# MAT 350
## Engineering mathematics

- Second order ODE with const. Coefficients.

**Lecture: 4**

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## Higher Order linear differential equations:

For a linear differential equation an ***nth-order initial-value problem is***

***Solve***

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The initial-value problem

$$3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

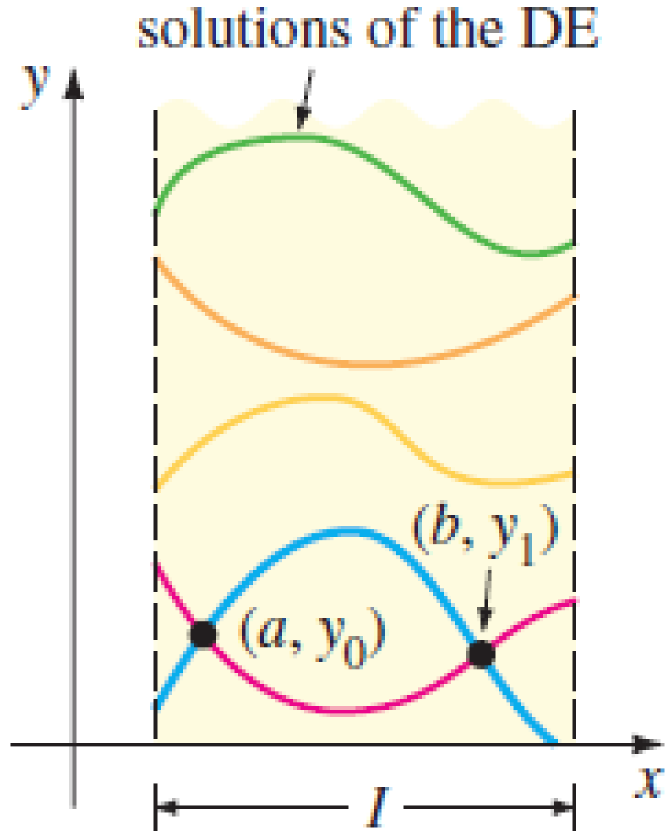
**THEOREM 4.1.1 Existence of a Unique Solution**

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval $I$ and let $a_n(x) \neq 0$ for every $x$ in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

## Boundary Value Problem (BVP)

*Solve:* $$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2)$$

*Subject to:* $y(a) = y_0, \quad y(b) = y_1$



**Homogeneous ODE of n-th order**:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (3)$$

Recall: Nonhomogeneous ODE of n-th order:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

$2y'' + 3y' - 5y = 0$ is a homogeneous

$2y'' + 3y' - 5y =$ 6/ $x^2$ / $e^x$ Nonhomogeneous

## Differential operator

The symbol $D$ (for Differentiation) is called **differential operator**.

example, $D(\cos 4x) = -4 \sin 4x$

$$D(5x^3 - 6x^2) = 15x^2 - 12x.$$

For higher order-

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y$$

and, in general, $\dfrac{d^ny}{dx^n} = D^ny,$

## THEOREM 4.1.2 Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous $n$th-order differential equation on an interval $I$. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x),$$

where the $c_i$, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

## DEFINITION 4.1.1 Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval $I$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every $x$ in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

Let us, consider (1) up to its second order derivatives,

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Considering $a_2(x) \neq 0$,and divide both sides with $a_2(x)$

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = r(x) \qquad (4)$$

where $a(x)=a_1(x)/a_2(x)$, $b(x)=a_0(x)/a_2(x)$, and $r(x)=g(x)/a_2(x)$.

A second-order ODE is called **linear if it can be written** as (4), and **nonlinear if it cannot be written in this form.**

If $r(x)=0$, it is Homogeneous, if $r(x) \neq 0$, it is Nonhomogeneous.

Consider second-order homogeneous linear ODEs whose coefficients *a and b are constant,*

$$y'' + ay' + by = 0. \tag{5}$$

A trial solution of (5) can be considered of the form

$$y = e^{\lambda x}. \tag{6}$$

Then,

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

Substituting into (5) gives,

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Since the exponential is never zero for any real (and complex) λ,

$$\lambda^2 + a\lambda + b = 0$$

**Characteristic Equation.**

$$\lambda^2 + a\lambda + b = 0 \tag{7}$$

Roots of the Chr. Eqn. (Auxiliary Equation) are:

$$\lambda_1 = \tfrac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \tfrac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

Equation (7) has the roots of three different types.

**(Case I)** *Two real roots if* $a^2 - 4b > 0$,

**(Case II)** *A real double root if* $a^2 - 4b = 0$,

**(Case III)** *Complex conjugate roots if* $a^2 - 4b < 0$.

## Case I. Two Distinct Real-Roots $\lambda_1$ and $\lambda_2$

The general solution is:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

## Case II. Real Double Root $\lambda = -a/2$

The general solution is:

$$y = (c_1 + c_2 x) e^{-ax/2}.$$

## Case III. Complex Roots $-\frac{1}{2}a + i\omega$ and $-\frac{1}{2}a - i\omega$

The general solution is:

$$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$$

$(A, B$ arbitrary).

Note:

$$y_1 = e^{\lambda_1 x}$$

and

$$y_2 = e^{\lambda_2 x}$$

called Basic solution of (7). Their linear Combination is called General Solution

Example: Two distinct real roots:

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution:

***General solution.*** The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0.$$

Roots are:

$$\lambda_1 = \tfrac{1}{2}(-1 + \sqrt{9}) = 1$$
$$\lambda_2 = \tfrac{1}{2}(-1 - \sqrt{9}) = -2$$

The general solution is: $y = c_1 e^{x} + c_2 e^{-2x}$.

***Particular solution.*** Since $y'(x) = c_1 e^{x} - 2c_2 e^{-2x}$,

$$y(0) = c_1 + c_2 = 4,$$

$$y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 1$ and $c_2 = 3$. This gives the *answer* $y = e^{x} + 3e^{-2x}$

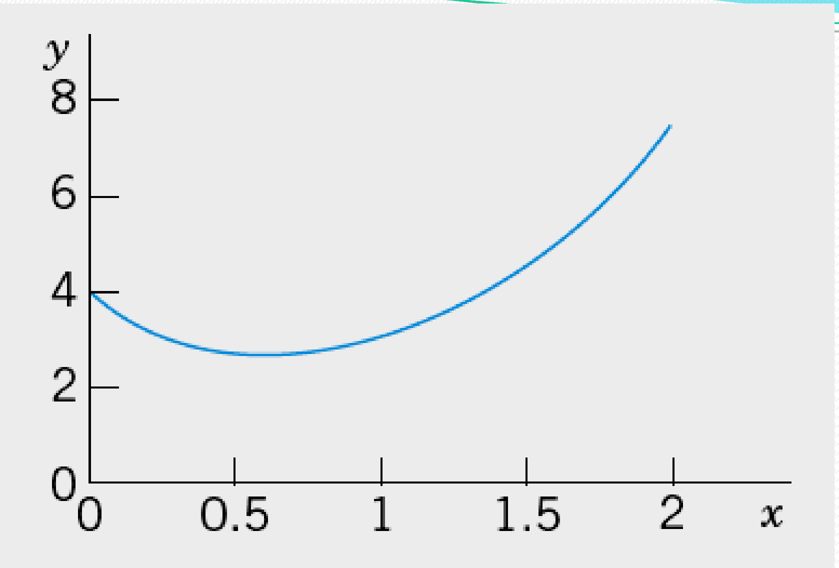



Figure shows that the curve begins at 4 with a negative slope ( -5 but note that the axes have different scales!), in agreement with the initial conditions.

Example: Two same real roots

$$\boxed{y'' + y' + 0.25y = 0,} \quad y(0) = 3.0, \quad y'(0) = -3.5.$$

***Solution.*** The characteristic equation is

$$\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0.$$

It has the double root $\lambda = -0.5$.

This gives the general solution

$$y = (c_1 + c_2 x)e^{-0.5x}.$$

To apply the initial conditions, we need to evaluate

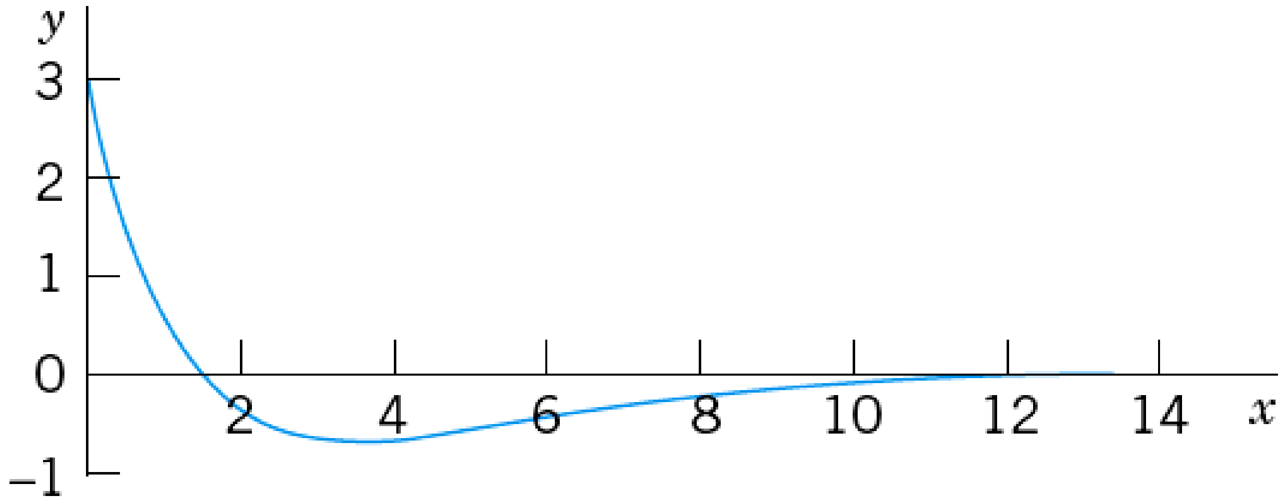
$$y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x}.$$

From this and the initial conditions we obtain

$$y(0) = c_1 = 3.0,$$

$$y'(0) = c_2 - 0.5c_1 = 3.5; \qquad \text{hence} \qquad c_2 = -2.$$

The particular solution of the initial value problem is $y = (3 - 2x)e^{-0.5x}$




Graph starts at 3 of y-axis and the slope there is -3, Example: Two complex roots:

Solve $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

Solution:

Chr. Equation is:

$$4m^2 + 4m + 17 = 0$$

Solving we have two complex roots:

$$m_1 = -\tfrac{1}{2} + 2i \text{ and } m_2 = -\tfrac{1}{2} - 2i.$$

Hence, Gen. Sol. is: $y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)$.

Applying the condition $y(0) = -1$,

$$e^0(c_1 \cos 0 + c_2 \sin 0) = -1 \qquad c_1 = -1.$$

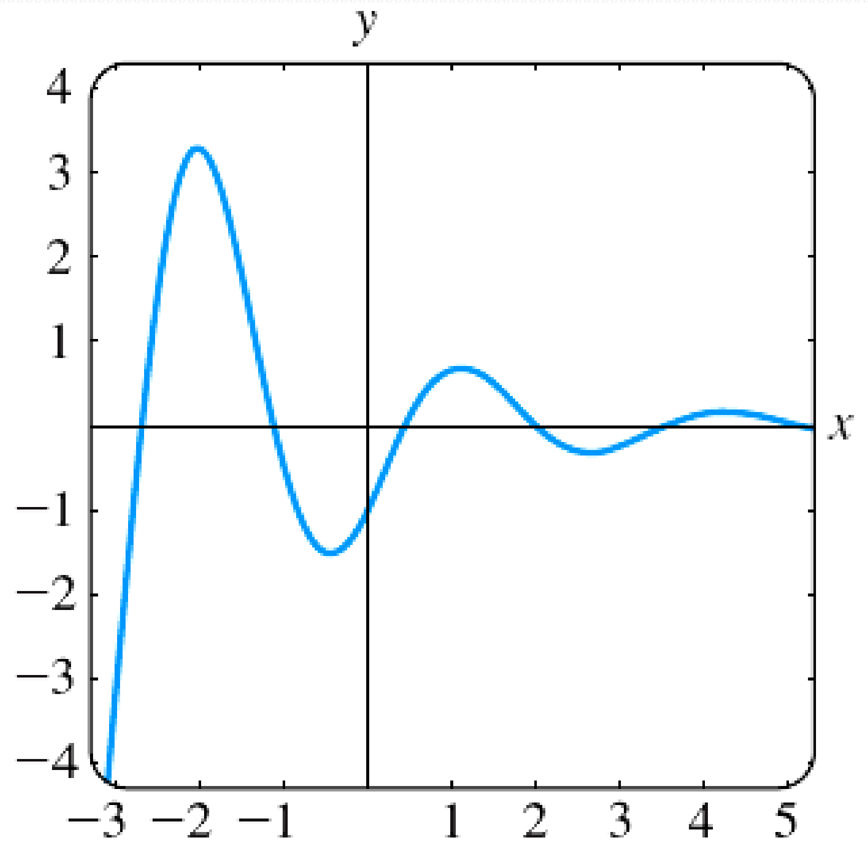
$$y = e^{-x/2}(-\cos 2x + c_2 \sin 2x)$$

and then using $y'(0) = 2$ gives $2c_2 + \frac{1}{2} = 2$ or $c_2 = \frac{3}{4}$.

Example: Two complex roots:

Hence the solution of the IVP is

$$y = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right).$$



# Summary of Cases I–III

| Case | Roots of (2) | Basis of (1) | General Solution of (1) |
|---|---|---|---|
| I | Distinct real $\lambda_1, \lambda_2$ | $e^{\lambda_1 x}, e^{\lambda_2 x}$ | $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ |
| II | Real double root $\lambda = -\frac{1}{2}a$ | $e^{-ax/2}, xe^{-ax/2}$ | $y = (c_1 + c_2 x)e^{-ax/2}$ |
| III | Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$ | $e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$ | $y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$ |

Solve the following differential equations.

**(a)** $2y'' - 5y' - 3y = 0$

**(b)** $y'' - 10y' + 25y = 0$

**(c)** $y'' + 4y' + 7y = 0$

Solve $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

## Exercise 4.3 (Zill 10th ed.)

Find the general solution of the given second-order differential equation.

**5.** $y'' + 8y' + 16y = 0$

**6.** $y'' - 10y' + 25y = 0$

**7.** $12y'' - 5y' - 2y = 0$

**14.** $2y'' - 3y' + 4y = 0$

Solve the given initial-value problem.

**30.** $\dfrac{d^2y}{d\theta^2} + y = 0, \quad y(\pi/3) = 0, y'(\pi/3) = 2$

**31.** $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$

**32.** $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

**34.** $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$

Solve the given boundary-value problem

**37.** $y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0$