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Servo control and Stabilization of Linear Inverted Pendulum on a Cart using LQG

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Abstract- **In this work, non-linear Mathematical model of Inverted Pendulum(IP)-cart system is obtained using Lagrange's equation. The linearization of the system about the vertical position is achieved by Tayler's series approximation. Linear Quadratic Gaussian(LQG) control strategy is used for solving servo problem and stabilization of angle of pendulum rod about vertically straight position. Reference signal used for tracking problem is, square wave. White Gaussian noises are injected as process/plant noise and sensor/measurement noise. Simulated results are obtained with and without noises in MATLAB. The performance of LQG is based on time response specification and noise rejection level.**

Keywords- ***Inverted Pendulum(IP)-cart system, LQG, Servo problem, Stabilization***.

I. Introduction

Inverted pendulum(IP) on a cart is an open loop unstable system with higher order non-linear dynamics. The modelling and controlling of IP system give solution to numerous real-time challenges which make it a fascinating field of research in control system engineering [1][2]. Pendulum rod angle stabilization about vertical position and cart position tracking a reference signals are two pivotal problems faced by control engineers [3][4].

In literature, many control strategies have been proposed for stabilization about upright position and tracking problem. Using conventional PID control there are several endeavour to control IP on a cart system [5]-[8] . Fractional order PID controller was used to stabilize the IP system but outcomes were not satisfactory [4]. Double PID was used for obtaining desired results of IP-cart system [9]. In [10], modified particle swarm optimization (PSO) techniques was used to obtain the gain of PID controller for stabilizing IP-cart system. In [11] [12], attempt was made to stabilize IP system by tuning PID controller using genetic algorithm(GA). LQR strategy was used for attaining desired results of inIP system [14]. In [13]; researcher tried to obtain satisfactory performance of IP-cart with disturbances using Pole placement and LQR control strategy. Response of IP on a cart system obtained and compared using Ant colony based optimized LQR and PID controller [15]. Robust control applied to Inverted pendulum with uncertain disturbances [16].

High performance control of electro-pneumatic actuator by (Linear Quadratic Gaussian) LQG, where the weighting matrices were guessed [17]. LQR, LQG and integral LQG controller were used and responses are compared for frequency control of inter connected smart grid [18]. A variable gain LQG controller was used to improve dynamic performance of double inverted pendulum [19]. In [20], a hybrid system controller neural-LGQ was utilized to acquire stabilization of inverted pendulum system but tracking problem faced Peak overshoot problem. LQG optimal controller was employed to stabilize rod angle of inverted pendulum about the upright position and a Kalman-Bucy estimator is employed to estimate the unmeasurable states [21]. Extended Kalman filter algorithm is used for weight update of neural network controller based on PID (NNPID) for controlling dynamic system like inverted pendulum [22] [23]. In [24], LGQ is used for tight set point control of steam temperature in power plant. LQG provided better results than optimized PID during transients.

In this paper, the performance like stabilization and reference signal tracking of inverted pendulum on a cart system attempted using linear quadratic gaussian(LQG) control strategy. System time response such as Maximum peak overshoot(M_p), rise time(T_r), maximum angle of pendulum (θ_{max} (rad)), steady state error(e_{ss}) are obtained with system injected process noise and sensor noise separately.

Six sections are ordered in this paper. Section I structures introduction and literature review. IP-cart System description, derivation of system dynamics, linearization about vertical position (equilibrium position) and then state space model was derived in section II. LQG control strategy is conferred in section III. Section IV contains MATLAB simulation of IP-cart system with process noise and sensor noise. Results and discussion are offered in section V. At the end, final remarks are offered in section VI.

II. Inverted Pendulum

In Fig.1; IP-cart system, where the cart is restricted to move horizontally along X-axis. One end is permitted to oscillate vertically in X-Y plane while the other end is hinged at pivot point of the cart. When the pendulum rod is at vertically straight to the cart it is called IP-cart system. The upright equilibrium point is unstable for the system therefore it needs a perpetual force supplied by motor to balance the rod at vertical straight position on the cart.

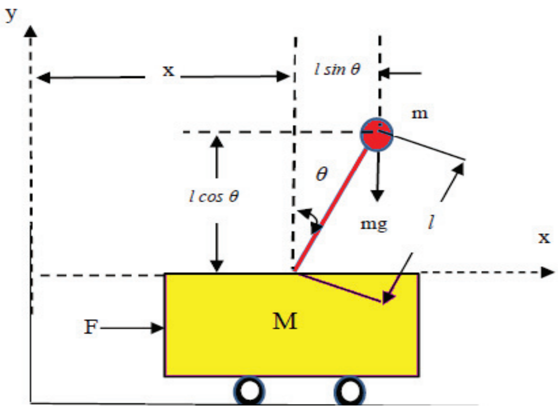

Fig. 1 The inverted pendulumsystem[15]

Non-linear Mathematical modelling of IP-cart system is done with few assumptions made as follows[13],

- The coefficient of friction between pendulum rod and cart is zero.
- Mass of the pendulum rod is neglected.

Kinetic energy(KE) of IP-cart system (1)

$$KE = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2l\dot{\theta}\dot{x}\cos\theta + \dot{x}^2 l^2\right] \quad (1)$$

Where,

M is mass of cart in Kg.

m is mass of pendulum in Kg.

l is length of rod of pendulum in meter.

θ is the angle subtended by vertical axis and the rod of pendulum.

$g$is the Gravity of theEarth.

Potential energy(T) of the system (2)

$$T = -mgl\cos\theta \quad (2)$$

Using Lagrange's equation, the non-linear mathematical model is obtained in (3) and (4)

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F - \mu x \quad (3)$$

$$ml\ddot{x}\cos\theta - ml^2\ddot{\theta} + mgl\sin\theta = 0 \quad (4)$$

Where,

F external force given to the system to stabilize the angle of rod.

μ is the coefficient of friction between cart and the surface.

Equation (3) & (4) are linearized using Tayler's series approximation about the vertical straight position i.e. $\theta = 0$.

Following assumptions are made

$$\cos\theta \cong 1 \quad \sin\theta \cong \theta\dot{\theta}^2 = \dot{\phi}^2 \cong 0 \quad (5)$$

Substituting values of (5) in (3) & (4) and rearranging them,

$$\ddot{x} = \frac{F - \mu\dot{x} + mg\theta}{M} \quad (6)$$

$$\ddot{\theta} = \frac{-F + \mu\dot{x} - g(m+M)\theta}{Ml} \quad (7)$$

Let state variable be

$$x_1 = \theta x_2 = \dot{\theta} x_3 = x x_4 = \dot{x} \quad (8)$$

Suppose output are

$$y_1 = \theta = x_1 y_2 = x = x_3 \quad (9)$$

State space representation is shown in (10)

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (10)$$

$$y(t) = Cx(t) + Du(t) \quad (11)$$

Where,

A is state transition matrix.

$x(t)$ is state vector.

B is control matrix.

$u(t)$ is input vector.

$y(t)$ is output vector.

C is output matrix.

D is feed through matrix.

Using (6) -(11) the state space matrices of IP-cart system are obtained below

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ {-g(m+M)}/{Ml} & 0 & 0 & {\mu}/{Ml} \\ 0 & 0 & 0 & 1 \\ {mg}/{M} & 0 & 0 & {-\mu}/{M} \end{bmatrix} \quad (12)$$

$$B = \begin{bmatrix} 0 \\ -1/_{Ml} \\ 0 \\ 1/_{M} \end{bmatrix} \tag{13}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14}$$

TABLE I.PARAMETERSOF THE IP SYSTEM [25]

Parameter	Value	Unit
Mass of the Cart 'M'	2.4	kg
Mass of the Pendulum 'm'	0.23	kg
Lengthof therod 'l'	0.36	m
Friction Coefficientbetween the Cart& base 'μ' (assumed)	0.1	$Nm^{-1}s^{-1}$
Force of Gravity on system 'g'	9.81	m/s^2
Length of the Track 'D'	±0.5	m
Applied force onthe cart'F'	-	N
Position of the cart 'x'	-	m
Angle of Pendulum 'θ'	-	rad

Substituting Table-I parameters in (12) -(14) state space matrix becomes

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -29.86 & 0 & 0 & 0.1157 \\ 0 & 0 & 0 & 1 \\ 0.94 & 0 & 0 & -0.0417 \end{bmatrix} \tag{15}$$

$$B = \begin{bmatrix} 0 \\ -1.157 \\ 0 \\ 0.4167 \end{bmatrix} \tag{16}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

III. LINEAR QUADRATIC GAUSSIAN

It is one of the fundamental optimal control strategy. It is the combination of Linear Quadratic Regulator(LQR) and Kalman filter which is Linear Quadratic Estimator(LQE). These two can be autonomously computed and designed which is ensured by Separation principle [24]. LQG useful in dealing with uncertain system corrupted by additive white Gaussian noise and incomplete states (i.e. not all the states are accessible for feedback).

It is presumed that stochastic disturbance (white Gaussian noise) are added to the system dynamics and output measurement of IP-cart system.

System dynamics becomes

$$\dot{x}(t) = Ax(t) + Bu(t) + W(t) \tag{18}$$

Where,

$W(t)$ is process/plant noise.

Mathematical properties of $W(t)$ are shown below

$$E(W) = 0 \tag{19}$$

$$E(W(t)W(\zeta)^T) = Q\delta(t - \zeta) \tag{20}$$

$$W(t) = W(t)^T \geq 0 \tag{21}$$

Where,

E() is expectation operator(mean or average value).

$\delta(\,)$ is Dirac delta function.

Q is process noise covariance matrix.

Measurement equation after injection of sensor noise to the IP-cart system becomes (22)

$$y(t) = Cx(t) + V(t) \tag{22}$$

Where,

$V(t)$ is sensor noise.

Mathematical properties of $V(t)$ are shown below,

$$E(V) = 0 \tag{23}$$

$$E(V(t)V(\zeta)^T) = R\delta(t - \zeta) \tag{24}$$

$$V(t) = V(t)^T > 0 \tag{25}$$

Where,

R is sensor noise covariance matrix.

Measurement noise vector and process noise vector are not correlated because they are white Gaussian noise.

$$E(V(t)W(\zeta)^T) = N\delta(t - \zeta) = 0 \tag{26}$$

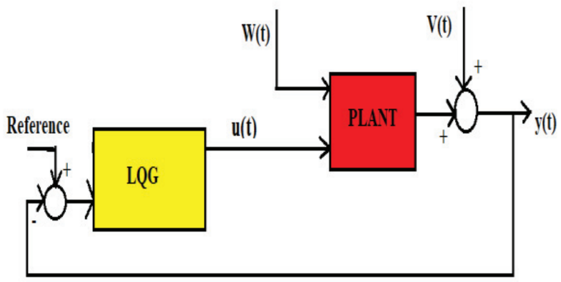

Fig. 2 Block diagram of LQG Servo control

Cost function which to be minimised of LQG Servo control shown below,

$$J = E\left\{\lim_{t\to\infty} \int_0^t \left([x\ \ u]Q_{xu}\,[x^T u^T] + x_i\, Q_i {x_i}^T\right) dt\right\} \tag{27}$$

Where,

J is cost function

Q_{xu} and Q_i are weighting matrices.

$x_i\ (= \int[y(t) - r(t)]dt)$ tracking error integral.

IV. Simulation

Weighting matrices Q_{xu} and Q_i are initialize as below,

$$Q_{xu} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5000 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.01 \end{bmatrix} \tag{28}$$

$$Q_i = [1] \tag{29}$$

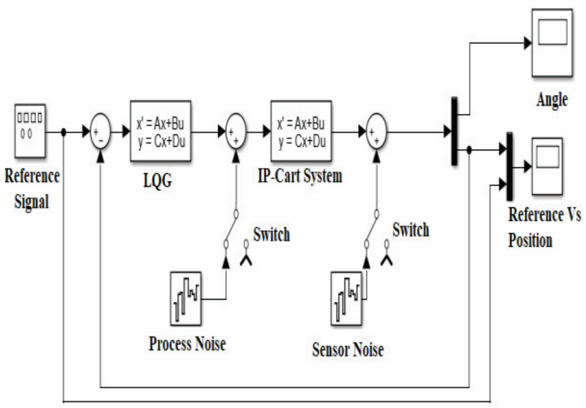

Fig. 3 LQG Simulink block in MATLAB

V. Results

In this section; responses plot of servo control and stabilization of IP-cart system with & without noises are obtained.

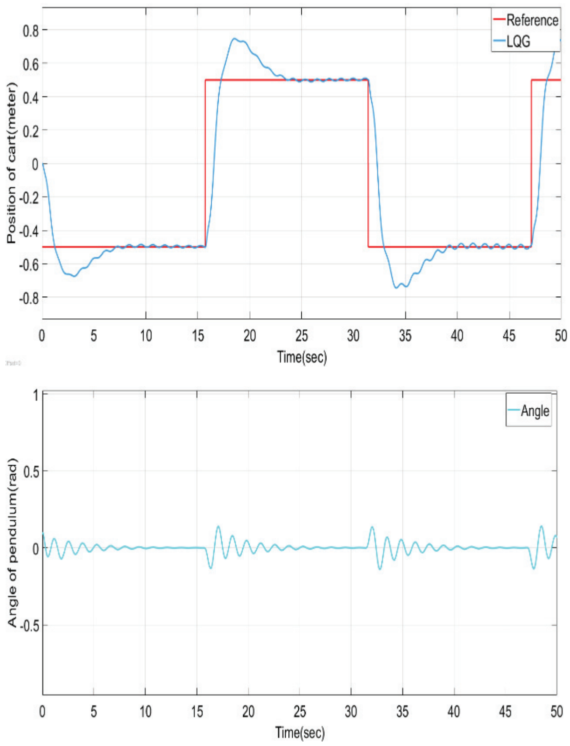

Fig. 4 Response of IP-cart system without noise

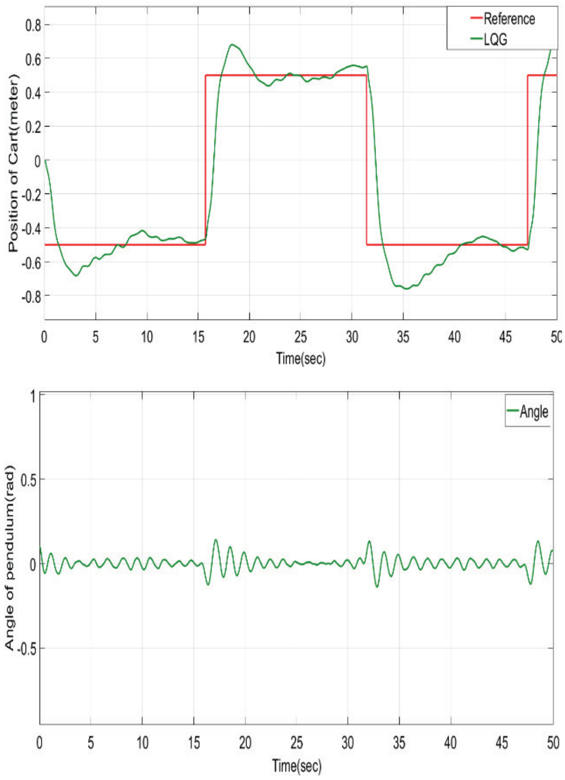

Fig. 5 Response of IP-cart system with process noise

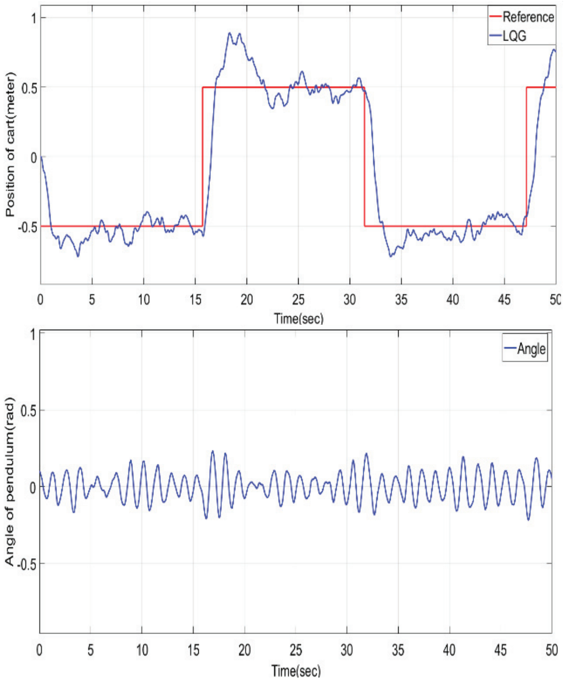

Fig. 6 Response of IP-cart system with sensor noise and process noise

TABLE II. STSTEM TIME RESPONCE

Noise	System Parameters			
	$\theta_{max}(rad)$	T_r(sec)	M_p(mtr)	$e_{ss}(\%)$
No noise	0.18	2	0.72	2.5
Process noise	0.20	2.15	0.69	6.25
Sensor & process noise	0.30	2.15	0.81	10.5

VI. CONCLUSION

In this work, servo problem and stabilization about upright position of IP-cart system were examined using LGQ controller. The responses were obtained without external noise, with process noise, with sensor noise in Simulink MATLAB. Without external noise; the IP-cart system perfectly tracking the reference signal with no steady state error. When process noise and sensor noise are injected the value of steady state error increases but stay within limit. The maximum peak overshot has significant value in the tracking response using LQG controller which is suggested to be reduced in future attempts.

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