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Overall properties of a cracked solid

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Abstract. The differential-integral equation of motion for the mean wave in a solid material containing embedded cavities or inclusions is derived. It consists of a series of terms of ascending powers of the scattering operator, and is here truncated after the third term. This implies the second-order interactions between scatterers are included but those of the third order are not.

The formulae are specialized to the case of thin cracks, either aligned in a single direction or randomly oriented. Expressions for the overall elastic constants are derived for the case of long wavelengths. These expressions are accurate to the second order in the number density of scatterers.

1. *Introduction*

The overall properties of cracked solids have been studied by several different workers and from various points of view. (For a review and bibliography, see Watt, Davies and O'Connell, (14)). The static elastic moduli of a solid with a dilute concentration of dry, randomly oriented, circular cracks were given by Bristow (2) and the equivalent formulae for fluid-filled cracks were given by Walsh (13). These expressions are correct to the first order in νa^3 ($\nu a^3 \ll 1$) where ν is the number-density of cracks, and a is the mean radius of the cracks. Budiansky & O'Connell (3) derived more accurate formulae by use of the self-consistent method. They also investigated elliptic cracks, and random mixtures of dry and fluid-filled cracks.

Equivalent formulae for aligned, dry or fluid-filled, circular cracks accurate to the first order in νa^3 may be obtained from expressions given by Garbin and Knopoff (7), (8), (9) and corresponding formulae were derived by the self-consistent method by Hoenig (10). In this case the overall properties are anisotropic (transversely isotropic in fact).

It may be expected that the static moduli can be derived from the wavespeeds in the material when the wavelengths ($2\pi/k$) are long compared with the scale-length of the cracks ($ka \ll 1$); this was the method employed by Garbin and Knopoff and the expectation is confirmed at this order of approximation. Alternatively, the wavespeed at long wavelengths may be derived from the static moduli, and numerical values derived in this way were given by Anderson, Minster and Cole (1) for wavespeeds in a

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material containing a dilute concentration of aligned, fluid-filled, circular cracks using various values (small but non-vanishing) for the aspect ratios.

In all the results mentioned so far, except for those of Garbin and Knopoff, a crack has been modelled as an ellipsoid with one semi-axis much smaller than the other two (usually vanishingly small). Budiansky and O'Connell, in their formulae for fluid-filled cracks, include terms which arise when the thinness of the ellipsoid is compensated by the low compressibility of the fluid.

In this paper we derive formulae for the overall elastic properties of a material containing aligned or randomly oriented cracks, correct to the second order in the concentration. This means that crack–crack interactions are accounted for, as opposed to the single-scattering formulae which are correct to the first order in (νa^3); we neglect third-order interactions. We consider the dynamic problem of a plane wave propagating through the medium, but work to the zeroth order in frequency. Hence our formulae should be equivalent to those for the static problem.

The self-consistent method used by Budiansky and O'Connell (3) and by Hoenig (10) (for the static problem) provides expressions for the overall elastic constants which appear to be valid over a wider range of values of the concentration than the first-order expressions. In general, a series expansion in ν of the self-consistent formulae agrees up to the first order with the approximations given by neglecting second-order interactions between scatterers. However, in one case where a comparison has been made, the expansion of the self-consistent formula (for a random distribution of embedded spheres) did not agree in the term in ν^2 with a series expansion to the same order, given by including second-order interactions (4).

It is clear, in any case, that since no explicit assumptions are made in the self-consistent method about the statistical distribution of one scatterer relative to another, the results will not in general agree with formulae which presuppose a specific statistical model.

Here we show how the probability distribution of scatterers arises in the derivation. However, we make certain general assumptions (chosen mainly for the simplifications produced) and arrive at a fairly simple expression for the second-order term in which only rather broad properties of the distribution of scatterers are involved.

2. *The differential-integral equation for the mean field*

We begin by setting up an operator equation for the mean field, using both differential and integral operators. In doing this, we follow closely the method of Keller (11). The formulation is not restricted to the problem of a cracked solid, but may be applied to wave propagation through a solid with a dilute concentration of any kind of scatterer.

Let $\mathscr{L}$ be the elastodynamic differential operator for a time-harmonic disturbance in a region $\mathscr{D}$ of the solid material containing no scatterers:

$$\mathscr{L}_i(\mathbf{u};\,\mathbf{x}) \equiv \frac{\partial}{\partial x_j} c^0_{ijpq} \frac{\partial u_p(\mathbf{x})}{\partial x_q} + \rho\omega^2 u_i(\mathbf{x}), \quad \mathbf{x} \in \mathscr{D}, \tag{1}$$

and let $\mathbf{u}^0$ represent the incident or unperturbed wave, so that

$$\mathscr{L}(\mathbf{u}^0;\,\mathbf{x}) = 0, \quad \mathbf{x} \in \mathscr{D}. \tag{2}$$

We now suppose that $\mathscr{D}$ contains a distribution of embedded scatterers. If the part of $\mathscr{D}$ outside the scatterers is $\mathscr{D}'$, then

$$\mathscr{L}(\mathbf{u};\, \mathbf{x}) = 0, \quad \mathbf{x} \in \mathscr{D}', \tag{3}$$

where $\mathbf{u}$ is the solution of the scattering problem.

Now let $\epsilon \mathbf{S}^n$ be the scattering operator associated with the nth scatterer, so that $\epsilon \mathbf{S}^n(\mathbf{u}^n;\, \mathbf{x})$ is the scattered wave when a disturbance $\mathbf{u}^n$ is incident upon the scatterer; ϵ is a small scalar which indicates that the scattered field is, in some way, small. Now, the incident field $\mathbf{u}^n$ is given by

$$\mathbf{u}^n(\mathbf{x}) + \epsilon \mathbf{S}^n(\mathbf{u}^n;\, \mathbf{x}) = \mathbf{u}(\mathbf{x}), \tag{4}$$

since $\mathbf{u}$ is the total field.

For the same reason, if there are N scatterers

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \epsilon \sum_{n=1}^{N} \mathbf{S}^n(\mathbf{u}^n;\, \mathbf{x}), \quad \mathbf{x} \in \mathscr{D}'. \tag{5}$$

Equations (4) and (5) together give

$$\mathbf{u}^n = \mathbf{u}^0 + \epsilon \sum_{m \neq n} \mathbf{S}^m \mathbf{u}^m, \quad n = 1, 2, \ldots, N, \quad \mathbf{x} \in \mathscr{D}'_n, \tag{6}$$

where $\mathscr{D}'_n$ is the region $\mathscr{D}$ excluding all scatterers but the nth. Successive approximations lead to the following series in powers of ϵ for $\mathbf{u}^n$:

$$\mathbf{u}^n = \mathbf{u}^0 + \epsilon \sum_{m \neq n} \mathbf{S}^m \mathbf{u}^0 + \epsilon^2 \sum_{m \neq n} \sum_{p \neq m} \mathbf{S}^m \mathbf{S}^p \mathbf{u}^0 + \ldots, \quad \mathbf{x} \in \mathscr{D}'_n.$$

Finally, substitution into equation (4) gives

$$\mathbf{u} = \mathbf{u}^0 + \epsilon \sum_{n=1}^{N} \mathbf{S}^n \mathbf{u}^0 + \epsilon^2 \sum_{n=1}^{N} \sum_{m \neq n} \mathbf{S}^n \mathbf{S}^m \mathbf{u}^0 + \ldots, \quad \mathbf{x} \in \mathscr{D}'. \tag{7}$$

We assume that, by some means, we can extend this field to all points of $\mathscr{D}$. For flat cracks this means assigning values of $\mathbf{u}$ to the points of the crack surface. For inclusions, it implies the use of the interior field of each inclusion.

We now regard the distribution of scatterers as one of a statistical ensemble and take the expectation of equation (7). Denoting expectation by angular brackets $\langle\ \rangle$, we get

$$\langle \mathbf{u} \rangle = \mathbf{u}^0 + \epsilon \sum_{n=1}^{N} \langle \mathbf{S}^n \rangle \mathbf{u}^0 + \epsilon^2 \sum_{n=1}^{N} \sum_{m \neq n} \langle \mathbf{S}^n \mathbf{S}^m \rangle \mathbf{u}^0 + O(\epsilon^3). \tag{8}$$

If we assume that the statistical distribution of the properties of each crack is the same, we may write

$$\langle \mathbf{S}^n \rangle = \langle \mathbf{S} \rangle, \quad n = 1, 2, \ldots, N.$$

This statistical similarity also implies that $\langle \mathbf{S}^n \mathbf{S}^m \rangle$ is independent of m and n, and we write

$$\langle \mathbf{S}^n \mathbf{S}^m \rangle = \langle \mathbf{S}^1 \mathbf{S}^2 \rangle, \quad \text{all } m \neq n.$$

Equation (8) now becomes

$$\langle \mathbf{u} \rangle = \mathbf{u}^0 + \epsilon N \langle \mathbf{S} \rangle \mathbf{u}^0 + \epsilon^2 N(N-1) \langle \mathbf{S}^1 \mathbf{S}^2 \rangle \mathbf{u}^0 + O(\epsilon^3), \tag{9}$$

and successive iterations may be used to invert this equation to give

$$\mathbf{u}^0 = \{1 - \epsilon N\langle \mathbf{S}\rangle + \epsilon^2[N^2\langle \mathbf{S}\rangle^2 - N(N-1)\langle \mathbf{S}^1\mathbf{S}^2\rangle] + O(\epsilon^3)\}\langle \mathbf{u}\rangle. \tag{10}$$

Application of the operator $\mathscr{L}$ gives the final equation governing $\langle \mathbf{u}\rangle$:

$$\{\mathscr{L} - \epsilon N\mathscr{L}\langle \mathbf{S}\rangle + \epsilon^2[N^2\mathscr{L}\langle \mathbf{S}\rangle^2 - N(N-1)\mathscr{L}\langle \mathbf{S}'\mathbf{S}^2\rangle] + O(\epsilon^3)\}\langle \mathbf{u}\rangle = 0. \tag{11}$$

We shall use this equation to determine the properties of the mean displacement field when it takes the form of a plane wave. But first we compare equation (11) with the results achieved by an alternative approach.

If we take the ensemble average of equation (5) we get

$$\langle \mathbf{u}\rangle = \mathbf{u}^0(\mathbf{x}) + \epsilon \sum_{n=1}^{N} \int_{\mathscr{D}} \bar{\mathbf{S}}^n(\langle \mathbf{u}^n\rangle_n;\, \mathbf{x})\, p(\mathbf{x}^n)\, dV_n, \tag{12}$$

where $\langle\ \rangle_n$ implies an average taken over the parameters of all the cracks except for the nth, and the bar over implies an averaging over all the parameters of the nth crack (orientation, radius, etc.) except its position. The final average over the position $\mathbf{x}^n$ of the nth crack is achieved by means of the probability density function $p(\mathbf{x}^n)$.

An approximation suggested by Foldy (6) is to assume that $\langle \mathbf{u}^n\rangle_n$, the average field at $\mathbf{x}^n$ in the absence of the nth obstacle, is equal to the overall mean field $\langle \mathbf{u}\rangle$. Thus equation (12) becomes

$$\langle \mathbf{u}\rangle = \mathbf{u}^0(\mathbf{x}) + \epsilon N\langle \mathbf{S}\rangle\langle \mathbf{u}\rangle,$$

and, in place of equation (11), we have

$$\{\mathscr{L} - \epsilon N\mathscr{L}\langle \mathbf{S}\rangle\}\langle \mathbf{u}\rangle = 0. \tag{13}$$

Foldy's approximation is therefore equivalent to neglecting all terms of order ϵ^2 and higher.

A more accurate approach is suggested by attempting to evaluate $\langle \mathbf{u}^n\rangle_n$ for substitution into (12). We take the statistical average of equation (6) keeping the position of the nth particle fixed:

$$\langle \mathbf{u}^n\rangle_n = \mathbf{u}^0 + \epsilon \sum_{m\neq n} \int \bar{\mathbf{S}}^m(\langle \mathbf{u}^m\rangle_{m,n};\, \mathbf{x})\, p(\mathbf{x}^m|\mathbf{x}^n)\, d\mathbf{V}_m, \tag{14}$$

where $\langle \mathbf{u}^m\rangle_{m,n}$ is the average of $\mathbf{u}^m$ keeping the positions of both mth and nth particles fixed and $p(\mathbf{x}^m|\mathbf{x}^n)$ is the conditional probability density for the position of the mth particle keeping the nth particle fixed. A whole hierarchy of equations can be constructed in this way, the average of $\mathbf{u}^n$ with j particles fixed is given in terms of the average with $(j+1)$ fixed. We need to truncate the series at some stage.

One method would be to construct from equation (14) an ascending series for $\langle \mathbf{u}^n\rangle$ in powers of ϵ and to truncate at a suitable level. In doing this we once more get equation (8) for $\langle \mathbf{u}\rangle$, as might be expected. An alternative procedure is to use the quasi-crystalline approximation of Lax (12) who put

$$\langle \mathbf{u}^m\rangle_{m,n} = \langle \mathbf{u}^m\rangle_m. \tag{15}$$

In a close-packed crystalline solid, fixing one crystal determines the positions of all the others, in which case equation (15) will be exact. In other circumstances, however, the assumption is rather unsatisfactory in that $\langle \mathbf{u}^m \rangle_{m,n}$ has a singularity in the neighbourhood of the nth particle, whereas $\langle \mathbf{u}^m \rangle$ does not.

Substitution of equation (15) into (14) gives

$$\langle \mathbf{u}^n \rangle_n = \mathbf{u}^0 + \epsilon \sum_{m \neq n} \int \bar{\mathbf{S}}^m(\langle u^m \rangle_m; \mathbf{x})\, p(\mathbf{x}^m | \mathbf{x}^n)\, dV_m$$

and a series expansion of $\langle \mathbf{u} \rangle$ in powers of ϵ derived from this equation is identical with equation (8) up to the term in ϵ^2. The next term is

$$\epsilon^3 \sum_n \sum_{m \neq n} \sum_{s \neq m} \left(\iiint \bar{\mathbf{S}}^n \bar{\mathbf{S}}^m \bar{\mathbf{S}}^s p(\mathbf{x}^s | \mathbf{x}^m)\, p(\mathbf{x}^m | \mathbf{x}^n)\, p(\mathbf{x}^n)\, dV_s\, dV_m\, dV_n \right) \mathbf{u}^0, \tag{16}$$

whereas the correct expression (the next highest term in equation (8)) is

$$\epsilon^3 \sum_n \sum_{m \neq n} \sum_{s \neq m} \left(\iiint \bar{\mathbf{S}}^n \bar{\mathbf{S}}^m \bar{\mathbf{S}}^s p(\mathbf{x}^s | \mathbf{x}^m, \mathbf{x}^n)\, p(\mathbf{x}^m | \mathbf{x}^n)\, p(\mathbf{x}^n)\, dV_s\, dV_m\, dV_n \right) \mathbf{u}^0. \tag{17}$$

The difference between these two expressions depends on the difference between the two probability density functions $p(\mathbf{x}^s|\mathbf{x}^m, \mathbf{x}^n)$ and $p(\mathbf{x}^s|\mathbf{x}^m)$. The quasi-crystalline approximation is accurate to third order if the probability distribution of the position of a scatterer is the same whether two particles or just one are fixed. In fact, it is clear that, if this condition is satisfied, equation (15) is correct and the approximation is exact to all orders.

We shall work here to terms of order ϵ^2 and therefore the results will be as accurate as would be obtained by the quasi-crystalline approximation.

One advantage of the present method is that the series expansion in ϵ in equation (11) shows clearly that, in ignoring terms of ϵ^3 and higher, we are neglecting interactions between scatterers of the third and higher orders. The term in ϵ includes the effect of single scattering only, while the term in ϵ^2 includes double scattering.

3. *The equation for the mean field for a distribution of cracks*

We now make explicit the fact that the scatterers are cracks. In addition we shall for the time being assume that all the cracks are aligned in a single direction and all have the same internal condition (dry, fluid-filled, etc.). For cracks orientated in a range of directions, we need to superpose one more averaging process; similarly, a distribution of cracks with a variety of internal conditions will require another average. We assume that, after averaging over crack shape, etc., $\bar{\mathbf{S}}^n(\mathbf{x}^n)$ is the scattering operator for a flat, circular crack with (mean) radius a and orientation given by the fixed normal vector $\mathbf{n}$, and with centre at $\mathbf{x}^n$. We may drop the superscript n on $\bar{\mathbf{S}}$; $\langle \mathbf{S} \rangle$ is given in terms of $\bar{\mathbf{S}}(\mathbf{x}^n)$ by

$$\langle \mathbf{S} \rangle = \int_{\mathscr{D}} \bar{\mathbf{S}}(\boldsymbol{\xi})\, p(\boldsymbol{\xi})\, dV_\xi. \tag{18}$$

Similarly

$$\langle \mathbf{S}^1 \mathbf{S}^2 \rangle = \iint_{\mathscr{D}} \bar{\mathbf{S}}(\boldsymbol{\xi}^1)\, \bar{\mathbf{S}}(\boldsymbol{\xi}^2)\, p(\boldsymbol{\xi}^2 | \boldsymbol{\xi}^1)\, p(\boldsymbol{\xi}^1)\, dV_{\xi 1}\, dV_{\xi 2}. \tag{19}$$

We now assume that the statistical distribution of each scatterer is homogeneous:

$$p(\boldsymbol{\xi}) = 1/V, \tag{20}$$

where V is the volume of the region $\mathscr{D}$, and we write

$$N/V = \nu. \tag{21}$$

So, in equation (11), we have

$$N\langle \mathbf{S}\rangle = \nu\int_{\mathscr{D}} \bar{\mathbf{S}}(\boldsymbol{\xi})\,dV_\xi,$$

$$N(N-1)\langle \mathbf{S}^1\mathbf{S}^2\rangle = \nu\iint_{\mathscr{D}} \bar{\mathbf{S}}(\boldsymbol{\xi}^1)\,\bar{\mathbf{S}}(\boldsymbol{\xi}^2)\,n(\boldsymbol{\xi}^2|\boldsymbol{\xi}^1)\,dV_{\xi 1}\,dV_{\xi 2}, \tag{22}$$

where $n(\boldsymbol{\xi}^2|\boldsymbol{\xi}^1)$ is the number density of cracks at $\boldsymbol{\xi}^2$, given that there is a crack centred on $\boldsymbol{\xi}^1$:

$$n(\boldsymbol{\xi}^2|\boldsymbol{\xi}^1) = (N-1)\,p(\boldsymbol{\xi}^2|\boldsymbol{\xi}^1). \tag{23}$$

Equation (11) now becomes

$$\mathscr{L}\langle\mathbf{u}\rangle - \epsilon\nu\int_{\mathscr{D}} \mathscr{L}\bar{\mathbf{S}}(\boldsymbol{\xi})\langle\mathbf{u}\rangle\,dV_\xi + \epsilon^2\nu\iint_{\mathscr{D}} \mathscr{L}\bar{\mathbf{S}}(\boldsymbol{\xi}^1)\,\bar{\mathbf{S}}(\boldsymbol{\xi}^2)\langle\mathbf{u}\rangle[\nu - n(\boldsymbol{\xi}^2|\boldsymbol{\xi}^1)]\,dV_{\xi 1}\,dV_{\xi 2}. \tag{24}$$

We are now in a position to substitute for $\langle\mathbf{u}\rangle$. We have assumed from the beginning that the mean properties of the cracked material will be the same as those of a homogeneous solid. Plane waves can propagate through such a medium and we may write

$$\langle\mathbf{u}\rangle = \mathbf{b}\,e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{25}$$

where $\mathbf{b}$ and $\mathbf{k}$ are constants.

We shall in the end, of course, make the assumption that the mean wave is a good approximation to the actual wave propagating through a material with a frequency distribution of cracks which approximates to the statistical distribution we have described.

If the uncracked material is isotropic,

$$\mathscr{L}_i\mathbf{u} = (\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j\,\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \rho\omega^2 u_i,$$

and

$$\mathscr{L}\mathbf{b}\,e^{i\mathbf{k}\cdot\mathbf{x}} = [(\rho\omega^2 - \mu k^2)\,\mathbf{b} - (\lambda+\mu)(\mathbf{b}\,.\,\mathbf{k})\,\mathbf{k}]\,e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{26}$$

thus giving the first term in equation (24).

In order to evaluate $\bar{\mathbf{S}}\langle\mathbf{u}\rangle$, we use Green's function $G_i^k(\mathbf{x},\mathbf{x}')$ for the uncracked medium, which satisfies

$$\mathscr{L}_i\mathbf{G}^k(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')\,\delta_{ik}. \tag{27}$$

Then

$$\epsilon\bar{S}_i(\boldsymbol{\xi})\,\mathbf{v} = \int_\Sigma [V_k](\boldsymbol{\xi},\mathbf{x}')\,c^0_{kjpq}\frac{\partial G_i^p(\mathbf{x},\mathbf{x}')}{\partial x'_q}\,n_j\,dS', \tag{28}$$

where integration is over points $\mathbf{x}'$ of the crack surface Σ centred on $\boldsymbol{\xi}$, and $[\mathbf{V}]$ is the jump in displacement on crossing the crack in the direction of $\mathbf{n}$ due to the incident field $\mathbf{v}$; $\mathbf{c}^0$ is, as before, the tensor of elastic moduli of the uncracked material,

$$c^0_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{29}$$

In any crack problem, the solution depends only on the tractions imposed on the plane of the crack by the incident field. Let the direction of $\mathbf{n}$ be the x_3 axis of the coordinate system. Then the traction on the plane of the crack due to an incident field $\mathbf{v}$ is given by

$$t_i(\mathbf{v};\, \mathbf{x}') = c^0_{i3lm} \frac{\partial v_l}{\partial x_m}\bigg|_{\mathbf{x}=\boldsymbol{\xi}+\mathbf{X}}, \tag{30}$$

where $\mathbf{X} \equiv (X_1, X_2, 0)$ and $\mathbf{x}' = \boldsymbol{\xi}+\mathbf{X}$ is a point of the crack. Since the problem is linear, we may superpose solutions for different components of traction on the crack, and the discontinuity in displacement may be written as

$$[V_k](\boldsymbol{\xi}, \mathbf{x}') = \frac{a}{\mu} \mathscr{U}_{ki}(t_i(\mathbf{v};\, \boldsymbol{\xi}+\mathbf{X});\, \mathbf{X}) \quad (\text{no sum over } i), \tag{31}$$

where the non-dimensional linear operator $\mathscr{U}_{ki}$ will depend on the internal conditions of the crack.

Putting $\qquad \mathbf{v} = \langle \mathbf{u} \rangle = \mathbf{b}\, e^{i\mathbf{k}.\mathbf{x}},$

we have $\qquad t_i = ic^0_{i3lm} b_l k_m e^{i\mathbf{k}.(\boldsymbol{\xi}+\mathbf{X})},$

and the displacement discontinuity is

$$[V_k](\boldsymbol{\xi}, \mathbf{x}') = \frac{ia}{\mu} e^{i\mathbf{k}.\boldsymbol{\xi}} b_l k_m c^0_{i3lm} \mathscr{U}_{ki}(e^{i\mathbf{k}.\mathbf{X}};\, \mathbf{X}). \tag{32}$$

Thus

$$\epsilon \bar{S}_i(\boldsymbol{\xi}) \langle \mathbf{u} \rangle = \frac{ia}{\mu} e^{i\mathbf{k}.\boldsymbol{\xi}} b_m k_n c^0_{k3pq} c^0_{l3mn} \int_\Sigma \mathscr{U}_{kl}(e^{i\mathbf{k}.\mathbf{X}};\, \mathbf{X}) \frac{\partial G^p_i}{\partial \xi_q}(\mathbf{x};\, \boldsymbol{\xi}+\mathbf{X})\, dS_X. \tag{33}$$

To get the second term in equation (24), we use equation (27) to get

$$\begin{aligned}\epsilon\nu \int_{\mathscr{D}} &\mathscr{L}_i \bar{\mathbf{S}}(\boldsymbol{\xi}) \langle \mathbf{u} \rangle\, dV_\xi \\ &= -\frac{i\nu a}{\mu} \int_{\mathscr{D}} dV_\xi \left\{ e^{i\mathbf{k}.\boldsymbol{\xi}} b_m k_n c^0_{k3pq} c^0_{l3mn} \delta_{ip} \int_\Sigma \mathscr{U}_{kl}(e^{i\mathbf{k}.\mathbf{X}};\, \mathbf{X}) \frac{\partial}{\partial \xi_q} \delta(\mathbf{x}-\boldsymbol{\xi}-\mathbf{X})\, dS_X \right\} \\ &= -\frac{\nu a^3}{\mu} b_m k_q k_n c^0_{k3pq} c^0_{l3mn} e^{i\mathbf{k}.\mathbf{x}} \bar{\mathscr{U}}_{kl}(\mathbf{k}a),\end{aligned} \tag{34}$$

where $\qquad \bar{\mathscr{U}}_{kl}(\mathbf{k}a) = \dfrac{1}{a^2} \displaystyle\int_\Sigma \mathscr{U}_{kl}(e^{i\mathbf{k}.\mathbf{X}};\, \mathbf{X})\, e^{-i\mathbf{k}.\mathbf{X}}\, dS_X.$

The quantities $\bar{\mathscr{U}}_{kl}$ are non-dimensional and therefore depend on $(\mathbf{k}a)$ and Poisson's ratio of the uncracked material only. In addition, the symmetry of the problem shows that $\bar{\mathscr{U}}_{kl}$ is diagonal, and that $\bar{\mathscr{U}}_{11} = \bar{\mathscr{U}}_{22}$.

Comparison of equations (26) and (34) shows that the second term in equation (24) is of order (νa^3) times the first. In truncating the series in ϵ we assume that interactions between scatterers are small (we ignore triple scattering), so (νa^3) is a small quantity.

Returning to calculate the third term in equation (24), we have for any incident field $\mathbf{v}(\boldsymbol{\xi}', \mathbf{x})$, where $\boldsymbol{\xi}'$ is just a parameter at this stage,

$$\epsilon\nu \int_{\mathscr{D}} \mathscr{L}_i \bar{\mathbf{S}}(\boldsymbol{\xi}^1) \mathbf{v}(\boldsymbol{\xi}^1, \mathbf{x})\, dV_{\xi_1} = \frac{a\nu}{\mu} c^0_{j3lm} c^0_{k3iq} \frac{\partial}{\partial x_q} \int_\Sigma \mathscr{U}_{kj}\left(\frac{\partial v_l}{\partial x_m}(\boldsymbol{\xi}^1, \boldsymbol{\xi}^1+\mathbf{X});\, \mathbf{X}\right)\bigg|_{\boldsymbol{\xi}'=\mathbf{x}-X} dS_X. \tag{35}$$

We need to substitute

$$\mathbf{v}(\boldsymbol{\xi}^1, \mathbf{x}) = \epsilon \int_{\mathscr{D}} \mathbf{S}(\boldsymbol{\xi}^2) \langle \mathbf{u} \rangle [\nu - n(\boldsymbol{\xi}^2|\boldsymbol{\xi}^1)]\, dV_{\xi_2}. \tag{36}$$

However, these expressions are rather complicated and we make further simplifying assumptions before attempting to evaluate them.

4. *The case where the radii and separation distances of the cracks are small compared with a wavelength*

If the radius a of the mean crack is small compared with a wavelength of the plane waves;

$$\frac{\omega a}{\beta} \ll 1, \quad \beta^2 = \mu/\rho, \tag{37}$$

then the incident field may be regarded as being uniform across the crack. Equation (35) becomes

$$\epsilon\nu \int_{\mathscr{D}} \mathscr{L}_i \mathbf{S}(\boldsymbol{\xi}^1) \mathbf{v}(\boldsymbol{\xi}^1, \mathbf{x})\, dV_{\xi_1} = \frac{\nu a^3}{\mu} c^0_{j3lm} c^0_{k3iq} \frac{\partial}{\partial x_q} \left(\left. \frac{\partial v_l(\boldsymbol{\xi}^1, \mathbf{x})}{\partial x_m} \right|_{\boldsymbol{\xi}^1 = \mathbf{x}} \right) \overline{\mathscr{U}}_{kj}(0), \tag{38}$$

while, from equations (36) and (33),

$$v_l(\boldsymbol{\xi}^1, \mathbf{x}) = \int_{\mathscr{D}} [\nu - n(\boldsymbol{\xi}^2|\boldsymbol{\xi}^1)] \frac{ia^3}{\mu} e^{i\mathbf{k}.\boldsymbol{\xi}^2} c^0_{n3pr} c^0_{s3uv} b_u k_v \overline{\mathscr{U}}_{ns}(0) \frac{\partial G^p_l(\mathbf{x}; \boldsymbol{\xi}^2)}{\partial \xi^2_r} dV_{\xi_2}. \tag{39}$$

Green's function $\mathbf{G}(\mathbf{x}; \boldsymbol{\xi})$ for a homogeneous medium is a function of $\mathbf{y} = \mathbf{x} - \boldsymbol{\xi}$ only. We assume that the number density $n(\boldsymbol{\xi}|\mathbf{x})$ is uniform throughout the medium and so it also is a function of $\mathbf{y}$ only. The right-hand side of equation (38) becomes

$$-\frac{i\nu a^6}{\mu^2} c^0_{n3pr} c^0_{s3uv} c^0_{j3lm} c^0_{k3iq} b_u k_v \overline{\mathscr{U}}_{ns}(0)\, \overline{\mathscr{U}}_{kj}(0) \frac{\partial}{\partial x_q} \left\{ e^{i\mathbf{k}.\mathbf{x}} \int_{\mathscr{D}} [\nu - n(\mathbf{y})] e^{-i\mathbf{k}\mathbf{y}} \frac{\partial^2 G^p_l(\mathbf{y})}{\partial y_r \partial y_m} dV_y \right\} \tag{40}$$

$$= -\frac{\nu^2 a^4}{\mu^3} (k_v a)(k_q a)\, c^0_{n3pr} c^0_{s3uv} c^0_{j3lm} c^0_{k3iq} b_u e^{i\mathbf{k}.\mathbf{x}} \overline{\mathscr{U}}_{ns}(0)\, \overline{\mathscr{U}}_{kl}(0)\, \chi_{lmpr},$$

where

$$\chi_{lmpr} = -\mu \int_{\mathscr{D}} \left[1 - \frac{n(\mathbf{y})}{\nu} \right] e^{-i\mathbf{k}.\mathbf{y}} \frac{\partial^2 G^p_l(\mathbf{y})}{\partial y_r \partial y_m} dV_y. \tag{41}$$

Now, χ_{lmpr} is non-dimensional and depends on Poisson's ratio and the ratio of the wavelength to the scale of n.

It seems reasonable to suppose that $n(\mathbf{y})$ will be approximately equal to the overall number density for $|\mathbf{y}|$ much larger than some separation distance d. If d is much shorter than a wavelength, then the factor $e^{-i\mathbf{k}.\mathbf{y}}$ varies little within the range of the integral in equation (40). Putting this equal to unity, and integrating by parts, we have

$$\chi_{lmpr} = -\frac{\mu}{\nu} \int_{\mathscr{D}} \frac{\partial n(\mathbf{y})}{\partial y_r} \frac{\partial G^p_l(\mathbf{y})}{\partial y_m} dV_y. \tag{42}$$

The fact that d is short compared with a wavelength means that we need only specify $\mathbf{G}$ near the origin. From Eason, Fulton and Sneddon(5) we have

$$\mu G^p_l(\mathbf{y}) = \frac{1}{4\pi y} \left[\frac{y_l y_p}{y^2} g_1(y) - \delta_{lp} g_2(y) \right], \tag{43}$$

where

$$g_1(y) = 3\left(\frac{\beta}{\omega y}\right)^2 (e^{i\omega y/\beta} - e^{i\omega y/\alpha}) - 3i\left(\frac{\beta}{\omega y}\right)\left(e^{i\omega y/\beta} - \frac{\beta}{\alpha} e^{i\omega y/\alpha}\right) - \left(e^{i\omega y/\beta} - \frac{\beta^2}{\alpha^2} e^{i\omega y/\alpha}\right)$$

$$= \frac{1}{2}\left(1 - \frac{\beta^2}{\alpha^2}\right) + O\left(\frac{\omega y}{\beta}\right),$$

$$g_2(y) = \left(\frac{\beta}{\omega y}\right)^2 (e^{i\omega y/\beta} - e^{i\omega y/\alpha}) - i\left(\frac{\beta}{\omega y}\right)\left(e^{i\omega y/\beta} - \frac{\beta}{\alpha} e^{i\omega y/\alpha}\right) - e^{i\omega y/\beta}$$

$$= -\frac{1}{2}\left(1 + \frac{\beta^2}{\alpha^2}\right) + O\left(\frac{\omega y}{\beta}\right),$$

$\alpha^2 = (\lambda + 2\mu)/\rho$, and $y = |\mathbf{y}|$. With the additional assumption that n is a function of y only, equation (42) becomes, to the lowest order,

$$\chi_{lmpr} = \frac{1}{4\pi\nu} \int \frac{\partial n}{\partial y} \left\{ \frac{1}{2}\left(1 - \frac{\beta^2}{\alpha^2}\right)\left(3\frac{y_l}{y}\frac{y_p}{y}\frac{y_r}{y}\frac{y_m}{y} - \delta_{lm}\frac{y_p}{y}\frac{y_r}{y} - \delta_{pm}\frac{y_l}{y}\frac{y_r}{y}\right.\right.$$
$$\left.\left. + \frac{1}{2}\left(1 + \frac{\beta^2}{\alpha^2}\right)\delta_{lp}\frac{y_m}{y}\frac{y_r}{y}\right\} dy \sin\theta \, d\theta \, d\phi, \quad (44)$$

where (y, θ, ϕ) are spherical polar coordinates. Integration gives

$$\chi_{lmpr} = \frac{1}{15}\left\{\partial_{lp}\delta_{mr}\left(4 + \frac{\beta^2}{\alpha^2}\right) - (\delta_{pm}\delta_{lr} + \delta_{lm}\delta_{pr})(1 - \beta^2/\alpha^2)\right\}. \quad (45)$$

It should be noticed that the distribution function $n(y)$ has now explicitly disappeared from the formula. We have assumed that the statistical distribution of cracks is homogeneous and isotropic and that $n(y)$ approaches ν as y tends to infinity. However, the assumption that the separation distance d of the cracks is small compared with the wavelength has meant that all other characteristics of $n(y)$ do not affect the overall properties to the order of magnitude to which we are working. This result is in contrast with that of Willis & Acton (15) for the overall elastostatic properties of a distribution of spherical inclusions, who showed that a certain integral of the function equivalent to $n(y)$ over its whole range enters into the formulae in the terms of order $(\nu a^3)^2$. This term arises from an integration over the finite volume of a representative sphere and it is perhaps not surprising that no similar term has arisen here.

Equation (24) may now be written as

$$b_j\{\rho\omega^2\delta_{ij} - c^0_{ipjq}k_p k_q - c^1_{ipjq}k_p k_q - c^2_{ipjq}k_p k_q + O\{(\nu a^3)^2\} = 0, \quad (46)$$

where, to lowest order in $(\mathbf{k}a)$,

$$\left.\begin{aligned} c_{ipjq} &= -\frac{\nu a^3}{\mu} c^0_{k3ip} c^0_{l3jq} \overline{\mathscr{U}}_{kl}(0), \\ c^2_{ipjq} &= \frac{(\nu a^3)^2}{\mu^3} c^0_{n3vr} c^0_{s3jq} c^0_{u3lm} c^0_{k3ip} \overline{\mathscr{U}}_{ns}(0)\, \overline{\mathscr{U}}_{kn}(0)\, \chi_{lmvr}, \\ &= \frac{1}{\mu} c^1_{vrjq} c^1_{iplm} \chi_{lmvr}. \end{aligned}\right\} \quad (47)$$

These relations are more easily expressed by writing the fourth-order tensors as (6×6) second-order tensors. The first and second pairs of suffixes are separately reduced to one by the arrangement

$$\left.\begin{array}{ll} (1,1)\rightarrow 1 & (2,3)\rightarrow 4 \\ (2,2)\rightarrow 2 & (3,1)\rightarrow 5 \\ (3,3)\rightarrow 3 & (1,2)\rightarrow 6, \end{array}\right\} \tag{48}$$

and we write

$$\left.\begin{array}{l} c_{11} = c_{1111}, \quad c_{12} = c_{1122}, \quad c_{13} = c_{1133}, \\ c_{21} = c_{2211}, \quad c_{22} = c_{2222}, \quad c_{23} = c_{2233}, \\ c_{31} = c_{3311}, \quad c_{32} = c_{3322}, \quad c_{33} = c_{3333}, \\ c_{14} = c_{1123}+c_{1132} = 2c_{1123}, \\ c_{15} = c_{1131}+c_{1113} = 2c_{1131}, \\ c_{16} = c_{1112}+c_{1121} = 2c_{1112}, \end{array}\right\} \tag{49}$$

and similar formulae for $c_{24}, c_{25}, \ldots, c_{36}$.

$$c_{41} = c_{2311},$$

and similarly c_{42}, c_{43}, etc.,

$$c_{44} = c_{2323}+c_{2332} = 2c_{2323},$$
$$c_{45} = 2c_{2331},$$
$$c_{46} = 2c_{2312}, \quad \text{etc.}$$

Equations (47) may thus be written as

$$\mathbf{c}^0 \equiv \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}, \tag{50}$$

$$\mathbf{c}^1 \equiv -\frac{\nu a^3}{\mu}\begin{bmatrix} \lambda^2\overline{\mathscr{U}}_3 & \lambda^2\overline{\mathscr{U}}_3 & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_3 & 0 & 0 & 0 \\ \lambda^2\overline{\mathscr{U}}_3 & \lambda^2\overline{\mathscr{U}}_3 & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_3 & 0 & 0 & 0 \\ \lambda(\lambda+2\mu)\overline{\mathscr{U}}_3 & \lambda(\lambda+2\mu)\overline{\mathscr{U}}_3 & (\lambda+2\mu)^2\overline{\mathscr{U}}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^2\overline{\mathscr{U}}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu^2\overline{\mathscr{U}}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{51}$$

where

$$\overline{\mathscr{U}}_1 = \overline{\mathscr{U}}_{11}(0) = \overline{\mathscr{U}}_{22}(0),$$
$$\overline{\mathscr{U}}_3 = \overline{\mathscr{U}}_{33}(0),$$

all other components of $\overline{\mathscr{U}}_{ij}$ being zero.

The tensor χ_{lmvr} is not symmetric in (l, m) and (v, r), which precludes a representation like equation (49), but we write

$$c^2_{ipjq} = \frac{1}{\mu}\, c^1_{iplm}\overline{\chi}_{lmvr}\, c^1_{vrjq},$$

where
$$4\bar{\chi}_{lmvr} = \chi_{lmvr} + \chi_{mlvr} + \chi_{lmrv} + \chi_{mlrv},$$
and $\bar{\chi}_{lmvr}$ is symmetric in the required way.

Our choice of representation (equations (49)) means that the second-order tensors that we construct may be multiplied together to give the correct result:
$$c^2_{ij} = \frac{1}{\mu} c^1_{ik} \bar{\chi}_{kl} c^1_{lj},$$
where
$$\bar{\boldsymbol{\chi}} = \frac{1}{15}\begin{bmatrix} \frac{3\beta^2}{\alpha^2}+2 & \frac{\beta^2}{\alpha^2}-1 & \frac{\beta^2}{\alpha^2}-1 & 0 & 0 & 0 \\ \frac{\beta^2}{\alpha^2}-1 & \frac{3\beta^2}{\alpha^2}+2 & \frac{\beta^2}{\alpha^2}-1 & 0 & 0 & 0 \\ \frac{\beta^2}{\alpha^2}-1 & \frac{\beta^2}{\alpha^2}-1 & \frac{3\beta^2}{\alpha^2}+2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\beta^2}{\alpha^2}+3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\beta^2}{\alpha^2}+3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2\beta^2}{\alpha^2}+3 \end{bmatrix}. \tag{52}$$

We get
$$\boldsymbol{\chi}\mathbf{c}^1 \equiv -\frac{\nu a^3}{15}\times$$
$$\begin{bmatrix} \frac{\lambda(3\lambda-2\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_3 & \frac{\lambda(3\lambda-2\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_3 & (3\lambda-2\mu)\bar{\mathscr{U}}_3 & 0 & 0 & 0 \\ \frac{\lambda(3\lambda-2\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_3 & \frac{\lambda(3\lambda-2\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_3 & (3\lambda-2\mu)\bar{\mathscr{U}}_3 & 0 & 0 & 0 \\ \frac{\lambda(9\lambda+14\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_3 & \frac{\lambda(9\lambda+14\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_3 & (9\lambda+14\mu)\bar{\mathscr{U}}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\mu(3\lambda+8\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\mu(3\lambda+8\mu)}{\lambda+2\mu}\bar{\mathscr{U}}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{53}$$

and so
$$\mathbf{c}^2 \equiv \frac{(\nu a^3)^2}{15}\times$$
$$\begin{bmatrix} \frac{\lambda^2 q}{\lambda+2\mu}(\bar{\mathscr{U}}_3)^2 & \frac{\lambda^2 q}{\lambda+2\mu}(\bar{\mathscr{U}}_3)^2 & \lambda q(\bar{\mathscr{U}}_3)^2 & 0 & 0 & 0 \\ \frac{\lambda^2 q}{\lambda+2\mu}(\bar{\mathscr{U}}_3)^2 & \frac{\lambda^2 q}{\lambda+2\mu}(\bar{\mathscr{U}}_3)^2 & \lambda q(\bar{\mathscr{U}}_3)^2 & 0 & 0 & 0 \\ \lambda q(\bar{\mathscr{U}}_3)^2 & \lambda q(\bar{\mathscr{U}}_3)^2 & (\lambda+2\mu)\,q(\bar{\mathscr{U}}_3)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4\mu(3\lambda+8\mu)}{\lambda+2\mu}(\bar{\mathscr{U}}_1)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4\mu(3\lambda+8\mu)}{\lambda+2\mu}(\bar{\mathscr{U}}_1)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{54}$$

where $$q = 15\left(\frac{\lambda}{\mu}\right)^2 + 28\left(\frac{\lambda}{\mu}\right) + 28.$$

Clearly, both $\mathbf{c}^1$ and $\mathbf{c}^2$ have the symmetries appropriate to transverse isotropy with the x_3 axis as axis of symmetry.

5. *The overall properties of a material containing randomly orientated cracks*

If the cracks are randomly orientated, we need to apply one more averaging process (over all possible directions of the normal to a crack) to each ensemble average.

Instead of equation (34), we have

$$\epsilon\nu\int_{\mathscr{D}} \mathscr{L}_i\,\overline{\mathbf{S}}(\boldsymbol{\xi})\,\langle\mathbf{u}\rangle\,dV_\xi = -\frac{\nu a^3}{\mu}\,b_m k_q k_n e^{i\mathbf{k}\cdot\mathbf{x}}\langle c^0_{k3iq}c^0_{l3mn}\overline{\mathscr{U}}_{kl}(0)\rangle, \tag{55}$$

where the angular brackets on the right now imply simply an averaging over the direction of the normal to the crack. This averaging process results in an isotropic tensor with $(i \leftrightarrow q)$ and $(m \leftrightarrow n)$ symmetry, and so we may write

$$-\frac{\nu a^3}{\mu}\,\langle c^0_{k3iq}c^0_{l3mn}\overline{\mathscr{U}}_{kl}(0)\rangle = \lambda_1\delta_{iq}\delta_{mn} + \mu_1(\delta_{im}\delta_{qn} + \delta_{in}\delta_{qm}). \tag{56}$$

It follows that

$$\begin{aligned}-\frac{\nu a^3}{\mu}\langle c^0_{k3ii}c^0_{l3nn}\overline{\mathscr{U}}_{kl}(0)\rangle &= -3(3\lambda_1 + 2\mu_1)\\ &= \frac{\nu a^3}{\mu}(3\lambda + 2\mu)^2\,\overline{\mathscr{U}}_3,\end{aligned} \tag{57}$$

and

$$-\frac{\nu a^3}{\mu}\langle c^0_{k3iq}c^0_{l3iq}\mathscr{U}_{kl}(0)\rangle = -3(\lambda_1 + 4\mu_1) = \frac{\nu a^3}{\mu}[4\mu^2\overline{\mathscr{U}}_1 + (3\lambda^2 + 4\lambda\mu + 4\mu^2)\,\overline{\mathscr{U}}_3], \tag{58}$$

where we again make use of the fact that $\overline{\mathscr{U}}_{ij}(0)$ is diagonal with $\overline{\mathscr{U}}_{11} = \overline{\mathscr{U}}_{22} = \overline{\mathscr{U}}_1$, $\overline{\mathscr{U}}_{33} = \overline{\mathscr{U}}_3$.

Equations (57) and (58) give

$$\left.\begin{aligned}\frac{\mu_1}{\mu} &= -\frac{(\nu a^3)2}{15}(3\overline{\mathscr{U}}_1 + 2\overline{\mathscr{U}}_3),\\ \frac{3\lambda_1 + 2\lambda\mu_1}{3\lambda + 2\mu} &= -(\nu a^3)\left(\frac{3\lambda + 2\mu}{3\mu}\right)\overline{\mathscr{U}}_3.\end{aligned}\right\} \tag{59}$$

Thus the first-order modifications to the isotropic material parameters λ and μ are given by λ_1, and μ_1 respectively. We denote the second-order modification by λ_2 and μ_2.

Assuming that the orientation of a crack is not affected by the orientation of any nearby crack, the average over directions of the crack normal may be performed separately in equations (35) and (36). This means that we have, instead of the second of equations (47),

$$\lambda_2\delta_{ip}\delta_{jq} + \mu_2(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}) = \frac{1}{\mu}\langle c^1_{vrjq}\rangle\langle c^1_{iplm}\rangle\,\overline{\chi}_{lmvr}. \tag{60}$$

From equation (45) we see that

$$\overline{\chi}_{lmvr} = \frac{1}{15}\left[-\frac{\lambda+\mu}{\lambda+2\mu}\delta_{lm}\delta_{vr} + \frac{3\lambda+8\mu}{2(\lambda+\mu)}(\delta_{lv}\delta_{mr} + \delta_{lr}\delta_{mv})\right]$$

and so

$$\lambda_2 \delta_{ip}\delta_{jq} + \mu_2(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp})$$
$$= \left(\frac{1}{\lambda+2\mu}\right)\left\{\left[\lambda_1^2 + \tfrac{4}{3}\lambda_1\mu_1 - \frac{4\mu_1^2}{15}\left(\frac{\lambda+\mu}{\mu}\right)\right]\delta_{ip}\delta_{jq} + \frac{2\mu_1^2}{15}\left(\frac{3\lambda+8\mu}{\mu}\right)(\delta_{ij}\delta_{pq}+\delta_{iq}\delta_{jp})\right\}.$$

Thus

$$\frac{3\lambda_2+2\mu_2}{3\lambda+2\mu} = \frac{(3\lambda_1+2\mu_1)^2}{3(\lambda+2\mu)(3\lambda+2\mu)}$$
$$= (\nu a^3)^2 \frac{(3\lambda+2\mu)^3}{27\mu^2(\lambda+2\mu)}(\overline{\mathscr{U}}_3)^2, \tag{61}$$

$$\frac{\mu_2}{\mu} = \frac{2}{15}\left(\frac{\mu_1}{\mu}\right)^2 \frac{(3\lambda+8\mu)}{(\lambda+2\mu)}$$
$$= (\nu a^3)^2\left(\frac{2}{15}\right)^3\left(\frac{3\lambda+8\mu}{\lambda+2\mu}\right)(3\overline{\mathscr{U}}_1+2\overline{\mathscr{U}}_3)^2. \tag{62}$$

6. *Fluid-filled cracks*

In order to compare our formulae with earlier results, we take as illustration a distribution of fluid-filled cracks. It is assumed that the bulk modulus of the fluid is sufficiently large and the aspect ratio sufficiently small that the crack opening displacement can be ignored. Thus $\overline{\mathscr{U}}_3$ is zero. From Garbin and Knopoff (7) we have

$$\mathscr{U}_{11}(1;\mathbf{X}) = \frac{8}{\pi(1+\gamma)}\frac{(a^2-r^2)^{\frac{1}{2}}}{a}, \tag{63}$$

where $r = |\mathbf{X}|$ and $\gamma = 2(\lambda+\mu)/(\lambda+2\mu)$, and so

$$\overline{\mathscr{U}}_1 = \frac{16}{3(1+\gamma)}. \tag{64}$$

From equations (59) we have

$$\frac{\mu_1}{\mu} = -(\nu a^3)\frac{32}{15}\frac{(\lambda+2\mu)}{(3\lambda+4\mu)},$$
$$\frac{3\lambda_1+2\mu_1}{3\lambda+2\mu} = 0, \tag{65}$$

which agrees with Walsh (13).

From equations (61) and (62) we have

$$\frac{\mu_2}{\mu} = \left(\frac{8}{15}\right)^3 4(\nu a^3)^2\frac{(3\lambda+8\mu)(\lambda+2\mu)}{(3\lambda+4\mu)^2},$$
$$\frac{3\lambda_2+2\mu_2}{3\lambda+2\mu} = 0. \tag{66}$$

The self-consistent method gives, for the same problem (3), the following expression for the overall shear modulus $\overline{\mu}$:

$$\frac{\overline{\mu}}{\mu} = 1 - \frac{32}{15}\frac{\kappa+\frac{4}{3}\overline{\mu}}{3\kappa+2\overline{\mu}}(\nu a^3), \tag{67}$$

where κ is the bulk modulus (which remains unchanged in the presence of the cracks). Expanding $\bar{\mu}/\mu$ in an ascending power series in (νa^3), we get, to order $(\nu a^3)^2$,

$$\begin{aligned}\frac{\bar{\mu}}{\mu} &= 1-\frac{32}{15}(\nu a^3)\left(\frac{\kappa+\frac{4}{3}\mu}{3\kappa+2\mu}\right)+\left(\frac{32}{15}\right)^2(\nu a^3)^2\frac{2\mu\kappa\kappa(+\frac{4}{3}\mu)}{(3\kappa+2\mu)^3}\\ &= 1-\frac{32}{15}(\nu a^3)\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)+\left(\frac{8}{15}\right)^2 4(\nu a^3)^2\,5\mu\frac{(3\lambda+2\mu)(\lambda+2\mu)}{(3\lambda+4\mu)^3}.\end{aligned} \tag{68}$$

As expected, equations (65), (66) and (68) agree to order (νa^3), but not to order $(\nu a^3)^2$.

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