Course Name : Physics – I
Course # PHY 107

Notes-4 : Motion in Two Dimensions

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Topics to be covered

1. Definitions of Physical Quantities in Two Dimensions
2. Mathematical Properties: Equations of Motions
3. Motion on a Horizontal Plane
4. Projectile Motion
5. Uniform Circular Motion
6. Examples

- In two dimensions, there two coordinate axes: x- and y-axes. The time t is a positive definite parameter (not an axis).
- All physical quantities are functions of time.
- The definitions are similar to the one dimensional case, but with an additional coordinate.
- Definitions:

Position, $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}.$

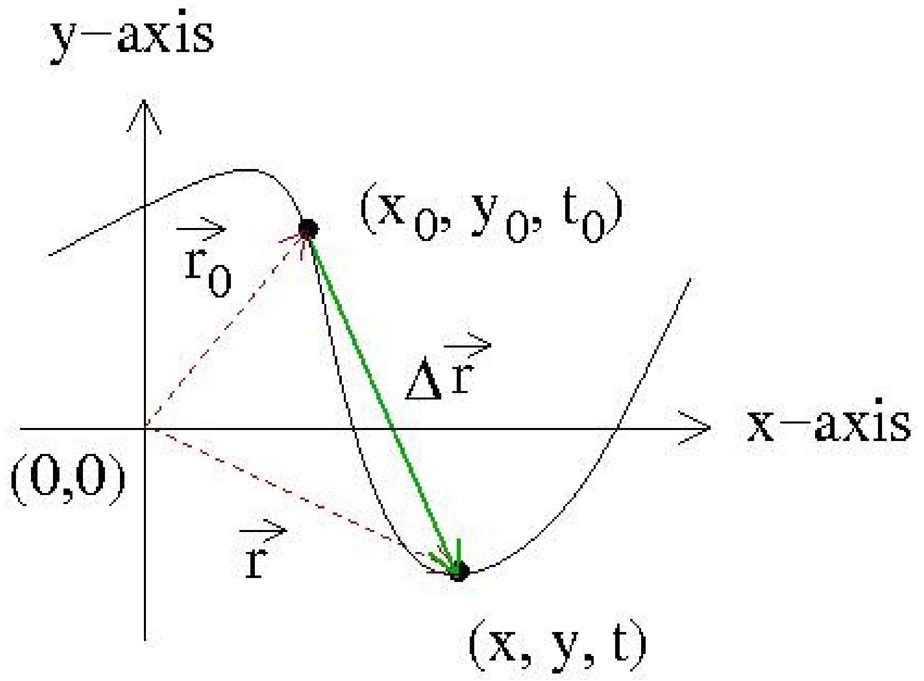
Displacement, $\Delta r(t) = \vec{r} - \vec{r}_0 = \Delta x\,\hat{i} + \Delta y\,\hat{j}.$

$$= (x - x_0)\,\hat{i} + (y - y_0)\,\hat{j}.$$

Distance, $\mathrm{d} = |\Delta r(t)| = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$

or the actual path.

Direction, $\theta = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right).$



- In polar form, the displacement is: $\Delta\vec{r}=(|\Delta\vec{r}|,\,\theta_r)$.
- Similarly, the velocity and acceleration at time t can be defined as

$$\vec{v}(t)=\frac{d\vec{r}}{dt}=\frac{dx}{dt}\,\hat{i}+\frac{dy}{dt}\,\hat{j}\equiv v_x\hat{i}+v_y\hat{j}=(v_x,v_y)=(v,\theta_v).$$

$$\vec{a}(t)=\frac{d\vec{v}}{dt}=\frac{dv_x}{dt}\,\hat{i}+\frac{dv_y}{dt}\,\hat{j}\equiv a_x\hat{i}+a_y\hat{j}=(a_x,a_y)=(a,\theta_a),$$

$$=\frac{d^2\vec{r}}{dt^2}=\frac{d^2x}{dt^2}\,\hat{i}+\frac{d^2y}{dt^2}\,\hat{j}\equiv a_x\hat{i}+a_y\hat{j}.$$

- The magnitude and directions of velocity (*i.e.* speed) and the accelerations are deriv

$$v=\sqrt{v_x^2+v_y^2},\;\theta_v=\tan^{-1}\left(\frac{v_y}{v_x}\right)\text{ and }a=\sqrt{a_x^2+a_y^2},\;\theta_a=\tan^{-1}\left(\frac{a_y}{a_x}\right).$$

Example: A rabbit is running on a floor. The position as a function of time are given by:

$$x(t) = -0.31t^2 + 7.2t + 28,$$
$$y(t) = 0.22t^2 - 9.1t + 30.$$

where x and y are in meters and t is in second.

a) At t = 1.5 sec, find $\vec{r}$ in Cartesian and in Polar forms.

Solution: The position is,

$$\begin{aligned}
\vec{r}(1.5\,\text{sec}) &= \left[x(1.5)\,\hat{i} + y(1.5)\,\hat{j}\right], \\
&= \left[\left(-0.31(1.5)^2 + 7.2(1.5) + 28\right)\hat{i} + \left(0.22(1.5)^2 - 9.1(1.5) + 30\right)\hat{j}\right], \\
&= \left(66\,\hat{i} - 57\,\hat{j}\right)\text{m}. \Rightarrow \text{This is the Cartesian form.} \\
&= \left(87\,\text{m}, 319^o\right). \Rightarrow \text{This is the Polar form.}
\end{aligned}$$

b) Find the velocity at t = 2 sec.

Solution: The velocity is given by,

$$\vec{v}(t=2\,\text{sec}) = \left(\frac{d\vec{r}}{dt}\right)_{t=2} = \left[\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right]_{t=2},$$

$$= \left[(-0.62t+7.2)\hat{i} + (0.44t-9.1)\hat{j}\right]_{t=2} = (5.96\hat{i} - 8.66\hat{j})\,\text{m/s}.$$

Note that this is the instantaneous velocity at time t=2sec.

By using the Pythagorean Theorem, the Polar form of the velocity at t=2sec can be easily calculated, and the result is

$$v = |\vec{v}| = \sqrt{v_x^2+v_y^2} = \sqrt{5.96^2+(-8.22)^2}\,\text{m/s} = 103.09\,\text{m/s} \approx 103\,\text{m/s}.$$

$$\theta_v = \tan^{-1}\left(\frac{v_y}{v_x}\right) = 360^o - \tan^{-1}\left(\frac{8.22}{5.96}\right) \approx 306^o.$$

c) Find the average velocity between t = 1sec to t = 2sec.

Answer: Note that there are two formula to find the average velocity:

$$v_{\text{av}} = \frac{\Delta \vec{r}}{\Delta t} \quad \text{or} \quad \frac{\vec{v}_2 + \vec{v}_1}{2}.$$

The first formula is the definition and hence can always be used. But the second formula is only valid when the acceleration is constant. In this example, both the x- and y-components of the position are quadratic function of time t, and hence their double derivatives are constant. This means that the acceleration is also constant. Hence, any of these two formulas can be used. Both formula will give the same answer.

- Using the definition, we find,

$$\begin{aligned}
\vec{v}_{\text{av}} &= \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}}{t_2 - t_1}. \\
\text{Now, } \vec{r}_2 &= x(2)\hat{i} + y(2)\hat{j} = (41.78\,\hat{i} + 12.24\,\hat{j})\,\text{m}, \\
\text{and } \vec{r}_1 &= x(1)\hat{i} + y(1)\hat{j} = (34.89\,\hat{i} + 21.12\,\hat{j})\,\text{m}. \\
\text{Therefore, } \Delta \vec{r} &= (6.89\,\hat{i} - 8.88\,\hat{j})\,\text{m} \quad \text{and} \quad t_2 - t_1 = 1\,\text{sec}. \\
\text{Hence, } \vec{v}_{\text{av}} &= (6.89\,\hat{i} - 8.88\,\hat{j})\,\text{m/s}.
\end{aligned}$$

- It can also be checked that the other formula also gives the same answer.

Projectile Motion:

- This is a vertical two dimensional motion. For projectile motion, the acceleration is fixed by the force of gravity, namely,

$$\vec{a} = -g\hat{j} \Rightarrow a_x = 0 \text{ and } a_y = -g = -9.80 \text{ m/s}^2.$$

- Since, $a_x = 0, \quad v_x = \text{constant}. \quad \text{So, } v_x = v_{0x}.$
- But vertically the motion is exactly like free-fall motion.
- The equations of motions are:

$$\left.\begin{aligned} x &= x_0 + v_{0x} t \\ v_x &= v_{0x}. \end{aligned}\right\}$$

These are the only two equation for the x-component, because the acceleration is zero horizontally.

- The other equations are exactly like free-fall equations:

$$
\left.\begin{aligned}
y &= y_0 + v_{0y} t, \\
v &= v_{0y} + a_y t, \\
y &= y_0 + v_{0y} t + \frac{1}{2} a_y t^2, \\
\text{and } v_y^2 &= v_{0y}^2 + 2 a_y (y - y_0).
\end{aligned}\right\}
$$

These are the y-components of the equations.

- These six equations form the projectile motions. Along the x-axis the velocity is constant, and along y-axis, it is not constant. The path on the xy-plane is a parabola.

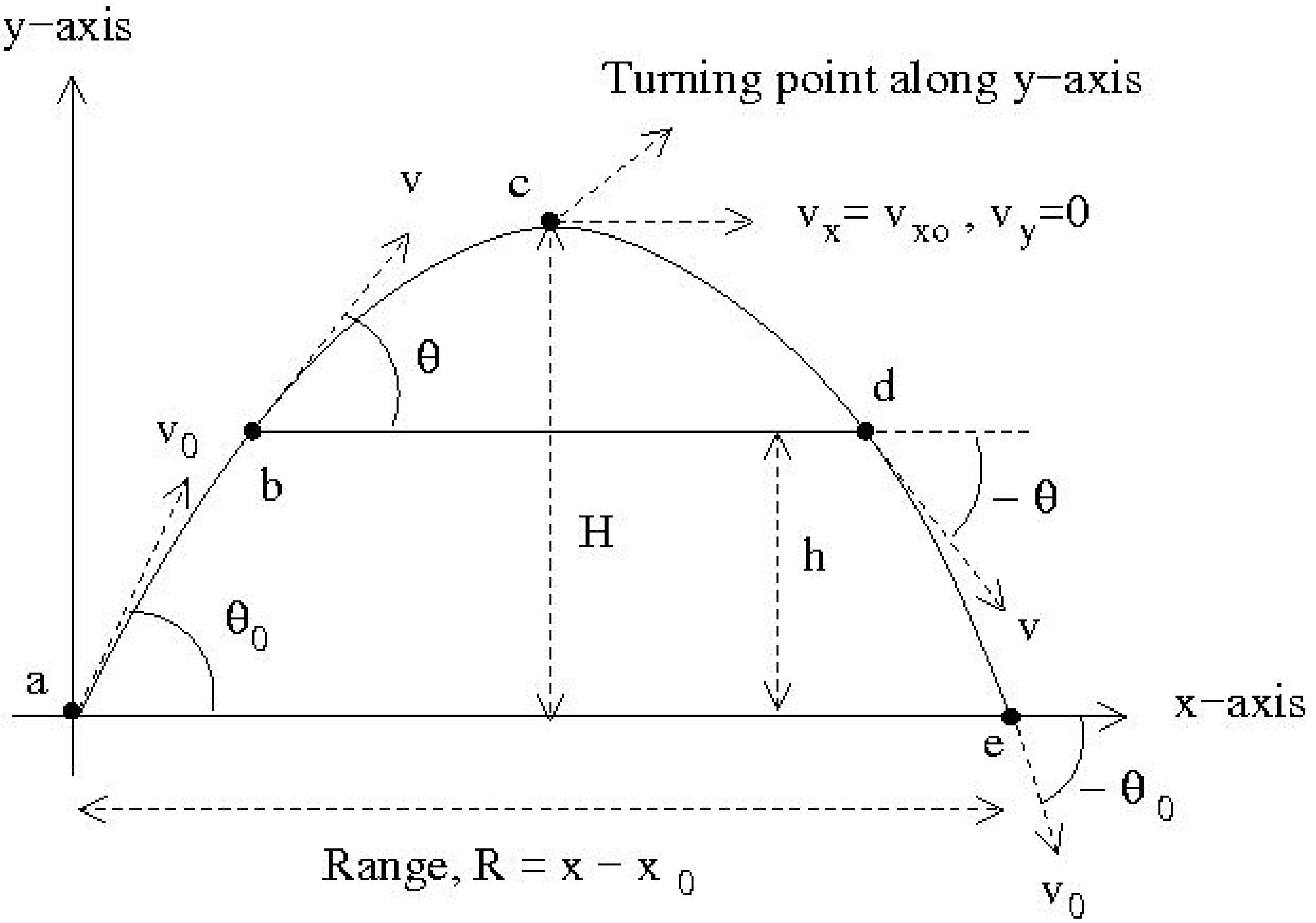
- The path of a projectile: Here $\vec{v}=(v_x,v_y)=(v,\theta)$ where $v_x=v\cos\theta$, $v_y=v\sin\theta$. Note that:

$\vec{v}_a=(v_{0,}\theta_0)$,
$\vec{v}_e=(v_0,-\theta_0)$,
$\vec{v}_b=(v,\theta)$,
$\vec{v}_d=(v,-\theta)$,
$\vec{v}_c=v_y\hat{i}$,

$$H=|y-y_0|=\frac{-v_{0y}^2}{2a_y}=\frac{v_{0y}^2}{2g},$$

$$T=2t_{\text{up}}=\frac{-2v_{0y}}{a_y}=\frac{2v_{0y}}{g},$$

$R=x-x_0=v_{0x}T$.

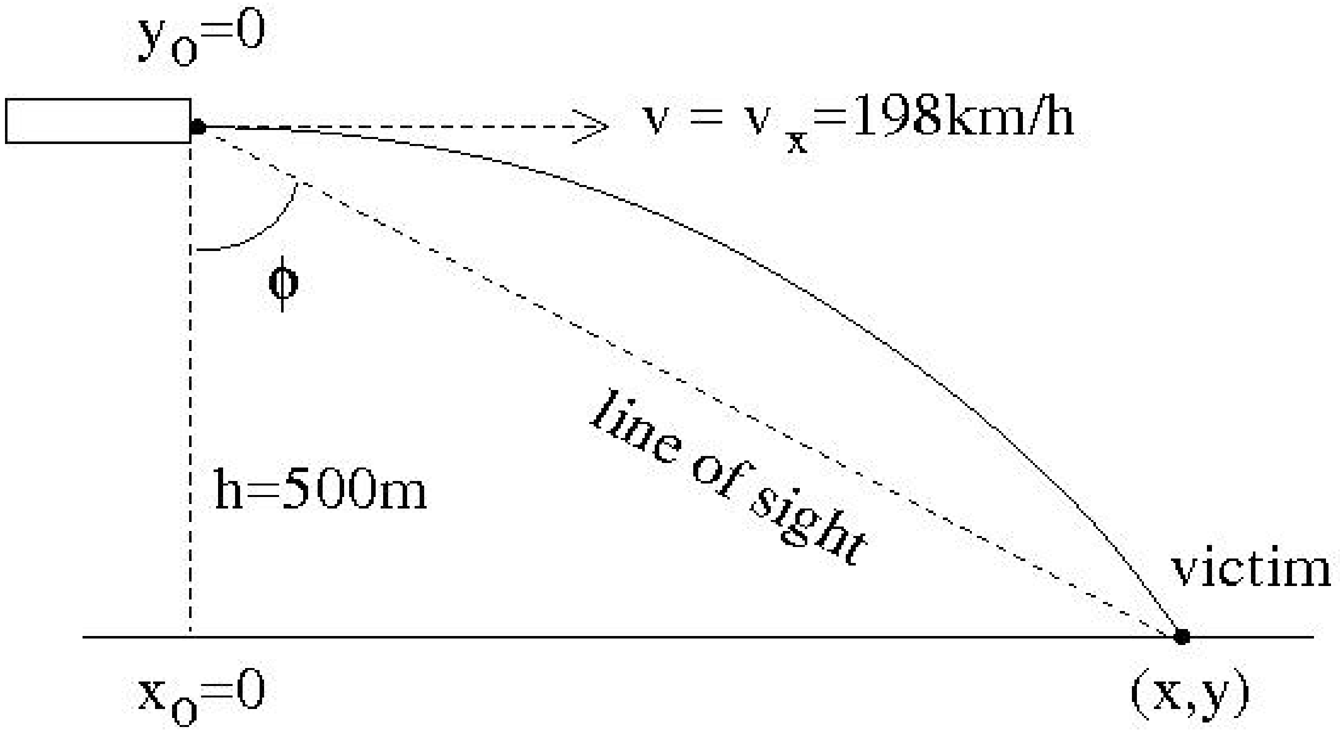


Example:

- A rescue plane flying at 198km/h at a constant height of h=500m above the ground to deliver a load for a victim on a the ground. The pilot drops the load at an angle ϕ of the line of sight. (a) Find the angle ϕ. (b) At what velocity the load reaches the victim?
- Solution:

  Here, we have,

$$v_{0y}=0,$$
$$v_x=v_{0x}=198\,\text{km/h}=55\,\text{m/s}.$$
$$x_0=0,\quad y_0=0,$$
$$y=-500\,\text{m}.$$



(a) From the y-component, we find,

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \;\Rightarrow\; -500 = 0 + 0 \times t + \frac{1}{2}(-g) t^2 \;\Rightarrow\; t = 10.1 \text{ sec}.$$

Therefore, $x = x_0 + v_{0x} t = (0 + 55 \times 10.1)\,\text{m} = 555.6\,\text{m}.$

By Trigonometric relation, we obtain,

$$\tan\phi = \frac{x}{h} \;\Rightarrow\; \phi = \tan^{-1}\left(\frac{x}{h}\right) = \tan^{-1}\left(\frac{555.6}{500}\right) = 48^o.$$

(b) After time t = 10.1 sec,

$$v_y = v_{0y} + a_y t = 0 + (-9.80) \times (10.1) \text{ m/s} = -99 \text{ m/s}.$$

$$v_x = v_{0x} \text{ (because horizontally velocity remains same)}$$

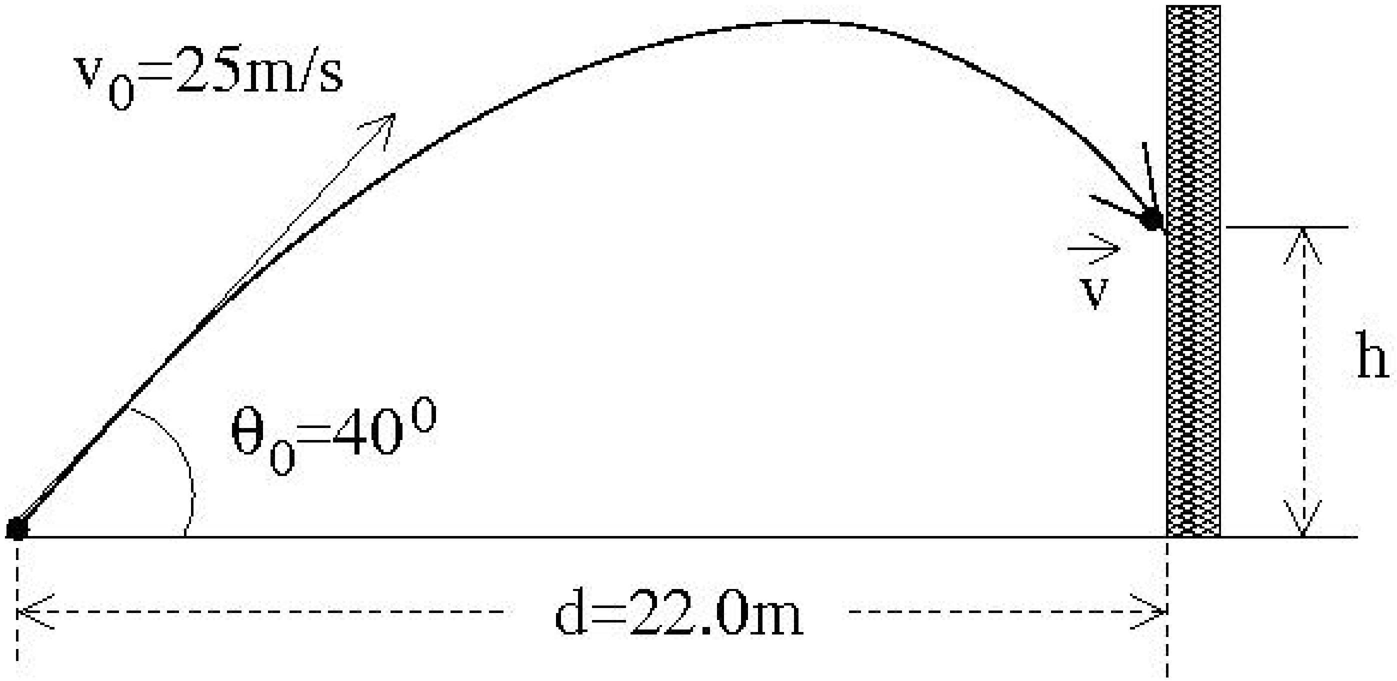
Therefore, $\vec{v} = v_x \hat{i} + v_y \hat{j} = (55\,\hat{i} - 99\,\hat{j})\,\text{m/s} = (113.3\,\text{m/s},\; 299.1^o).$

Example-2:

- A ball is thrown with speed of 25 m/s at an angle of 40° above the ground towards a wall which is 22.0 m away. (a) How high above the ground will it hit the wall? (b) Find the velocity of the ball when it hit the wall. (c) Does the ball pass the point of maximum height before hitting the wall? Explain or show calculation.

- **Solution**: We have,

$$\vec{v}_0 = (25\,\text{m/s}\,,\,40^o).$$
$$v_{0x} = 25\cos 40^o\,\text{m/s} = 19.15\,\text{m/s}.$$
$$v_{0y} = 25\sin 40^o\,\text{m/s} = 16.07\,\text{m/s}.$$
$$x_0 = 0, \quad y_0 = 0.$$
$$y = h \text{ and } x = d = 22.0\,\text{m}.$$



(a) From the horizontal part of the motion we find the time of flight t,

$$d = x - x_0 = v_{0x} t \;\Rightarrow\; t \;=\; \frac{d}{v_{0x}} \;=\; \frac{22.0\,\mathrm{m}}{19.15\,\mathrm{m/s}} \;=\; 1.15\,\mathrm{sec}\,.$$

So, in time t=1.15 sec, the ball reaches a height h. Hence,

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \;\Rightarrow\; h \;\equiv\; y - y_0 = v_{0y} t + \frac{1}{2}(-g) t^2$$

$$= \left[16.07 \times 1.15 - \frac{1}{2} \times 9.80 \times (1.159)^2 \right] \mathrm{m} = 12.0\,\mathrm{m}\,.$$

(b) After time t=1.15 sec, $v_y = v_{0y} + a_y t = (16.07 - 9.80 \times 1.15)\,\mathrm{m/s} = 4.80\,\mathrm{m/s}\,.$

Therefore, the velocity after time t=1.15 sec is

$$\vec{v} \;=\; v_x \hat{i} + v_y \hat{j} \;=\; (19.15\,\hat{i} + 4.80\,\hat{j})\,\mathrm{m/s} \;=\; (19.74\,\mathrm{m/s}\,,\, 14.1^o).$$

(c) This can be answered two different ways. Firstly by inspecting the sign of the y-component of the velocity. v_y will be positive, negative or zero if the ball is going upward, downward or horizontal respectively at the time of impact. Since it is positive, the ball did not cross the point of maximum height.

Alternatively, we can also compute the time to reach the maximum height. If it is more than 1.15 sec, then the ball did not cross the point of maximum height. If it is less than, it has crossed. Finally, if equal , then it is at the maximum height.

Uniform Circular Motion

- The conditions of a uniform circular motions are:

$$1)\ \text{Radius},\ R=|\vec{r}|=\text{constant and}\ \ 2)\ \text{Speed},\ \mathrm{v}=|\vec{v}|=\text{constant}.$$

- At time t, the position and velocity are:

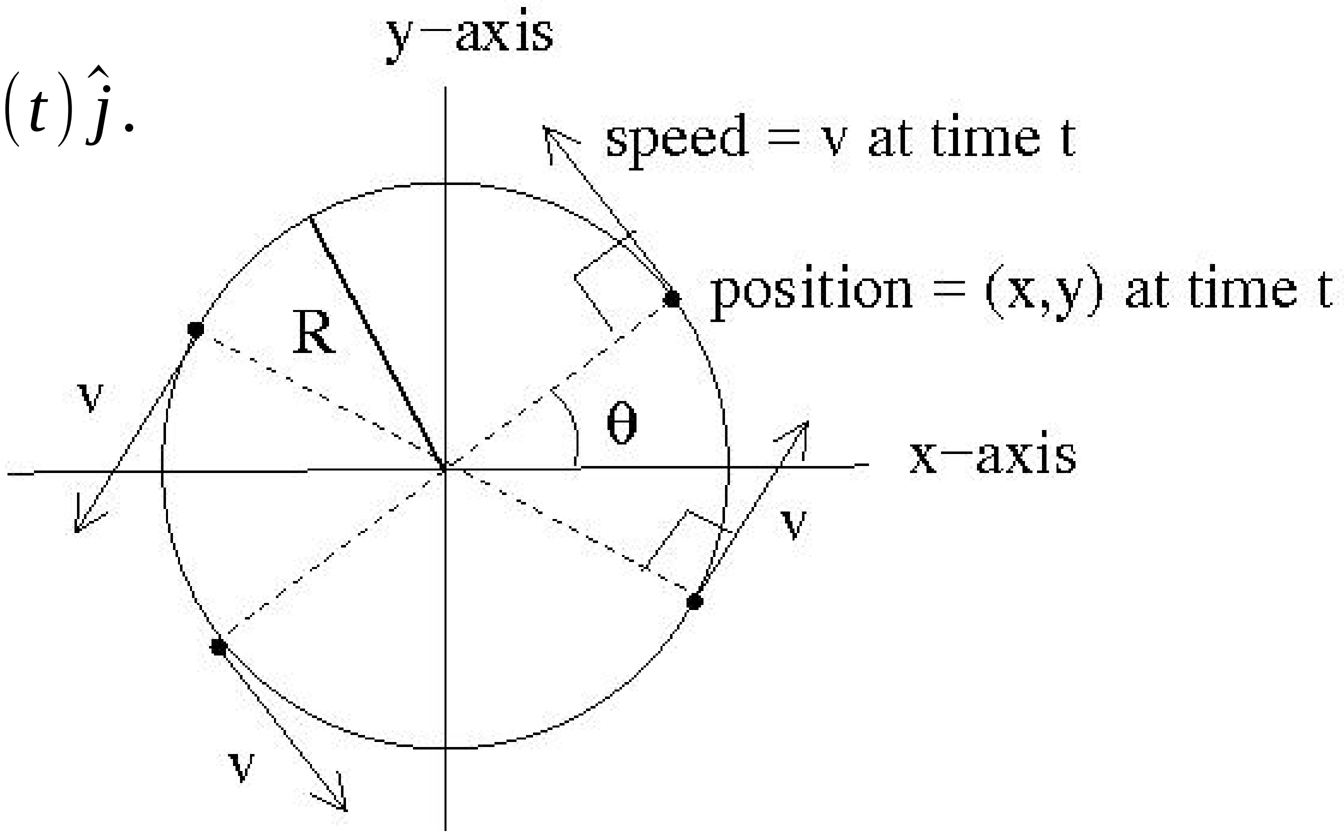
$$\vec{r}(t)=(R,\theta(t))=x(t)\hat{i}+y(t)\hat{j}$$
$$\vec{v}(t)=(v,90^o+\theta(t))=v_x(t)\hat{i}+v_y(t)\hat{j}.$$

- Note that, since speed is fixed,

$\theta \propto t.$

So, $\theta(t)=\omega t$, where ω is the proportionality constant.

ω is called the angular speed, and measured in rad/sec.



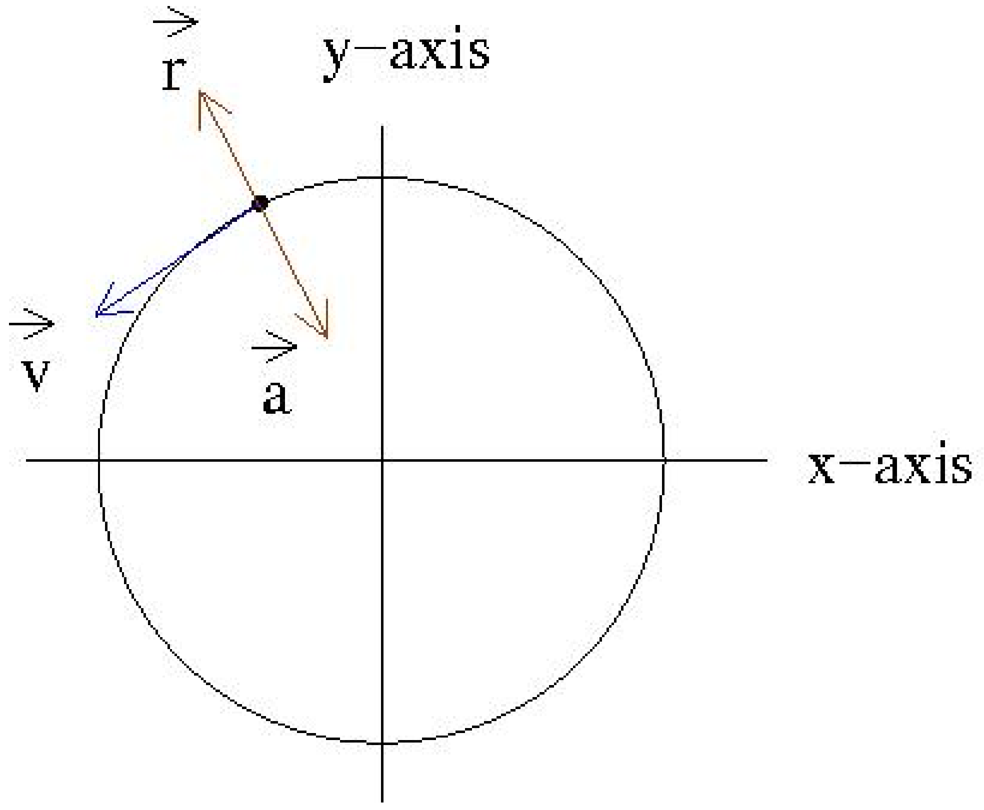
- Now the position, velocity and acceleration at time t are,

$$\begin{aligned}
\vec{r}(t) &= \left(R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}\right),\\
\vec{v}(t) &= \frac{d\vec{r}}{dt} = \omega R\left(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}\right)\\
\vec{a}(t) &= \frac{d^2\vec{r}}{dt^2} = -\omega^2 R\left(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right) = -\omega^2\vec{r}(t).
\end{aligned}$$

- Note that the position is always radially outward (i.e. positive), and hence the acceleration is always radially inward or negative. That is, the acceleration is always towards the center of the circle, and so it is called the centripetal acceleration.
- Note also that at any time t,

$$\vec{r}\cdot\vec{v}=0,\;\; \vec{v}\cdot\vec{a}=0,\; \text{and}\; \vec{r}\cdot\vec{a}=-\omega^2 R^2<0\,.$$

- This shows that the velocity is perpendicular to both the position and acceleration, so it is always tangential.
- If at time t, the position is directed towards the 2nd quadrant, then the velocity and acceleration must be directed towards the 3rd and 4th quadrant respectively, as shown in the adjacent diagram (rotating counter-clockwise only).



- It can be easily checked that,

$$|\vec{r}(t)| \;=\; \sqrt{R^2(\cos^2\omega t+\sin^2\omega t)} \;=\; R \;=\; \text{constant}.$$
$$|\vec{v}(t)| \;=\; \sqrt{(\omega R)^2(\sin^2\omega t+\cos^2\omega t)} \;\Rightarrow\; v \;=\; \omega R \;=\; \text{constant}.$$
$$|\vec{a}(t)| \;\equiv\; a_c=\sqrt{\omega^4 R^2(\cos^2\omega t+\sin^2\omega t)} \;\Rightarrow\; a_c=\omega^2 R \;=\; \frac{v^2}{R}.$$

- Note that this property of the uniform circular motion will be very relevant when we will study the Gravitational force and Simple Harmonic Motion in due time later in this course.