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Wave speeds and attenuation of elastic waves in material containing cracks

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Summary. Expressions now exist from which may be calculated the propagation constants of elastic waves travelling through material containing a distribution of cracks. The cracks are randomly distributed in position and may be randomly orientated. The wavelengths involved are assumed to be large compared with the size of the cracks and with their separation distances so that the formulae, based on the mean taken over a statistical ensemble, may reasonably be used to predict the properties of a single sample. The results are valid only for small concentrations of cracks.

Explicit expressions, correct to lowest order in the ratio of the crack size to a wavelength, are derived here for the overall elastic parameters and the overall wave speeds and attenuation of elastic waves in cracked materials where the mean crack is circular, and the cracks are either aligned or randomly orientated. The cracks may be empty or filled with solid or fluid material. These results are achieved on the basis of simply the static solution for an ellipsoidal inclusion under stress.

The extension to different distributions of orientation or to mixtures of different types of crack is quite straightforward.

1 Introduction

The mean elastodynamic properties of a cracked solid form a topic of some interest in several different fields. For instance, the rock structure at and near the Earth's surface may contain intersecting families of parallel cracks which divide the material into a rough block structure. The local seismic wave speeds are strongly influenced by such fracturing (Morland 1974). The cracks may be dry or permeated with water.

To take another example, the extent of fracturing in the region of a borehole is a vital factor in the extraction of oil (Babcock 1978) or of geothermal heat (Batzle & Simmons 1976). Also, the dilatancy theory for earthquake prediction is based on a hypothesis that, under increasing shear strain, certain rocks dilate by the opening up of small cracks. Although largely dry at this stage, the cracks become permeated with water which partially relieves the hydrostatic pressure and also, therefore, the frictional resistance to catastrophic failure (see, for instance, Griggs *et al.* 1975). Identification, by seismic sounding, of all or any of these types of fracture would be a very useful achievement.

If, in some kind of seismic exploration, the incident wave has wavelengths which are short compared with the scale-size of the cracks, the chief means of identification of the extent and nature of the fractures would be by the observation of scattered waves, as in non-destructive testing of solid materials (see, for instance, Doyle & Scala 1978). If the wavelengths are long compared with the size of the cracks, observations will be made primarily of the overall properties of the fractured solid. We shall be concerned here with providing some theoretical results for the latter problem.

We shall take as our starting point the equations derived by Hudson (1980) from Keller's (1964) equations for the mean field in a solid with a random distribution of inclusions; in this case, cracks. We shall work to the first order only in the quantity (νa^3), where ν is the number density of cracks, and a the mean radius of the cracks, although Hudson's formulae are accurate to second order in (νa^3). Furthermore, we shall retain only the lowest order terms in (ka), where k is the wavenumber of the wave.

Expressions which are accurate for higher values of (νa^3) than the first-order formulae, have been derived for the overall static moduli by the self-consistent method by Budiansky & O'Connell (1976) and by Hoenig (1979). However, there does not seen to be a way at present of extending this method to dynamic problems where, for instance, attenuation is important.

The attenuation of seismic waves by variations in structural properties was first treated by Sir Harold Jeffreys almost exactly 50 years ago (Jeffreys 1931). However, his analysis applies to materials having a continuous, or piece-wise continuous, variation in wave speeds rather than isolated cracks.

2 Basic equations

We consider the problem of a plane harmonic wave

$$\mathbf{u} = \mathbf{b} \exp (i\mathbf{k} \cdot \mathbf{x}) \qquad (\mathbf{b}, \mathbf{k} \text{ constants}) \tag{1}$$

travelling through a solid containing a random distribution of cracks. The displacement field given by equation (1) is to be regarded as a statistical mean over the fields which arise in an ensemble of such solids, and we assume that fluctuations about the mean are such that observations of the field in a single member of the ensemble give results which are close to this statistical mean. The number density of cracks is ν, and we assume that the mean crack shape is circular with radius a. (The assumption of an elliptic rather than circular shape for the mean crack does not appear to change the results very much (O'Connell & Budiansky 1974) while substantially increasing the complexity of the analysis.)

If the concentration of cracks is dilute,

$$(\nu a^3) \ll 1,$$

and substitution of equation (1) into the equations for the mean field give, to first order in (νa^3),

$$b_j(\rho\omega^2\delta_{ij} - c^0_{ipjq}k_p k_q - c^1_{ipjq}k_p k_q) = 0, \tag{2}$$

(Hudson 1980) where ω is the frequency, ρ the density, and $\mathbf{c}^0$ is the matrix of elastic constants for the solid uncracked material:

$$c^0_{ipjq} = \lambda\delta_{ip}\delta_{jq} + \mu(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{pj}) \tag{3}$$

and $\mathbf{c}^1$ is the first-order correction.

If the cracks are aligned in the x_3 direction,

$$c^1_{ipjq} = -\frac{(\nu a^3)}{\mu} c^0_{k3ip} c^0_{l3jq} \overline{\mathscr{U}}_{kl}(\mathbf{k}a), \tag{4}$$

where

$$\overline{\mathscr{U}}_{kl}(\mathrm{k}a) = \frac{1}{a^2}\int_{\Sigma} \mathscr{U}_{kl}(\exp i\mathbf{k}\cdot\mathbf{X};\mathbf{X}) \exp(-i\mathbf{k}\cdot\mathbf{X})\, dS_x, \tag{5}$$

the integration being performed over the points $\mathbf{X} \equiv (X_1, X_2, 0)$ of the face Σ of a crack of radius a, centred on the origin $\mathbf{X} = 0$. The quantity $\mathscr{U}_{kl}(\exp i\mathbf{k}\cdot\mathbf{X};\mathbf{X})$ in the integrand is defined to be (μ/a) times the k component of the discontinuity in displacement across the crack due to imposed tractions in the x_l direction of amplitude $\exp(i\mathbf{k}\cdot\mathbf{X})$.

The symmetry of the problem implies that $\overline{\mathscr{U}}_{kl}$ is diagonal, and $\overline{\mathscr{U}}_{11} = \overline{\mathscr{U}}_{22}$, and we may in fact write $\mathbf{c}^1$ as a second-order (6×6) matrix:

$$\mathbf{c}^1 = -\frac{(\nu a^3)}{\mu}\begin{pmatrix} \lambda^2\overline{\mathscr{U}}_{33} & \lambda^2\overline{\mathscr{U}}_{33} & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_{33} & 0 & 0 & 0 \\ \lambda^2\overline{\mathscr{U}}_{33} & \lambda^2\overline{\mathscr{U}}_{33} & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_{33} & 0 & 0 & 0 \\ \lambda(\lambda+2\mu)\overline{\mathscr{U}}_{33} & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_{33} & (\lambda+2\mu)^2\overline{\mathscr{U}}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^2\overline{\mathscr{U}}_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu^2\overline{\mathscr{U}}_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

where

$c_{11} = c_{1111}, \quad c_{12} = c_{1122}, \quad c_{13} = c_{1133}$, etc.,

$c_{44} = 2c_{2323}, \quad c_{45} = 2c_{2331}$, etc.

The quantities $\overline{\mathscr{U}}_{11}(\mathrm{k}a)$ and $\overline{\mathscr{U}}_{33}(\mathrm{k}a)$ depend primarily on the conditions imposed on the surface of the crack; we may take the crack to be dry (stress-free conditions), fluid-filled (shear traction and discontinuity of the normal component of displacement both zero), or some other model. We shall consider specific models later.

If there is a mixture of cracks with different internal conditions, we simply calculate $\mathbf{c}^1$ for each type of crack separately (with the appropriate values of $\overline{\mathscr{U}}_{kl}$ and ν) and add the results to obtain the final value of $\mathbf{c}^1$.

Similarly, if there are two or more sets of cracks aligned in different directions, we again calculate $\mathbf{c}^1$ for each set transformed appropriately (as a fourth-order tensor) so that the x_3 axis rotates into the axis of alignment, and add the results. If there were two sets, for instance, each with the same number density $(\nu/2)$ and aligned at right-angles, we obtain $\mathbf{c}^1$ by adding the matrix on the right-hand side of equation (6) to the same quantity with certain rows and columns interchanged, and dividing by 2:

$$\mathbf{c}^1 = -\frac{(\nu a^3)}{\mu} \tag{7}$$

$$\times\begin{pmatrix} \lambda^2\overline{\mathscr{U}}_{33} & \lambda(\lambda+\mu)\overline{\mathscr{U}}_{33} & \lambda(\lambda+\mu)\overline{\mathscr{U}}_{33} & 0 & 0 & 0 \\ \lambda(\lambda+\mu)\overline{\mathscr{U}}_{33} & (\lambda^2+2\lambda\mu+2\mu^2)\overline{\mathscr{U}}_{33} & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_{33} & 0 & 0 & 0 \\ \lambda(\lambda+\mu)\overline{\mathscr{U}}_{33} & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_{33} & (\lambda^2+2\lambda\mu+\mu^2)\overline{\mathscr{U}}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^2\overline{\mathscr{U}}_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^2\overline{\mathscr{U}}_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^2\overline{\mathscr{U}}_{11} \end{pmatrix}$$

where both sets of crack normals are perpendicular to the x_1 direction.

Finally, the result for randomly orientated cracks is obtained by an extension of the above process; we take the mean value of $\mathbf{c}^1$ for all possible orientations of the axis of alignment. The final form for $\mathbf{c}^1$ is a fourth-order isotropic tensor:

$$c^1_{ipjq} = \lambda_1 \delta_{ip} \delta_{jq} + \mu_1(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}) \tag{8}$$

with the same scalar invariants as the form of $\mathbf{c}^1$ given by equation (4). This is enough to enable us to find λ_1 and μ_1 (Hudson 1980):

$$\left(\frac{3\lambda_1 + 2\mu_1}{3\lambda + 2\mu}\right) = -(\nu a^3)\left(\frac{3\lambda + 2\mu}{3\mu}\right)\overline{\mathscr{U}}_{33}$$
$$\frac{\mu_1}{\mu} = -(\nu a^3)\frac{2}{15}(3\overline{\mathscr{U}}_{11} + 2\overline{\mathscr{U}}_{33}). \tag{9}$$

We shall now consider the different formulae arising for $\overline{\mathscr{U}}_{kl}$ from the different internal conditions on the crack.

3 Results for specific crack conditions

All expressions for the overall properties of cracked solids which have appeared in the literature so far have been derived as lowest order approximations for (ka) or $(\omega a/\beta)$ small (where $\beta = (\mu/\rho)^{1/2}$ is the shear wave speed of the uncracked solid). In addition, expressions for $\overline{\mathscr{U}}_{kl}$ are available only under the same assumption. We therefore restrict ourselves in this section to calculating $\mathbf{c}^1$ to the lowest order in ka. This means that, in the various formulae above, we use

$$\overline{\mathscr{U}}_{kl}(0) = \frac{1}{a^2}\int_\Sigma \mathscr{U}_{kl}(1;\mathbf{X})\,dS_x, \tag{10}$$

where inertia terms are neglected in the calculation of $\mathscr{U}_{kl}$.

3.1 FLUID-FILLED CRACKS

The standard model for a fluid-filled crack is, as stated earlier, one in which the shear traction on the crack is zero, and the displacement discontinuity is confined to the transverse direction only. This immediately implies

$$\overline{\mathscr{U}}_{33}(0) = 0. \tag{11}$$

From Garbin & Knopoff (1973) we have

$$\mathscr{U}_{11}(1;\mathbf{X}) = \frac{8}{\pi}\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right)(1 - r^2/a^2)^{1/2},$$

where $r^2 = X_1^2 + X_2^2$, and so

$$\overline{\mathscr{U}}_{11}(0) = \frac{16}{3}\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right). \tag{12}$$

It follows, therefore, that for aligned cracks,

$$c^1_{44} = c^1_{55} = -\frac{32}{3}(\nu a^3)\mu\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right), \tag{13}$$

all other components being zero. These equations agree, to first order in (νa^3) with those of Hoenig (1979). Wave-speeds $c(=\omega/k)$ in the composite material may be found by multiplying equation (2) by b_j and summing over i:

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{\rho}\,[\mu + (\lambda+\mu)\,(\hat{\mathbf{b}}\cdot\hat{\mathbf{k}})^2 + c^1_{ipjq}\hat{b}_i\hat{b}_j\hat{k}_p\hat{k}_q], \tag{14}$$

where $k = |\mathbf{k}|$, $b = |\mathbf{b}|$, $\hat{\mathbf{k}} = \mathbf{k}/k$, $\hat{\mathbf{b}} = \mathbf{b}/b$. Given a wave normal $\hat{\mathbf{k}}$, the unit vector $\hat{\mathbf{b}}$ may be written as

$$\hat{\mathbf{b}} = \hat{\mathbf{b}}^0 + \boldsymbol{\epsilon}, \quad |\boldsymbol{\epsilon}| = O(\nu a^3),$$

where $\hat{\mathbf{b}}^0$ is a value of $\hat{\mathbf{b}}$ associated with $\hat{\mathbf{k}}$ in the uncracked solid. If the wave is approximately a compressional wave, $\hat{\mathbf{b}}^0 = \hat{\mathbf{k}}$, and since $\hat{\mathbf{b}}$ is a unit vector

$$\hat{\mathbf{k}}\cdot\boldsymbol{\epsilon} = O\,[(\nu a^3)^2].$$

If the wave is approximately a shear wave, $\hat{\mathbf{k}}\cdot\hat{\mathbf{b}}^0 = 0$, and so

$$\hat{\mathbf{k}}\cdot\hat{\mathbf{b}} = O\,(\nu a^3).$$

In either case, we may replace $\hat{\mathbf{b}}$ by $\hat{\mathbf{b}}^0$ in the second term on the right-hand side of equation (14) with errors or order $(\nu a^3)^2$. We may, of course, make the same substitution with the same order of magnitude of error in the third term as well.

To the first order in (νa^3) then,

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{\rho}\,[\mu + (\lambda+\mu)\,(\hat{\mathbf{b}}^0\cdot\hat{\mathbf{k}})^2 + c^1_{ipjq}\hat{b}^0_i\hat{b}^0_j\hat{k}_p\hat{k}_q]. \tag{15}$$

So, for a compressional-type wave travelling in a direction θ with the axis of alignment,

$$\left(\frac{\omega}{k}\right)^2 = \alpha^2\left[1 - \frac{64}{3}(\nu a^3)\left(\frac{\mu}{3\lambda+4\mu}\right)\sin^2\theta\,\cos^2\theta\right], \tag{16}$$

where $\alpha^2 = (\lambda+2\mu)/\rho$, which agrees with formulae given by Garbin & Knopoff (1975b).

The overall properties of the cracked solid are transversely isotropic with symmetry axis in the x_3 direction. The waves of shear type in such a medium separate into two sets with different wave speeds. One (*SH*) is polarized parallel to the x_1x_2 plane, and the other (*SV*) parallel to the plane of $\mathbf{k}$ and the x_3 axis. For an *SH* wave,

$$\left(\frac{\omega}{k}\right)^2 = \beta^2\left[1 - \frac{16}{3}(\nu a^3)\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)\cos^2\theta\right], \tag{17}$$

and for an *SV* wave

$$\left(\frac{\omega}{k}\right)^2 = \beta^2\left[1 - \frac{16}{3}(\nu a^3)\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)(\cos^2\theta - \sin^2\theta)^2\right]. \tag{18}$$

These formulae too agree with Garbin & Knopoff (1975b), although their formulae are expressed as if the wave speed varies continuously with polarization angle, which is not strictly true.

For randomly orientated cracks, we have

$$(3\lambda_1 + 2\mu_1) = 0$$

$$\frac{\mu_1}{\mu} = -\frac{32}{15}(\nu a^3)\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right), \tag{19}$$

which agrees with Walsh (1969).

3.2 DRY CRACKS

The model usually used for dry cracks is one in which all components of the traction on the crack face vanish. The displacement discontinuity due to an imposed shear traction remains the same as for a fluid-filled crack, and so $\mathscr{U}_{11}(0)$ is once more given by equation (12). On the other hand,

$$\mathscr{U}_{33}(1;\mathbf{X}) = \frac{2}{\pi}\left(\frac{\lambda + 2\mu}{\lambda + \mu}\right)(1 - r^2/a^2)^{1/2}$$

(Garbin & Knopoff 1973), and so

$$\overline{\mathscr{U}}_{33}(0) = \frac{4}{3}\left(\frac{\lambda + 2\mu}{\lambda + \mu}\right). \tag{20}$$

It follows from equation (6) that, for aligned cracks,

$$\begin{aligned}
c^1_{11} = c^1_{12} = c^1_{21} = c^1_{22} &= -\frac{4}{3}(\nu a^3)\frac{\lambda^2(\lambda + 2\mu)}{\mu(\lambda + \mu)}\\
c^1_{13} = c^1_{23} = c^1_{31} = c^1_{32} &= -\frac{4}{3}(\nu a^3)\frac{\lambda(\lambda + 2\mu)^2}{\mu(\lambda + \mu)}\\
c^1_{33} &= -\frac{4}{3}(\nu a^3)\frac{(\lambda + 2\mu)^3}{\mu(\lambda + \mu)}\\
c^1_{44} = c^1_{55} &= -\frac{32}{3}(\nu a^3)\frac{\mu(\lambda + 2\mu)}{(3\lambda + 4\mu)},
\end{aligned} \tag{21}$$

all other components being zero, again agreeing to first order in (νa^3) with Hoenig (1979).

From equation (15) we have the wave speeds of compressional-type waves given by

$$\left(\frac{\omega}{k}\right)^2 = \alpha^2\left[1 - \frac{4}{3}(\nu a^3)\left\{\frac{(\lambda + 2\mu\cos^2\theta)^2}{\mu(\lambda + \mu)} + \frac{16\mu}{3\lambda + 4\mu}\sin^2\theta\cos^2\theta\right\}\right], \tag{22}$$

that of *SH* waves is given once again by equation (17), and that of *SV* waves by

$$\left(\frac{\omega}{k}\right)^2 = \beta^2\left[1 - \frac{16}{3}(\nu a^3)\left\{\frac{\lambda + 2\mu}{\lambda + \mu}\cos^2\theta\sin^2\theta + \frac{\lambda + 2\mu}{3\lambda + 4\mu}(\cos^2\theta - \sin^2\theta)^2\right\}\right]. \tag{23}$$

With the proviso mentioned above, these formulae agree with Garbin & Knopoff (1973, 1975a) except for a factor of 2 in the first term in the curly brackets of equation (23).

For randomly orientated cracks, we have

$$\begin{aligned}
\left(\frac{3\lambda_1 + 2\mu_1}{3\lambda + 2\mu}\right) &= -\frac{4}{9}(\nu a^3)\frac{(3\lambda + 2\mu)(\lambda + 2\mu)}{\mu(\lambda + \mu)}\\
\frac{\mu_1}{\mu} &= -\frac{16}{45}(\nu a^3)\frac{(\lambda + 2\mu)(9\lambda + 10\mu)}{(3\lambda + 4\mu)(\lambda + \mu)},
\end{aligned} \tag{24}$$

which differs from the expressions given by Bristow (1960) in that he appears to have dropped a factor of $2(\lambda + \mu)/(3\lambda + 4\mu)$ in his expression for μ_1/μ. However, the above expression agrees with O'Connell & Budiansky (1974).

3.3 CRACKS FILLED WITH WEAK MATERIAL

Budiansky & O'Connell (1974) pointed out that, if the cracks are filled with fluid with a relatively small bulk modulus, the crack opening displacement due to an imposed normal traction may not be negligible. The high compressibility of the fluid leads to large strains and these strains may be integrated over the thickness of the crack to give the discontinuity in displacement. In the analysis of a fluid-filled crack in Section 3.1, we assumed that the thickness of the crack is sufficiently small for the discontinuity in normal displacement to be neglected. We now take the fact that the thickness of the crack is non-zero into account and allow the material within the crack to have small bulk and shear moduli. We need to re-calculate $\overline{\mathscr{U}}_{11}(0)$ and $\overline{\mathscr{U}}_{33}(0)$.

Eshelby (1957) has shown that the strain field inside a single ellipsoidal inclusion is uniform when the matrix is subjected to uniform strain at infinity. In addition, he provides a formula to calculate one from the other. If we use an ellipsoid with two semi-axes equal and one vanishingly small as a model for a circular crack (thus following most workers in this field) we may use Eshelby's results to calculate the strains in the material of the crack.

We begin by investigating the modification to the formula for $\overline{\mathscr{U}}_{11}(0)$ (which arises from the discontinuity in transverse displacement due to an imposed shear stress) when the material within the crack has non-zero rigidity μ'; up to now we have assumed that all shear stresses vanish within the crack.

Under a uniform shear field σ_{13}^{∞} at infinity, the shear stress in the inclusion is (Eshelby 1957)

$$\sigma_{13}^{\mathrm{I}} = \sigma_{13}^{\infty}\left[1 + \frac{\pi}{4}\left(\frac{c\mu}{a\mu'}\right)\left(\frac{3\lambda + 4\mu}{\lambda + 2\mu}\right)\right]^{-1} \tag{25}$$

where a, a and c are the semi-axes of the ellipsoid, and we have neglected terms of order (c/a) and $(\mu c^2/\mu' a^2)$, but retained those of order $(\mu c/\mu' a)$. We may find the jump in displacement across the faces of the crack by prescribing σ_{13}^{I} as the value of the shear traction on a flat circular disc of radius a and normal in the x_3 direction and solving for the displacements in the matrix material. By subtracting the uniform field σ_{13}^{I}, we regain the problem of the stress-free crack but with the stress field $\sigma_{13}^{\infty} - \sigma_{13}^{\mathrm{I}}$ at infinity. Thus $\mathscr{U}_{11}(1; X)$ is now

$$\begin{aligned}\mathscr{U}_{11}(1;\mathbf{X}) &= \frac{8}{\pi}\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right)(1 - r^2/a^2)^{1/2}\left(1 - \frac{\sigma_{13}^{\mathrm{I}}}{\sigma_{13}^{\infty}}\right)\\ &= \frac{8}{\pi}\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right)\frac{(1 - r^2/a^2)^{1/2}}{1 + M},\end{aligned} \tag{26}$$

where

$$M = \frac{4}{\pi}\left(\frac{a\mu'}{c\mu}\right)\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right).$$

The overall properties of a material containing a random distribution of inclusions with non-zero rigidity may now be found from the above formulae for cracked materials simply by multiplying the original value of $\overline{\mathscr{U}}_{11}(0)$ by $(1 + M)^{-1}$ so that

$$\overline{\mathscr{U}}_{11}(0) = \frac{16}{3}\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right)(1 + M)^{-1}.$$

If the bulk modulus of the material of the inclusions is sufficiently large that we may take $\overline{\mathscr{U}}_{33}(0)$ to be zero, we have the first-order perturbation of the elastic constants to be (from equation 13)

$$c_{44}^1 = c_{55}^1 = -\frac{32}{3}(\nu a^3)\mu\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)(1+M)^{-1}, \tag{27}$$

all other components of $\mathbf{c}^1$ being zero.

We see that, unless μ'/μ is small, the expression in equation (27) is of order c/a and therefore negligible. It is also clear that if we put μ'/μ equal to zero, we regain the result for a crack filled with an inviscid fluid. So equations (27) provide us with a generalization of equations (13). Similarly, we may extend equations (16) to (18) for the wave speeds, and equation (19) for the perturbation of the rigidity when the inclusions are randomly orientated, by multiplying each term in (νa^3) by $(1+M)^{-1}$.

If the material of the inclusions is viscous rather than elastic, we may replace μ' with an effective rigidity

$$\mu' = -i\omega\eta$$

where η is the modulus of viscosity. This means that all terms in (νa^3) are multiplied by the factor

$$\left[1 - \frac{4ia\omega\eta}{\pi c\mu}\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)\right]^{-1}. \tag{28}$$

This result may be compared with that of Chatterjee *et al.* (1980) who give formulae for the overall wave speeds in a material with aligned fluid-filled cracks. The effect of the viscosity of the fluid within the cracks is given by multiplying the terms in (νa^3) in their expressions by

$$1 + \frac{6ia\omega\eta}{\pi\mu\delta}\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)$$

when $(a\omega\eta/\mu\delta)$ is small, and by

$$\frac{3\pi i}{32}\frac{\mu\delta}{a\omega\eta}\left(\frac{3\lambda+4\mu}{\lambda+2\mu}\right)$$

when $(a\omega\eta/\mu\delta)$ is large; where δ is the thickness of the crack. These may be compared with the approximations to expression (28) for $(a\omega\eta/\mu c)$ small or large, which are

$$1 + \frac{4ia\omega\eta}{\pi\mu c}\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)$$

and

$$\frac{\pi i}{4}\frac{\mu c}{a\omega\eta}\left(\frac{3\lambda+4\mu}{\lambda+2\mu}\right)$$

respectively, if we also assume that $\omega\eta/\mu$ is small.

The expressions for small $(a\omega\eta/\mu\delta)$ are the same if we put $\delta = (3/2)c$. Those for large $(a\omega\eta/\mu\delta)$ are the same if we put $\delta = (8/3)c$. The difference lies in the fact that Chatterjee *et al.* assumed a viscous stress on the face of a crack of uniform thickness δ, whereas the thickness of the ellipsoid which we have used as the model of a crack varies from $2c$ to zero.

We may, in fact, obtain equation (26) by a simpler, but rather more heuristic, route. The dominant strain within the inclusion is

$$e_{13}^{\mathrm{I}} \triangleq \frac{1}{2}\frac{\partial \mu_1}{\partial x_3},$$

and so the jump in transverse displacement across the crack is

$$\begin{aligned}[u_1] &\triangleq 2e_{13}^{\mathrm{I}}\cdot 2c(1 - r^2/a^2)^{1/2},\\ &= \frac{\sigma_{13}^{\mathrm{I}}}{\mu'}\cdot 2c(1 - r^2/a^2)^{1/2},\\ &= \frac{a}{\mu}\,\sigma_{13}^{\infty}\mathscr{U}_{11}(1;\mathbf{X}).\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}\mathscr{U}_{11}(1;\mathbf{X}) &= \frac{8}{\pi}\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)(1 - r^2/a^2)^{1/2}\left(1 - \frac{\sigma_{13}^{\mathrm{I}}}{\sigma_{13}^{\infty}}\right)\\ &= \frac{8}{\pi}\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)(1 - r^2/a^2)^{1/2} - \frac{4}{\pi}\left(\frac{a\mu'}{c\mu}\right)\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)\mathscr{U}_{11}(1;\mathbf{X}).\end{aligned}$$

and equation (26) is regained.

Since the evaluation of the strain field in the inclusion arising from an axial stress σ_{33}^{∞} at infinity is rather complicated, we shall use this latter method to find a new expression for $\overline{\mathscr{U}}_{33}(0)$.

The dominant strain in the ellipsoid in this case is

$$e_{33}^{\mathrm{I}} = \frac{\partial u_3}{\partial x_3}$$

and so

$$\begin{aligned}[u_3] &= e_{33}^{\mathrm{I}}.2c(1 - r^2/a^2)^{1/2}\\ &\triangleq \frac{\sigma_{33}^{\mathrm{I}}}{[\kappa' + (4/3)\mu']}\cdot 2c(1 - r^2/a^2)^{1/2}\\ &= \frac{a}{\mu}\,\sigma_{33}^{\infty}\mathscr{U}_{33}(1;\mathbf{X})\end{aligned} \tag{30}$$

where κ' is the bulk modulus of the material of the inclusion. The discontinuity $[u_3]$ in normal displacement may be found, as before, by solving for a stress-free crack under a stress field $\sigma_{33}^{\infty} - \sigma_{33}^{\mathrm{I}}$ at infinity. Therefore,

$$\begin{aligned}\mathscr{U}_{33}(1;\mathbf{X}) &= \frac{2}{\pi}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)(1 - r^2/a^2)^{1/2}(1 - \sigma_{33}^{\mathrm{I}}/\sigma_{33}^{\infty})\\ &= \frac{2}{\pi}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)(1 - r^2/a^2)^{1/2} - \frac{1}{\pi}\,\frac{a[\kappa' + (4/3)\mu']}{c\mu}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\mathscr{U}_{33}(1;\mathbf{X}),\end{aligned} \tag{31}$$

and so

$$\overline{\mathscr{U}}_{33}(0)=\frac{4}{3}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)(1+K)^{-1}, \tag{32}$$

where

$$K=\frac{1}{\pi}\,\frac{a[\kappa'+(4/3)\mu']}{c\mu}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right),$$

and expressions (21) for the elastic moduli c^1_{11}, $c^1_{12}\ldots, c^1_{33}$ (but not c^1_{44} and c^1_{55}) must be multiplied by the factor $(1+K)^{-1}$ to allow for the fact that $c\mu/a[\kappa'+(4/3)\mu']$ may not be small.

We notice once again that, if κ'/μ, μ'/μ are not both small, $\overline{\mathscr{U}}_{33}(0)$ is $O(c/a)$ and may be neglected; we regain the result (11) originally obtained for a fluid-filled crack. If κ' and μ' are zero we have, as expected, the result (20) for dry cracks.

Combining the new formulae for $\overline{\mathscr{U}}_{11}(\mathrm{o})$ and $\overline{\mathscr{U}}_{33}(0)$, we have the wave speeds to be

$$\left(\frac{\omega}{k}\right)^2=\alpha^2\left[1-\frac{4}{3}(\nu a^3)\left(\frac{(\lambda+2\mu\cos^2\theta)^2}{\mu(\lambda+\mu)(1+K)}+\frac{16\mu\sin^2\theta\cos^2\theta}{(3\lambda+4\mu)(1+M)}\right)\right] \tag{33}$$

for compressional-type waves,

$$\left(\frac{\omega}{k}\right)^2=\beta^2\left\{1-\frac{16}{3}(\nu a^3)\left[\frac{(\lambda+2\mu)\cos^2\theta\sin^2\theta}{(\lambda+\mu)(1+K)}+\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)\frac{(\cos^2\theta-\sin^2\theta)^2}{(1+M)}\right]\right\} \tag{34}$$

for SV-type waves, and

$$\left(\frac{\omega}{k}\right)^2=\beta^2\left[1-\frac{16}{3}(\nu a^3)\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)\frac{\cos^2\theta}{(1+M)}\right] \tag{35}$$

for SH-type waves.

The first-order changes in the isotropic elastic moduli in a material containing randomly orientated inclusions are given by

$$\frac{3\lambda_1+2\mu_1}{3\lambda+2\mu}=-\frac{4}{9}(\nu a^3)\,\frac{(3\lambda+2\mu)(\lambda+2\mu)}{(\lambda+\mu)\mu(1+K)}, \tag{36}$$

$$\frac{\mu_1}{\mu}=-\frac{16}{45}(\nu a^3)(\lambda+2\mu)\left[\frac{6(1+M)^{-1}}{(3\lambda+4\mu)}+\frac{(1+K)^{-1}}{(\lambda+\mu)}\right], \tag{37}$$

and equations (21) for the anisotropic moduli are replaced by

$$\begin{aligned}
c^1_{11}=c^1_{12}=c^1_{21}=c^1_{22}&=-\frac{4}{3}(\nu a^3)\,\frac{\lambda^2(\lambda+2\mu)}{\mu(\lambda+\mu)(1+K)},\\
c^1_{13}=c^1_{23}=c^1_{31}=c^1_{32}&=-\frac{4}{3}(\nu a^3)\,\frac{\lambda(\lambda+2\mu)^2}{\mu(\lambda+\mu)(1+K)},\\
c^1_{33}&=-\frac{4}{3}(\nu a^3)\,\frac{(\lambda+2\mu)^3}{\mu(\lambda+\mu)(1+K)},\\
c^1_{44}=c^1_{55}&=-\frac{32}{3}(\nu a^3)\,\frac{\mu(\lambda+2\mu)}{(3\lambda+4\mu)(1+M)},
\end{aligned} \tag{38}$$

all other components being zero.

If the material of the inclusion is a perfect fluid, $\mu' = 0$, and so

$$M = 0, \quad K = \frac{1}{\pi}\left(\frac{a\kappa'}{c\mu}\right)\left(\frac{\lambda + 2\mu}{\lambda + \mu}\right).$$

This agrees exactly with Budiansky & O'Connell (1976) and Hoenig (1979).

Equations (36) and (37) also agree with expressions given by O'Connell & Budiansky (1977). The latter give the overall complex moduli for a material containing cracks filled with visco-elastic fluid in the case where $(\kappa c/\kappa' a)$ is small and so, for isolated cracks, $(1 + K)^{-1}$ is zero. However, these results were derived by the self-consistent method and are, as usual, valid over a wider range of concentration than equations (36) and (37). In addition, O'Connell & Budiansky (1977) allow for the possibility of flow of fluid between cracks. This does not alter the overall bulk modulus, since no flow occurs under hydrostatic stress, and their results remain $3\lambda_1 + 2\mu_1 = 0$. But μ_1 is changed and agrees, to first order in (νa^3), with equation (37) if

$$M = \frac{4 i a \omega \eta}{\pi c \mu}\left(\frac{\lambda + 2\mu}{3\lambda + 4\mu}\right)$$

(as in equation (28) with the opposite sign convention for the complex representation) and

$$K = \frac{i\omega}{\omega_1}\,\frac{4}{9}\,\frac{(3\lambda + 2\mu)(\lambda + 2\mu)}{(\lambda + \mu)\mu},$$

where ω_1 is a frequency determined by consideration of the effects of flow between cracks.

3.4 WAVE PROPAGATION THROUGH REGULARLY JOINTED MATERIAL

Certain rock formations contain series of plane parallel joints or fracture planes. These are often regularly spaced and may be filled with fluid or weak material such as fault gouge. Different joint sets may occur, with different orientations and spacings, in the same region. Overall elastic parameters for wave propagation in such material when wavelengths are large compared with the joint spacing have been given by Morland (1974). Although the geometry of joints is rather different from that of circular cracks, we shall show that, under certain conditions, the results are very similar.

For material with a single set of regularly spaced joints (with spacing distance d) orientated in the x_3 direction, the overall elastic parameters are given by Morland to be

$$\mathbf{c} = \mathbf{c}^0 + \mathbf{c}^1,$$

where

$$\begin{aligned}
c^1_{11} &= c^1_{22} = c^1_{12} = c^1_{21} = -\lambda^2/(\lambda + 2\mu + \Lambda_d),\\
c^1_{13} &= c^1_{23} = c^1_{31} = c^1_{32} = -\lambda(\lambda + 2\mu)/(\lambda + 2\mu + \Lambda_d),\\
c^1_{33} &= -(\lambda + 2\mu)^2/(\lambda + 2\mu + \Lambda_d),\\
c^1_{44} &= c^1_{55} = -2\mu^2/(\mu + \Lambda_s),
\end{aligned} \tag{39}$$

and all other components are zero. The quantities Λ_d and Λ_s are given approximately by

$$\Lambda_d = \frac{d}{\delta}\left(\kappa' + \frac{4}{3}\mu'\right), \quad \Lambda_s = \frac{d}{\delta}\mu', \tag{40}$$

where δ is the mean thickness of the joint, and κ', μ' are the bulk and shear moduli of the material within. It is assumed that $\delta/d \ll 1$.

Equations (39) are exactly the same as equation (6) (for the first-order changes in the elastic parameters for a dilute distribution of cracks) if

$$\begin{aligned} \nu a^3 \overline{\mathscr{U}}_{33} &= \frac{\mu}{\lambda + 2\mu + \Lambda_d} = \left[\frac{\lambda + 2\mu}{\mu} + \left(\frac{\lambda' + 2\mu'}{\mu}\right)\frac{d}{\delta}\right]^{-1}, \\ \nu a^3 \overline{\mathscr{U}}_{11} &= \frac{\mu}{\mu + \Lambda_s} = \left(1 + \frac{\mu'}{\mu}\frac{d}{\delta}\right)^{-1}. \end{aligned} \tag{41}$$

Equations (26) and (32) give, for circular cracks filled with weak material,

$$\begin{aligned} \overline{\mathscr{U}}_{33} &= \left[\frac{3}{4}\left(\frac{\lambda + \mu}{\lambda + 2\mu}\right) + \frac{3}{4\pi}\left(\frac{\lambda' + 2\mu'}{\mu}\right)\frac{a}{c}\right]^{-1}, \\ \overline{\mathscr{U}}_{11} &= \left[\frac{3}{16}\left(\frac{3\lambda + 4\mu}{\lambda + 2\mu}\right) + \frac{3}{4\pi}\frac{\mu'}{\mu}\frac{a}{c}\right]^{-1}. \end{aligned} \tag{42}$$

Comparison of the two sets of equations shows that they are equivalent only when $(a\mu'/c\mu)$, or equivalently $(d\mu'/\delta\mu)$, is large. It is clear that equations (39) are unlikely to hold for $(d_1\mu'/\delta\mu)$ small, since c^1 in that case is largely independent of the joint spacing. In addition, the results for circular cracks are true only to the first order in the perturbation due to the cracks. For the joints, this again implies that $(d\mu'/\delta\mu)$ must be large.

In this approximation we have for joints,

$$\begin{aligned} (\nu a^3)\,\overline{\mathscr{U}}_{33} &\simeq \frac{\mu}{\lambda' + 2\mu'}\frac{\delta}{d}, \\ (\nu a^3)\,\overline{\mathscr{U}}_{11} &\simeq \frac{\mu}{\mu'}\frac{\delta}{d}, \end{aligned} \tag{43}$$

and for cracks,

$$\begin{aligned} (\nu a^3)\,\overline{\mathscr{U}}_{33} &\simeq \frac{\mu}{\lambda' + 2\mu'}\,\nu(\pi a^2)\,\frac{4c}{3}, \\ (\nu a^3)\,\overline{\mathscr{U}}_{11} &\simeq \frac{\mu}{\mu'}\,\nu(\pi a^2)\,\frac{4c}{3}. \end{aligned} \tag{44}$$

Since the crack area per unit volume is $1/d$ for joints and $(\nu\pi a^2)$ for circular cracks, then equations (43) and (44) are equivalent if we regard $(4c/3)$ as representing the mean thickness δ of the crack.

4 Attenuation

Apart from the case where the cracks are filled with viscous fluid, the overall elastic properties we have investigated so far have shown no attenuation. It is clear that a wave travelling through a cracked solid will, in general, lose energy by means of scattering, but the effect is given by terms in (ka) of higher order than those considered so far.

In order to estimate the attenuation of a wave by scattering of energy out of the principal wave direction, we use the expression (15) for the wave speeds. This is correct to first order in (νa^3), and the deductions from it (e.g. equations 16–18) correct to lowest order in $(\omega a/\beta)$. We shall now want to work to higher order in $(\omega a/\beta)$.

Calculation of $\overline{\mathscr{U}}_{11}$ and $\overline{\mathscr{U}}_{33}$ to higher orders is a rather formidable task, but we may avoid the need for it by concentrating on the attenuation only and ignoring higher order effects on the phase velocity. If we write

$$k = \frac{\omega}{v} + i\gamma$$

where γ is the attenuation coefficient and v the phase velocity, then

$$\mathrm{Im}\,(\omega/k)^2 = -\frac{2v^3}{\omega}\,\gamma,$$

and so, from equations (15) and (4) we have

$$\gamma = -\frac{\omega}{2\rho v^3}\,\mathrm{Im}\,(c^1_{ipjq}\hat{b}^0_i\hat{b}^0_j\hat{k}_p\hat{k}_q) = \frac{\omega\nu a}{2\mu\rho v^3}\,\mathrm{Im}\,(c^0_{k3ip}c^0_{l3jq}\hat{b}^0_i\hat{b}^0_j\hat{k}_p\hat{k}_q)$$

$$\times \int_\Sigma \mathscr{U}_{kl}\left(\exp\frac{i\omega}{v}\hat{\mathbf{k}}\cdot\mathbf{X}\,;\ \mathbf{X}\right)\exp\left(-\frac{i\omega}{v}\hat{\mathbf{k}}\cdot\mathbf{X}\right)dS_X. \tag{45}$$

In equation (45), to lowest order, $\hat{\mathbf{b}}^0$ is the displacement orientation for a wave in uncracked material, and $v=\alpha$ or β for compressional and shear waves respectively. We now write the expression on the right-hand side in terms of the energy scattered by a single crack from an incident (unperturbed) wave with wavenumber $(\omega/v)\hat{\mathbf{k}}$ and displacement vector $\hat{\mathbf{b}}^0$.

The tractions imposed on the crack face by such a wave are

$$\sigma^0_{i3} = \frac{i\omega}{v}c^0_{i3pq}\hat{b}^0_p\hat{k}_q\exp\left(\frac{i\omega}{v}\hat{\mathbf{k}}\cdot\mathbf{X}\right),$$

and so, using the definition of $\mathscr{U}_{kl}$, we have

$$\gamma = \frac{\nu}{2\rho v\omega}\,\mathrm{Im}\left(\int_\Sigma [u_k]\,\sigma^{0*}_{k3}\,dS_X\right), \tag{46}$$

where $[\mathbf{u}]$ is the discontinuity of displacement across the crack, and the asterisk denotes the complex conjugate.

Considering the crack now as an obstacle, we may treat the scattered wave $\mathbf{u}^s$ both inside and outside the obstacle as being generated by a discontinuity in traction on the boundary S_0 of the obstacle of $(-\sigma^0_{ij}n_j)$, where $\mathbf{n}$ is the outward normal. Outside the obstacle we have

$$\mathrm{Im}\left(\int_{S_\infty} u^s_k\,\sigma^{s*}_{kj}n_j\,dS - \int_{S_0} u^s_k\,\sigma^{s*}_{kj}n_j\,dS\right) = 0,$$

where S_∞ is a sphere of large radius enclosing the obstacle and $\mathbf{n}$ the outward normal again. Within the obstacle

$$\mathrm{Im}\left(\int_{S_0} u^s_k\,\sigma^{s*}_{kj}n_j\,dS\right) = 0,$$

and so, combining these equations together, we have

$$\mathrm{Im}\left(\int_{S_\infty} u^s_k\,\sigma^{s*}_{kj}n_j\,dS\right) = -\,\mathrm{Im}\left(\int_{S_0} u^s_k\,\sigma^{0*}_{kj}n_j\,dS\right) = -\,\mathrm{Im}\left(\int_\Sigma [u_k]\,\sigma^{0*}_{k3}\,dS\right), \tag{47}$$

the last step being achieved by regarding the obstacle once more as having very small aspect ratio.

Equations (46) and (47) give

$$\gamma = \frac{\nu}{\rho v \omega^2}\left[\mathrm{Re}\left(\frac{1}{2}\int_{S_\infty}(i\omega u_k^s \sigma_{kj}^{s*} n_j)\, dS\right)\right]. \tag{48}$$

But the quantity in square brackets is the energy scattered from the crack. The energy flux per unit area in the incident wave is $(\rho v \omega^2/2)$ and so

$$\gamma = \frac{\nu\sigma}{2}, \tag{49}$$

where σ is the scattering cross-section of the crack.

Equation (49) is exactly the result of assuming that the energy scattered away from the incident wave may be calculated from the sum of the individual cross-sections of the cracks, neglecting any correlation between the radiation fields of separate cracks. It is not clear that this assumption would be true when the cracks are aligned and wavelengths are large compared with separation distances. However, equation (49) holds whether the cracks are aligned or not.

The corresponding result for acoustic waves was proved by Waterman & Truell (1961) and was assumed to be true for elastic waves by Piau (1979).

The scattering cross-section for an individual crack may be calculated, to lowest order in (ka) by using the static approximation for $[\mathbf{u}]$ on Σ. The radiation from the crack is given by (Hudson 1980)

$$u_i^s(\mathbf{x}) = -\int_\Sigma [u_k](\mathbf{X}) c_{k3pq}^0 \frac{\partial G_i^p}{\partial x_q}(\mathbf{x}, \mathbf{X})\, dS_X, \tag{50}$$

where $\mathbf{G}$ is Green's function for an unbounded medium. For small (ka) we may regard $\mathbf{G}$ as constant across the face of the crack, and so

$$\begin{aligned} u_i^s(\mathbf{x}) = &-\frac{ia^3\omega}{\mu v} c_{i3mn}^0 \hat{b}_m^0 \hat{k}_n \overline{\mathscr{U}}_{kl}(0) c_{k3pq}^0 \frac{\partial G_i^p}{\partial x_q}(\mathbf{x}, 0) = -\frac{ia^3\omega}{\mu v}\Bigg\{\mu^2(\hat{b}_1^0\hat{k}_3 + \hat{b}_3^0\hat{k}_1) \\ &\times\left(\frac{\partial G_i^3}{\partial x_1} + \frac{\partial G_i^1}{\partial x_3}\right)\overline{\mathscr{U}}_{11} + \mu^2(\hat{b}_2^0\hat{k}_3 + \hat{b}_3^0\hat{k}_2)\left(\frac{\partial G_i^2}{\partial x_1} + \frac{\partial G_i^1}{\partial x_2}\right)\overline{\mathscr{U}}_{11} \\ &+ [(\lambda + 2\mu)\hat{b}_3^0\hat{k}_3 + \lambda(\hat{b}_1^0\hat{k}_1 + \hat{b}_2^0\hat{k}_2)]\left[(\lambda + 2\mu)\frac{\partial G_i^3}{\partial x_3} + \lambda\left(\frac{\partial G_i^1}{\partial x_1} + \frac{\partial G_i^2}{\partial x_2}\right)\right]\overline{\mathscr{U}}_{33}\Bigg\}. \end{aligned} \tag{51}$$

For *SH* waves ($\hat{\mathbf{b}}^0 \equiv (0, 1, 0)$, $\hat{\mathbf{k}} \equiv (\sin\theta, 0, \cos\theta)$, $v = \beta$) we have

$$\begin{aligned} u_i^s(\mathbf{x}) &= -\frac{ia^3\omega}{\beta}\mu\hat{k}_3\overline{\mathscr{U}}_{11}\left(\frac{\partial G_i^2}{\partial x_1} + \frac{\partial G_i^1}{\partial x_2}\right) \\ &= \frac{a^3\omega^2\mu\cos\theta}{4\pi\rho\beta}\overline{\mathscr{U}}_{11}\left[\frac{2l_i l_1 l_2}{\alpha^3}\frac{\exp(i\omega r/\alpha)}{r} + \frac{(\delta_{i1}l_2 + \delta_{i2}l_1 - 2l_i l_1 l_2)}{\beta^3}\frac{\exp(i\omega r/\beta)}{r}\right], \end{aligned}$$

using the far-field expressions for $\mathbf{G}$ given in the Appendix, where $|\mathbf{x}| = r$ and $l_i = x_i/r$. The cross-section is

$$\sigma_{SH} = \left(\frac{a^3\omega^2}{4\pi\beta^2}\right)^2(\overline{\mathscr{U}}_{11})^2\cos^2\theta\left[\left(\frac{\beta}{\alpha}\right)^5\frac{16\pi}{15} + \frac{8\pi}{5}\right]$$

and the attenuation coefficient

$$\gamma_{SH} = \frac{\omega}{\beta}(\nu a^3)\left(\frac{\omega a}{\beta}\right)^3 \frac{(\overline{\mathscr{U}}_{11})^2 \cos^2\theta}{30\pi}\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right). \tag{52}$$

The corresponding quality factor Q is, in general, given by

$$Q^{-1} = \left(\frac{\upsilon\nu}{\omega}\right)\sigma = \frac{2\upsilon\gamma}{\omega}, \qquad \upsilon = \alpha,\ \beta, \tag{53}$$

and so, for SH waves,

$$Q_{SH}^{-1} = (\nu a^3)\left(\frac{\omega a}{\beta}\right)^3 \frac{(\overline{\mathscr{U}}_{11})^2 \cos^2\theta}{15\pi}\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right). \tag{54}$$

This agrees exactly with the expression given by Chatterjee *et al.* (1980) for attenuation by cracks filled with a weakly viscous fluid if the argument of Section 3.3 is followed. The results of Chatterjee *et al.* are valid for small M only, whereas expressions (52) and (54) for the attenuation and quality factor, with $\overline{\mathscr{U}}_{11}$ substituted from equation (27), hold for all values of M.

For SV waves ($\hat{\mathbf{b}}^0 \equiv (\cos\theta, 0, -\sin\theta)$, $\hat{\mathbf{k}} \equiv (\sin\theta, 0, \cos\theta)$, $\upsilon = \beta$) we have

$$\begin{aligned} u_i^s(\mathbf{x}) &= -\frac{ia^3\omega\mu}{\beta}\left\{\overline{\mathscr{U}}_{11}\cos 2\theta\left(\frac{\partial G_i^3}{\partial x_1}+\frac{\partial G_i^1}{\partial x_3}\right) - \overline{\mathscr{U}}_{33}\sin 2\theta\left[\frac{\alpha^2}{\beta^2}\frac{\partial G_i^3}{\partial x_3} - 2\left(\frac{\partial G_i^1}{\partial x_1}+\frac{\partial G_i^2}{\partial x_2}\right)\right]\right\} \\ &= \frac{a^3\omega^2\mu}{4\pi\rho\beta}\left\{\left[\overline{\mathscr{U}}_{11}\cos 2\theta(2l_1l_3) - \overline{\mathscr{U}}_{33}\sin 2\theta\left(\frac{\alpha^2}{\beta^2} - 2(l_1^2+l_2^2)\right)\right]\frac{l_i\exp(i\omega r/\alpha)}{\alpha^3 r}\right. \\ &\quad + \left[\overline{\mathscr{U}}_{11}\cos 2\theta(\delta_{i1}l_3+\delta_{i3}l_1 - 2l_il_1l_3) + 2\,\overline{\mathscr{U}}_{33}\sin 2\theta\,[(\delta_{i1}-l_il_1)l_1 + (\delta_{i3}-l_il_3)l_3]\right] \\ &\quad \left.\times\frac{\exp(i\omega r/\beta)}{\beta^3 r}\right\}, \end{aligned} \tag{55}$$

and the cross-section

$$\sigma_{SV} = \left(\frac{a^3\omega^2}{\beta^2}\right)^2\frac{1}{15\pi}\left[\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{11})^2\cos^2 2\theta + \left(2+\frac{15}{4}\frac{\beta}{\alpha} - 10\frac{\beta^3}{\alpha^3} + 8\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{33})^2\sin^2 2\theta\right]. \tag{56}$$

Thus

$$\begin{aligned} \gamma_{SV} &= \frac{\omega}{\beta}(\nu a^3)\left(\frac{\omega a}{\beta}\right)^3\frac{1}{30\pi}\left[\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{11})^2\cos^2 2\theta\right. \\ &\quad \left. + \left(2+\frac{15}{4}\frac{\beta}{\alpha} - 10\frac{\beta^3}{\alpha^3} + 8\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{33})^2\sin^2 2\theta\right]. \end{aligned} \tag{57}$$

This again agrees with Chatterjee *et al.* (1980) for fluid-filled cracks ($\overline{\mathscr{U}}_{33} = 0$, and $\overline{\mathscr{U}}_{11}$ given by equation 27). In general we may substitute for $\overline{\mathscr{U}}_{11}$ from equation (27) and for $\overline{\mathscr{U}}_{33}$ from equation (32).

For a shear wave of general orientation ($\hat{\mathbf{b}}^0 \equiv (0, 1, 0) \cos\phi + (\cos\theta, 0, -\sin\theta)\sin\phi$) the cross-sections for the SH and SV components are simply added together to give

$$\gamma_S = \frac{\omega}{\beta}(\nu a^3)\left(\frac{\omega a}{\beta}\right)^3 \frac{1}{30\pi}\left[\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{11})^2(\cos^2\theta\cos^2\phi + \cos^2 2\theta \sin^2\phi)\right.$$

$$\left. + \left(2 + \frac{15}{4}\frac{\beta}{\alpha} - 10\frac{\beta^3}{\alpha^3} + 8\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{33})^2 \sin^2 2\theta \sin^2\phi\right]. \tag{58}$$

If the cracks are randomly orientated we have

$$\gamma_S = \frac{\omega}{\beta}(\nu a^3)\left(\frac{\omega a}{\beta}\right)^3 \frac{1}{75\pi}\left[\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{11})^2 + \frac{1}{3}\left(2 + \frac{15}{4}\frac{\beta}{\alpha} - 10\frac{\beta^3}{\alpha^3} + 8\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{33})^2\right]. \tag{59}$$

Finally we consider the attenuation of compressional waves ($\hat{\mathbf{b}}^0 = \hat{\mathbf{k}} \equiv (\sin\theta, 0, \cos\theta)$, $v = \alpha$). In this case

$$u_i^s(\mathbf{x}) = -\frac{ia^3\omega\mu}{\alpha}\left[\sin 2\theta\, \overline{\mathscr{U}}_{11}\left(\frac{\partial G_i^3}{\partial x_1} + \frac{\partial G_i^1}{\partial x_3}\right)\right.$$

$$\left. + \left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)\overline{\mathscr{U}}_{33}\left(\frac{\alpha^2}{\beta^2}\frac{\partial G_i^k}{\partial x_k} - 2\frac{\partial G_i^1}{\partial x_1} - 2\frac{\partial G_i^2}{\partial x_2}\right)\right]$$

$$= \frac{a^3\omega^2\mu}{4\pi\rho\alpha}\left\{\left[\overline{\mathscr{U}}_{11}\sin 2\theta(2l_1l_3) + \overline{\mathscr{U}}_{33}\left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)\left(\frac{\alpha^2}{\beta^2} - 2l_1^2 - 2l_2^2\right)\right]\frac{\exp(i\omega r/\alpha)}{\alpha^3 r}\right.$$

$$+ \left[\overline{\mathscr{U}}_{11}\sin 2\theta(\delta_{i1}l_3 + \delta_{i3}l_1 - 2l_il_1l_3) - 2\,\overline{\mathscr{U}}_{33}\left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)\right.$$

$$\left.\left. \times (\delta_{i1}l_1 + \delta_{i2}l_2 - l_il_1^2 - l_il_2^2)\right]\frac{\exp(i\omega r/\beta)}{\beta^3 r}\right\} \tag{60}$$

and the cross-section σ_P is

$$\sigma_P = \left(\frac{a^3\omega^2}{\alpha^2}\right)^2 \frac{(\alpha/\beta)}{15\pi}\left[(\overline{\mathscr{U}}_{11})^2\sin^2 2\theta\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right) + (\overline{\mathscr{U}}_{33})^2\left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)^2\right.$$

$$\left. \times \left(2 + \frac{15}{4}\frac{\beta}{\alpha} - 10\frac{\beta^3}{\alpha^3} + 8\frac{\beta^5}{\alpha^5}\right)\right] \tag{61}$$

The attenuation coefficient is, therefore,

$$\gamma_P = \frac{\omega}{\beta}(\nu a^3)\left(\frac{\omega a}{\alpha}\right)^3 \frac{1}{30\pi}\left[\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{11})^2\sin^2 2\theta\right.$$

$$\left. + \left(2 + \frac{15}{4}\frac{\beta}{\alpha} - 10\frac{\beta^3}{\alpha^3} + 8\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{33})^2\left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)^2\right] \tag{62}$$

for aligned cracks, and

$$\gamma_P = \frac{\omega}{\beta}(\nu a^3)\left(\frac{\omega a}{\alpha}\right)^3 \frac{4}{(15)^2\pi}\left[\left(\frac{3}{2}+\frac{\beta^5}{\alpha^5}\right)(\overline{\mathscr{U}}_{11})^2 + \frac{1}{2}\left(2+\frac{15}{4}\frac{\beta}{\alpha} - 10\frac{\beta^3}{\alpha^3} + 8\frac{\beta^5}{\alpha^5}\right)\frac{\alpha^4}{\beta^4}\right.$$
$$\left.\times\left(\frac{15}{4} - 10\frac{\beta^2}{\alpha^2} + 8\frac{\beta^4}{\alpha^4}\right)(\overline{\mathscr{U}}_{33})^2\right] \qquad (63)$$

for random orientations.

When substitution is made for $\overline{\mathscr{U}}_{11}$ and $\overline{\mathscr{U}}_{33}$, these results agree exactly with those given by Piau (1979) for $M = K = 0$.

The variation of γ as the fourth power of the frequency ω is characteristic of expressions for energy loss in long wavelength (Rayleigh) scattering problems. At shorter wavelengths, different relations arise. For instance, Jeffreys (1931, 1970) gets a second-power law in his treatment of attenuation by a granular type of structure. However, at short wavelengths, the mean wave is not necessarily a good approximation to the observed wave.

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Appendix

Green's function **G** for time-harmonic disturbances in an unbounded homogeneous medium satisfies

$$\rho\omega^2 G_i^j(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\frac{\partial^2 G_k^j(\mathbf{x},\mathbf{y})}{\partial x_k\,\partial x_i} + \mu\frac{\partial^2}{\partial x_k^2}G_i^j(\mathbf{x},\mathbf{y}) = -\delta_{ij}\delta(\mathbf{x}-\mathbf{y}), \tag{A1}$$

where λ, μ are the Lamé constants, ρ the density and ω the frequency. The function itself is

$$\begin{aligned} G_i^j(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi\rho}\Bigg\{&\frac{l_il_j}{\alpha^2 r}\exp(i\omega r/\alpha) + \frac{(\delta_{ij}-l_il_j)}{\beta^2 r}\exp(i\omega r/\beta) \\ &-\frac{(\delta_{ij}-3l_il_j)}{\beta^2 r}\left(\frac{\beta}{\omega r}\right)\left[\left(\frac{\beta}{\omega r}\right)[\exp(i\omega r/\beta)-\exp(i\omega r/\alpha)]\right. \\ &\left.-i\left(\exp(i\omega r/\beta)-\frac{\beta}{\alpha}\exp(i\omega r/\alpha)\right)\right]\Bigg\}, \end{aligned} \tag{A2}$$

where $r = |\mathbf{x}-\mathbf{y}|$, $\mathbf{l} = (\mathbf{x}-\mathbf{y})/r$ and α and β are the two elastic wave speeds.

Equation (A2) gives the displacement in the i-direction at $\mathbf{x}$ due to a time-harmonic point force at $\mathbf{y}$. The far-field ($\beta/\omega r \ll 1$) disturbance due to a dipole is

$$\frac{\partial G_i^j(\mathbf{x},\mathbf{y})}{\partial x_k} = \frac{i\omega}{4\pi\rho}\left\{l_il_jl_k\frac{\exp(i\omega r/\alpha)}{\alpha^3 r} + (\delta_{ij}-l_il_j)l_k\frac{\exp(i\omega r/\beta)}{\beta^3 r}\right\}. \tag{A3}$$

Similarly, the far-field radiation from a point dilatation is

$$\frac{\partial G_i^j(\mathbf{x},\mathbf{y})}{\partial x_j} = \frac{i\omega}{4\pi\rho}l_i\frac{\exp(i\omega r/\alpha)}{\alpha^3 r}, \tag{A4}$$

and that from a double couple is

$$\frac{\partial G_i^j}{\partial x_k} + \frac{\partial G_i^k}{\partial x_j} = \frac{i\omega}{4\pi\rho}\left\{2l_il_jl_k\frac{\exp(i\omega r/\alpha)}{\alpha^3 r} + (\delta_{ij}l_k+\delta_{ik}l_j-2l_il_jl_k)\frac{\exp(i\omega r/\beta)}{\beta^3 r}\right\}. \tag{A5}$$