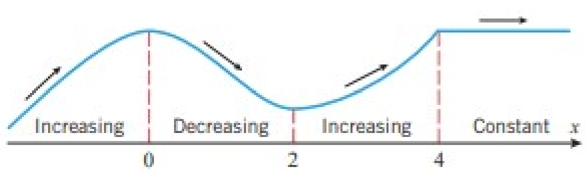
Chapter # 04
(The Derivative in Graphing and Applications)

In this chapter we will study various applications of the derivative. For example, we will use methods of calculus to analyze functions and their graphs. In the process, we will show how calculus and graphing utilities, working together, can provide most of the important information about the behavior of functions. Another important application of the derivative will be in the solution of optimization problems. Mathematically, optimization problems can be reduced to finding the largest or smallest value of a function on some interval, and determining where the largest or smallest value occurs. Using the derivative, we will develop the mathematical tools necessary for solving such problems. We will also use the derivative to study the motion of a particle moving along a line, and we will show how the derivative can help us to approximate solutions of equations

4.1 Analysis of Functions (Increase, Decrease and Concavity): Although graphing utilities are useful for determining the general shape of a graph, many problems require more precision than graphing utilities are capable of producing. The purpose of this section is to develop mathematical tools that can be used to determine the exact shape of a graph and the precise locations of its key features.

Increasing and Decreasing Functions: The terms increasing, decreasing, and constant are used to describe the behavior of a function as we travel left to right along its graph. For example, the function graphed in the following figure can be described as increasing to the left of ***x*** = ***0***, decreasing from ***x*** = ***0*** to ***x*** = ***2***, increasing from ***x*** = ***2*** to ***x*** = ***4***, and constant to the right of ***x*** = ***4***.

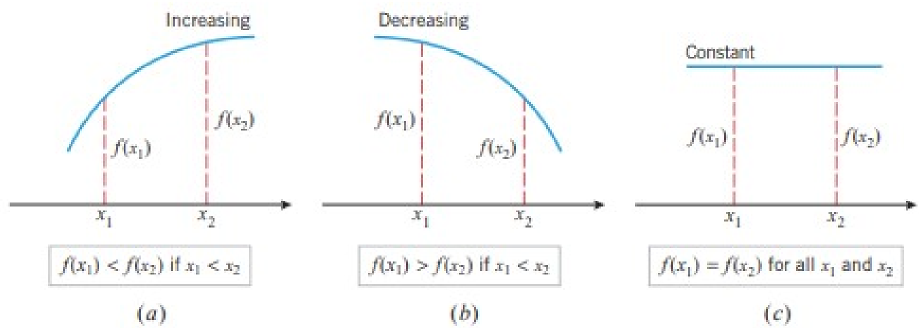


Definition: Let ***f*** be defined on an interval, and let x_1 and x_2 denote points in that interval.

(***a***) ***f*** is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(***b***) ***f*** is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(***c***) ***f*** is constant on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



Theorem: Let $\boldsymbol{f}$ be a function that is continuous on a closed interval $[\boldsymbol{a}, \boldsymbol{b}]$ and differentiable on the open interval $(\boldsymbol{a}, \boldsymbol{b})$.

(*a*) If $\boldsymbol{f}'(\boldsymbol{x}) > 0$ for every value of $\boldsymbol{x}$ in $(\boldsymbol{a}, \boldsymbol{b})$, then $\boldsymbol{f}$ is increasing on $[\boldsymbol{a}, \boldsymbol{b}]$.

(*b*) If $\boldsymbol{f}'(\boldsymbol{x}) < \boldsymbol{0}$ for every value of $\boldsymbol{x}$ in $(\boldsymbol{a}, \boldsymbol{b})$, then $\boldsymbol{f}$ is decreasing on $[\boldsymbol{a}, \boldsymbol{b}]$.

(*c*) If $\boldsymbol{f}'(\boldsymbol{x}) = \boldsymbol{0}$ for every value of $\boldsymbol{x}$ in $(\boldsymbol{a}, \boldsymbol{b})$, then f is constant on $[\boldsymbol{a}, \boldsymbol{b}]$.

Example 1: Find the intervals on which $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{2}} - \boldsymbol{4x} + \boldsymbol{3}$ is increasing and the intervals on which it is decreasing.

Solution: Given, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{2}} - \boldsymbol{4x} + \boldsymbol{3}$

$$f'(x) = 2x - 4 = 2(x - 2)$$

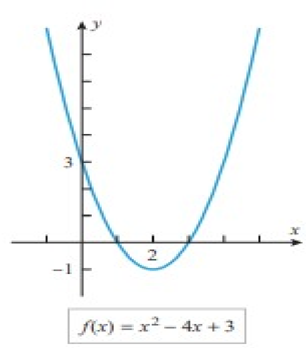
It follows that,

$$f'(x) < 0 \quad if \;\; x < 2$$

$$f'(x) > 0 \quad if \;\; x > 2$$

Since $\boldsymbol{f}$ is continuous everywhere, it follows that

f is decreasing on $(-\infty, 2]$
f is increasing on $[2, +\infty)$



Example 2: Find the intervals on which $\boldsymbol{f(x) = x^3}$ is increasing and the intervals on which it is decreasing.

Solution: Given, $\boldsymbol{f(x) = x^3}$

$$f'(x) = 3x^2$$

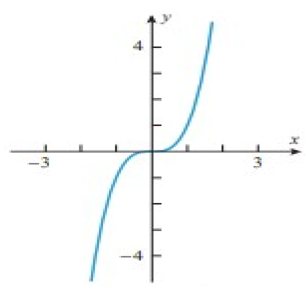
It follows that,

$$f'(x) > 0 \quad if \;\; x < 0$$

$$f'(x) > 0 \quad if \;\; x > 0$$

Since $\boldsymbol{f}$ is continuous everywhere, it follows that

f is increasing on $(-\infty, 0]$
f is increasing on $[0, +\infty)$



$f(x) = x^3$

Example 3: Find the intervals on which $\boldsymbol{f(x) = 3x^4 + 4x^3 - 12x^2 + 2}$ is increasing and the intervals on which it is decreasing.

Solution: Given,

$$\boldsymbol{f(x) = 3x^4 + 4x^3 - 12x^2 + 2}$$

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x-1)(x+2)$$

For critical values, $f'(x) = 0 \Rightarrow 12x(x-1)(x+2) = 0 \qquad \therefore x = 0,\; x = -2 \;\; and \;\; x = 1.$

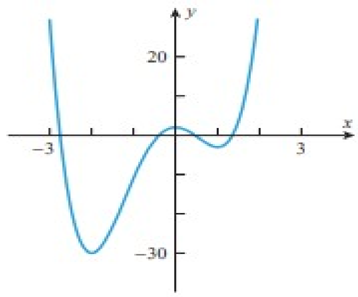
Using the critical values, the domain divides $(-\infty,\; 2),\; (-2, 0),\; (0, 1) \;\; and \;\; (1, \infty)$.

For $(-\infty,\; 2)$: $f'(x) < 0 \;\; \Rightarrow f(x) \; is \; decreasing \; on \;\; (-\infty, -2]$

For $(-2,\; 0)$: $f'(x) > 0 \;\; \Rightarrow f(x) \; is \; increasing \; on \;\; [-2, 0]$

For $(0,\; 1)$: $f'(x) < 0 \;\; \Rightarrow f(x) \; is \; decreasing \; on \;\; [0, 1]$

For $(1,\; \infty)$: $f'(x) > 0 \;\; \Rightarrow f(x) \; is \; increasing \; on \;\; [1, \infty)$



$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

Concavity: Characterize the concavity of a differentiable function ***f*** on an open interval

- ***f*** is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.
- ***f*** is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.

Definition: If ***f*** is differentiable on an open interval, then ***f*** is said to be concave up on the open interval if $\boldsymbol{f'}$ is increasing on that interval, and ***f*** is said to be concave down on the open interval if $\boldsymbol{f'}$ is decreasing on that interval.

Theorem: Let ***f*** be twice differentiable on an open interval.

(***a***) If $\boldsymbol{f''(x) > 0}$ for every value of ***x*** in the open interval, then ***f*** is concave up on that interval

(***b***) If $\boldsymbol{f''(x) < 0}$ for every value of ***x*** in the open interval, then ***f*** is concave down on that interval

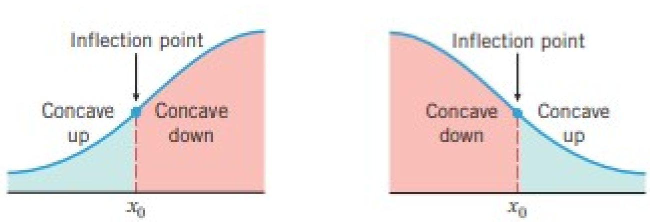
Example 4: Find the intervals on which $\boldsymbol{f(x) = x^2 - 4x + 3}$ is concave up and the intervals on which it is concave down.

Solution: Given,

$$\boldsymbol{f(x) = x^2 - 4x + 3 \quad \therefore \quad f'(x) = 2x - 4 \quad and \quad f''(x) = 2}$$

Since $\boldsymbol{f''(x) > 0}$ for all ***x***, so the given function $\boldsymbol{f(x)}$ is concave up on $(-\infty, \infty)$.

Inflection Points: If ***f*** is continuous on an open interval containing a value $\boldsymbol{x_0}$, and if ***f*** changes the direction of its concavity at the point $(\boldsymbol{x_0}, \boldsymbol{f(x_0)})$, then we say that ***f*** has an ***inflection point at*** $\boldsymbol{x_0}$, and we call the point $(\boldsymbol{x_0}, \boldsymbol{f(x_0)})$ on the graph of ***f*** an ***inflection point*** of ***f***.



Example 5: Given the function, $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points.

Solution: Calculating the first two derivatives of f we obtain

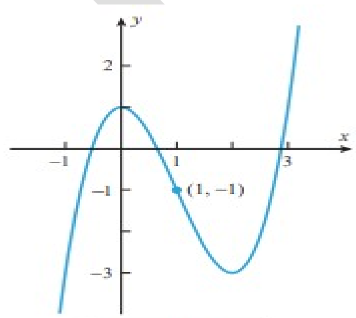
$$f'(x) = 3x^2 - 6x = 3x(x-2)$$
$$f''(x) = 6x - 6 = 6(x-1)$$

The sign analysis of these derivatives is shown in the following tables:

INTERVAL	$(3x)(x-2)$	$f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

INTERVAL	$6(x-1)$	$f''(x)$	CONCLUSION
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$

The second table shows that there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. The inflection point is $(1, f(1)) = (1, -1)$.



$f(x) = x^3 - 3x^2 + 1$

Home Work: Exercise 4.1: Problem No. 7-10, 15-20, 25-31 and 33-36