

ON THE DEPENDENCE OF THE ELASTIC PROPERTIES OF A POROUS ROCK ON THE COMPRESSIBILITY OF THE PORE FLUID

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An equation is derived for the dependence of the elastic properties of a porous material on the compressibility of the pore fluid. More generally, the elastic properties of a container of arbitrary shape are related to the compressibility of the fluid filling a cavity in the container. If the pore system or cavity under consideration is filled with a fluid of compressibility κ_F , the compressibility κ^* of the closed container is given by

$$(\kappa^* - \kappa_M)^{-1} = (\kappa_A - \kappa_M)^{-1} + [(\kappa_F - \kappa_\phi)\phi]^{-1}.$$

Here κ_A is the compressibility of the container with the fluid pressure held constant in the interconnected pore system or cavity. Fluids in

other pores or cavities not connected with the one in question contribute to the value of κ_A . ϕ is the porosity, i.e., the volume fraction corresponding to the pore system or cavity in question. The equation contains two distinct effective compressibilities, κ_M and κ_ϕ , of the material exclusive of the pore fluid. When this material is homogeneous, one has $\kappa_M = \kappa_\phi$, and the equation reduces to a well-known relation by Gassmann. For the other elastic properties, we also obtain expressions which generalize Gassmann's work and which also differ from it only in the appearance of κ_ϕ instead of κ_M in one term. Our result is intimately related to the reciprocity theorem of elasticity. Special cases are discussed.

INTRODUCTION

More than twenty years ago, Gassmann (1951) derived a relation between the elastic properties of a porous medium with fluid-filled pores and the elastic properties of that same medium with empty pores. His equation has become an important tool in the interpretation of seismic data on sedimentary materials. We have investigated the extent to which it is valid for these applications. This question arises because the assumptions on which Gassmann's equation rests do not apply fully to many real materials of interest.

Gassmann considered a material which is homogeneous on a macroscopic scale. It consists of a microhomogeneous and microisotropic elastic solid out of which a pore is carved. The pore space is interconnected and has an irregu-

lar shape to make it possible for the concept of macrohomogeneity to apply. The solid need not be continuous. The fluid has a compressibility κ_F , which is the explicit variable in Gassmann's theoretical considerations.

Sedimentary materials, to which his theory is most often applied, do not even approximately satisfy the conditions of microhomogeneity and microisotropy. For this reason we tried to estimate the departures from Gassmann's equation to be expected for various models. To our surprise we found that, with a minor generalization involving only *one* new parameter, the requirements of microhomogeneity and microisotropy can be dropped entirely. This gives a new equation which can be applied with much more confidence, although it is of course still based on the assumptions that the solid is elastic

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and that measurements are made over a time scale long enough for pressure equilibration throughout the pore. However, fluids in other cavities, not connected with the pore, may be present and must be incorporated in the theory as part of the "solid."

We also succeeded in removing the requirement of macrohomogeneity. Macroisotropy was not required by Gassmann. We will show that, with appropriate definitions of elastic constants of an arbitrary piece of rock, the equations have the same form whether applied to a macrohomogeneous medium or to any piece of rock.

The analysis is presented in two stages. In the next section we give, without derivation, an equation for the compressibility applicable to any finite piece of porous elastic material with interconnected pore space; this equation generalizes Gassmann's equation in the above sense. In the third section this result is derived. It contains only one parameter more than Gassmann's equation, a result of the fact that two distinct "effective compressibilities" of the heterogeneous solid material occur. If this material is strictly homogeneous, these two are identical, and Gassmann's equation results.

The derivation of a Gassmann-type equation for an "effective elastic tensor" of a piece of a heterogeneous porous material is given in the fourth section. The reasoning runs parallel to that in the scalar case and can be followed without reference to Gassmann's original paper. As in the scalar case, microheterogeneity introduces only one extra scalar parameter. The last section contains discussion and interpretation of the results.

Our results could have been obtained starting from some of Biot's results, cf. Biot (1972) and also Geertsma and Smit (1961), but we give a derivation from first principles for completeness and clarity and in order to establish the conditions under which the results are valid.

COMPRESSIBILITY, STATEMENT OF RESULT

We consider a piece of porous material which we assume to be perfectly elastic for sufficiently small deformations. The total volume V is made up of the volume of "solid" material V_M and the connected pore volume $V_\phi = \phi V$, where ϕ is the porosity. Any liquid inclusions not connected with the pore are to be considered as

part of V_M . We regard the pore volume as sufficiently well connected that a single pressure p_F exists throughout the pore fluid. We require that the time scale of the measurements is long enough that the sample may be regarded as in equilibrium. The pore is filled with a fluid with compressibility κ_F . Σ is the macroscopic outer surface of the sample; where Σ cuts the pore it is extended so that the pore is sealed off from the outside, permitting the application of an external pressure p , which can be different from the fluid pressure p_F .

When an incremental pressure δp is applied, the total volume changes by an amount δV , which depends on many details of the internal structure, such as pore size and shape, the distribution in position as well as in orientation of solid material of different types, and the compressibility of the pore fluid. Our objective is to obtain an explicit equation for the fractional volume change per unit pressure increment, $(1/V)(\delta V/\delta p)$, as a function of κ_F . We will derive an exact relation of this type that contains, besides the porosity ϕ , three parameters which take values characteristic of the solid part of the sample and are independent of κ_F .

Defining the effective compressibility κ^* by

$$\kappa^* = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_f, \quad (1)$$

where the subscript f indicates that the fluid is not allowed to flow across the outer boundary of the sample, we will derive the equation

$$\kappa_A - \kappa^* = (\kappa_A - \kappa_M)^2 / [(\kappa_F - \kappa_\phi)\phi + (\kappa_A - \kappa_M)]. \quad (2)$$

Here κ_A , κ_M , and κ_ϕ are independent of κ_F . This result can also be written as

$$\frac{1}{\kappa^* - \kappa_M} = \frac{1}{\kappa_A - \kappa_M} + \frac{1}{(\kappa_F - \kappa_\phi)\phi}. \quad (3)$$

Gassmann's equation has only two parameters to represent the properties of the rock exclusive of the pore fluid. It can be cast in the form (2) or (3) with $\kappa_M = \kappa_\phi$. This special case follows theoretically, as shown below, when one assumes that the solid part is strictly homogeneous, made of a material with compressibility κ_M .

κ_A , κ_M , and κ_ϕ have simple physical meanings, but that of course does not necessarily help in

knowing their values. These quantities are obtained by allowing the pore pressure p_F and the differential pressure $p_d = p - p_F$ to vary independently. We have adopted the definitions,

$$\kappa_A = -\frac{1}{V} \left(\frac{\partial V}{\partial p_d} \right)_{p_F}, \quad (4a)$$

$$\kappa_M = -\frac{1}{V} \left(\frac{\partial V}{\partial p_F} \right)_{p_d}, \quad (4b)$$

$$\kappa_\phi = -\frac{1}{V_\phi} \left(\frac{\partial V_\phi}{\partial p_F} \right)_{p_d}. \quad (4c)$$

From these it is immediately obvious that $\kappa_M = \kappa_\phi$ if the solid part is homogeneous, because changing p_F with constant p_d means applying the same incremental pressure to the outer and inner (pore) surface, and in the case of homogeneous bodies (however complex and whether or not isotropic) the application of such incremental pressure leads to a linear mapping, which does not change the porosity ϕ . When ϕ is not a variable, we can get equation (4c) from (4b) by multiplying the differentiated V by ϕ and dividing by the same quantity.

Elementary considerations, presented in the next section, could lead one to believe that an additional constant, viz.,

$$\kappa' = -\frac{1}{V} \left(\frac{\partial V_\phi}{\partial p_d} \right)_{p_F} \quad (4d)$$

is involved. We will show, however, that on account of the reciprocity theorem in the theory of elasticity, one has

$$\kappa' = \kappa_A - \kappa_M. \quad (5)$$

The proof is given in the Appendix. Equation (5) can also be derived from a thermodynamic argument similar to one used by Biot (1972). We will deal with the thermodynamic aspects of the theory in a separate publication. Biot (1973) also distinguished κ_ϕ (his C_m) from κ_M (his C_s).

COMPRESSIBILITY, DERIVATION OF RESULT

If we apply an incremental external pressure δp , the confined fluid will undergo a change of pressure which we denote by δp_F . The change in volume δV due to these two pressure changes can also be obtained by adding the change in V resulting from an increase in differential pressure by the amount $\delta p_d = \delta p - \delta p_F$ with the pore pressure constant to that resulting from the

application of a pressure δp_F inside and outside, i.e., with the differential pressure constant:

$$\delta V = \left(\frac{\partial V}{\partial p_d} \right)_{p_F} \delta p_d + \left(\frac{\partial V}{\partial p_F} \right)_{p_d} \delta p_F. \quad (6)$$

Using (4) and (1), we can write equation (6) as follows:

$$\kappa^* \delta p = \kappa_A (\delta p - \delta p_F) + \kappa_M \delta p_F. \quad (7)$$

This can also be written in the form

$$(\kappa_A - \kappa^*) \delta p = (\kappa_A - \kappa_M) \delta p_F. \quad (8)$$

A second relation between δp and δp_F is obtained by requiring that the amount of fluid is conserved upon compressing it with pressure δp_F :

$$\delta V_\phi = -\kappa_F \delta p_F V_\phi. \quad (9)$$

For δV_ϕ we have, by a derivation analogous to that of equation (6),

$$\delta V_\phi = \left(\frac{\partial V_\phi}{\partial p_d} \right)_{p_F} \delta p_d + \left(\frac{\partial V_\phi}{\partial p_F} \right)_{p_d} \delta p_F \quad (10)$$

or, from (4d) and (4c),

$$-\frac{1}{V} \delta V_\phi = \kappa' \delta p_d + \phi \kappa_\phi \delta p_F. \quad (11)$$

Combining equations (9) and (11) and substituting $\delta p_d = \delta p - \delta p_F$, one obtains

$$\kappa' \delta p = [(\kappa_F - \kappa_\phi) \phi + \kappa'] \delta p_F. \quad (12)$$

Eliminating δp and δp_F from equations (8) and (12) gives

$$\begin{aligned} \kappa_A - \kappa^* \\ = (\kappa_A - \kappa_M) \kappa' / [(\kappa_F - \kappa_\phi) \phi + \kappa']. \end{aligned} \quad (13)$$

Substitution of κ' from equation (5) gives the announced result, equation (2). Equation (3) is obtained from equation (2) by writing the left-hand side in the form $(\kappa_A - \kappa_M) - (\kappa^* - \kappa_M)$, bringing the first term to the right-hand side, and inverting.

TENSOR THEORY

We now consider external forces more general than a pressure. For our purpose we consider the case in which a uniform stress exists at the external surface Σ . This is defined in terms of a tension (force per unit surface area) $\boldsymbol{\tau}(\hat{x})$, with components τ_1 , τ_2 , and τ_3 , of the special form

$$\tau_i(\hat{x}) = \sigma_{ij} n_j(\hat{x}). \quad (14)$$

The circumflex indicates that x is a point of Σ . Here σ_{ij} is a constant symmetric tensor, and $n_1(\hat{x})$, $n_2(\hat{x})$, and $n_3(\hat{x})$ are the components of the outward normal to the surface. In equation (14) and following, *summation over repeated indices is implied*.

The boundary condition (14) produces a stress field $\sigma_{ij}^V(x)$ inside the body which would have the value σ_{ij} at all points of Σ if the body were homogeneous and which, in any case, has

$$\sigma_{ij}^V(\hat{x}) n_j(\hat{x}) = \sigma_{ij} n_j(\hat{x}).$$

We will call σ_{ij} the boundary stress.

We wish to study the effect of small \hat{x} -independent changes in σ_{ij} . To this end it is convenient to imagine that the state with boundary stress σ_{ij} was obtained by elastic deformation from some prior state, defined by an initial boundary stress σ_{ij}^0 . We denote by $u_i(\hat{x})$ the displacement that a point \hat{x} of the final surface Σ has undergone as the boundary stress changed from σ_{ij}^0 to σ_{ij} . We stipulate that the difference between σ_{ij}^0 and σ_{ij} is small, so that u_i is small compared with the linear dimensions of the system and the η_{ij} defined below are small compared with V . From the detailed change of shape given by $u_i(\hat{x})$, we now construct the "overall" change of size and shape by defining a symmetric tensor with components

$$\eta_{ij} = \int_{\Sigma} \frac{1}{2} (u_i n_j + u_j n_i) d^2 \hat{x}. \quad (15)$$

To get an idea of the physical meaning of η_{ij} , note that if we were to substitute for our sample any solid elastic body whatever that has the same outside shape and size, the same surface displacements, and is continuous so that displacements are defined everywhere inside as well as on the outer surface, this surface integral could be transformed by Gauss' theorem, giving

$$\eta_{ij} = \int_V \frac{1}{2} (\partial_i u_j + \partial_j u_i) d^3 x. \quad (16)$$

The integrand is the strain; therefore, η_{ij}/V would be the average over V of the strain produced by changing the boundary stress from σ_{ij}^0 to σ_{ij} .

It follows immediately from the definition (15) and the smallness of u_i that the variation of

the trace of η_{ij} corresponding to infinitesimal variations δu_i of the displacements u_i is equal to the variation of the volume:

$$\delta \eta_{ii} = \int_{\Sigma} \delta u_i n_i d^2 \hat{x} = \delta V. \quad (17)$$

η_{ij} is a function of σ_{kl} . Defining

$$S_{ijkl}^* = \frac{1}{V} \left(\frac{\partial \eta_{ij}}{\partial \sigma_{kl}} \right), \quad (18)$$

we call s^* the effective compliance tensor of the piece of matter under consideration. When our definition is applied to a completely homogeneous piece of material with compliance tensor s , s^* equals s . The effective compressibility κ^* , defined in the second section, is related to s^* by

$$\kappa^* = S_{iikk}^* \quad (\text{sum over } i \text{ and } k). \quad (19)$$

We obtain this from (15), (17), and (18) by taking $\sigma_{kl} = -p \delta_{kl}$, where δ_{kl} is the Kronecker delta.

We now evaluate the change in η_{ij} caused by a small increment $\delta \sigma_{ij}$ of the boundary stress. An equation analogous to equation (8) is obtained by applying $\delta \sigma_{ij}$ in two stages. If δp_F is the increase in pore pressure resulting from the application of $\delta \sigma_{ij}$, we first apply an external differential stress

$$\delta \sigma_{ij}^d = \delta \sigma_{ij} + \delta p_F \delta_{ij}, \quad (20)$$

leaving the pore pressure constant. Subsequently we apply a pressure δp_F (a stress, $-\delta p_F \delta_{ij}$) outside and inside, i.e., on the outer surface Σ and on the pore surface Σ_p . The resulting change in the value of the tensor η_{ij} is the sum of two terms, obtained in the two stages. We define, in analogy with (4a), a tensor with components S_{ijkl}^A and, in analogy with (4b), a quantity S_{ij}^M :

$$S_{ijkl}^A = \frac{1}{V} \left(\frac{\partial \eta_{ij}}{\partial \sigma_{kl}^d} \right)_{p_F}, \quad (21a)$$

$$S_{ij}^M = - \frac{1}{V} \left(\frac{\partial \eta_{ij}}{\partial p_F} \right)_{\sigma_{kl}^d}. \quad (21b)$$

From these we obtain, using the definition (18),

$$\begin{aligned} S_{ijkl}^* \delta \sigma_{kl} &= S_{ijkl}^A (\delta \sigma_{kl} + \delta p_F \delta_{kl}) - S_{ij}^M \delta p_F \quad (22) \end{aligned}$$

or

$$(S_{ijkl}^* - S_{ijkl}^A) \delta \sigma_{kl} = (S_{ij}^A - S_{ij}^M) \delta p_F. \quad (23)$$

Here we define S_{ii}^A by

$$S_{ii}^A = S_{ijkk}^A = S_{ijkl}^A \delta_{kl}. \quad (24)$$

It follows from equations (21a) and (17) that $S_{ii}^A \delta \sigma_{ii}$ is the fractional change in V due to the boundary stress $\delta \sigma_{ii}$ with constant p_F .

Equation (23) is a generalization of equation (8), which is obtained with $\delta \sigma_{kl} = -\delta p \delta_{kl}$ by taking $i = j$ and summing. From (4b), (17), and (21b), one sees that

$$S_{ii}^M = \kappa_M, \quad S_{ijkk}^A = S_{ii}^A = \kappa_A, \quad (25)$$

as in equation (19).

To find the consequences of the conservation of fluid, we write in analogy with equation (10):

$$\delta V_\phi = \left(\frac{\partial V_\phi}{\partial \sigma_{ij}^d} \right)_{p_F} \delta \sigma_{ij}^d + \left(\frac{\partial V_\phi}{\partial p_F} \right)_{\sigma_{ij}^d} \delta p_F. \quad (26)$$

Defining

$$S'_{ij} = \frac{1}{V} \left(\frac{\partial V_\phi}{\partial \sigma_{ij}^d} \right)_{p_F} \quad (21d)$$

and, in analogy with (4c),

$$\kappa_\phi = - \frac{1}{V_\phi} \left(\frac{\partial V_\phi}{\partial p_F} \right)_{\sigma_{ij}^d}, \quad (21c)$$

we write equation (26) in the form

$$- \frac{1}{V} \delta V_\phi = - S'_{ij} \delta \sigma_{ij}^d + \phi \kappa_\phi \delta p_F. \quad (27)$$

Combining this with equation (9) and substituting equation (20) gives the desired generalization of equation (12), viz.,

$$- S'_{ij} \delta \sigma_{ij} = [(\kappa_F - \kappa_\phi)\phi + \kappa'] \delta p_F, \quad (28)$$

where

$$\kappa' = S'_{ii}. \quad (29)$$

Eliminating $\delta \sigma_{ij}$ and δp_F from (28) and (23), we obtain

$$S_{ijkl}^A - S_{ijkl}^{*A} = (S_{ii}^A - S_{ii}^M) S'_{kl} / [(\kappa_F - \kappa_\phi)\phi + \kappa'], \quad (30)$$

which is completely analogous to equation (13) and yields equation (13) if we set $i = j$, $k = l$ and sum.

As before, the reciprocity theorem leads to a

great simplification. As is shown in the Appendix, one has

$$S'_{kl} = S_{kl}^A - S_{kl}^M. \quad (31)$$

The final result thus takes the form

$$S_{ijkl}^A - S_{ijkl}^{*A} = (S_{ii}^A - S_{ii}^M) (S_{kl}^A - S_{kl}^M) / [(\kappa_F - \kappa_\phi)\phi + (\kappa_A - \kappa_M)], \quad (32)$$

which is the desired extension of equation (2). Equation (3) can also be extended to tensor form, but this is not particularly useful.

DISCUSSION

Note that we have defined the various macroscopic parameters of the system without the requirement of macroscopic homogeneity or isotropy. The principal restrictions on the validity of equation (32) are that the pore system be completely connected and the time scale of a measurement long enough for pressure equalization and short enough that nonelastic deformations are not important. We also require that variations in the stresses and strains being considered be small enough to insure linearity of the system.

Rock samples with various degrees of macroscopic homogeneity and isotropy give various degrees of simplification of equation (32). We first discuss symmetry under reflections and rotations of the elastic properties of a given piece of rock, without assuming homogeneity.

If the sample has a threefold, fourfold, or sixfold axis of symmetry, or if it has two or three perpendicular planes of symmetry, or if it belongs to the rhombic, trigonal, tetragonal, hexagonal or cubic syngony, or if it is isotropic, then $S_{ijkl}^{*A} = S_{ijkl}^A$ for all elements with $i \neq j$ and/or $k \neq l$. This is due to the fact (Lekhnitskii, 1963) that in these cases $S_{ijkk}^A = S_{ijkk}^M = 0$ ($i \neq j$, sum over k) which, when used in equation (32), immediately demonstrates the above statement. With the exception of the cases of axes of threefold or fourfold symmetry and of trigonal or tetragonal syngony, these sums over k are zero because all terms (S_{ijkl} , with $i \neq j$ and $k = l$) are separately zero. For the exceptional cases, the sums are zero because the nonzero terms cancel.

Similar statements can be made concerning materials, provided that they are macrohomog-

eneous. They apply to a sufficiently large piece of such material or to the average properties of an ensemble of samples cut at random from a larger piece.

Of all these cases, axial symmetry (which for elastic properties is indistinguishable from the existence of a sixfold axis) and isotropy are the most likely to apply to rock samples. In the case of isotropy, S_{iijj} ($i \neq j$; no sum) = $2/\mu$, where μ is the shear modulus. Thus, we have $\mu^* = \mu^A$, which means that the shear modulus does not depend on the compressibility of the pore fluid.

In the case of axial symmetry, the velocity of shear waves traveling parallel or perpendicular to the axis of symmetry, with polarization either parallel or perpendicular to the axis of symmetry, is likewise independent of the compressibility of the pore fluid. This is so despite the fact that four of the five linearly independent compliances do, in general, depend on κ_F . These are, taking the three-direction along the axis, S_{1111} ($= S_{2222}$), S_{3333} , S_{1122} , and S_{1133} ($= S_{2233}$). The fifth, namely S_{1313} ($= S_{2323}$), which determines the velocity of shear waves parallel to the axis, regardless of polarization, and perpendicular to the axis with polarization parallel to the axis, is κ_F -independent as shown above. The same holds for the velocity of shear waves traveling perpendicular to the axis with polarization perpendicular to the axis. This velocity is determined by S_{1212} , and even though it is equal to $2(S_{1111} - S_{1122})$ (Lekhnitskii, 1963), it is κ_F -independent according to our argument. The various modes of wave propagation in axially symmetric media are given by Grant and West (1965). Regardless of macroscopic symmetry, we have $\kappa_A = S_{iijj}^A = S_{iij}^A$, $\kappa^* = S_{iijj}^*$, and $\kappa_M = S_{ii}^M$. Using these in equation (32) gives equation (2), with no restriction on microscopic or macroscopic symmetry or homogeneity.

Gassmann considered porous media that were macroscopically homogeneous and microscopically homogeneous and isotropic. The microscopic isotropy would not have been needed for his treatment. In effect Gassmann noted that geometric similarity is maintained in detail if the solid part of a porous medium is microhomogeneous and microisotropic and equal pressure increments are applied to the pore fluid and to the outside of the sample. This gives $\kappa_M = \kappa_\phi$. As we showed above, this would still apply if the material were anisotropic but

uniformly oriented, that is, if the porous medium were carved out of a single crystal, although this is perhaps not a likely situation. Gassmann then referred to a result given by Love (1944) and derived from that result the equivalent of equation (5). If he had noted that in the following paragraph a "vessel" can be a porous medium, he could have taken this result directly from Love. Thus, Love's theorem

$$-\delta V_M = (V \delta p - V_\phi \delta p_F) \kappa_M,$$

which is restricted to a microhomogeneous porous medium (or vessel with homogeneous solid part), is easily seen to be equivalent to

$$\kappa_\phi = \kappa_M = \kappa_A - \kappa'.$$

The main purpose of this paper is to include the case of microscopic heterogeneity, because many porous media are composites of materials with quite varied elastic properties. In this case, Love's theorem does not apply, but the reciprocity theorem, equation (31), permits a result nearly as simple. The only "penalty" for microscopic heterogeneity is the appearance of κ_ϕ instead of κ_M in the first term of the denominator of equation (32).

Natural rocks and porous materials in general come in infinite variety. We will not attempt a general discussion of the effects of composition and texture on the tensors s^A and s^M and on κ_ϕ .

However, we will consider a few special cases of conditions found in some rocks. Consider rocks which are macroscopically homogeneous over some distance scale; by this we mean that the rock can be arbitrarily split into a few comparable sized samples without changing the intensive properties of interest to us enough to matter for our purposes. Furthermore, assume that there is an axis of symmetry for macroscopically anisotropic samples. Let us now consider the effects of the presence or absence of several additional restrictions on the nature of the rock:

(A) Various geometrical features of the pore and framework structures are randomly oriented in direction.

(B) Anisotropic grains are randomly oriented.

(C) A constituent with a markedly different-from-average compressibility is not preferentially in positions in the elastic framework

where this constituent would be subjected to more or less stress than it would in the average positions in the elastic framework.

First, what would we expect to be the properties of s^A and s^M if all three of the above conditions are satisfied? We have by conditions A and B ruled out directional properties, and we would expect s^A and s^M to be isotropic. Condition C is only semiquantitative in form but probably makes κ_M approximately the same as κ_ϕ . Biot (1973) came to the same conclusion. In addition, κ_M should be somewhere between the compliance-average and the modulus-average of the solid components. For rocks subject to conditions A, B, and C, the mean of these averages is probably a reasonable guess, namely the Voigt-Reuss-Hill (Hill, 1952) average. The Hashin-Shtrikman (1963) bounds should apply also.

If we have conditions A and B, but not C, we still have isotropy of s^A and s^M , but we no longer have $\kappa_M = \kappa_\phi$. The Voigt-Reuss or Hashin-Shtrikman bounds probably still apply to a quantity such as

$$\kappa'' = -(1/V_M)(\partial V_M/\partial p_F)_{p_d},$$

where $V_M = V - V_\phi$. This is the compressibility of the solid material under all-sided compression. It follows from the definition that κ'' is related to κ_M and κ_ϕ by

$$\kappa_M = \phi\kappa_\phi + (1 - \phi)\kappa'',$$

and we have $\kappa_M = \kappa_\phi = \kappa''$ if the solid is homogeneous. These bounds for κ'' do not apply to κ_M . We can see this easily by noting that we could have a single one of many solid components of a porous material form the stress-bearing solid framework, while the rest are merely attached here and there and just "go along for the ride". We can have values of κ_M anywhere between the extremes of those of the components. If one wishes to make estimates of κ_M for some particular model of a porous material, a reasonable approach might be to weight the contribution of a solid component according to the product of the amount present and a factor related to its participation in the stress-bearing framework, then compute a VRH average. For computing κ'' , we could compute a VRH average without the "participation" weighting. If the nonparticipating part is in the form of material on pore walls, it might be appropriate to compute a

compliance as the weighted average of that of the nonparticipating part and the VRH compliance of the participating part. If the nonparticipating part is relatively soft, we should anticipate $\kappa'' > \kappa_M$. Note that, although cements in the vicinities of grain contacts can drastically affect s^A , they do not necessarily contribute greatly to s^M , and the cement is not necessarily an underparticipating or overparticipating component.

If we have condition A but not B, both s^A and s^M could be anisotropic and would have the same axis. If we have condition B but not A, we would probably have s^A anisotropic, but have s^M still isotropic.

From the fact that an increase in pore pressure causes the total volume to increase, we have

$$\begin{aligned} (\partial V/\partial p_F)_p &= -(\partial V/\partial p_d)_{p_F} + (\partial V/\partial p_F)_{p_d} \\ &= V(\kappa_A - \kappa_M) > 0, \quad \text{i.e., } \kappa_A > \kappa_M. \end{aligned}$$

If we can make the stronger assumption that the porosity decreases with increased differential pressure, we have a lower bound to κ_A that is somewhat larger. We have

$$V_\phi = \phi V,$$

and

$$(\partial V_\phi/\partial p_d)_{p_F} = \phi(\partial V/\partial p_d)_{p_F} + V(\partial\phi/\partial p_d)_{p_F}.$$

If the last term is negative, we have

$$\kappa_A > \kappa_M/(1 - \phi).$$

This assumption excludes the situation where spherical pores are "protected" by ultrastiff rock components.

A common source of macroscopic anisotropy is a laminated structure, on a scale from millimeters to meters. If we are interested in seismic waves of wavelengths of hundreds of meters, a laminated formation may still be effectively homogeneous. However, whether or not the individual laminae are isotropic, the resultant large-scale formation will usually be anisotropic. If the laminae are parallel, there should be a likelihood of axial symmetry. The large-scale elastic constants can be calculated from the distribution and small-scale constants of the laminae.

If our assumption of pressure equalization throughout the pore system is satisfied, we do not require that the pore fluid be a single phase.

If there are several fluids, such as brine, oil, and gas, the effective fluid compressibility is the volume-weighted average of the compressibilities of the components. If gas is one of the phases, its very high compressibility is likely to dominate, even if only a small amount is present.

If a small amount of gas is present and is not uniformly distributed, there may not be sufficient time for flow of liquid through the various channels to compress the gas when pressure is applied to a sample. This should be more of a problem at high frequencies, such as those employed in laboratory measurements, than at exploration seismic frequencies.

For clay present in a sandstone, the fluid in the clay may be slow to migrate into larger pores to compress gas bubbles when a sample is subjected to pressure. At high frequencies it may thus be necessary to regard fluid-containing clay as a part of the framework of the rock. At this point it should be noted that our definition of κ_A is the sample compressibility with pore-fluid pressure held constant. We could regard this as the same as having an infinitely compressible fluid in the pores or as having the pores empty. However, the removal of all fluid from some clay components may well change the nature of the solid framework, giving values of κ_A not valid for the fluid-filled rock.

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APPENDIX

Equation (5) and, more generally, equation (31) are consequences of the reciprocity theorem of elasticity. To state this theorem, we consider any piece of an elastic solid, subject to surface tensions $\tau_i(\hat{x})$ acting on the surface Σ . A small increment $\delta\tau_i(\hat{x})$ of these tensions causes small displacements of the points of Σ ; we denote the displacements by $\delta u_i(\hat{x})$. The reciprocity theorem relates the displacements δu_i , caused by $\delta\tau_i$, to displacements $\delta'u_i$, caused by a different increment $\delta'\tau_i$, as follows:

$$I \equiv \int_{\Sigma} [\delta u_i(\hat{x}) \delta'\tau_i(\hat{x}) - \delta'u_i(\hat{x}) \delta\tau_i(\hat{x})] d^2\hat{x} = 0, \quad (A-1)$$

subject to the condition that Hooke's law applies. For a proof we refer to Love (1944).

In order to derive equation (5), we apply (A-1) to the solid part of our porous sample, taking for $\delta\tau_i$ and $\delta'\tau_i$ small increments of the existing stress. Assume for the moment that there is only one pore, that is, that all pore space is adequately connected for pressure equalization; the total surface is given by $\Sigma + \Sigma_\phi$. We let the unprimed quantities correspond to a change of p with constant p_F ; the primed quantities correspond to a change of p_F with constant p . In other words,

$$\begin{aligned} \delta\tau_i &= -\delta p n_i \quad \text{on } \Sigma, \\ \delta\tau_i &= 0 \quad \text{on } \Sigma_\phi, \end{aligned} \quad (A-2)$$

and

$$\begin{aligned} \delta'\tau_i &= 0 \quad \text{on } \Sigma \\ \delta'\tau_i &= -\delta p_F n_i \quad \text{on } \Sigma_\phi. \end{aligned} \quad (A-3)$$

Using (A-2) and (A-3) in (A-1) gives

$$\begin{aligned} \delta p_F \int_{\Sigma_\phi} \delta u_i n_i d^2\hat{x} \\ = \delta p \int_{\Sigma} \delta'u_i n_i d^2\hat{x}. \end{aligned} \quad (A-4)$$

On Σ_ϕ , n_i is directed into the pore volume; so the first integral is the negative of the change in V_ϕ for the unprimed displacement, caused by an application of δp with constant p_F , namely $-(\partial V_\phi/\partial p)_{p_F} \delta p$. On Σ , n_i is directed outward; so the second integral is the change of V for constant p , namely $(\partial V/\partial p_F)_p \delta p_F$. Therefore, one obtains the identity

$$-\left(\frac{\partial V_\phi}{\partial p}\right)_{p_F} = \left(\frac{\partial V}{\partial p_F}\right)_p. \quad (A-5)$$

From this, equation (5) follows if we change from the independent variables p_F and p to p_F and $p_d = p - p_F$. One has

$$\left(\frac{\partial V}{\partial p_F}\right)_p = \left(\frac{\partial V}{\partial p_F}\right)_{p_d} - \left(\frac{\partial V}{\partial p_d}\right)_{p_F}. \quad (A-6)$$

Substituting in (A-5) and using equations (4a), (4b), and (4d) yields equation (5).

To obtain equation (31), one must replace (A-2) with

$$\begin{aligned} \delta \tau_i &= \delta \sigma_{ij} n_j \quad \text{on } \Sigma, \\ \delta \tau_i &= 0 \quad \text{on } \Sigma_\phi. \end{aligned} \quad (A-7)$$

This gives, instead of (A-4),

$$\begin{aligned} \delta p_F \int_{\Sigma_\phi} \delta u_i n_i d^2 \hat{x} \\ = -\delta \sigma_{ij} \int_{\Sigma} \delta' u_i n_j d^2 \hat{x}. \end{aligned} \quad (A-8)$$

Because of the smallness of u_i , the integral on the left is

$$-(\partial V_\phi/\partial \sigma_{ij})_{p_F} \delta \sigma_{ij},$$

while the integral on the right is, from equation (15),

$$(\partial \eta_{ij}/\partial p_F) \sigma_{kl} \delta p_F.$$

Since this holds for any $\delta \sigma_{ij}$, we have

$$\left(\frac{\partial V_\phi}{\partial \sigma_{ij}}\right)_{p_F} = \left(\frac{\partial \eta_{ij}}{\partial p_F}\right)_{\sigma_{kl}}. \quad (A-9)$$

Equation (31) follows by change of variables from p_F and σ_{ij} to p_F and $\sigma_{ij}^d = \sigma_{ij} + p_F \delta_{ij}$.

If there are other pores not connected to our "pore" (or connected pore system), then we may regard the unconnected pores and their fluid filling as a part of the framework, that is, as part of the volume V_M . To show this, we extend the surface integral I in equation (A-1) over the surfaces Σ_α of the unconnected pores. Now, the total surface is given by

$$\Sigma + \Sigma_\phi + \sum \Sigma_\alpha.$$

If δp_α and δV_α are the increments of fluid pressure and volume of a pore α for the boundary conditions (A-2), and similarly $\delta p'_\alpha$ and $\delta V'_\alpha$ for (A-3), (A-4) acquires extra terms, $\Sigma_\alpha \delta p'_\alpha \delta V_\alpha$ on the left and $\Sigma_\alpha \delta p_\alpha \delta V'$ on the right. But if κ_α is the compressibility of the fluid in pore α , one has

$$\delta V_\alpha = -\kappa_\alpha \delta p_\alpha, \quad \delta V'_\alpha = -\kappa_\alpha \delta p'_\alpha,$$

so that these contributions cancel. This justifies our regarding isolated pores as part of V_M .