S & \\ (0,0,0), & (0,0,1) \\ (0,1,0) & (0,1,1) \\ (1,0,0) & (1,0,1) \\ (1,1,0) & (1,1,1) \end{array}}$$

Probability Values

A set of probability values for an experimental outcomes with a sample space $S = \{O_1, O_2, O_3, \ldots O_n\}$ consists of probability $P_1, P_2, P_3, \ldots P_n$

That satisfy,

$0 \leq P_1 \leq 1$

$0 \leq P_2 \leq 1$

$0 \leq P_3 \leq 1$

$\vdots$

$0 \leq P_n \leq 1$

0 being 0% chance

1 being 100% chance

and,

$$P_1 + P_2 + P_3 + \cdots + P_n = 1$$

For example, if a dice is thrown then there are total of six possibility. So, P_1 (i) $= \frac{1}{6}$; (ii) $P_2 = \frac{1}{6}$; (iii) $P_3 = \frac{1}{6}$

(iv) $P_4 = \frac{1}{6}$; (v) $P_5 = \frac{1}{6}$; (vi) $P_6 = \frac{1}{6}$

So, (i) P_1 + (ii) P_2 + (iii) P_3 + (iv) P_4 + (v) P_5 + (vi) P_6

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= \frac{6}{6}$

$= 1$

Problems (1.1)

Ex: 1 what is the sample space ~~of~~ when a coin is tossed three times?

Let, 0 = Tail

1 = Head

$\therefore S = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

Ex : 02

What is the sample space for counting the number of females in a group of n peoples ?

$$S = \{0 \text{ female}, 1 \text{ female}, 2 \text{ females}, \ldots\ldots, n \text{ females}\}$$

Ex : 03

What is the sample space for the number of aces in a hand of 13 playing cards ?

$$S = \{0 \text{ aces}, 1 \text{ aces}, 2 \text{ aces}, 3 \text{ aces}, 4 \text{ aces}\}$$

Ex : 04

What is the sample spaces for a person's birthday ?

Sol^n :

$$S = \{1^{st} \text{ Jan}, \ldots, 29^{th} \text{ Feb}, \ldots\ldots 25^{th} \text{ Mar} \ldots, 2^{nd} \text{ Apr}, \ldots\ldots, 31^{st} \text{ Dec.}\}$$

Ex : 05

A car repair in performed either on time or late and either satisfactory or unsatisfactory. What is the sample space for car repair ?

Sol^n:

let,

On time $= 1$ and, satisfactory $= x$

late $= 0$ unsatisfactory $= y$

$$S = \{ (1,x), (0,x), (1,y), (0,y) \}$$

Ex: 06

A bag contains balls that are either red or blue and either shiny or dull. What is the sample space when a ball is choosen from the bag?

Sol^n:

let,

red $= 1$ and, shiny $= a$

blue $= 0$ dull $= b$

$$\therefore S = \{ (1,a), (0,a), (1,b), (0,b) \}$$

Ex: 07

A probability value p is often reported as an odds ratio, which is p (1-p). This is the ratio of the probability that the event happens to the probability that the event does not happen.

(a) If the odds ratio is 1, what is P ?

$$\frac{P}{1-P} = 1$$

$\Rightarrow P = 1-P$

$\Rightarrow 2P = 1$

$\therefore P = \frac{1}{2}$

(b) If the odds ratio is 2, what is P ?

$$\frac{P}{1-P} = 2$$

$\Rightarrow P = 2 - 2P$

$\Rightarrow 3P = 2$

$\therefore P = \frac{2}{3}$

(c) If P = 0.25 ; what is the odd ratio ?

$$\frac{P}{1-P} = \frac{0.25}{1-0.25}$$

$$= \frac{0.25}{0.75}$$

$$= \frac{1}{3}$$

Ex : 08

An experiment has five outcomes I, ii, iii, iv, v

If $P(i) = 0.13$; $P(ii) = 0.24$; $P(iii) = 0.07$ and $P(iv) = 0.38$; So, what is $P(v)$?

Soln :

$$P(i) + P(ii) + P(iii) + P(iv) + P(v) = 1$$

$$\Rightarrow 0.13 + 0.24 + 0.07 + 0.38 + P(v) = 1$$

$$\Rightarrow P(v) = 1 - 0.82$$

$$\therefore P(v) = 0.18$$

(Ans:

Ex : 09

An experiment has five outcomes. I, II, III, IV, V.

If $P(I) = 0.08$, $P(II) = 0.20$ and $P(III) = 0.33$.

What are the possible values for the probability of outcomes V ? If outcomes IV and V are equally likely. What probability value ?

Solⁿ:

$$P(I) + P(II) + P(III) + P(IV) + P(V) = 1$$

$$\Rightarrow 0.08 + 0.20 + 0.33 + P(IV) + P(V) = 1$$

$$\Rightarrow P(IV) + P(V) = 0.39 \quad \text{---(i)}$$

So, $P(V)$ is $[0, 0.39]$

So, $0 \leq P(V) \leq 0.39$

Since, outcome $P(IV)$ and $P(V)$ equally likely.

Hence, we can write $P(IV) = P(V)$

$$\therefore P(V) + P(V) = 0.39$$

$$\Rightarrow 2P(V) = 0.39 \quad [\because P(V) = P(IV)]$$

$$\Rightarrow P(V) = 0.195$$

So, $P(IV) = P(V) = 0.195$

(Ans:)

Ex : 10

An experiment has three outcomes. I, II and III. If outcome I is twice as likely as outcome II and outcome II is three times as likely as outcome III. What are the probability values of the three outcomes ?

Solⁿ :

given, $P(I) = 2P(II)$ ——— (i)

$P(II) = 3P(III)$ ——— (ii)

Now, from (i) and (ii) we get,

$P(I) = 2 \times 3 \times P(III)$

$\Rightarrow P(I) = 6P(III)$ ——— (iii)

So,

$P(I) + P(II) + P(III) = 1$

$\Rightarrow 6P(III) + 3P(III) + P(III) = 1$

$\Rightarrow 10 \quad P(III) = 1$

$\therefore P(III) = \frac{1}{10}$ ——————— (iv)

Now, putting the values of (iv) we get from (ii) and (iii).

$P(II) = 3 \times \frac{1}{10}$

$\therefore P(II) = \frac{3}{10}$

and, $P(I) = 2 \times \frac{3}{10}$

$= \frac{6}{10}$

$= \frac{3}{5}$

So, $P(I) = \frac{3}{5}$

$P(II) = \frac{3}{10}$

and, $P(III) = \frac{1}{10}$ (Ans

Ex : 11

A company's advertising expenditure is either low with probability 0.28, average with probability 0.55 or high with probability P. What is the value of P ?

Sol^n :

$$P(\text{low}) + P(\text{avg}) + P(\text{high}) = 1$$

$$\Rightarrow 0.28 + 0.55 + P(\text{high}) = 1$$

$$\Rightarrow P(\text{high}) = 1 - 0.83$$

$$\therefore P(\text{high}) = 0.17$$

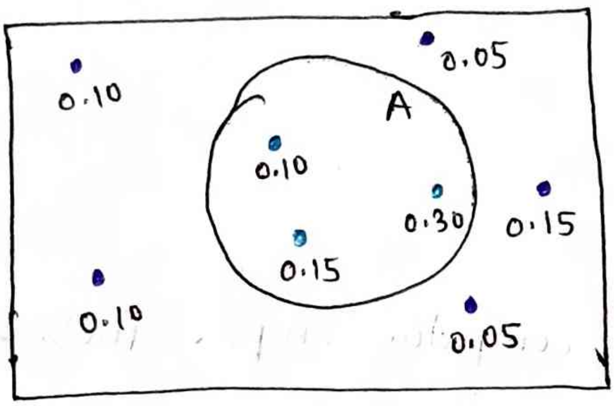
(Ans)

Events and Components

Event :

An event A is a subset of the Sample space S. It collects outcomes of perticular interest.

For Example :



$$P(A) = 0.10 + 0.15 + 0.30$$
$$= 0.55$$

Compliments of Event :

The event A', is the compliment of an event A, is the event consisting of everything in the sample space S is that is not contained within the event A. So, in all cases

$$P(A) + P(A') = P(S)$$

$P(S) = 1$; so, $P(A) + P(A') = 1$

from fig 1,

$$P(A') = 0.10 + 0.05 + 0.15 + 0.10 + 0.05$$

$$= 0.45$$

so, $P(A) + P(A')$

$= 0.55 + 0.45$

$= 1$

$= P(S)$

Example : 1

Defective computer chips. Probabilities value for the number of defective chips in a box of 500 chips.

$P(0 \text{ defective}) = 0.02$

$P(1 \text{ defective}) = 0.11$

$P(2 \text{ defective}) = 0.16$

$P(3 \text{ defective}) = 0.21$

$P(4 \text{ defective}) = 0.13$

$P(5 \text{ defective}) = 0.08$

So, company claims that,

$P(A) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$; has no more than 5 defective chips.

and, $P(S) = P(A) + P(A')$

$\Rightarrow 1 = P(A) + P(A')$

$\Rightarrow P(A') = 1 - P(A)$; has at least 6 or more defective chips.

So, $P(A) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= 0.02 + 0.11 + 0.16 + 0.21 + 0.13 + 0.08$

$= 0.71$

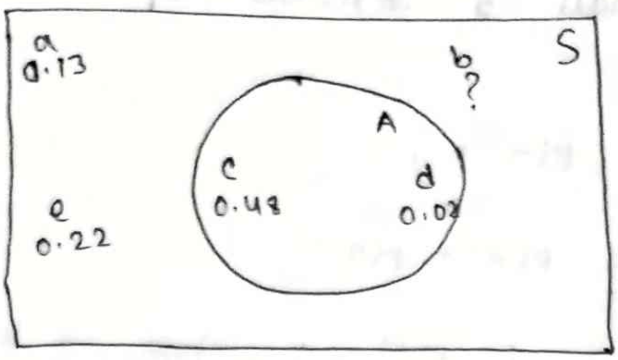
$\therefore P(A') = 1 - P(A)$

$= 1 - 0.71$

$= 0.29$

So, 71% of computer chips box has 5 or less defective chips and 29% of computer chips box has 6 or more defective chips as company claims.

Problems : 1.2.1



consider the sample space S with outcomes a, b, c, d and e. (a) $P(b) = ?$ (b) $P(A) = ?$ (c) $P(A') = ?$

Soln:

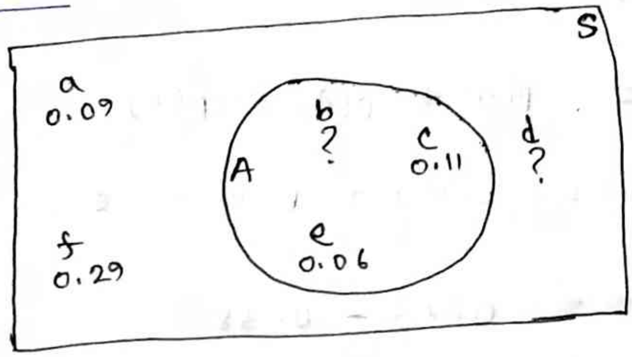
(a)
$$P(S) = P(a) + P(b) + P(c) + P(d) + P(e)$$
$$\Rightarrow 1 = 0.13 + P(b) + 0.48 + 0.02 + 0.22$$
$$\Rightarrow P(b) = 1 - 0.85$$
$$\therefore P(b) = 0.15$$

(b)
$$P(A) = P(c) + P(d)$$
$$= 0.48 + 0.02$$
$$= 0.50$$

(c)
$$P(A') = 1 - P(A)$$
$$= 1 - 0.50$$
$$= 0.50$$

(Ans)

problem : 1.2.2



consider the sample space S with outcomes a, b, c, d, e and f. If $P(A) = 0.27$

(a) $P(b) = ?$ (b) $P(A') = ?$ (c) $P(d) = ?$

Soln:

(a)

$$P(A) = P(b) + P(c) + P(e)$$

$$\Rightarrow 0.27 = P(b) + 0.11 + 0.06$$

$$\Rightarrow P(b) = 0.27 - 0.17$$

$$\therefore P(b) = 0.10$$

(b)

$$P(A') = 1 - P(A)$$

$$= 1 - 0.27$$

$$= 0.73$$

(c) $p(d) = ?$

$$p(A') = p(a) + p(d) + p(f)$$

$$\Rightarrow 0.73 = 0.09 + p(d) + 0.29$$

$$\Rightarrow p(d) = 0.73 - 0.38$$

$$\therefore p(d) = 0.35$$

(Ans)

Problem : 1.2.3

If Birthdays are equally likely to fall on any day what is the probability that a person choosen at random has a birthday on January? What about February?

Soln :

As given the birthdays are equally likely to fall on anyday and that include 29th of February o leap year, so the number of days is.

$$(3 \times 365) + 366$$

$$= 1461 \text{ days}$$

Now,

in 4 years the number of days in January is $4 \times 31 = 124$ days

and,
days in February is $(3\times 28) + 29 = 113$ days

So, the probability for birthday to fall on

January is $= \frac{124}{1461}$

and, February is $= \frac{113}{1461}$

$$\boxed{\text{Event} = \frac{\text{Possible outcomes}}{\text{Total outcomes}}}$$

Problem 1.2.4

When a company introduces initiatives to reduce its carbon footprint, it costs will either increase, stay the same or decrese. Suppose that the probability that the costs increases is 0.03 and the probability that the costs stay same is 0.18. What is the probability that the cost will decrese and will not increase ?

Sol^n:

$$S = \{ \text{increase, stay same, decrease} \}$$

$\Rightarrow p(S) = p(\text{increase}) + p(\text{stay same}) + p(\text{decrease})$

$\Rightarrow 1 = 0.03 + 0.18 + p(\text{decrease})$

$\Rightarrow p(\text{decrease}) = 1 - 0.21$

$\therefore p(\text{decrease}) = 0.79$

Now,

$p(\text{cost will not increase}) = p(\text{cost decrease}) + p(\text{cost stays the same})$

$\Rightarrow p(\text{will not increase}) = 0.79 + 0.18$

$\Rightarrow p(\text{will not increase}) = 0.97$

(Ans.)

Problem 1.2.5

An investor is monitorying stocks from company A and B which each either increasing or decreasing each day. On a given day, suppose that there is a probability of 0.38 that the both stock will increase in price, and probability of 0.11 the both stock will decrease in price. Also, there is a possibility of 0.16 that the stock from company A will decrease while the stock from company B will increase.

ⓐ What is the probability that the stock from company A will increase while the stock from company B will decrease?

Sl^n:

$S = \{$ both increase, both decrease, A decrease B increase, A increase B decrease $\}$

$\therefore P(S) = P(\text{both increase}) + P(\text{both decrease}) + P(\text{A decrease B increase}) + P(\text{A increase B decrease})$

$\Rightarrow 1 = 0.38 + 0.11 + 0.16 + P(\text{A increase B decrease})$

$\Rightarrow P(\text{A increase B decrease}) = 1 - 0.65$

$\therefore P(\text{A increase B decrease}) = 0.35$

(b) what is the probability that at least one company will incrase in the stock price ?

$\therefore$ $P(A) = P(\text{both increase}) + P(A \text{ increase } B \text{ decrease}) + P(A \text{ decrease } B \text{ increase})$

$\Rightarrow P(A) = 0.38 + 0.35 + 0.16$

$\Rightarrow P(A) = 0.89$ (Ans)

1.2.6

Two fair dice are thrown, one red and one blue. What is the probability that the red dice has a score strictly greater then the score of the blue dice ?

(a) why is probability less tham 0.5 ?

(b) what is the compliment of this event ?

[red, blue]

Soln :

$S = \{ (1,1), (1,2)\ (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

So, total possible outcome $6 \times 6 = 36$

and, total possible outcome where red is greater that blue $= 15$

So, the probability of red is greater than blue is $\frac{15}{36} = 0.41667$

Soln :

The probability is less than 0.5 or 50%

because there are situation where both blue and red dice are equal.

Sol^n:

The compliment of this event is

$$P(A) + P(A') = 1$$

$$\Rightarrow P(A') = 1 - P(A)$$

$$\Rightarrow P(A') = 1 - 0.41667$$

$$\therefore P(A') = 0.58333$$

1.2.7:

If a card is choosen at random from a pack of cards, what is the probability that the card is from one of the two black suits?

Sol^n:

$$S = \{ \underset{\text{out of } 52}{13 \text{ cards of clovers}}, \underset{\text{out of } 52}{13 \text{ cards of pikes}} \}$$

$$\Rightarrow P(S) = \frac{13}{52} + \frac{13}{52}$$

$$\Rightarrow P(S) = \frac{26}{52}$$

$$\therefore P(S) = 1/2$$

problem : 1.2.8

If a card is choosen at random from pack of cards, what is probability is it an ace ?

Sol^n:

Each ~~of~~ set of cards have one ace in them.

13 cards of Hearts have 1 ace.

13 cards of Diamonds have 1 ace.

13 card of clovers have 1 ace.

13 card of pikes have 1 ace.

So,

probability of the random card is Hearts ace is $\frac{1}{52}$

" " " " " " Diamond's " " $\frac{1}{52}$

" " " " " " Clover's " " $\frac{1}{52}$

" " " " " " pikes " " $\frac{1}{52}$

∴ So, the probability of random card to be any ace,

is, $\frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52}$

$$= \frac{4}{52}$$

$$= \frac{1}{13}$$

(Ans)

problem 1.2.9

A winner and a runner-up are decided in a Tournament of four players, one of whom is Terica. If all outcomes are equally likely, what is the ~~problem~~ probability that

(a) Terica is the winner ?

Soln:

let,
the players are A, B, C, T

So, the sample space of winner and runner up is

$$S = \{ (A,B), (A,C), (A,T), (B,A), (B,C), (B,T), (C,A), (C,B), (C,T), (T,A), (T,B), (T,C) \}$$

So, the probability of Terica is a winner is 3 out of 12 cases.

So, $p(s) = \frac{3}{12}$

$= \frac{1}{4}$

(b) Terica is either winner or runner up ?

Sol^n :

There are total 6 cases out of 12 that Terica is either winner or runner up.

So, $P(s) = \frac{6}{12}$

$= \frac{1}{2}$

Problem : 1.2.10

Three types of batteries are being tested, type 1, type 11 and tye III. The outcomes denotes that the batteries of type I fail first, the battery of type II next and the battery of type III lasts the longest. The probabilities of the six outcomes are given.

(I, II, III)	(I, III, II)
0.11	0.07
(II, I, III)	(II, III, I)
0.24	0.39
(III, I, II)	(III, II, I)
0.16	0.03

ⓐ the type I battery lasts lonest ?

$$P(I) = P(II, III, I) + P(III, II, I)$$

$$= 0.39 + 0.03$$

$$= 0.42$$

ⓑ the type I battery lasts shortest ?

Sol^n:

$$P(I) = P(I, II, III) + P(I, III, II)$$

$$= 0.11 + 0.07$$

$$= 0.18$$

(c) The type I battery does not lasts the longest ?

Sol^n:

$$P(I) = P(I, II, III) + P(I, III, II) + P(II, I, III) + P(III, I, II)$$

$$= 0.11 + 0.07 + 0.24 + 0.16$$

$$= 0.58$$

(d) The type I battery lasts longest than the type II battery ?

Sol^n:

$$P(I) = P(II, I, III) + P(II, III, I) + P(III, II, I)$$

$$= 0.24 + 0.39 + 0.03$$

$$= 0.66$$

(Ans)

Problem : 1.2.11

A factory has two assembly lines each of which is shut down (s) at partial capacity (p) or at full capacity (f). The sample space is given for (sp) denotes that the first assembly line is shoutdown and the second one is operating is partial capacity.

(S,S) 0.02	(S,P) 0.06	(S,F) 0.05
(P,S) 0.07	(P,P) 0.14	(P,F) 0.20
(F,S) 0.06	(F,P) 0.21	(F,F) 0.19

what is the probability that, (a) Both assembly lines are shut down ?

Sol^n:

$P(S) = P(\text{both assembly lines are shut down})$

$= 0.02$

$= 2\%$

(b) Neither of the assembly lines are shut down ?

Sol^n:

$P(S) = P(\text{neither of the assembly lines are shut down})$

$= 0.14 + 0.20 + 0.21 + 0.19$

$= 0.74$

(c) At least one assembly line is at full capacity ?

$P(S) = P(S,F) + P(P,F) + P(F,S) + P(F,F) + P(F,P)$

$= 0.05 + 0.20 + 0.06 + 0.19 + 0.21$

$= 0.50 + 0.21$

$= 0.71$

(d) Exactly one assembly line is at full capacity ?

$$P(S) = P(S,F) + P(P,F) + P(F,S) + P(F,P)$$

$$= 0.05 + 0.20 + 0.06 + 0.21$$

$$= 0.52$$

Problem : 1.2.12

A fair coin tossed three times, what is the probabity that two heads will be attained in successtion?

Soln :

let,

Head = H = 1

Tail = T = 0

$$P(S) = \{ (0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1) \}$$

So, the probability of two head will be obtain is $\frac{3}{8}$

(Ans