2 First-Order Differential Equations

Direction Field If we systematically evaluate f over a rectangular grid of points in the xy-plane and draw a line element at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all these line elements is called a **direction fiel** or a **slope fiel** of the differential equation $dy/dx = f(x, y)$. Visually, the direction field suggests the appearance or shape of a family of solution curves of the differential equation, and consequently, it may be possible to see at a glance certain qualitative aspects of the solutions—regions in the plane, for example, in which a

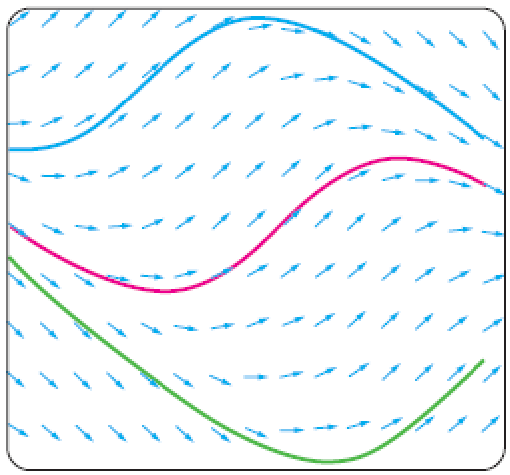

FIGURE 2.1.2 Solution curves following flow of a direction fie

solution exhibits an unusual behavior. A single solution curve that passes through a direction field must follow the flow pattern of the field; it is tangent to a lineal element when it intersects a point in the grid. Figure 2.1.2 shows a computer-generated direction field of the differential equation $dy/dx = \sin(x + y)$ over a region of the xy-plane. Note how the three solution curves shown in color follow the flow of the fiel

2.1.2 AUTONOMOUS FIRST-ORDER DEs

Autonomous First-Order DEs In Section 1.1 we divided the class of ordinary differential equations into two types: linear and nonlinear. We now consider briefly another kind of classification of ordinary differential equations, a classification that is of particular importance in the qualitative investigation of differential equations. An ordinary differential equation in which the independent variable does not appear explicitly is said to be **autonomous.** If the symbol x denotes the independent variable, then an autonomous first-order differential equation can be written as $f(y, y') = 0$ or in normal form as

$$\frac{dy}{dx} = f(y). \tag{2}$$

We shall assume throughout that the function f in (2) and its derivative f' are continuous functions of y on some interval I. The first-order equation

$$\frac{dy}{dx} = \overbrace{1 + y^2}^{f(y)} \qquad \text{and} \qquad \frac{dy}{dx} = \overbrace{0.2xy}^{f(x,\,y)}$$

are autonomous and nonautonomous, respectively.

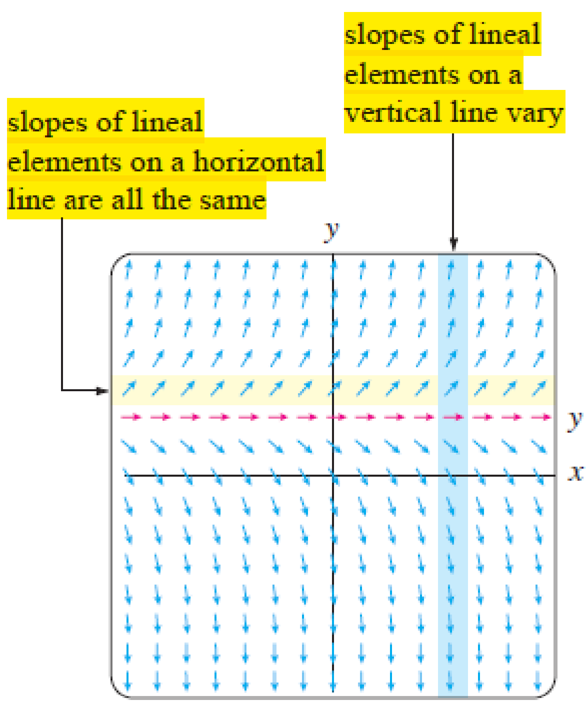

FIGURE 2.1.10 Direction field for a autonomous DE

2.2 SEPARABLE EQUATIONS

DEFINITION 2.2.1 Separable Equation

A first-order di ferential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables.**

EXAMPLE 1 Solving a Separable DE

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

EXAMPLE 2 Solution Curve

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad y(4) = -3.$

SOLUTION Rewriting the equation as $y\,dy = -x\,dx$, we get

$$\int y\,dy = -\int x\,dx \qquad \text{and} \qquad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

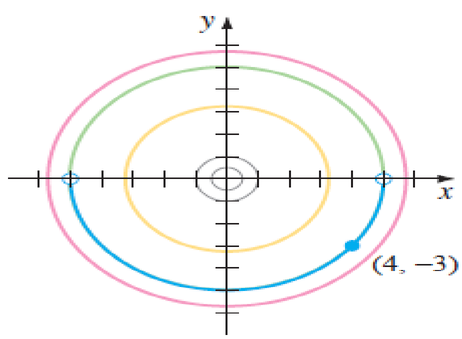

FIGURE 2.2.1 Solution curve for the IVP in Example 2

EXAMPLE 4 An Initial-Value Problem

Solve $(e^{2y} - y)\cos x \dfrac{dy}{dx} = e^{y} \sin 2x, \quad y(0) = 0.$

SOLUTION Dividing the equation by $e^{y} \cos x$ gives

$$\frac{e^{2y} - y}{e^{y}}\,dy = \frac{\sin 2x}{\cos x}\,dx.$$

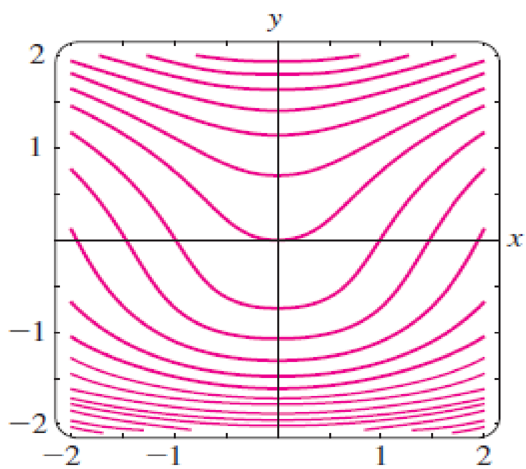

FIGURE 2.2.2 Level curves of $G(x, y) = e^{y} + ye^{-y} + e^{-y} + 2\cos x$

EXERCISES 2.2

In Problems 1–22 solve the given differential equation by separation of variables.

15. $\dfrac{dS}{dr} = kS$

16. $\dfrac{dQ}{dt} = k(Q - 70)$

17. $\dfrac{dP}{dt} = P - P^2$

18. $\dfrac{dN}{dt} + N = Nte^{t+2}$

19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

20. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

22. $(e^{x} + e^{-x})\dfrac{dy}{dx} = y^2$

In Problems 23–28 find an explicit solution of the given initial-value problem.

27. $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0, \quad y(0) = \dfrac{\sqrt{3}}{2}$

28. $(1 + x^4)\,dy + x(1 + 4y^2)\,dx = 0, \quad y(1) = 0$

In Problems 45–50 use a technique of integration or a substitution to find an explicit solution of the given differential equation or initial-value problem.

45. $\dfrac{dy}{dx} = \dfrac{1}{1 + \sin x}$

46. $\dfrac{dy}{dx} = \dfrac{\sin\sqrt{x}}{\sqrt{y}}$

47. $(\sqrt{x} + x)\dfrac{dy}{dx} = \sqrt{y} + y$

48. $\dfrac{dy}{dx} = y^{2/3} - y$

2.3 LINEAR EQUATIONS

DEFINITION 2.3.1 Linear Equation

A first-order di ferential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{1}$$

is said to be a linear equation in the variable y.

Standard Form By dividing both sides of (1) by the lead coefficient $a_1(x)$, we obtain a more useful form, the **standard form**, of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x). \tag{2}$$

SOLVING A LINEAR FIRST-ORDER EQUATION

(*i*) Remember to put a linear equation into the standard form (2).

(*ii*) From the standard form of the equation identify $P(x)$ and then find th integrating factor $e^{\int P(x)\,dx}$. No constant need be used in evaluating the indefinite integral $\int P(x)\,dx$.

(*iii*) Multiply the both sides of the standard form equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the product of the integrating factor $e^{\int P(x)\,dx}$ and y:

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = e^{\int P(x)\,dx}f(x).$$

(*iv*) Integrate both sides of the last equation and solve for y.

EXAMPLE 1 Solving a Linear Equation

Solve $\dfrac{dy}{dx} - 3y = 0$.

SOLUTION This linear equation can be solved by separation of variables. Alternatively, since the differential equation is already in standard form (2), we identify $P(x) = -3$, and so the integrating factor is $e^{\int(-3)\,dx} = e^{-3x}$. We then multiply the given equation by this factor and recognize that

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}\cdot 0 \qquad \text{is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 0.$$

EXAMPLE 2 Solving a Linear Equation

Solve $\dfrac{dy}{dx} - 3y = 6$.

SOLUTION This linear equation, like the one in Example 1, is already in standard form with $P(x) = -3$. Thus the integrating factor is again e^{-3x}. This time multiplying the given equation by this factor gives

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x} \qquad \text{and so} \qquad \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

EXAMPLE 3 General Solution

Solve $x\dfrac{dy}{dx} - 4y = x^6e^x$.

SOLUTION Dividing by x, the standard form of the given DE is

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x. \tag{6}$$

From this form we identify $P(x) = -4/x$ and $f(x) = x^5e^x$ and further observe that P and f are continuous on $(0, \infty)$. Hence the integrating factor is

$$e^{-4\int dx/x} = \underbrace{e^{-4\ln x}}_{\text{we can use } \ln x \text{ instead of } \ln|x| \text{ since } x > 0} = e^{\ln x^{-4}} = x^{-4}.$$

EXAMPLE 4 General Solution

Find the general solution of $(x^2 - 9)\dfrac{dy}{dx} + xy = 0$.

SOLUTION We write the differential equation in standard form

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0 \tag{7}$$

and identify $P(x) = x/(x^2 - 9)$. Although P is continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$, we shall solve the equation on the first and third intervals. On these intervals the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{\frac{1}{2}\int 2x\,dx/(x^2-9)} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2 - 9}.$$

After multiplying the standard form (7) by this factor, we get

$$\frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = 0.$$

EXAMPLE 5 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = x, \quad y(0) = 4.$

SOLUTION The equation is in standard form, and $P(x) = 1$ and $f(x) = x$ are continuous on $(-\infty, \infty)$. The integrating factor is $e^{\int dx} = e^x$, so integrating

$$\frac{d}{dx}[e^x y] = xe^x$$

gives $e^x y = xe^x - e^x + c$. Solving this last equation for y yields the general solution $y = x - 1 + ce^{-x}$. But from the initial condition we know that $y = 4$ when $x = 0$. Substituting these values into the general solution implies that $c = 5$. Hence the solution of the problem is

$$y = x - 1 + 5e^{-x}, \quad -\infty < x < \infty. \tag{8}$$

EXAMPLE 6 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = f(x), \quad y(0) = 0 \quad \text{where} \quad f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$

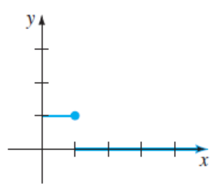

FIGURE 2.3.3 Discontinuous $f(x)$ in Example 6

SOLUTION The graph of the discontinuous function f is shown in Figure 2.3.3. We solve the DE for $y(x)$ first on the interval $[0, 1]$ and then on the interval $(1, \infty)$. For $0 \le x \le 1$ we have

$$\frac{dy}{dx} + y = 1 \qquad \text{or, equivalently,} \qquad \frac{d}{dx}[e^x y] = e^x.$$

Integrating this last equation and solving for y gives $y = 1 + c_1 e^{-x}$. Since $y(0) = 0$, we must have $c_1 = -1$, and therefore $y = 1 - e^{-x}$, $0 \le x \le 1$. Then for $x > 1$ the equation

$$\frac{dy}{dx} + y = 0$$

leads to $y = c_2 e^{-x}$. Hence we can write

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ c_2 e^{-x}, & x > 1. \end{cases}$$

By appealing to the definition of continuity at a point, it is possible to determine c_2 so that the foregoing function is continuous at $x = 1$. The requirement that $\lim_{x \to 1^+} y(x) = y(1)$ implies that $c_2 e^{-1} = 1 - e^{-1}$ or $c_2 = e - 1$. As seen in Figure 2.3.4, the function

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ (e - 1)e^{-x}, & x > 1 \end{cases} \tag{9}$$

is continuous on $(0, \infty)$.

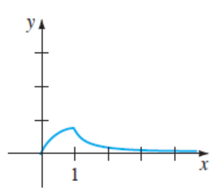

FIGURE 2.3.4 Graph of (9) in Example 6

EXERCISES 2.3

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–24 find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $\dfrac{dy}{dx} = 5y$

2. $\dfrac{dy}{dx} + 2y = 0$

3. $\dfrac{dy}{dx} + y = e^{3x}$

4. $3\dfrac{dy}{dx} + 12y = 4$

5. $y' + 3x^2y = x^2$

6. $y' + 2xy = x^3$

18. $\cos^2 x \sin x \dfrac{dy}{dx} + (\cos^3 x)y = 1$

19. $(x + 1)\dfrac{dy}{dx} + (x + 2)y = 2xe^{-x}$

20. $(x + 2)^2 \dfrac{dy}{dx} = 5 - 8y - 4xy$

21. $\dfrac{dr}{d\theta} + r \sec\theta = \cos\theta$

22. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

In Problems 37–40 proceed as in Example 6 to solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

37. $\dfrac{dy}{dx} + 2y = f(x),\ y(0) = 0,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

38. $\dfrac{dy}{dx} + y = f(x),\ y(0) = 1,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

39. $\dfrac{dy}{dx} + 2xy = f(x),\ y(0) = 2,$ where

$$f(x) = \begin{cases} x, & 0 \le x \quad 1 \\ 0, & x \ge 1 \end{cases}$$

2.4 EXACT EQUATIONS

INTRODUCTION Although the simple first-order equation

$$y\,dx + x\,dy = 0$$

is separable, we can solve the equation in an alternative manner by recognizing that the expression on the left-hand side of the equality is the differential of the function $f(x, y) = xy$; that is,

$$d(xy) = y\,dx + x\,dy.$$

Differential of a Function of Two Variables

If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{1}$$

DEFINITION 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order di ferential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

THEOREM 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a \quad x \quad b$, $c \quad y \quad d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

EXAMPLE 1 Solving an Exact DE

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so by Theorem 2.4.1 there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating

$$f(x, y) = x^2 y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2 y - y$, so the solution of the equation in implicit form is $x^2 y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$.

EXAMPLE 2 Solving an Exact DE

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

EXAMPLE 3 An Initial-Value Problem

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$

that $4(1) - \cos^2(0) = c$, and so $c = 3$. An implicit solution of the problem is then $y^2(1 - x^2) - \cos^2 x = 3$.

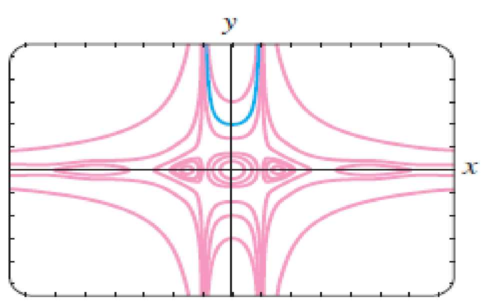

FIGURE 2.4.1 Solution curves of DE in Example 3

Integrating Factors

We summarize the results for the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{12}$$

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for (12) is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}. \tag{13}$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for (12) is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}. \tag{14}$$

EXAMPLE 4 A Nonexact DE Made Exact

The nonlinear first-order di ferential equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$, we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere, since

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

depends on x and y. However, (14) yields a quotient that depends only on y:

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}.$$

The integrating factor is then $e^{\int 3dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$. After we multiply the given DE by $\mu(y) = y^3$, the resulting equation is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0.$$

You should verify that the last equation is now exact as well as show, using the method of this section, that a family of solutions is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$.

EXERCISES 2.4

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$

2. $(2x + y)\,dx - (x + 6y)\,dy = 0$

3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$

4. $(\sin y - y \sin x)\,dx + (\cos x + x \cos y - y)\,dy = 0$

5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$

7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$

17. $(\tan x - \sin x \sin y)\,dx + \cos x \cos y\,dy = 0$

18. $(2y \sin x \cos x - y + 2y^2e^{xy^2})\,dx$
$$= (x - \sin^2 x - 4xye^{xy^2})\,dy$$

In Problems 21–26 solve the given initial-value problem.

21. $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$

22. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$

23. $(4y + 2t - 5)\,dt + (6y + 4t - 1)\,dy = 0, \quad y(-1) = 2$

In Problems 29 and 30 verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor $\mu(x, y)$ and verify that the new equation is exact. Solve.

29. $(-xy \sin x + 2y \cos x)\, dx + 2x \cos x\, dy = 0;$
$\mu(x, y) = xy$

30. $(x^2 + 2xy - y^2)\, dx + (y^2 + 2xy - x^2)\, dy = 0;$
$\mu(x, y) = (x + y)^{-2}$

2.5 SOLUTIONS BY SUBSTITUTIONS

Homogeneous Equations If a function f possesses the property $f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α, then f is said to be a **homogeneous function** of degree α. For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3, since

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

whereas $f(x, y) = x^3 + y^3 + 1$ is not homogeneous. A first-order DE in differential form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is said to be **homogeneous*** if both coefficient functions M and N are homogeneous functions of the *same* degree. In other words, (1) is homogeneous if

$$M(tx, ty) = t^{\alpha} M(x, y) \qquad \text{and} \qquad N(tx, ty) = t^{\alpha} N(x, y).$$

EXAMPLE 1 Solving a Homogeneous DE

Solve $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$.

SOLUTION Inspection of $M(x, y) = x^2 + y^2$ and $N(x, y) = x^2 - xy$ shows that these coefficients are homogeneous functions of degree 2. If we let $y = ux$, then $dy = u\,dx + x\,du$, so after substituting, the given equation becomes

$$
\begin{aligned}
(x^2 + u^2x^2)\,dx + (x^2 - ux^2)[u\,dx + x\,du] &= 0\\
x^2(1 + u)\,dx + x^3(1 - u)\,du &= 0\\
\frac{1 - u}{1 + u}\,du + \frac{dx}{x} &= 0\\
\left[-1 + \frac{2}{1 + u}\right] du + \frac{dx}{x} &= 0. \quad \leftarrow \text{long division}
\end{aligned}
$$

After integration the last line gives

$$
\begin{aligned}
-u + 2\ln|1 + u| + \ln|x| &= \ln|c|\\
-\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| &= \ln|c|. \quad \leftarrow \text{resubstituting } u = y/x
\end{aligned}
$$

Using the properties of logarithms, we can write the preceding solution as

$$
\ln\left|\frac{(x + y)^2}{cx}\right| = \frac{y}{x} \qquad \text{or} \qquad (x + y)^2 = cxe^{y/x}.
$$

Bernoulli's Equation The differential equation

$$
\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{4}
$$

where n is any real number, is called **Bernoulli's equation.** Note that for $n = 0$ and $n = 1$, equation (4) is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form (4) to a linear equation.

EXAMPLE 2 Solving a Bernoulli DE

Solve $x\dfrac{dy}{dx} + y = x^2y^2$.

SOLUTION We first rewrite the equation a

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

by dividing by x. With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$. We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \qquad \leftarrow \text{Chain Rule}$$

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor for this linear equation on, say, $(0, \infty)$ is

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

Integrating
$$\frac{d}{dx}[x^{-1}u] = -1$$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$. Since $u = y^{-1}$, we have $y = 1/u$, so a solution of the given equation is $y = 1/(-x^2 + cx)$.

EXERCISES 2.5

Answers to selected odd-numbered problems begin on page ANS-2.

Each DE in Problems 1–14 is homogeneous.

In Problems 1–10 solve the given differential equation by using an appropriate substitution.

1. $(x - y)\,dx + x\,dy = 0$
2. $(x + y)\,dx + x\,dy = 0$
3. $x\,dx + (y - 2x)\,dy = 0$
4. $y\,dx = 2(x + y)\,dy$
5. $(y^2 + yx)\,dx - x^2\,dy = 0$
6. $(y^2 + yx)\,dx + x^2\,dy = 0$
7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$
8. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$
9. $-y\,dx + \left(x + \sqrt{xy}\right)dy = 0$

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20 solve the given differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$
16. $\dfrac{dy}{dx} - y = e^xy^2$
17. $\dfrac{dy}{dx} = y(xy^3 - 1)$
18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$
19. $t^2\dfrac{dy}{dt} + y^2 = ty$
20. $3(1 + t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$

In Problems 21 and 22 solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$
22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$