The Graph of Rational Function

lec-13

Problem: 1

Analyze the graph of a rational function

$$R(x) = \frac{x-1}{x^2-4}$$

Solution: Given $R(x) = \frac{x-1}{x^2-4}$

Step: 1 Factor numerator and denominator. Find domain of the rational function

$$R(x) = \frac{x-1}{x^2-4} = \frac{(x-1)}{(x+2)(x-2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$

Step: 2 Write R in lowest terms.

there are no common factors betn numerator and denominator, R is in lowest terms.

Step: 3 Find intercepts. Determine the behavior of R near each x-intercepts.

y-intercept, let $x=0$, $\therefore y = R(0) = \frac{1}{4}$.

Thus y intercept is $1/4$

x-intercept, let $R(x) = 0$

$$\Rightarrow \frac{x-1}{x^2-4} = 0 \Rightarrow x-1=0 \Rightarrow x=1$$

Thus x-intercept is 1.

Near 1: $R(x) = \frac{x-1}{(x+2)(x-2)} \approx \frac{x-1}{(1+2)(1-2)} = -\frac{1}{3}(x-1)$

linear function with negative slope.

Step: 4 Find vertical asymptotes.

To find vertical asymptotes let $q(x)=0$ if $R(x)=\frac{p(x)}{q(x)}$

$$\Rightarrow x^2-4=0$$
$$\Rightarrow x=-2 \text{ and } x=+2.$$

Step: 5 Find horizontal or oblique asymptotes. Find if R intersects the asymptotes.

We have $R(x)=\frac{x-1}{x^2-4}$

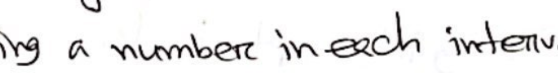
Since the degree of the numerator is less than the degree of the denominator, the line $y=0$ is the horizontal asymptotes.

To determine, if the graph intersects the horizontal asymptotes solve the eqn $R(x)=0$

$$\frac{x-1}{x^2-4}=0 \Rightarrow x-1=0 \Rightarrow x=1$$

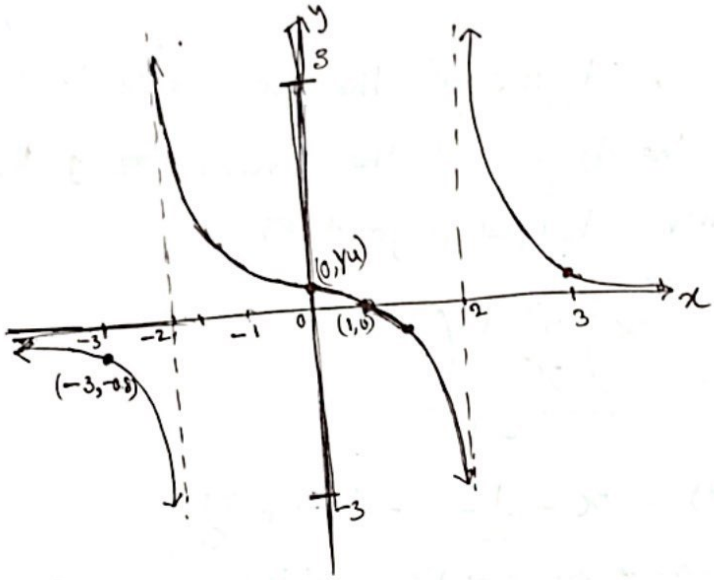
The only solution is $x=1$, so the graph of R intersects the horizontal asymptote at $(1,0)$.

Step: 6 Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluate R there.



Interval	$(-\infty,-2)$	$(-2,1)$	$(1,2)$	$(2,\infty)$
Number chosen	-3	0	$\frac{3}{2}$	3
$R(x)$	-0.8	$\frac{1}{4}$	$-\frac{2}{7}$	0.4
location on graph	Below x-axis.	above x-axis	below x-axis	above x-axis

$\boxed{\text{Step-7:}}$ Analyze the behavior of the graph R near each asymptote and indicate the behavior of the graph.



Problem: 2

Analyze the graph of rational function $R(x) = \frac{x^2-1}{x}$

Soln:

$\boxed{\text{Step 1:}}$ $R(x) = \frac{(x+1)(x-1)}{x}$. The domain is $\{x \mid x \neq 0\}$

$\boxed{\text{Step 2:}}$ R is in lowest terms.

$\boxed{\text{Step 3:}}$ x cannot equal to zero, so there is no y-intercept.

x-intercept, let $R(x) = 0$

$\therefore \frac{x^2-1}{x} = 0 \Rightarrow x^2-1=0 \Rightarrow x=-1$ and $x=1$

$\therefore$ x-Intercept is -1 and 1.

Near -1: $R(x) = \frac{(x+1)(x-1)}{x} \approx \frac{(x+1)(-1-1)}{-1} = 2(x+1)$

Near 1: $R(x) = \frac{(x+1)(x-1)}{x} \approx \frac{(1+1)(x-1)}{1} = 2(x+1)$

linear function with the slope.

Step 4 To find vertical asymptote let $q(x)=0$

$$\Rightarrow x=0$$

The line $x=0$ is the vertical asymptote.

Step 5 Since the degree of the numerator is 2, which is one greater than the degree of the denominator, 1, the rational function will have oblique asymptotes.

$$x\,\overline{)\,x^2-1\,}\,(\,x$$
$$\underline{-x^2}$$
$$-1$$

$\therefore$ $R(x) = x - \dfrac{1}{x} = f(x) + \dfrac{r(x)}{q(x)}$

As $x \to -\infty$ or $x \to +\infty$, $\dfrac{r(x)}{q(x)} = \dfrac{1}{x} \to 0$.

$\therefore$ $R(x) \to f(x) = x$

Thus $y=x$ line is the oblique asymptote.

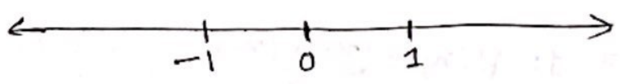
To determine whether the graph of R intersects the asymptote $y=x$, we solve the eqn $R(x)=x$

$$\therefore \quad \frac{x^2-1}{x} = x$$

$$\Rightarrow x^2-1 = x^2 \Rightarrow -1=0 \quad \text{(impossible)}$$

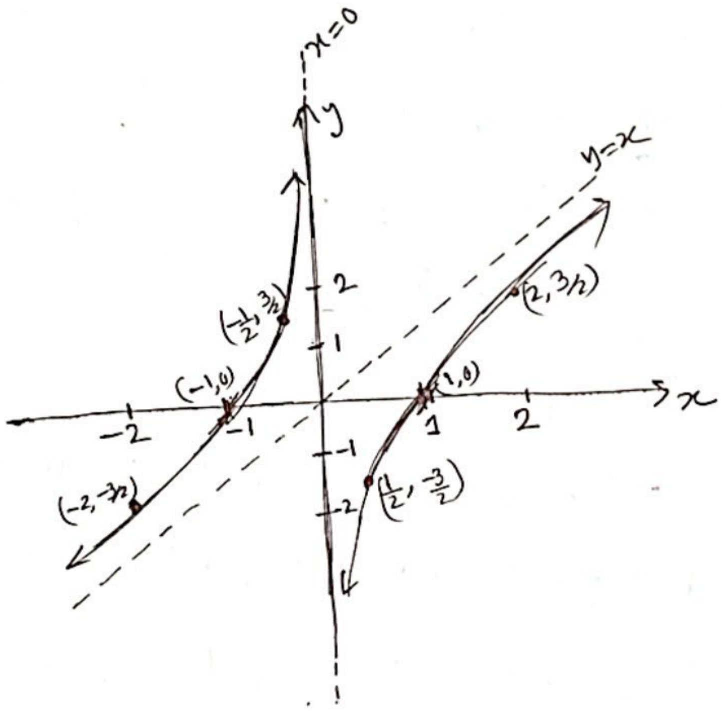
So $R(x)=x$ doesnot have any solution, so the graph of R doesn't intersects the line $y=x$.

Step 6: Divide the x-axis into four intervals.



$(-\infty,-1)$ $(-1,0)$ $(0,1)$ $(1,\infty)$

interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
number chosen	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
$R(x)$	$R(-2)=-\frac{3}{2}$	$R(-\frac{1}{2})=\frac{3}{2}$	$R(\frac{1}{2})=-\frac{3}{2}$	$R(2)=\frac{3}{2}$
location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis



Problem: 3

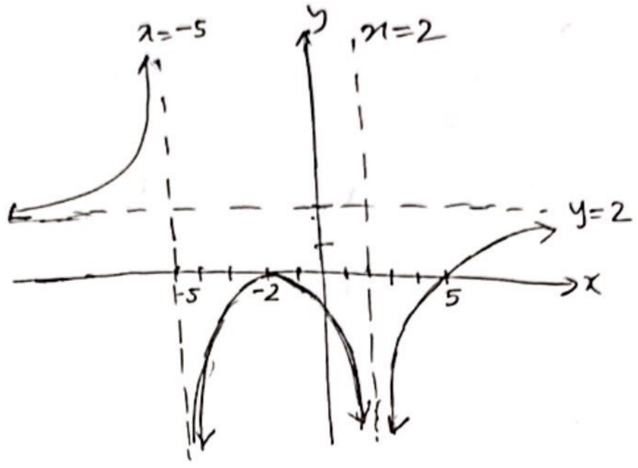
Analyze the graph of $R(x) = \frac{x^4+1}{x^2}$

Problem: 4

Analyze the graph of $R(x) = \frac{2x^2-5x+2}{x^2-4}$

Problem: 6 Find a rational function that might have the following graph.

Solution:



$$R(x) = \frac{P(x)}{q(x)}$$

$P(x)$ in lowest terms determines the x-intercepts of the graph

$q(x)$ in lowest terms determines the vertical asymptotes.

x-intercept -2, even multiplicity, graph touches the x-axis.

5, odd " graph crosses "

$$\therefore P(x) = (x+2)^2(x-5)$$

vertical asymptotes:

$x = -5$, so $(x+5)$ is a factor of odd multiplicity since

$\lim_{x\to -5^-} R(x) = +\infty$ and $\lim_{x\to -5^+} R(x) = -\infty$.

$x = 2$, so $(x-2)$ is a factor of even multiplicity since

$\lim_{x\to 2^-} R(x) = -\infty$ and $\lim_{x\to 2^+} R(x) = -\infty$

$$\therefore q(x) = (x+5)(x-2)^2$$

$$\therefore R(x) = \frac{(x+2)^2(x-5)}{(x+5)(x-2)^2}$$

Horizontal asymptote: $y=2$ is the Horizontal asymptote.

So we know the degree of the numerator must equal the degree of denominator and the quotient of leading coefficients must be $\frac{2}{1}$. This leads to

$$R(x) = \frac{2(x+2)^2(x-5)}{(x+5)(x-2)^2}$$