# Series Solutions of Linear Equations

Up to this point in our study of differential equations we have primarily solved linear equations of order two (or higher) that have constant coefficients. The only

In this chapter we shall study two infinite-series methods for finding solutio of homogeneous linear second-order DEs $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, where the variable coefficients $a_2(x)$, $a_1(x)$, and $a_0(x)$ are, for the most part, simple polynomial functions.

**Power Series** Recall from calculus that **power series** in $x - a$ is an infinit series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

- **Convergence** A power series is **convergent** at a specified value of $x$ if its sequence of partial sums $\{S_N(x)\}$ converges, that is, $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n(x-a)^n$ exists. If the limit does not exist at $x$, then the series is said to be **divergent**.

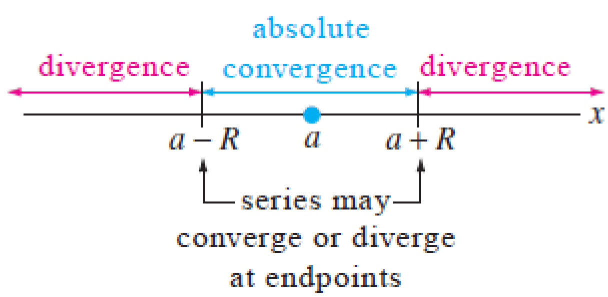


**FIGURE 6.1.1** Absolute convergence within the interval of convergence and divergence outside of this interval

- **A Power Series Defines a Functio** A power series defines a function that is, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. If the radius of convergence is $R > 0$ or $R = \infty$, then $f$ is continuous, differentiable, and integrable on the intervals $(a - R, a + R)$ or $(-\infty, \infty)$, respectively. Moreover, $f'(x)$ and $\int f(x)\,dx$ can be found by term-by-term differentiation and integration. Convergence at an endpoint may be either lost by differentiation or gained through integration. If

- **Analytic at a Point** A function $f$ is said to be **analytic at a point $a$** if it can be represented by a power series in $x - a$ with either a positive or an infinite radius of conve gence. In calculus it is seen that infinitel

You might remember some of the following Maclaurin series representations.

| Maclaurin Series | Interval of Convergence | |
|---|---|---|
| $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ | $(-\infty, \infty)$ | |
| $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ | $(-\infty, \infty)$ | |
| $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ | $(-\infty, \infty)$ | |
| $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ | $[-1, 1]$ | (2) |
| $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$ | $(-\infty, \infty)$ | |
| $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$ | $(-\infty, \infty)$ | |
| $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ | $(-1, 1]$ | |
| $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ | $(-1, 1)$ | |

# 6.2 SOLUTIONS ABOUT ORDINARY POINTS

## DEFINITION 6.2.1 Ordinary and Singular Points

A point $x = x_0$ is said to be an **ordinary point** of the differential of the differential equation (1) if both coefficients $P(x)$ and $Q(x)$ in the standard form (2) are analytic at $x_0$. A point that is *not* an ordinary point of (1) is said to be a **singular point** of the DE.

## EXAMPLE 2 Singular Points

**(a)** The differential equation

$$y'' + xy' + (\ln x)y = 0$$

is already in standard form. The coefficient functions ar

$$P(x) = x \quad \text{and} \quad Q(x) = \ln x.$$

Now $P(x) = x$ is analytic at every real number, and $Q(x) = \ln x$ is analytic at every *positive* real number. However, since $Q(x) = \ln x$ is discontinuous at $x = 0$ it cannot be represented by a power series in $x$, that is, a power series centered at 0. We conclude that $x = 0$ is a singular point of the DE.

**(b)** By putting $xy'' + y' + xy = 0$ in the standard form

$$y'' + \frac{1}{x}y' + y = 0,$$

we see that $P(x) = 1/x$ fails to be analytic at $x = 0$. Hence $x = 0$ is a singular point of the equation.

## EXAMPLE 3 Ordinary and Singular Points

**(a)** The only singular points of the differential equation

$$(x^2 - 1)y'' + 2xy' + 6y = 0$$

are the solutions of $x^2 - 1 = 0$ or $x = \pm 1$. All other values of $x$ are ordinary points.

**(b)** Inspection of the Cauchy-Euler

$$\downarrow a_2(x) = x^2 = 0 \text{ at } x = 0$$

$$x^2y'' + y = 0$$

shows that it has a singular point at $x = 0$. All other values of $x$ are ordinary points.

**(c)** Singular points need not be real numbers. The equation

$$(x^2 + 1)y'' + xy' - y = 0$$

has singular points at the solutions of $x^2 + 1 = 0$—namely, $x = \quad i$. All other values of $x$, real or complex, are ordinary points.

## THEOREM 6.2.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (1), we can always find two linearly independent solutions in the form of a power series centered at $x_0$, that is,

$$y = \sum_{n=0}^{\infty} c_n(x - x_0)^n.$$

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where $R$ is the distance from $x_0$ to the closest singular point.

## EXAMPLE 6 Power Series Solution

Solve $(x^2 + 1)y'' + xy' - y = 0$.

**SOLUTION** As we have already seen on page 240, the given differential equation has singular points at $x = \pm i$, and so a power series solution centered at 0 will converge at least for $|x| < 1$, where 1 is the distance in the complex plane from 0 to either $i$ or $-i$. The assumption $y = \sum_{n=0}^{\infty} c_n x^n$ and its first two derivatives lead t

$$(x^2 + 1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2x^0 - c_0x^0 + 6c_3x + c_1x - c_1x + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n}$$

$$+ \underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=2}^{\infty} c_n x^n}_{k=n}$$

$$= 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty} [k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k$$

$$= 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty} [(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0.$$

From this identity we conclude that $2c_2 - c_0 = 0$, $6c_3 = 0$, and

$$(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0.$$

Thus

$$c_2 = \frac{1}{2}c_0$$

$$c_3 = 0$$

$$c_{k+2} = \frac{1-k}{k+2}c_k, \qquad k = 2, 3, 4, \ldots .$$

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9 + c_{10}x^{10} + \cdots$$

$$= c_0\left[1 + \frac{1}{2}x^2 - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^6 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^8 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^{10} - \cdots\right] + c_1x$$

$$= c_0y_1(x) + c_1y_2(x).$$

**Assignment: Find two power series solutions of the differential equation**

$$(x^2 + k)y'' + kxy' - y = 0 \qquad \text{at } x = 0.$$

**where $k$ is the last 6th digit of your id# .**

## 6.3 SOLUTIONS ABOUT SINGULAR POINTS

**DEFINITION 6.3.1 Regular and Irregular Singular Points**

A singular point $x = x_0$ is said to be a **regular singular point** of the differential equation (1) if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are both analytic at $x_0$. A singular point that is not regular is said to be an **irregular singular point** of the equation.

**EXAMPLE 1 Classification of Singula Points**

It should be clear that $x = 2$ and $x = -2$ are singular points of

$$(x^2 - 4)^2y'' + 3(x - 2)y' + 5y = 0.$$

After dividing the equation by $(x^2 - 4)^2 = (x - 2)^2(x + 2)^2$ and reducing the coefficients to lowest terms, we find th

$$P(x) = \frac{3}{(x-2)(x+2)^2} \qquad \text{and} \qquad Q(x) = \frac{5}{(x-2)^2(x+2)^2}.$$

### THEOREM 6.3.1 Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation (1), then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \tag{4}$$

where the number $r$ is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

### EXAMPLE 2 Two Series Solutions

Because $x = 0$ is a regular singular point of the differential equation

$$3xy'' + y' - y = 0, \tag{5}$$

we try to find a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Now

$$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1} \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2},$$

$$\begin{aligned}
3xy'' + y' - y &= 3\sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty} (n + r)(3n + 3r - 2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= x^r\left[r(3r - 2)c_0 x^{-1} + \underbrace{\sum_{n=1}^{\infty} (n + r)(3n + 3r - 2)c_n x^{n-1}}_{k = n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k = n}\right] \\
&= x^r\left[r(3r - 2)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k + r + 1)(3k + 3r + 1)c_{k+1} - c_k]x^k\right] = 0,
\end{aligned}$$

which implies that $\qquad r(3r - 2)c_0 = 0$

and $\qquad (k + r + 1)(3k + 3r + 1)c_{k+1} - c_k = 0, \qquad k = 0, 1, 2, \ldots .$

Because nothing is gained by taking $c_0 = 0$, we must then have

$$r(3r - 2) = 0 \tag{6}$$

and

$$c_{k+1} = \frac{c_k}{(k + r + 1)(3k + 3r + 1)}, \qquad k = 0, 1, 2, \ldots . \tag{7}$$

When substituted in (7), the two values of $r$ that satisfy the quadratic equation (6), $r_1 = \frac{2}{3}$ and $r_2 = 0$, give two different recurrence relations:

$$r_1 = \tfrac{2}{3}, \qquad c_{k+1} = \frac{c_k}{(3k+5)(k+1)}, \qquad k = 0, 1, 2, \ldots \tag{8}$$

$$r_2 = 0, \qquad c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \qquad k = 0, 1, 2, \ldots . \tag{9}$$

solutions are

$$y_1(x) = x^{2/3}\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,5 \cdot 8 \cdot 11 \cdots (3n+2)} x^n\right] \tag{10}$$

$$y_2(x) = x^0\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,1 \cdot 4 \cdot 7 \cdots (3n-2)} x^n\right]. \tag{11}$$

## EXAMPLE 3 Two Series Solutions

Solve $2xy'' + (1 + x)y' + y = 0$.

**Three Cases** For the sake of discussion let us again suppose that $x = 0$ is a regular singular point of equation (1) and that the indicial roots $r_1$ and $r_2$ of the singularity are real. When using the method of Frobenius, we distinguish three cases corresponding to the nature of the indicial roots $r_1$ and $r_2$. In the first two cases the symbol $r_1$ denotes the largest of two distinct roots, that is, $r_1 > r_2$. In the last case $r_1 = r_2$.

**Case I:** If $r_1$ and $r_2$ are distinct and the difference $r_1 - r_2$ is not a positive integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0, \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0.$$

This is the case illustrated in Examples 2 and 3.

Next we assume that the difference of the roots is $N$, where $N$ is a positive integer. In this case the second solution *may* contain a logarithm.

**Case II:** If $r_1$ and $r_2$ are distinct and the difference $r_1 - r_2$ is a positive integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0, \tag{19}$$

$$y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad b_0 \neq 0, \tag{20}$$

where $C$ is a constant that could be zero.

**Assignment: Use the method of Frobenius to obtain a series solution of the differential equations**

**i)** $x^2y'' + xy' + (x^2 - k^2)y = 0$

**ii)** $(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$

**about the point $x = 0$. Where $k$ is the last 5th digit of your id# .**

**Show at least five significant terms of each series.**