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Crack Models for a Transversely Isotropic Medium

C. H. CHENG

Earth Resources Laboratory, Department of Earth, Atmospheric, and Planetary Sciences Massachusetts Institute of Technology, Cambridge

A commonly used model for a transversely isotropic cracked rock is that by Hudson (1980, 1981). This model is based on a simplified analysis of a thin circular crack, with displacement and stress conditions specified on the boundary. These papers have a second-order correction in addition to the first-order term in porosity/crack density. In this paper we compare the results of Hudson with those of Anderson et al. (1974) and Cheng (1978) using the long-wavelength static approximation and the ellipsoidal crack model first proposed by Eshelby (1957). We show that the Hudson model and those based on the complete Eshelby theory agree for small-aspect-ratio cracks and small crack densities, as expected, provided the "weak material" version of Hudson's (1981) model is used. For larger crack densities but small aspect ratios, Hudson's first-order term agrees with the Eshelby solution. The expansion in the second-order term in crack density is an asymptotic series and not a uniformly converging series. Thus there is no general statement one can make about the accuracy of the second-order expansion that is valid for a variety of situations. A new expansion based on the Padé approximation is proposed which is identical to Hudson's expansion up to second-order in density. This expansion avoids some of the problems associated with Hudson's second-order expansion such as increasing moduli with crack density at relatively small crack densities.

INTRODUCTION

Crack-induced anisotropy is of extreme importance in a large number of geophysical applications ranging from earthquake prediction to petroleum and geothermal exploration. The concept of shear wave splitting through a cracked or fractured medium is widely accepted and is considered to be diagnostic of such a medium. Furthermore, given the observed shear wave splitting and velocity anisotropy, one can attempt to obtain crack or fracture parameters such as crack density by the use of a crack model. The commonly used crack-induced seismic-anisotropy model is that of *Hudson* [1980, 1981], which is based on a scattering formulation for very small-aspect-ratio (thickness over length) cracks. In order to better understand quantitatively the limits and accuracy of this model, we will examine the Hudson model closely in this paper, especially in comparison with the first-order static model of*Anderson et al.* [1974] and *Cheng* [1978] derived directly from the Eshelby theory, without the limitation of small-aspect-ratio cracks. We will also examine the issue of the second-order term in the expansion of the effective properties (seismic velocities) in terms of the crack density. This study is done with an emphasis on the implications to the interpretation of crack density and saturation in seismic anisotropy observed from field data.

Previous work in comparing the Hudson model with a model based on the Eshelby theory was done by *Douma* [1988]. In his paper, Douma compared the results of the second- order Hudson model to that of *Nishizawa* [1982] in which a differential embedding scheme was applied to the Eshelby theory to model higher crack densities. Here we will compare the first-order Hudson model with the equivalent first-order Eshelby model for various aspect ratio cracks in order to quantitatively assess the limit of accuracy of Hudson's assumption of very small aspect ratio cracks. We will also compare the differences between the "fluid-filled" (zero-aspect-ratio) and the "weak-material" (fluid-filled) models of Hudson.

It is well known that at higher crack densities, the second-order term in Hudson's expansion begins to dominate over the first-order term, causing the effective moduli to increase instead of decrease with increasing crack density. This effect can occur at relatively small crack densities (about 0.19 for the compressive moduli with empty cracks). The whole idea of having a higher order expansion is to try to extend the first-order theory to higher crack densities. However, because of the nature of the expansion, this is not possible under the present form. By examining this nature and by comparison with a well known exact solution for a spherical cavity, we propose a new expansion based on the Padé approximation. The Padé approximation is frequently used to evaluate functions which are divergent if expanded in a power series representation [*Bender and Orszag*, 1978; *Press and Teukolsky*, 1992]. We demonstrate that this expansion is monotonically decreasing with crack density for all crack densities and is identical to the Hudson expansion up to the second order in crack density. We suggest that this new form should be more accurate to a higher crack density.

THEORY

For a transversely isotropic material with the axis of symmetry the three-axis, the five independent elastic constants are usually represented by $c_{11}, c_{13}, c_{33}, c_{44}$, and c_{66}, corresponding to the terms $c_{1111}, c_{1133}, c_{3333}, c_{2323}$, and c_{1212} in the fourth order tensor notation. For the compressional and shear wave velocities, respectively, the solutions of the wave equation are given by [*Love*, 1927]:

$$\begin{aligned} 2\rho v^2 &= c_{11} sin2\theta + c_{33} cos2\theta + c_{44} \\ &\pm\{(c_{11}-c_{44}) sin2\theta - (c_{33}-c_{44}) cos2\theta)^2 \\ &+4 sin2\theta cos2\theta (c_{13}+c_{44})^2\}^{1/2} \end{aligned} \tag{1}$$

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where ρ is the effective density of the medium, v the phase velocity. The + sign corresponds to the quasi-P wave, the - sign the quasi-SV wave. The incidence angle θ is measured from the axis of symmetry (three-axis). The phase velocity of the quasi-SH wave is given by

$$\rho v^2 = c_{66} sin2\theta + c_{44} cos2\theta. \tag{2}$$

The problem now is to find the effective c_{ij} for a circular or ellipsoidal crack of radius a imbedded in an isotropic background medium. The crack is assumed to have an axis of symmetry along the three-axis. With the effective c_{ij} found, the corresponding P and S wave velocities as a function of crack parameters and incidence angles can be easily calculated. In the next sections we will outline the two crack models under study in this paper

Eshelby's Model

Eshelby [1957] developed an exact solution for strain inside an ellipsoidal inclusion in an isotropic matrix under applied constant stress or strain. It is a static solution. From this we can get the effective elastic moduli by considering the potential energy of the total system with volume V under a constant applied strain e^A:

$$\frac{1}{2} c^*_{ij} e^A_i e^A_j V = \frac{1}{2} c^0_{ij} e^A_i e^A_j V + E_{int} \tag{3}$$

where E_{int} is the change in the energy due to presence of the crack. It is given by

$$E_{int} = -\frac{1}{2} c^0_{ij} e^A_i e^T_j V_{int}. \tag{4}$$

and e^T_j is called the "stress-free" strain of the inclusion. It is the strain necessary to restore the inclusion to its original form under the applied strain e^A_j. It is related to the applied strain through a matrix which is a function of the volume of the inclusion and the aspect ratio of the inclusion, as well as its elastic properties. V_{int} is the volume of the inclusion. The upshot is that

$$c^*_{ij} = c^0_{ij} - \phi c^1_{ij} \tag{5}$$

where ϕ is the porosity. *Anderson et al.* [1974] presented a calculation of velocity anisotropy as a function of saturation, porosity and aspect ratio of an ellipsoidal crack based on the Eshelby formulation. *Cheng* [1978] gave the complete explicit form for the coefficients c^1_{ij}. The terms are given in the Appendix.

Hudson's Model

Hudson, in a series of papers [*Hudson*, 1980, 1981], developed a second-order expansion for the effective moduli of a crack-induced transversely isotropic medium by means of a scattering approach. Very small aspect ratio cracks were assumed. The results can be expressed in the following form:

$$c^*_{ij} = c^0_{ij} + c^1_{ij} + c^2_{ij} \tag{6}$$

c^*_{ij} are the effective elastic moduli, c^0_{ij} the background moduli, c^1_{ij} the first-order correction, and c^2_{ij} the second-order correction. The background moduli are given by

$$\begin{aligned} c^0_{11} &= c^0_{33} = \lambda + 2\mu \\ c^0_{13} &= \lambda \\ c^0_{44} &= c^0_{66} = \mu, \end{aligned} \tag{7}$$

where λ, μ are the Lamé constants. First-order corrections are given by

$$\begin{aligned} c^1_{11} &= -\frac{\lambda^2}{\mu} \epsilon U_3 \\ c^1_{13} &= -\frac{\lambda(\lambda+2\mu)}{\mu} \epsilon U_3 \\ c^1_{33} &= -\frac{(\lambda+2\mu)2}{\mu} \epsilon U_3 \\ c^1_{44} &= -\mu \epsilon U_1 \\ \text{and } c^1_{66} &= 0; \end{aligned} \tag{8}$$

and second-order corrections are given by

$$\begin{aligned} c^2_{11} &= \frac{q}{15} \frac{\lambda^2}{\lambda+2\mu} (\epsilon U_3)^2 \\ c^2_{13} &= \frac{q}{15} \lambda (\epsilon U_3)^2 \\ c^2_{33} &= \frac{q}{15} (\lambda+2\mu)(\epsilon U_3)^2 \\ c^2_{44} &= \frac{2}{15} \frac{\mu(3\lambda+8\mu)}{\lambda+2\mu} (\epsilon U_1)^2 \\ \text{and } c^2_{66} &= 0; \end{aligned} \tag{9}$$

$$\begin{aligned} \theta &= 15\frac{\lambda^2}{\mu^2} + 15\frac{\lambda}{\mu} + 28 \\ \epsilon &= \text{crackdensity} = \frac{3\phi}{4\pi\alpha}, \\ \alpha &= \text{aspect ratio} \\ &= \text{minor/major axis of ellipsoid,} \end{aligned} \tag{10}$$

and U_3 and U_1 are to be defined for various inclusions. For fluid-filled cracks, *Hudson* [1980,1981] gave the following expressions for U_3 and U_1, using the assumption that fluid saturation does not affect the compressional moduli, which is equivalent to saying that the crack thickness is zero (i.e., aspect ratio, α, is zero):

$$\begin{aligned} U_3 &= 0 \\ U_1 &= \frac{16(\lambda+2\mu)}{3(3\lambda+4\mu)} \end{aligned} \tag{11}$$

For dry cracks, the expressions are

$$\begin{aligned} U_3 &= \frac{4(\lambda+2\mu)}{3(\lambda+\mu)} \\ U_1 &= \frac{16(\lambda+2\mu)}{3(3\lambda+4\mu)}. \end{aligned} \tag{12}$$

In (11) and (12) the aspect ratio of the crack does not appear because the cracks are assumed to be vanishingly thin; namely, with an aspect ratio equal to zero.

In addition, *Hudson* [1981] provided expressions for "weak" inclusions; namely, materials with bulk and shear moduli very much smaller than the matrix or background moduli. In particular, a fluid-filled crack can also be considered as a special case where the inclusion shear modulus vanishes. For this case, the expressions become

$$\begin{aligned} U_3 &= \frac{4(\lambda+2\mu)}{3(\lambda+\mu)}\frac{1}{(1+K)} \\ U_1 &= \frac{16(\lambda+2\mu)}{3(3\lambda+4\mu)} , \end{aligned} \qquad (13)$$

with

$$K = \frac{\kappa'(\lambda+2\mu)}{\pi\alpha\mu(\lambda+\mu)} , \qquad (14)$$

where κ' is the bulk modulus of the fluid. *Hudson* [1981] derived (12) using a simplified version of *Eshelby's* [1957] equations. For this case, U_3 does depend on the aspect ratio, but U_1 remains independent of the aspect ratio. Thus for a fluid-saturated crack, (13) is a more complete expression for U_3 than (11), and the latter can be recovered from the former by letting the aspect ratio α go to zero.

NUMERICAL EXAMPLES

In this section we compare the results from the complete first-order expansion based on Eshelby's theory and those of Hudson. *Douma* [1988] compared the second-order Hudson theory with that of *Nishizawa* [1982] which is based on a differential embedding of Eshelby's formulation. There is no direct comparison between Hudson's first-order term with that of Eshelby. Since Hudson used a small aspect ratio assumption, it is important to know when this assumption breaks down. One way to determine this is by comparing Hudson's approximate solution with the complete solution on which it is based (i.e., Eshelby), and not by comparing models with other approximations built in, as it was done by *Douma* [1988].

We should point out that the formal limit quoted by Hudson is for crack densities less than 0.1 (or a crack porosity of about 0.42 times the aspect ratio) for both first- and second- order terms. However, in practice, many authors have pushed the application to beyond the formal limit, especially for the second-order expansion. It was assumed by many authors that the second-order expansion allows for the extension of the validity of the theory to a higher crack density. In fact, the whole idea of having a second-order expansion is for precisely that reason, since rocks in the earth tend to have higher crack densities. It should be pointed out that the formal limit for the exact Eshelby-based theories is for a crack porosity equal to the aspect ratio, or equivalent to a crack density of 0.24.

Fluid-Filled (Zero-Aspect-Ratio) Cracks

The first comparison is for the fluid-filled (zero-aspect-ratio) crack model. Figure 1 shows the velocities for the quasi-P, quasi-SV, and quasi-SH wave as a function of incident angle (measured from the axis of symmetry) for both Hudson's and Eshelby's model. In this case, $U_3 = 0$ for the Hudson theory and all the changes in the quasi-P wave velocities comes about through the changes in the shear modulus term. A crack porosity of 0.005 (crack density of 0.12) and an aspect ratio of 0.01 (for the Eshelby model) is used. The background material is a Poisson solid, with $\lambda = \mu =$ 39 GPa, with water-saturation and a fluid bulk modulus of 2.2 GPa. As seen in the figure, the differences in the two results are significant. This difference is due to the assumption used by Hudson that the fluid saturation does not affect the compressional moduli, which is only strictly true for an infinitely thin crack.

Weak Inclusions (Fluid-Filled, Including Dry Cracks)

Figure 2 shows the same situation as Figure 1 except now the weak-inclusion (fluid-filled) formulation (equations(13) – (14)) is used instead of the zero-aspect-ratio, fluid-filled crack assumption. The first-order theory of Hudson is used to compare with the first-order expansion of Eshelby. The two results are almost identical. Figure 3 shows the results for the dry crack. All other parameters remain constant.

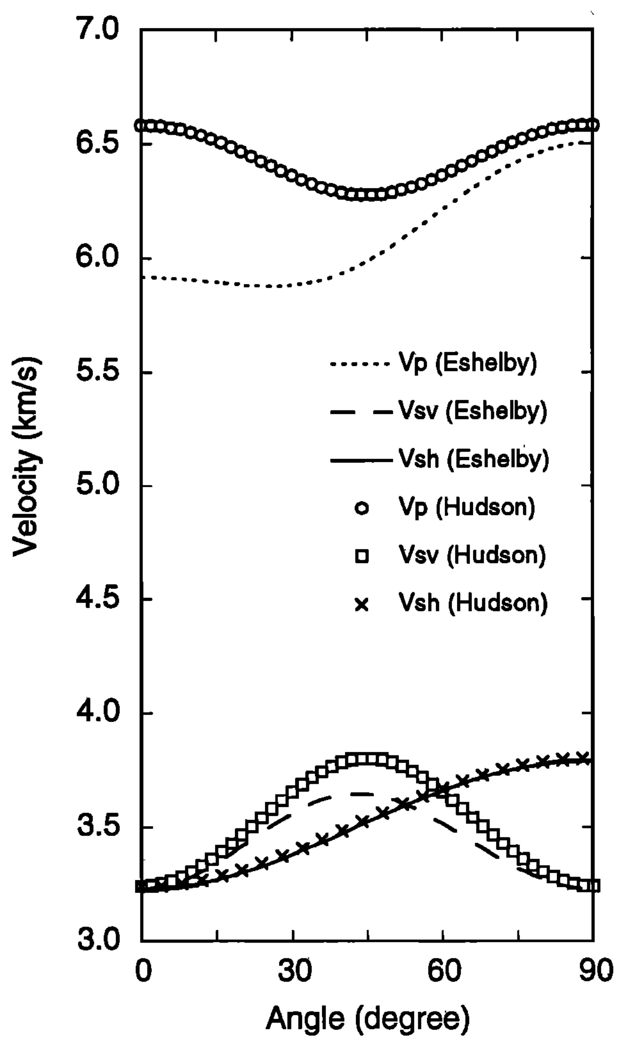

Fig. 1. Comparison of quasi-P, quasi-SV and quasi-SH velocities as a function of angle for Hudson's fluid-filled crack model and Eshelby's model. The background medium is taken to be a Poisson solid with $\lambda = \mu = 39$ GPa. The bulk modulus of water is taken to be 2.2 GPa. The crack aspect ratio is 0.01 and the crack porosity 0.005.

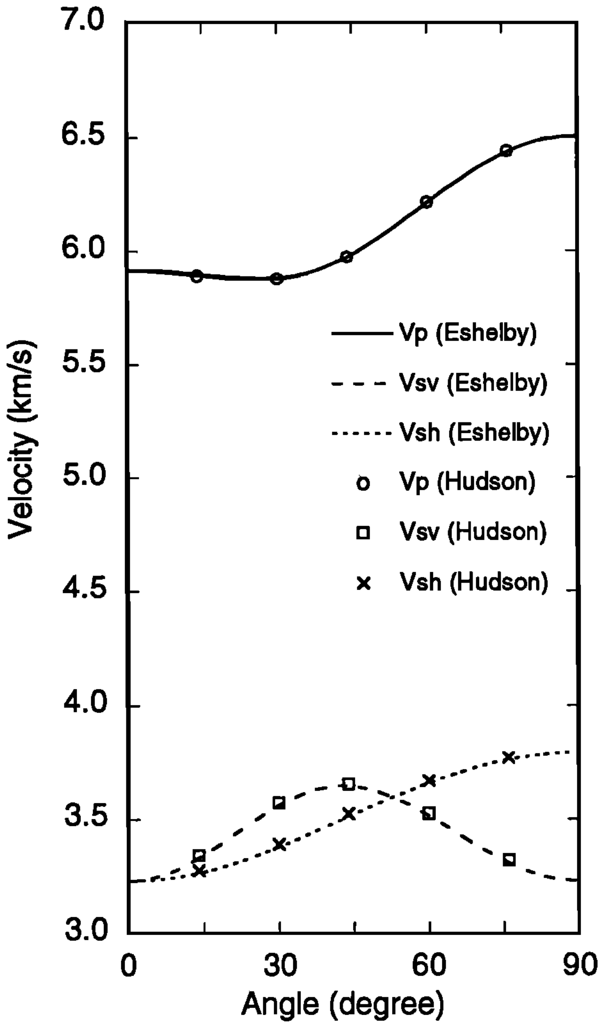

Fig. 2. Same as Figure 1 for Hudson's first order weak inclusion model and Eshelby's model. The cracks are water-saturated.

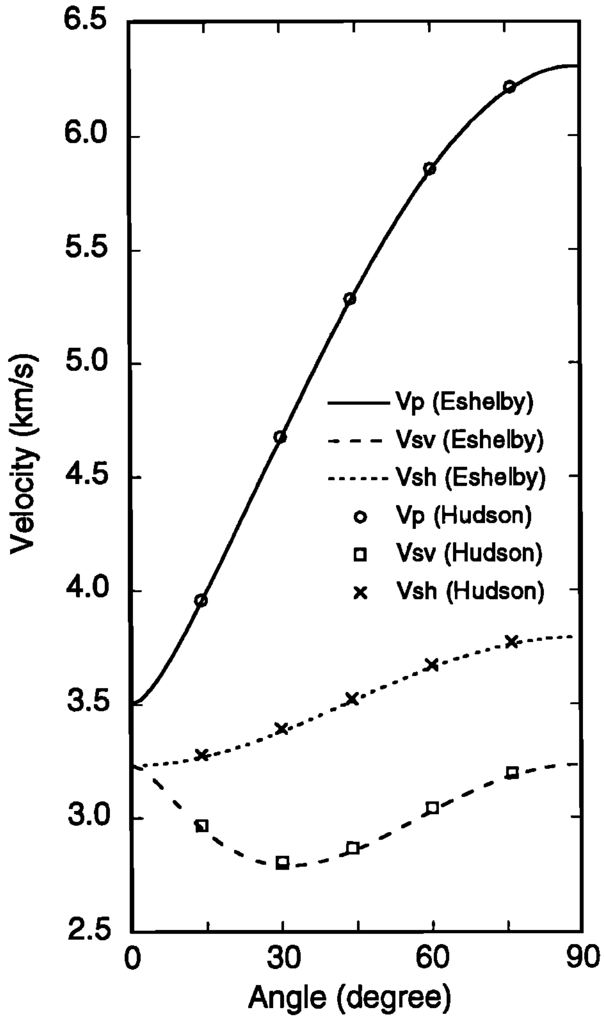

Fig. 3. Same as Figure 2 for dry cracks.

Again, the results of the first-order theory of Hudson agree well with those from the more complete theory of Eshelby. We thus can expect that for aspect ratios less than or equal to 0.01, the first-order weak inclusion theory of Hudson is essentially the same as the much more complicated complete expansion of Eshelby's theory.

Large Aspect Ratio Cracks

Figures 4 and 5 show the result for water-saturated and dry cracks with aspect ratio of 0.1. All other parameters remain the same as in the previous figures. The crack density is held at 0.12. As we can see from Figures 4 and 5, the difference between the first-order Hudson theory and the Eshelby expansion is a little larger than that for smaller aspect ratio cracks. This difference is expected since the Hudson expansion is specifically for small aspect ratio cracks.

Second-Order Hudson Theory

By using a scattering approach, *Hudson* [1980] developed the second-order expansion for the effective moduli for rocks with crack-induced anisotropy. Figures 6 and 7 compare the effects of Hudson's second and first-order expansions for water-saturated and dry cracks. The weak inclusion formulation is used for the water-saturated crack. The other parameters are the same as those used in Figures 2 and 3. The overall effect of the second- order term is to reduce the degree of crack induced anisotropy. Although the second-order corrections for the water-saturated crack case are small, those for the dry crack case are quite significant, especially for the quasi-SV wave.

In order to understand the relative behavior of the first and second-order expansion for the Hudson model, we take a closer look at the equations. We must first keep in mind that the expansion in terms of small crack densities is an asymptotic expansion. Mathematically this means that the expansion is approaching the true solution faster for the second-order than for the first-order expansion as crack density approaches zero. It does not guarantee that the second-order expansion at higher crack densities are better than the first-order (for a complete discussion on the difference between an asymptotic series and a convergent series, see *Bender and Orszag* [1978]). To see this consider the following example for dry cracks in a Poisson solid: with $\lambda = \mu$ we have the following values for the various terms:

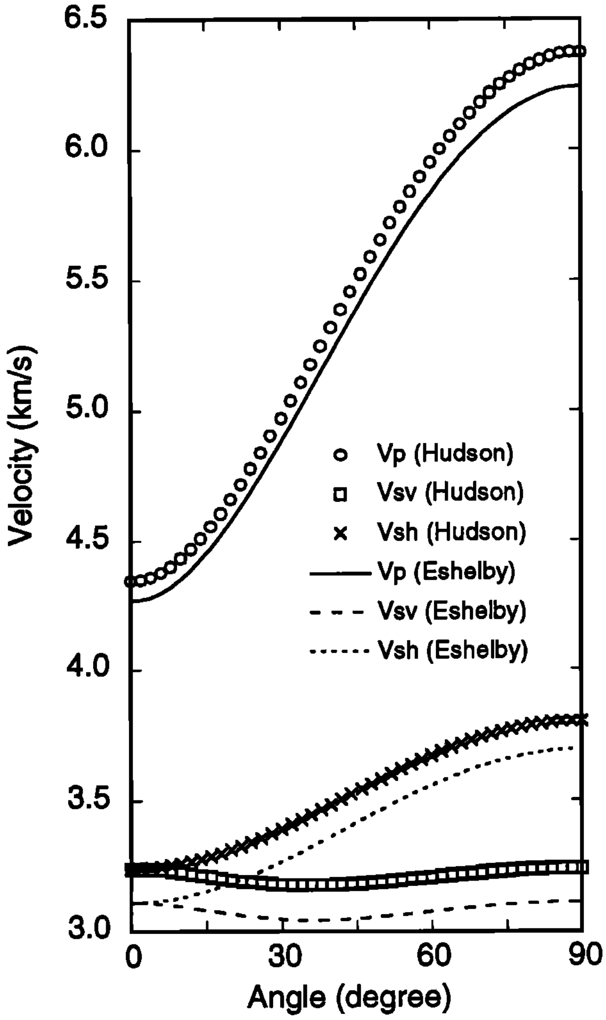

Fig. 4. Same as Figure 2 for crack aspect ratio of 0.1 and crack porosity of 0.05.

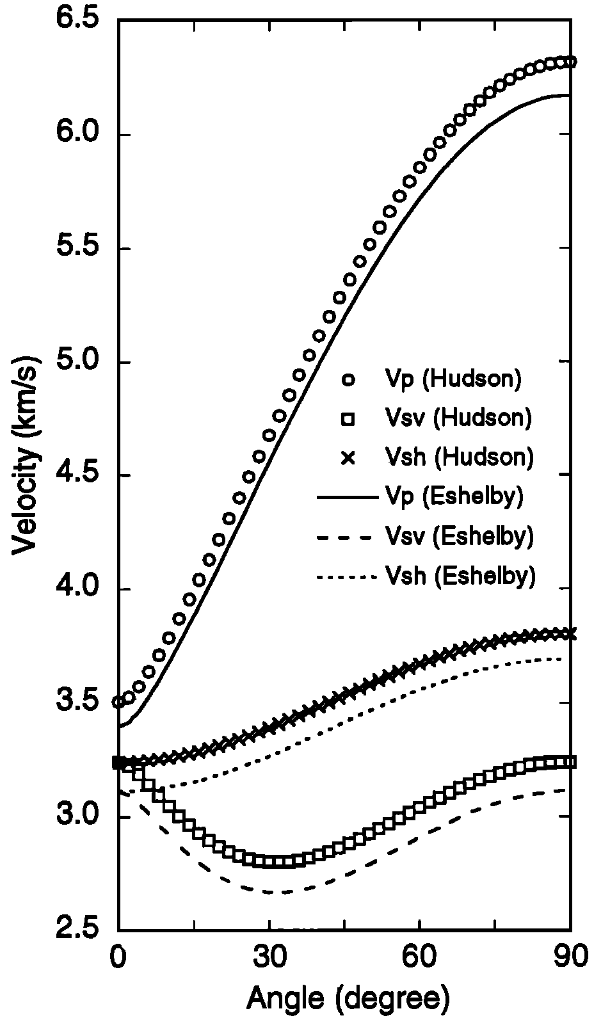

Fig. 5. Same as Figure 3 for crack aspect ratio of 0.1 and crack porosity of 0.05.

$$
\begin{aligned}
U_3 &= 2, U_1 = \frac{16}{7}, q = 58 \\
c_{11}^1 &= -\frac{\lambda^2}{\mu}\epsilon U_3 = -2\lambda\epsilon \\
c_{11}^2 &= \frac{q}{15}\frac{\lambda^2}{\lambda+2\mu}(\epsilon U_3)^2 = \frac{232}{45}\lambda\epsilon^2 \\
c_{11}^* &= c_{11}^0 + c_{11}^1 + c_{1}^2 1 \\
&= 3\lambda - 2\lambda\epsilon + \frac{232}{45}\lambda\epsilon^2 \\
&= 3\lambda - 2\lambda\epsilon(1 - \frac{116}{45}\epsilon).
\end{aligned} \tag{15}
$$

This means that the second-order term completely cancels out the first-order term when

$$\epsilon = \frac{45}{116} \approx 0.39. \tag{16}$$

Worse yet, it implies that c_{11}^* as a function of crack density will have a turning point, i.e., c_{11}^* will increase instead of decrease with crack density, at

$$\epsilon = \frac{45}{232} \approx 0.19. \tag{17}$$

All other moduli have similar behavior. Figure 8 shows the plot of normalized moduli for the first and second-order expansion as a function of crack density for a dry rock. The effective moduli are normalized with respect to the background moduli. The turning points discussed above are clearly demonstrated.

It is only fair to point out that the formal limit of validity for Hudson's expansions are for crack densities less than 0.1 [*Hudson,* 1980, 1981]. Within this limit, the second-order expansion is well behaved. However, this limit is usually ignored, even by the original author [e.g., *Hudson and Crampin,* 1991]. Moreover, the idea of a second-order expansion is to try to extend the theory to high crack densities. Clearly, in the present form, this is not achieved, at least for dry cracks. This is because the expansion is asymptotic in crack density and is of alternating sign. The series diverges at relatively low crack densities. It is invalid for crack densities greater than 0.19, at least in dry cracks.

PROPOSED NEW SECOND-ORDER MODEL

In order to get around the difficulties encountered with the turning point and a diverging series, we look for an alternative expansion to the effective moduli. We look to the

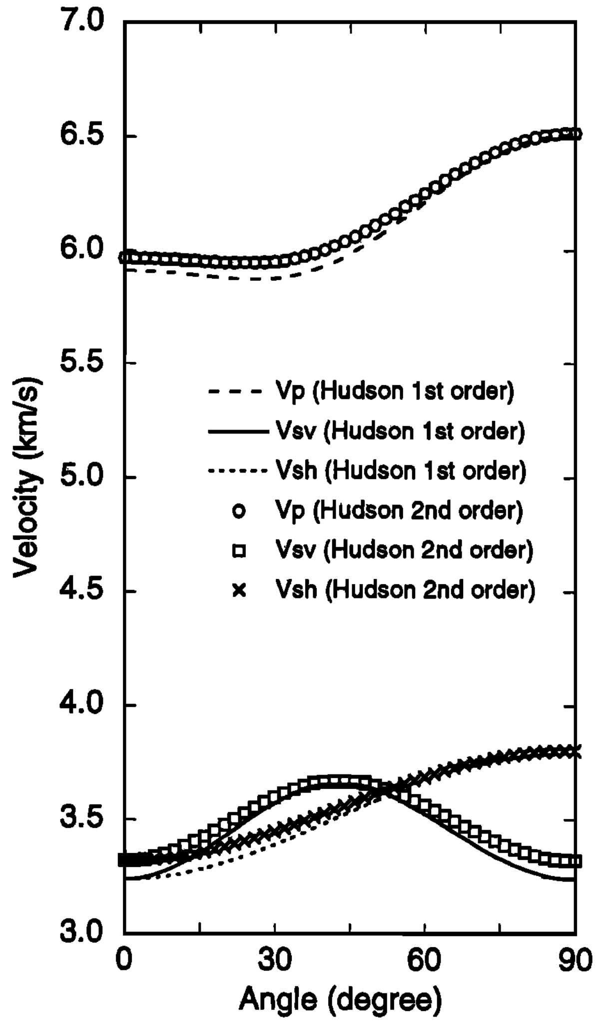

Fig. 6. Same as Figure 2 for Hudson's first and second order models. The cracks are water-saturated.

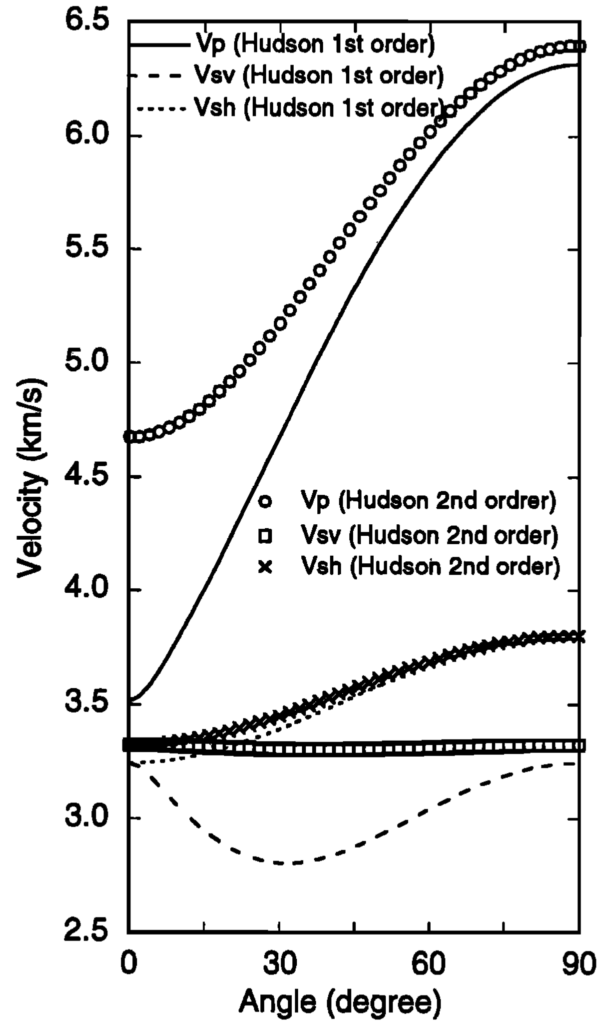

Fig. 7. Same as Figure 6 for dry cracks.

more basic expressions for a form for the expansion. A well known example is the effective moduli for empty spherical pores in a solid. *MacKenzie* [1950] developed the exact solution for the effective bulk modulus of a sphere with a single concentric spherical pore:

$$\frac{\kappa^*}{\kappa} = \frac{1-\phi}{1+\dfrac{3\kappa\phi}{4\mu}} \tag{18}$$

where κ^* and κ are the effective and background bulk modulus, respectively. The first- order power series expansion of eq. (18) in terms of porosity is given by

$$\frac{\kappa^*}{\kappa} = 1 - \phi\left(1+\frac{3\kappa}{4\mu}\right), \tag{19}$$

and the second order expansion by

$$\frac{\kappa^*}{\kappa} = 1 - \phi\left(1+\frac{3\kappa}{4\mu}\right) + \phi^2\left(1+\frac{3\kappa}{4\mu}\right)\frac{3\kappa}{4\mu}. \tag{20}$$

Figure 9 shows a plot of the normalized bulk modulus as a function of porosity for the spherical pores for the exact, first-order and second-order solution. The shape of the first and second-order curves are very similar to those for the anisotropic moduli shown in Figure 8. The expansion in a power series form is of alternating signs and is divergent. However, the exact solution is well behaved, all the way up to a porosity of 1. We take note of the form of the exact solution given in eq. (18) and thus propose a new second- order expansion for the anisotropic model. Instead of expanding in a power series:

$$c_{ij}^* = c_{ij}^0 + c_{ij}^1 + c_{ij}^2,$$

we model using the form of the exact solution for the spherical inclusion and expand in a Padé approximation [*Bender and Orszag,* 1978; *Press and Teukolsky,* 1992], namely:

$$c_{ij}^* = c_{ij}^0 \frac{1-a_{ij}\epsilon}{1+b_{ij}\epsilon}. \tag{21}$$

A Padé approximation is a rational function of two polynomials. It is frequently used to model the behavior of func-

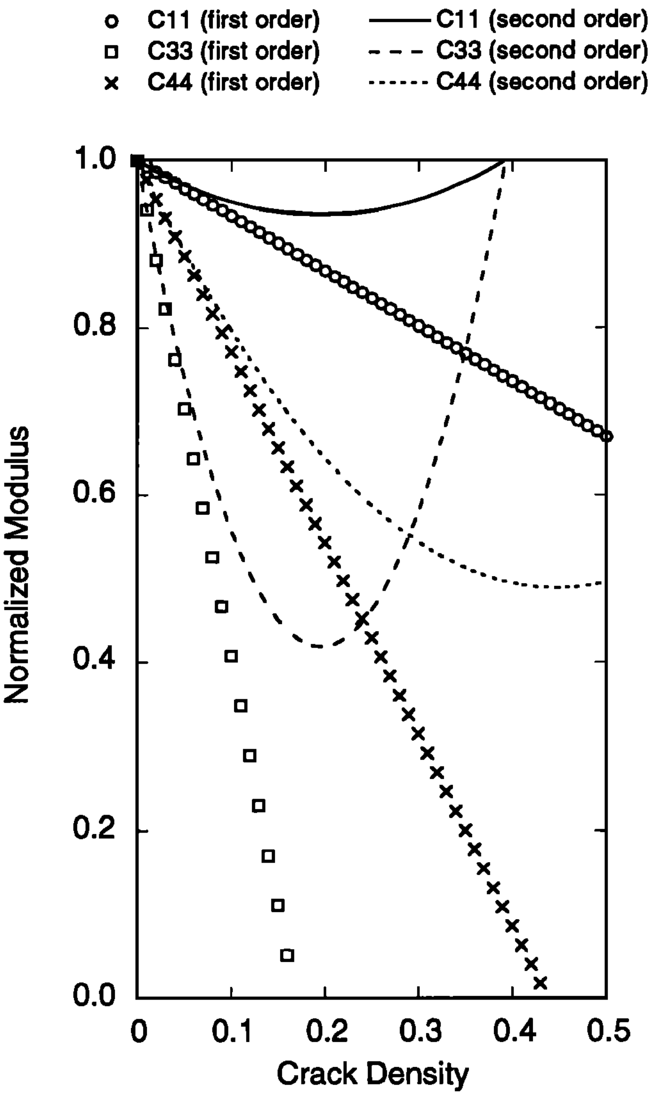

Fig. 8. Effective moduli normalized with background moduli for first and second order Hudson models with dry cracks as a function of crack density.

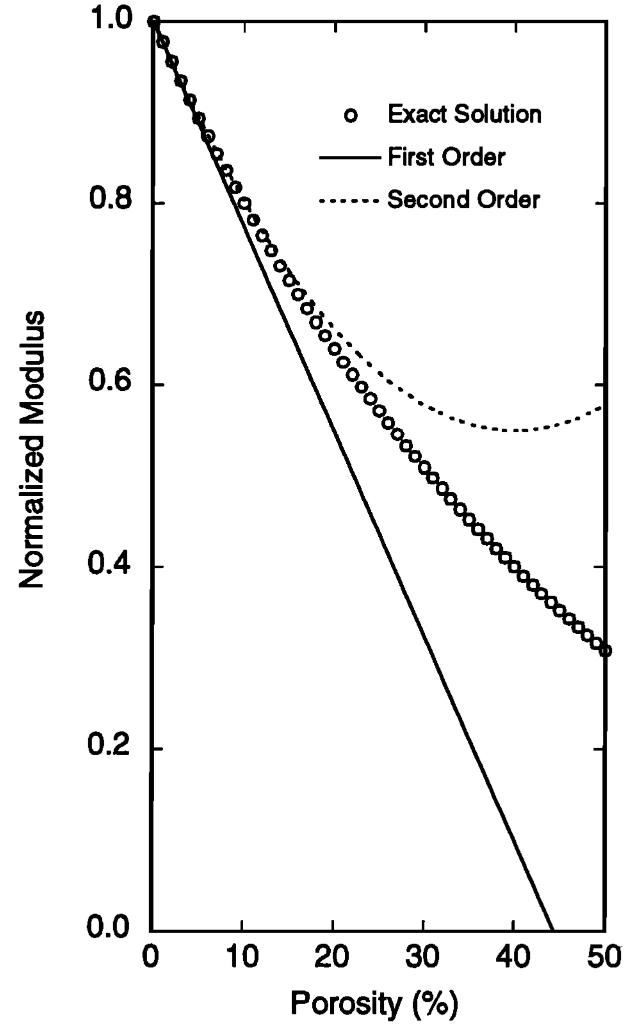

Fig. 9. Effective bulk modulus normalized with background modulus for empty spherical inclusions as a function of porosity.

tions which are divergent if expressed in a power series. The complete mathematical basis of the Padé approximation is not yet well understood [*Bender and Orszag,* 1978; *Press and Teukolsky,* 1992]. This expansion can be made to be identical to Hudson's model up to second-order in crack density ϵ by setting the Padé coefficients:

$$b_{ij} = -\frac{c_{ij}^2}{c_{ij}^1 \epsilon},$$

and

$$a_{ij} = -\frac{c_{ij}^1}{c_{ij}^0 \epsilon} - b_{ij}. \qquad (22)$$

Figures 10 and 11 show the comparison of the proposed model with the first- and second- order expansions of Hudson for water-saturated and dry rock. The parameters are the same as in Figures 2 and 3. The new expansion falls between the first- and second-order expansions of Hudson. As a matter of fact, it can be shown that this is always the case even for large crack densities (less than 1). Moreover, the expansion appears to be monotonically decreasing with crack density, thus it is more physical and avoids the problem of the second-order expansion outlined above. Perhaps the most important point is that mathematically it is as valid as Hudson's second-order expansion since the coefficients in the Padé expansion are chosen so that they match Hudson's. By using the Padé form, this new model has implicit higher order terms in crack density than the second-order power series expansion. These higher order terms, in particular the third order, are at least of the proper sign, even though they may not be the exact terms obtained from a more rigorous calculation. This new expansion thus has all the rigorous validity of the power series expansion while avoiding its divergent behavior.

CONCLUSIONS

In this paper we closely examined Hudson's model for crack induced seismic anisotropy. We found that Hudson's first-order corrections agree with Eshelby's first-order complete expansions for cracks with aspect ratio of 0.01 and less, provided the weak-material (fluid- filled) formulation is used for the former in the case of fluid saturation and not

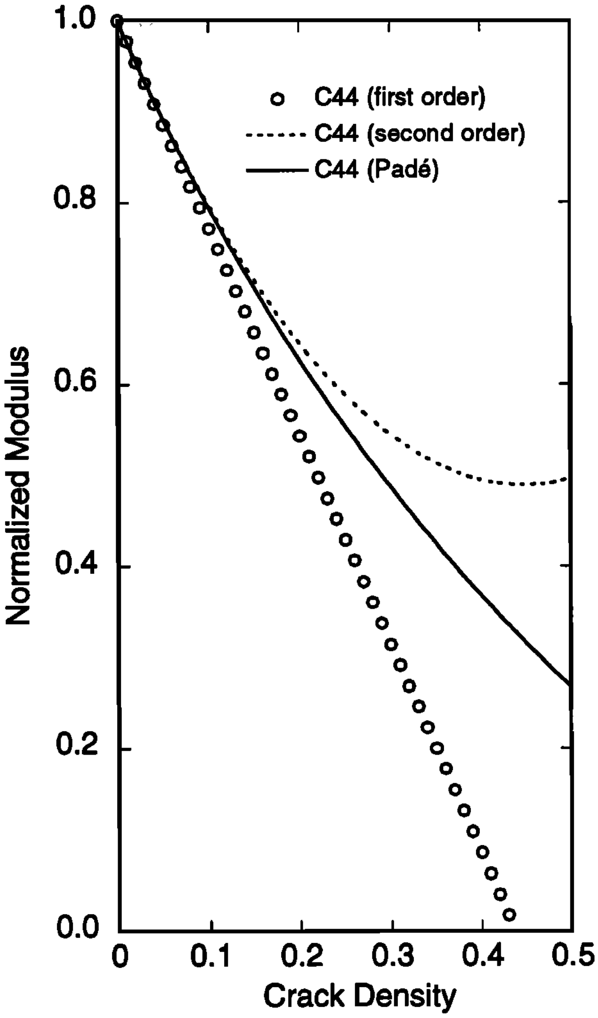

Fig. 10. Comparison of first order, second order, and Padé expansion for normalized c_{44} for saturated cracks. For c_{11} and c_{33} there is little difference between the three expansions.

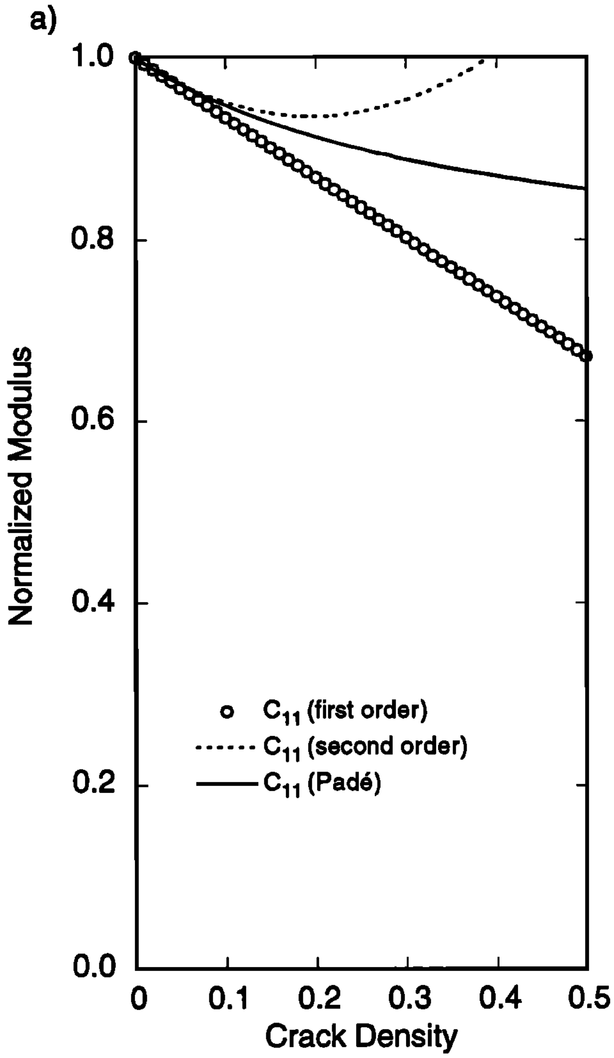

Fig. 11. Comparison of first order, second order, and Padé expansion for normalized (a) c_{11}, (b) c_{33}, (c) c_{44} for dry cracks.

the fluid- filled, zero-aspect-ratio, crack model listed by *Hudson* [1981]. As expected, Hudson's first-order results depart from the complete expansion of Eshelby's theory for larger aspect ratios. In addition, Hudson's second-order term does not necessarily extend the validity of the expansion to higher crack densities. In fact, this term is shown to break down at quite low crack densities. This is not surprising considering the asymptotic nature of the expansion.

A new expansion based on the Padé approximation is proposed. This expansion is identical to that of Hudson up to the second-order in crack density but avoids the problem of divergence at higher crack densities. Because of its implicit inclusion of higher order terms, this expansion should be applicable to higher crack densities than the existing model, although, just as in the case of the existing theories, the formal limits of validity are very difficult to establish.

APPENDIX

The effective moduli for fluid-filled ellipsoidal crack (of aspect ratio α, porosity ϕ, and axis of symmetry the three-axis) induced seismic anisotropy using the complete Eshelby expansion are given by (*Eshelby*, 1957; *Cheng*, 1978)

$$c^*_{ij} = c^0_{ij} - \phi c^1_{ij}$$

where

$$\begin{aligned}
c^1_{11} &= \lambda(s_{31} - s_{33} + 1) \\
&\quad + (2\mu(s_{33}s_{11} - s_{31}s_{13} - (s_{33} + s_{11} - 2C - 1) \\
&\quad + C(s_{31} + s_{13} - s_{11} - s_{33})))/D(s_{12} - s_{11} + 1) \\
c^1_{33} &= \frac{(\lambda + 2\mu)(-s_{12} - s_{11} + 1) + 2\lambda s_{13} + 4\mu C}{D} \\
c^1_{13} &= ((\lambda + 2\mu)(s_{13} + s_{31}) - 4\mu C \\
&\quad + \lambda(s_{13} - s_{12} - s_{11} - s_{33} + 2))/2D \\
c^1_{44} &= \frac{\mu}{1 - 2s_{1313}} \\
c^1_{66} &= \frac{\mu}{1 - 2s_{1212}}
\end{aligned}$$

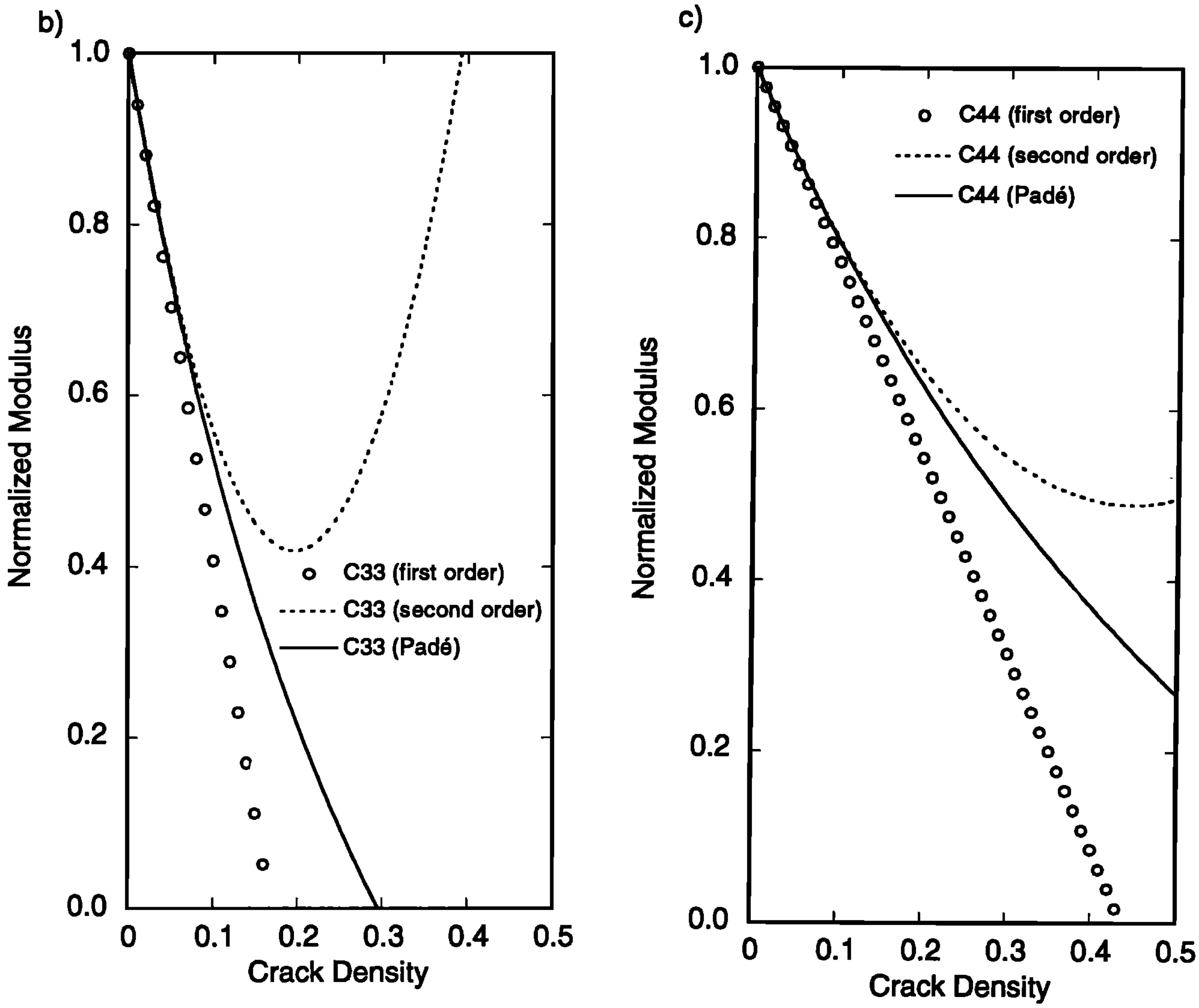

Fig. 11. (continued)

with

$$
\begin{aligned}
D &= s_{33}s_{11} + s_{33}s_{12} - 2s_{31}s_{13} - (s_{11} + s_{12} + s_{33} \\
&\quad -1 - 3C) - C(s_{11} + s_{12} + 2(s_{33} - s_{13} - s_{31})) \\
s_{11} &= QI_{aa} + RI_a \\
s_{33} &= Q(\frac{4\pi}{3} - 2I_{ac}\alpha^2) + I_c R \\
s_{12} &= QI_{ab} - RI_a \\
s_{13} &= QI_{ac}\alpha^2 - RI_a \\
s_{31} &= QI_{ac} - RI_c \\
s_{1212} &= QI_{ab} + RI_a \\
s_{1313} &= \frac{Q(1+\alpha^2)I_{ac}}{2} + \frac{R(I_a + I_c)}{2} \\
C &= \frac{\kappa'}{3(\kappa - \kappa')}
\end{aligned}
$$

and

$$
\begin{aligned}
I_a &= \frac{2\pi\alpha(cos^{-1}\alpha - \alpha S_a)}{S_a^3} \\
I_c &= 4\pi - 2I_a \\
I_{ac} &= \frac{I_c - I_a}{3S_a^2} \\
I_{aa} &= p - \frac{3I_{ac}}{4} \\
I_{ab} &= \frac{I_{aa}}{3}
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= \frac{3\kappa - 2\mu}{6\kappa + 2\mu} \\
S_a &= \sqrt{1 - \alpha^2} \\
R &= \frac{1 - 2\sigma}{8\pi(1 - \sigma)} \\
Q &= \frac{3R}{1 - 2\sigma}
\end{aligned}
$$

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C. H. Cheng, Earth Resources Laboratory, MIT, 42 Carleton St., Cambridge, MA 02142.

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