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Confirmation of Biot's theory

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(Received 4 April 1980; accepted for publication 9 June 1980)

Plona's recent measurements of elastic-wave speeds in a water-saturated porous structure of sintered glass beads are compared quantitatively to the predictions of Biot's theory. The theoretical predictions lie within the bounds of experimental error (3%) for the fast compressional wave and for the shear wave in all cases. For the slow compressional wave, the theoretically predicted speeds lie within about 10% of the experimental values and increase with increase in porosity as observed. Our model achieves this agreement with no significant free parameters. The frame moduli are estimated using a recently developed self-consistent theory of composite materials. The induced mass of the frame in a water environment is also estimated theoretically.

PACS numbers: 43.20.Hq, 43.35 Cg, 03.40.Kf, 62.30. + d

In recent experiments, Plona[1] has observed two distinct compressional waves in a water-saturated porous structure of sintered glass beads: (i) a fast compressional wave with speed comparable to the compressional wave speed of the constituent glass beads and (ii) a slow compressional wave with speed *less than the speed of sound in water*. Plona suggests that the observed slow compressional wave corresponds qualitatively to the *wave of the second kind* predicted by Biot's theory[2-7] of elastic wave propagation in fluid-saturated porous media. However, no quantitative comparison with the predictions of Biot's theory were presented. The purpose of this letter is to demonstrate quantitative agreement between Biot's theory and Plona's measurements.

Our approach departs from current standard practice as reviewed by Stoll[6] in two important respects. First, independent measurements of the frame moduli are not required. Instead a new method[8,9] of estimating elastic constants of composite materials is used to estimate the moduli of the porous frame. (Although the method developed in Ref. 8 is very similar in philosophy to that used in the work of Hill,[10] Budiansky,[11] and Wu,[12] the final results for needle-shaped inclusions used in this letter have been shown[9] to differ substantially from the results of these earlier papers.) Second, the parameter α (related to the induced mass of the porous frame in a fluid environment) is estimated theoretically. Consequently, our model has only one free parameter (the pore size a). For reasonable values of a ($\frac{1}{6}-\frac{1}{7}$ of the glass bead size), the calculated elastic-wave speeds are insensitive to the precise value of this parameter in the pertinent range of frequency ($f \sim 500$ kHz). Since our model has virtually no free parameters, the quantitative agreement found between theory and experiment is strong evidence that the observed slow compressional wave is indeed the one predicted by Biot.

Since Biot's theory is widely used, the agreement which we find between theory and experiment will not surprise those who have already accepted the theory as a correct description of the behavior of elastic waves in fluid-saturated porous media. However, for those who believe that a theory is not proven correct until all of its predictions are verified experimentally, we believe the present results are sufficiently strong to eliminate grounds for further skepticism.

To summarize the analysis, first recall the dispersion relations[3,6,7] for the compressional wave speeds $\bar{v}_c^2 = \omega^2/k^2$,

$$\begin{vmatrix} \rho - H/\bar{v}_c^2 & \rho_f - C/\bar{v}_c^2 \\ \rho_f - C/\bar{v}_c^2 & q_f - M/\bar{v}_c^2 \end{vmatrix} = 0, \tag{1}$$

and for the shear wave speed $\bar{v}_s^2 = \omega^2/s^2$,

$$\begin{vmatrix} \rho - \mu^*/\bar{v}_s^2 & \rho_f \\ \rho_f & q_f \end{vmatrix} = 0. \tag{2}$$

In terms of the porosity β, bulk moduli of glass K_g and fluid K_f, and densities of the glass ρ_g and the fluid ρ_f, the various terms appearing in (1) and (2) are defined by[6]

$$H = K^* + \tfrac{4}{3}\mu^* + (K_g - K^*)^2/(D - K^*), \tag{3}$$

$$C = K_g(K_g - K^*)/(D - K^*), \tag{4}$$

$$M = K_g^2/(D - K^*), \tag{5}$$

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$$D = K_g[1 + \beta(K_g/K_f - 1)], \tag{6}$$

$$\rho = (1-\beta)\rho_g + \beta\rho_f, \tag{7}$$

$$q_f = \alpha\rho_f/\beta + iF(\zeta)\eta/\kappa\omega, \tag{8}$$

where η is the kinematic viscosity of the fluid, κ is the permeability of the porous frame, and ω is the angular frequency of the plane wave proportional to $\exp i(kx - \omega t)$ for a compressional wave or $\exp i(sx - \omega t)$ for a shear wave; $F(\zeta)$ is a dynamic viscosity factor defined by

$$F(\zeta) = \frac{1}{4}\frac{\zeta T(\zeta)}{1 - 2T(\zeta)/i\zeta}, \tag{9}$$

where

$$T(\zeta) = \frac{\mathrm{ber}'(\zeta) + i\,\mathrm{bei}'(\zeta)}{\mathrm{ber}(\zeta) + i\,\mathrm{bei}(\zeta)}, \tag{10}$$

$$\zeta = (\omega/\omega_c)^{1/2} = (\omega a^2/\eta)^{1/2}. \tag{11}$$

The functions $\mathrm{ber}(\zeta)$ and $\mathrm{bei}(\zeta)$ are the real and imaginary parts of the Kelvin function. The factor $\alpha \geqslant 1$ will presently be shown to be simply related to the induced mass of the frame in a fluid environment. The bulk K^* and shear μ^* moduli of the porous frame are also related to the bulk K_g and shear μ_g moduli of the constituent glass beads in a manner which will now be described.

Methods of experimentally determining the bulk modulus K^* of the free-draining porous frame using a "jacketed" test have been discussed by Geertsma,[13] Biot and Willis,[14] and Stoll.[6] However, it follows[15] from the physical interpretation of K^* and μ^* as the moduli of the porous (drained) frame that these moduli can be estimated by treating the frame as a two-phase composite; one phase has the moduli of the glass $K_1 = K_g$ and $\mu_1 = \mu_g$, while the other phase is vacuum $K_2 = 0 = \mu_2$. The recently developed self-consistent theory of composite materials[8,9] then predicts that

$$\sum_{n=1}^{2} c_n(K_n - K^*)\tfrac{1}{3}T_{iijj}^{(n)} = 0, \tag{12}$$

$$\sum_{n=1}^{2} c_n(\mu_n - \mu^*)\tfrac{1}{5}(T_{ijij}^{(n)} - \tfrac{1}{3}T_{iijj}^{(n)}) = 0. \tag{13}$$

The volume concentrations are clearly $c_1 = 1 - \beta$ and $c_2 = \beta$. The values of the scalars T_{iijj} and T_{ijij} depend on the K_n's, μ_n's, K^*, μ^*, and also on the shapes of the inclusions, ellipsoidal-shaped inclusions being treated most readily by this approach.[12] For $\beta \ll 1$, the ellipsoidal shape which the pores should most closely resemble is that of a prolate spheroidal void with very small aspect ratio. Then, to a good approximation, the scalars for needle-shaped inclusions may be used to represent the pore space. Similarly, for $1 - \beta \ll 1$, the frame becomes so porous that only thin filments of solid material remain; these thin filaments again may be represented by needle-shaped inclusions. Thus the scalars for needle-shaped inclusions[12] may be used for both constituents in (12) and (13). Equations (12) and (13) may be solved easily by a numerical iteration scheme and provide estimates of the frame moduli K^* and μ^*. Although these self-consistent estimates are subject to the criticism that they are only approximations to the true values of K^* and μ^*, it has been shown elsewhere[9] that these estimates satisfy all the known rigorous bounds on elastic moduli of composites. In light of this other work, we expect the proposed estimates to be quite good approximations to K^* and μ^* in the range of small porosities relevant in Plona's experiments.

To estimate α, it is useful to consider the physical interpretation of certain densities $\rho_{11}, \rho_{12}, \rho_{22}$ introduced in Biot's original work.[2] Biot notes the relations

$$\rho_1 = \rho_{11} + \rho_{12} = (1-\beta)\rho_g, \tag{14}$$

$$\rho_2 = \rho_{22} + \rho_{12} = \beta\rho_f. \tag{15}$$

Furthermore, ρ_{11} represents the total effective density of the solid moving in the fluid while $-\rho_{12}$ is the additional apparent density ($\rho_{12} \leqslant 0$). If we make the definition $\rho_{22} = \alpha\beta\rho_f$, where α is a parameter to be determined (this definition is consistent with both Stoll[6] and Biot[16]), we find

$$\rho_{12} = -(\alpha - 1)\beta\rho_f, \tag{16}$$

and ρ_{12} satisfies the inequality $\rho_{12} \leqslant 0$ if $\alpha \geqslant 1$. But from its physical interpretation it follows that ρ_{11} must be given by

$$\rho_{11} = (1-\beta)(\rho_g + r\rho_f), \tag{17}$$

where $r\rho_f$ is the induced mass[17] due to the oscillation of solid particles in the fluid. In particular, $r = \frac{1}{2}$ for spheres. In general, the value of r must be calculated from a microscopic model of the frame moving in the fluid. Substituting (16) and (17) into (19), we find that

$$\alpha = 1 - r(1 - 1/\beta). \tag{18}$$

Observe that $\alpha = 1$ for $\beta = 1$ and that $\rho_{11}\rho_{22} \geqslant \rho_{12}^2$ as expected. Also note that $\alpha \to \infty$ as $\beta \to 0$. This latter behavior is particularly important for the slow compressional wave since it can be shown[15] that, when the second term on the right-hand side of (8) may be neglected relative to the first (i.e., at high frequencies on the order of 500 kHz),

$$v_c^2 \simeq K_f/\alpha\rho_f = \beta K_f/\rho_{22} \tag{19}$$

as $\beta \to 0$. Thus the speed of the slow compressional wave is strongly dependent upon α for small β.

Further assumptions concerning the dependence of the permeability κ and the pore size parameter a on porosity are required to complete the model. It turns out that the speeds are largely insensitive to the precise values of κ and a chosen. For simplicity, we have chosen to let κ obey the Kozeny-Carman relation[18]

$$\kappa(1-\beta)^2/\beta^3 = \kappa_0(1-\beta_0)^2/\beta_0^3 = \text{const}, \tag{20}$$

and similarly

$$a^2/\kappa = a_0^2/\kappa_0 = \text{const}. \tag{21}$$

Plona's experimental value of the permeability was $\kappa_0 \simeq 1$ $\mu\mathrm{m}^2$ ($= 1$ darcy), while the range of values was approximately $0.5\kappa_0 \leqslant \kappa \leqslant 1.5\kappa_0$ for samples like those used in the experiments.[19] The pore size parameter cannot be measured but is generally estimated[6] to be in the range $\frac{1}{6} - \frac{1}{7}$ of the grain size (0.21–0.29-mm diameter).

The input parameters to our model (obtained from Ref. 1) were $K_g = 0.407$ Mb, $\mu_g = 0.297$ Mb, $\rho_g = 2.48$ g/cc, $K_f = 0.022$ Mb, $\rho_f = 1.00$ g/cc, $\eta = 1.00$ centistoke, and $f = 500$ kHz. The frame moduli K^* and μ^* are computed

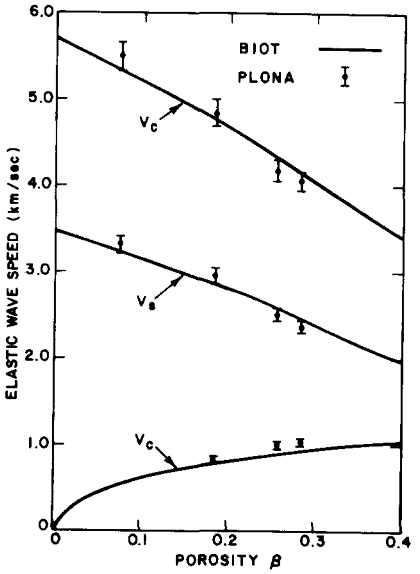

FIG. 1. Comparison of theoretical predictions and experimental results for speeds of the fast compressional wave, the shear wave, and the slow compressional wave. Input parameters to the model are $K_g = 0.407$ Mb, $\mu_g = 0.297$ Mb, $\rho_g = 2.48$ g/cc, $K_f = 0.022$ Mb, $\rho_f = 1.00$ g/cc, $\eta = 1.00$ centistoke, $f = 500$ kHz, $\kappa_0 = 1\ \mu\text{m}^2$, and $a_0 = 0.02$ mm.

from (12) and (13) as previously explained. The induced mass factor α is calculated from (18) with $r = \frac{1}{2}$ since we expect many of the glass beads to retain their spherical shape during sintering for the porosities studied.[1] (A better estimate of r based on the true frame geometry would be desirable and will be attempted in future work.) The permeability was chosen so that $\kappa = \kappa_0 = 1\ \mu\text{m}^2$ at $\beta_0 = 0.20$ and other values of κ were estimated from (20). The value of a_0 at $\beta_0 = 0.20$ was varied from $a_0 = 0.01$–0.05 mm with no significant variation found in the computed speeds. The real speeds v_c and v_s are determined from $\bar{v}_c$ and $\bar{v}_s$ using

$$\frac{1}{\bar{v}} = \frac{1}{v}\left(1 + \frac{i}{2Q}\right), \tag{22}$$

where Q^{-1} is the attenuation factor (our calculated values of Q^{-1} are always less than 10^{-6} for the porosities in Plona's experiments). The results of the calculations are illustrated in Fig. 1.

Observe that the theoretical results for the fast compressional wave and the shear wave agree with Plona's measurements within experimental error (3%) in all four cases. Agreement is not quite so good for the slow compressional wave. The value at $\beta = 0.185$ agrees within experimental error but the theoretical values at $\beta = 0.258$ and 0.283 are lower than the experimental values by about 10%. Nevertheless, we consider this to be very good agreement. It is important to emphasize that the agreement observed in Fig. 1 was obtained using *a model with no significant free parameters*. To the author's knowledge, this result is the first quantitative confirmation of the wave of the second kind predicted by Biot's theory. The accuracy of the results presented in Fig. 1 also lends further support both to the effective medium theory used to estimate K^* and μ^* and to the estimate (18) of the parameter α. For the shear wave speed in Fig. 1, the agreement between theory and experiment is a direct check on the accuracy of the self-consistent estimate of μ^* [see Eq. (2)]. For the two compressional wave speeds, the accuracy check is less direct because it involves K^* and α simultaneously in a quadratic formula [see Eq. (1)]. Nevertheless, the results are quite satisfactory.

We conclude that Biot's theory provides an accurate description of elastic wave propagation in fluid-saturated porous media and also that the various assumptions and approximations made in this letter provide a useful prescription for evaluating the parameters appearing in Biot's theory.

The author thanks R. L. Holford, P. R. Ogushwitz, and T. J. Plona for enlightening discussions.

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