Solving Non-Linear Equation Bisection Method

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Roots of Equation

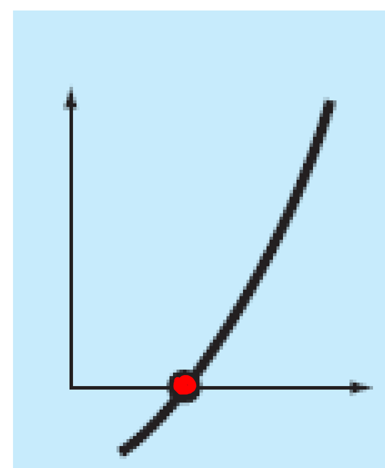
- Equation : $f(x) = ax^2 + bx + c$
  - Roots of equation represent the values of x that make $f(x)$ =0 ➔

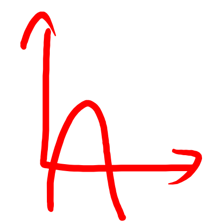
also called the *zeros* of the equation

- **<u>Analytical Method-> Exact solutions:</u>**
  - The handy Quadratic formula for solving $f(x)$ is :
    - X= $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  - There are many other functions for which the root cannot be determined so easily ➔ numerical methods

* - Ex: $f(x) = e^x - x$ =0

```
a= input('a=');
 b= input('b=');
 c= input('c=');
%Analytical (Exact solution)
 x1=-b+sqrt(b^2-4*a*c)/(2*a)
 x2=-b-sqrt(b^2-4*a*c)/(2*a)
```





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- **<u>Numerical Method-> approximation solutions:</u>**

Graphical techniques	MATLAB

• Plot the function and determine where it crosses the x axis.
• This point, which represents the x value for which $f(x)$ =0 , is the root.
• Although graphical methods are useful for obtaining rough estimates of roots, they are limited because of their lack of precision.
• However, these estimates can be employed as starting guesses for numerical methods discussed
	a= input('a=');
	b= input('b=');
	c= input('c=');
	
	%Graphically (app. solution)
	X=linspace(1,10,100);
	Y=5x.^2+3x-4
	Plot(X,Y)

Analytical solving

- $F(x)=5x-10=0$
  - root(f(x)) is x=2 ➔ f(2)=5*2-10=0
- $F(X) = 5x^2+3x-4=0$
  - There are two roots
  - $X_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = -2.0566$
  - $X_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = -3.9434$

P1.m Matlab code

```
a= input('a=');
b= input('b=');
c= input('c=');
x1=-b+sqrt(b^2-4*a*c)/(2*a)
x2=-b-sqrt(b^2-4*a*c)/(2*a)
```

Problem Statement. Newton's second law

- Problem
  - Newton's second law, used for calculation the parachutist's velocity:
    - $v = \frac{gm}{c}(1 - e^{-\left(\frac{c}{m}\right)t})$ v as a function of t
  - Given
    - Parachutist's mass m
    - drag coefficient c
    - freefalling time t
    - gravity is 9.81 m/s2.
  - You can easily find the parachutist's velocity after freefalling for time t
    - Direct substitution

$$\boxed{\frac{gm}{c}\left(\quad t \right) - v = 0}$$

- What happens if we know the parachutist's velocity, and we want to determine the drag coefficient *c* ??????

$y = 20$

$\frac{y-6}{5} = x$

$y = 5x + 6$

$x = 1$ $y = 11$

Problem Statement. Newton's second law

- Problem
  - Use the graphical approach to determine the drag coefficient *c* needed for a parachutist of mass *m*=68.1 kg to have a velocity of v=40 m/s after freefalling for time *t*=10 s. Where, the parachutist's velocity
  - $v = \frac{gm}{c}(1 - e^{-\left(\frac{c}{m}\right)t})$ v as a function of t
  - *Note:* The acceleration due to gravity is 9.81 m/s^2.
- Solution.
  - This problem can be solved by determining the root of Eq.
    - $f(c) = \frac{gm}{c}\left(1 - e^{-\left(\frac{c}{m}\right)t}\right) - v = 0$
  - using the parameters *t*=10, *g*=9.81, v=40, and *m*=68.1:
- See the following graphical method…

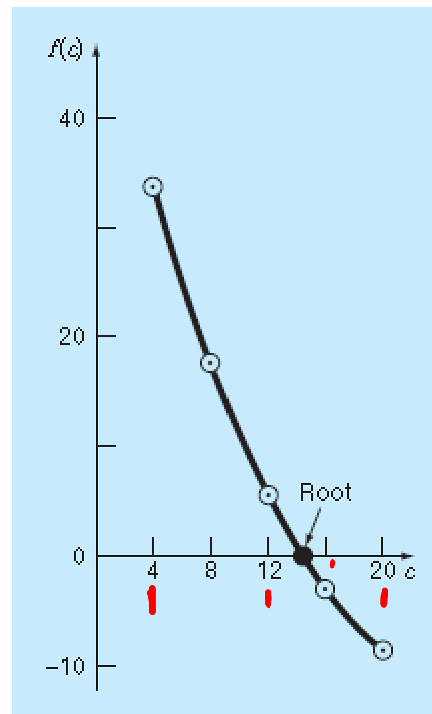
$y = f(c)$

Problem Statement. Newton's second law

```
t=10;
g=9.81
v=40;
m=68.1;
c=-4:4:20    % suggest a range of c
%c=-4:0.1:20    % suggest a range of c
%c=-4:0.01:20    % suggest a range of c
fc=g*m./c.*(1-exp(-(c./m)*t))-v
h=plot(c,fc)
%to make x axis passes through
original point
ax = gca;  %get current axis
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
```

- $f(c) = \frac{gm}{c}\left(1 - e^{-\left(\frac{c}{m}\right)t}\right) - v = 0$

c	f(c)
4	34.115
8	17.653
12	6.067
16	−2.269
20	−8.401



Visual inspection of the plot provides a rough estimate of the root of 14.8.

c=14.8

fc=g*m./c.*(1-exp(-(c./m)*t))-v

➔fc= 0.0022 which is close to zero.

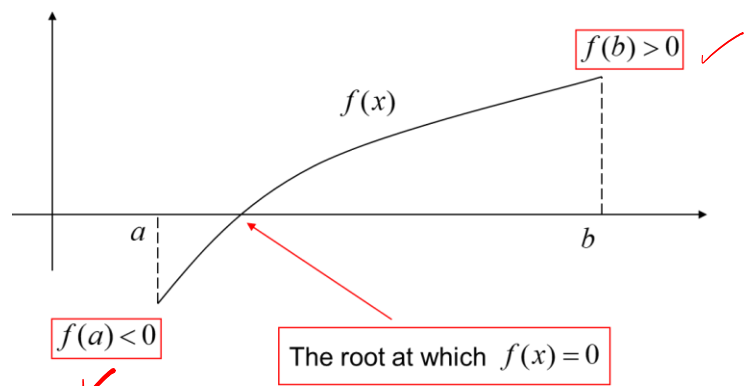
$$f(14.75) = \frac{667.38}{14.75}\left(1 - e^{-0.146843(14.75)}\right) - 40 = 0.059$$

$$v = \frac{9.8(68.1)}{14.75}\left(1 - e^{-(14.75/68.1)10}\right) = 40.059$$

which is very close to the desired fall velocity of 40 m /s.

Solving non-linear equation using numerical methods

- **Bracketing Methods:**
  - To find roots of equations.
  - They deals with methods that exploit the fact that
    - **a function typically changes sign in the vicinity (neighborhood) of a root.**
  - These techniques are called *bracketing methods* because two initial guesses for the root are required. Each guess makes
  - Ex: **The Bisection Method**



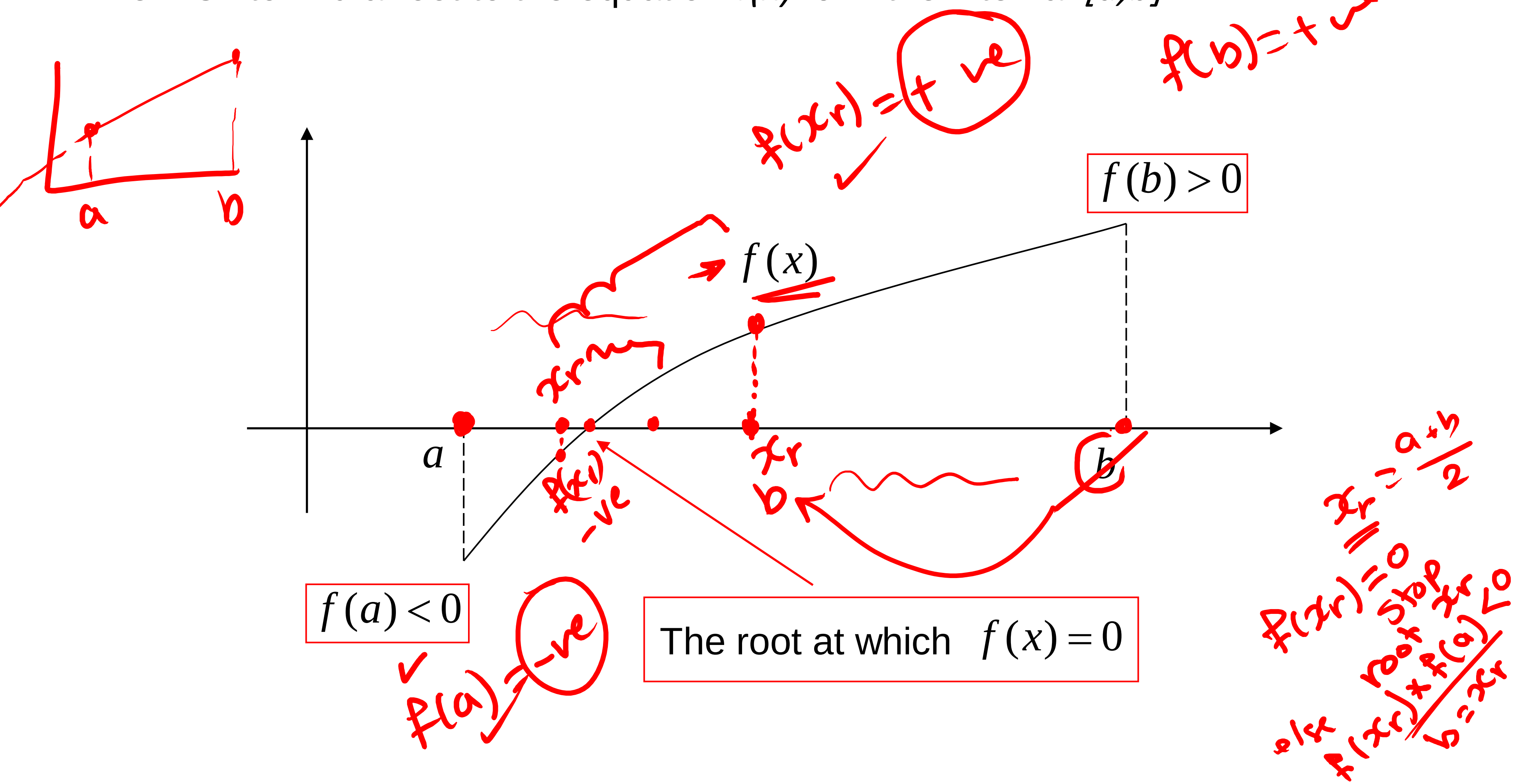
-ve

+ve

- **Open Methods**
  - The Newton-Raphson Method
  - The Secant Method

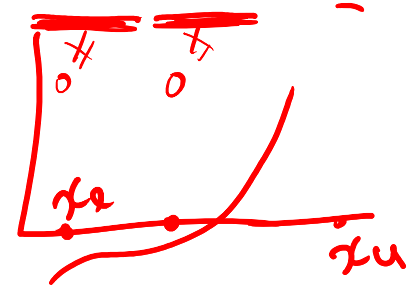
Solving non-linear equation using Bisection Method

We wish to find a root to the equation *f(x)=0* in the interval *[a,b]*



Solving non-linear equation using Bisection Method

Step 1: Choose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_l)\,f(x_u) < 0$.

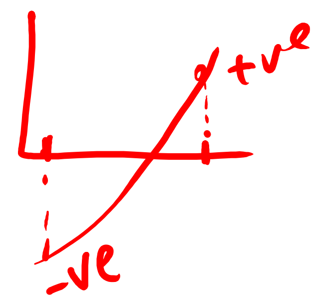


Step 2: An estimate of the root x_r is determined by

$$x_r = \frac{(x_l + x_u)}{2}$$

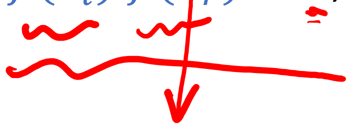
Step 3: Make the following evaluations to determine in which subinterval the root lies:

(a) If $f(x_l)\,f(x_r) < 0$,
- the root lies in the lower subinterval.
- therefore, set $x_u = x_r$ and **return to step 2.**



(b) If $f(x_r)\,f(x_u) > 0$
- the root lies in the upper subinterval.
- therefore, set $x_l = x_r$ and **return to step 2.**

(c) If $f(x_l)\,f(x_r) = 0$, the root equals x_r; **terminate the computation.**



l → lower
u → upper

•Using bisection, solve the $f(x) = x^3 - x - 11$ on the interval [2,3] until the approximate error falls below a stopping criterion of ε_s = ~~0.5~~%. 0.05%

$2^3 - 2 - 11 = 8 - 2 - 11 = -5$

$3^3 - 3 - 11 = 27 - 3 - 11 = 13$

$\boxed{-ve}$ $\boxed{+ve}$

x_l	x_u	$x_r = \frac{(x_l + x_u)}{2}$	$\varepsilon_a = \left\lvert\frac{(x_r - x_{r-1})}{x_r}\right\rvert * 100$	$f(x_r)$
2	3	2.5	------	+ve
2	2.5	2.25	$\frac{2.25-2.5}{2.25} \times 100$ =11.111%	−ve
2.25	2.5	2.375	$\frac{2.375-2.25}{2.375}$ 5.2%	+ve
2.25	2.375	2.3125	2.7%	−ve
2.3125	2.375	2.34375	1.33%	−ve
2.34375	2.375	2.359375	0.6%	−ve
2.359375	2.375	2.3671875	$\boxed{0.33\%}$	−ve
2.3671875	2.375	2.3710938	0.164%	−ve
2.3710938	2.375	2.3730469	0.0823%	−ve
2.3730469	2.375	$\boxed{2.3740235}$ root	0.0411%	

2.373

Bisection Method

$$f(x) = x^3 - x - 11$$

```
clear
clc
syms x;
fx=x^3-x-11;
xl=2;
xu=3;
e=0.05;
numberOfIteration=100;
 if subs(fx,xl)*subs(fx,xu)<0
    names={'-ve','+ve','xn','error'};
    for i=1:numberOfIteration
        xr=(xl+xu)/2;
        if i==1
            data(i,:)=[xl,xu,xr,9999];% no error
        else
            error=abs((xr-xrold)/xr)*100;
            if (error<e)
                 break
            end
            data(i,:)=[xl,xu,xr,error];
        end

        if subs(fx,xl)*subs(fx,xr)<0
            xu=xr;
        else
            xl=xr;
        end
       xrold=xr;
    end
    t=uitable('ColumnName',names,'Data',data,'Position',[100
100 410 200]);
 else
     disp('no solution')
 end
```

Problem Statement. Newton's second law

- Problem
  - Using Bisection Method on the interval [12,16], determine the drag coefficient c needed for a parachutist of mass m=68.1 kg to have a velocity of v=40 m/s after freefalling for time t=10 s. Where, the parachutist's velocity
  - $v = \frac{gm}{c}\left(1 - e^{-\left(\frac{c}{m}\right)t}\right)$ v as a function of t
  - *Note:* The acceleration due to gravity is 9.81 m/s^2.
  - Error= 0.5%

$\varepsilon_s = 0.5\%$

Iteration	x_l	x_u	x_r	app ε_a (%)	true error ε_t (%)
1	12	16	14		5.279
2	14	16	15	6.667	1.487
3	14	15	14.5	3.448	1.896
4	14.5	15	14.75	1.695	0.204
5	14.75	15	14.875	0.840	0.641
6	14.75	14.875	14.8125	0.422	0.219

$\frac{x_r - x_{r-1}}{x_r}$

$\frac{true - x_r}{true}$

Drawbacks of Bisection Method

Drawbacks

- Converges very slowly
- Requires two guess points

Notes

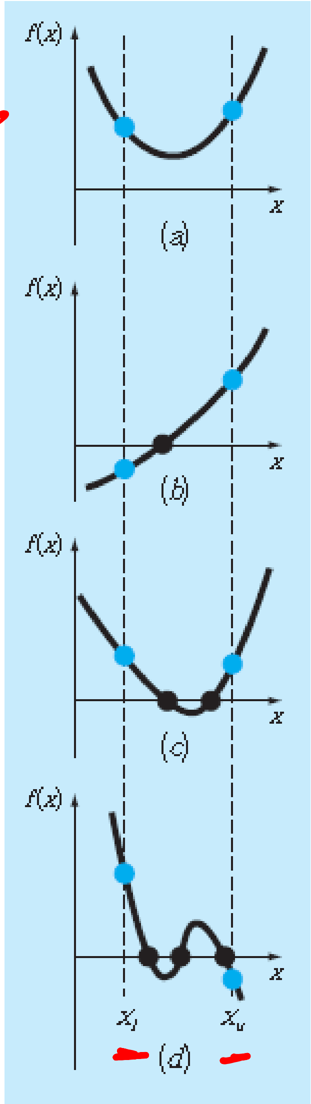


Illustration of a number of general ways that a root may occur in an interval prescribed by a lower bound x_l and an upper bound x_u.

- Parts (*a*) and (*c*) indicate that if both $f(x_l)$ and $f(x_u)$ have the same sign, either there will be no roots or there will be an even number of roots within the interval.

- Parts (*b*) and (*d*) indicate that if the function has different signs at the end points, there will be an odd number of roots in the interval.

Notes

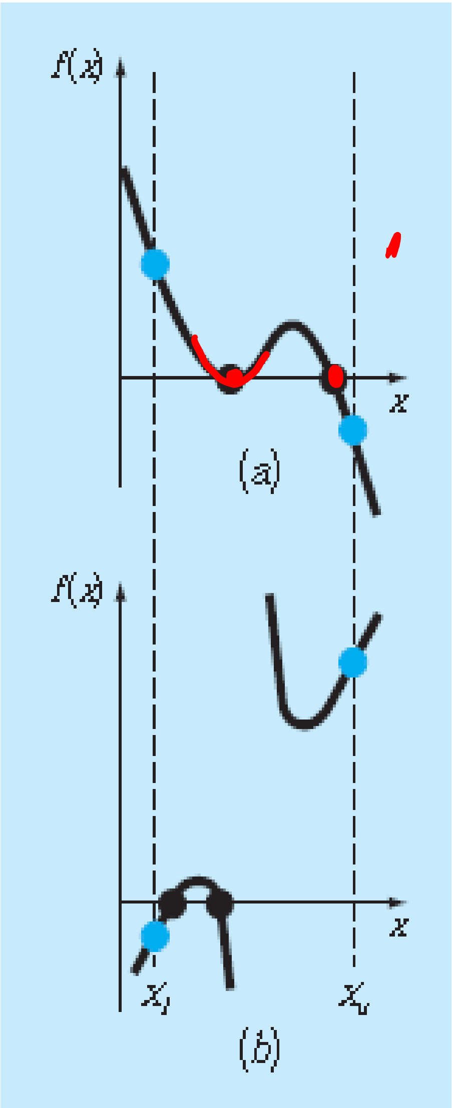
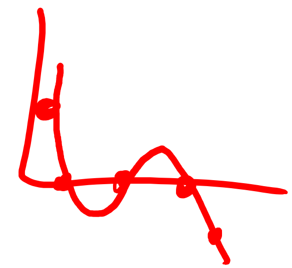
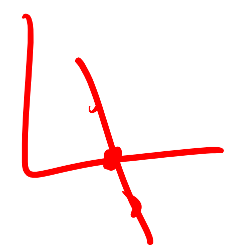


Illustration of some exceptions to the general cases depicted in

Part (a) Multiple root that occurs when the function is tangential to the x axis. For this case, although the end points are of opposite signs, there are an even number of axis intersections for the interval.



(b) Discontinuous function where end points of opposite sign bracket an even number of roots. Special strategies are required for determining the roots for these cases.



Problem 5.1

Determine the real roots of $f(x) = -0.6x^2 + 2.4x + 5.5$:

(a) Graphically.

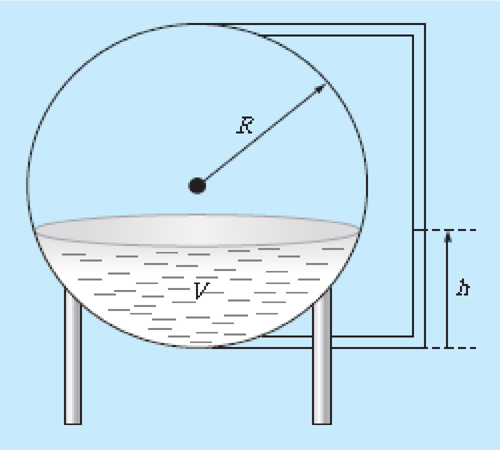
(b) Using the quadratic formula.

(c) Using three iterations of the bisection method to determine the highest root. Employ initial guesses of $xl = 5$ and $xu = 10$.

Compute the estimated error ε_a and the true error ε_t after each iteration.

Problem 5.17

- You are designing a spherical tank to hold water for a small village in a developing country. The volume of liquid it can hold can be computed as
  - $V = \pi h^2 [3R - h]/3$
- where V = volume [m^3], h = depth of water in tank [m], and R = the tank radius [m].
- If R = 3 m, to **what depth** must the tank be filled so that it holds 30 m^3?
  - Use three iterations of the bisection method to determine your answer.
  - Determine the approximate relative error after each iteration. Employ initial guesses **of 0 and *R***.



$\boxed{V - \pi h^2(3R-h)/3 = 0}$

$[0, R]$
$\quad 0 \quad 3$

Problem 1

- By using Bisection method, find an iterative formula to find the value of where is a positive real number and hence use this formula to approximate the following square roots with an error less than 0.001 %
  - (a) $\sqrt{5}$ (b) $\sqrt{12}$

$\sqrt{4} = \pm 2$

$\sqrt{5} =$

$y = \sqrt{m}$

$\boxed{y - \sqrt{5} = 0}$

$\sqrt{4} < \sqrt{5} < \sqrt{9}$

$2 < \sqrt{5} < 3$

3, 4

$a, b = [2, 3]$