Numerical Analysis
(Solving non-linear equation)
Secant and Modified Secant

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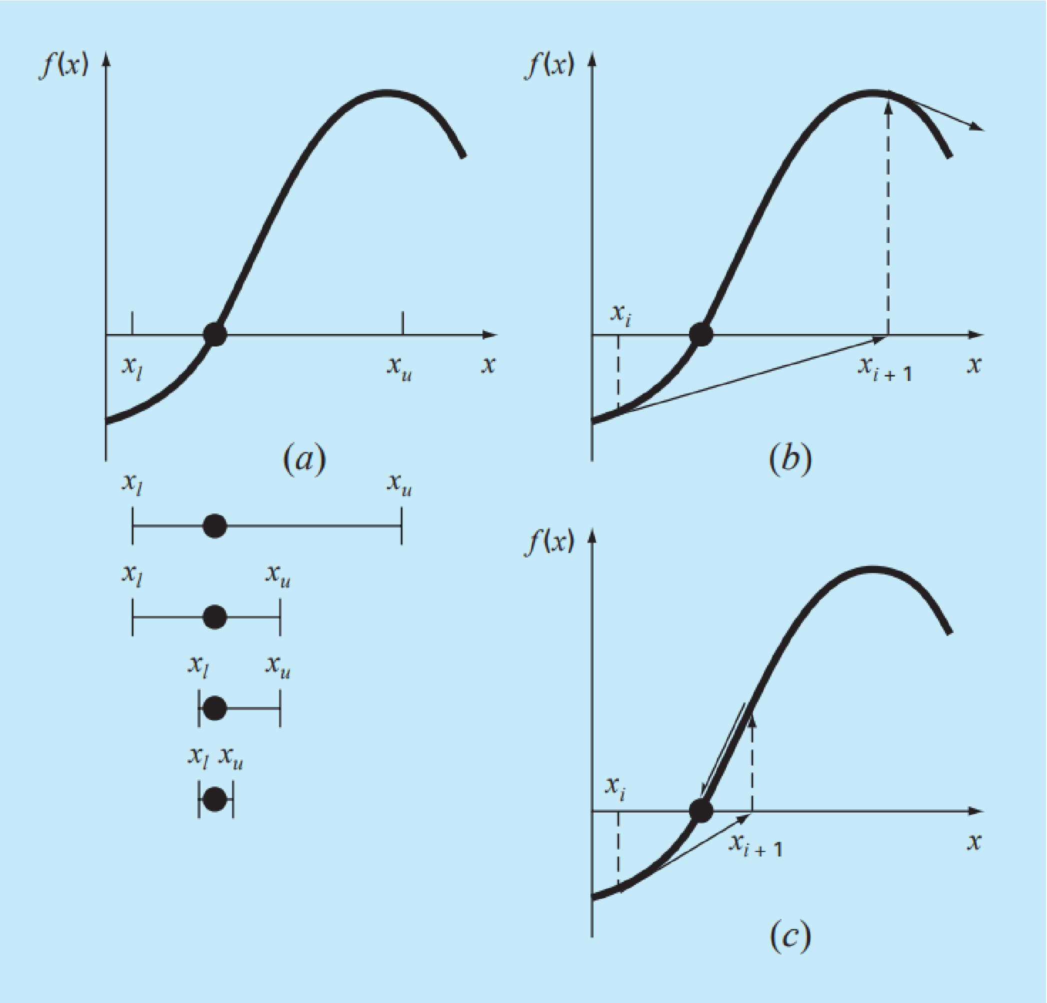
Solving non-linear equation using numerical methods

- **Bracketing Methods: (Figure a)**
  - The Bisection Method
- **Open Methods (Figure b, c)**
  - Simple Fixed-Point Iteration
  - The Newton-Raphson Method
  - **The Secant Method**



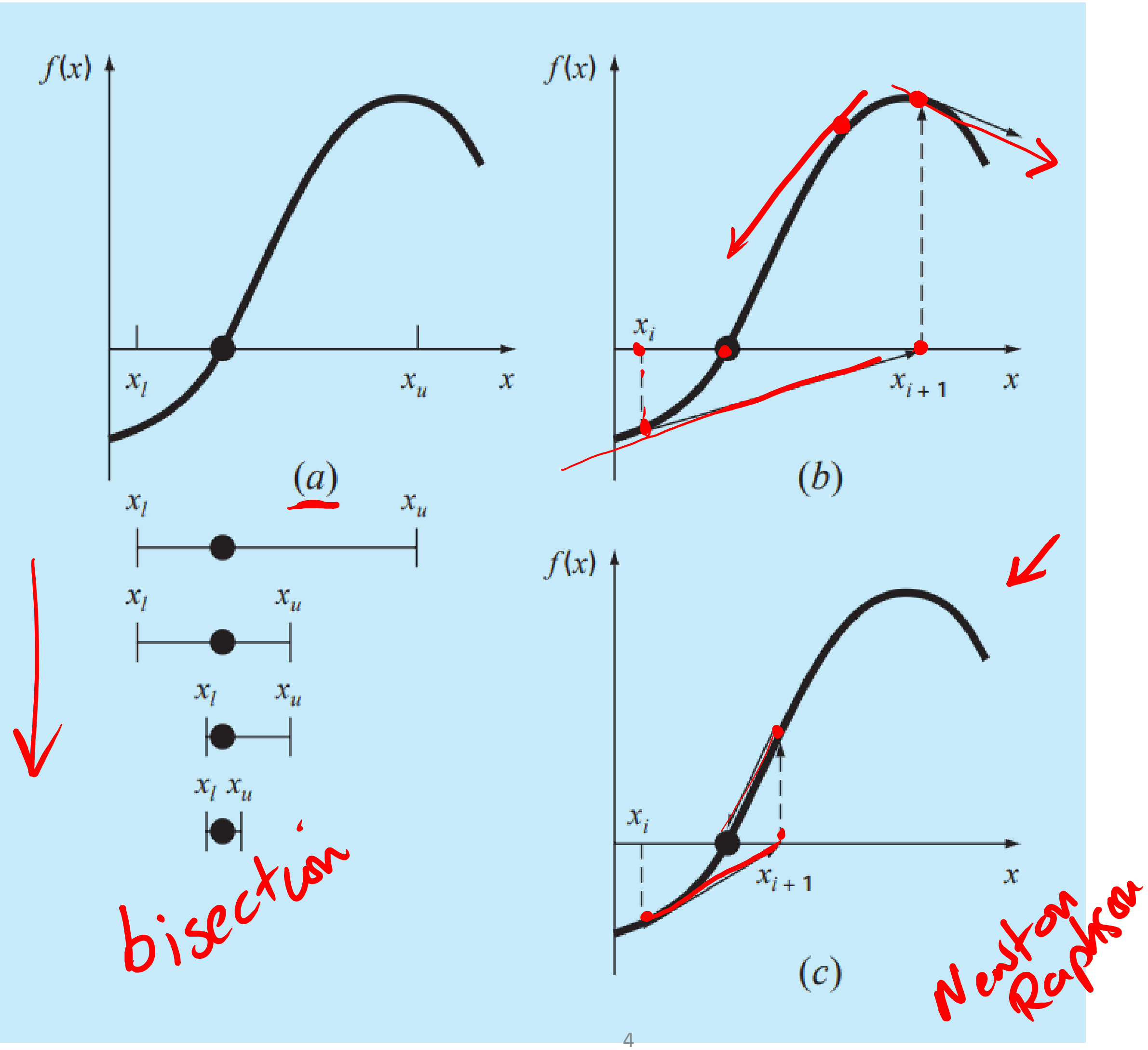
Open Method

- For the bracketing methods, the root is located within an interval prescribed by a lower and an upper bound.
- Repeated application of these methods always results in closer estimates of the true value of the root.
- Such methods are said to be convergent because they move closer to the truth as the computation progresses.
- **In contrast, the open methods are based on formulas that require only a** single starting value of x **or** two starting values that do not necessarily bracket **the root.**

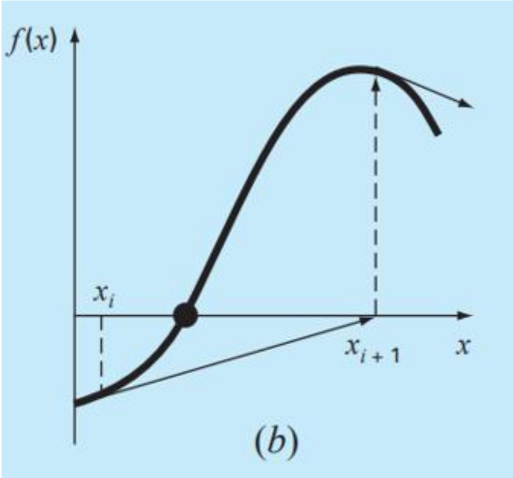
the fundamental difference between the bracketing and open methods for root location

- (a) which is the bisection method, the root is constrained within the interval prescribed by xl and xu

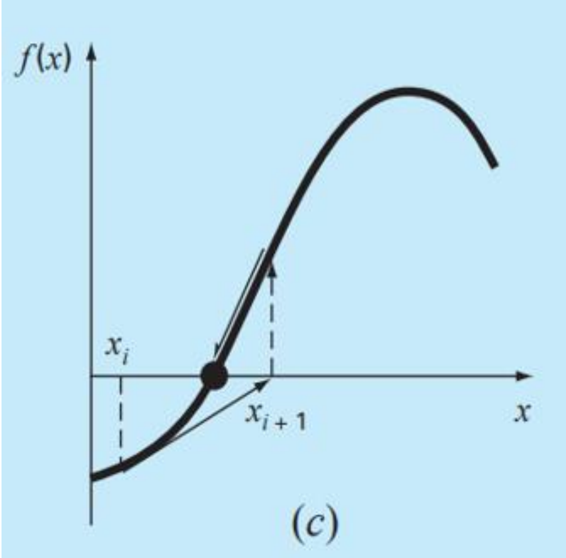
- In contrast, for the open method depicted in (b) and (c), a formula is used to project from xi to xi+1 in an iterative fashion.
- Thus, the method can **either (b) diverge** or **(c) converge rapidly**, depending on the value of the initial guess.



- As noted, **open methods** sometimes **diverge** or **move away from the true root**.



- However, when the open methods **converge**, they usually do so much **more quickly than the bracketing methods**.



THE SECANT METHOD

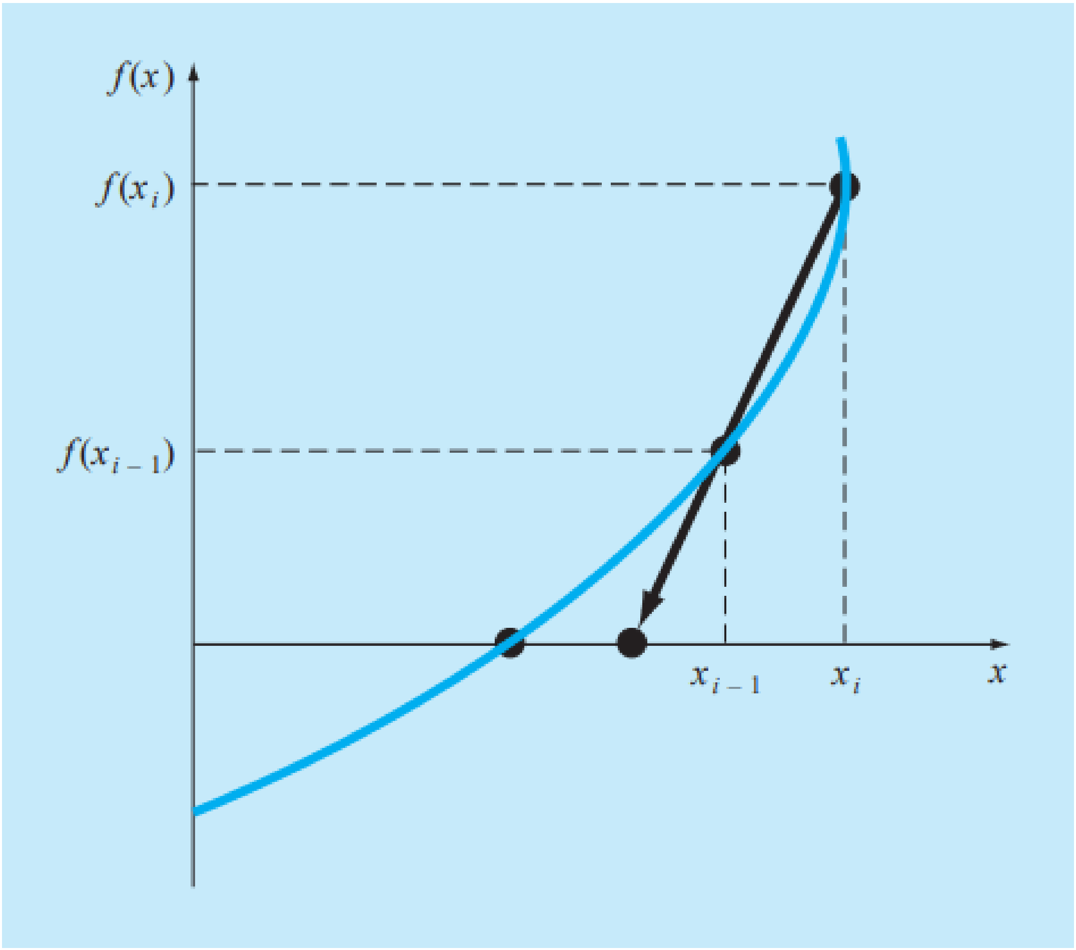
- A potential problem in implementing the Newton-Raphson method is

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

- polynomials and many other
  s whose derivatives may be
  evaluate.
- For these cases, the derivative can be approximated by a backward finite divided difference, as in (Fig. 6.7) see next slide

FIGURE 6.7

Graphical depiction of the secant method. This technique is similar to the Newton-Raphson technique (Fig. 6.5) in the sense that an estimate of the root is predicted by extrapolating a tangent of the function to the x axis. However, the secant method uses a difference rather than a derivative to estimate the slope.



THE SECANT METHOD

The approximation of derivative is

$$f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i} \quad \rightarrow 1$$

Using newton Raphson equation

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \rightarrow 2$$

Substitute 2 in 1 to get the following iterative equation

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

THE SECANT METHOD

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

- Problem Statement. Use the secant method to estimate the root of $f(x) = e^{-x} - x$.

  Start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$.

Solution. Recall that the true root is 0.56714329. . . .

First iteration:

$$x_{-1} = 0 \qquad f(x_{-1}) = 1.00000$$

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_1 = 1 - \frac{-0.63212(0-1)}{1-(-0.63212)} = 0.61270 \qquad \varepsilon_t = 8.0\%$$

Second iteration:

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

(Note that both estimates are now on the same side of the root.)

$$x_2 = 0.61270 - \frac{-0.07081(1-0.61270)}{-0.63212-(-0.07081)} = 0.56384 \qquad \varepsilon_t = 0.58\%$$

Third iteration:

$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

$$x_2 = 0.56384 \qquad f(x_2) = 0.00518$$

$$x_3 = 0.56384 - \frac{0.00518(0.61270-0.56384)}{-0.07081-(-0.00518)} = 0.56717 \qquad \varepsilon_t = 0.0048\%$$

Algorithm for the Secant Method

- The same as Newton Raphson by simply modifying the following algorithm so that two initial guesses are input

```
FUNCTION Fixpt(x0, es, imax, iter, ea)
  xr = x0
  iter = 0
  DO
    xrold = xr
    xr = g(xrold)
    iter = iter + 1
    IF xr ≠ 0 THEN
      ea = | (xr - xrold) / xr | · 100
    END IF
    IF ea < es OR iter ≥ imax EXIT
  END DO
  Fixpt = xr
END Fixpt
```

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

In addition, the options suggested for the Newton-Raphson method can also be applied to good advantage for the secant program.

Modified Secant Method

- an alternative approach to estimate $f'(x)$

$$f'(x_i) \cong \frac{f(x_i+\delta x_i)-f(x_i)}{\delta x_i}$$

Where δ =a small fraction. ➔

$$x_{i+1} = x_i - \frac{\delta x_i f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

- Problem Statement. Use the modified secant method to estimate the root of $f(x) = e^{-x} - x$. Use a value of 0.01 for δ and start with $x_0 = 1.0$. Recall that the true root is 0.56714329. . . .

$$x_{i+1} = x_i - \frac{\delta x_i f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

Solution.

First iteration:

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_0 + \delta x_0 = 1.01 \qquad f(x_0 + \delta x_0) = -0.64578$$

$$x_1 = 1 - \frac{0.01(-0.63212)}{-0.64578 - (-0.63212)} = 0.537263 \qquad |\varepsilon_t| = 5.3\%$$

Second iteration:

$$x_0 = 0.537263 \qquad f(x_0) = 0.047083$$

$$x_0 + \delta x_0 = 0.542635 \qquad f(x_0 + \delta x_0) = 0.038579$$

$$x_1 = 0.537263 - \frac{0.005373(0.047083)}{0.038579 - 0.047083} = 0.56701 \qquad |\varepsilon_t| = 0.0236\%$$

Third iteration:

$$x_0 = 0.56701 \qquad f(x_0) = 0.000209$$

$$x_0 + \delta x_0 = 0.572680 \qquad f(x_0 + \delta x_0) = -0.00867$$

$$x_1 = 0.56701 - \frac{0.00567(0.000209)}{-0.00867 - 0.000209} = 0.567143 \qquad |\varepsilon_t| = 2.365 \times 10^{-5}\%$$

- The choice of a proper value for δ is not automatic.
- If δ is too small, the method can be swamped by round-off error caused by subtractive cancellation in the denominator of Eq

$$x_{i+1} = x_i - \frac{\delta x_i f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

- If it is too big, the technique can become inefficient and even divergent.
- However, if chosen correctly, it provides a nice alternative for cases where evaluating the derivative is difficult and developing two initial guesses is inconvenient

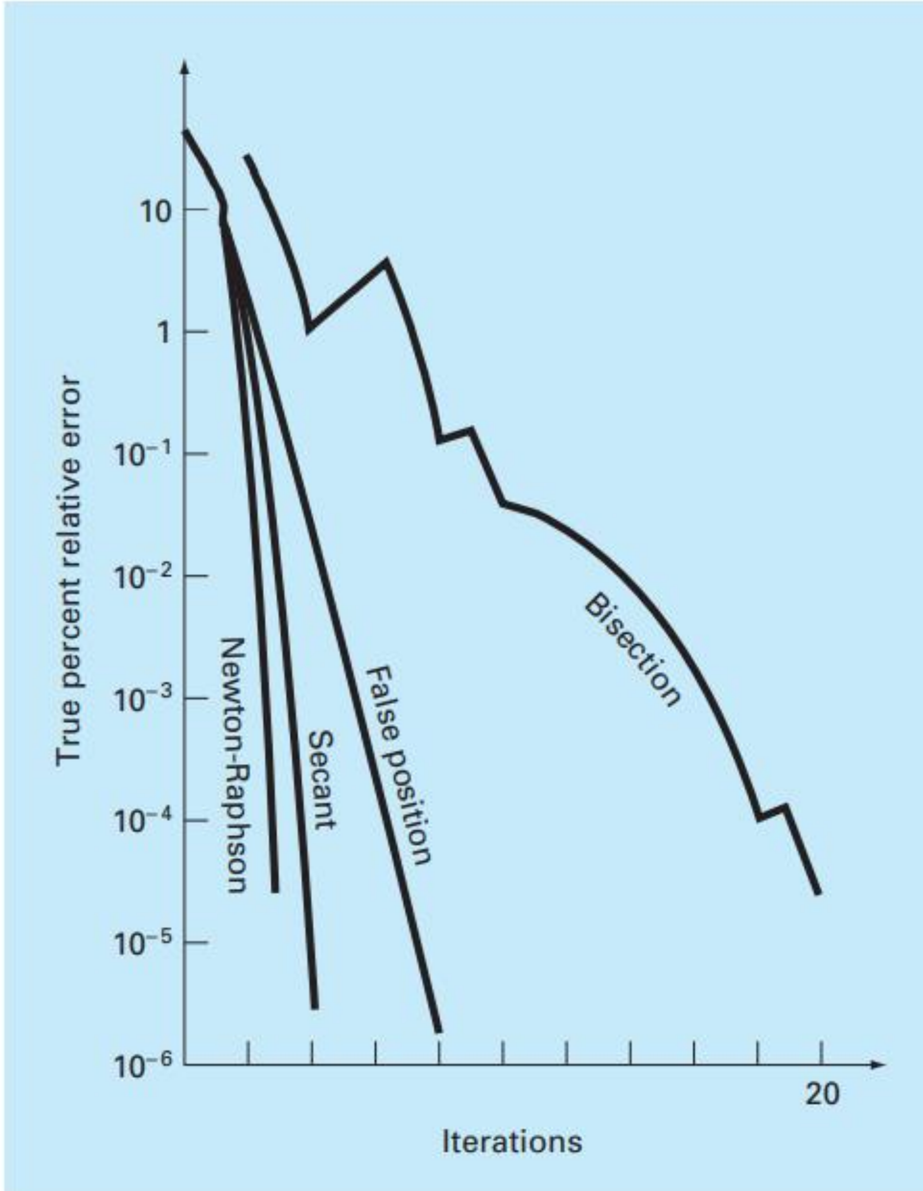


FIGURE 6.9

Comparison of the true percent relative errors ε_t for the methods to determine the roots of $f(x) = e^{-x} - x$.