

# Calculus 3: Homework 2

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## Problem 1

Give an appropriate positive constant  $c$  such that  $f(n) \leq c \cdot g(n)$  for all  $n > 1$ .

1.  $f(n) = n^2 + n + 1, g(n) = 2n^3$
2.  $f(n) = n\sqrt{n} + n^2, g(n) = n^2$
3.  $f(n) = n^2 - n + 1, g(n) = n^2/2$

### Solution

We solve each solution algebraically to determine a possible constant  $c$ .

#### Part One

$$\begin{aligned} n^2 + n + 1 &= \\ &\leq n^2 + n^2 + n^2 \\ &= 3n^2 \\ &\leq c \cdot 2n^3 \end{aligned}$$

Thus a valid  $c$  could be when  $c = 2$ .

#### Part Two

$$\begin{aligned} n^2 + n\sqrt{n} &= \\ &= n^2 + n^{3/2} \\ &\leq n^2 + n^{4/2} \\ &= n^2 + n^2 \\ &= 2n^2 \\ &\leq c \cdot n^2 \end{aligned}$$

Thus a valid  $c$  is  $c = 2$ .

#### Part Three

$$\begin{aligned} n^2 - n + 1 &= \\ &\leq n^2 \\ &\leq c \cdot n^2/2 \end{aligned}$$

Thus a valid  $c$  is  $c = 2$ .

## Problem 2

Let  $\Sigma = \{0, 1\}$ . Construct a DFA  $A$  that recognizes the language that consists of all binary numbers that can be divided by 5.

Let the state  $q_k$  indicate the remainder of  $k$  divided by 5. For example, the remainder of 2 would correlate to state  $q_2$  because  $7 \bmod 5 = 2$ .

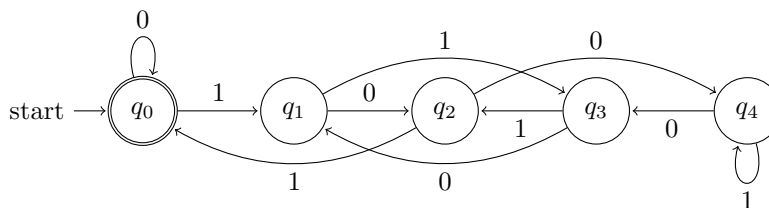


Figure 1: DFA,  $A$ , this is really beautiful, ya know?

### Justification

Take a given binary number,  $x$ . Since there are only two inputs to our state machine,  $x$  can either become  $x0$  or  $x1$ . When a 0 comes into the state machine, it is the same as taking the binary number and multiplying it by two. When a 1 comes into the machine, it is the same as multiplying by two and adding one.

Using this knowledge, we can construct a transition table that tell us where to go:

	$x \bmod 5 = 0$	$x \bmod 5 = 1$	$x \bmod 5 = 2$	$x \bmod 5 = 3$	$x \bmod 5 = 4$
$x0$	0	2	4	1	3
$x1$	1	3	0	2	4

Therefore on state  $q_0$  or ( $x \bmod 5 = 0$ ), a transition line should go to state  $q_0$  for the input 0 and a line should go to state  $q_1$  for input 1. Continuing this gives us the Figure 1.

## Problem 3

Write part of **Quick-Sort**( $list, start, end$ )

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1: function QUICK-SORT( $list, start, end$ )
2:   if  $start \geq end$  then
3:     return
4:   end if
5:    $mid \leftarrow \text{PARTITION}(list, start, end)$ 
6:   QUICK-SORT( $list, start, mid - 1$ )
7:   QUICK-SORT( $list, mid + 1, end$ )
8: end function

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Algorithm 1: Start of QuickSort

## Problem 4

Suppose we would like to fit a straight line through the origin, i.e.,  $Y_i = \beta_1 x_i + e_i$  with  $i = 1, \dots, n$ ,  $E[e_i] = 0$ , and  $\text{Var}[e_i] = \sigma_e^2$  and  $\text{Cov}[e_i, e_j] = 0, \forall i \neq j$ .

### Part A

Find the least squares estimator for  $\hat{\beta}_1$  for the slope  $\beta_1$ .

### Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^n (Y_i - \hat{\beta}_1 x_i)^2 \end{aligned}$$

By taking the partial derivative in respect to  $\hat{\beta}_1$ , we get:

$$\frac{\partial}{\partial \hat{\beta}_1} (RSS) = -2 \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^n x_i Y_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^n x_i Y_i - \hat{\beta}_1 \sum_{i=1}^n x_i^2 \end{aligned}$$

Solving for  $\hat{\beta}_1$  gives the final estimator for  $\beta_1$ :

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

**Part B**

Calculate the bias and the variance for the estimated slope  $\hat{\beta}_1$ .

**Solution**

For the bias, we need to calculate the expected value  $E[\hat{\beta}_1]$ :

$$\begin{aligned} E[\hat{\beta}_1] &= E\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i E[Y_i]}{\sum x_i^2} \\ &= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\ &= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \end{aligned}$$

Thus since our estimator's expected value is  $\beta_1$ , we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned} \text{Var}[\hat{\beta}_1] &= \text{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \sigma^2 \\ &= \frac{\sigma^2}{\sum x_i^2} \end{aligned}$$

## Problem 5

Prove a polynomial of degree  $k$ ,  $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$  is a member of  $\Theta(n^k)$  where  $a_k \dots a_0$  are nonnegative constants.

*Proof.* To prove that  $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ , we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are,  $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$  even if  $c_1 = 1$  and  $n_0 = 1$ . This is because  $n^k \leq c_1 \cdot a_k n^k$  for any nonnegative constant,  $c_1$  and  $a_k$ .

Taking the second inequality, we prove it in the following way. By summation,  $\sum_{i=0}^k a_i$  will give us a new constant,  $A$ . By taking this value of  $A$ , we can then do the following:

$$\begin{aligned} a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0 &= \\ &\leq (a_k + a_{k-1} \dots a_1 + a_0) \cdot n^k \\ &= A \cdot n^k \\ &\leq c_2 \cdot n^k \end{aligned}$$

where  $n_0 = 1$  and  $c_2 = A$ .  $c_2$  is just a constant. Thus the proof is complete.  $\square$

**Problem 18**

Evaluate  $\sum_{k=1}^5 k^2$  and  $\sum_{k=1}^5 (k-1)^2$ .

**Problem 19**

Find the derivative of  $f(x) = x^4 + 3x^2 - 2$

**Problem 6**

Evaluate the integrals  $\int_0^1 (1-x^2)dx$  and  $\int_1^\infty \frac{1}{x^2} dx$ .