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Rigorous connection between physical properties of porous rocks

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Abstract. Rigorous cross-property bounds that connect the effective thermal conductivity k_* (or the electrical conductivity σ_*) and the effective bulk modulus K_* of any isotropic, two-phase composite were recently derived by the authors. Here we reformulate these bounds and apply them to porous rocks with dry or fluid-filled pores. It is shown that knowledge of the effective conductivity can yield sharp estimates of the effective bulk modulus (and vice versa), even in cases where there is a wide disparity in the phase properties. The bounds yield, in particular, relations between the formation factor and the bulk modulus of the porous medium. By using the same approach we obtain new relations between the bulk moduli of a dry porous material and the bulk modulus of the same material with fluid-filled pores that are more general than the traditional Gassmann equation. The Gassmann formula for the bulk modulus of the fluid-saturated porous medium is shown to correspond to a lower bound on this quantity. Limiting cases that we consider include cracked materials with dry and fluid-saturated pores. Theoretical results are tested against experimental measurements of the effective bulk modulus of dry and water-saturated Westerly granite and sandstone samples. We found good agreement between our cross-property bounds and the experimental data, even when the experimental data depart from the Gassmann formula. Our results add new insight to understanding of the properties of the porous media. They show that the Gassmann approximation works well for rocks with high porosity but needs to be corrected for rocks with high crack-type porosity.

1. Introduction

All natural rocks and minerals contain pores, cracks, and microcracks that greatly affect their elastic, conduction, and diffusion properties. The effect of such defects on the overall behavior of the rocks has been extensively discussed in the literature; see, for example, the article by *O'Connell and Budiansky* [1974], and the recent book by *Mavko et al.* [1998] for a comprehensive review and references. One can estimate the properties of the rocks knowing the porosity, crack density, and crack distribution. However, these microstructural quantities are often unknown.

The goal of this paper is to suggest an alternative approach. Instead of extracting microstructural information to estimate or bound the properties, we suggest that measurements of one physical property of the rocks, such as the thermal or electrical conductivity, be used to bound other properties of the same composite, such as the bulk or shear moduli, or fluid permeability. Rigorous cross-property relations for a variety of properties were established by *Prager* [1969], *Milton* [1984], *Berryman and Milton* [1988], *Torquato* [1990, 1992], *Avellaneda and Torquato* [1991], *Cherkaev and Gibiansky* [1992], and *Gibiansky and Torquato* [1993, 1995, 1996a, b]. In this paper, we will apply cross-property relations between the thermal or electrical conductivity and the bulk modulus of isotropic rocks.

We will also develop relations between the bulk modulus of the fluid-saturated rocks and the bulk modulus of the same rocks with dry pores or pores saturated with a different fluid. This issue has been well studied in the geophysics community. For example, the *Gassmann* [1951] formulas allow one to express the bulk and the shear moduli of the fluid-saturated rock in terms of the elastic moduli of the dry rocks and the compressibility of the saturating fluid. *Brown and Korringa* [1975] have generalized the Gassmann results to cases when more than one solid phase is present. *Berryman and Milton* [1991] found exact relations for porous rocks containing only two different solid constituents (see also *Norris* [1992] who discusses an analogy between exact relations in poroelasticity and thermoelasticity). *Biot* [1956] extended the theory to include dynamic effects and to take into account viscoelastic effects.

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These results are often treated as exact ones for general conditions; however, they are rigorously applicable only under certain restrictions. Specifically, it is assumed that the fluid pressure has the same value at all points. This is a reasonable assumption for a system of well-connected pores. However, fluid pressure in thin pores (or cracks) may differ from the fluid pressure in larger pores, especially in dynamic measurements of rocks saturated with a viscous liquid.

Experimental data also provide evidence that Gassmann equations fail to predict moduli correctly for some rocks. For example, *Gregory* [1976] reports the shear modulus of the rocks depends on the degree of the fluid saturation, whereas the Gassmann's theory predicts that saturation should not influence the shear modulus. Similarly, *Biot*'s [1956] theory does not always agree with the experimental data of *Gregory* [1977] on the shear waves velocities in rocks subject to low confining pressure.

In this paper we develop relations between the bulk moduli of the rocks saturated with different fluids, that are valid independently of the rock microstructure, even in the presence of cracks. We show that the Gassmann's bulk modulus relation corresponds to our lower bound. On the other hand, our upper bound corresponds to the *Hashin and Shtrikman* [1963] upper bound on the effective bulk modulus of the fluid-saturated porous material.

To obtain our cross-property relations, we use the methods and results of the theory of composite materials. Indeed, porous media, such as rocks, can be viewed as special types of composites. A dry porous medium can be considered to be a two-phase composite in which one of the phases is taken to be a void phase (with zero bulk and shear moduli). A fluid-saturated porous medium is a composite in which one of the phases is a fluid phase (with finite bulk modulus and zero shear modulus). These limits are difficult to study theoretically because of the large or infinite ratio of the phase properties. Indeed, dry pore regions can be viewed as a phase with vanishing elastic moduli and zero conductivity. Therefore even infinitesimal porosity can dramatically change the effective behavior of the composite. This is the case for solids containing cracks. The arrangement, shapes and density of the pores play a crucial role in defining the effective properties of the cracked bodies. Yet such crack statistics are often unknown thus preventing any reasonable estimate of the effective behavior. Conventional bounds on the effective properties of the composites [*Hashin and Shtrikman*, 1962, 1963; *Berryman and Milton,* 1988] fail in this situation, due to the special limiting nature of the problem, that is, infinite contrast of the phase properties and effectively zero volume fraction of the cracks. Recently, the authors obtained cross-property bounds that do not degenerate even in this special limiting situation, thus providing a means to estimate the effective properties of the porous or cracked solids [*Gibiansky and Torquato*, 1996a]. Such bounds remain meaningful even for the case of dry porous or cracked materials. They can be especially useful for geophysical applications when it is difficult or impossible to measure all of the properties of the rocks, but it is important to know their effective moduli.

In section 2 we review and reformulate the original and general cross-property bounds [*Gibiansky and Torquato*, 1993, 1996a] on the effective bulk modulus K_* of three-dimensional isotropic composites consisting of two isotropic phases in terms of the effective thermal conductivity k_* of such a medium. We then evaluate them in the special case of porous or cracked materials. As yet another application of our bounds, we found the relations between the formation factor of the fluid-saturated medium and the bulk modulus of the same medium.

In section 3 we further explore the idea of cross-property bounds. We discuss the fundamental *Gassmann* [1951] equation that relate the bulk modulus of the fluid-filled porous rock and the bulk modulus of the same dry rock or rock saturated with the fluid of a different compressibility. Then we derive new relations between these bulk moduli, without the constant-fluid-pressure assumption that lies at the root of the Gassmann's theory. In the limiting case when one of the fluids has an infinite compressibility (i.e., voids), our relations connect the effective bulk modulus of a dry porous medium to the bulk modulus of the same medium with fluid-filled pores. We show that Gassmann formula is in fact a lower bound on the bulk modulus of the fluid-saturated rock. We consider situations when one can measure the porosity of the samples, as well as those in which the porosity is unknown. We also apply our relations to cracked rocks saturated with a so-called soft fluid. This is a special limit of fluid-saturated rocks when the ratio γ of the fluid bulk modulus to the solid phase bulk modulus is small, the porosity of the sample f_2 is small, but the ratio $\beta = \gamma/f_2$ is finite and given. All of the results of section 3 are new.

In section 4 we summarize our theoretical findings and describe how to apply them in a variety of specific cases. In section 5 we apply our results to dry and water-saturated samples of Westerly granite and sandstones. We test our bounds against known experimental data for the effective bulk modulus of these rocks with dry or fluid-filled pores. We find good agreement between the theory and experiment.

2. Conductivity-Bulk Modulus Bounds

We start with the derivation of the cross-property relations that bound the effective bulk modulus of a porous composite given the effective thermal (or electrical) conductivity of the same sample. The physical idea of connecting elastic properties of porous media to the conductivity of the same media is not new. For example, *Brace et al.* [1965] studied the dependencies of elastic and electrical properties of rocks on the confining pressure. They attributed the observed dependence to a common cause: the change in the porosity due to the closing of cracks. Recently, we gave rigorous meaning to this intuitive idea and obtained optimal bounds on the effective bulk modulus of two-phase composites or porous materials in terms of their conductivity [*Gibiansky and Torquato*, 1993, 1996a, b]. In this section we summarize our results and recast them into a more explicit and useable form.

First let us introduce notation. We denote the thermal conductivity of a material by the constant k. Elastic properties of an isotropic body can be characterized by two constants. In the mathematical literature, the bulk modulus K and the shear modulus G are used to describe stiffness of the material. In the engineering literature the Young's modulus E and the Poisson's ratio ν are used more frequently. Only two of these constants are independent; the other two can be found by using the following interrelations:

$$K = \frac{E}{3(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)},$$
$$E = \frac{9KG}{3K+G}, \quad \nu = \frac{3K-2G}{6K+2G}. \tag{1}$$

Although there is no physical law that requires the Poisson's ratio to have a positive value, this coefficient is positive for most geological materials and we will assume in this paper that

$$\nu > 0. \tag{2}$$

As a remark, we emphasize that (2) is assumed for the Poisson's ratio of the solid constituent/grain of the porous material. Although the effective Poisson's ratio of dry porous rocks with high porosity may be negative [*Gregory*, 1976], the grain Poisson's ratio is always positive.

Let phase 1 have bulk modulus K_1, shear modulus G_1 and conductivity k_1, and phase 2 have the corresponding properties K_2, G_2, and k_2. The composite moduli are denoted by K_*, G_*, and k_*. Let also k_{1*} and k_{2*} be given by

$$k_{1*} = f_1k_1 + f_2k_2 - \frac{f_1f_2(k_1-k_2)^2}{f_2k_1 + f_1k_2 + 2k_1},$$
$$k_{2*} = f_1k_1 + f_2k_2 - \frac{f_1f_2(k_1-k_2)^2}{f_2k_1 + f_1k_2 + 2k_2}, \tag{3}$$

$k_{1\#}$ and $k_{2\#}$ denote the expressions

$$k_{1\#} = f_1k_1 + f_2k_2 - \frac{f_1f_2(k_1-k_2)^2}{f_2k_1 + f_1k_2 - 2k_1},$$
$$k_{2\#} = f_1k_1 + f_2k_2 - \frac{f_1f_2(k_1-k_2)^2}{f_2k_1 + f_1k_2 - 2k_2}, \tag{4}$$

and K_{1*}, K_{2*} denote the expressions

$$K_{1*} = f_1K_1 + f_2K_2 - \frac{f_1f_2(K_1-K_2)^2}{f_2K_1 + f_1K_2 + 4G_1/3},$$
$$K_{2*} = f_1K_1 + f_2K_2 - \frac{f_1f_2(K_1-K_2)^2}{f_2K_1 + f_1K_2 + 4G_2/3}. \tag{5}$$

In these formulas f_1 and f_2 are the volume fractions of phases 1 and 2, respectively. Moreover, let k_a and k_h, respectively, denote the arithmetic and harmonic averages of the phase conductivities

$$k_a = f_1k_1 + f_2k_2, \quad k_h = \left(\frac{f_1}{k_1} + \frac{f_2}{k_2}\right)^{-1}, \tag{6}$$

and K_a and K_h, respectively, denote the arithmetic and harmonic averages of the phase bulk moduli

$$K_a = f_1K_1 + f_2K_2, \quad K_h = \left(\frac{f_1}{K_1} + \frac{f_2}{K_2}\right)^{-1}. \tag{7}$$

Note that (3) and (5) coincide with the *Hashin and Shtrikman* [1962] bounds on the effective conductivity and *Hashin and Shtrikman* [1963] bounds on the effective bulk modulus of isotropic composites, respectively. The formulas (6) and (7) coincide with the Weiner-Reuss-Voigt bounds on the effective conductivity and the Hill's bounds on the effective bulk modulus, respectively. To our knowledge, the relations (4) do not have any physical meaning.

The cross-property bounds that were found by *Gibiansky and Torquato* [1993, 1996a, b] are given by segments of hyperbolas in the k_*-K_* plane. We denote by $HYP[(x_1,y_1),(x_2,y_2),(x_3,y_3)]$ the segment AB of a hyperbola that passes through the points $A=(x_1,y_1)$, $B=(x_2,y_2)$, and $C=(x_3,y_3)$ and may be parametrically described in the x_*-y_* plane as follows

$$x_* = \gamma_1x_1 + \gamma_2x_2 - \frac{\gamma_1\gamma_2(x_1-x_2)^2}{\gamma_2x_1 + \gamma_1x_2 - x_3},$$
$$y_* = \gamma_1y_1 + \gamma_2y_2 - \frac{\gamma_1\gamma_2(y_1-y_2)^2}{\gamma_2y_1 + \gamma_1y_2 - y_3}, \tag{8}$$

where $\gamma_1 = 1-\gamma_2 \in [0,1]$. Now we are ready to state the main results of this section.

2.1. General case

In a recent paper *Gibiansky and Torquato* [1996a] found the following prescription to obtain cross-property bounds between the effective bulk modulus and the effective conductivity:

Statement 1. To find cross-property bounds on the set of the pairs (k_*, K_*) of effective conductivity and effective bulk modulus of an isotropic composite at a fixed volume fraction $f_1 = 1 - f_2$, one should enscribe in the conductivity-bulk modulus plane the following five segments of hyperbolas:

$H_1 = HYP[(k_{1*},K_{1*}),(k_{2*},K_{2*}),(k_a,K_a)]$,
$H_2 = HYP[(k_{1*},K_{1*}),(k_{2*},K_{2*}),(k_1,K_1)]$,
$H_3 = HYP[(k_{1*},K_{1*}),(k_{2*},K_{2*}),(k_2,K_2)]$,
$H_4 = HYP[(k_{1*},K_{1*}),(k_{2*},K_{2*}),(k_{1\#},K_h)]$,
$H_5 = HYP[(k_{1*},K_{1*}),(k_{2*},K_{2*}),(k_{2\#},K_h)]$.

The outermost pair of these curves gives us the desired bounds.

The bounds of Statement 1 are depicted in Figure 1 for the hypothetical case where the phase moduli and volume fractions are given by

$$K_1 = 1, \quad \nu_1 = 0.3, \quad k_1 = 1, \quad f_1 = 0.2,$$
$$K_2 = 20, \quad \nu_2 = 0.3, \quad k_2 = 20, \quad f_2 = 0.8. \qquad (9)$$

Such cross-property relationss allow us to obtain restrictive bounds on the effective bulk modulus of a composite given the conductivity of the same composite, and vice versa. For example, according to Figure 1, if the experimentally measured dimensionless conductivity k_*^E/k_1 equal 10 then the dimensionless bulk modulus K_*/K_1 of this sample should lie in the interval $[5.7, 7.6]$.

Statement 1 is precise but at first glance may appear to be too complicated to implement. A routine but lengthy calculation leads to the following step-by-step procedure to find the bounds on the effective bulk modulus given effective conductivity of this composite, which is equivalent to Statement 1 but easier to implement.

Statement 1a. Given the value of the effective conductivity of a composite k_*, the effective bulk modulus K_* of the same composite is restricted by the following relations:

$$K_* \in [F(\alpha_{\min}, k_*), F(\alpha_{\max}, k_*)], \quad \text{for any } k_* \in [k_{2*}, k_{1*}], \qquad (10)$$

where $F(\alpha, k_*)$ denotes the function

$$F(\alpha, k_*) = \qquad (11)$$

$$\frac{\alpha K_{1*}(k_{2*} - k_*)(k_{1*} - k_{2*}) - K_{2*}(k_{1*} - k_*)(K_{1*} - K_{2*})}{\alpha(k_{2*} - k_*)(k_{1*} - k_{2*}) - (k_{1*} - k_*)(K_{1*} - K_{2*})},$$

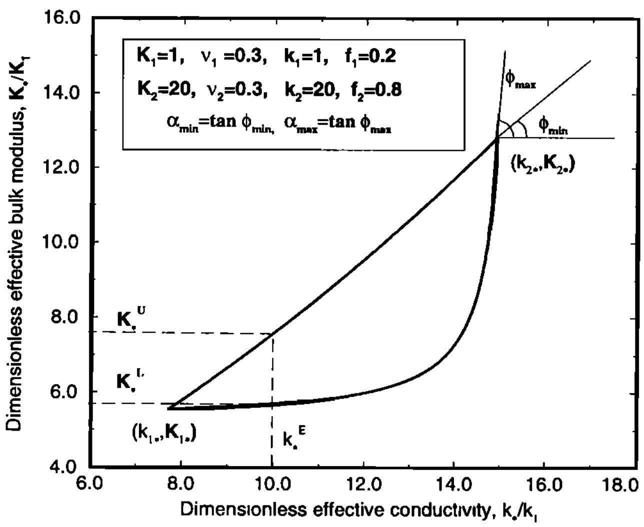

Figure 1. The cross-property bounds of Statement 1 in the conductivity-bulk modulus plane for the two-phase composite with phase moduli (9). The constants k_{1*}, k_{2*}, and K_{1*}, K_{2*} are defined in the text. The constants $\alpha_{\min}$ and $\alpha_{\max}$ are equal to the slope of the lines tangent to the bounds at the point (k_{2*}, K_{2*}); the values K_*^L and K_*^U are the bounds on the effective bulk modulus if the conductivity of the composite is equal to k_*^E.

$\alpha_{\max}$ and $\alpha_{\min}$ are the maximal and the minimal values

$$\alpha_{\max} = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\},$$
$$\alpha_{\min} = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \qquad (12)$$

of the following five coefficients:

$$\alpha_1 = \frac{6(G_1 - G_2)(f_1 k_2 + f_2 k_1 + 2k_2)^2 (K_1 - K_2)^2}{(k_1 - k_2)^3 (3 f_1 K_2 + 3 f_2 K_1 + 4G_2)^2},$$

$$\alpha_2 = \alpha_1 \frac{3k_1(3K_1 + 4G_2)}{(k_1 + 2k_2)(3K_1 + 4G_1)},$$

$$\alpha_3 = \alpha_1 \frac{(2k_1 + k_2)(3K_2 + 4G_2)}{3k_2(3K_2 + 4G_1)}, \qquad (13)$$

$$\alpha_4 = \alpha_1 \frac{2k_1 G_2}{(k_1 + k_2)G_1}, \quad \alpha_5 = \alpha_1 \frac{(k_1 + k_2)G_2}{2k_2 G_1}.$$

Note that

$$F(\alpha_{\min}, k_*) \le K_* \le F(\alpha_{\max}, k_*), \quad k_1 \ge k_2,$$
$$F(\alpha_{\min}, k_*) \ge K_* \ge F(\alpha_{\max}, k_*), \quad k_1 \le k_2, \quad (14)$$

and α_i is the tangent of the angle between the k_* axes in the k_*-K_* plane and the line tangent to the hyperbola H_i at the point (k_{2*}, K_{2*}) (see Figure 1).

For illustration purposes, we consider computing cross-property bounds on the bulk modulus of the composite with phase moduli (9) (where it should be noted that $k_1 \le k_2$ for this example). For such materials, $G_1 = 0.4615$, $G_2 = 9.231$, and the values of the parameters are equal

$$k_{1*} = 7.706, \quad k_{2*} = 14.91, \quad K_{1*} = 5.534,$$
$$K_{2*} = 12.82, \quad \alpha_1 = 2.110, \quad \alpha_2 = 1.272, \quad (15)$$
$$\alpha_3 = 1.212, \quad \alpha_4 = 4.019, \quad \alpha_5 = 22.16.$$

Therefore we have

$$\alpha_{\min} = \alpha_3 = 1.212, \qquad \alpha_{\max} = \alpha_5 = 22.16, \qquad (16)$$

and the effective bulk modulus is restricted by the inequalities

$$\frac{5.642k_* + 0.0623}{9.260 - 0.1808k_*} \ge K_* \ge \frac{62.26 - 3.950k_*}{11.62 - 0.7617k_*}, \quad k_* \in [7.706, 14.91]. \qquad (17)$$

Thus if the measured value of the conductivity is equal to $k_s^E/k_1 = 10$, then the bulk modulus of such a material are restricted by the interval $K_*/K_1 \in [5.686, 7.579]$, in agreement with Figure 1.

2.2. Porous Materials

One case of particular interest to us in this study is when the fluid phase 2 "moduli" k_2, K_2, and G_2 are much smaller than those of the stiffer phase 1, that is,

$$k_2/k_1 \ll 1, \quad K_2/K_1 \ll 1, \quad G_2/G_1 \ll 1, \qquad (18)$$

compared with the moduli of the stiffer phase 1. In this particular limit,

$$\alpha_1 = \frac{2G_1}{3k_1}, \quad \alpha_2 = \frac{6K_1G_1}{k_1(3K_1+4G_1)}, \quad \alpha_3 = \frac{3K_2+4G_2}{9k_2}, \quad \alpha_4 = 0, \quad \alpha_5 = \frac{G_2}{3k_2}. \tag{19}$$

Obviously, $\alpha_{\min} = \alpha_4 = 0$. To find $\alpha_{\max}$ we mention first that $\alpha_3 \geq \alpha_5$, and $\alpha_2 \geq \alpha_1$ if the Poisson's ratio of the phase 1 $\nu_1 > 0$ is positive. As we mentioned, rocks typically have a positive Poisson's ratio, and thus $\alpha_{\max} = \max\{\alpha_2, \alpha_3\}$. One can see that the the coefficient α_3 depends on the ratio of the properties of the void phase. At first glance, this appears counterintuitive, but actually it is natural to expect dependence of the composite effective properties on the properties of the void phase for special microstructures. Consider, for example, a cellular composite with the matrix of the phase 2 (with zero properties) surrounding the inclusions of the material with finite moduli. If we assume that the thickness δ of the cell walls is of the order of k_2, or K_2, or G_2, then the effective properties of such a composite will explicitly depend on the ratios δ/k_2, δ/K_2, or δ/G_2. Therefore the ratio of the effective conductivity to the effective bulk modulus of such a composite will depend on ratios similar to the α_3. Obviously, we can safely assume that the rocks' microstructures are not of this type. It is natural to restrict our consideration to composites with effective properties that are independent of the exact values of the void phase "moduli" k_2, K_2, and G_2 as long as these moduli are small compared with the moduli of the stiffer phase 1.

In light of the discussion above, we may assume without loss of generality that the moduli of the void phase satisfy the relation

$$\frac{3K_2+4G_2}{9k_2} \leq \frac{6K_1G_1}{k_1(3K_1+4G_1)}, \tag{20}$$

that guarantee the relation $\alpha_2 > \alpha_3$. Then

$$\alpha_{\max} = \alpha_2 = \frac{6K_1G_1}{k_1(3K_1+4G_1)} \tag{21}$$

and the following statement holds.

Statement 2. For a given value of the effective thermal conductivity k_* of a dry porous medium with a porosity f_2, the bounds on the effective bulk modulus K_* of the same medium are given by the inequalities:

$$0 \leq K_* \leq \frac{\alpha_2 K_{1*} k_* k_{1*}}{\alpha_2 k_* k_{1*} + (k_{1*} - k_*)K_{1*}}, \tag{22}$$

$$k_* \in [0, k_{1*}],$$

where α_2 is given by (19), and k_{1*} and K_{1*} are given by (3) and (5), respectively.

The bounds (22) can be recast as

$$\frac{1}{K_*} \geq \frac{1}{K_{1*}} + \frac{k_1(3K_1+4G_1)}{6K_1G_1}\left(\frac{1}{k_*} - \frac{1}{k_{1*}}\right), \tag{23}$$

$$k_* \in [0, k_{1*}].$$

From this inequality, one can get cross-property bounds on the effective bulk modulus of the porous medium with unknown porosity. Indeed, to find such relations, one needs only to find the union over the porosity $f_2 \in [0,1]$ of the sets that are described by the inequalities of Statement 2. The following result immediately follows.

Statement 3. The bulk modulus of a dry porous medium with arbitrary porosity is bounded by the inequality

$$\frac{1}{K_*} \geq \frac{1}{K_1} + \frac{k_1(3K_1+4G_1)}{6K_1G_1}\left(\frac{1}{k_*} - \frac{1}{k_1}\right), \tag{24}$$

$$k_* \in [0, k_1],$$

where k_* is the effective conductivity of the same medium.

One can see that in such a form, the bounds are applicable to cracked solids when $f_2 = 0$. The inequalities (23) and (24) were first obtained by *Gibiansky and Torquato* [1996a] and used for cracked solids by *Gibiansky and Torquato* [1996b]. It is convenient to express inequality (24) in the following dimensionless form:

$$\frac{K_1}{K_*} \geq 1 + \frac{3K_1+4G_1}{6G_1}\left(\frac{k_1}{k_*} - 1\right) = 1 + \frac{1-\nu_1}{1-2\nu_1}\left(\frac{k_1}{k_*} - 1\right), \tag{25}$$

$$k_* \in [0, k_1].$$

In section 5 we will apply these inequalities to get bounds on the effective thermal conductivity of dry samples of Westerly granite by using bulk modulus data obtained by *Brace et al.* [1965].

2.3. Fluid-Saturated Porous Materials

Statements 1 and 1a can be used to describe relations between the effective electrical conductivity and the effective bulk modulus of fluid-saturated porous solids. The electrical conductivity of the fluid is assumed to be much larger than the conductivity of the frame. Therefore we examine the important limiting case when the phases have the properties

$$\sigma_1 = 0, \quad G_2 = 0, \tag{26}$$

where σ_1 is the electrical conductivity of the frame. In this particular limit it is customary to introduce the dimensionless formation factor

$$F = \sigma_2/\sigma_* \tag{27}$$

where σ_2 is the electrical conductivity of the fluid, and σ_* is the effective electrical conductivity of the rock. In this limit we also have

$$\alpha_1 = -\frac{2G_1(f_1+2)^2(K_1-K_2)^2}{3\sigma_2(f_1K_2+f_2K_1)^2}, \quad \alpha_2 = \alpha_4 = \alpha_5 = 0, \quad \alpha_3 = \alpha_1\frac{K_2}{3K_2+4G_1}. \tag{28}$$

Obviously, $\alpha_{\min} = \alpha_1$, $\alpha_{\max} = 0$ and our bounds can be reformulated to yield the following statement.

Statement 4. For given values of the porosity f_2 and the formation factor F, the bounds on the effective bulk modulus K_* of the same fluid-filled medium are given by the inequalities:

$$\left[\frac{f_1}{K_1} + \frac{f_2}{K_2}\right]^{-1} \le K_* \le K_{1*} - \frac{2f_1^2 f_2 G_1 (K_1 - K_2)^2}{a(3a(f_2F-1) - 2f_1G_1)},$$

$$F \ge \frac{2+f_1}{2f_2}, \tag{29}$$

where K_{1*} is given by (5) and

$$a = f_2K_1 + f_1K_2 + 4G_1/3. \tag{30}$$

3. Relation Between Moduli of Dry and Fluid-Saturated Porous Materials

In this section we discuss traditional and obtain new relations between the effective bulk moduli of two isotropic fluid-filled porous media that have identical microstructures, the same moduli of matrix materials, but different fluid compressibilities. For example, such bounds can be applied to estimate the properties of oil-saturated rocks if one knows the effective moduli of the same rocks saturated by water. One of the fluids is allowed to have zero bulk modulus, corresponding to dry pores.

3.1. Gassmann Theory

Fundamental equations that are used to relate the effective bulk and shear moduli of an isotropic fluid-filled porous medium in terms of the moduli of a dry medium with the same microstructure were derived by *Gassmann* [1951]. According to the Gassmann theory, the bulk and shear moduli of the fluid-saturated rock (K_{sat} and G_{sat}, respectively) are given by

$$K_{\text{sat}} = K_m \frac{f_2K_{\text{dry}} - (1+f_2)K_{\text{dry}}K_f/K_m + K_f}{f_1K_f + f_2K_m - K_fK_{\text{dry}}/K_m},$$

$$G_{\text{sat}} = G_{\text{dry}}. \tag{31}$$

Here K_{dry} and G_{dry} are the bulk and shear moduli of the dry rock, K_m and G_m are the bulk and shear moduli of the mineral matrix phase, K_f is the bulk modulus of the saturating fluid, and f_2 is the porosity.

The inverse equation

$$K_{\text{dry}} = K_m \frac{1 - f_1K_{\text{sat}}/K_m - f_2K_{\text{sat}}/K_f}{1 + f_2 - f_2K_m/K_f - K_{\text{sat}}/K_m}, \tag{32}$$

gives the dry-rock bulk modulus in terms of the bulk modulus of the fluid-saturated rock.

Applying the relations (32) to the same rock saturated by the different fluids and excluding the bulk modulus of the dry rock, one can get the relations

$$K_{\text{sat}}^{(2)} = K_m + \left[\frac{K_f^{(1)} - K_f^{(2)}}{f_2(K_m - K_f^{(1)})(K_m - K_f^{(2)})} + \frac{1}{K_{\text{sat}}^{(1)} - K_m}\right]^{-1}, \quad G_{\text{sat}}^{(2)} = G_{\text{sat}}^{(1)}, \tag{33}$$

where $K_{\text{sat}}^{(1)}$ and $K_{\text{sat}}^{(2)}$ are the bulk moduli of the rock saturated by two different fluids, and $K_f^{(1)}$ and $K_f^{(2)}$ are the bulk moduli of two saturating fluids.

The "fluid-substitution" equations (31) and (32) have been used by geophysicists for decades as exact relations, sometimes without realizing the nature of the assumptions that are built into the Gassmann theory. Specifically, it is assumed that the fluid pressure in all the pores is the same. This assumption holds with a high degree of accuracy for static measurements of the rocks possessing a well-connected network of pores. However, for materials with a high crack-type porosity and in situations of low-frequency dynamic measurements (when the fluid cannot flow easily within a system of pores separated by thin cracks), the assumption of equal pore pressure may be too restrictive. One of the obvious drawbacks of the formulas (31) and (32) can be easily seen by studying a cracked body with zero porosity. In such a limit the formula (31) correctly predicts that the effective bulk modulus of the cracked body is equal to the bulk modulus of the uncracked matrix phase $K_{\text{sat}} = K_m$. This illustrates the well-known effect [*O'Connell and Budiansky*, 1974] that the saturation by the fluid strongly increases the effective bulk moduli of the cracked material by "gluing" the sides of the cracks. However, (32) incorrectly states that the bulk modulus of the cracked body is equal to the bulk modulus of the uncracked matrix, which is obviously wrong. Since any rock can have part of the porosity in the form of cracks, the Gassmann formulas may fail in specific situations when crack-type porosity is important in defining the effective properties of the rocks.

There are experimental data that support this last point. For example, *Gregory* [1976] reports the dependence of the shear modulus of the rocks on the degree of the fluid saturation that contradicts (32). Similarly, *Biot*'s [1956] theory, which is based on the same assumption as the Gassmann theory, does not agree with the experimental data of *Gregory* [1977] on the shear wave velocities in rocks subject to low confining pressure. One reason for such disagreement is the presence of cracks and microcracks where the fluid pressure may differ from the fluid pressure in a large pores, especially for dynamic measurements.

The goal of this section is to study relations between the effective properties of the fluid-saturated rocks without making any restrictive assumptions. This will allow us better understand the limit of applicability of Gassmann's equations.

3.2. New relations

The results presented in this section are derived by using the so-called translation method. Consideration of this specialized mathematical technique is beyond the scope of the present paper. We mention only that the derivation of the bounds is almost identical to the one given in our earlier papers on cross-property conductivity-bulk modulus bounds [*Gibiansky* *and Torquato*, 1996a]. Here we present the final results only.

Consider two porous media of identical microstructure (or two samples of the same medium) saturated by two different fluids. We assume that the fluids are not allowed to flow out of the solid frame. This means that the finite compressibility of the fluids contributes to the overall elastic moduli of the composite. Denote by K_m and G_m the bulk and shear moduli of the matrix material, respectively, by $K_f^{(1)}$ the bulk modulus of the fluid that fills the pores in the first composite ($K_f^{(1)} = 0$ in case of voids), and by $K_f^{(2)}$ the bulk modulus of the fluid that saturates the pores of the second composite. The fluid phases have zero shear moduli, that is, $G_f^{(1)} = 0$ and $G_f^{(2)} = 0$. Finally, we denote by $K_{\text{sat}}^{(1)}$ and $K_{\text{sat}}^{(2)}$ the effective bulk moduli of the first and second media, respectively. We also assume without loss of generality that the ratio $G_f^{(1)}/G_f^{(2)}$ lies within the interval

$$\frac{G_f^{(1)}}{G_f^{(2)}} \in \left[1, \frac{K_f^{(1)}(3K_f^{(2)}+4G_m)}{K_f^{(2)}(3K_f^{(1)}+4G_m)}\right]. \tag{34}$$

This is a technical assumption (similar to the inequality (20) of the previous section) that is required to prove the results of this section.

Let us define $K_{1*}^{(1)}$ and $K_{1*}^{(2)}$ by the following expressions:

$$K_{1*}^{(1)} = f_1K_m + f_2K_f^{(1)} - \frac{f_1f_2\left(K_m - K_f^{(1)}\right)^2}{f_2K_m + f_1K_f^{(1)} + 4G_m/3},$$

$$K_{1*}^{(2)} = f_1K_m + f_2K_f^{(2)} - \frac{f_1f_2\left(K_m - K_f^{(2)}\right)^2}{f_2K_m + f_1K_f^{(2)} + 4G_m/3} \tag{35}$$

Let also $K_h^{(1)}$ and $K_h^{(2)}$ denote the corresponding harmonic means

$$K_h^{(1)} = \left[\frac{f_1}{K_m} + \frac{f_2}{K_f^{(1)}}\right]^{-1}, \quad K_h^{(2)} = \left[\frac{f_1}{K_m} + \frac{f_2}{K_f^{(2)}}\right]^{-1} \tag{36}$$

of the phase moduli. Now we are ready to state the new results.

Statement 5. The pair $(K_{\text{sat}}^{(1)}, K_{\text{sat}}^{(2)})$ of effective bulk moduli associated with any two isotropic porous fluid-saturated composites at a fixed porosity f_2 having identical microstructures and matrix phase moduli but different fluid compressibilities, belong to a region in the $K_{\text{sat}}^{(1)}$-$K_{\text{sat}}^{(2)}$ plane restricted by two segments of the hyperbolas:

$H_1 = HYP[(K_{1*}^{(1)}, K_{1*}^{(2)}), (K_h^{(1)}, K_h^{(2)}), (K_m, K_m)]$,

$H_2 = HYP[(K_{1*}^{(1)}, K_{1*}^{(2)}), (K_h^{(1)}, K_h^{(2)}), (K_f^{(1)}, K_f^{(2)})]$.

Statement 5 can be rephrased according to the following computational prescription.

Statement 5a. Given the effective bulk modulus $K_{\text{sat}}^{(1)}$ of a fluid-saturated porous medium with porosity f_2, the effective bulk modulus $K_{\text{sat}}^{(2)}$ of the same medium filled with a different fluid is restricted by the inequalities

$$K_{\text{sat}}^{(2)} \in \left[F(\alpha_1, K_{\text{sat}}^{(1)}), F(\alpha_2, K_{\text{sat}}^{(1)})\right], \tag{37}$$

$$K_{\text{sat}}^{(1)} \in [K_h^{(1)}, K_{1*}^{(1)}],$$

where

$$F(\alpha, K_{\text{sat}}^{(1)}) = \frac{\alpha K_{1*}^{(2)}A + K_h^{(2)}B}{\alpha A + B}, \tag{38}$$

$$A = (K_h^{(1)} - K_{\text{sat}}^{(1)})(K_{1*}^{(1)} - K_h^{(1)}),$$

$$B = (K_{1*}^{(1)} - K_{\text{sat}}^{(1)})(K_h^{(2)} - K_{1*}^{(2)}),$$

$$\alpha_1 = \frac{(K_m - K_f^{(2)})^2(f_1K_f^{(1)} + f_2K_m)^2}{(K_m - K_f^{(1)})^2(f_1K_f^{(2)} + f_2K_m)^2},$$

$$\alpha_2 = \alpha_1\frac{K_f^{(2)}(3K_f^{(1)} + 4G_m)}{K_f^{(1)}(3K_f^{(2)} + 4G_m)}. \tag{39}$$

Note that one of the bounds

$$K_{\text{sat}}^{(2)} = F(\alpha_1, K_{\text{sat}}^{(1)}) \tag{40}$$

coincides with (33) derived by using the Gassmann formula (32). Thus, although the Gassmann formula in general does not represent the exact relation, it always gives us a bound on the effective bulk modulus.

One needs to know the porosity f_2 of the samples to evaluate the bounds of Statement 5. If the porosity is unknown then one should use the bounds that are independent of the phase volume fractions. Such bounds can be found as the union of the bounds (37) over the porosity $f_f \in [0,1]$ [*Gibiansky and Lakes*, 1993, 1997; *Gibiansky and Torquato*, 1995]. The corresponding volume fraction independent result is given by the following statement:

Statement 6. The pair $(K_{\text{sat}}^{(1)}, K_{\text{sat}}^{(2)})$ of effective bulk moduli associated with any two fluid-saturated isotropic composites at arbitrary porosity having identical microstructures matrix phase moduli but different fluid compressibilities, belongs to the region in the $K_{\text{sat}}^{(1)}$-$K_{\text{sat}}^{(2)}$ plane restricted by two segments of the hyperbolas:

$H_1 = HYP[(K_m, K_m), (K_f^{(1)}, K_f^{(2)}), -4(G_m, G_m)/3]$,

$H_2 = HYP[(K_m, K_m), (K_f^{(1)}, K_f^{(2)}), (0, 0)]$.

The extremal structures that precisely satisfy the bounds of Statement 6 are the *Hashin and Shtrikman* [1962, 1963] assemblages of the coated spheres with the external coating of the matrix material that surrounds the fluid phase (curve H_1) or with fluid spheres containing solid inclusions (curve H_2). Statement 6 is equivalent to the inequality given by the following statement.

Statement 6a. Given the effective bulk modulus $K_{\text{sat}}^{(1)}$ of a fluid-filled porous medium at arbitrary porosity, the effective bulk modulus $K_{\text{sat}}^{(2)}$ of the same medium saturated with a different fluid is restricted by the inequalities

$$K_{\text{sat}}^{(2)} \in \left[F_1(K_{\text{sat}}^{(1)}), F_2(K_{\text{sat}}^{(1)})\right], \tag{41}$$

$$K_{\text{sat}}^{(1)} \in [K_f^{(1)}, K_1],$$

where

$$F_1(K_{\text{sat}}^{(1)}) = K_m - \frac{a}{b}, \tag{42}$$

$$a = (K_m - K_{\text{sat}}^{(1)})(K_m - K_f^{(2)})(K_f^{(1)} + 4G_m/3),$$

$$b = 4G_m(K_m - K_f^{(1)})/3 + K_f^{(1)}(K_m - K_f^{(2)}) + K_{\text{sat}}^{(1)}(K_f^{(2)} - K_f^{(1)}),$$

$$F_2(K_{\text{sat}}^{(1)}) = \frac{K_{\text{sat}}^{(1)} K_f^{(2)}(K_m - K_f^{(1)})}{K_f^{(1)}(K_m - K_f^{(2)}) + K_{\text{sat}}^{(1)}(K_f^{(2)} - K_f^{(1)})}. \tag{43}$$

3.3. Dry Porous Medium Versus Fluid-Filled Porous Medium

In this section we study a particular limit of the bounds of the previous section when the bulk modulus of one of the fluids is equal to zero, that is, $K_f^{(1)} = 0$, corresponding to dry pores. In this case, the following bounds hold.

Statement 7. Given the effective bulk modulus K_{dry} of a dry porous medium with porosity f_2, the bounds on the effective bulk modulus K_{sat} of the same medium filled with a fluid are given by the following inequalities:

$$F(K_{\text{dry}}) \leq K_{\text{sat}} \leq K_{1*}^{(2)}, \qquad K_{\text{dry}} \in [0, K_{1*}^{(1)}], \tag{44}$$

where

$$F(K_{\text{dry}}) = K_m \frac{f_2 K_{\text{dry}} - (1 + f_2)K_{\text{dry}} K_f / K_m + K_f}{f_1 K_f + f_2 K_m - K_f K_{\text{dry}}/K_m} \tag{45}$$

coincides with the Gassmann equation (31), and $K_{1*}^{(1)}$ and $K_{1*}^{(2)}$ are given by (35).

Note that the Gassmann equation (31) turns out to be the lower bound on the effective bulk modulus of a fluid-saturated porous media. By varying the porosity of the composite and keeping track of the bounds of Statement 7, one can estimate the effective moduli of a composite with arbitrary porosity.

Statement 8. Given the effective bulk modulus K_{dry} of a dry porous medium with arbitrary porosity, the effective bulk modulus K_{sat} of the same medium filled with a fluid is restricted by the inequalities

$$K_m - \frac{4G_m(K_m - K_{\text{dry}})(K_m - K_f^{(2)})}{3K_f^{(2)} K_{\text{dry}} + 4K_m G_m} \leq K_{\text{sat}} \leq K_m,$$

$$K_{\text{dry}} \in [0, K_m]. \tag{46}$$

The other case of interest is a porous medium saturated with a soft fluid [*O'Connell and Budiansky*, 1974], that is, when the ratio $K_f^{(2)}/K_m$ of the bulk moduli of the filling fluid and matrix phase is small, $K_f^{(2)}/K_m \ll 1$, the porosity of the sample is small, $f_2 \ll 1$, but the ratio

$$\lim_{f_2 \to 0} \frac{K_f^{(2)}}{K_m f_2} \equiv \beta \tag{47}$$

has finite value. In this limit, Statement 7 leads to the following bounds.

Statement 9. Given the effective bulk modulus K_{dry} of a composite with dry pores and the value of the parameter β (see (47)), the effective bulk modulus K_{sat} of a soft fluid saturated porous medium is restricted by the inequalities:

$$K_m - \frac{K_m(K_m - K_{\text{dry}})}{K_m + \beta(K_m - K_{\text{dry}})} \leq K_*^{(2)} \leq K_m, \tag{48}$$

$$K_*^{(1)} \in [0, K_m].$$

4. Summary of the Theoretical Results

Given the number of different statements and bounds, it is useful to summarize our results in words and provide recommendations when and how to use each of the bounds. The following results were obtained in sections 2 and 3.

1. Cross-property bounds (10)-(13) relate the effective thermal conductivity of a composite to the effective bulk modulus of the same composite. Statement 1 gives the general form of the bounds, whereas Statement 1a reduces the bounds to a simple computational procedure. Such bounds can be used to estimate the effective bulk modulus of a two-phase media by using thermal conductivity measurements, and vice versa.

2. Cross-property conductivity-bulk modulus bounds are specialized to the case of porous media with dry pores by Statements 2 and 3. The bounds of Statement 2 should be used if one can measure the porosity of the sample. On the other hand, the bounds of Statement 3 are wider than those of Statement 2 since they do not require porosity measurements. They can be used as a rough estimate of the bulk modulus by using conductivity measurements if the porosity of the sample is unknown.

3. Cross-property electrical conductivity-bulk modulus bounds are specialized to the case of fluid-saturated porous media by Statement 4. They can be used to estimate the bulk modulus of the fluid-saturated medium if the formation factor is known.

4. Statements 5 and 5a describe relations between the bulk moduli of two rock samples of the same structure but saturated with different fluids. It is shown that Gassmann's relation corresponds to one of the bounds of Statement 5. Such bounds allow one to estimate the effective bulk modulus of the fluid-saturated rock (with given porosity) by measuring the bulk modulus of the same rock saturated by a different fluid. Statements 6 and 6a can be applied for the same purposes but when the porosity of the sample cannot be or has not been measured.

5. Statements 7 and 8 specialize the results of the Statements 5 and 6, respectively, to the case when one of the fluids is absent, that is, when one of the rocks is dry. It is shown that the lower bound of Statement 5 on the bulk modulus of the fluid-saturated porous rock exactly corresponds to the Gassmann's formula, whereas the upper bound is given by the corresponding Hashin-Shtrikman bound.

6. Statement 9 describes the relations between the effective bulk modulus of a dry porous body and the bulk modulus of the same body saturated by a soft fluid. Such a bound can be applied in the situation when the fluid that saturates the pores of the rocks is "weak" compared with the matrix and the porosity is low.

Now we shall apply the aforementioned results to study specific examples of geological and geophysical interest.

5. Applications

In this section we apply our bounds to examine data available in the literature for porous rocks. We will discuss two different examples. One concerns the properties of samples of dry and fluid-saturated Westerly granite which is often used as a benchmark to check approximate theories [see, e.g., *O'Connell and Budiansky*, 1974]. The other concerns the properties of dry and water-saturated sandstones presented by *Gregory and Podio* [1996].

5.1. Effective Properties of Westerly Granite

To analyze the effective properties of the Westerly granite we use the data found in *Brace et al.* [1965] and *Takeuchi and Simmons* [1973]. In order to apply our bounds, we need to have the phase properties, that is, physical properties of the rocks in an ideal state without pores and inclusions. When the applied pressure is zero, Westerly granite has a porosity $f_2 = 0.009$. Although the porosity is small, it influences the measurements of the properties due to the presence of cracks. When the large pressures are applied to such samples, the small cracks close, the porosity decreases, and the elastic moduli of the samples increase. In order to decrease further the influence of pores on the measurement of the granite properties we use data for the water-saturated samples, subjected to the large pressure, and then frozen. The ice that fills the pores in such frozen samples further decreases the influence of presence of pores. The bulk and the shear moduli of the Westerly granite samples at the pressure $p = 200$ MPa and temperature $T = -26°$C are given by

$$K_g = 51.5 \text{ GPa}, \quad G_g = 39.7 \text{ GPa}, \tag{49}$$

as reported by *Takeuchi and Simmons* [1973]. We will use these values to characterize the properties of pure uncracked Westerly granite. We also will need to use the elastic moduli of water. At pressure $p = 100$ MPa and room temperature the elastic moduli of water are equal to [*CRC Press*, 1972]

$$K_w = 2.8 \text{ GPa}, \quad G_w = 0. \tag{50}$$

As we already stated, the porosity of the granite depends on the applied pressure. Therefore the elastic moduli of granite also depend on applied pressure. Elastic moduli can be measured by two different methods. The dependence of the volume V of a sample versus the applied pressure P was reported by *Brace et al.* [1965]. We approximate their data by cubic splines to obtain the dependence of the bulk modulus $K_{\text{dry}} = -dV/dP$ on the applied pressure, as presented in Table 1. Although in the experiment [*Brace et al.*, 1965], the pores of the granite were filled with water, the measured quantity was in fact the bulk modulus of the solid frame, that is, of granite with dry pores. Indeed, the water was allowed to flow out of the samples, resulting in zero resistance to the volume change of the sample. In a different experiment [*Takeuchi and Simmons*, 1973], the speed of compressional waves through the water-saturated sample of Westerly granite was measured. By using these data and the known density of granite, the bulk modulus of the sample was calculated. In these experiments water had no time to flow from the pores, and thus it contributes to the overall bulk modulus. We treat these data K_{sat} as the bulk moduli of the water-saturated granite (see Table 1). Both types of measurements show that the elastic moduli increase with applied pressure, which can be explained by the fact that there is a closing of the cracks and pores.

Note that in the dynamic experiment, the water had no time to flow from pore to pore, especially if the pores were of the crack type. Under such conditions one may expect that the Gassmann assumption of equal pore pressure fails, and the resulting effective moduli will disagree with those given by the Gassmann's expression (31).

Now we will apply the conductivity-bulk modulus bounds of section 1. First, we will use data K_{dry} from Table 1 and Statement 3 to get the upper bound on the thermal conductivity of the dry sample of Westerly

Table 1. Pressure Dependence of Bulk Modulus and Bounds on Conductivity for Westerly Granite

P, MPa	K_{dry}, GPa	K_{sat}, GPa	k_{dry}/k_g
10	16.1	47.0	0.374
20	26.9	48.8	0.589
40	32.5	49.5	0.692
100	38.9	50.3	0.802
200	45.9	50.1	0.915

K_{dry} is interpolated from data by *Brace et al.* [1965]; K_{sat} is taken from *Takeuchi and Simmons* [1973].

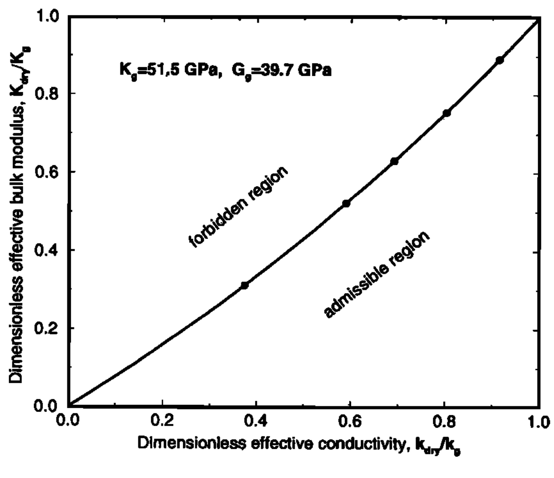

Figure 2. Cross-property bounds of Statement 3 for dry samples of Westerly granite of arbitrary porosity. The solid circles correspond to data of Table 1.

granite with arbitrary porosity. Granite has a positive Poisson's ratio

$$\nu_g = \frac{3K_g - 2G_g}{6K_g + 2G_g} = 0.193 \geq 0 \qquad (51)$$

and therefore by inverting (25) we have the inequality:

$$\frac{k_g}{k_{\text{dry}}} \leq 1 + \frac{1-2\nu_g}{1-\nu_g}\left(\frac{K_g}{K_{\text{dry}}} - 1\right). \qquad (52)$$

Substituting the values K_{dry} from Table 1, we get the bound on the effective thermal conductivity of such a medium; see Table 1. The set of possible pairs $(k_{\text{dry}}/k_g, K_{\text{dry}}/K_g)$ defined by the inequalities (25) is shown in Figure 2. The circles correspond to the lower bounds on the effective conductivity given in Table 1.

The relation between the formation factor and the bulk modulus of the water-saturated granite is illustrated by Figure 3. This figure shows the plane of admissible values of the pairs (F, K_{sat}), where $F = \sigma_w/\sigma_*$ is the formation factor equal to the inverse effective conductivity of water-saturated sample normalized by the conductivity of the fluid. We assume that the electrical conductivity of the granite is negligible compared to the electrical conductivity of the water. This is an excellent assumption for rocks saturated with brine.

Our next example illustrates the relation between the bulk moduli of dry and water-saturated samples of Westerly granite. Figure 4 shows the plane of admissible values of the pairs $(K_{\text{dry}}, K_{\text{sat}})$. The solid circles in this plane correspond to the data of Table 1. One can see that by increasing the applied pressure, one can dramatically increase the bulk modulus of the granite with dry cracks, but the effective bulk modulus of water-saturated samples is essentially unchanged. Indeed, even the very thin dry cracks can dramatically reduce the bulk modulus of the material, but the same fluid-filled cracks resist compression and will not affect the bulk modulus of the fluid-saturated sample, if these cracks are sufficiently thin. Therefore closing of small cracks will not affect bulk modulus of the fluid-saturated sample but will increase the stiffness of dry samples, as we see in Figure 4.

The bold solid curves show the bounds of Statement 8. Here we use the bulk modulus data for the dry samples of the Westerly granite and the inequality (46) to get a lower bound on the bulk modulus of a fluid-saturated sample of the same structure and unknown porosity. As we see, this bound is far below the real data. The main reason for this behavior is that it does not incorporate porosity data that are essential for the problem under study. We get much better results when the volume fraction of the pores is taken into account. The upper bound on the bulk modulus of fluid-saturated samples is independent of the bulk moduli of the dry samples and equal to the bulk modulus of the matrix phase K_m.

The porosity of natural samples of Westerly granite is equal to $f_2 = 0.009$ [*Takeuchi and Simmons,* 1973]. The porosity of the samples under the pressure is smaller, but we will not take this difference into account and will substitute $f_2 = 0.009$ in the bounds. The bounds (44) for such samples are shown by the bold dashed curves in Figure 4. It is seen that the lower bound is much better than the one that does not take into account porosity information. The upper bound is independent of the bulk moduli of the dry samples and equal to $K_{1*}^{(2)}$.

Note that the lower bound of Statement 7 (the bold dashed curve in Figure 4) coincides with the Gassmann's formula. One can see that the experimental data lie between the upper and the lower bounds, but closer to the upper bound. This confirms our expectations that Gassmann's assumptions are not valid for rocks with high crack-type porosity such as Westerly granite. Corrections should be made to the Gassmann's formula (31) that take into account presence of cracks.

Let us now apply the bounds of Statement 9 to the same sample. First, we check whether it is appropri-

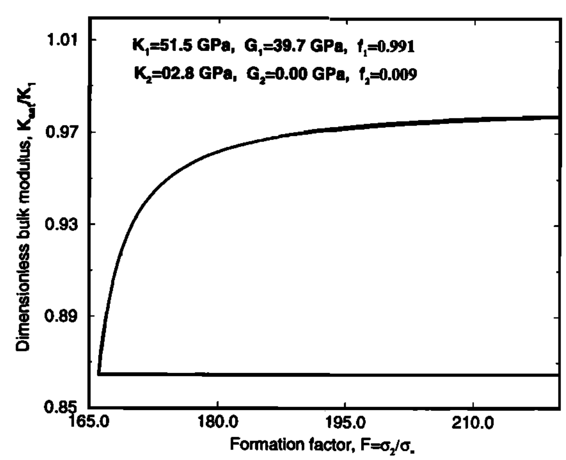

Figure 3: Cross-property bounds of Statement 4 in the formation factor-bulk modulus plane (F, K_{sat}) for water-saturated samples of Westerly granite.

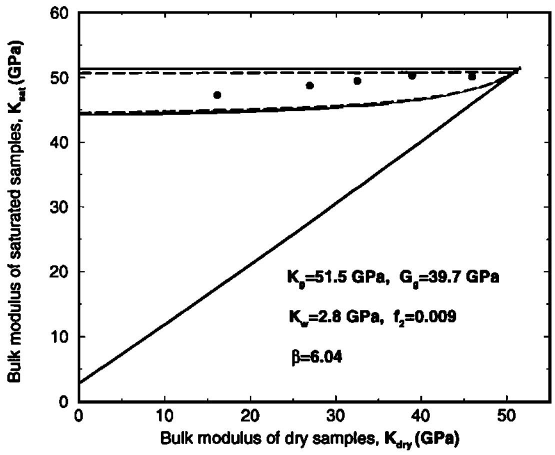

Figure 4. Cross-property bounds in the bulk moduli plane ($K_{\rm dry}, K_{\rm sat}$) for dry and water-saturated samples of Westerly granite. The bold solid, bold dashed, and light solid curves correspond to the bound of Statement 8 (for rocks of arbitrary porosity), Statement 7 (for rocks with known porosity), and Statement 9 (for soft-fluid saturated rocks), respectively. The solid circles correspond to the bulk moduli data of Table 1.

ate to use the soft fluid approximation for such rocks. The porosity of the Westerly granite is $f_2 = 0.009$ [*O'Connell and Budiansky*, 1974]. Therefore

$$\beta = \frac{K_w}{f_2 K_g} = \frac{0.028}{0.009 \times 0.515} = 6.04, \tag{53}$$

and we may safely use this approximation. The lower bound (48) of Statement 9 is given for this case by the thin solid curve in the Figure 4. It almost exactly follows the bound for $f_2 = 0.009$. The upper bound is equal to K_m.

5.2. Sandstone Samples

Our next goal is to compare the predictions of our bounds with experimental measurements of sandstone properties [*Gregory and Podio*, 1996]. Since these results are not widely available, we will first summarize the experimental data. By using measurements of the speed of different modes of elastic waves, these authors found the bulk and shear moduli of dry- and fluid-saturated rocks. The most comprehensive set of data exist for the sandstone samples taken from different places either on the Earth's surface or underground and studied in the laboratory to measure dependence of the elastic properties on the pressure and degree of saturation. In order to exclude the dependence of the properties on the cracks, we choose to use the data on the elastic properties of the samples subjected to the high pressure $P = 69$MPa when most of the small cracks are closed and do not influence the elastic moduli. Table 2 contains all of the data that will be used in this section.

To obtain the phase moduli of the sandstone (i.e., the moduli of the pure sandstone without damaging effect of pores and cracks), we study the experimental data of Table 2 concerning dependence of the bulk and shear moduli of the water-saturated sandstone samples on the porosity and extrapolate this dependence to the origin, that is, to the value $f_2 = 0$ of the volume fraction of water.

There are several possible approximations $K_*(f_2)$, $G_*(f_2)$ that allow for the extrapolation to the origin $K_m = K_*(0)$, $G_m = G_*(0)$. The simplest is the linear one, but it does not take into account specific behavior of the effective moduli as a function of the volume fraction at zero porosity and at the other end $f_2 = 1$ of the interval $f_2 \in [0, 1]$. Instead, we have used the following approximations:

$$K_*(f_2) = (1 - f_2)K_m + f_2 K_w - \frac{f_2(1-f_2)(K_m - K_w)^2}{f_2 K_m + (1-f_2)K_w + B_K},$$
$$G_*(f_2) = (1 - f_2)G_m - \frac{f_2(1-f_2)G_m^2}{f_2 G_m + B_G}, \tag{54}$$

where f_2 is the porosity of the sample, K_m and G_m are the unknown bulk and shear moduli of the sandstone, K_w is the bulk modulus of water, and B_K and B_G are some unknown parameters.

This dependence closely resembles the dependence of the moduli of the Hashin-type structures [*Hashin and Shtrikman*, 1963] on the porosity. The functions $K_*(f_2)$, $G_*(f_2)$ have the correct values at the end points of the interval $f_2 \in [0, 1]$ where we have

$$K_*(0) = K_m, \qquad K_*(1) = K_w,$$
$$G_*(0) = G_m, \qquad G_*(1) = G_w = 0. \tag{55}$$

By using the data of Table 2 and the nonlinear regression program of the graphic package XMgr (Turner, P.J., XMgr computer program, 1995), we found that

$$K_m = 43.6, \qquad B_K = 5.3,$$
$$G_m = 38.5, \qquad B_G = 6.1. \tag{56}$$

Table 2. Elastic Moduli of the Water-Saturated Sandstone Samples

N	Origin	f_2	$K_{\rm dry}$ GPa	$G_{\rm dry}$ GPa	$K_{\rm sat}$ GPa	$G_{\rm sat}$ GPa
1	Travis Peak	0.046	24.0	26.2	35.6	26.1
2	Travis Peak	0.080	18.0	27.8	24.1	28.6
3	Chugwater	0.110	15.2	16.3	26.4	14.1
4	Green River	0.117	22.2	24.7	28.3	24.3
5	Cabinda	0.124	18.2	21.0	30.2	18.7
6	Tensleep	0.152	17.8	20.4	21.2	19.9
7	Bandera	0.179	14.5	12.5	18.9	12.1
8	Berea	0.191	14.2	15.7	18.7	14.8
9	Gulf Coast	0.217	12.7	13.1	18.1	12.7
10	Nichols Buff	0.225	11.4	13.0	18.4	11.6
11	Boise	0.268	10.7	08.8	14.6	09.3

According to *Gregory and Podio*, [1996].

Table 3. Bounds on the Elastic Moduli of the Water-Saturated Sandstone Samples

N	Origin	f_2	K_{dry} GPa	K^L_{sat} GPa	K_{sat} GPa	K^U_{sat} GPa
1	Travis Peak	0.046	24.0	31.9	35.6	40.4
2	Travis Peak	0.080	17.8	26.5	24.1	38.2
3	Chugwater	0.110	15.2	23.4	26.4	36.3
4	Green River	0.117	22.2	27.0	28.3	35.9
5	Cabinda	0.124	18.2	24.4	30.2	35.5
6	Tensleep	0.152	17.9	23.3	21.2	33.8
7	Bandera	0.179	14.5	20.4	18.9	32.3
8	Berea	0.191	14.2	19.9	18.7	31.6
9	Gulf Coast	0.217	12.7	18.3	18.0	30.2
10	Nichols Buff	0.225	11.1	17.1	18.4	29.8
11	Boise	0.268	10.7	16.0	14.6	27.6

The experimental data K_{dry} [*Gregory and Podio*, 1996] (see also Table 2) and the bounds of Statement 7 are used to get the lower bound K^L_{sat} and upper bound K^U_{sat} on the bulk modulus K_{sat} of the water-saturated sandstones.

Note that in the analyses we use the values (50) of the compressibility of water at high pressure used in these experiments. The experimental points and the result of the approximation (54) with the parameters (56) are presented in Figures 5 and 6 for the bulk and shear modulus, respectively.

Now our aim is to apply the bounds of Statement 8 on the bulk modulus of the water-saturated sandstone samples given the bulk modulus of the dry sandstone samples with arbitrary porosity. Such bounds are depicted by the bold curves in Figure 7 and compared with the experimental results (solid circles in Figure 7) described by Table 2. As we see, the theoretical bounds agree with the experiment with sufficient accuracy even if the porosity data are not used.

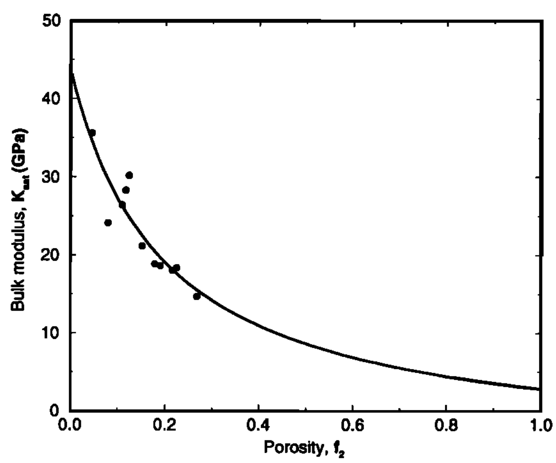

Figure 5. Experimental data (solid circles) which describe the dependence of the bulk modulus of water-saturated sandstone samples on the porosity, as given in Table 2. The solid curves represent approximation (54) with the values of the parameters given by (56).

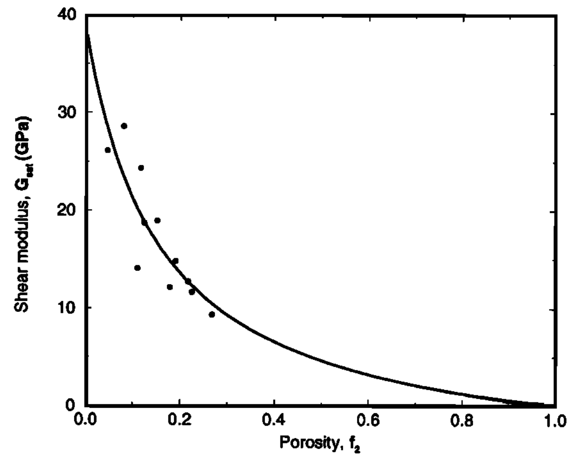

Figure 6. Experimental data (solid circles) which describe the dependence of the shear modulus of water-saturated sandstone samples on the porosity, as given in Table 2. The solid curves represent approximation (54) with the values of the parameters given by (56).

Let us now apply the bounds of Statement 7 on the bulk modulus of the water-saturated sandstone samples given the bulk modulus of the dry sandstone samples and the porosity of the samples. The results are described in Table 3 where the lower bound K^L_{sat} and the upper bound K^U_{sat} on the bulk modulus of the water-saturated sandstone samples are presented and compared against the experimental measurement K_{sat} of the same quantity. Again we see that the theoretical bounds agree quite well with the experiment: the experimental points scattered around the lower bound with the largest error being less than 10% from the lower bound. The difference can be easily explained by the variations in the properties of the matrix phase in the samples taken from different locations. The bounds in Table 3 provide better estimates of the elastic properties because they utilize porosity data.

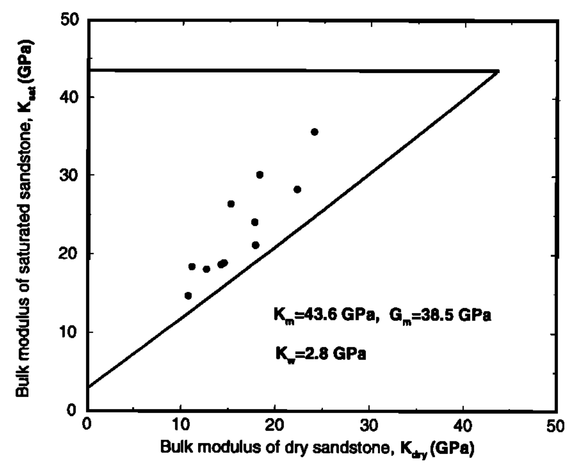

Figure 7. Cross-property bounds of Statement 8 for dry and water-saturated samples of sandstones with arbitrary porosity. The solid circles correspond to the bulk moduli data summarized in Table 2.

Note that for the sandstones (unlike the Westerly granite samples), the data are closer to our lower bound that corresponds to the Gassmann's formula (31). This is not surprising since sandstones are rocks with high porosity and a well-connected system of pore channels. For such rocks the assumption of equal pore pressure that lies in the foundation of the Gassmann theory is in good agreement with reality.

To summarize, it is remarkable that our bounds agree so well with the experiments. Indeed, since the rock samples differ from place to place, our attempt to find the pure sandstone moduli can be considered only as a statistical estimate of the average value of such moduli over the large and diverse set of samples. Nevertheless, the resulting bounds are confirmed by the experiments with reasonable accuracy. This shows that our approach based on cross-property relations has the potential to be useful in studying rock properties. Our results add new insight to understanding of the properties of the porous media. They show the limitations of the traditional Gassmann's relations (that coincides with our lower bounds). It is shown that the Gassmann approximation works well for rocks with high porosity but needs to be corrected for rocks with high crack-type porosity.

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