Higher Order Linear ODEs : Non-Homogeneous

- ***Linear differential equation of order TWO : constant coefficients*** (a_0, a_1, a_2)

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x) \Rightarrow a_0 D^2 y + a_1 Dy + a_2 y = R(x) \Rightarrow f(D)y = R(x)$$

$$Where\, f(D) = a_0 D^2 + a_1 D + a_2$$

- ***Linear differential equation of order Three : constant coefficients*** (a_0, a_1, a_2, a_3)

$$a_0 \frac{d^3y}{dx^3} + a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = R(x) \Rightarrow a_0 D^3 y + a_1 D^2 y + a_2 Dy + a_3 y = R(x)$$

$$\Rightarrow f(D)y = R(x)$$

$$Where\, f(D) = a_0 D^3 + a_1 D^2 + a_2 D + a_3$$

Example.

$$D^3y - D^2y = 3e^x \qquad [\text{Non}-\text{Homogeneous, third order}]$$

$$(D^2+1)y = \sin x \qquad [\text{Non}-\text{Homogeneous, second order}]$$

$$(D^3 - D)y = 4e^{-x} + 3e^{2x} \qquad [\text{Non}-\text{Homogeneous, third order}]$$

$$D^2y - 6Dy + 9y = e^x \qquad [\text{Non}-\text{Homogeneous, second order}]$$

Higher Order Linear ODEs : Non-Homogeneous

Example. Find the general solution of the following higher order homogeneous ODE

(a) $(\mathrm{D}^3 + 3D^2 - 4)y = 0$ (b) $(\mathrm{D}^4 + 2\mathrm{D}^2 + 1)\, y = 0$ (c) $\mathrm{D}^2(\mathrm{D} + 4)^2 y = 0$

Solution. **(a)** The auxiliary equation for the ODE is,

$$m^3 + 3m^2 - 4 = 0 \Rightarrow (m-1)(m+2)^2 = 0 \Rightarrow m = -2, -2, 1$$

Thus, the general solution is, $y = (c_1 + c_2 x)e^{-2x} + c_3 e^x$.

(b) The auxiliary equation for the ODE is,

$$m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2+1)^2 = 0 \Rightarrow m^2 + 1 = 0 \;\; or \;\; m^2 + 1 = 0 \Rightarrow m = \pm i, \pm i$$

Thus, the general solution is, $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$.

(c) The auxiliary equation for the ODE is,

$$m^2(m+4)^2 = 0 \Rightarrow m^2 = 0 \;\; or \;\; (m+4)^2 = 0 \Rightarrow m = 0, 0 \text{ or } m = -4, -4$$

Thus, the general solution is, $y = (c_1 + c_2 x) + (c_3 + c_4 x)e^{-4x}$.

Higher Order Linear ODEs : Non-Homogeneous

General solutions of a non-homogeneous Linear ODEs

The solution of the non-homogeneous linear differential equation $f(D)y = R(x)$ is of the form

$y = y_c + y_p$ where y_c: general solution of $f(D)y = 0$ and

y_p: particular solution of $f(D)y = R(x)$.

Here, y_c is called the complementary function for $f(D)y = R(x)$.

A number of methods are used to obtain particular integrals for non-homogeneous differential equations. Some of the standard methods are

1. **Variation of Parameters**
2. **Inverse Operator method**
3. The method of Undetermined Coefficients

Higher Order Linear ODEs : Non-Homogeneous

Inverse Differential Operators $\left(\frac{1}{D}\right)$

In differential calculus, the symbol D is often used to denote the differentiation $\frac{d}{dx}$, i.e.,

$$Dy = \frac{dy}{dx} = y', \qquad D^2 y = \frac{d^2y}{dx^2} = y'', \qquad D^3 y = \frac{d^3y}{dx^3} = y''', \dots \; \dots \dots, D^n y = \frac{d^n y}{dx^n} = y^n$$

Reversely, in integral calculus , the symbol $\frac{1}{D}$ is used to denote the integration $\int \quad dx$, i.e.,

$$\frac{1}{D} y = \int y\, dx, \qquad \frac{1}{D^2} y = \int\int y\, dx\, dx, \qquad \frac{1}{D^3} y = \int\int\int y\, dx\, dx\, dx, \dots$$

For example,

$$\frac{1}{D}(\sin 4x) = \int \sin 4x\, dx = -\frac{1}{4}\cos 4x$$

$$\frac{1}{D^2}(\sin 4x) = \int\int \sin 4x\, dx\, dx = \int\left(-\frac{1}{4}\cos 4x\right) dx = \left(-\frac{1}{4^2}\right)\sin 4x$$

$$\frac{1}{D}(e^{mx}) = \int e^{mx}\, dx = \frac{1}{m} e^{mx}$$

$$\frac{1}{D^2}(e^{mx}) = \int\int e^{mx}\, dx\, dx = \frac{1}{m^2} e^{mx}$$

$$\boxed{\begin{aligned} \frac{1}{D^2}(\sin mx) &= \left(-\frac{1}{m^2}\right)\sin mx \\ \frac{1}{D^2}(\cos mx) &= \left(-\frac{1}{m^2}\right)\cos mx \\ \frac{1}{D^n}(e^{mx}) &= \frac{1}{m^n} e^{mx} \end{aligned}}$$

Higher Order Linear ODEs : Non-Homogeneous

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

In seeking a particular solution of $f(D)y = R(x)$, we can write

$$y_p = \frac{1}{f(D)} R(x)$$

Type-1: $R(x) = e^{ax}$

If $R(x) = e^{ax}$ and $f(a) \neq 0$ then the particular solution of the ODE yields,

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

Type-2: $R(x) = e^{ax} \cdot V(x)$

i. If $V(x) = 1.0$ and $f(a) = 0$ then the particular solution of the ODE yields,

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^n \phi(D)} e^{ax} = \frac{x^n e^{ax}}{n!\,\phi(a)} \qquad [\phi(a) \neq 0]$$

ii. If $V(x) \neq 1.0$ then the particular solution of the ODE yields

$$y_p = \frac{1}{f(D)} e^{ax} \cdot V(x) = e^{ax} \frac{1}{f(D+a)} V(x)$$

Higher Order Linear ODEs : Non-Homogeneous

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

Example. Find the general solution of the following non-homogeneous ODE:

(a) $(D^2+1)y = e^{2x}$ (b) $D^2(D+4)^2 y = 96e^{-4x}$ (c) $(D^2-2D+1)y = xe^{x}$

Solution. (a) Here, the auxiliary equation is, $m^2+1=0 \Rightarrow m = \pm i$.

Therefore, the complementary solution yields, $y_c = c_1\cos x + c_2\sin x$.

For the particular solution, $y_p = \frac{1}{D^2+1}e^{2x} = \frac{1}{2^2+1}e^{2x} = \frac{1}{5}e^{2x}$.

The general solution becomes, $y = y_c + y_p = c_1\cos x + c_2\sin x + \frac{1}{5}e^{2x}$.

(b) Here, the auxiliary equation is, $m^2(m+4)^2 = 0 \Rightarrow m = 0,0,-4,-4$.

Therefore, the complementary solution yields, $y_c = c_1 + c_2x + (c_3+c_4x)e^{-4x}$.

For the particular solution,

$$y_p = \frac{1}{D^2(D+4)^2}[96e^{-4x}] = 96\cdot\frac{1}{D^2(D+4)^2}[e^{-4x}] = 96\cdot\frac{x^2e^{-4x}}{(-4)^2 2!} = 3x^2e^{-4x}$$

The general solution becomes, $y = y_c + y_p = c_1 + c_2x + (c_3+c_4x)e^{-4x} + 3x^2e^{-4x}$

Higher Order Linear ODEs : Non-Homogeneous

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

Example. Find the general solution of the following non-homogeneous ODE:

(a) $(D^2+1)y = e^{2x}$ (b) $D^2(D+4)^2 y = 96e^{-4x}$ (c) $(D^2-2D+1)y = xe^x$

Solution. (c) Here, the auxiliary equation is, $m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1.$

Therefore, the complementary solution yields, $y_c = (c_1+c_2x)e^x$.

For the particular solution, $y_p = \frac{1}{(D-1)^2}[xe^x] = e^x\frac{1}{(D+1-1)^2}x = e^x\frac{1}{D^2}(x) = \frac{1}{6}x^3e^x$.

The general solution becomes, $y = y_c + y_p = (c_1+c_2x)e^x + \frac{1}{6}x^3e^x$.

Higher Order Linear ODEs : Non-Homogeneous

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

Example. Find the general solution of the following differential equations:

(a) $y'' - 4y' + 4y = (12 + 9x)e^{-x}$ (b) $y'' + y' - 12y = 14e^{-4x}$

Solution. (a) Here, the auxiliary equation is, $m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$.

Therefore, the complementary solution yields, $y_c = (c_1 + c_2 x)e^{2x}$.

For the particular solution, $y_p = \frac{1}{(D-2)^2}(12 + 9x)e^{-x} = e^{-x}\frac{1}{(D-1-2)^2}(12 + 9x)$

$$= e^{-x}\frac{1}{(D-3)^2}(12 + 9x) = e^{-x}(D-3)^{-2}(12 + 9x)$$

$$= \frac{e^{-x}}{9}\left(1 - \frac{D}{3}\right)^{-2}(9x + 12) = \frac{e^{-x}}{9}\left(1 + \frac{2}{3}D\right)(9x + 12)$$

$$= \frac{e^{-x}}{9}\left(9x + 12 + \frac{2}{3}\cdot 9\right) = e^{-x}(x + 2)$$

The general solution becomes, $y = y_c + y_p = (c_1 + c_2 x)e^{2x} + (x + 2)e^{-x}$

Higher Order Linear ODEs : Non-Homogeneous

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

Example. Find the general solution of the following differential equations:

(a) $y'' - 4y' + 4y = (12 + 9x)e^{-x}$ (b) $y'' + y' - 12y = 14e^{-4x}$

Solution. (b) Here, the auxiliary equation is, $m^2 + m - 12 = 0 \Rightarrow (m + 4)(m - 3) = 0$

$$\Rightarrow m = -4, 3$$

Therefore, the complementary solution yields, $y_c = c_1 e^{-4x} + c_2 e^{3x}$.

For the particular solution,

$$y_p = \frac{1}{(D+4)(D-3)} 14e^{-4x} = 14 \frac{1}{(D+4)^1 (D-3)} [e^{-4x}] = 14 \frac{x^1 e^{-4x}}{1!\,(-4-3)} = -2xe^{-4x}$$

The general solution becomes, $y = y_c + y_p = c_1 e^{-4x} + c_2 e^{3x} - 2xe^{-4x}$

Higher Order Linear ODEs : Non-Homogeneous

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

Type-3: $R(x) = \sin(ax)$ or $R(x) = \cos(ax)$

If $R(x) = \sin(ax)$ or $R(x) = \cos(ax)$ and suppose $f(D) = F(D^2)$ then the particular solution of the ODE yields,

$$y_p = \frac{1}{f(D)}\sin ax = \frac{1}{F(D^2)}\sin ax = \frac{1}{F(-a^2)}\sin ax \qquad [F(-a^2) \neq 0]$$

$$y_p = \frac{1}{f(D)}\cos ax = \frac{1}{F(D^2)}\cos ax = \frac{1}{F(-a^2)}\cos ax \qquad [F(-a^2) \neq 0]$$

If $F(-a^2) = 0$ then the particular solution of the ODE yields,

$$y_p = \frac{1}{f(D)}\sin ax = \frac{1}{F(D^2)}\sin ax = x\frac{1}{F'(D^2)}\sin ax \qquad [F'(-a^2) \neq 0]$$

$$y_p = \frac{1}{f(D)}\cos ax = \frac{1}{F(D^2)}\cos ax = x\frac{1}{F'(D^2)}\cos ax \qquad [F'(-a^2) \neq 0]$$

For the special case of $F\left(D^2\right) = D^2 + a^2$ and if $F(-a^2) = 0$, then

$$y_p = \frac{1}{D^2 + a^2}\sin ax = -\frac{x}{2a}\cos ax \quad and \quad y_p = \frac{1}{D^2 + a^2}\cos ax = \frac{x}{2a}\sin ax$$

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

Example. Find the general solution of the following non-homogeneous ODE:

(a) $(D^2+1)y = 3\sin 2x$ (b) $(D^2+25)y = \sin 5x$

(c) $(D^2-2D+5)y = e^x \cos 2x$

Solution. (a) Here, the auxiliary equation is, $m^2+1=0 \Rightarrow m = \pm\, i$.

Therefore, the complementary solution yields, $y_c = c_1\cos x + c_2 \sin x$.

For the particular solution, $y_p = \frac{1}{D^2+1}[3\sin 2x] = \frac{3}{-2^2+1}\sin 2x = -\sin 2x$.

The general solution becomes, $y = y_c + y_p = c_1\cos x + c_2\sin x - \sin 2x$.

(b) Here, the auxiliary equation is, $m^2+25=0 \Rightarrow m = \pm 5i$.

Therefore, the complementary solution yields, $y_c = c_1\cos 5x + c_2\sin 5x$.

For the particular solution, $y_p = \frac{1}{D^2+25}[\sin 5x] = -\frac{x}{2\cdot 5}\cos 5x = -\frac{x}{10}\cos 5x$.

The general solution becomes, $y = y_c + y_p = c_1\cos 5x + c_2\sin 5x - \frac{x}{10}\cos 5x$.

Higher Order Linear ODEs : Non-Homogeneous

Inverse Operator Method $\left(\frac{1}{f(D)}\right)$

Example. Find the general solution of the following non-homogeneous ODE:

(a) $(D^2+1)y = 3\sin 2x$ (b) $(D^2+25)y = \sin 5x$

(c) $(D^2-2D+5)y = e^x \cos 2x$

Solution. (c) Here, the auxiliary equation is,

$$m^2 - 2m + 5 = 0 \Rightarrow (m-1)^2 + 4 = 0 \Rightarrow m = 1 \pm 2i$$

Therefore, the complementary solution yields, $y_c = (c_1 \cos 2x + c_2 \sin 2x)e^x$.

For the particular solution, $y_p = \frac{1}{D^2-2D+5}[e^x \cos 2x] = \frac{1}{(D-1)^2+4}[e^x \cos 2x]$

$$= e^x \frac{1}{(D+1-1)^2+4} \cos 2x = e^x \frac{1}{D^2+4} \cos 2x$$

$$= e^x \cdot \frac{x}{2 \cdot 2} \sin 2x = \frac{1}{4} x e^x \sin 2x$$

The general solution becomes, $y = y_c + y_p = (c_1 \cos 2x + c_2 \sin 2x)e^x + \frac{1}{4} x e^x \sin 2x$.

Higher Order Linear ODEs : Non-Homogeneous

Exercise Problems:

1. Solve the following differential equations and justify your answers:

(a) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = x^2 + x$, (b) $\dfrac{d^2y}{dx^2} + 4y = x^3$,

(c) $(D^2 - 2D - 3)y = 5$, given that $y = -1, y' = 1$ when $x = 0$.

2. Solve the following differential equations and justify your answers:

(a) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^{-2x}$, (b) $\dfrac{d^2y}{dx^2} - 9y = 6e^{-3x}$,

(c) $(\mathrm{D}^2 - 4\mathrm{D} + 4)y = e^{2x}$ given that $y(0) = 1$ and $y'(0) = 5$,

(d) $(D^2 + 4D + 4)y = e^{-2x}\cos(2x + 1)$, (e) $(D^2 - 6D + 9)y = (x^2 + \sin 2x)e^{3x}$.

3. Solve the following differential equations and justify your answers:

(a) $\dfrac{d^2y}{dx^2} - 9y = \cos 3x$, (b) $\dfrac{d^2y}{dx^2} + 4y = 5\sin(3x + 2)$,

(c) $(D^2 + 25)y = \cos 5x$, (d) $(D^2 + 9)y = \cos x - \sin 3x$,

(e) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = \sin 3x$, (f) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \cos 2x$,

(g) $(D^2 - 1)(D^2 - 9)y = \cos 2x$, (h) $(D^4 - 1)y = \sin x$,

(i) $4y'' - 4y' + 5y = 17\cos x$, given that $y = 2$, $y' = -7\frac{1}{2}$ when $x = 0$.