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Differential Effective Medium Modeling of Rock Elastic Moduli with Critical Porosity Constraints

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Abstract. Rocks generally have a percolation porosity at which they lose rigidity and fall apart. Percolation behavior is a purely geometrical property, independent of any physical properties, and is a powerful constraint on any valid velocity-porosity relation. We show how the conventional Differential Effective Medium (DEM) theory can be modified to incorporate percolation of elastic moduli in rocks by taking the material at the critical porosity as one of the constituents of a two-phase composite. Any desired percolation porosity can be specified as an input. In contrast, the conventional DEM model always predicts percolation at a porosity of either 0 or 100 percent. Most sedimentary rocks however have intermediate percolation porosities and are therefore not well represented by the conventional theory. The modified DEM model incorporates percolation behavior, and at the same time is always consistent with the Hashin-Shtrikman bounds. The predictions compare favorably with laboratory sandstone data.

Introduction

Any model that accurately relates elastic wave velocities to porosity in rocks must incorporate two simple and almost universal observations: (1) Rocks generally have their highest elastic moduli and seismic velocities in the limit of zero porosity, approaching the moduli of the mineral constituents, and (2) there is almost always a finite *critical porosity* or percolation porosity, ϕ_c, at which the rock loses its rigidity and falls apart (Nur, 1992). In addition, it is desirable that the model predictions always lie within known, rigorous bounds such as the Hashin-Shtrikman (1963) bounds. While most models have the correct mineral limit at zero porosity, only a few, including the self-consistent (SC) models, also incorporate percolation behavior. The conventional differential effective medium (DEM) formulations do not have this percolation property. Furthermore, only the DEM and one of the SC models based on the coherent potential approximation (Berryman, 1980) have been shown to be realizable for ellipsoidal inclusions (Milton, 1985; Norris, 1985; Avellaneda, 1987), and are therefore always consistent with the Hashin-Shtrikman bounds.

The SC models are parametrized by spectra of aspect ratios of the ellipsoidal inclusions. It is not easy to prespecify the critical porosity as an input. While almost any velocity-porosity trend can be obtained by simply varying the aspect ratio, in real rocks the appropriate aspect ratio spectrum is rarely easy to determine. Furthermore, the aspect ratio of an ellipsoid is not always a natural descriptor of the complicated pore microgeometry.

In contrast, the critical porosity is more readily observable from the general trend of the modulus-porosity data within restricted classes of rock samples (Nur, 1992). The percolation porosity is determined by the way the rock and pore space are formed and is a more natural descriptor of the pore space. Moreover, the percolation behavior of porous rocks is a purely geometrical phenomenon, independent of electrical or elastic properties of rocks. Therefore it is a powerful, model-independent input to any valid, physically reasonable velocity-porosity relation. Hence it is advantageous to have models that specifically incorporate this knowledge of ϕ_c, and at the same time are guaranteed never to violate bounds, including the Hashin-Shtrikman bounds.

In this paper we show how the DEM formulation can be modified to include any percolation porosity as a convenient input for modeling rock elastic properties. The estimates are always consistent with the Hashin-Shtrikman bounds, since DEM is a realizable model. The predictions of the modified DEM model fit laboratory data for sandstones reasonably well.

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Critical Porosity

The critical porosity marks a transition in the state of the porous rock, with a distinct change in physical properties. Below ϕ_c, the mineral grains are load-bearing. Above ϕ_c, the pore fluid becomes load-bearing, and the material behaves mechanically like a suspension. A dry rock falls apart above ϕ_c. At the percolation porosity, the material is a loose packing of grains barely touching each other. We define this to be the critical phase. Here "phase" has no chemical connotations.

The critical porosity is a geometric property determined by the process of formation of the rock and the pore microgeometry (Nur, 1992; Guégen and Palciauskas, 1994). Each class of rocks, depending on its genesis and evolution, has a particular critical porosity. Clastics have a critical porosity representative of the originally deposited sediments before diagenesis, typically $\phi_c \sim 0.36 - 0.40$. This is close to the porosity for a random close packing of spheres (0.36). Vesicular volcanics with nearly spherical pores formed by outgassing can stay intact to a $\phi_c \sim 0.9$ or more. Crystalline rocks with crack porosity can have a ϕ_c of only $0.02 - 0.05$.

The elastic moduli of the critical phase at ϕ_c depend on factors such as sorting and confining pressure. For very low confining pressures, the moduli at the percolation point are equal to the Reuss (harmonic) average of the constituent moduli.

The simplest curve through the mineral modulus at $\phi = 0$ and the critical phase modulus at $\phi = \phi_c$ is a straight line. This *modified Voigt* line amounts to redefining the end members and the weighting factors for the Voigt (arithmetic) average (Nur et al., 1991; Chen, 1992). Instead of pure mineral and pure fluid, the pure mineral and the critical phase at ϕ_c are taken as the two constituents. It is a heuristic model, and for sufficiently large ϕ_c, can sometimes violate the Hashin-Shtrikman bounds. Its appeal is its simplicity and good fit to high pressure data.

The idea of taking the critical phase as one of the end members has been used to modify various other existing analytical and empirical relations such as the Hashin-Shtrikman upper bound, Wyllie's relation and Mori-Tanaka's inclusion model (Chen, 1992). We use the same concept and modify the DEM formulation to model rocks with percolation behavior. The modification is implemented by 1) replacing the inclusion phase by the critical phase and 2) renormalizing the porosity to ϕ/ϕ_c. Sheng (1991) introduced a somewhat similar DEM method for modeling electrical and elastic properties of rocks. His starting material is a mixture of fluid and cement, with the fluid fraction restricted to lie between 0.4 and 0.6. Furthermore, his model is restricted to very specific grain shapes, motivated by a desire to agree with Archie's law for electrical conductivity. This implementation is very different from the one presented here, leading to restrictions on the range and ease of applicability that our theory does not share. Using the critical phase as one of the constituents does not model in all detail the complex microstructure of the pores, pore-to-pore interaction, and the effects of cementation and diagenesis. It is just a convenient way to incorporate critical porosity as an input to existing models.

DEM with Critical Porosity Inputs

Differential effective medium theory models two-phase composites by incrementally adding inclusions of one phase to the matrix phase. The matrix begins as phase 1 and is changed at each step as a new increment of phase 2 material is added. The process is continued until the desired proportion of the constituents is reached. For reviews of the DEM theory see e.g. Norris (1985), and Zimmerman (1991). The process of incrementally adding inclusions to the matrix is a thought experiment and not an accurate description of the true evolution of rock porosity in nature.

The coupled differential equations for the effective bulk and shear moduli, K^* and μ^*, respectively, are (Berryman, 1992):

$$(1-y)\frac{d}{dy}[K^*(y)] = (K_2 - K^*)P^{(*2)}(y) \tag{1}$$

$$(1-y)\frac{d}{dy}[\mu^*(y)] = (\mu_2 - \mu^*)Q^{(*2)}(y) \tag{2}$$

with initial conditions $K^*(0) = K_1$ and $\mu^*(0) = \mu_1$, where K_1, μ_1 are the bulk and shear moduli of the initial host material, K_2, μ_2 are the moduli of the phase 2 end member, and y is the concentration of phase 2. The factors $P^{(*2)}$ and $Q^{(*2)}$ depend on the inclusion aspect ratio. For spherical inclusions of phase 2, they are:

$$P^{(*2)}(y) = \frac{K^*(y) + \frac{4}{3}\mu^*(y)}{K_2 + \frac{4}{3}\mu^*(y)} \tag{3}$$

$$Q^{(*2)}(y) = \frac{\mu^*(y) + F^*(y)}{\mu_2 + F^*(y)} \tag{4}$$

where

$$F^*(y) = \frac{\mu^*(y)[9K^*(y) + 8\mu^*(y)]}{6[K^*(y) + 2\mu^*(y)]}. \tag{5}$$

Expressions for other ellipsoidal inclusions [based on Eshelby (1957)] may be found in Berryman (1980). In the usual DEM model, starting from a solid initial host, a porous material falls apart only at 100 percent porosity. This is because the solid host remains connected and therefore load bearing.

While DEM is a good model for materials such as glass foam and oceanic basalts (Berge et al., 1992, 1993), most reservoir rocks have a ϕ_c significantly less than 1.0 and are not represented well by the conventional DEM theory. We modify the DEM model to incorporate percolation behavior at any desired ϕ_c by redefining the phase 2 end member. The inclusions are now no longer pure fluid (the original phase 2) but are composite inclusions of the critical phase at ϕ_c with elastic moduli (K_c, μ_c). With this definition, y denotes the concentration of the critical phase in the matrix,

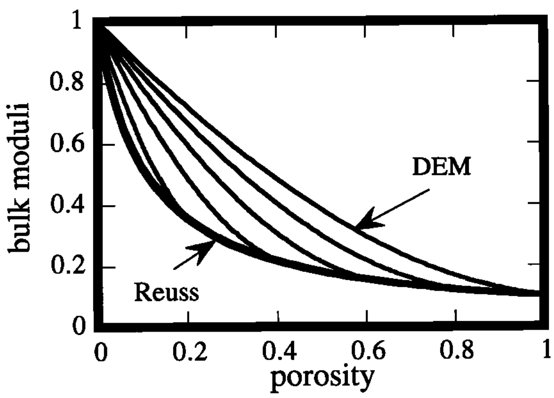

Figure 1. Normalized bulk moduli computed from the conventional and modified DEM theories.

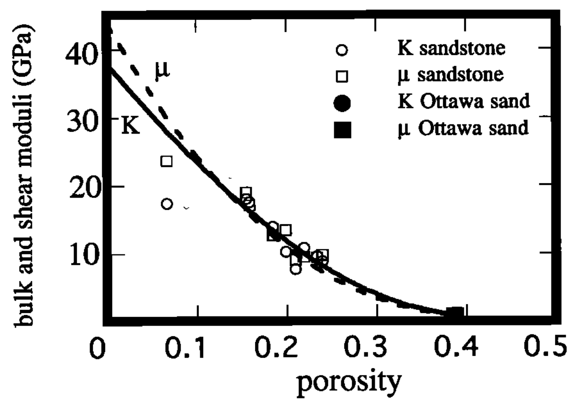

Figure 2. Comparison of theoretical and experimental bulk and shear moduli at 5 MPa confining pressure.

and now the total porosity $\phi = y\phi_c$. The computations are implemented by replacing (K_2, μ_2) by (K_c, μ_c) everywhere in equations (1)— (5). Integrating along the reverse path, from $\phi = \phi_c$ to $\phi = 0$, gives lower moduli as now the softer critical phase is the matrix. The moduli of the critical phase may be taken as the Reuss average value at ϕ_c of the pure end member moduli. For $\phi > \phi_c$, the material is a suspension best characterized by the Reuss average.

Figure 1 shows normalized bulk moduli curves for the conventional DEM theory (percolation at $\phi = 1$) and for the modified DEM (percolation at $\phi_c < 1.0$) for a range of ϕ_c values. When $\phi_c = 1$, the modified DEM coincides with the conventional DEM curve. The inclusions were taken to be spheres. The integration path was from 0 to ϕ_c and (K_c, μ_c) were taken as the Reuss average values at ϕ_c. Better estimates of K_c and μ_c may be obtained from data on loose sands or granular material models.

The elastic properties of the granular, unconsolidated critical phase are dominated by grain-to-grain contact interactions. For a brief review of contact theories see chapter 1 in Wang and Nur (1992). In our modeling we used Walton and Digby's theories [equation 4.10 in Walton (1987), and equations 33 and 34 from Digby (1981)], in addition to laboratory data, to get estimates of the critical phase moduli. However, we do not claim that this is always the best estimate of the critical phase properties. Other models exist (Yoshioka and Scholz, 1989; Palciauskas, 1992), but require several parameters that are difficult to determine.

Comparison with Laboratory Data

Figure 2 shows ultrasonic bulk and shear moduli in clean, dry sandstones under 5 MPa confining pressure (Han, 1986). The solid and dashed curves are the modified DEM predictions of bulk and shear moduli, respectively. The mineral moduli at $\phi = 0$ were those of quartz: K_1 = 38 GPa, and μ_1 = 44 GPa. The critical phase end point (K_c, μ_c) was obtained from ultrasonic measurements on loose Ottawa sand, also at 5 MPa confining pressure (Yin, 1992) before pressure cycling. The critical porosity (0.387), is the porosity of the loose sand. Bulk and shear moduli calculated from Walton's (1987) (infinitely rough spheres) and Digby's (1981) sphere pack models are close to the Ottawa sand moduli and would likewise be good estimates for the critical phase properties. For the Digby model the ratio of initial bonding radius to grain radius, b/R, was taken as 0.01. The path of integration was from $\phi = 0$ to $\phi = \phi_c$. Calculations along the reversed path give lower moduli and do not fit the data very well. We found no obvious reasons why the rock at 6 percent porosity lies below the general trend.

Figure 3a shows experimental and theoretical normalized bulk moduli for dry sandstones under 40 MPa confining pressure. The data includes clean and clay-bearing sandstones (with up to 10 percent clay) from Han (1986) and Jizba (1991). For the clean sandstones, the mineral moduli were those of quartz; for the clay-bearing ones, they were obtained from the zero porosity intercept of Han's (1986) linear regression for shaly

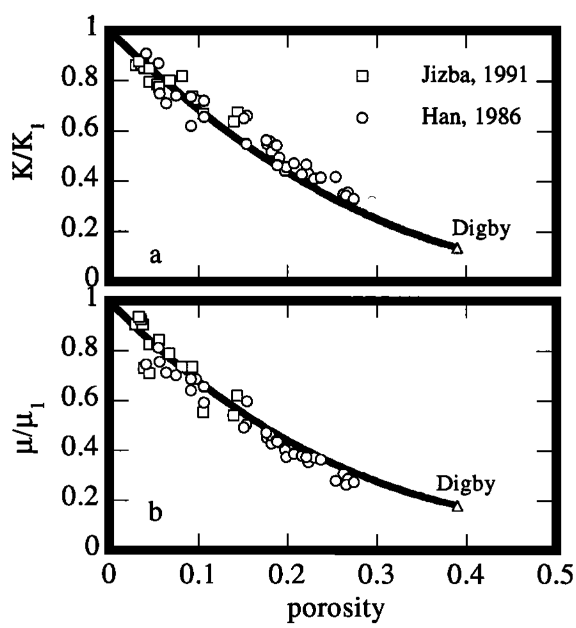

Figure 3. a) Experimental and theoretical bulk moduli for sandstones under 40 MPa confining pressure. b) as in a) but for the shear moduli.

sandstones ($K_1 = 31$ GPa, $\mu_1 = 34$ GPa). Figure 3b is a similar plot for the shear modulus. In this case, (K_c, μ_c) calculated from Digby's (1981) sphere pack model (with $b/R = 0.15$) gave good results. In fact, for these high pressure data, the modified Hashin-Shtrikman or the modified Voigt captures the linear trend of the observations equally well (Yin et al., 1993).

Summary

Seismic velocities of porous crustal rocks almost always show a percolation or critical porosity behavior. For most sedimentary rocks this percolation porosity is much less than 100 percent. The percolation behavior is a purely geometrical property, independent of physical properties of the rocks. Any valid velocity-porosity model for rocks must incorporate this critical porosity behavior and must also give estimates that always lie within the theoretically established bounds for elastic moduli. A simple way to include critical porosity as an input in existing analytical relations for two-phase composites is by redefining one of the end members to be the critical phase. We have used this simple concept to modify the conventional DEM theory to have a percolation behavior at any desired critical porosity. The predictions of the modified DEM model compare favorably with ultrasonic sandstone data. Since the DEM model is realizable, the estimates are always consistent with the Hashin-Shtrikman bounds.

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References

Avellaneda, M., Iterated homogenization, differential effective medium theory and applications, *Commun. Pure Appl. Math., 40*, 527-554, 1987.

Berge, P. A., G. J. Fryer, and R. H. Wilkens, Velocity-porosity relationships in the upper oceanic crust: Theoretical considerations, *J. Geophys. Res., 97*, 15239-15254, 1992.

Berge, P. A., J. G. Berryman, and B. P. Bonner, Influence of microstructure on rock elastic properties, *Geophys. Res. Lett. 20*, 2619-2622, 1993.

Berryman, J. G., Long-wavelength propagation in composite elastic media I. Spherical inclusions and II. Ellipsoidal inclusions, *J. Acoust. Soc. Amer., 68*, 1809-1831, 1980.

Berryman, J. G., Single-scattering approximations for coefficients in Biot's equations of poroelasticity, *J. Acoust. Soc. Amer., 91*, 551-571, 1992.

Chen, Q., Critical concentration models for the mechanical and acoustic properties of porous materials, Ph. D. thesis, Stanford University, Stanford, 1992.

Digby, P. J., The effective elastic moduli of porous granular rocks, *ASME J. Appl. Mech., 48*, 803-808, 1981.

Eshelby, J. D., The determination of the elastic field of an ellipsoidal inclusion, and related problems, *Proc. Roy. Soc. London, A241*, 376-396, 1957.

Guéguen, Y., and V. Palciauskas, *Introduction to the Physics of Rocks*, 294 pp., Princeton University Press, Princeton, 1994.

Hashin, Z., and S. Shtrikman, A variational approach to the elastic behavior of multiphase materials, *J. Mech. Phys. Solids, 11*, 127-140, 1963.

Han, D.-h., Effect of porosity and clay content on acoustic properties of sandstones and unconsolidated sediments, Ph. D. thesis, Stanford University, Stanford, 1986.

Jizba, D. L., Mechanical and acoustical properties of sandstones and shales, Ph. D. thesis, Stanford University, Stanford, 1991.

Milton, G. W., The coherent potential approximation is a realizable effective medium theory, *Commun. Math. Phys., 99*, 463-500, 1985.

Norris, A. N., A differential scheme for the effective moduli of composites, *Mech. of Mater., 4*, 1-16, 1985.

Nur, A., Critical porosity and the seismic velocities in rocks (abstract), *EOS Trans. AGU, 73*, 66, 1992.

Nur, A., D. Marion, and H. Yin, Wave velocities in sediments, in *Shear waves in marine sediments*, edited by J. M. Hovem, M. D. Richardson, and R. D. Stoll, pp. 131-140, Kluwer Academic, Dordrecht, 1991.

Palciauskas, V. V., Compressional to shear wave velocity ratio of granular rocks: role of rough grain contacts, *Geophys. Res. Lett., 19*, 1683-1686, 1992.

Sheng, P., Consistent modeling of the electrical and elastic properties of sedimentary rocks, *Geophysics, 56*, 1236-1243, 1991.

Walton, K., The effective elastic moduli of a random packing of spheres, *J. Mech. Phys. Solids, 35*, 213-226, 1987.

Wang, Z., and A. Nur (Eds.), *Seismic and acoustic velocities in reservoir rocks vol. 2, Theoretical and model studies*, Soc. Explor. Geophys., Tulsa, 1992.

Yin, H., Acoustic velocity and attenuation of rocks: Isotropy, Intrinsic Anisotropy, and Stress-Induced Anisotropy, Ph. D. thesis, Stanford University, Stanford, 1992.

Yin, H., A. Nur, and G. Mavko, Critical porosity—A physical boundary in poroelasticity, *Int. J. Rock Mech. Min. Sci. Geomech. Abstr., 30*, 805-808, 1993.

Yoshioka, N., and C. H. Scholz, Elastic properties of contacting surfaces under normal and shear loads: 1. Theory, *J. Geophys. Res., 94*, 17681-17690, 1989.

Zimmerman, R. W., *Compressibility of sandstones*, 173 pp. Elsevier Science, Amsterdam, 1991.

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