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THE EFFECT OF INCLUSION SHAPE ON THE ELASTIC MODULI OF A TWO-PHASE MATERIAL*

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Abstract—A self-consistent model is used to calculate the elastic moduli of two-phase materials with the assumption that the inclusions are spheroidal. The limiting cases of needle and disk shaped inclusions are explicitly shown. It is found that the latter has the greatest effect in changing the elastic moduli of the matrix.

INTRODUCTION

VARIOUS calculations concerning the properties of heterogeneous materials have been reviewed recently [1]. An exact calculation appears beyond any possibility since the precise distribution of the different phases cannot be specified. A low concentration of spherical inclusions was first studied by Einstein [2], but his method is difficult to extend to the study of high concentrations. Upper and lower bounds were obtained with the application of energy methods [3] and subsequently refined [4]. The exact bounds are still too far apart to provide predictions of experiment results. Curve fitting method was developed [5], but an arbitrary parameter was left unspecified.

In order to study the effect of inclusion shapes at relatively high concentrations, the method of self-consistent field [6] provides a useful estimation. An explicit calculation requires the assumption of the shape of the inclusions [7]. This shape-effect on elastic moduli is studied in the present paper.

STRAIN FIELD WITHIN THE INCLUSION

The strain field within an elastic ellipsoidal inclusion embedded inside an elastic matrix subjected to a displacement with uniform strain field at infinity has been obtained by Eshelby [8]. That a closed form solution is possible in this special case can be suspected from the integral equation formulation [5], i.e.

$$u_i(\mathbf{x}) = u_i^0(\mathbf{x}) + \frac{1}{4\pi G_0}\int_{V_2}\left[\frac{\partial}{\partial x_j'}\frac{\delta_{li}}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{4(1-\nu_0)}\frac{\partial^3}{\partial x_j'\partial x_l'\partial x_i'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] [\alpha\varepsilon_{lj}(\mathbf{x}') + \beta\delta_{lj}\varepsilon_{\kappa\kappa}(\mathbf{x}')]\, d^3\mathbf{x}', \tag{1}$$

where $u_i(\mathbf{x})$ represents the displacement field, $u_i^0(\mathbf{x})$ that at infinity. The shear modulus and Poisson's ratio of the matrix are designated by G_0 and ν_0, V_2 is the volume occupied by the inclusion, $\varepsilon_{ij}(\mathbf{x})$ is the strain field derived from $u_i(\mathbf{x})$,

$$\alpha = 2(G_0 - G_2), \tag{2}$$

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and

$$\beta = 2\left(\frac{G_0}{1-2\nu_0} - \frac{G_2}{1-2\nu_2}\right), \tag{3}$$

where G_2 and ν_2 are the shear modulus and Poisson's ratio of the inclusion. If it is assumed that ε_{ij} is homogeneous, the second bracket can be put in front of the integral sign. For the special case of V_2 being ellipsoidal, the integration has been calculated and found to depend on the displacement linearly [8], so that ε_{ij} is indeed homogeneous. Therefore, the integral equation (1) can be reduced to an algebraic equation. For the ellipsoidal axes coincide with the specimen axes,

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + \frac{1}{2G_0}(\alpha\varepsilon_{lk} + \beta\delta_{lk}\varepsilon_{qq})\left(S_{ijlk} - \frac{\nu_0}{1+\nu_0}\delta_{lk}S_{ijpp}\right), \tag{4}$$

where ε_{ij}^0 is the uniform strain field at infinity and S_{ijlk} is the tensor introduced in Reference [8].

Equation (4) can be solved to give

$$\varepsilon_{ij} = T_{ijpq}\varepsilon_{pq}^0, \tag{5}$$

where

$$T_{1212} = T_{2121} = \frac{1}{2(1+2AS_{1212})}, \tag{6}$$

and similar expression for $T_{2323} = T_{3232}$, and $T_{3131} = T_{1313}$;

$$[T] \equiv \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} \\ T_{2211} & T_{2222} & T_{2233} \\ T_{3311} & T_{3322} & T_{3333} \end{bmatrix}$$

$$= \begin{bmatrix} 1+AS_{1111}+BS_1 & AS_{1122}+BS_1 & AS_{1133}+BS_1 \\ AS_{2211}+BS_2 & 1+AS_{2222}+BS_2 & AS_{2233}+BS_2 \\ AS_{3311}+BS_3 & AS_{3322}+BS_3 & 1+AS_{3333}+BS_3 \end{bmatrix}^{-1};$$

and the other components of T_{ijpq} vanish. In equations (6) and (7), the following abbreviations have been used,

$$A = \frac{G_2}{G_0} - 1, \tag{8}$$

$$B = \frac{1}{3}\left(\frac{K_2}{K_0} - \frac{G_2}{G_0}\right), \tag{9}$$

$$S_1 = S_{1111} + S_{1122} + S_{1133}, \tag{10}$$

and similar expressions for S_2 and S_3, where K_0 and K_2 are the bulk moduli of the matrix and inclusion respectively.

Elastic moduli of the two-phase material

In a composite, let the matrix be specified by subscript 1, and the inclusions subscript 2, and the composite by subscript 0. Consider a uniform surface traction $T_i^0 = \sigma_{ij}^0 n_j$ prescribed at the boundary of the composite which is infinitely large. The total elastic energy is [8]

$$\begin{aligned} \mathrm{E} &= \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\, dV \\ &= \frac{1}{2}\int_V \sigma_{ij}^0\varepsilon_{ij}^0\, dV + \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} - \sigma_{ij}^0\varepsilon_{ij}^0)\, dV \end{aligned} \tag{11}$$

where $\varepsilon_{ij}^0 = L_1(\sigma_{ij}^0)$, and L_1 is the linear Hookean operator with elastic constants of the matrix. Since

$$\int_V \sigma_{ij}\varepsilon_{ij}\, \mathrm{d}V = \int_S T_i u_i\, \mathrm{d}S = \int_S T_i^0 u_i\, \mathrm{d}S = \int_V \sigma_{ij}^0 \varepsilon_{ij}\, \mathrm{d}V, \tag{12}$$

and

$$\int_V \sigma_{ij}^0\varepsilon_{ij}^0\, \mathrm{d}V = \int_V \sigma_{ij}\varepsilon_{ij}^0\, \mathrm{d}V = \int_V \sigma_{ij} L_1(\sigma_{ij})\, \mathrm{d}V, \tag{13}$$

the energy expression (11) can be simplified to

$$\mathrm{E} = \frac{1}{2}\int_V \sigma_{ij}^0\varepsilon_{ij}^0\, \mathrm{d}V + \frac{1}{2}\int_{V_2} \sigma_{ij}^0[\varepsilon_{ij} - L_1(\sigma_{ij})]\, \mathrm{d}V. \tag{14}$$

On the other hand, the total energy is also given by $\frac{1}{2}\sigma_{ij}^0 L_0(\sigma_{ij}^0)V$, where L_0 is the linear Hookean operator with elastic constants of the composite. Therefore,

$$\sigma_{ij}^0 L_0(\sigma_{ij}^0) = \sigma_{ij}^0 L_1(\sigma_{ij}^0) + C_2\sigma_{ij}^0\{(\varepsilon_{ij})_{\mathrm{ave}} - L_1[L_2^{-1}(\varepsilon_{ij})_{\mathrm{ave}}]\}, \tag{15}$$

where C_2 is the volume concentration of the inclusions, L_2 is the linear Hookean operator with elastic constants of the inclusions, and

$$(\varepsilon_{ij})_{\mathrm{ave}} = \frac{1}{V_2}\int_{V_2} \varepsilon_{ij}\, \mathrm{d}V. \tag{16}$$

It is therefore clearly indicated that in order to find the elastic moduli of the composite, there is no requirement to calculate $\varepsilon_{ij}(\mathbf{x})$ explicitly.

In order to estimate $(\varepsilon_{ij})_{\mathrm{ave}}$, an inclusion of a specified shape is assumed to be embedded in a matrix made of homogeneous material with the gross properties of the composite. If the inclusion is ellipsoidal, the strain field is given in equation (5) and assumed to be a sufficiently close estimation for the true $(\varepsilon_{ij})_{\mathrm{ave}}$. The case of spherical inclusions has been considered previously [9] and is particularly simply due to the lack of orientation dependence. In the case of ellipsoids, the estimated strain depends on the relative orientations between the ellipsoidal and specimen axes, so that an additional average is required.

According to Kröner [10], for random distribution, the two unknown constants, K_0 and G_0, can be determined from the scalar forms of T_{ijpq}. In fact, there are only two scalar forms, i.e. T_{iijj} and T_{ijij}. This argument can be verified by direct integration over all orientations. Equation (5) may be generalized to include relative orientations, and substituting this general form into equation (15) gives

$$
\begin{aligned}
&\sigma_{ij}^{0}\sigma_{pq}\frac{1}{E_0}[(1+\nu_0)\delta_{ip}\delta_{jq}-\nu_0\delta_{ij}\delta_{pq}] \\
&= \sigma_{ij}^{0}\sigma_{pq}^{0}\frac{1}{E_1}[(1+\nu_1)\delta_{ip}\delta_{jq}-\nu_1\delta_{ij}\delta_{pq}] \\
&\quad +\sigma_{ij}^{0}\sigma_{pq}C_2\Bigg\{(T_{ijrs})_{\text{ave}}\frac{1}{E_0}[(1+\nu_0)\delta_{rp}\delta_{sq}-\nu_0\delta_{rs}\delta_{pq}] \\
&\qquad -\frac{1}{E_1}[(1+\nu_1)\delta_{ia}\delta_{jb}-\nu_1\delta_{ij}\delta_{ab}]\frac{E_2}{1+\nu_2} \\
&\qquad \left[\delta_{ac}\delta_{bd}+\frac{\nu_2}{1-2\nu_2}\delta_{ab}\delta_{cd}\right](T_{cdrs})_{\text{ave}}\frac{1}{E_0} \\
&\qquad [(1+\nu_0)\delta_{rp}\delta_{sq}-\nu_0\delta_{rs}\delta_{pq}]\Bigg\}.
\end{aligned}
\tag{17}
$$

where E_0, E_1 and E_2 are Young's moduli of the composite, matrix and inclusions respectively. Since $(T_{ijij})_{\text{ave}} = T_{ijij}$ and $(T_{iijj})_{\text{ave}} = T_{iijj}$, with $\sigma_{11}^{0} = \sigma_{22}^{0} = \sigma_{33}^{0} = \sigma$ and the other components of σ_{ij}^{0} vanishing, equation (17) is reduced to

$$\frac{1}{K_0} = \frac{1}{K_1}\left[1+\frac{C_2}{3}T_{iijj}\left(\frac{K_1-K_2}{K_0}\right)\right]. \tag{18}$$

Similarly, the shear modulus of the composite is given by

$$\frac{1}{G_0} = \frac{1}{G_1}\left[1+\frac{C_2}{15}(3T_{ijij}-T_{iijj})\left(\frac{G_1-G_2}{G_0}\right)\right]. \tag{19}$$

Equations (18) and (19) are coupled through T_{iijj} and T_{ijij}.

For simplicity, only the case of spheroids is considered so that the limiting cases of needle and disk shaped inclusions can be calculated explicitly. The values of the scalar quantities, T_{iijj} and $T_{ijij}-\frac{1}{3}T_{iijj}$, for composites with spheroidal inclusions are given in the Appendix.

The shape effect

The result in the Appendix can be simplified in the case of needle and disk shaped inclusions. For needle-shaped ones, the elastic moduli are governed by

$$\frac{1}{K_0} = \frac{1}{K_1}\left[1+\frac{C_2}{3}\,\frac{3+RA}{1+(1-R)A+(3-4R)B}\,\frac{K_1-K_2}{K_0}\right], \tag{20}$$

and

$$
\begin{aligned}
\frac{1}{G_0} = \frac{1}{G_1}\Bigg(1+\frac{C_2}{5}\Bigg\{&\frac{2}{1+(A/2)}+\frac{1}{1+(1+R)(A/2)}+\frac{(1-\frac{4}{3}R)(A+3B)}{1+(1-R)A+(3-4R)B} \\
&+\frac{2[1+(3-R)(A/4)+(3-4R)(B/2)]}{[1+(1+R)(A/2)][1+(1-R)A+(3-4R)B]}\Bigg\}\frac{G_1-G_2}{G_0}\Bigg),
\end{aligned}
\tag{21}
$$

where A and B are given by equations (8) and (9), and

$$R = \frac{3G_0}{3K_0+4G_0}. \tag{22}$$

Similar, for disk-shaped inclusions,

$$\frac{1}{K_0} = \frac{1}{K_1}\left[1+\frac{C_2}{3}\frac{3+4RA}{1+A+(3-4R)B}\frac{K_1-K_2}{K_0}\right], \tag{23}$$

and

$$\frac{1}{G_0} = \frac{1}{G_1}\left\{1+\frac{C_2}{5}\left[\frac{2}{1+A}+1+\frac{2+(2-\frac{4}{3}R)A+2(3-4R)B}{1+A+(3-4R)B}\right]\frac{G_1-G_2}{G_0}\right\}. \tag{24}$$

For comparison, the governing equations for spherical inclusion are [9]

$$\frac{1}{K_0} = \frac{1}{K_1}\left[1+\frac{C_2}{1+(1-\frac{4}{3}R)(A+3B)}\frac{K_1-K_2}{K_0}\right], \tag{25}$$

and

$$\frac{1}{G_0} = \frac{1}{G_1}\left[1+\frac{C_2}{1+(2+\frac{4}{3}R)(A/5)}\frac{G_1-G_2}{G_0}\right]. \tag{26}$$

In equations (25) and (26), the roles of K_1 and K_2, G_1 and G_2, $C_1 = 1-C_2$ and C_2, i.e. the roles of matrix and inclusions, can be interchanged without alteration of the governing equations. This is not true for the case of equations (20) and (21), or equations (23) and (24). Such symmetry property seems to indicate that the present scheme of estimation may not be adequate in distinguishing different types of distributions.

In general, equations (20) to (26) are still too complicated to give any indication of the shape effect of the inclusions. In order to simplify these relations, the special case of $\nu_1 = \nu_2 = 0{\cdot}2$ is considered. It can be shown that $\nu_0 = 0{\cdot}2$ for a composite with disk-shaped and spherical inclusions, and that ν_0 is approximately equal to $0{\cdot}2$ for that with needle-shaped inclusions. Then, Young's modulus of the composite can be explicitly

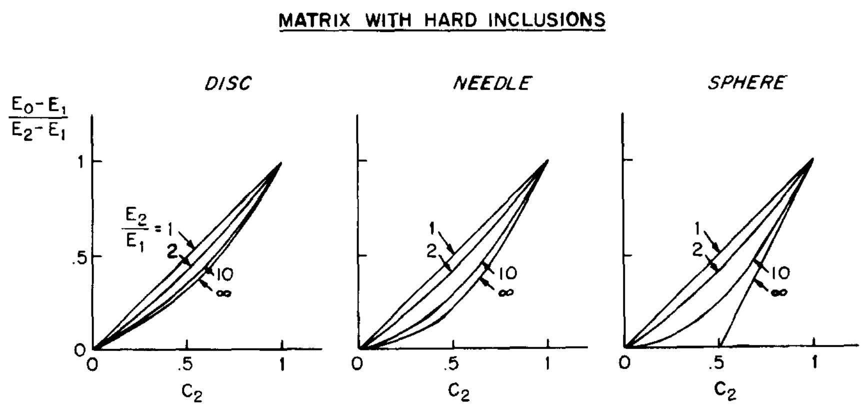

FIG. 1.

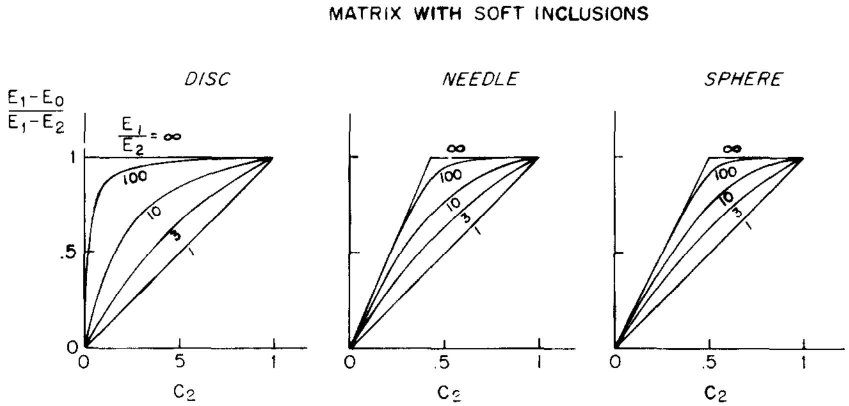

FIG. 2.

calculated for different values of C_2 with the use of the standard relations between E_0 and G_0, K_0, ν_0. In Figs. 1 and 2, the effects of hardening and softening are shown by plotting $(E_0-E_1)/(E_2-E_1)$ and $(E_1-E_0)/(E_1-E_2)$ against C_2 respectively. The symmetrical nature of spherical inclusions is again indicated.

Physically, if we want to render a material completely useless by the introduction of holes, this will require 50% spherical ones, 43% needle-shaped ones, and an infinitesimal amount of disk-shaped ones (Fig. 2). Intuitively, disk-shaped inclusions are similar to plane cuts, so that their effects are most pronounced. However, due to the inherent nature of approximations involved in the self-consistent field, the above numbers should not be taken too seriously. A similar behavior is observed in increasing Young's modulus of the composite (Fig. 1). This result can be shown more clearly by plotting E_0/E_2 against C_2 with $E_2/E_1 = 10$ (Fig. 3) or E_0/E_1 against C_2 with $C_1/E_2 = 10$ (Fig. 4).

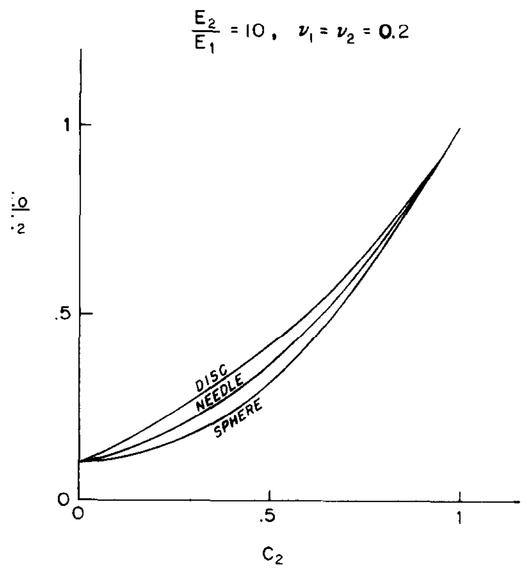

FIG. 3.

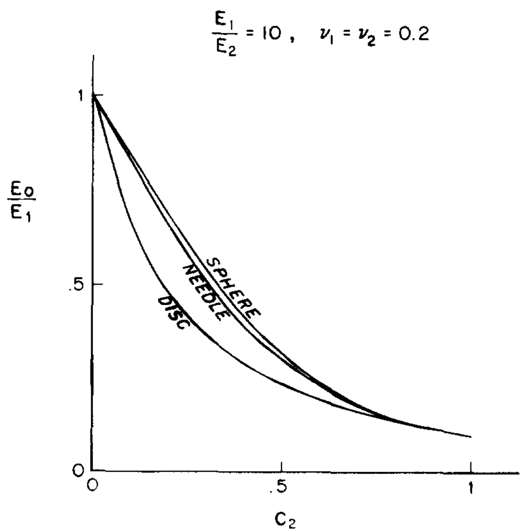

FIG. 4.

CONCLUSION

One of the practical applications of two-phase materials is to increase the elastic moduli of a relatively soft and light material by dispersing inclusions of a different material in the former. In the present study, it has been found that the shape of the inclusions plays an important role. The disk-shaped inclusions give by far the most significant increase in Young's modulus. The validity of the present analysis, however, awaits experimental verification.

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APPENDIX

In this appendix, the scalars, T_{iijj} and $T_{ijij}-\frac{1}{3}T_{iijj}$, involved in equations (18) and (19) for spheroidal inclusions are summarized. Their derivation is straight forward but involves tedious algebraic manipulations. Let the radii of the ellipsoid be denoted by a, b and c. For prolate spheroids, $a \geq b = c$,

$$\begin{aligned} T_{iijj} = {} & 3\{1+A[\tfrac{3}{2}(f+\theta)-R(\tfrac{3}{2}f+\tfrac{5}{2}\theta-\tfrac{4}{3})]\} \\ & \{1+A[1+\tfrac{3}{2}(f+\theta)-R(\tfrac{3}{2}f+\tfrac{5}{2}\theta)]+B(3-4R) \\ & \quad -\tfrac{1}{2}A(A+3B)(3-4R)[f+\theta-R(f-\theta+2\theta^2)]\}^{-1}, \end{aligned} \tag{A1}$$

and

$$\begin{aligned} T_{ijij}-\tfrac{1}{3}T_{iijj} = {} & 2\left\{1+\tfrac{1}{2}A\left[-\frac{a^2+c^2}{c^2}f+R\left(2-\theta+\frac{a^2+c^2}{c^2}f\right)\right]\right\}^{-1} \\ & +\{1+\tfrac{1}{4}A[3\theta+f-R(f-\theta)]\}^{-1} \\ & +\Big(\{1+\tfrac{1}{4}A[3\theta+f-R(f-\theta)]\} \end{aligned}$$

$$\times\{A[-f+R(f+\theta-\tfrac{4}{3})]+B\theta(3-4R)\}$$
$$+2\{1+A[1+f-R(f+\theta)]+B(1-\theta)(3-4R)\}$$
$$\times\{1+\tfrac{1}{8}A[9\theta+3f-R(5\theta+3f)]+\tfrac{1}{2}B\theta(3-4R)\}$$
$$-2\{A[1-\tfrac{3}{2}\theta-\tfrac{1}{2}f+\tfrac{1}{2}R(5\theta+f-4)]+B(1-\theta)(3-4R)\}$$
$$\times\{A[-\tfrac{1}{2}f+\tfrac{1}{2}R(f-\theta)]+\tfrac{1}{2}B\theta(3-4R)\}\Big)$$
$$\times\{1+\tfrac{1}{4}A[3\theta+f-R(f-\theta)]\}^{-1}$$
$$\times\{1+A[1+\tfrac{3}{2}(f+\theta)-R(\tfrac{3}{2}f+\tfrac{5}{2}\theta)]$$
$$+B(3-4R)+\tfrac{1}{2}A(A+3B)(3-4R)[f+\theta-R(f-\theta+2\theta^2)]\}^{-1}, \tag{A2}$$

where A, B and R are given by equations (8), (9) and (22),

$$\theta = \frac{ac^2}{(a^2-c^2)^{\frac{3}{2}}}\left[\frac{a}{c}\left(\frac{a^2}{c^2}-1\right)^{\frac{1}{2}}-\cosh^{-1}\frac{a}{c}\right], \tag{A3}$$

and

$$f = \frac{a^2}{a^2-c^2}(2-3\theta). \tag{A4}$$

For prolate spheroids, $a = b \geq c$, θ and f in equations (A1) and (A2) are replaced by ϕ and g respectively, where

$$\phi = \frac{a^2c}{(a^2-c^2)^{\frac{3}{2}}}\left[\cos^{-1}\frac{c}{a}-\frac{c}{a}\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}\right], \tag{A5}$$

and

$$g = \frac{c^2}{a^2-c^2}(3\phi-2). \tag{A6}$$

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Résumé—Un modèle auto-uniforme sert à calculer les coéfficients d'élasticité de matières biphasées en supposant que les inclusions soient sphéroidales. Les cas limites d'inclusions en forme d'aiguille ou de disque sont nettement indiqués. On a trouvé que cette dernière cause le plus grand changement de coéfficients de la matrice.

Zusammenfassung—Ein in sich konsistentes Modell wird zur Errechnung des Elastizitätsmoduls zweiphasiger Werkstoffe benützt unter der Voraussetzung, dass die Einschlüsse sphäroider Form sind. Grenzfälle nadel-und scheibenförmiger Einschlüsse werden ausdrücklich gezeigt. Es wurde festgestellt, dass die letzteren die stärkste Wirkung in Bezug auf die Veränderung des Elastizitätsmoduls der Matrix ausüben.

Абстракт—Дается применение самосовместимой модели для вычисления модуля упругости двухфазовых веществ, предполагая сфероидность вложений. Подробно разработанны крайние случаи иглы и диска. Установленно, что максимальное изменение модуля упругости матрисы постигается в лимитирующем случае диска.