The real zeros of polynomials

Division algorithm for Polynomials:

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

$$\Rightarrow \underset{\text{dividend}}{\underset{\uparrow}{f(x)}} = \underset{\text{quotient}}{\underset{\uparrow}{q(x)}}\, \underset{\text{divisor}}{\underset{\uparrow}{g(x)}} + \underset{\text{remainder}}{\underset{\uparrow}{r(x)}}$$

Remainder Theorem:

Let f be a polynomial function. If $f(x)$ is divided by $x-c$ then the remainder is $f(c)$.

Example:

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x-3$ (b) $x+2$

Solution:

a) Using remainder theorem, the remainder will be

$$f(3) = 3^3 - 4.3^2 - 5 = -14.$$

b) Using remainder theorem, the remainder will be

$$f(-2) = (-2)^3 - 4(-2)^2 - 5 = -29$$

Factor theorem:

Let f be a polynomial function. Then $x-c$ is a factor of $f(x)$ if and only if $f(c)=0$

Example:

Use factor theorem to determine whether the function $f(x)=2x^3-x^2+2x-3$ has the factor

(a) $x-1$ (b) $x+3$.

Soln: a. Since $x-1$ is of the form $x-c$ with $c=1$, so we have to find $f(1)$ and if $f(1)=0$ then $x-1$ will be a factor of $f(x)$.

Given $f(x)=2x^3-x^2+2x-3$

$\therefore f(1)=2\cdot 1^3-1^2+2\cdot 1-3 = 2-1+2-3 = 4-4 = 0$

$\therefore (x-1)$ is a factor of $f(x)$.

Rational zeros theorem:

Let f be a polynomial of degree 1 or higher of the form

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0;\quad a_n\neq 0,\ a_0\neq 0$$

Where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

Example:

List the potential rational zeros of $f(x) = 2x^3 + 11x^2 - 7x - 6$

Sol^n: Given. $f(x) = 2x^3 + 11x^2 - 7x - 6$

Here $a_0 = -6$, $a_n = 2$

$\therefore$ P = factor of $a_0 = \pm 1, \pm 2, \pm 3, \pm 6$

q = factor of $a_n = \pm 1, \pm 2$

Now form all possible ratios $\frac{p}{q}$.

$$\frac{p}{q} : \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$$

If f has a rational zero, it will be found in this list which contains 12 possibilities.

Note: If a function has a rational zero, it is one of those listed. It may be the case that the function doesnot have any rational zeros.

Number of real zeros:

A polynomial function cannot have more real zeros than its degree.

Example: Find the real zeros of $f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$. write f in factored form.

Solution: $f(x)$ has degree 5, so there are atmost five real zeros.

Here $a_5 = 1$ and $a_0 = -16$.

$\therefore$ Potential rational zeros $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$.

Use factor theorem to test if $x-1$ is a factor of $f(x)$.

$$f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$$

$$\therefore f(1) = 1 - 5 + 12 - 24 + 32 - 16 = 45 - 45 = 0$$

$\therefore$ $(x-1)$ is a factor of $f(x)$.

$$\therefore f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$$

$$= x^5 - x^4 - 4x^4 + 4x^3 + 8x^3 - 8x^2 - 16x^2 + 16x + 16x - 16$$

$$= x^4(x-1) - 4x^3(x-1) + 8x^2(x-1) - 16x(x-1) + 16(x-1)$$

$$= (x-1)(x^4 - 4x^3 + 8x^2 - 16x + 16)$$

Now work on the $q_1(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$; depressed eq^n.

The potential rational zeros of q_1 are still $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$.

We test 1 first, since it may be a repeated zero of f.

$$q_1(1) = 1 - 4 + 8 - 16 + 16 = 5 \neq 0.$$

Next we try 2.

$$q_1(2) = 16 - \cancel{32} + \cancel{32} - 32 + 16 = 32 - 32 = 0$$

$\therefore (x-2)$ is a factor of f.

Again we find,

$$f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$$

$$= (x-1)(x^4 - 4x^3 + 8x^2 - 16x + 16)$$

$= (x-1)(x^4-2x^3-2x^3+4x^2+4x^2-8x-8x+16$

$= (x-1)\{x^3(x-2)-2x^2(x-2)+4x(x-2)-8(x-2)\}$

$= (x-1)(x-2)(x^3-2x^2+4x-8)$

The remaining zeros satisfy the new depressed eq^n

$$q_2(x) = x^3-2x^2+4x-8$$

Now $q_2(x)$ can be factored by grouping. Then

$$q_2(x) = x^3-2x^2+4x-8$$
$$= x^2(x-2)+4(x-2)$$
$$= (x-2)(x^2+4)$$

Since $x^2+4=0$ has no real solutions, the real zero of f are 1 and 2 with 2 being a zero of multiplicity 2. The factored form of f is

$$f(x) = x^5-5x^4+12x^3-24x^2+32x-16$$
$$= (x-1)(x-2)^2(x^2+4)$$

Example: Find the real solutions of the equations

$$x^5-5x^4+12x^3-24x^2+32x-16=0.$$

Sol^n: The real solution of $f(x)$ are the real zeros of the polynomial function.

Using the result of the previous example we can say that the real zeros of f are 1 and 2.

So $\{1,2\}$ is the solution set of the equation.

Exercise: $f(x) = 2x^3 + 11x^2 - 7x - 6$. Find the real zeros of the Polynomial.

Find bounds on the real zeros of Polynomial function:

If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Then a bound M on the real zeros of f is the smaller of the two numbers

$\max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}$, $1 + \max\{|a_0|, |a_1| \cdots |a_{n-1}|\}$

Then $-M \leq$ any real zero of $f \leq M$.

Example:

a. Find a bound on the real zero of $f(x) = x^5 + 3x^3 - 9x^2 + 5$

Solⁿ: Given, $f(x) = x^5 + 3x^3 - 9x^2 + 5$

$a_0 = 5,\ a_1 = 0,\ a_2 = -9,\ a_3 = 3,\ a_4 = 0$

$\therefore \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} = \max\{1, |5| + |0| + |-9| + |3| + |0|\}$

$= \max\{1, 17\} = 17.$

$1 + \max\{|a_0|, |a_1| \cdots |a_{n-1}|\} = 1 + \max\{|5|, |0|, |-9|, |3|, |0|\}$

$= 1 + 9 = 10$

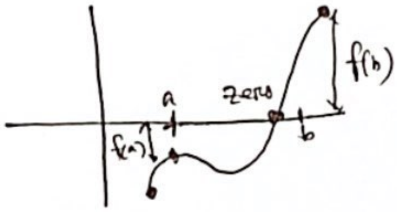
The smaller of the two numbers 10, is the bound.

$\therefore$ Every real zero of f lies betⁿ -10 and 10.

b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1.$

Intermediate value theorem:

Let f denote a polynomial function. If $a<b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.



Example.

Show that $f(x) = x^5 - x^3 - 1$ has a zero between 1 and 2.

Soln: $f(1) = -1$ and $f(2) = 23$

Because $f(1) < 0$ and $f(2) > 0$, it follows from the intermediate theorem that f has at least one zero between 1 and 2.