Goal (for next 4 lectures): Develop techniques for making a quadrotor hover in place.

Hovering is arguably the most basic capability a quadrotor should have. The techniques we develop will also be broadly applicable to many other robotics applications (e.g., getting a humanoid robot to balance).

Approach we will pursue for making quadrotor hover:

(1) Figure out how the quadrotor will behave when you apply different propeller commands [**Dynamics**].
(2) Figure out a mechanism for taking corrective actions when the quadrotor moves away from the desired hover configuration [**Feedback Control**].

Plan for today: Tackle part (1) above: dynamics of quadrotor.

**Plan for next 3 lectures:** Tackle part (2): feedback control.

Note: Entire courses can be (and are) devoted to dynamics (e.g., MAE 206 and MAE 542 at Princeton) and control (e.g., MAE 433, 434, 544). We will cover just the basics: enough to make a quadrotor hover and give you a good understanding of how control works for robotic systems. Discussing control theory will also allow us to develop terminology we will make extensive use of throughout the course.

1. Planar Quadrotor

We will first consider a highly simplified model of a quadrotor; this system resides in the x-y plane and is referred to as the "planar quadrotor" (see Figure 1(a)).
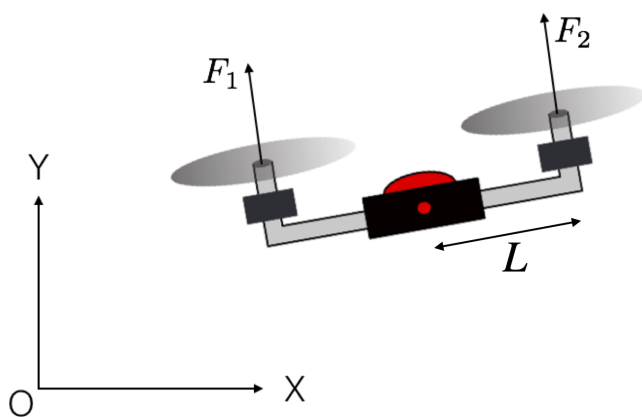
(a) Planar Quadrotor
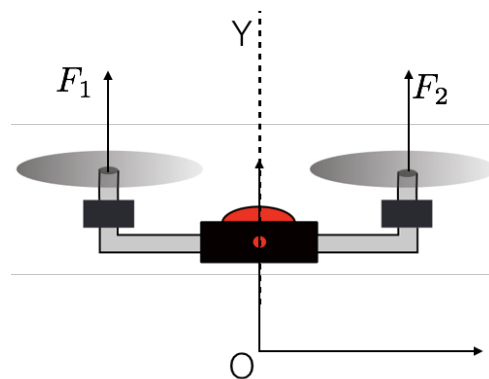
(b) Planar Quadrotor constrained to move along y-axis

Figure 1

In fact, we will make things even simpler and begin by only considering the motion in the y direction (i.e., up and down); see Figure 1(b). In this case, we can describe the configuration of

the entire quadrotor at any point in time with a single number: the y position of the center of mass (COM). This system thus has *one degree of freedom (DoF)*.

Degrees of freedom (DoF): The number of independent ways a system can move.

Equivalently, one can think of the number of DoFs as the minimal number of coordinates we need to completely specify the configuration of the system.

How do we describe the motion of this system (still considering only the motion in the y direction)? We can derive the equations of motion (EoMs) for the system using Newton's second law:

$$m\ddot{y} = F_1 + F_2 - mg, \tag{1}$$

where m is the total mass of the quadrotor, and F_1 and F_2 are the thrust forces from the propeller (we will say more about how motors produce these forces later). Thus,

$$\ddot{y} = \frac{F_1 + F_2}{m} - g. \tag{2}$$

Now, denote the total thrust force produced by the propellers as:

$$u_1 \triangleq F_1 + F_2. \tag{3}$$

We then have:

$$\ddot{y} = \frac{u_1}{m} - g. \tag{4}$$

Now, let's consider the planar quadrotor (no longer constrained to move only along the y-direction). How many DoFs does this system have? Answer: 3 DoFs corresponding to x-position, y-position, and orientation θ.

How do we describe the system's motion? Again, we can use Newton's second law to derive the equations of motion:

$$\ddot{x} = -\frac{u_1}{m} \sin \theta, \tag{5}$$

$$\ddot{y} = \frac{u_1}{m} \cos \theta - g, \tag{6}$$

$$\ddot{\theta} = \frac{(F_2 - F_1)L}{I}. \tag{7}$$

Here, u_1 is the same as above (total thrust force), I corresponds to the moment of inertia of the system, and L is the length of the quadrotor's arm (see Figure 1(a)). The moment of inertia is a physical property of the system that can be estimated from the geometry (and mass distribution) of the quadrotor or via physical experiments.

Next, denote the total moment due to the propeller forces as:

$$u_2 \triangleq (F_2 - F_1)L. \tag{8}$$

Our equations of motion for the planar quadrotor can then be expressed as:

$$\ddot{x} = -\frac{u_1}{m}\sin\theta, \tag{9}$$

$$\ddot{y} = \frac{u_1}{m}\cos\theta - g, \tag{10}$$

$$\ddot{\theta} = \frac{u_2}{I}. \tag{11}$$

As a sanity check, Equation (10) matches Equation (2) when $\theta = 0$.

Note: I will not expect you to derive equations of motion in this course (although students in the MAE department should be able to do this!). We will typically just assume that we are provided the EoMs. But, it is good to go through the exercise of deriving EoMs at least once (as we did with the planar quadrotor above).

The equations (9) – (11) are three *second-order* ordinary differential equations (ODEs). Recall from your ODE course that we can always take a system of second-order ODEs and convert them to a system of first-order ODEs by introducing new variables. Let us see how this works for the planar quadrotor's EoMs.

Define:

$$v_x \triangleq \dot{x}, \tag{12}$$

$$v_y \triangleq \dot{y}, \tag{13}$$

$$\omega \triangleq \dot{\theta}. \tag{14}$$

We can then re-express the EoMs (9) – (11) as six *first-order* ODEs:

$$\dot{x} = v_x, \tag{15}$$

$$\dot{y} = v_y, \tag{16}$$

$$\dot{\theta} = \omega, \tag{17}$$

$$\dot{v}_x = \ddot{x} = -\frac{u_1}{m}\sin\theta, \tag{18}$$

$$\dot{v}_y = \ddot{y} = \frac{u_1}{m}\cos\theta - g, \tag{19}$$

$$\dot{\omega} = \ddot{\theta} = \frac{u_2}{I}. \tag{20}$$

We can then define the **state** of the system as: $\bar{x} = [x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}]^T$. [Note: we will use bars to denote vectors.] The state defines everything we need to know in order to predict the motion of the system. If we specify the initial state of the system (i.e., the state at time $t = 0$), we can solve (either analytically or numerically) the EoMs and obtain the state of the system at any future time.

Number of states: The dimensionality of the state vector (i.e., the number of components of the state). The planar quadrotor thus has 6 states. The planar quadrotor constrained to move in the y-direction has 2 states (convince yourself of this).

Configuration variables: $[x, y, \theta]$. This is the portion of the state that describes the configuration of the system at any given time (this does not include the components of the state corresponding to time derivatives).

Control input: $\bar{u} = [u_1, u_2]^T$. These are things that we can directly control/command (the propeller thrusts for the planar quadrotor).

The equations of motion can then be written as:

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}). \tag{21}$$

We will always be able to write EoMs in this form.

2. Motor Model

So far, we have treated the propellers as thrusters. In reality, we do not directly control the thrust produced; we directly control the speed of the motor. It turns out that there is a nice relationship between the speed of the motor and the generated thrust.

Thrust (produced by propeller spinning): $F = k_f \times (\text{rotational speed of motor})^2$.

The constant k_f is referred to as the *thrust coefficient*. How can we measure this quantity? Here is a simple method: place the drone on a scale and sweep through different values of motor speeds. Measure the readings of the scale (which you can convert from kilograms to Newtons).

We have performed this experiment for a Crazyflie quadrotor. In the first assignment, you will use this data to estimate k_f.

The planar quadrotor thus has 3 physical parameters: m, I, k_f. In order to fully specify our dynamics model, we need to specify the value of these parameters.
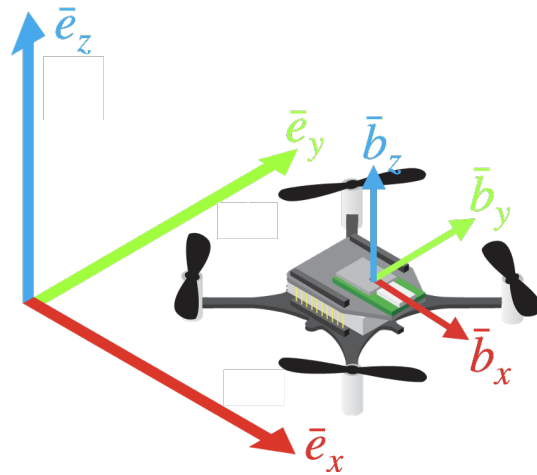
3. 3D Quadrotor

FIGURE 2. Axes for (Crazyflie) quadrotor; source: Bitcraze.io.

How many degrees of freedom does a full three-dimensional quadrotor have (see Figure 2)?

Position: 3 numbers x, y, z;

Orientation: 3 numbers;

$\implies$ 6 DoFs.

For the planar quadrotor, the orientation was easy to describe using θ. For the 3D quadrotor, it is slightly more complicated. We will use *Euler angles* to describe the orientation of a 3D quadrotor (and 3D rigid bodies in general).

3.1. **Euler Angles.** Euler angles allow us to describe orientations in 3D. Suppose we attach axes $\bar{b}_x, \bar{b}_y, \bar{b}_z$ to the body of our quadrotor (see Figure 2).

$\bar{b}_x$: Points in the forward direction; $\bar{b}_y$: Points in the left direction; $\bar{b}_z$: Points up.

We will denote this frame by B (corresponding to "body"). We also define $\bar{e}_x, \bar{e}_y, \bar{e}_z$ associated with an *inertial reference frame I*.

One way to describe the orientation of B relative to I is the following:

(1) Rotate B about $\bar{e}_x$ by ϕ ("roll");
(2) Then rotate B about $\bar{e}_y$ by θ ("pitch");
(3) Then rotate B about $\bar{e}_z$ by ψ ("yaw").

It turns out that *any* configuration of the quadrotor can be specified by specifying (ϕ, θ, ψ).

The convention above is only one of many possibilities for defining Euler angles. This convention (which is the one we will use in this course) is referred to as the **space** $1 - 2 - 3$ **convention**, since we first rotate about $\bar{e}_x$ (Axis 1), then about $\bar{e}_y$ (Axis 2), and then $\bar{e}_z$ (Axis 3).

There are other possibilities. For example, one could:

- Change the order of rotations, e.g., 2-3-1, or 3-2-1, etc.
- Perform the rotations about the intermediate axes, i.e., first rotate about $\bar{e}_x$, then perform a rotation about the *new* $\bar{b}_y$, and then perform a rotation about the *new* $\bar{b}_z$. This is called the **body 1-2-3 convention**.

In practice, the fact that we have so many different conventions is a source of many headaches/errors. Be careful! Make sure you know which convention you're using (especially if you're working in a team, make sure everyone is using the same convention!). In this course, however, we won't worry too much about different conventions. We will work with the space 1-2-3 convention.

4. States of the 3D quadrotor

Using Euler angles, the states of the 3D quadrotor are: $\bar{x} = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, p, q, r]^T$.

Here, p, q, r correspond to *angular velocities*; these are directly related to $\dot{\phi}, \dot{\theta}, \dot{\psi}$. It is slightly more convenient to work with the angular velocities than with the time derivates of the Euler angles.

Control inputs: we can use the four propeller thrusts as our control inputs. However, as with the planar quadrotor, it will be slightly more convenient to work with the following control input vector:

$$\bar{u} = [F_{\text{tot}}, M_1, M_2, M_3]^T, \tag{22}$$

where:

F_{tot} : Total thrust (from 4 propellers);

M_1 : Moment about x-axis;

M_2 : Moment about y-axis;

M_3 : Moment about z-axis.

It is easy to go back and forth between (F_1, F_2, F_3, F_4) and $(F_{\text{tot}}, M_1, M_2, M_3)$.

In the next lecture, we will write down the EoMs for the 3D quadrotor and begin our discussion of control.