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The Effective Elastic Moduli of Porous Granular Rocks

A porous granular rock is modeled by an aggregate of identical, randomly stacked, spherical particles. Contacting particles are initially bonded together across small areas. A theory is developed for the deformation of two such spherical particles under equal and opposite forces acting through the line joining their centers. The theory is used to calculate the effective elastic moduli of the medium. The dependence of the derived elastic wave speeds on the confining pressure and adhesion radius of the contacting particles is then predicted.

1 Introduction

In the present paper, we consider the calculation of the effective elastic moduli of porous, granular rocks. The rock will always be elastically isotropic and homogeneous over lengths large compared with the radii of the individual particles of which the rock is composed. Many earlier papers treating this problem (for example, Deresiewicz [2] and Duffy and Mindlin [3]) concentrate on regular packings of identical solid spherical particles. The results obtained in such special cases are very useful. All predict effective elastic moduli whose values are directly proportional to the cube root of the confining pressure, no matter what kind of regular packing model is selected. One limitation of this model is, of course, that the material will not be elastically isotropic. Another is that values of the effective elastic moduli have so far only been predicted for a few descrete values of the porosity.

Brandt [1], in his elegant treatment of the porous, granular rock proposed an alternative model. A random packing of spherical particles of different radii was used. He was able to calculate explicitly the effective bulk modulus of an elastically isotropic, homogeneous porous rock as a function of the confining pressure, porosity, and liquid saturation. Values of the effective shear modulus, and the pressure and shear wave speeds were then derived, assuming that the effective Poisson's ratio for the rock was known.

In the present paper, we shall calculate explicitly *both* the effective bulk and the effective shear modulus of a porous granular rock modeled by a random packing of identical solid spherical particles. We shall also consider the effect of the initial bonding of the particles on the calculated values of the effective bulk and effective shear moduli. In all of the earlier papers just described, the particles (grains) composing the rock were not actually bonded together initially. Rather, the particles were forced to come into contact over small areas by subjecting a specimen of the modeled rock to large confining pressures.

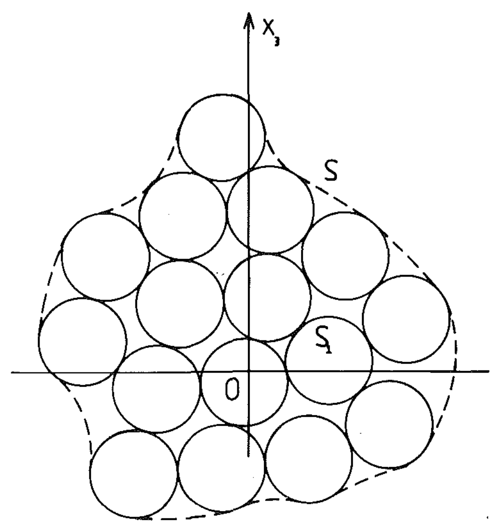

Fig. 1 Set of spheres in the medium intersected by an arbitrary plane S_1 normal to the Ox_3-axis; externally applied stress is transmitted to the particles across a surface S enclosing a large number of particles in the medium

We shall deal with the more general case in which the granular rock is modeled by a random packing of bonded spheres of different radii in a later paper. This exercise will enable the effects of porosity and liquid saturation to be included in the calculations.

2 Model of the Rock

We consider a random packing of identical solid spherical particles of radius R. Each particle (grain) is elastically isotropic and homogeneous with shear modulus, Poisson's ratio, and density, μ, ν, and ρ, respectively. A bar over any of these quantities denotes the corre-

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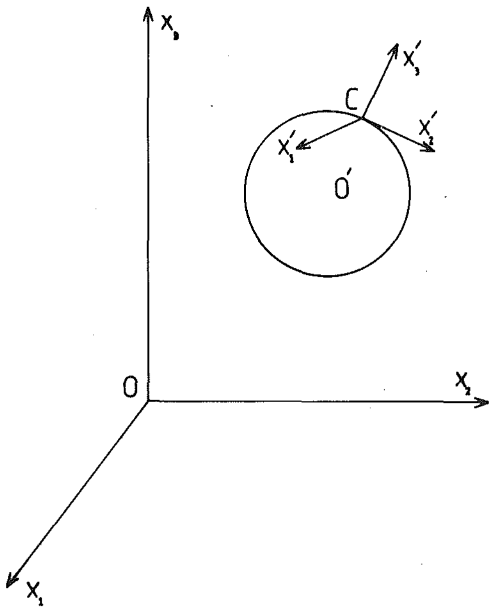

Fig. 2 $Ox_1x_2x_3$ **is a right-handed set of axes embedded in the rock;** C **is a contact point on the surface with center** $0'$**;** $Cx'_1x'_2x'_3$ **is a right-handed set of axes on the surface of the sphere with** Cx'_3 **normal to the surface**

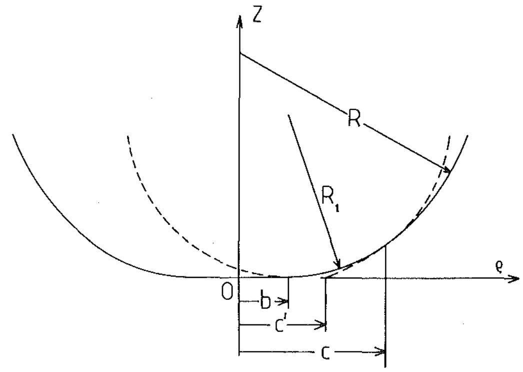

Fig. 3 Cross section of the surface profile of a spherical particle near an adhesion region before deformation

sponding effective elastic modulus or density of the medium we are considering. Neighboring particles are initially firmly bonded across small, flat, circular regions of the same average radius b. Outside the adhesion surfaces, the particles are assumed to be smooth. Consider an aggregate of these particles enclosed by the surface S shown in Fig. 1. The dimensions of S are large compared with the radius of any particle. The surface S transmits an external pressure P to the particles. Under such a purely hydrostatic stress condition, we suppose that the particles deform in such a way that the contact regions of all neighboring particles remain flat, circular, and have the same average radius $a(>b)$ small compared with R. Under purely hydrostatic externally applied stresses we assume that the contact forces are normal to the contact surfaces and have the same average value. Each sphere has the same average number of contacts K.

3 Some Preliminary Definitions

We define a right-handed set of orthogonal unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ along axes Ox_1, Ox_2, Ox_3, respectively, embedded in the rock we are considering, as shown in Fig. 2. At a given contact point C on the sphere with center O' we define an analogous set of unit vectors $\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3$ along the axes Cx'_1, Cx'_2, Cx'_3 with $\mathbf{i}'_3$ normal to the surface of the sphere. Referred to axes Ox_1, Ox_2, Ox_3, the unit vectors $\mathbf{i}'_r$ $(r = 1, 2, 3)$ have the components

$$\mathbf{i}'_r = (\alpha_{r1}, \alpha_{r2}, \alpha_{r3}) \tag{1}$$

The position and displacement vectors have components x_i and u_i, respectively, and the components of the applied stress and strain tensors are written p_{ij}, e_{ij} $(i, j = 1, 2, 3)$ referred to the same axes. Corresponding primed quantities denote components referred to the axes $Cx'_1x'_2x'_3$. $u_i{}^c$ and $u_i{}^0$ $(i = 1, 2, 3)$ denote the displacement components of the points C and O', respectively, for the sphere center O'. Contact forces at C have the components N'_{31}, N'_{32}, and N'_{33} along axes $\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3$, respectively.

4 Calculation Procedure

We suppose the medium is initially subjected to a uniform hydrostatic pressure $P = -p_{kk}/3$. e_{kk} is the corresponding uniform strain field. We now superimpose on this uniform field, small, arbitrary applied stresses δp_{ij} $(i, j = 1, 2, 3)$. Let δe_{ij} be the corresponding strain field. δp_{ij} and δe_{ij} are supposed uniform over lengths large compared with R. We suppose that under the given applied stresses, each grain undergoes zero rotation, that is, no couple stresses act in the medium. We then have

$$\delta p_{ij} = \bar{\kappa}\delta_{ij}\delta e_{kk} + 2\bar{\mu}(\delta e_{ij} - \tfrac{1}{3}\,\delta_{ij}\delta e_{kk}) \tag{2}$$

As usual, δ_{ij} is the Kronecker delta symbol and, unless stated otherwise, summation is performed over repeated subscripts.

Our problem is to calculate explicitly the effective bulk and the effective shear moduli $\bar{\kappa}$ and $\bar{\mu}$ in equation (2). A similar problem has been solved by Duffy and Mindlin [3] for a special type of regular packing. However, rigorous application of their methods, for rocks modeled by random packings of spheres, would be a hopelessly complex task. We must therefore construct an approximate method.

We suppose first that the increment of displacement of a contact point on the surface of any sphere in the medium, relative to its center, $\delta(u_j{}^c - u_j{}^o)$, $(j = 1, 2, 3)$ can be uniquely determined from the uniform applied strain field δe_{ij}, that is,

$$\delta(u_j{}^c - u_j{}^o) = R\alpha_{3k}\delta e_{jk} \tag{3}$$

In Section 6, we consider the number of spheres in the medium intersected by an arbitrary plane. Variations in contact forces, $\delta N'_{3i}$ $(i = 1, 2, 3)$ are related to the arbitrary variations in the applied stresses δp_{ij} by deriving a force equilibrium equation for all spheres cut by the plane. This problem is considered in Section 7. It will be seen that the values of the effective elastic moduli (given in equations (33) and (34) later) can be explicitly calculated, if the relative displacements of the centers of neighboring, bonded spheres can be related to the contact forces. This problem is considered in the following section.

5 Contact Problem With Adhesion

(a) Boundary Conditions. Fig. 3 shows a plane section normal to the contact surface and through the center of a typical spherical particle (grain) in the medium before deformation has occurred. The surface profile in the adhesion region consists of two circular arcs of radii R and $R_1(R_1 < R)$ joining smoothly at a point distant c from the axis of symmetry. The circular arc of radius R intersects the adhesion region (of radius b) produced at a point distant c' from the axis of symmetry. If c is sufficiently small, the equation of the surface profile in Fig. 3 can be written locally as

$$\begin{aligned} z &= 0 && \text{for} \quad \rho \leqslant b \\ &= (\rho - b)^2/2R_1 && \text{for} \quad c \geqslant \rho \geqslant b \\ &= (\rho^2 - c'^2)/2R && \text{for} \quad \rho \geqslant c \end{aligned} \tag{4}$$

Each surface profile is smooth in the neighborhood of the bonding surface. We therefore have the condition

$$R_1 = R(c - b)/c \tag{5}$$

It can then be assumed that when purely normal forces acting through the centers of the spheres are applied, there are no singularities in the normal stress $p_{zz}(\rho)$ at the boundaries of the adhesion and contact regions. Furthermore, the normal stress vanishes at the boundary of the contact region. We must calculate the distance 2δ by which the centers of two adhering spheres approach one another when a purely normal force Y acting through the center of each sphere is applied. We approximate this by a "half plane" problem, as in the classical Hertz theory. The normal displacements in the contact region are prescribed. From equations (4) one obtains the following

$$\begin{aligned} u &= \delta - \frac{(\rho^2 - c'^2)}{2R} \quad \text{for } a \geqslant \rho \geqslant c \geqslant b \\ &= \delta - \frac{(\rho - b)^2}{2R_1} \quad \text{for } a \geqslant c \geqslant \rho \geqslant b \\ &= \delta \quad \text{for } a \geqslant c \geqslant b \geqslant \rho \end{aligned} \tag{6}$$

Outside the contact region ($\rho \geqslant a$) all stresses $p_{zz}(\rho)$, $p_{\rho z}(\rho)$ vanish. Inside the contact region ($\rho \leqslant a$) the shear stress $p_{\rho z}(\rho)$ again vanishes since we have two identical spheres pushed together by purely normal forces acting along the line joining their centers. The normal stress $p_{zz}(\rho)$ has a resultant equal to Y.

(*b*) Solution of the Boundary-Value Problem. The adhesion and contact surfaces of the adhering spheres in Fig. 3 join smoothly for all $Y \geqslant 0$. We can therefore assume that the normal stress $p_{zz}(\rho)$ is continuous for all values of ρ, $0 \leqslant \rho < \infty$. We can therefore follow the procedure given in Sneddon's book [5]. One obtains the following expression for the normal displacement of the contact surface $u_z(\rho, 0)$, $(0 \leqslant \rho \leqslant a)$

$$\pi \mu u_z(\rho, 0) = 2(1-\nu) \int_0^\rho \frac{dt}{(\rho^2 - t^2)^{1/2}} \int_t^a \frac{\rho' p_{zz}(\rho') d\rho'}{(\rho'^2 - t^2)^{1/2}} \tag{7}$$

From equations (6) and (7) we then have

$$\begin{aligned} 2(1-\nu) \int_0^\rho \frac{dt}{(\rho^2 - t^2)^{1/2}} \int_t^a \frac{\rho' p_{zz}(\rho') d\rho'}{(\rho'^2 - t^2)^{1/2}} &= \pi\mu \left\{\delta - \frac{(\rho^2 - c'^2)}{2R}\right\} \quad \text{if } \ c \leqslant \rho \leqslant a \\ &= \pi\mu \left\{\delta - \frac{(\rho - b)^2}{2R_1}\right\} \quad \text{if } \ b \leqslant \rho \leqslant c \leqslant a \\ &= \pi\mu\delta \quad \text{if } \ 0 \leqslant \rho \leqslant b \end{aligned} \tag{8}$$

We then find, for $0 \leqslant \rho \leqslant a$,

$$p_{zz}(\rho) = -\frac{2a^2}{\pi\rho} \frac{d}{d\rho} \int_{\rho/a}^1 \frac{tF(t)dt}{\left[t^2 - \frac{\rho^2}{a^2}\right]^{1/2}} \tag{9}$$

where

$$\begin{aligned} F(t) &= \frac{\mu\delta}{a(1-\nu)} + \frac{\mu b t}{R_1(1-\nu)} \left\{\sin^{-1}\left(\frac{c}{at}\right) - \sin^{-1}\left(\frac{b}{at}\right)\right\} \\ &\quad - \frac{\mu a t}{R_1(1-\nu)} \left(t^2 - \frac{b^2}{a^2}\right)^{1/2} - \frac{\mu a t}{(1-\nu)} \left(\frac{1}{R} - \frac{1}{R_1}\right) \left(t^2 - \frac{c^2}{a^2}\right)^{1/2} \\ &\quad \text{if } \ 1 \geqslant t \geqslant c/a > b/a > 0 \\ &= \frac{\mu\delta}{a(1-\nu)} + \frac{\mu b t}{R_1(1-\nu)} \left\{\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{at}\right)\right\} \\ &\quad - \frac{\mu a t}{R_1(1-\nu)} \left(t^2 - \frac{b^2}{a^2}\right)^{1/2} \quad \text{if } \ c/a \geqslant t \geqslant b/a > 0 \\ &= \frac{\mu\delta}{a(1-\nu)} \quad \text{if } \ b/a \geqslant t > 0 \end{aligned} \tag{10}$$

We can show from equations (5)and (9) that the condition that the normal stress $p_{zz}(\rho)$ has no singularity as $\rho \to a - 0$ (and in fact vanishes at $\rho = a$) is that $F(1)$ vanishes. We then obtain,

$$\begin{aligned} \frac{\delta}{a} + \frac{b}{R_1} \left\{\sin^{-1}\left(\frac{c}{a}\right) - \sin^{-1}\left(\frac{b}{a}\right)\right\} \\ + \frac{a}{cR_1} \left\{b\left(1 - \frac{c^2}{a^2}\right)^{1/2} - c\left(1 - \frac{b^2}{a^2}\right)^{1/2}\right\} = 0 \quad \text{if } \ a \geqslant c > b > 0 \end{aligned} \tag{11}$$

(*Cont.*)

An explicit expression for the resultant normal force, Y, can also be derived. We find

$$\begin{aligned} Y &= \frac{4\mu a\delta}{(1-\nu)} + \frac{2\mu b a^3}{3R_1 c(1-\nu)} \left(2 + \frac{c^2}{a^2}\right) \left(1 - \frac{c^2}{a^2}\right)^{1/2} \\ &\quad - \frac{2\mu a^3}{3R_1(1-\nu)} \left(2 + \frac{b^2}{a^2}\right) \left(1 - \frac{b^2}{a^2}\right)^{1/2} \\ &\quad + \frac{2\mu a^2 b}{R_1(1-\nu)} \left\{\sin^{-1}\left(\frac{c}{a}\right) - \sin^{-1}\left(\frac{b}{a}\right)\right\} \quad \text{if } \ a \geqslant c > b > 0 \end{aligned} \tag{12}$$

For the case $c \geqslant a > b > 0$ one obtains the following corresponding results:

$$\frac{\delta}{a} + \frac{b}{R_1} \left\{\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{a}\right)\right\} - \frac{a}{R_1} \left(1 - \frac{b^2}{a^2}\right)^{1/2} = 0 \tag{13}$$

and

$$\begin{aligned} Y &= \frac{4\mu a\delta}{(1-\nu)} - \frac{2\mu a^3}{3R_1(1-\nu)} \left(2 + \frac{b^2}{a^2}\right) \left(1 - \frac{b^2}{a^2}\right)^{1/2} \\ &\quad + \frac{2\mu a^2 b}{R_1(1-\nu)} \left\{\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{a}\right)\right\} \end{aligned} \tag{14}$$

From equations (11) and (12) (or (13) and (14)) together, we find by differentiation that

$$\frac{dY}{d\delta} = \frac{4\mu a}{(1-\nu)} \tag{15}$$

for either $a \geqslant c > b > 0$ or $c \geqslant a > b > 0$.

Suppose that a small tangential force dT is now superimposed on the normal force Y. Suppose the centers of the spheres undergo a relative tangential displacement $2\ ds$. Mindlin's results [4], show that

$$\frac{dT}{ds} = \frac{8\mu b}{(2-\nu)} \tag{16}$$

We now present the limiting forms of the results given in equations (11) and (12) (valid for $a \geqslant c > b > 0$). We let $c \to b + 0$. We find that

$$\delta = a(a^2 - b^2)^{1/2}/R \tag{17}$$

and

$$Y = \frac{4\mu}{(1-\nu)} \left\{a\delta - \frac{(a^2 - b^2)^{3/2}}{3R}\right\} \tag{18}$$

The use of the limiting forms for Y and δ (equations (17) and (18)) in the following sections of this paper will simplify the subsequent analysis considerably. Equations (17) and (18) reduce to those of the classical Hertz theory when we put b equal to zero. Equations (15)–(18) derived in the foregoing now enable us to relate relative displacements of the centers of neighboring bonded spheres, to small, arbitrary variations in the contact forces.

6 Some Statistical Modeling

We now consider in Fig. 1, the set of spheres in the medium intersected by an arbitrary plane S_1 normal to the Ox_3-axis. We consider the portions of these intersected spheres lying on the side of S_1 whose unit normal points along the positive Ox_3-axis, as shown in Fig. 4. We suppose that the surface area of S_1, denoted by A is large compared with the surface area of any particle. Suppose that the circle of intersection of S_1 with a given sphere subtends an angle 2θ at the center of this sphere (Fig. 4). Suppose there are $n(\theta)$ such spheres. $A(1-\alpha)$ is the total area bounded by all the circles of intersection of S_1 with the spheres, where α is the porosity of the aggregate. The number of spheres per unit bulk volume of the medium is $3(1-\alpha)/4\pi R^3$. The number of spheres cut by S_1 at an angle greater than or equal to θ is therefore $3A(1-\alpha)(1+\cos\theta)/4\pi R^2$. The total number of spheres $n(\theta)$

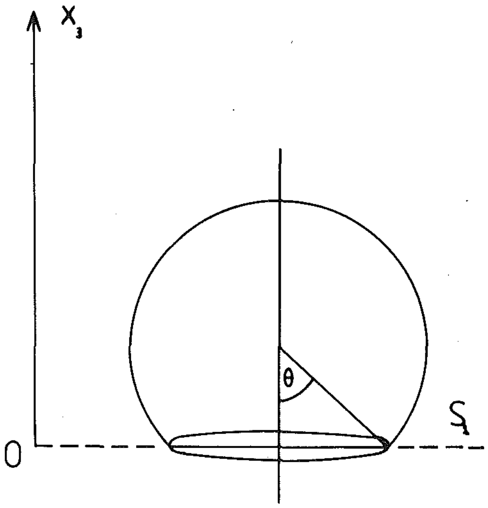

Fig. 4 Portion of intersected spherical particle lying on side of S_1 whose unit normal points along the positive Ox_3-axis; circle of intersection of S_1 with the sphere subtends an angle 2θ at the center of the sphere

cut at between θ and $\theta + \delta\theta$ is therefore equal to $3A(1 - \alpha) \sin \theta\delta\theta/4\pi R^2$. We then have

$$\sum_\theta n(\theta) \sin^4 \theta = \sum_\theta 3A(1 - \alpha) \sin^5 \theta\delta\theta/4\pi R^2 = \frac{3A(1 - \alpha)}{4\pi R^2} \int_0^\pi \sin^5 \theta d\theta = \frac{4A(1 - \alpha)}{5\pi R^2} \tag{19}$$

Similarly, we have

$$\sum_\theta n(\theta) \sin^2 \theta = A(1 - \alpha)/\pi R^2 \tag{20}$$

Equation (20) can in any case be derived independently of the arguments just given. In later calculations we shall suppose that the K contact points on the surface of a given sphere are distributed over the surface with uniform probability, or at any rate approximately. There are then $K/4\pi$ contacts per unit solid angle distributed over the surface of the sphere. Considering now the portion of the given sphere cut by S_1 (as previously defined) we then easily derive the following results (for $r = 1, 2, 3$) which are also needed in later sections of this paper,

$$\sum_{\text{contacts}} \alpha_{3r} = \tfrac{1}{4}\, \delta_{3r} K \sin^2 \theta \tag{21}$$

$$\sum_{\text{contacts}} \alpha_{33}{}^2 \alpha_{3r} = \frac{\delta_{3r} K \sin^2 \theta}{8} (2 - \sin^2 \theta) \tag{22}$$

$$\sum_{\text{contacts}} \alpha_{3p}{}^2 \alpha_{3r} = \frac{\delta_{3r} K \sin^4 \theta}{16} \quad \text{for} \quad p = 1, 2 \tag{23}$$

$$\sum_{\text{contacts}} \alpha_{31} \alpha_{32} \alpha_{33} = 0 \tag{24}$$

7 Force Equilibrium Equations

We now derive force equilibrium equations for the set of spheres intersected by the plane S_1. The force increments acting on the portions of the spheres (defined in Section 6) are first due to the force increments acting on the plane S_1 of area A. We suppose that the force increments are the resultant of the uniform applied stress increments δp_{ij} distributed over the plane S_1. Thus the resultant force acting on S_1 has the components $A\delta p_{3r}$, ($r = 1, 2, 3$). Equilibrium of this set of spheres will be maintained if each of these components is equal to the corresponding component of resultant contact forces acting on the same set of spheres, that is,

$$A\delta p_{3r} = \sum_{\substack{\text{all}\\ \text{spheres}}} \sum_{\text{contacts}} \alpha_{1r} \delta N'_{31} + \alpha_{2r} \delta N'_{32} - \alpha_{3r} \delta N'_{33} \tag{25}$$

for $r = 1, 2, 3$. We can also write (from equations (1) and (3)), for $r = 1, 2, 3$:

$$\delta(u_r{}^{c'} - u_r{}^{0'}) = R\alpha_{rp}\alpha_{3k}\delta e_{pk} \tag{26}$$

After some algebra, it can then be shown, from the results of Section 5 (equations (15) and (16)) and equations (25) and (26), that, for $r = 1, 2, 3$,

$$A\delta p_{3r} = \sum_\theta \sum_{\text{contacts}} \frac{8\mu b n(\theta)}{(2 - \nu)} \alpha_{3p}\delta e_{rp} - \alpha_{3r}(\alpha_{3p}\alpha_{3q})\delta e_{pq} + \sum_\theta \sum_{\text{contacts}} \frac{4\mu a n(\theta)}{(1 - \nu)} \alpha_{3r}(\alpha_{3p}\alpha_{3q})\delta e_{pq} \tag{27}$$

where θ and $n(\theta)$ have been defined in Section 6. We note that only the components of the unit normals to the surfaces of the spheres intersected by the plane S_1 arise in the equilibrium equation (27). It should also be noted that the coefficients of the terms α_{3p} and $\alpha_{3r}(\alpha_{3p}\alpha_{3q})$ for $p, q, r = 1, 2, 3$ are constant over the surface of any sphere. Using equations (21)–(24) derived in Section 5, one obtains from equation (27)

$$A\delta p_{3r} = \sum_\theta \frac{K\mu b R n(\theta) \sin^2 \theta}{(2 - \nu)} \delta e_{3r}(2 - \sin^2 \theta) + \sum_\theta \frac{K\mu a R n(\theta) \sin^4 \theta}{2(1 - \nu)} \delta e_{3r} \quad \text{for} \quad r = 1, 2 \tag{28}$$

and

$$A\delta p_{33} = \sum_\theta \frac{K\mu b R n(\theta) \sin^4 \theta}{2(2 - \nu)} (3\delta e_{33} - \delta e_{kk}) + \sum_\theta \frac{K\mu a R n(\theta) \sin^2 \theta}{4(1 - \nu)} \{\delta e_{33}(4 - 3 \sin^2 \theta) + \delta e_{kk} \sin^2 \theta\} \tag{29}$$

From equation (29) we then derive the following results:

$$\bar{\lambda} = \sum_\theta \frac{K\mu R n(\theta) \sin^4 \theta}{2A} \left\{\frac{a}{2(1 - \nu)} - \frac{b}{(2 - \nu)}\right\} \tag{30}$$

and

$$2\bar{\mu} = \sum_\theta \frac{K\mu R n(\theta) \sin^2 \theta}{2A} \left\{\frac{a(4 - 3 \sin^2 \theta)}{2(1 - \nu)} + \frac{3b \sin^2 \theta}{(2 - \nu)}\right\} \tag{31}$$

From equation (28) one also obtains

$$2\bar{\mu} = \sum_\theta \frac{K\mu R n(\theta) \sin^2 \theta}{A} \left\{\frac{a \sin^2 \theta}{2(1 - \nu)} + \frac{b(2 - \sin^2 \theta)}{(2 - \nu)}\right\} \tag{32}$$

Equations (31) and (32) appear to be inconsistent. However, we assume that the material we are considering is isotropic. The coefficients of the terms $n(\theta) \sin^2 \theta$ and $n(\theta) \sin^4 \theta$ in equations (30)–(32) are then independent of θ. We can then show, with the help of equations (19) and (20), that equations (31) and (32) are identical. In fact, one finally obtains

$$\bar{\lambda} = \frac{\mu K(1 - \alpha)}{5\pi R} \left\{\frac{a}{(1 - \nu)} - \frac{2b}{(2 - \nu)}\right\} \tag{33}$$

and

$$2\bar{\mu} = \frac{\mu K(1 - \alpha)}{5\pi R} \left\{\frac{2a}{(1 - \nu)} + \frac{6b}{(2 - \nu)}\right\} \tag{34}$$

We can derive a further result from equation (25) with the help of our results given in Section 5. Under a purely hydrostatic increment of stress, δP, we have,

$$A\delta P = \sum_{\substack{\text{all}\\ \text{spheres}}} \sum_{\text{contacts}} \alpha_{33} \delta N'_{33} \tag{35}$$

Under the assumptions outlined in Section 2, and using equations (20)–(24) one obtains

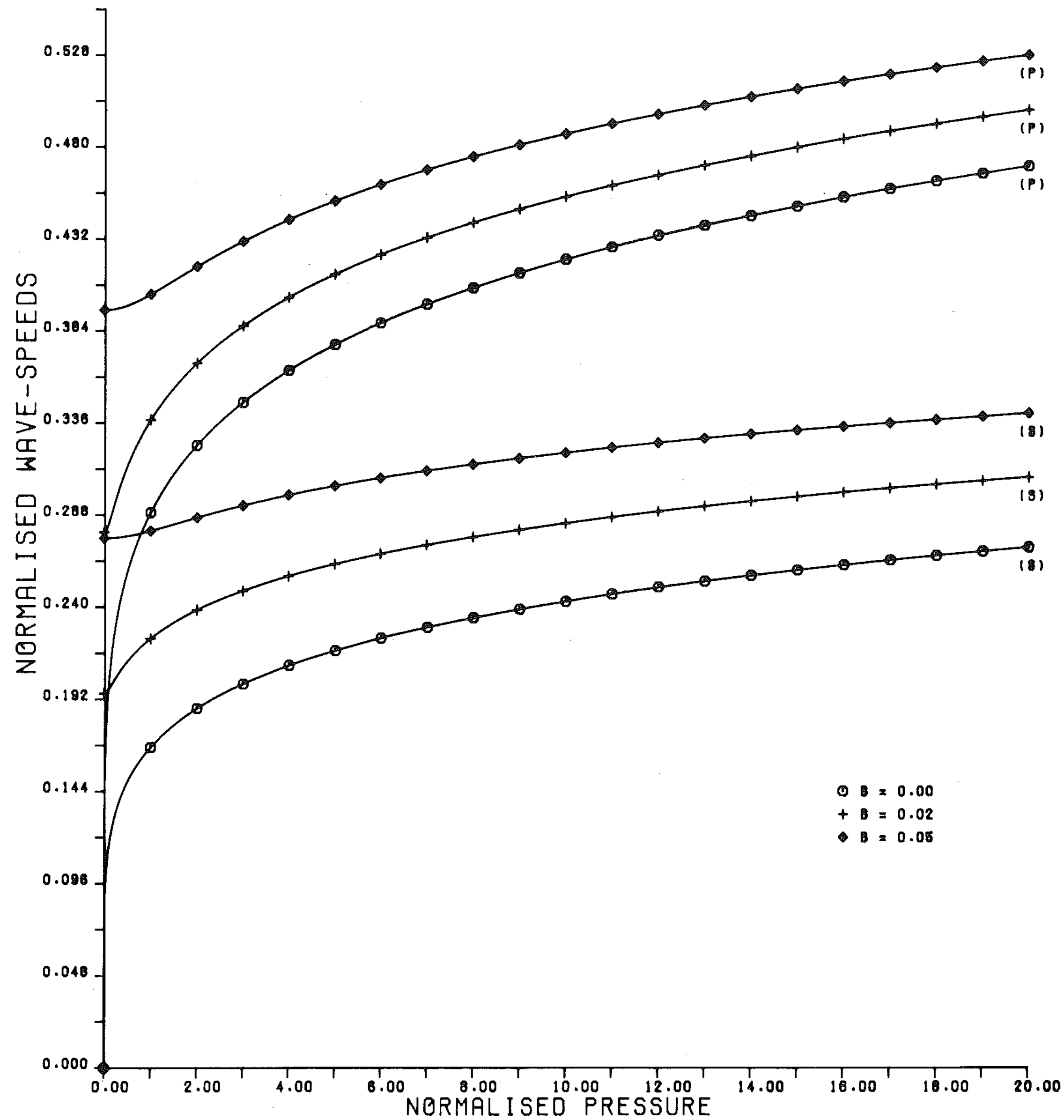

Fig. 5 Normalized wave speeds plotted as functions of the normalized pressure for some fixed values of the normalized adhesion radius $B = b/R$

$$\frac{4\pi R^2 \delta P}{(1-\alpha)} = K\delta N'_{33} = K\delta Y \tag{36}$$

where Y is defined in Section 4. We therefore have, if up to the pressure P, the loading was purely hydrostatic,

$$\frac{4\pi R^2 P}{(1-\alpha)} = KY \tag{37}$$

This result agrees exactly with equation (16) in Brandt's paper [1], for the special case of a "dry" packing of spheres, when, in Brandt's paper we put $C_{\phi\epsilon} = 1$ and $K = 8.84$. It can also be shown, in this case, that the effective density of the medium is given by

$$\bar{\rho} \simeq \rho(1-\alpha)\left(1 + 3\left(\frac{\delta}{R}\right)\right) \tag{38}$$

to order δ/R, where δ has been defined in Section 4 and ρ is the grain density.

From equations (17), (18), and (37) one obtains the following equation for the normalized contact radius, a/R,

$$a/R = \{(x/R)^2 + (b/R)^2\}^{1/2} \tag{39}$$

where x/R satisfies the cubic equation

$$\left(\frac{x}{R}\right)^3 + \frac{3}{2}\left(\frac{b}{R}\right)^2\left(\frac{x}{R}\right) - \frac{3\pi}{2K}\left(\frac{1-\nu}{1-\alpha}\right)\left(\frac{P}{\mu}\right) = 0 \tag{40}$$

Equations (39) and (40), together with equations (33) and (34), enable us to determine explicitly the effective bulk and effective shear modulus of the granular rock we are considering as a function of the confining pressure P and the adhesion radius b of the adhering particles. One can also derive the corresponding numerical values of the elastic wave speeds (using also equation (38))—as is done in the next section. Walton [6], however, has shown that this procedure is only strictly valid for seismic waves of sufficiently low frequency ω, namely, those for which $\rho\omega^2 R^2/(\lambda + 2\mu) < \rho\omega^2 R^2/\mu \ll 1$.

8 Results

Fig. 5 shows values of the normalized wave speeds $((\bar{\lambda} + 2\bar{\mu})\rho/\mu\bar{\rho})^{1/2}$ and $(\bar{\mu}\rho/\mu\bar{\rho})^{1/2}$ plotted as functions of the normalized confining pressure P/μ, for some fixed values of the normalized adhesion radius $B = b/R$. The values selected for the grain Poisson's ratio ν, and the grain shear modulus μ were 0.2 and 38 GPa, respectively. Values of P/μ range from 0 to 18.10×10^{-4}. The values of α and K are the same as those used in Brandt's paper [1]; namely, 0.392 and 8.84, respectively.

References

1 Brandt, H., "A Study of the Speed of Sound in Porous Granular Media," ASME JOURNAL OF APPLIED MECHANICS, Vol 22, 1955, pp. 479–486.

2 Deresiewicz, H., "Stress-Strain Relations for a Simple Model of a Granular Medium," ASME JOURNAL OF APPLIED MECHANICS, Vol 25, 1958, pp. 402–406.

3 Duffy, J., and Mindlin, R. D., "Stress-Strain Relations and Vibrations of a Granular Medium," ASME JOURNAL OF APPLIED MECHANICS, Vol. 24, 1957, pp. 585–593.

4 Mindlin, R. D., "Compliance of Elastic Bodies in Contact," ASME JOURNAL OF APPLIED MECHANICS, Vol. 16, 1949, pp. 259–268.

5 Sneddon, I. N., *Mixed Boundary-Value Problems in Potential Theory*, North Holland Publishing Company, Amsterdam, 1966.

6 Walton, W., "Elastic Wave Propagation in Model Sediments—I," *Geophysical Journal of the Royal Astronomical Society*, Vol. 48, pp. 461–478.