GEOPHYSICS, VOL. 40, NO. 2 (APRIL 1975), P. 224-232; 2 FIGS., 1 TABLE

COMPUTED SEISMIC SPEEDS AND ATTENUATION IN ROCKS WITH PARTIAL GAS SATURATION

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Calculations for an unconsolidated sand with partial gas saturation show a 20 percent increase in compressional wave velocity between 1 and 100 hz and attenuation of 27 db/1000 ft at 31 hz and 82 db/1000 ft at 123 hz.

Shear velocity and attenuation are not affected. Fluid-flow waves are shown to be responsible for the dispersion and attenuation at low frequencies; relations are derived by extending Gassmann's viewpoint to include coupling between fluid-flow waves and seismic body waves. This appears to be an important loss mechanism for heterogeneous porous rocks.

GENERAL FEATURES

Interest in partial saturation

At the gas-oil or gas-water contact in a homogeneous reservoir rock, capillary pressure is responsible for a transition zone in which the gas saturation varies through a wide range. When the reservoir rock is not homogeneous, it seems plausible that gas saturation may vary accordingly. Shale stringers may seal off local pockets of gas creating a multitude of gas-liquid contacts. During production of a field, gas may come out of solution and create distributed pockets of free gas. In view of these possibilities, a study of seismic wave propagation in porous rocks with mixed fluid saturation may well have practical application.

A mechanism and a model

Through the work of Frenkel (1944) and Biot (1956) and others, there is a good theoretical framework for treating seismic waves in porous rocks. The full expressions are complex, and for anything but the simplest geometry, their application is a formidable task. Gassmann (1951) derived elastic constants for porous rocks by straightforward application of static elasticity, obtaining speeds which agree with Biot's low-frequency values. White (1965, p. 133) extended Gassmann's approach to calculate the effect of fluid flow on shear-wave attenuation, obtaining the same expression as Biot's low-frequency attenuation. The present paper can be looked at as a further application of the low-frequency analysis to show how pressure differences at a boundary may cause substantial fluid flow and how the shifting fluid couples back into the seismic wave to a significant degree. The approach will be described qualitatively, with brief derivations to be found in the Appendix.

As a compressional wave travels through a porous rock, pressure gradients in the fluid cause some flow relative to the rock skeleton and hence some loss of energy. Within a single homogeneous rock, these gradients are small and attenuation due to fluid flow is negligible at prospecting frequencies. If the rock has mixed saturation, with segregated pockets of gas, for instance, then pressure gradients are high near the inhomogeneities and fluid flow will be high

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also. The loss of energy at these local spots may be averaged over the total volume and expressed as a large attenuation for the compressional wave.

For consideration of this loss mechanism mathematically, some idealized geometry must be adopted. Spherical gas pockets located at the corners of a cubic array are shown in Figure 1. Perhaps it should be repeated that the skeleton is uniform, and the pockets are spherical volumes saturated with gas in a rock which is elsewhere saturated with liquid. At low frequencies, an elementary cube with its enclosed sphere of gas is a typical volume, with average properties which are the same as the average properties of the composite medium. Even for computing average bulk modulus, however, the combination of a cube and a sphere is complicated. Hence the typical volume is considered to be equivalent to the concentric spheres shown in the lower part of the figure, where the volume of the outer sphere is the same as the volume of the original cube. A specified fractional volume change is impressed, at a low frequency, on the outer surface, and the resulting pressure amplitude at the surface is computed; the effects of fluid flow in the spherical shell are included. The gas pocket is assumed to provide a complete pressure release. The ratio of pressure amplitude to the fractional volume change yields the complex bulk modulus. With known shear modulus and easily computed density, the complex bulk modulus yields the speed and attenuation of compressional waves.

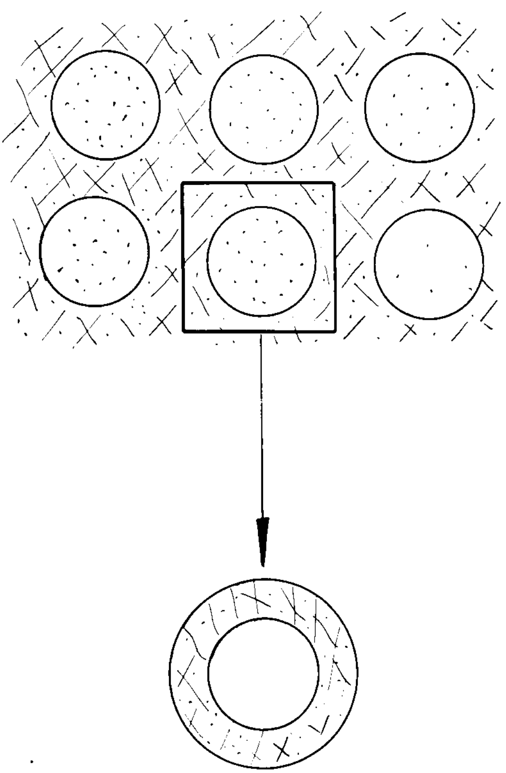

FIG. 1. Model of porous rock with mixed saturation—spherical pockets saturated with gas, intervening volume saturated with liquid. The typical volume considered in the calculations is the pair of concentric spheres shown in the lower part of the figure.

Resulting average properties

It is easy to see that this model exhibits a relaxation phenomenon. There is a characteristic time, dependent on the size of the gas pockets and the distance between them. When the period of oscillation is long compared to this time, the fluid pressure is zero and the bulk modulus is independent of frequency. When the oscillation period is short, the bulk modulus is substantially the value computed as if there were no fluid flow, but with a phase angle which is decreasing with frequency. In the intermediate frequency range, as the modulus varies between these two limits, the medium is characterized by both dispersion and attenuation.

These features are illustrated by the numerical example, for an unconsolidated sandstone at a fairly shallow depth. The dispersion is obvious at a glance. The attenuation is drastic (27 db/1000 ft at 31 hz), perhaps 30 times greater than normal in the frequency range 20 to 50 hz. The round trip through a few hundred feet of this composite medium would just about wipe out a deep reflection.

The rock for this example was chosen to emphasize the effect of fluid content. More competent rocks at greater depths must have the same general features, to a lesser degree. In the example shown, size and spacing of the gas pockets were chosen to place the transition zone in the seismic prospecting band of frequencies. Presumably, random combinations of sizes and spacings would give a blurred-out transition zone which could span this frequency band. At any rate, the results shown here support the suggestion that fluid flow is indeed an important source of attenuation for compressional waves in rocks saturated with a mixture of liquid and gas. Attenuation of shear waves is not affected by the partial saturation.

MATHEMATICAL DERIVATION

Static stress with no fluid flow

We are dealing with such low frequencies that our small elementary volume is in equilibrium locally, and we can, therefore, use the equations of static elasticity. We apply them to two concentric spheres, the inner of radius a and the outer of radius b.

On the outer surface we impress a displacement $u_r(b) = U_0 e^{i\omega t}$, which creates a pressure on the outside of $P_0 e^{i\omega t}$, equal to $-p_{rr}(b)$. A list of symbols and their definitions is given. We express the displacement as a fractional change in volume

$$D_0 e^{i\omega t} = \frac{4\pi b^2}{(4/3)\pi b^3} U_0 e^{i\omega t},$$

$$\text{or} \quad D_0 = 3U_0/b. \tag{1}$$

The bulk modulus is the ratio of $-P_0$ *to* D_0,

$$k_0 = -P_0/D_0. \tag{2}$$

This is the average bulk modulus of the two concentric spherical bodies with no fluid flow. The expression for k_0 in terms of the elastic constants for the two media is derived in Appendix A.

For this composite medium with no fluid flow the elastic constants and velocities are

$$M_0 = k_0 + 4\mu_0/3,$$

$$\rho_0 = S_G\rho_1 + (1 - S_G)\rho_2, \quad \text{and} \tag{3}$$

$$C_{P0} = (M_0/\rho_0)^{1/2}.$$

The subscripts 0, 1, and 2 refer to the average properties of the concentric spheres, the inner sphere, and the concentric shell, respectively. Since we have the same skeleton in the entire volume and since fluid content does not affect rigidity,

$$\mu_0 = \mu_1 = \mu_2 = \bar{\mu}, \quad \text{and}$$

$$C_{S0} = (\mu_0/\rho_0)^{1/2}. \tag{4}$$

LIST OF SYMBOLS

- a_P^*, a_S^*: attenuation of compressional and shear waves
- a: radius of inner sphere
- b: radius of outer sphere
- C_P, $\bar{C}_P$, C_P^*: compressional-wave speed
- C_S, $\bar{C}_S$, C_S^*: shear-wave speed
- D_0: dilatation amplitude
- E, E_1, E_2: Young's modulus
- k, k_S, $\bar{k}$, k^*, k_L: bulk modulus
- k_A, k_E: "bulk moduli", as defined
- M, $\bar{M}$, M_0, M^*: plane-wave modulus
- P_i, P_0: complex pressure amplitude
- p_{rr}: normal stress
- p_f: fluid pressure
- q: frequency parameter, $(\omega/\omega_0)^{\frac{1}{2}}$
- q': frequency parameter, $(1 - a/b)q$
- r: radial spherical coordinate
- S_G: fractional gas saturation, (a^3/b^3)
- t: time
- U_0: displacement amplitude
- u_r: radial displacement
- v: particle velocity in fluid relative to skeleton
- x, z: complex variables
- Z: acoustic impedance
- α: complex constant, $(i\omega\eta/k_E\kappa)^{\frac{1}{2}}$
- η: viscosity
- ϕ: porosity
- κ: permeability
- μ, $\bar{\mu}$, μ_1, μ_2, μ_0: shear modulus
- ρ, $\bar{\rho}$, ρ_0, ρ_s, ρ_f, ρ^*: density
- σ, σ_1, σ_2: Poisson's ratio
- θ_P^*: phase angle
- ω: angular frequency
- ω_0: reference frequency, $(2\kappa k_E/\eta b^2)$
- γ: complex propagation constant

Effect of fluid flow

The static stresses discussed above create a fractional volume change which is constant within the inner sphere and a different fractional volume change which is constant within the outer shell. Thus there is no tendency for fluid flow within either body when the boundary between the two is sealed. However, the values of fluid pressure (P_1 and P_2) created within the two bodies may be different; each is proportional to the impressed pressure P_0.

$$P_1 = R_1 P_0. \qquad P_2 = R_2 P_0. \tag{5}$$

The constants of proportionality R_1 and R_2 are derived in Appendix B. The pressure difference causes a fluid flow across the boundary; the flow velocity is given by

$$v = \frac{(P_1 - P_2)}{(Z_1 + Z_2)} = \frac{P_0(R_1 - R_2)}{(Z_1 + Z_2)}. \tag{6}$$

The acoustic impedances Z_1 and Z_2 are derived in Appendix C. They are the impedances for fluid-flow waves (or diffusion waves) looking inward and outward, respectively, from the boundary. As fluid flows out of medium 1, for instance, "unloading" occurs and the volume occupied by the saturated rock decreases. Similarly, the (equal) flow of fluid into medium 2 causes it to expand. However, the amount of rock expansion for a given fluid volume is different in the two media, as derived in Appendix D. The constants of proportionality are Q_1 and Q_2.

Since we have chosen to impress a dilatation D_0 on the outer sphere and compute the resulting pressure, we are in effect using an infinite-impedance source. It is appropriate, therefore, to compute the change in pressure ΔP_0 on the surface of a rigidly held sphere when a small change in volume is introduced by expansion of a thin spherical shell. This pressure change is independent of the radius of the thin shell and is simply proportional to fractional volume change;

$$\Delta P_0 = k^* D. \tag{7}$$

We shall use the asterisk to indicate parameters of the case which includes fluid flow. Since the pressure change is independent of the radial position of the fluid which causes the expansion (or contraction) of each spherical shell, we only need to know the total fluid volume injected (or withdrawn) through the boundary. This volume is $(4\pi a^2 v/i\omega)$, leading to a fractional volume change in the medium of

$$D = (4\pi a^2 v/i\omega)(-Q_1 + Q_2)/(4\pi b^3/3). \tag{8}$$

Combining these expressions with equation (6) we obtain

$$\Delta P_0 = k^* \frac{(R_1 - R_2)(4\pi a^2)(-Q_1 + Q_2)P_0}{(Z_1 + Z_2)(i\omega)(4\pi b^3/3)},$$

or

$$\Delta P_0 = k^* W P_0, \tag{9}$$

where

$$W = \frac{3a^2(R_1 - R_2)(-Q_1 + Q_2)}{b^3 i\omega(Z_1 + Z_2)}.$$

Without flow,

$$P_0 = -k_0 D_0.$$

With flow,

$$P_0 + \Delta P_0 = -k^* D_0, \quad \text{or}$$

$$P_0 + k^* W P_0 = -k^* D_0, \quad \text{with} \tag{10}$$

$$k^* = [k_0/(1 - k_0 W)] = k_r^* + i k_i^*.$$

Since fluid content does not affect shear distortions, the shear modulus for the partial saturation is

$$\mu^* = \mu_0 = \mu = \bar{\mu}. \tag{11}$$

Density does not depend on fluid flow, so

$$\rho^* = \rho_0 = (1 - \phi)\rho_S + \phi(1 - S_G)\rho_f. \tag{12}$$

Plane-wave modulus is

$$M^* = (M_r^* + iM_i^*) = (k_r^* + 4\mu^*/3 + ik_i). \tag{13}$$

Speeds and attenuation

For a plane compressional wave in this composite medium, we can write the wave equation

$$M^* \frac{\partial^2 u}{\partial x^2} = \rho^* \frac{\partial^2 u}{\partial t^2}$$

and its solution

$$u = U_0 e^{-\gamma x} e^{i\omega t}, \quad \text{where} \qquad \gamma = a_P^* + i\omega/C_P^* = i\omega(\rho^*/M^*)^{1/2}. \tag{14}$$

The velocity and attenuation are

$$C_P^* = (|M^*|/\rho^*)^{1/2}/\cos(\theta_P^*/2), \quad \text{and}$$

$$a_P^* = \omega \tan(\theta_P^*/2)/C_P^*, \quad \text{where}$$

$$\theta_P^* = \tan^{-1}(M_i^*/M_r^*).$$

There is no attenuation of the shear wave due to partial saturation. We write for its velocity and attenuation

$$C_S^* = (\mu^*/\rho^*)^{1/2} \quad \text{and} \qquad a_S^* = 0. \tag{15}$$

Partial gas saturation

It seems clear that the effect of fluid flow will be most pronounced when one medium is saturated with a very compressible fluid. We shall assume that the central sphere is saturated with a gas so light and compressible that R_1, Q_1, and Z_1 can all be neglected. In this case, equation (10) simplifies to

$$k^* = \frac{k_0}{1 + 3a^2R_2Q_2k_0/b^3i\omega Z_2}. \tag{16}$$

Defining $(\text{Re} + i\ \text{Im}) = (2ak_{E2}/b^2i\omega Z_2)$, we can write

$$k^* = \frac{k_0}{1 + (3k_0R_2Q_2/2k_{E2})(a/b)(\text{Re} + i\ \text{Im})}. \tag{17}$$

To simplify the expression for calculation, the following additional definitions are made:

$$\omega_0 = (2\kappa_2 k_{E2}/\eta_2 b^2);$$

$$\alpha_2 b = (1 + i)q \quad \text{and} \quad \alpha_2(b - a) = (1 + i)q';$$

$$q = (\omega/\omega_0)^{1/2}.$$

As a function of dimensionless frequency q,

$$(\text{Re} + i\ \text{Im}) = -\frac{i}{q^2}\left[\frac{(1 + q' + iq' - i2q^2a/b) - (1 - q' - iq' - i2q^2a/b)e^{2q'}(\cos 2q' + i \sin 2q')}{(1 + q + iq) - (1 - q - iq)e^{2q'}(\cos 2q' + i \sin 2q')}\right]. \tag{18}$$

Let us consider the limiting cases of zero gas saturation ($a/b \rightarrow 0$) and full gas saturation ($a/b \rightarrow 1.0$). In the first case, ($\text{Re} + i\text{Im}$) is not zero but it is multiplied by a/b in equation (17). Hence $k^* = k_0$, which is the bulk modulus for liquid-saturated rock. As a/b approaches unity, ($\text{Re} + i\text{Im}$) approaches zero, and again $k^* = k_0$. However, k_0 is now the bulk modulus for gas-saturated rock. Hence equation (17) gives the correct values for the bulk modulus at two limiting saturations.

At "high" frequencies (for which the wavelength of fluid-flow waves is small compared to sphere radius), ($\text{Re} + i\text{Im}$) approaches zero and $k^* = k_0$. Although the phase angle θ_P^* decreases with frequency, the attenuation (per unit distance) increases as the square root of frequency. At very low frequencies, ($\text{Re} + i\text{Im}$) approaches $+1$, and k^* approaches a value which depends on a/b. This sort of composite medium is a low-pass filter, with a transition frequency which is proportional to the inverse square of the typical dimension (b) of the inhomogeneity.

NUMERICAL EXAMPLE

Properties of skeleton

We start from measured speeds reported by Gardner et al (1964) for unconsolidated packings of sand under various confining pressures. At a pressure of 2500 psi on the skeleton, an air-filled sand of 30 percent porosity had a compressional speed of 1.5×10^5 cm/sec and a shear speed two-thirds as great. The solid material (quartz) has a density

$$\rho_S = 2.65 \text{ gm/cm}^3$$

and a bulk modulus

$$k_S = 35 \times 10^{10} \text{ dynes/cm}^2.$$

From this information we obtain the skeleton properties:

$$\phi = 0.30,$$

$$\bar{C}_P = 1.5 \times 10^5 \text{ cm/sec},$$

$$\bar{C}_S = 1.0 \times 10^5 \text{ cm/sec},$$

$$\bar{\rho} = (1 - \phi)\rho_S = 1.85 \text{ gm/cm}^3,$$

$$\bar{M} = \bar{\rho}\bar{C}_P^{\,2} = 4.17 \times 10^{10} \text{ dynes/cm}^2,$$

$$\bar{\mu} = \bar{\rho}\bar{C}_S^{\,2} = 1.85 \times 10^{10} \text{ dynes/cm}^2,$$

$$\bar{k} = \bar{M} - 4\bar{\mu}/3 = 1.71 \times 10^{10} \text{ dynes/cm}^2.$$

Properties of water-saturated rock

When the fluid in the pore space is pure water under normal conditions, then the constants of the fluid in region 2 are

$$\rho_{f2} = 1.0 \text{ gm/cm}^3,$$

$$k_{f2} = 2.25 \times 10^{10} \text{ dynes/cm}^2,$$

$$\eta_2 = 0.01 \text{ gm/cm sec (one centipoise)},$$

$$\kappa_2 = 10^{-9} \text{ cm}^2 \text{ (100 millidarcies)}.$$

The constants for the water saturated rock of region 2 are

$$\rho_2 = (1 - \phi)\rho_S + \phi\rho_{f2} = 2.155 \text{ gm/cm}^3,$$

$$k_2 = \bar{k} + \frac{(1 - \bar{k}/k_S)^2}{(\phi/k_{f2} + (1 - \phi)/k_S - \bar{k}/k_S{}^2)}$$

$$= 7.66 \times 10^{10} \text{ dynes/cm}^2,$$

$$\mu_2 = \bar{\mu} = 1.85 \times 10^{10} \text{ dynes/cm}^2,$$

$$M_2 = k_2 + 4\mu_2/3$$

$$= 10.12 \times 10^{10} \text{ dynes/cm}^2,$$

$$C_{P2} = (M_2/\rho_2)^{1/2} = 2.167 \times 10^5 \text{ cm/sec},$$

and

$$C_{S2} = (\mu_2/\rho_2)^{1/2} = 0.927 \times 10^5 \text{ cm/sec}.$$

Mixed saturation with no fluid flow

With $b/a = 2.0$, $a^3/b^3 = S_G = 0.125$ and for mixed saturation with no fluid flow we obtain the rock properties:

$$\sigma_1 = \frac{\bar{M} - 2\bar{\mu}}{2(\bar{M} - \bar{\mu})} = 0.10,$$

$$E_1 = 2\bar{\mu}(1 + \sigma_1) = 4.07 \times 10^{10} \text{ dynes/cm}^2,$$

$$\sigma_2 = \frac{M_2 - 2\mu_2}{2(M_2 - \mu_2)} = 0.39,$$

$$E_2 = 2\mu_2(1 + \sigma_2) = 5.14 \times 10^{10} \text{ dynes/cm}^2,$$

$$K_1 = 1.158/(2.158 - S_G) = 0.570,$$

$$K_2 = (0.240/S_G) + 0.760 = 2.680,$$

$$K_3 = 1.872 \times 10^{10}(1/S_G - 1)$$

$$= 13.11 \times 10^{10} \text{ dynes/cm}^2,$$

(see Appendix A)

$$k_0 = 6.21 \times 10^{10} \text{ dynes/cm}^2,$$

$$M_0 = (k_0 + 4\mu_0/3)$$

$$= 8.68 \times 10^{10} \text{ dynes/cm}^2,$$

$$\rho_0 = S_G\bar{\rho} + (1 - S_G)\rho_2 = 2.11 \text{ gm/cm}^3,$$

$$C_{P0} = (M_0/\rho_0)^{1/2} = 2.03 \times 10^5 \text{ cm/sec},$$

and

$$C_{S0} = (\mu_0/\rho_0)^{1/2} = 0.94 \times 10^5 \text{ cm/sec}.$$

Mixed saturation including fluid flow

For the case of mixed saturation including fluid flow, the pertinent constants are

$$R_2 = 0.867,$$

$$k_{A2} = 6.58 \times 10^{10} \text{ dynes/cm}^2,$$

$$k_{E2} = 2.71 \times 10^{10} \text{ dynes/cm}^2,$$

$$Q_2 = 0.618, \quad \text{and}$$

$$k^* = \frac{6.21 \times 10^{10}}{1 + 1.842(a/b)(\text{Re} + i \text{ Im})}.$$

For $a/b = 0.5$ and $q = (\pi/2)$, we obtain the complex quantities:

$$(\text{Re} + i \text{ Im}) = 0.684 - i0.535,$$

$$k^* = (3.45 + i1.056) \times 10^{10}, \quad \text{and}$$

$$M^* = (5.91 + i1.056) \times 10^{10} \text{ dynes/cm}^2.$$

Also we note that

$$\theta_P{}^* = 10.13 \text{ degrees and}$$

$$C_P{}^* = 1.69 \times 10^5 \text{ cm/sec}.$$

If we arbitrarily choose for the radius of the outer sphere $b = 8.3$ cm, $\omega_0 = 25\,\pi$. Then $q = \pi/2$ corresponds to 31 hz, and the attenuation

$$a_P{}^* = 1.03 \times 10^{-4} \text{ cm}^{-1}, \text{ or 27 db/1000 ft}.$$

Numerical results for the same rock and dimensions and for various frequencies are shown in the table. These results for $C_P{}^*$ and $\theta_P{}^*$ are plotted against frequency in Figure 2.

Table 1. Results for numerical example

q	f hz	$C_P{}^*$ m/sec	$\theta_P{}^*$ radians	$a_P{}^*$ m^{-1}	$a_P{}^*$ db/1000 ft
0.02	.005	1620	0.004	—	—
1.57	31	1690	0.179	0.0103	27
3.14	123	1910	0.152	0.0307	82
6.28	495	1980	0.063	0.0487	130

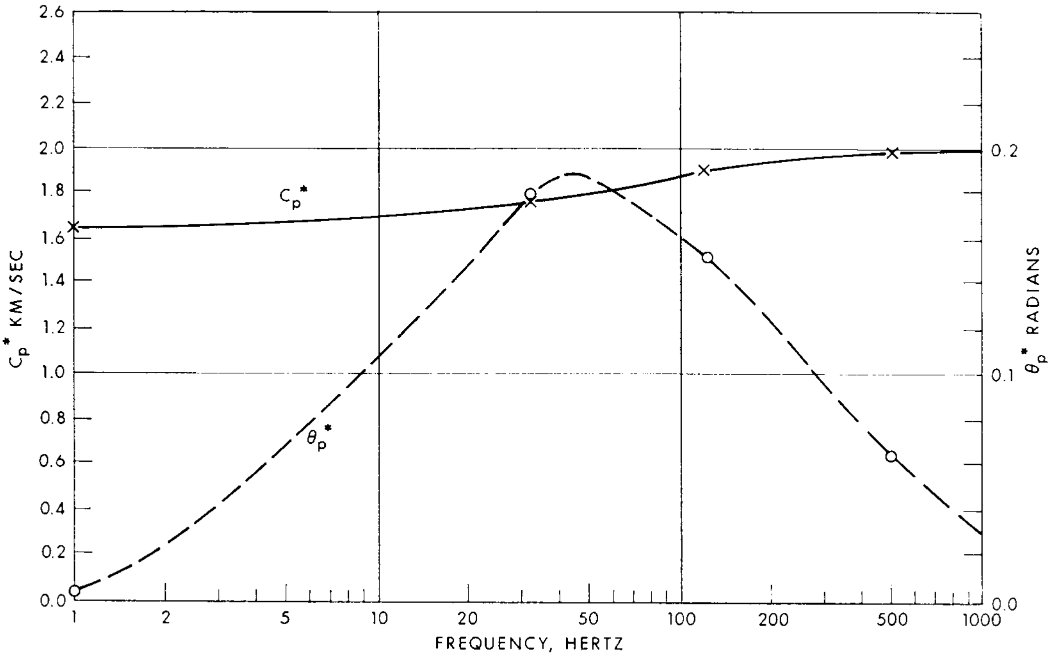

FIG. 2. Phase angle $\theta_P{}^*$ and compressional wave speed $C_P{}^*$ for a porous rock with mixed saturation. Diameter of the gas pockets is 8.3 cm and the distance between centers is 13.3 cm.

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APPENDIX A

BULK MODULUS WITHOUT FLUID FLOW

We wish to calculate the average bulk modulus of a composite elastic body consisting of a sphere of radius a and a spherical shell of outer radius b. A text such as Lamb (1960, p. 342–43) gives for radially symmetric stress and displacement

$$p_{rr} = A + B/r^3$$
$$u_r = \frac{(1-2\sigma)}{E} Ar - \frac{(1+\sigma)}{2E}\frac{B}{r^2}. \tag{A-1}$$

Consider application of a pressure P_0 at the outer boundary, resulting in a pressure P_i at the inner boundary. Designating the inner sphere by subscript 1 and the shell by 2, we characterize the two media by Young's modulus E and Poisson's ratio σ. For the inner body, $B = 0$ and $A = p_{rr}(a) = -P_i$. Continuity of displacement at the inner boundary yields

$$-\frac{(1-2\sigma_1)}{E_1} aP_i =$$
$$\frac{(1-2\sigma_2)}{E_2} a \frac{(a^3P_i - b^3P_0)}{(b^3-a^3)} - \frac{(1+\sigma_2)}{2E_2}\frac{1}{a^2}\frac{a^3b^3}{(b^3-a^3)}(P_0 - P_i). \tag{A-2}$$

Displacement at the outer boundary is

$$U_0 = \frac{(1 - 2\sigma_2)}{E_2} \frac{(a^3 P_i - b^3 P_0)}{(b^3 - a^3)} b$$

$$- \frac{(1 + \sigma_2)}{2E_2} \frac{1}{b^2} \frac{a^3 b^3}{(b^3 - a^3)} (P_0 - P_i). \qquad \text{(A-3)}$$

From these equations we obtain the result

$$k_0 = K_3/(K_2 - K_1), \qquad \text{(A-4)}$$

where (with $a^3/b^3 = S_G$)

$$K_1 = \frac{3(1 - \sigma_2)/2}{(1 - 2\sigma_1)(1 - S_G)E_2/E_1 + S_G(1 - 2\sigma_2) + (1 + \sigma_2)/2},$$

$$K_2 = \frac{(1 - 2\sigma_2)/S_G + (1 + \sigma_2)/2}{3(1 - \sigma_2)/2}, \quad \text{and}$$

$$K_3 = \frac{E_2(1/S_G - 1)}{3(1 - 2\sigma_2) + 3(1 + \sigma_2)/2}.$$

APPENDIX B
FLUID PRESSURE DUE TO DILATATION

Assume that a pressure p is applied to the faces of a cube of saturated rock, creating dilatation D, fluid pressure p_f, and skeleton pressure $\bar{p}$. By definition, $p = -kD$. Also

$$D = -\phi p_f/k_f - (1 - \phi)p_f/k_S - \bar{p}/k_S, \text{ and } p = p_f + \bar{p}.$$

From these relations we conclude that

$$p_f = -\frac{k_f}{\phi}\left[\frac{(1 - k/k_S)}{(1 - k_f/k_S)}\right] D. \qquad \text{(B-1)}$$

We need to apply this relation to the central sphere and to the spherical shell. For the same geometry as Appendix A, static analysis gives for the dilatation in the sphere $D_1 = -P_i/k_1 = -K_1P_0/k_1$ so that $p_{f1} = P_1 = R_1P_0$ with

$$R_1 = \frac{k_{f1}}{\phi}\left[\frac{(1 - k_1/k_{S1})}{(1 - k_{f1}/k_{S1})}\right]\frac{K_1}{k_1}. \qquad \text{(B-2)}$$

Dilatation in the shell is

$$D_2 = \frac{1}{k_2}\frac{(S_GP_i - P_0)}{(1 - S_G)} = \frac{1}{k_2}\frac{(S_GK_1 - 1)P_0}{(1 - S_G)},$$

and, hence, $p_{f2} = P_2 = R_2P_0$, with

$$R_2 = \frac{k_{f2}}{\phi k_2}\frac{(1 - k_2/k_{S2})(1 - S_GK_1)}{(1 - k_{f2}/k_{S2})(1 - S_G)}. \qquad \text{(B-3)}$$

APPENDIX C
FLUID-FLOW IMPEDANCES

In each fluid-saturated porous medium, we assume that the oscillatory particle velocity of the fluid relative to the skeleton is proportional to the pressure gradient (Darcy's law). For radial motion in spherical coordinates this relation is simply

$$v = -\frac{\kappa}{\eta}\frac{\partial p_f}{\partial r}. \qquad \text{(C-1)}$$

We also recognize that as fluid accumulates in any elementary volume of the medium (the rate of accumulation being proportional to the divergence of fluid velocity), the pressure increases. This linear relationship defines an "effective bulk modulus" k_E as folows:

$$p_f = k_E\, D_f, \qquad \text{(C-2)}$$

where D_f is the volume of fluid added divided by the volume of the element. This result may be derived through relations for an elementary cube in two steps. First, assume that the faces of the cube are held rigidly so that the volume of the injected fluid must be matched by compression of the fluid in the pore space ($p_{fB}\phi/k_f$), compression of the solid [$p_{fB}(1 - \phi)/k_s$], and expansion of the solid due to unloading of frame pressure ($-p_{fB}\bar{k}/k_s^2$). This defines an apparent bulk modulus k_A, yielding the pressure p_{fB} created by injection of fluid into a blocked elementary volume:

$$p_{fB} = k_A\, D_f, \qquad \text{(C-3)}$$

where

$$k_A = \left(\frac{\phi}{k_f} + \frac{1 - \phi}{k_S} - \frac{\bar{k}}{k_S^2}\right)^{-1}.$$

Because of shrinkage of the skeleton, skeleton pressure will take on the value $\bar{p}_B = -\bar{k}p_{fB}/k_s$, and the total rock pressure on the rigid container is $p_B = p_{fB} + \bar{p}_B = (1 - \bar{k}/k_s)p_{fB}$.

As the second step, we recognize that for a wave with displacement along only one coordinate, the blocked condition applies to the four faces parallel to the fluid flow, but the faces perpendicular to flow can move. At low frequencies, the constraining pressure on these faces is negligibly small. So consider a pressure $p_E = -p_B$ to be applied to these faces, the other four faces being blocked. The fluid pressure change is

$$p_{fE} = \frac{k_f}{\phi M}\frac{(1 - k/k_S)}{(1 - k_f/k_S)}\, p_E = -\frac{k_f}{\phi M}\frac{(1 - k/k_S)(1 - \bar{k}/k_S)}{(1 - k_f/k_S)}\, p_{fB}. \tag{C-4}$$

The combined fluid pressure from both steps is

$$p_f = p_{fB} + p_{fE} = \left[1 - \frac{k_f(1 - k/k_S)(1 - \bar{k}/k_S)}{\phi M(1 - k_f/k_S)}\right] k_A\, D_f. \tag{C-5}$$

Using equation (C-1), we obtain

$$\frac{\partial^2 p_f}{\partial r^2} + \frac{2}{r}\frac{\partial p_f}{\partial r} = \frac{\eta}{\kappa k_E}\frac{\partial p_f}{\partial t}. \tag{C-8}$$

In the central sphere,

$$p_1 e^{i\omega t} = (A/r)(e^{\alpha_1 r} - e^{-\alpha_1 r})\, e^{i\omega t} \text{ and}$$

$$v_1 e^{i\omega t} = -(\kappa_1 A/\eta_1 r^2)[(\alpha_1 r - 1)e^{\alpha_1 r} + (\alpha_1 r + 1) \cdot e^{-\alpha_1 r}]e^{i\omega t}, \tag{C-9}$$

where $\alpha_1 = (i\omega\eta_1/\kappa_1 k_{E1})^{1/2}$.

The impedance looking inward is $-p_1/v_1$, or

$$Z_1 = \frac{\eta_1 a}{\kappa_1}\left[\frac{(1 - e^{-2\alpha_1 a})}{(\alpha_1 a - 1) + (\alpha_1 a + 1)e^{-2\alpha_1 a}}\right]. \tag{C-10}$$

In the spherical shell,

$$p_2 e^{i\omega t} = (A/r)[e^{\alpha_2 r} + e^{2\alpha_2 b}(\alpha_2 b - 1)e^{-\alpha_2 r}/(\alpha_2 b + 1)]e^{i\omega t} \quad \text{and}$$

$$v_2 e^{i\omega t} = -(\kappa_2 A/\eta_2 r^2)[(\alpha_2 r - 1)e^{\alpha_2 r} - e^{2\alpha_2 b}(\alpha_2 b - 1)(\alpha_2 r + 1)e^{-\alpha_2 r}/(\alpha_2 b + 1)]e^{i\omega t}. \tag{C-11}$$

The impedance looking outward is p_2/v_2, or

$$Z_2 = -\frac{\eta_2 a}{\kappa_2}\left[\frac{(\alpha_2 b + 1) + (\alpha_2 b - 1)e^{2\alpha_2(b-a)}}{(\alpha_2 b + 1)(\alpha_2 a - 1) - (\alpha_2 b - 1)(\alpha_2 a + 1)e^{2\alpha_2(b-a)}}\right]. \tag{C-12}$$

Since $p_f = k_E D_f$, the "effective bulk modulus" is

$$k_E = \left[1 - \frac{k_f(1 - k/k_S)(1 - \bar{k}/k_S)}{\phi M(1 - k_f/k_S)}\right] k_A. \tag{C-6}$$

For radial fluid flow, the rate of fluid influx is $1/r^2 \partial/\partial r(r^2 v)$ so that

$$\frac{\partial p_f}{\partial t} = -k_E\left(\frac{\partial v}{\partial r} + \frac{2}{r}v\right). \tag{C-7}$$

APPENDIX D
SKELETON ADJUSTMENT TO FLUID INFLUX

As discussed in Appendix C, an elementary volume expands or contracts as a consequence of the fluid-flow wave. In the example, the fractional expansion required to achieve zero stress is

$$D_E = p_B/M = [(1 - \bar{k}/k_S)k_A/M]\, D_f = Q\, D_f,$$

where

$$Q = (1 - \bar{k}/k_S)k_A/M. \tag{D-1}$$