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Overall elastic properties of isotropic materials with arbitrary distribution of circular cracks

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SUMMARY

The transmission properties of the mean field in elastic material with a random distribution of circular cracks of small aspect ratio are presented here for the general case where the crack normals are distributed in any pre-determined way in space. Special distributions, where the crack normals lie in one direction only, or lie at a fixed angle to a fixed direction, reproduce established results. Expressions are given for the case of crack normals lying close to a given direction in a Gaussian distribution. All results are valid to second order in the crack number density.

Key words: cracks, elastic properties.

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1 INTRODUCTION

The method of smoothing (Keller 1964) has been developed extensively for the calculation of overall elastic properties of materials with an internal distribution of cracks (see Hudson & Knopoff 1989). Formulae exist for a variety of conditions within the crack—dry, partially or completely fluid-filled or filled with weak material (Hudson 1981, 1988)—while the cracks are assumed to be of small aspect ratio. In addition, the number density v of cracks must be small, although the formulae are accurate to second order in (va^3), where a^3 is the mean cubed radius of the cracks. The application of these results to seismological observations has also developed rapidly and there has been considerable success in the estimation of crack parameters and conditions from the apparent wavespeeds and attenuation (see, for instance, Crampin 1985; Booth, Crampin & Chesnokov 1987; Sayers 1988b).

In the simplest form (Hudson 1981), the theory was applied to aligned crack distributions; that is, all crack normals were assumed to lie in a fixed direction. However, the extension to distributions with normals in more than one direction is straightforward (Hudson 1986). The elastic parameters are given as an expansion in ascending powers of va^3. The zeroth-order term is, of course, the elastic tensor for the uncracked matrix material. The first-order term ignores all crack–crack interactions and so the effect on this term of a superposition of crack distributions is simply the sum of the effects of each distribution existing on its own. Expressions for general distribution of crack orientations are given by Sayers (1988a). The second-order term in $(va^3)^2$ can be found in a straightforward way from the first-order term (Hudson 1986).

In this paper we give formulae for any given distribution of orientations accurate to second order and in a form which is in line with earlier papers in this series. In addition, we correct certain expressions given earlier (Hudson 1986) for systems of cracks with normals at a fixed angle to a given direction.

2 OVERALL ELASTIC PARAMETERS FOR MATERIALS WITH CRACKS OF ARBITRARY ORIENTATION

The overall or effective elastic tensor for an isotropic material with cracks aligned in more than one direction is (Hudson 1986)

$$\boldsymbol{c} = \boldsymbol{c}^0 + \boldsymbol{c}^1 + \boldsymbol{c}^2, \tag{1}$$

where $\boldsymbol{c}^0$ is the elasticity tensor for the isotropic matrix material:

$$c^0_{ijpq} = \lambda_0 \delta_{ij} \delta_{pq} + \mu_0(\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}), \tag{2}$$

$\boldsymbol{c}^1$ is the first-order perturbation for small crack densities, and $\boldsymbol{c}^2$ is the second-order term; $\boldsymbol{c}^2$ is given in terms of $\boldsymbol{c}^1$ by

$$c^2_{ijpq} = \frac{1}{\mu_0} c^1_{ijrs} \chi_{rskl} c^1_{klpq}, \tag{3}$$

where

$$\chi_{ijpq} = [\delta_{ip}\,\delta_{jq}(4+\beta^2/\alpha^2) - (\delta_{iq}\,\delta_{jp} + \delta_{ij}\,\delta_{pq})(1-\beta^2/\alpha^2)]/15.$$

The first-order term is given by

$$c^1_{ijpq} = -\sum_I \frac{\nu^I(a^I)^3}{\mu_0}[\lambda_0^2 \bar{U}^I_{33}\,\delta_{ij}\,\delta_{pq} + 2\lambda_0\mu_0\bar{U}^I_{33}(\delta_{ij}n^I_p n^I_q + \delta_{pq}n^I_i n^I_j) + 4\mu_0^2\bar{U}^I_{33}n^I_i n^I_j n^I_p n^I_q$$
$$+ \mu_0^2 \bar{U}^I_{11}(n^I_i n^I_p\,\delta_{jq} + n^I_i n^I_q\,\delta_{jp} + n^I_j n^I_p\,\delta_{iq} + n^I_j n^I_q\,\delta_{ip} - 4n^I_i n^I_j n^I_p n^I_q)], \tag{4}$$

where ν^I, a^I, $\mathbf{n}^I$ are the number density, radius and orientation of normals for the Ith set of cracks, and $\bar{U}^I_{33}$ and $\bar{U}^I_{11}$ are expressions governed by the crack conditions for the Ith set. If the cracks of the Ith set are not all of the same radius, then a^I should be taken to be the cube root of the mean cubed radius. If the cracks are not circular but rather elliptic in shape, then a^3 should be replaced by $2A^2/\pi P$ (O'Connell & Budiansky 1974), where A is the area and P the perimeter of the cracks—averaged if necessary over the crack population.

If there is a continuous distribution of orientations $\mathbf{n}$, then the sum in equation (4) is replaced by an integral over the polar angles ϑ, ϕ:

$$c^1_{ijpq} = -\frac{1}{\mu_0}\int_0^{2\pi}\int_0^{\pi/2} \nu(\vartheta,\phi)a^3(\vartheta,\phi)[\lambda_0^2\bar{U}_{33}(\vartheta,\phi)\,\delta_{ij}\,\delta_{pq}$$
$$+ 2\lambda_0\mu_0\bar{U}_{33}(\vartheta,\phi)(\delta_{ij}n_p n_q + \delta_{pq}n_i n_j) + 4\mu_0^2\bar{U}_{33}(\vartheta,\phi)n_i n_j n_p n_q$$
$$+ \mu_0^2\bar{U}_{11}(\vartheta,\phi)(n_i n_p\,\delta_{jq} + n_i n_q\,\delta_{jp} + n_j n_p\,\delta_{iq} + n_j n_q\,\delta_{ip} - 4n_i n_j n_p n_q)]\sin\vartheta\,d\vartheta\,d\phi, \tag{5}$$

where

$$\mathbf{n} \equiv (\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$$

and the sense of the normal $\mathbf{n}$ to the cracks is taken to lie in $0 \le \vartheta \le \pi/2$ to eliminate ambiguity; $\nu(\vartheta,\phi)\sin\vartheta\,d\vartheta\,d\phi$ is the number density of cracks lying in the solid angle given by the ranges $[\vartheta, \vartheta + d\theta]$, $[\phi, \phi + d\phi]$ and $a(\vartheta,\phi)$ is the distribution of crack radii.

If we take the conditions at the cracks to be uniform, so that $\bar{U}_{11}$ and $\bar{U}_{33}$ are no longer functions of ϑ and ϕ, and write

$$\epsilon(\vartheta,\phi) = \nu(\vartheta,\phi)a^3(\vartheta,\phi),$$

we get

$$c^1_{ijpq} = -\frac{A}{\mu_0}\bar{U}_{33}[\lambda_0^2\,\delta_{ij}\,\delta_{pq} + 2\lambda_0\mu_0(\delta_{ij}\bar{\epsilon}_{pq} + \delta_{pq}\bar{\epsilon}_{ij}) + 4\mu_0^2\bar{\epsilon}_{ijpq}] - A\mu_0\bar{U}_{11}(\delta_{jq}\bar{\epsilon}_{ip} + \delta_{jp}\bar{\epsilon}_{iq} + \delta_{iq}\bar{\epsilon}_{jp} + \delta_{ip}\bar{\epsilon}_{jq} - 4\bar{\epsilon}_{ijpq}), \tag{6}$$

where

$$A = \int_0^{2\pi}\int_0^{\pi/2}\epsilon(\vartheta,\phi)\sin\vartheta\,d\vartheta\,d\phi, \quad \bar{\epsilon}_{ij} = \frac{1}{A}\int_0^{2\pi}\int_0^{\pi/2}\epsilon(\vartheta,\phi)n_i n_j\sin\vartheta\,d\vartheta\,d\phi, \quad \bar{\epsilon}_{ijpq} = \frac{1}{A}\int_0^{2\pi}\int_0^{\pi/2}\epsilon(\vartheta,\phi)n_i n_j n_p n_q\sin\vartheta\,d\vartheta\,d\phi. \tag{7}$$

3 SPECIAL CASES

(a) Suppose a^3 is fixed and the normals are aligned in the direction $\vartheta = \vartheta_0$, $\phi = \phi_0$:

$$\nu(\vartheta,\phi) = \nu_0\frac{\delta(\vartheta - \vartheta_0)}{\sin\vartheta}\delta(\phi - \phi_0), \tag{8}$$

then

$$A = \nu_0 a^3,$$
$$\bar{\epsilon}_{ij} = n^0_i n^0_j, \tag{9}$$
$$\bar{\epsilon}_{ijpq} = n^0_i n^0_j n^0_p n^0_q,$$

where $n^0_1 = \sin\vartheta_0\cos\phi_0$, $n^0_2 = \sin\vartheta_0\sin\phi_0$, $n^0_3 = \cos\vartheta_0$. Substituting back into equation (6) we get equation (4) once more with a single term only in the sum.

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If, further,

$$\mathbf{n}^0 = (0, 0, 1),$$

then

$$\begin{aligned} &A = v_0 a^3, \\ &\bar{\epsilon}_{33} = 1, \qquad \bar{\epsilon}_{ij} = 0 \qquad \text{otherwise}, \\ &\bar{\epsilon}_{3333} = 1, \qquad \bar{\epsilon}_{ijpq} = 0 \qquad \text{otherwise}, \end{aligned} \tag{10}$$

and equation (6) gives

$$\begin{aligned} c^1_{1111} &= -\frac{v_0 a^3}{\mu_0} \bar{U}_{33}(\lambda_0^2) = c^1_{1122} = c^1_{2222}, \\ c^1_{3333} &= -\frac{v_0 a^3}{\mu_0} \bar{U}_{33}(\lambda_0 + 2\mu_0)^2, \\ c^1_{1133} &= -\frac{v_0 a^3}{\mu_0} \bar{U}_{33}[\lambda_0(\lambda_0 + 2\mu_0)] = c^1_{2233}, \\ c^1_{2323} &= -v_0 a^3 \bar{U}_{11}\mu_0 = c^1_{3131}, \end{aligned} \tag{11}$$

and, apart from the elements related to these by the symmetry relations, all others are zero, restoring the result originally given by Hudson (1981).

(b) If, on the other hand, v is a function of ϑ only; that is, normals are symmetrically distributed about the direction $\vartheta = 0$,

$$\begin{aligned} &A = 2\pi \int_0^{\pi/2} \epsilon(\vartheta) \sin \vartheta \, d\vartheta, \\ &\bar{\epsilon}_{12} = \bar{\epsilon}_{23} = \bar{\epsilon}_{31} = 0, \\ &\bar{\epsilon}_{11} = \bar{\epsilon}_{22} = \frac{\pi}{A} \int_0^{\pi/2} \epsilon \sin^3 \vartheta \, d\vartheta = \tfrac{1}{2}(1 - \bar{\epsilon}_{33}), \\ &\bar{\epsilon}_{1111} = \bar{\epsilon}_{2222} = \frac{3\pi}{4A} \int_0^{\pi/2} \epsilon \sin^5 \vartheta \, d\vartheta = 3\bar{\epsilon}_{1122} = 3\bar{\epsilon}_{1212}, \qquad \text{etc.}, \\ &\bar{\epsilon}_{3333} = \tfrac{8}{3}\bar{\epsilon}_{1111} - 4\bar{\epsilon}_{11} + 1, \\ &\bar{\epsilon}_{1133} = \bar{\epsilon}_{11} - \tfrac{4}{3}\bar{\epsilon}_{1111} = \bar{\epsilon}_{2233} = \bar{\epsilon}_{1313} = \bar{\epsilon}_{2323}, \qquad \text{etc.} \end{aligned} \tag{12}$$

Apart from the elements related to these by symmetry, all others are zero.

Substitution in equation (6) shows that the overall elastic properties depend on the distribution of orientations of cracks through two parameters only; namely $\bar{\epsilon}_{11}$ and $\bar{\epsilon}_{1111}$. A wide variety of orientation distributions will give the same overall properties and measurement of wavespeeds and attenuation will give no further information.

If we now use the Fisher distribution (Mardia 1972) for $\epsilon(\vartheta)$:

$$\epsilon(\vartheta) = \frac{(\overline{va^3})}{2\pi} e^{(\cos\vartheta)/\sigma^2} / \sigma^2(e^{1/\sigma^2} - 1), \tag{13}$$

we have a model for a Gaussian distribution on the sphere; that is, for small σ^2, ϵ is approximately

$$\epsilon(\vartheta) \simeq (\overline{va^3}) e^{-\vartheta^2/2\sigma^2} / 2\pi\sigma^2. \tag{14}$$

The proportion of crack normals lying outside the range $0 \le \vartheta \le 2\sigma$ is approximately e^{-2}. Then

$$\begin{aligned} &A = (\overline{va^3}), \\ &\bar{\epsilon}_{11} = \frac{-1 + 2\sigma^2 e^{1/\sigma^2} - 2\sigma^4(e^{1/\sigma^2} - 1)}{2(e^{1/\sigma^2} - 1)} \simeq \sigma^2, \\ &\bar{\epsilon}_{1111} = \frac{3}{8}\left[\frac{-1 + 4\sigma^4(2e^{1/\sigma^2} + 1) - 24\sigma^6 e^{1/\sigma^2} + 24\sigma^8(e^{1/\sigma^2} - 1)}{(e^{1/\sigma^2} - 1)}\right] \simeq 3\sigma^4. \end{aligned} \tag{15}$$

This type of distribution is particularly suitable for a material with cracks whose normals are orientated randomly about a mean direction along the 3-axis with small variance.

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4 CRACKS WITH NORMALS RANDOMLY DISTRIBUTED AT A FIXED ANGLE AROUND A GIVEN DIRECTION

In this case ν is independent of ϕ and is zero unless $\vartheta = \vartheta_0$, $0 \le \phi \le 2\pi$. The first-order terms are given by (6), while $\bar{\epsilon}_{ij}$ and $\bar{\epsilon}_{ijpq}$ are given by equation (12). We may write

$$\epsilon = (\overline{\nu a^3})\,\frac{\delta(\vartheta - \vartheta_0)}{2\pi \sin\vartheta} \tag{16}$$

and so

$$\begin{aligned} A &= (\overline{\nu a^3}), \\ \bar{\epsilon}_{11} &= \tfrac{1}{2}\sin^2\vartheta_0, \\ \bar{\epsilon}_{111} &= \tfrac{3}{8}\sin^4\vartheta_0. \end{aligned} \tag{17}$$

The elastic parameters are given by equation (6):

$$\begin{aligned} c^1_{1111} &= \frac{-(\overline{\nu a^3})}{2\mu_0}[\bar{U}_{33}(2\lambda_0^2 + 4\lambda_0\mu_0\sin^2\vartheta_0 + 3\mu_0^2\sin^4\vartheta_0) + \bar{U}_{11}\mu_0^2\sin^2\vartheta(4 - 3\sin^2\vartheta_0)] \\ &= c^1_{2222}, \\ c^1_{3333} &= \frac{-(\overline{\nu a^3})}{\mu_0}[\bar{U}_{33}(\lambda_0 + 2\mu_0\cos^2\vartheta_0)^2 + \mu_0^2\bar{U}_{11}4\cos^2\vartheta_0\, sin^2\,\vartheta_0], \\ c^1_{1122} &= \frac{-(\overline{\nu a^3})}{2\mu_0}[\bar{U}_{33}(2\lambda_0^2 + 4\lambda_0\mu_0\sin^2\vartheta_0 + \mu_0^2\sin^4\vartheta_0) - \bar{U}_{11}\mu_0^2\sin^4\vartheta_0], \\ c^1_{1133} &= \frac{-(\overline{\nu a^3})}{\mu_0}[\bar{U}_{33}(\lambda_0 + \mu_0\sin^2\vartheta_0)(\lambda_0 + 2\mu_0\cos^2\vartheta_0) - \bar{U}_{11}\mu_0^2 2\sin^2\vartheta_0\cos^2\vartheta_0] \\ &= c^1_{2233}, \\ c^1_{2323} &= \frac{-(\overline{\nu a^3})}{2}\mu_0[\bar{U}_{33}4\sin^2\vartheta_0\cos^2\vartheta_0 + \bar{U}_{11}(\sin^2\vartheta_0 + 2\cos^2\vartheta_0 - 4\sin^2\vartheta_0\cos^2\vartheta_0)] \\ &= c^1_{1313}, \\ c^1_{1212} &= \frac{-(\overline{\nu a^3})}{2}\mu_0[\bar{U}_{33}\sin^4\vartheta_0 + U_{11}\sin^2\vartheta_0(2 - \sin^2\vartheta_0)], \end{aligned} \tag{18}$$

and all other components except those related to the above by the symmetry relations, are zero. These equations correct two separate errors in equations (41) of Hudson (1986).

5 CONCLUSIONS

The extension of the formulae for the overall parameters of a cracked material to models where the crack distribution has a number density which varies continuously with orientation is a useful one, as Sayers has clearly demonstrated (Sayers 1988a, b). In this paper we have provided this extension in a notation which is different from that of Sayers but is uniform with other papers in this series (e.g. Hudson 1981, 1986, 1988). As a result the formulae have the capacity to model cracks, not only of any orientation, but with a variety of internal conditions (Hudson 1981, 1988). In addition, there is the ability to work to second order in the number density.

The first-order derivation of the elastic parameters from those of the matrix material is given by equation (6) for the general case, and for various special cases in Sections 3 and 4. Once this first-order term is obtained, the second-order term is given by equation (3). Conditions governing the internal state of the crack influence the result through the quantities $\bar{U}_{11}$ and $\bar{U}_{33}$.

In the simplest model of a material with internal cracks, the cracks are aligned in a single direction. Even if this is nearly true in practice, there will always be some variation in orientation about this main direction and this variation may be observable. The appropriate formulae for an approximately Gaussian distribution about the mean is given in Section 3.

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