Second Order Linear ODEs

- ***Linear differential equation of order TWO :***

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

Example.

$$xy'' + 2y' + xy = 4x \qquad \text{[Non-Homogeneous]}$$

$$y'' - 4y = 12x \qquad \text{[Non-Homogeneous]}$$

$$xy'' + y' + xy = 0 \qquad \text{[Homogeneous]}$$

$$y'' + 25y = e^{-x}\cos x \qquad \text{[Non-Homogeneous]}$$

$$2y'' - 5y' - 3y = 0 \qquad \text{[Homogeneous]}$$

Homogeneous Linear ODE of Second Order :

$$y'' + P(x)y' + Q(x)y = 0.$$

Homogeneous Second Order Linear ODEs

Superposition Principle of Solutions

If y_1 and y_2 are solutions for the homogeneous linear second order ODEs on an interval I, then any linear combination of two solutions is also a solution of the ODEs on that interval I.

For example, $y = \sin x$ and $y = \cos x$ are solutions of the homogeneous linear ODE,

$$y'' + y = 0$$

$\therefore \frac{d^2}{dx^2}(\sin x) + \sin x = -\sin x + \sin x = 0$ and $\frac{d^2}{dx^2}(\cos x) + \cos x = -\cos x + \cos x = 0.$

According to the superposition principle, any linear combination of $y_1 = \sin x$ and $y_2 = \cos x$ such as $y = c_1 y_1 + c_2 y_2 = c_1 \sin x + c_2 \cos x$, where c_1 and c_2 are arbitrary constants, is also a solution, called **general solution** of the above ODE. Since,

$$\frac{d^2 y}{dx^2} + y = \frac{d^2}{dx^2}(c_1 \sin x + c_2 \cos x) + (c_1 \sin x + c_2 \cos x)$$

$$= \left[\frac{d^2}{dx^2}(c_1 \sin x) + c_1 \sin x\right] + \left[\frac{d^2}{dx^2}(c_2 \cos x) + c_2 \cos x\right] = 0 + 0 = 0.$$

Homogeneous Second Order Linear ODEs

Differential Operators (D)

In differential calculus, the symbol D is often used to denote the differentiation $\frac{d}{dx}$, i.e.,

$$Dy = \frac{dy}{dx} = y', \qquad D^2y = \frac{d^2y}{dx^2} = y'', \qquad D^3y = \frac{d^3y}{dx^3} = y''', \dots \dots \dots, D^ny = \frac{d^ny}{dx^n} = y^n$$

The symbol D is called a differential operator because it transforms a differentiable function into another function. For example,

$$D(\sin 4x) = \frac{d}{dx}(\sin 4x) = 4\cos 4x\,, D(5x^3 - 6x^2) = 15x^2 - 12x$$

$$D^2(e^{mx}) = D[D(e^{mx})] = D[me^{mx}] = m^2e^{mx}$$

$$\boxed{D^k(e^{mx}) = m^ke^{mx}}$$

$$D^2(\sin mx) = D[D(\sin mx)] = D[m\cos mx] = -m^2\sin mx$$

$$D^2(\cos mx) = D[D(\cos mx)] = D[-m\sin mx] = -m^2\cos mx$$

Polynomial expression of D:

$$\frac{dy}{dx} + 6y = Dy + 6y = (D+6)y,$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = D^2y - Dy - 6y = (D^2 - D - 6)y = (D-3)(D+2)y$$

$$\frac{d^2y}{dx^2} - 4y = (D^2 - 4)y = (D+2)(D-2)y$$

Homogeneous Second Order Linear ODEs

Solutions when $P(x)$ and $Q(x)$ are constants

The second-order homogeneous linear ODEs whose coefficient are constants can be written as,

$$ay'' + by' + cy = 0 \Rightarrow (aD^2 + bD + c)y = 0.$$

To find the solution of the above ODE, let us consider a first order linear homogeneous ODE with constant coefficient b and c,

$$by' + cy = 0 \Rightarrow y' = -\frac{c}{b}y \Rightarrow y' = my \Rightarrow y = ce^{mx} \quad \left[\text{where } m = -\frac{c}{b}\right]$$

For example, $2y' + 5y = 0 \Rightarrow y' = -\frac{5}{2}y \Rightarrow \int \frac{1}{y}dy = -\frac{5}{2}\int xdx \Rightarrow y = ce^{-\frac{5}{2}x}$

Therefore, $y = e^{mx}$ can be considered as a **trial solution** for the first order homogeneous linear ODEs and **the general solution** can be written as $y = ce^{mx}$.

Alternatively, by considering $y = e^{mx}$ as a trial solution of the above example, we get

$$2y' + 5y = 0 \Rightarrow (2D + 5)y = 0 \Rightarrow (2D + 5)e^{mx} = 0 \Rightarrow (2m + 5)e^{mx} = 0$$

Now, for non-trivial solution of the above ODE, we must consider, $2m + 5 = 0 \Rightarrow m = -\frac{5}{2}$.

Therefore, the general solution of the above example yields, $y = ce^{-\frac{5}{2}x}$.

Homogeneous Second Order Linear ODEs

Solutions when $P(x)$ and $Q(x)$ are constants

Now substituting the trial solution $y = e^{mx}$ into the second order homogeneous ODE, we get

$$ay'' + by' + cy = 0 \Rightarrow (aD^2 + bD + c)y = 0 \Rightarrow (aD^2 + bD + c)e^{mx} = 0$$

$$\Rightarrow (am^2 + bm + c)e^{mx} = 0 \Rightarrow am^2 + bm + c = 0 \qquad [\because e^{mx} \neq 0]$$

Here, $am^2 + bm + c = 0$ is called the auxiliary equation of the differential equation.

Two roots of the auxiliary equation yields,

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Depending on the nature of m_1 and m_2, the general solution of the ODE can be obtained as:

Case I	If m_1 and m_2 are distinct real roots	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Case II	If m_1 and m_2 are equal real roots	$y = (c_1 + c_2 x)e^{m_1 x}$
Case III	If m_1 and m_2 are conjugate complex roots i.e. $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$	$y = (c_1 \cos \beta x + c_2 \sin \beta x)e^{\alpha x}$

Homogeneous Second Order Linear ODEs

Linear equations with constant coefficients.

Example. Find the general solution of

(a) $2y'' - 5y' - 3y = 0$ (b) $y'' - 10y' + 25y = 0$ (c) $y'' + 4y' + 8y = 0$

Solution. **(a)** The auxiliary equation for the ODE is,

$$2m^2 - 5m - 3 = 0 \Rightarrow 2m^2 - 6m + m - 3 = 0 \Rightarrow (m-3)(2m+1) = 0 \Rightarrow m = -\frac{1}{2}, 3$$

Thus, the general solution is, $y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$.

(b) The auxiliary equation for the ODE is,

$$m^2 - 10m + 25 = 0 \Rightarrow (m-5)^2 = 0 \Rightarrow m = 5, 5$$

Thus, the general solution is, $y = (c_1 + c_2 x)e^{5x}$.

(c) The auxiliary equation for the ODE is,

$$m^2 + 4m + 8 = 0 \Rightarrow (m+2)^2 + 4 = 0 \Rightarrow (m+2)^2 = -4 \Rightarrow m = -2 \pm 2i$$

Thus, the general solution is, $y = (c_1 \cos 2x + c_2 \sin 2x)\mathrm{e}^{-2x}$.

Homogeneous Second Order Linear ODEs

Linear equations with constant coefficients.

Example. Find the general solution of

$$4y'' + 4y' + 17y = 0, \qquad y(0) = -1, y'(0) = 2$$

Solution. The auxiliary equation for the ODE is,

$$4m^2 + 4m + 17 = 0 \Rightarrow (2m+1)^2 + 4 = 0 \Rightarrow m = -\frac{1}{2} \pm 2i$$

Thus, the general solution is, $y(x) = (c_1 \cos 2x + c_2 \sin 2x)\mathrm{e}^{-0.5x}$.

Where, $y'(x) = (-2c_1 \sin 2x + 2c_2 \cos 2x)\mathrm{e}^{-0.5x} - 0.5(c_1 \cos 2x + c_2 \sin 2x)\mathrm{e}^{-0.5x}$

Given that, $y(0) = -1 \Rightarrow c_1 = -1$ and $y'(0) = 2 \Rightarrow 2c_2 - 0.5c_1 = 2 \Rightarrow c_2 = \frac{3}{4}$.

Thus the particular solution solution is,

$$y(x) = \left(-\cos 2x + \tfrac{3}{4}\sin 2x\right)\mathrm{e}^{-0.5x}.$$

Homogeneous Second Order Linear ODEs

Exercises 4.3

H.W. from the text book

Find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$

2. $y'' - 36y = 0$

3. $y'' - y' - 6y = 0$

4. $y'' - 3y' + 2y = 0$

5. $y'' + 8y' + 16y = 0$

6. $y'' - 10y' + 25y = 0$

7. $12y'' - 5y' - 2y = 0$

8. $y'' + 4y' - y = 0$

9. $y'' + 9y = 0$

10. $3y'' + y = 0$

11. $y'' - 4y' + 5y = 0$

12. $2y'' + 2y' + y = 0$

13. $3y'' + 2y' + y = 0$

14. $2y'' - 3y' + 4y = 0$

Solve the given initial-value problem.

29. $y'' + 16y = 0, \quad y(0) = 2, y'(0) = -2$

30. $\dfrac{d^2y}{d\theta^2} + y = 0, \quad y(\pi/3) = 0, y'(\pi/3) = 2$

31. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$

32. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

33. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$

34. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$