Stabilization of Inverted Pendulum on Cart Based on LQG Optimal Control

Ramashis Banerjee
Department of Applied Physics
University of Calcutta
Kolkata, India
ramashisgnitee@rediffmail.com

Arnab Pal
Department of Applied Physics
University of Calcutta
Kolkata, India
amiarnab007@gmail.com

Abstract—An inverted pendulum on cart is an object which is a nonlinear, unstable system, is used as a standard for designing the control methods and finds most versatile application in the field of control theory. To achieve the stabilization of an inverted pendulum system state observer based linear quadratic Gaussian optimal control has been applied. As the separation principal of LQG states, first the control law has been obtained to design a state feedback controller and when the system's all states are not measurable as well as system is affected by process and measurement noise, Kalman Filter has been designed for the inverted pendulum system. The results shown that the new designed controller stabilizes the inverted pendulum system as well as eliminates process and measurement noise. Simulation has been carried out to show the approach.

Keywords— Inverted pendulum; optimal control; LQG; Kalman Filter.

I. INTRODUCTION

Inverted Pendulum on Cart is a good platform for researchers for validation of different control strategies in the field of control system. Most of the modern day technologies use the concept of Inverted Pendulum, such as attitude control of satellites and rockets, aircraft landing, ship balancing against turbulent tide, Seismometer etc. An inverted pendulum has its mass above it's pivoted point is placed on a cart which may be moved horizontally. An inverted pendulum is inherently nonlinear, unstable system that needs to be stabilized[1]. In this case the inverted pendulum system has one input - the force is being applied to the cart, and two output –position of the cart and angle of the pendulum, making it as a Single Input Multiple Output System. There are mainly three ways of balancing an inverted pendulum i.e.

(i) The application of a torque at the pivot point
(ii) The movement of the cart in horizontal direction.

Swinging the support continuously up and down[4]. For achieving the three ways, proper control strategy is required. The linear mathematical methods cannot model the non-linear system which makes the system more critical for analysis Inverted pendulum is known for it's unstable equilibrium point, many different control strategies are applied for the problem. The Proportional-Integral-Derivative (PID)and Proportional-Derivative (PD) controllers [2] and [3] and fuzzy control [5] to mention a few.

But one of the problems faced by conventional industrial controllers are that they cannot control all the pendulum states[3]. A linear state feedback controller is used which is designed based on the linearized mathematical model of the inverted pendulum system.

When the disturbance is present in the system as process noise and measurement noise then disturbance observer (Kalman Filter) is designed to improve the performance in terms of disturbance rejection. The control strategy is used to stabilize the Inverted Pendulum in upright position using optimal state feedback control and state estimator based state feedback control. Here the performance objectives of a control system is directly addressed in terms of cost function and then cost function is minimized to find the state feedback gain matrix. When all states are not available due to lack of sensors, then few states need to be estimated by designing Kalman filter which also rejects the process and measurement noise presents in the system.

First the mathematical model of the system has been derived. Then optimal state feedback controller has been designed by minimizing quadratic cost function feedback has been designed then Kalman filter for the system having a known control input, a output and white noise with power spectral densities has been designed to estimate unknown states[1].

II. SYSTEM MODELLING

A. Mathematical Modelling

Inverted pendulum system is put on cart, shown in fig 1.
Before the derivation few assumptions has been made:

1. The whole system is a rigid body.
2. There is no relative sliding presents in the whole system
3. Friction which exits between cart and rail is directly proportional to the speed of the cart.

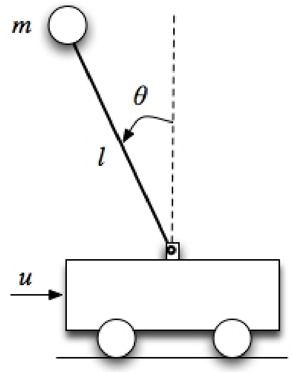

Fig. 1: Configuration of inverted pendulum on cart

Force balance equation along x axis is

$$M(d^2y)/dt^2 + m(d^2x_G)/(dt^2) = u \quad (1)$$

where the time-dependent center of gravity (COG) of the point mass is given by coordinates (x_G, y_G). For the point mass assumed here, the location of the center of gravity of the pendulum mass is

$$x_G = x + l\sin\theta \quad (2)$$

$$y_G = l\cos\theta \quad (3)$$

Substituting (2) and (3) in (1) it is obtained

$$(M+m)\ddot{x} - ml\sin\theta\dot{\theta}^2 + ml\cos\theta\ddot{\theta} = u \quad (4)$$

In a similar way, a torque balance on the system is performed. Finally cart position dynamics and pendulum angle dynamics have been derived, which can be written as

$$\ddot{x} = \frac{u + ml(\sin\theta)\dot{\theta}^2 - mg\sin\theta\cos\theta}{M + m - m(\cos\theta)^2} \quad (5)$$

$$\ddot{\theta} = \frac{u\cos\theta - (M+m)g\sin\theta + ml\dot{\theta}^2\sin\theta\cos\theta}{ml(\cos\theta)^2 - (M+m)l} \quad (6)$$

State space model of the system Let the states are θ and $\dot{\theta}$

The typical parameters of inverted pendulum-cart system setup are selected as mass of the cart (*M*) is 1 kg, mass of the pendulum (*m*) is 0.1 kg, length of the pendulum (*l*) is 0.5m, Acceleration due to gravity (g) is 9.8m/sec^2,the friction coefficient of the cart and pole rotation is assumed negligible. After linearization, the state space model is

$$\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 15.8180 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0 \\ 1.4674\end{bmatrix}u \quad (7)$$

output state equations

$$y_1 = \theta \quad (8)$$

$$y_2 = 0 \quad (9)$$

Where u is known control input, y_1 and y_2 is output, *and* θ and $\dot{\theta}$ is angular position and angular velocity of inverted pendulum.

III. CONTROLLER DESIGN

A. LQR CONTROLLER

To get rid of some problems that are faced by conventional PID controller, the another control strategy is used, which is Linear-Quadratic Regulator (LQR) optimal control. LQR is a control strategy which operates the system with minimum cost when the system dynamics is expressed by differential equations which are linear. The performance measurement is expressed with a cost function which is quadratic in nature and made of state vector and control input control defines the pole location which are optimal in location depends on two cost function. For obtaining the gains, which are optimal. The optimal performance index must be expressed first and then solve the State Dependent Algebraic Riccati Equation(ARE). The LQR design method consists of obtaining a state feedback controller K such a way that the quadratic performance index J is minimized. In the control strategy a feedback gain matrix is obtained which aids to minimize the quadratic performance index and makes the system stable.

For a continuous-time linear system described by:

$$\dot{x} = Ax + Bu \quad (10)$$

Quadratic cost function is written below:

$$J = \int (x^T Q x + u^T R u)dt \quad (11)$$

Now the model is linearized, assume θ *is very small.* Where Q and R are weight matrices, Q must be positive definite or positive semi-definite symmetric matrix. R must be positive definite symmetric matrix. Weighting matrices should be diagonal matrix. The value of the elements of the weighting matrices are linked to the impact on the performance index J. The feedback control law is:

$$u = -Kx \quad (12)$$

Where, *K* is expressed as $K = R^{-1}B^T P$ (13)

And P can be obtained by the solution of SDRE equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (14)$$

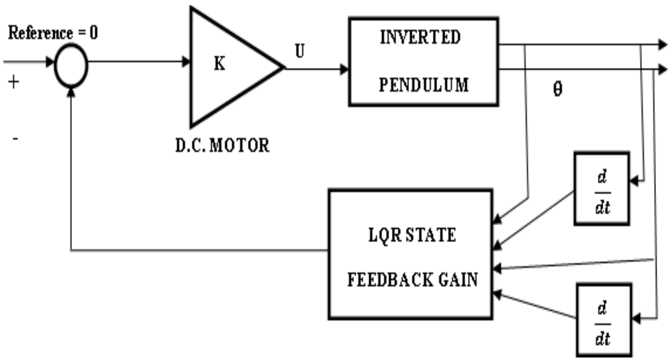

Fig. 2: Block Diagram of LQR Controller

LQR is just an automatic method to find out an appropriate state. This is common to find that control engineers prefer different methods as full state feedback controller to find a controller over the LQR controller. With these the researcher has a much clearer link between adjusted parameters and the resulting changes in the behavior of the controller. Difficulties are faced in finding out the right weighting matrices, which limits the application of the LQR Controller design [4].
Here

Optimal state feedback matrix is

$$K = [44.1897 \quad 8.3804]$$

B. LQG CONTROLLER

In reality, all state variables can not be fully measured, to solve this problem, estimator is needed. The system state equation are given as below. Here process noise and measurement noise is also present in the state equation, this is also called Linear Quadratic Gaussian(LQG) controller, This is the combination of LQR controller and the Kalman Filter, which is used to estimate the unmeasured states.

$$\dot{x} = Ax + Bu + Tw \tag{15}$$

$$y = Cx + Du + v \tag{16}$$

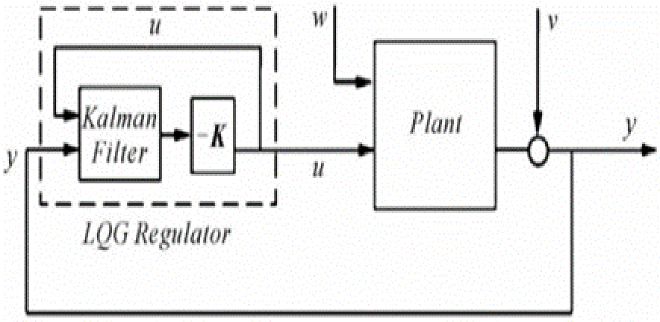

Fig. 3: Block Diagram of LQG Controller

B. *Design of Kalman Filter*

From equation (15) T is noise input matrix is process noise vector, v is measurement noise vector.
So, the state estimation which is made by Kalman Filter is

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly \tag{17}$$

$$\hat{y} = Cx + v \tag{18}$$

Where $\hat{x}$ is state estimation of Kalman filter and Y is output estimation of the filter.
Kalman filter gain is written as

$$L = PC^T R^{-1} \tag{19}$$

The expectation

$$Exp = \lim_{t \to \infty} E\left[(x - \hat{x})(x - \hat{x})^T\right] \tag{20}$$

When the expectation is minimum, then the filter is used for optimal state estimation. Kalman filter minimizes estimated error, i.e

$e = x - \hat{x}$

Here $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $v = 10CC^T$

and $Q = \begin{bmatrix} 1000 & 0 \\ 0 & 10 \end{bmatrix}$ $R = 1$

So, Kalman filter gain is obtained as

$$L = \begin{bmatrix} 7.9611 & 31.6393 \end{bmatrix}$$

IV. SIMULATION RESULT

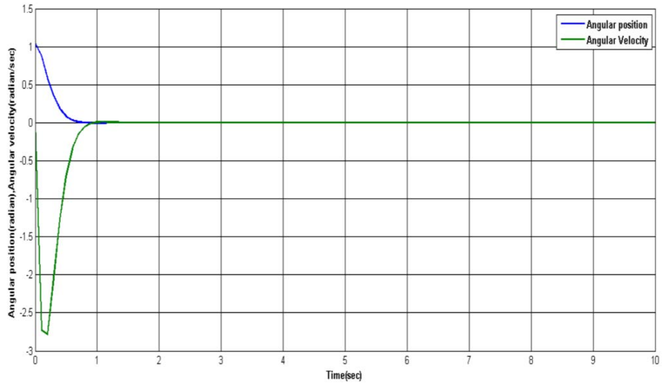

Fig. 4: Time response of inverted pendulum system for angular position and angular velocity after applying LQR

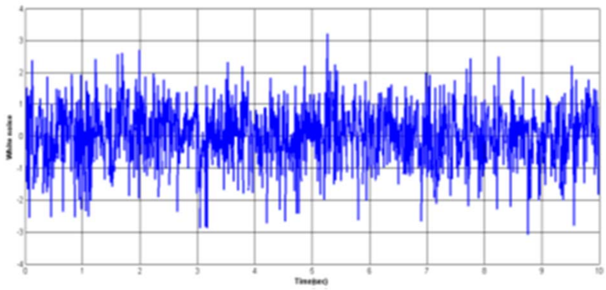

Fig. 5: Nature of white noise given to the inverted pendulum system

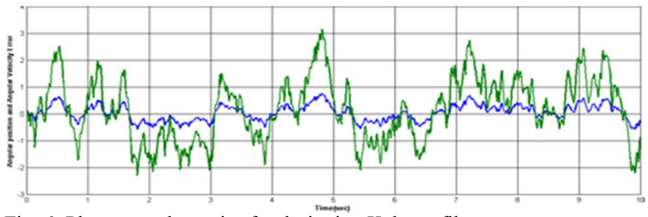

Fig. 6: Plant error dynamics for designing Kalman filter

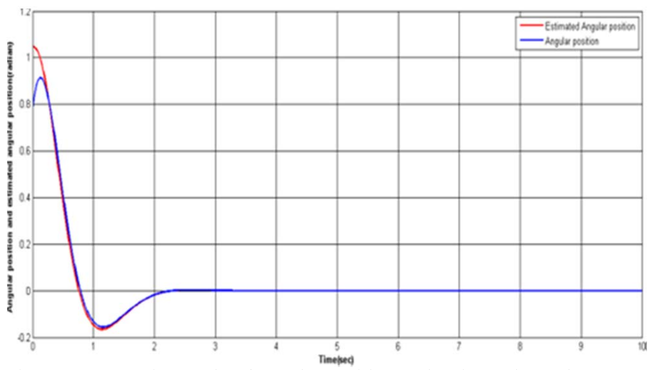

Fig. 7: Comparative study of angular position and estimated angular position by Kalman filter model.

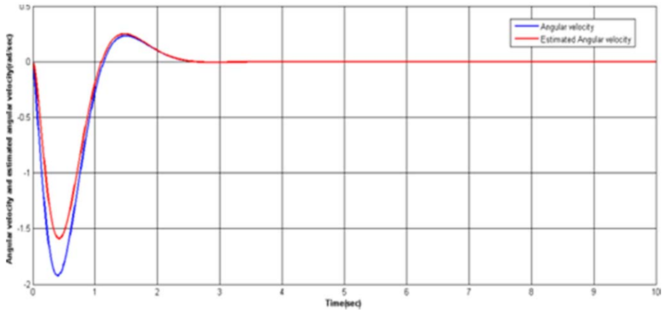

Fig. 8: Comparative study of angular velocity and estimated angular velocity by Kalman filter model.

V.CONCLUSIONS

The simulation of inverted pendulum system using LQR,LQG controller has been done, it has been observed from the response curve that system is stable and the difference between two controllers has been stated. By changing the weighting matrices and noise covariance matrices LQR and LQG will show different result and this result has some limitations ,this result shows the difference between two controllers.

REFERENCES

[1] J. Pati, “Cart-Pole System Modelling and Controller Design for an Inverted Pendulum System”, M.Tech Thesis, Department of Electronics and Communication Engineering, National Institute of Technology Rourkela, May 2014.

[2] J. F. Hauser, A. Saccon, “On the Driven Inverted Pendulum,” proceedings of the 5th International Conference on Information, Communication and Signal Processing, Bangkok

[3] K. Tanaka, T. Ikada and H.O.Wany, “Fuzzy Regulator & Fuzzy observers; Relaxed Stability Condition and Limibased Designs,” IEEE transaction on Fuzzy Systems, doi:10.1109/91.669023

[4] N.Lenka “Modeling and Controller Design for an Inverted Pendulum System” B.Tech Thesis, Department of Electrical Engineering National Institute of Technology Rourkela

[5] N.Singh Bhangal “Design and Performance of LQR and LQR based Fuzzy Controller for Double Inverted Pendulum System”, Journal of Image and Graphics Vol. 1, No. 3, 2013,pp-143-146.