Elastic wave propagation in fluid-saturated porous media

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A new identification of the coefficients in the strain energy functional for Biot's theory is presented. Two types of porous media are distinguished: (1) With the granular constituents fully consolidated so the porous frame acts as a cohesive unit, and (2) with the granular constituents only partially consolidated so a fraction of the solid particles compose the porous frame while the remaining particles are (essentially) suspended in the saturating fluid. For complete consolidation, an exact identification of the coefficients is found. This identification differs from the standard identification of Geertsma [Trans. AIME **210**, 331 (1957)] and of Biot and Willis [J. Appl. Mech. **24**, 594 (1957)] for frames whose effective elastic moduli are not related to the grain moduli by the Voigt average. For partial consolidation, an exact identification of the coefficients is not known, but the standard identification is a good approximation. A method from the theory of composites is used to estimate the frame moduli and a theoretical model of the frame inertia in a fluid environment is developed. The predictions of the resulting model are compared to Plona's [Appl. Phys. Lett. **36**, 259 (1980)] recent measurements on a water-saturated porous structure of sintered glass beads. Good agreement between the theory for a fully consolidated frame and the experiment is found. Measured fast and slow compressional wave speeds agree with the theoretical predictions within experimental error (3%) in all cases.

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INTRODUCTION

Theories of consolidation and of elastic-wave propagation in fluid-saturated porous media were initiated by Biot in a series of papers[1-6] spanning more than 20 years of research. In the case of elastic-wave propagation, contributions to Biot's theory have been made by many authors[7-10] through the years. However, a thorough review of the extensive prior work on Biot's theory is beyond the scope of this paper. Instead, we will present a brief review of a recent experimental confirmation of Biot's theory as well as some new theoretical advances in the subject.

Plona[11] has performed laboratory pulse propagation experiments using a fluid-saturated porous structure consisting of fused glass beads immersed in water. At typical frequencies of 500 kHz, he unambiguously identifies three modes of propagation: (1) a fast compressional wave, (2) a slow compressional wave with speed slower than that of sound in water, and (3) a shear wave. Biot's theory predicts three such modes: (1) a fast compressional wave corresponding (approximately) to in-phase motion of the porous frame and pore fluid, (2) a slow compressional wave corresponding (approximately) to out-of-phase motion of frame and fluid, and (3) a shear wave propagating in the frame with an inertial contribution from the pore fluid. The qualitative agreement between the theory and the experiment was noted by Plona.[11] Quantitative agreement was established by the present author[12] using Biot's theory with the identification of coefficients given by Geertsma,[13] Biot and Willis,[4] and Stoll.[10]

Continuing this effort in Sec. I, we will derive a rigorous identification of the coefficients appearing in the strain energy functional of Biot's theory. This new identification depends only on the bulk and shear moduli of the porous frame, the bulk modulus of the pore fluid, and the porosity. The difference between the present identification and that of previous authors is shown to arise from the imprecise identification of the "unjacketed compressibility" with the compressibility of the granular constituents. The new identification is exact for porous media with fully consolidated frames but inappropriate for media which are only partially consolidated. On the other hand, the old identification is a valid approximation for media at any level of consolidation.

Section II summarizes our methods[12] of calculating the frame moduli and frame inertia. Section III provides a discussion of frequency dependence and porosity dependence of speeds for plane waves using the new identification of the coefficients. Theoretical and experimental results are compared in Sec. IV. We find that the good agreement reported previously[12] using the old version of Biot's theory has been improved by the new version—especially at the largest values of porosity. The theoretical predictions now agree with Plona's experimental results in all cases within the stated experimental error. We conclude with a summary of our results and with suggestions for future research in Sec. V.

I. IDENTIFYING THE COEFFICIENTS

Consider a porous structure whose connected pore space is saturated with a viscous fluid. The fraction of the volume occupied by the fluid is the porosity β which is assumed uniform. The bulk modulus and density of the fluid are K_f and ρ_f, respectively. The bulk and shear moduli of the (dry) porous frame are K^* and μ^*. If the frame is composed of a single type of solid constituent grain, the bulk and shear moduli and density of the grains are K_g, μ_g, and ρ_g. If the frame is itself a composite containing a distribution of granular constituents with different elastic properties (K_n, μ_n, ρ_n), then K^* and μ^* can be estimated using a theory of composite elastic materials.[14]

For long-wavelength disturbances ($\lambda \gg a$ where a is a typical pore size) propagating through such a porous medium, it makes sense to define average values of the (local) displacements in the solid and also in the fluid. The average displacement vector for the solid frame is

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u while that for the pore fluid is $\mathbf{u}_f$. The average fluid displacement relative to the frame is $\mathbf{w}=\beta(\mathbf{u}_f-\mathbf{u})$. For small (infinitesimal) strains, the frame dilatation is

$$e=e_x+e_y+e_z=\nabla\cdot\mathbf{u}, \tag{1}$$

where e_x, e_y, and e_z are the usual Cartesian strain components. Similarly, the average fluid dilatation is

$$e_f=\nabla\cdot\mathbf{u}_f \tag{2}$$

and the increment of fluid content[5] is

$$\zeta=-\nabla\cdot\mathbf{w}=\beta(e-e_f). \tag{3}$$

With these definitions, Biot[5] shows for an isotropic, linear medium that the strain energy functional E is a quadratic function of the strain invariants[15] $I_1=e$, I_2, and of ζ having the form

$$2E=He^2-2Ce\zeta+M\zeta^2-4\mu^*I_2, \tag{4}$$

where

$$I_2=e_xe_y+e_ye_z+e_ze_x-\tfrac{1}{4}(\gamma_x^2+\gamma_y^2+\gamma_z^2) \tag{5}$$

and γ_x, γ_y, γ_z are the shear strain components. The coefficients H, C, and M are undetermined at this point in the analysis. In fact, the purpose of this section of the paper is to determine the physical significance of these coefficients.

From (4) with E treated as a function of the e_i's, γ_i's, and ζ, it follows from the definition of the partial derivatives that the stress-strain relations are given by

$$\begin{aligned}
\tau_{xx}&=\frac{\partial E}{\partial e_x}=He-2\mu^*(e_y+e_z)-C\zeta,\\
\tau_{yy}&=\frac{\partial E}{\partial e_y}=He-2\mu^*(e_z+e_x)-C\zeta,\\
\tau_{zz}&=\frac{\partial E}{\partial e_z}=He-2\mu^*(e_x+e_y)-C\zeta,\\
\tau_{yz}&=\frac{\partial E}{\partial \gamma_x}=\mu^*\gamma_x,\\
\tau_{zx}&=\frac{\partial E}{\partial \gamma_y}=\mu^*\gamma_y,\\
\tau_{xy}&=\frac{\partial E}{\partial \gamma_z}=\mu^*\gamma_z,\\
p_f&=\frac{\partial E}{\partial \zeta}=M\zeta-Ce,
\end{aligned} \tag{6}$$

where the τ_{ij} are stresses in the frame and p_f is the isotropic pressure in the pore fluid. [Note that partial derivatives of ζ with respect the e_i's do not appear in (6) because ζ is one of the independent variables upon which the energy E is functionally dependent.]

Assuming a time dependence of the form $\exp(-i\omega t)$, the coupled wave equations which follow from (6) in the presence of dissipation[5,9] are

$$\begin{aligned}
&(H-\mu^*)\,\nabla e+\mu^*\nabla^2\mathbf{u}-C\nabla\zeta+\omega^2(\rho\mathbf{u}+\rho_f\mathbf{w})=0,\\
&C\nabla e-M\nabla\zeta+\omega^2(\rho_f\mathbf{u}+q\mathbf{w})=0,
\end{aligned} \tag{7}$$

where

$$\rho=(1-\beta)\rho_g+\beta\rho_f, \tag{8}$$

and

$$q=\alpha\rho_f/\beta+iF(\xi)\eta\rho_f/\kappa\omega. \tag{9}$$

The kinematic viscosity of the fluid is η; the permeability of the frame is κ; the dynamic viscosity factor[3] is given by

$$F(\xi)=\frac{1}{4}\,\frac{\xi T(\xi)}{1-2T(\xi)/i\xi}, \tag{10}$$

where

$$T(\xi)=\frac{\mathrm{ber}'(\xi)+i\,\mathrm{bei}'(\xi)}{\mathrm{ber}(\xi)+i\,\mathrm{bei}(\xi)}, \tag{11}$$

and

$$\xi=(\omega a^2/\eta)^{1/2}. \tag{12}$$

The functions $\mathrm{ber}(\xi)$ and $\mathrm{bei}(\xi)$ are the real and imaginary parts of the Kelvin function. The pore size parameter a is related to the average radius of the pores and is generally estimated[9] to be in the range $\frac{1}{6}-\frac{1}{7}$ of the grain size. The factor $\alpha\geqslant 1$ is a pure number related to the frame inertia in the fluid environment which we estimate later.[12]

To determine the coefficients H, C, and M, consider the significance of these coefficients as they appear in (6). Clearly the coefficients *may* depend on K^*, μ^*, K_g, μ_g, K_f, and β since the coefficients are themselves elastic constants determined by the constituents of the saturated porous structure. Furthermore, for studies of linear elastic waves, H, C, and M may be treated as constant; the equations in (6) provide the relationships between the pertinent infinitesimals τ_{ij}, e_i, γ_i, ζ. In fact the forms for H, C, and M we obtain are functions of the porosity β which is itself a function of the strain for finite strain.[13] Therefore, while determining these coefficients, we must remember that the β appearing in the expressions for the coefficients is the zeroth-order unstrained $\beta=\beta_0$. Contributions to the strain energy associated with changes in porosity are of higher order than quadratic in the strains and hence are properly neglected in the analysis of linear elastic waves. It must be stressed, however, that such neglect may not be appropriate in theories of consolidation of porous materials[4,13] where the pertinent stresses and strains may be finite.

Biot's theory is frequently applied in two physically distinct situations. One case is that of the fully consolidated frame. The second case is that of a partially consolidated frame or fluid suspension wherein a rigid frame may exist but at least some of the solid particles are not cemented to the frame. These two cases will be treated in turn and then compared.

A. Fully consolidated frame

To determine the coefficients in the case of a fully consolidated frame, consider an external applied isotropic pressure p such that

$$-p=\tau_{xx}=\tau_{yy}=\tau_{zz},$$

and

$$\tau_{xy}=\tau_{zx}=\tau_{xy}=0. \tag{13}$$

Then, by averaging the first three equations of (6), we find

$$-p=(H-\tfrac{4}{3}\mu^*)e-C\zeta$$
$$=(H-\tfrac{4}{3}\mu^*-\beta C)e+\beta Ce_f\,. \tag{14}$$

It is straightforward to invert (9) and the last equation of (6) to obtain

$$-e_f=\frac{(H-\frac{4}{3}\mu^*-\beta C)p_f+(\beta M-C)p}{\beta M(H-\frac{4}{3}\mu^*-C^2/M)}\,, \tag{15}$$

and

$$-e=\frac{-\beta Cp_f+\beta Mp}{\beta M(H-\frac{4}{3}\mu^*-C^2/M)}\,. \tag{16}$$

Equations (15) and (16) provide linear relations between the infinitesimals e, e_f and p, p_f. The coefficients are necessarily constant in the linear theory and are equal to the appropriate partial derivatives. Four partial derivatives are therefore determined by (15) and (16). Of these, the following two are of significance:

$$-\frac{\partial e}{\partial p}\bigg|_{p_f=\text{const}}=\frac{1}{H-\frac{4}{3}\mu^*-C^2/M}\,, \tag{17}$$

and

$$-\frac{\partial e_f}{\partial p}\bigg|_{p_f=\text{const}}=\frac{\beta M-C}{\beta M(H-\frac{4}{3}\mu^*-C^2/M)}\,. \tag{18}$$

This pair of partial derivatives expresses mathematically the concepts used by various authors[4,9,13] to describe the so-called "jacketed test" of a fluid-saturated porous material. In the jacketed test, the fluid-saturated porous material is placed in a flexible but impervious bag; then the external pressure p is applied; the pore fluid is allowed to escape through a tube. In this situation, the pressure of the pore fluid remains unchanged $p_f=\text{const}$. The modulus measured by this test is the bulk modulus of the free draining (or dry) porous frame K^* and is given by

$$-\frac{\partial e}{\partial p}\bigg|_{p_f=\text{const}}=\frac{1}{K^*}\,. \tag{19}$$

Combining (17) and (19) then gives one equation relating H, C, and M to known quantities

$$K^*=H-\tfrac{4}{3}\mu^*-C^2/M\,. \tag{20}$$

(This result is standard.[4,9]) Furthermore, we see that (18) supplies a second equation. During the jacketed test, the pore fluid remains both unstressed *and unstrained* so that

$$-\frac{\partial e_f}{\partial p}\bigg|_{p_f=\text{const}}=0\,. \tag{21}$$

Combining (18) and (21), we find

$$C=\beta M\,. \tag{22}$$

Equation (22) is new but it should be compared with the result of Biot and Willis[4] that $\beta M\leqslant C\leqslant M$.

We now have two equations relating H, C, and M to known quantities but we still need a third. The remaining relation is obtained from the last equation in (6) by using the same general method. Of the partial derivatives remaining, only one is both independent and known *a priori*, namely

$$-\frac{\partial p_f}{\partial e_f}\bigg|_{e=\text{const}}=\beta M\,. \tag{23}$$

Equation (23) gives the proportionality between the fluid stress and fluid strain when the solid frame is maintained at constant strain. Such a test is not physically realizable for any compressible frame but *it does not need to be* since (23) is just one definition of the bulk modulus of the fluid. We think of (23) as the mathematical statement of a hypothetical test wherein the true frame is replaced by an incompressible frame of the same shape; then the frame becomes the vessel (albeit a very complicated one) within which the bulk modulus of the fluid is measured. From this discussion, we conclude that

$$\beta M=K_f\,. \tag{24}$$

Finally, it follows from (20), (22), and (24) that the coefficients which were to be determined are given by

$$H^*=K^*+\tfrac{4}{3}\mu^*+\beta K_f\,, \tag{25}$$

$$C^*=K_f\,, \tag{26}$$

and

$$M^*=K_f/\beta\,. \tag{27}$$

The superscripts have been added to distinguish this identification from the standard identification presented later.

Equations (25)–(27) are the main results of this section. These results are valid as long as the frame is fully consolidated with finite values of K^* and μ^*. In particular, note that the present analysis fails if $K^*=0=\mu^*$ for some value of $\beta<1$. To allow the moduli to vanish for $\beta\neq 1$ would reduce the physical picture to one of a fluid with the fraction $(1-\beta)$ of its volume occupied by vacuum. Such a situation is of no physical interest. However, it is of interest to allow the frame moduli to vanish by supposing that the granular constituents of the frame break away to form a fluid suspension where the fraction $(1-\beta)$ of the volume is occupied by suspended solid particles. This situation is the subject of the next section.

B. Partially consolidated frame and fluid suspensions

If the granular constituents of the porous material are not consolidated into a structure with finite rigidity, then the arguments used to describe the jacketed test producing (17)–(22) are not valid. In particular, Eq. (17) becomes undefined as $K^*\to 0$.

For an unconsolidated porous structure, it is useful to consider the so-called[4,9,13] "unjacketed test." After immersing the entire fluid-saturated porous structure in a reservoir of the fluid, hydrostatic pressure is applied to the reservoir. Then *it is assumed* that the fluid pressure p_f and the pressure p in the granular constituents are equal. This assumption is strictly valid only in a fluid suspension but it is also a good approximation (corresponding to the Reuss average in the theory of composites[14]) for high-porosity structures when the frame moduli K^* and μ^* are very small. Setting $p=p_f$ in (15) and (16), we find

$$-\left.\frac{\partial e_f}{\partial p_f}\right|_{p=p_f}=\frac{H-\frac{4}{3}\mu^*-C+\beta(M-C)}{\beta M(H-\frac{4}{3}\mu^*-C^2/M)}, \tag{28}$$

and

$$-\left.\frac{\partial e}{\partial p_f}\right|_{p=p_f}=\frac{1-C/M}{H-\frac{4}{3}\mu^*-C^2/M}. \tag{29}$$

Of course, small changes in fluid strain with respect to small changes in fluid pressure are controlled by the compressibility of the fluid so

$$-\left.\frac{\partial e_f}{\partial p_f}\right|_{p=p_f}=\frac{1}{K_f}. \tag{30}$$

Within the stated assumptions (Reuss average), small changes in the strain of granular constituents with respect to small changes in hydrostatic pressure are given by

$$-\left.\frac{\partial e}{\partial p_f}\right|_{p=p_f}=\frac{1}{1-\beta}\sum_{n=1}^{N}\frac{c_n}{K_n}\equiv\frac{1}{\overline{K}_g}, \tag{31}$$

where the K_n's are the bulk moduli of N granular constituents and the c_n's are the volume concentrations of the various constituents satisfying

$$\sum_{n=1}^{N}c_n=1-\beta. \tag{32}$$

If only one type of grain is present, the second equality in (31) is a tautology; otherwise, (31) defines $\overline{K}_g$. Equation (30) is exact but Eq. (31) is an approximation in all cases except that of a dilute fluid suspension.

Combining (28) with (30) and (29) with (31) provides two equations relating H, C, and M to known quantities. A third equation is required. If $K^*\equiv 0$, then Biot's equations are not really appropriate to the physical situation. Thus, we may suppose that K^* is finite though it may be quite small. Then, (17), (19), and (20) are still valid. These results together with those already obtained produce the standard formulas for the coefficients[4,7,9,13]

$$H=K^*+\frac{4}{3}\mu^*+\frac{(\overline{K}_g-K^*)^2}{D-K^*}, \tag{33}$$

$$C=\frac{\overline{K}_g(\overline{K}_g-K^*)}{D-K^*}, \tag{34}$$

and

$$M=\frac{\overline{K}_g^2}{D-K^*}, \tag{35}$$

where

$$D=\overline{K}_g[1+\beta(\overline{K}_g/K_f-1)]. \tag{36}$$

We see that the standard results for H, C, and M differ from (25)–(27), although in many cases of practical interest the difference in numerical value may be quite small.

Having obtained (33)–(35) while assuming K^* is finite, we may now take the limit K^*, $\mu^*\to 0$ to see if we obtain correct results for a fluid suspension. We find

$$\frac{1}{H}=\frac{1}{C}=\frac{1}{M}=\frac{\beta}{K_f}+\frac{1-\beta}{\overline{K}_g} \tag{37}$$

which, together with the definition (31), is exactly the Reuss average for a fluid suspension as we might expect. Geertsma and Smit[7] have shown that (37) implies a single compressional wave in the suspension and that the speed of the wave is given by Wood's formula[16] at low frequencies when the mass coupling effects are neglected.

C. Discussion

To compare (33)–(35) with (25)–(27), suppose that, for a frame composed of a single type of granular constituent, the bulk modulus of the frame differs only slightly (ϵ) from the Voigt average (a rigorous upper bound[14])

$$K^*=(1-\beta)K_g-\epsilon, \tag{38}$$

where $\epsilon\geq 0$. Substituting (38) into (33)–(36), we find

$$H=H^*+\epsilon\frac{K_f}{K_g}\left(2-\frac{K_f}{K_g}\right)+O(\epsilon^2), \tag{39}$$

$$C=C^*+\epsilon\frac{K_f}{\beta K_g}\left(1-\frac{K_f}{K_g}\right)+O(\epsilon^2), \tag{40}$$

and

$$M=M^*-\epsilon\frac{K_f^2}{\beta^2K_g^2}+O(\epsilon^2). \tag{41}$$

Note that, when $\epsilon=0$, the two sets of formulas (33)–(35) and (25)–(27) are identical. When $\epsilon>0$, these formulas agree only in the trivial case with $K_f/K_g\to 0$.

The inaccuracy in the derivation of the standard formulas occurs in the analysis of the unjacketed test for a fully consolidated frame. The standard argument[4,9,13] gives the "unjacketed compressibility"

$$\delta=\frac{1-C/M}{H-\frac{4}{3}\mu^*-C^2/M}=\frac{1}{K_g} \tag{42}$$

and the "coefficient of fluid content"

$$\gamma=\frac{H-\frac{4}{3}\mu^*-C}{M(H-\frac{4}{3}\mu^*-C^2/M)}=\beta\left(\frac{1}{K_f}-\delta\right). \tag{43}$$

In contrast, our analysis for the fully consolidated frame gives instead

$$\delta^*=\frac{1-C^*/M^*}{H^*-\frac{4}{3}\mu^*-C^{*2}/M^*}=\frac{1-\beta}{K^*} \tag{44}$$

and

$$\gamma^*=\frac{H^*-\frac{4}{3}\mu^*-C^*}{M^*(H^*-\frac{4}{3}\mu^*-C^{*2}/M^*)}=\beta\left(\frac{1}{K_f}-\delta^*\right). \tag{45}$$

The only difference is that the compressibility of the constituent grains is used directly in (42) whereas the compressibility of the frame weighted by the solid volume fraction appears in (44). The magnitude of the difference between (42) and (44) depends upon the value of K^* and how it is calculated. This question is considered in Sec. II.

We conclude that it is important to make a clear distinction between the two types of porous media. Fully consolidated frames consist of granular constituents cemented together forming a cohesive whole with no significant fraction of stray particles floating in the fluid. For simplicity, we may label this situation the "clean frame porous medium." By contrast, the "dirty frame porous medium" consists of a partially consolidated frame with a significant fraction of stray particles muddying the pore fluid.

The fact that Biot's theory for the clean frame with coefficients specified by (25)–(27) depends only on K^*, μ^*, K_f, and β appeals to our physical intuition. The theory still depends implicitly on K_g and μ_g (see Sec. II and Ref. 12) but the explicit dependence has been eliminated. Contemplating the particle motion for waves in fully consolidated porous media, we see that, for both in-phase and out-of-phase compressional waves, the controlling parameters must be limited to K^*, μ^*, K_f, and an appropriate weighting factor depending upon the porosity β. The bulk modulus of the constituent grains cannot play a role independent of the frame moduli unless either (1) nonlinear (porosity dependent) effects are being considered or (2) the frame moduli are so weak that they are of secondary importance compared to the grain moduli of suspended grains. If porosity dependent effects are of interest, then K_g and μ_g determine how K^* and μ^* change as the porosity changes. However, as stressed previously, all effects resulting from changes in porosity are of higher order than those treated in linear elasticity and are therefore properly neglected in the determination of H^*, C^*, and M^*. We conclude qualitatively that the coefficients should depend only on K^*, μ^*, K_f, and β for Biot's theory with a clean frame—in agreement with the results (25)–(27).

II. FRAME MODULI AND FRAME INERTIA

In a previous publication,[12] the author has shown that the frame parameters K^*, μ^*, and α can be estimated theoretically. This analysis is summarized here for the sake of completeness.

Since the moduli K^* and μ^* are moduli of the porous (drained) frame, it follows that these moduli can be estimated by treating the frame as an $(N+1)$-phase composite: N phases are the granular constituents (K_n, μ_n, c_n) while the remaining phase is vacuum $(K_0, \mu_0, c_0)=(0, 0, \beta)$. (This method may not work so well for unconsolidated media at high porosities.) A variety of methods are available for estimating[14,17] K^* and μ^*.

The method preferred by the present author is the recently developed self-consistent theory of composite materials[18] which predicts that

$$\sum_{n=0}^{N} c_n(K_n - K^*)\tfrac{1}{3}T_{iijj}^{(n)} = 0 \tag{46}$$

and

$$\sum_{n=0}^{N} c_n(\mu_n - \mu^*)\tfrac{1}{5}\left(T_{ijij}^{(n)} - \tfrac{1}{3}T_{iijj}^{(n)}\right) = 0\,. \tag{47}$$

The volume concentrations must satisfy

$$\sum_{n=1}^{N} c_n = 1-\beta\,, \quad \text{and } c_0=\beta\,. \tag{48}$$

The values of the scalars $T_{iijj}^{(n)}$ and $T_{ijij}^{(n)}$ depend on the K_n's, μ_n's, K^*, μ^*, and also on the shapes of the inclusions. Since the theory has been studied in detail for inclusions of arbitrary ellipsoidal shape, we choose to approximate the pores for $\beta \ll 1$ by prolate spheroidal voids with very small aspect ratio. To a good approximation, the scalars for needle-shaped inclusions (zero aspect ratio prolate spheroids) may therefore be used to represent the pore space. These scalars are given by[19]

$$\tfrac{1}{3}T_{iijj}^{(n)} = \frac{K^* + \mu^* + \frac{1}{3}\mu_n}{K_n + \mu^* + \frac{1}{3}\mu_n}\,, \tag{49}$$

and

$$\tfrac{1}{5}\left(T_{ijij}^{(n)} - \tfrac{1}{3}T_{iijj}^{(n)}\right) = \frac{1}{5}\left(\frac{4\mu^*}{\mu^* + \mu_n} + 2\,\frac{\mu^* + \gamma^*}{\mu_n + \gamma^*} + \frac{K_n + \frac{4}{3}\mu^*}{K_n + \mu^* + \frac{1}{3}\mu_n}\right), \tag{50}$$

where

$$\gamma^* = \mu^*\left(\frac{3K^* + \mu^*}{3K^* + 7\mu^*}\right)\,. \tag{51}$$

For the pore space, $n=0$.

As $\beta \to 1$, the frame becomes so porous that only thin filaments of solid material remain, assuming K^* and μ^* remain finite. These thin filaments may again be represented by needle-shaped inclusions and (49)–(51) may be used in (46) and (47) for all $0 \leq n \leq N$.

Depending upon the true geometry of the pore space, other choices of the scalars may be used. In particular, we are not limited to zero aspect ratio prolate spheroids; rather, we could use the scalars appropriate for the finite aspect ratio case.[19]

Equations (46) and (47) may be solved easily by numerical iteration. The resulting estimates of the frame moduli K^* and μ^* are not exact. However, they are known[18] to satisfy all presently known rigorous bounds on the effective moduli and are therefore expected to be reasonably accurate estimates of the desired moduli. If more accurate theoretical or experimental values of K^* and μ^* are known, then of course they should be used instead of the self-consistent estimates.

The inertia of the frame in the fluid environment determines the value of the parameter α. To show this, consider the physical interpretation of the densities $\rho_{11}, \rho_{12}, \rho_{22}$ appearing in Biot's original work.[2] The following relations between known quantities and the unknown densities were noted by Biot:

$$\rho_{11} + \rho_{12} = (1-\beta)\rho_g\,, \tag{52}$$

and

$$\rho_{22} + \rho_{12} = \beta\rho_f\,, \tag{53}$$

where ρ_g and ρ_f are the densities of the constituent grains and the fluid, respectively. If more than one type of grain is present, then (52) is replaced by

$$\rho_{11}+\rho_{12}=\sum_{n=1}^{N} c_n\rho_n . \tag{54}$$

The physical interpretation of ρ_{11} given by Biot is that of the total effective density of the solid frame moving in the fluid. The additional apparent density is $(-\rho_{12}) \geq 0$.

Equations (52) and (53) provide two equations for the three unknowns. A third equation follows from the physical interpretation of ρ_{11}:

$$\rho_{11}=(1-\beta)(\rho_g+r\rho_f), \tag{55}$$

where $r\rho_f$ is the induced mass[20] due to the oscillation of solid particles in a fluid of density ρ_f. For spheres oscillating in a fluid, $r=\frac{1}{2}$; for other ellipsoidal shapes, Lamb[20] shows that $0\leq r\leq 1$ (and also provides tables of the value of r as a function of aspect ratio). Substituting (55) into (52) and making the standard definition $\rho_{22}=\alpha\beta\rho_f$, we find easily that

$$\alpha=1-r(1-1/\beta)\geq 1 . \tag{56}$$

Note that, although $\alpha\to\infty$ as $\beta\to 0$, all physical quantities remain finite because α always occurs in the combination $(\alpha\beta)$; in particular, $\rho_{22}\to r\rho_f$ as $\beta\to 0$. Also note that $\alpha\to 1$ as $\beta\to 1$.

We wish to stress that the derivation just given is heuristic. Equation (55) is expected to be a close approximation to the true value of ρ_{11} for some value of r. However, to calculate a more accurate estimate of ρ_{11} requires a microscopic model. Burridge and Keller[21] have recently made some significant advances with such a model but the results of these efforts are not yet available.

III. POROSITY AND FREQUENCY DEPENDENCE

The frequency dependence of the theory for plane waves in an infinite medium will now be treated. Although the frequency dependence has been studied before[7] with the standard identification of H, C, and M, the dependence on porosity of (25)–(27) and (56) is sufficiently different that we need to repeat some of this work here.

First, we must decouple the wave equations (7) into equations for the three modes of propagation. Deresiewicz[22] used the same method which we use to execute the decoupling although his analysis was for the nondissipative case. Burridge and Vargas[23] have performed a similar decoupling in the time domain assuming dissipation is present.

The displacements $\mathbf{u}$ and $\mathbf{w}$ can be decomposed according to

$$\begin{aligned}\mathbf{u}&=\nabla\theta+\nabla\times\boldsymbol{\phi},\\ \mathbf{w}&=\nabla\Psi+\nabla\times\boldsymbol{\chi},\end{aligned} \tag{57}$$

where θ,Ψ are scalar potentials and $\boldsymbol{\phi}$, $\boldsymbol{\chi}$ are vector potentials. Substituting (57) into (7), we find that (7) is satisfied if the potentials satisfy two pairs of coupled equations: one pair for the scalar potentials and one pair for the vector potentials.

The vector potentials satisfy

$$\boldsymbol{\chi}=-\rho_f\boldsymbol{\phi}/q , \tag{58}$$

and

$$(\nabla^2+k_s^2)\boldsymbol{\phi}=0 , \tag{59}$$

where

$$k_s^2=\omega^2(\rho-\rho_f^2/q)/\mu^* . \tag{60}$$

The single Helmholtz equation (59) determines the shear wave modes for Biot's theory.

The scalar potentials satisfy

$$(\nabla^2+k_\pm^2)\Gamma_\pm=0 , \tag{61}$$

where

$$\Gamma_\pm=A_\pm\theta+B_\pm\Psi , \tag{62}$$

$$k_\pm^2=\tfrac{1}{2}\{b+f\mp[(b-f)^2+4cd]^{1/2}\}, \tag{63}$$

$$A_\pm/B_\pm=d/(k_\pm^2-b)=(k_\pm^2-f)/c , \tag{64}$$

and

$$\begin{aligned}b&=\omega^2(\rho M-\rho_f C)/(MH-C^2),\\ c&=\omega^2(\rho_f M-qC)/(MH-C^2),\\ d&=\omega^2(\rho_f H-\rho C)/(MH-C^2),\\ f&=\omega^2(qH-\rho_f C)/(MH-C^2).\end{aligned} \tag{65}$$

The wave vectors $k_\pm$ are completely determined by (63) and (65) while $\Gamma_\pm$ is determined within a multiplicative factor by (62) and (64). The decoupling is therefore complete.

Since (59) and (61) are completely general, they may be applied to boundary value problems with arbitrary symmetry. For our present purposes, it is satisfactory to consider plane-wave solutions in an infinite medium. For plane waves proportional to $\exp i(kx-\omega t)$, the wave vectors for shear waves (k_s) and for both fast and slow compressional waves $(k_\pm)$ are given by (60) and (63).

The frequency dependence for shear waves is most easily treated first. The phase velocity for shear waves is given by

$$v_s^2(\omega)=\frac{\omega^2}{k_s^2}=\frac{\mu^*}{\rho-\rho_f^2/q} . \tag{66}$$

The frequency dependence of (66) is contained in q (9). Since $F(\xi)\to 1$ as ω, $\xi\to 0$ and $F(\xi)\propto\omega^{1/2}$ as ω, $\xi\to\infty$, we find that $q\to i\eta/\omega\kappa$ as $\omega\to 0$ and $q\to\alpha\rho_f/\beta$ as $\omega\to\infty$. Thus,

$$v_s^2(0)=\frac{\mu^*}{\rho} , \tag{67}$$

and

$$v_s^2(\infty)=\frac{\mu^*}{\rho-\beta\rho_f/\alpha} \tag{68}$$

Of course, Biot's theory is only valid for long wavelengths and low frequencies. However, frequencies in the MHz region appear (in Plona's experiments[11]) to be sufficiently low so that Biot's theory applies $(ka\ll 1)$, yet sufficiently high that (68) is still a good approximation ($\eta \ll \omega\,\kappa$). The long-wavelength shear wave speed for the dry porous frame is given correctly by (67) or (68) if $\dot{\rho}_f \to 0$. Therefore, the shear wave speed for the fluid-saturated frame is just the speed for the dry frame with a frequency-dependent correction for the inertia of the fluid.

At low frequencies, the other two modes behave, respectively, like a compressional wave (k_+) and a diffusion process (k_-). To see this, consider (63) as $\omega \to 0$ yielding

$$k_+^2 \to \omega^2 \frac{\rho}{H}, \tag{69}$$

and

$$k_-^2 \to i\omega\eta H/\kappa(MH - C^2). \tag{70}$$

The phase velocity of the compressional wave is

$$v_+^2(0) = \frac{\omega^2}{k_+^2} = \frac{H}{\rho}, \tag{71}$$

while (70) corresponds to a diffusion process in the time domain with

$$\mathscr{D}\nabla^2\Gamma_- = \frac{\partial}{\partial t}\Gamma_-, \tag{72}$$

where the diffusion coefficient is given by

$$\mathscr{D} = (\kappa/\eta)(M - C^2/H). \tag{73}$$

The interpretation of (71) is simple since $(H/\rho)^{1/2}$ closely approximates the low-frequency phase velocity in a nonporous composite material[14,18] where the constituents oscillate in phase. Differences between the values predicted by (71) and those predicted by a theory of composites are expected conceptually because of an inherent difference in the two models: the theory of composites generally assumes bounded inclusions imbedded in an infinite matrix whereas the theory for porous media assumes both matrix (frame) and inclusion (pore structure) are infinite in extent. For small values of porosity, the predicted value of the phase velocity for both theories should be comparable. For large porosities, substantial differences in their predictions may be expected.

The interpretation of (72) and (73) may be checked by another approach. In the theory of fluid flow through a rigid porous medium, a diffusion equation is obtained by the following argument.[24] The continuity equation (conservation of mass) is

$$\beta \frac{\partial}{\partial t}\rho_f + \nabla\cdot(\rho_f \mathbf{V}) = 0, \tag{74}$$

where $\mathbf{V}$ is the average flow velocity. Darcy's law is

$$\mathbf{V} = -(\kappa/\eta)\nabla p_f, \tag{75}$$

and the fluid equation of state satisfies

$$\nabla p_f = \frac{\partial p_f}{\partial \rho_f}\nabla\rho_f. \tag{76}$$

Combining (74)–(76) and eliminating terms higher than first order in the change in density, we find

$$\mathscr{D}\nabla^2\rho_f = \frac{\partial}{\partial t}\rho_f, \tag{77}$$

where the diffusion coefficient

$$\mathscr{D} = \left(\rho_f \frac{\partial p_f}{\partial \rho_f}\right)\frac{\kappa}{\beta\eta} = -\frac{\partial p_f}{\partial e_f}\frac{\kappa}{\beta\eta} \tag{78}$$

is evaluated at the equilibrium value of ρ_f. It follows from (23) that

$$\mathscr{D} = (\kappa/\eta)M. \tag{79}$$

Equation (79) should now be compared to (73) in the limit of an infinitely rigid frame where $H \to \infty$. Thus, (72) and (77) agree within a constant multiple relating small changes in ρ_f to small changes in Γ_-, as is usual for linear processes.

At high frequencies, these two modes both behave like compressional waves. As $\omega \to \infty$, $v_\pm^2(\omega) \to \text{const}_\pm$ since $q \to \alpha\rho_f/\beta$ in this limit. The limiting constants are quite complicated in general as may be inferred from (63) and (65). However, in the limit $\beta \ll 1$, these expressions simplify sufficiently to be instructive. Recalling that $\alpha = 0(1/\beta)$ as $\beta \to 0$, we find for $\beta \ll 1$ that

$$v_+^2(\infty) = H/\rho \tag{80}$$

and

$$v_-^2(\infty) = K_f/\alpha\rho_f = \beta K_f/\rho_{22}. \tag{81}$$

For low porosity ($\beta \ll 1$), the speed of the fast compressional wave (71) and (80) is virtually independent of frequency. This result is expected for a theory valid only for long wavelengths; as previously explained, the high frequencies of interest are no higher than a few MHz. For low porosity, $v_-^2(\infty)$ is essentially the bulk modulus of the fluid weighted by the porosity and divided by the inertia of the fluid in the pores ρ_{22}. We see that $v_- \to 0$ as $\beta \to 0$ which also agrees with physical intuition.

IV. PLONA'S EXPERIMENT

As discussed in the Introduction, Plona[11] has observed two distinct compressional waves in a water-saturated porous structure of sintered glass beads. Using the standard formulas for H, C, and M given by (33)–(36) and the estimates of K^*, μ^*, and α as discussed in Sec. II, the present author found good agreement[12] between theory and experiment. The theoretical predictions were within the bounds of experimental error (3%) for the fast compressional wave and for the shear wave in all cases. For the slow compressional wave, the theoretically predicted speed was within about 10% of the experimental values and was found to increase with increase in porosity as Plona observed. This agreement was sufficiently good to allow us to conclude that the slow wave observed by Plona was indeed the wave of the second kind predicted by Biot's theory.

To demonstrate the accuracy of our estimates of the frame parameters in Sec. II, we will now repeat the comparison between theory and experiment but this time we use the new expressions for H^*, C^*, and M^* given in (25)–(27). The input parameters to our model[11,25] are $K_g = 0.407$ Mb, $\mu_g = 0.297$ Mb, $\rho_g = 2.48$ g/cc, $K_f = 0.022$ Mb, $\rho_f = 1.00$ g/cc, $\eta = 1.00$ centistoke, and $f = 500$ kHz. The frame moduli K^* and μ^* are computed using (46)–(51). The frame inertia factor α is calculated from (56) with $r=\frac{1}{2}$ since we expect many of the glass beads to retain their spherical shape during sintering for the porosities studied.[11] Although Stoll[9] has shown that *attenuation* is highly dependent on permeability and pore size for sands and fine-grained sediments even at high frequency, we find that for Plona's porous glass structure the calculated *speeds* are insensitive to the precise values of permeability and pore size parameter. The reason for this difference is that the permeability of the porous glass was at least four orders of magnitude larger than the permeability for sands and fine-grained sediments. Since the attenuation depends inversely on the permeability, the effects of small changes in a large value of the permeability are negligible. Therefore, we choose for simplicity to parametrize these variables using the Kozeny–Carman relation[26]

$$\kappa \frac{(1-\beta)^2}{\beta^3} = \kappa_0 \frac{(1-\beta_0)^2}{\beta_0^3} = \text{const} \tag{82}$$

and similarly

$$a^2/\kappa = a_0^2/\kappa_0 = \text{const}. \tag{83}$$

The permeability was chosen[25] so that $\kappa=\kappa_0=2\ \mu\text{m}^2$ at $\beta_0=0.20$ and other values of κ are estimated from (82). The pore size parameter at $\beta_0=0.20$ was chosen to be $a_0=0.02$ mm (compared to glass bead size of 0.21–29 mm diameter). The real speeds v_c and v_s are determined from the complex speeds $\overline{v}_c$ and $\overline{v}_s$ using

$$\frac{1}{\overline{v}} = \frac{1}{v}\left(1+\frac{i}{2Q}\right), \tag{84}$$

where Q^{-1} is the attenuation factor. Calculated values of Q^{-1} are always less than 10^{-6} for the porosities in Plona's experiments.

The results of these calculations are illustrated in Fig. 1. The vertical error bars are ±3% of the measured speeds. The error in the porosity measurement[25] was ±0.005 so the horizontal bars indicate the probable range of error in the porosity value. We observe that the theoretical values agree in all cases within the stated experimental error. These results improve upon the results of Ref. 12 since the slow wave speeds only agreed within 10% in that work. We conclude that the formulas (25)–(27) do improve on the standard formulas (33)–(35) when the porous frame is fully consolidated. Furthermore, the various assumptions made to arrive at a complete computational model for Biot's theory provide good approximations to the physical quantities involved at least for this one experiment.

To provide some insight into the sensitivity of our results to the choice of model parameters, we will briefly discuss this question before concluding the present section. Although the values of K_g, μ_g, K_f, and η are fixed by the experimental configuration, we have some freedom in choosing how to calculate K^* and μ^* from (46) and (47). For example, instead of using (49)–(51) we could use the formulas for prolate spheroids.[18,19] With aspect ratio equal to 0.1 (instead of zero) used in calculating the frame moduli K^* and μ^*, the theoretical curves in Fig. 1 change by less than 0.5% and still

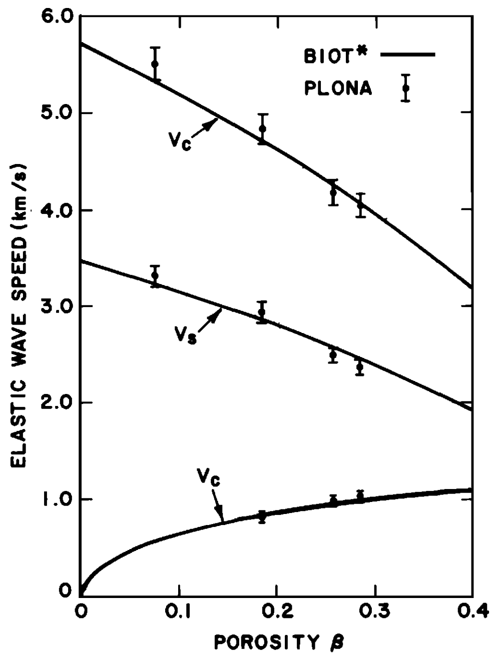

FIG. 1. Comparison of theoretical predictions and experimental results for speeds of the fast compressional wave, the shear wave, and the slow compressional wave. Input parameters to the model are $K_g=0.407$ Mb, $\mu_g=0.297$ Mb, $\rho_g=2.48$ g/cc, $K_f=0.022$ Mb, $\rho_f=1.00$ g/cc, $\eta=1.00$ cs, $f=500$ kHz, $\kappa_0=2\ \mu\text{m}^2$, $a_0=0.02$ mm, and $\beta_0=0.20$. Equations (25)–(27) were used to define the coefficients in Biot's theory.

provide good agreement with experiment. As discussed previously, the results are insensitive to the precise value of κ and a as long as these parameters have values of the same order of magnitude as those quoted. The final parameter over which we have any significant control is α; equivalently, it follows from (56) that r is a parameter which we have some freedom to specify. As explained in Sec. II, the physical interpretation of r indicates that its value is restricted to the range $0\le r\le 1$. It follows from (81) that, at high frequencies, the slow wave speed is strongly dependent on α. Therefore, the good agreement between theory and experiment in Fig. 1 is a direct check on the chosen value of $r=\frac{1}{2}$. A better theoretical estimate of r based on a microscopic model of the true frame geometry is desirable and will be attempted in future work.

In conclusion, we wish to emphasize that we have made no attempt to vary the parameters to obtain the best possible fit. Our goal has been to obtain a simple, yet complete, method of specifying the parameters in Biot's theory. For Plona's experiments, the only significant free parameter in our model is r (or α) but even for this parameter physical intuition suggests that its value should be $r=\frac{1}{2}$, to a good approximation. For sands and fine-grained sediments where the permeability and pore size parameter play a significant role, Eqs. (82) and (83) may or may not apply. In such circumstances, additional work may be required to specify the model parameters both fully and accurately.

V. CONCLUSION

We have shown that the coefficients H, C, and M in Biot's strain energy functional are related to the frame and fluid moduli precisely by Eqs. (25)–(27) for a fully consolidated ("clean") frame. For a partially consolidated ("dirty") frame or for a fluid suspension, Eqs. (25)–(27) are not valid and may be replaced by (33)–(35), giving the standard approximation for the coefficients. The predictions of our formulation of Biot's theory for a fully consolidated frame agree quantitatively with the wave speeds observed in Plona's experiments. This agreement is sufficiently good that we may attribute some validity to the various assumptions made in Secs. II and IV to prescribe the model parameters. However, a more accurate determination of the frame moduli and frame inertia is still desirable. Such a determination may be possible analytically in the context of a microscopic model.[21]

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