Homework 2

Eigen Value Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are eigen-values of matrix A. Prove :

- $\lambda_1 + ... + \lambda_n = trace(A)$
- $\lambda_1 ... \lambda_n = |A|$
- AB and BA have the set of eigen values.

Matrix derivative Prove the followings.

- $\frac{\partial a.x}{\partial x} = a^T$
- $\frac{\partial x^T A x}{\partial x} = 2x^T A$
- $\frac{\partial^2 x^T A x}{\partial x \partial x^T} = A + A^T$
- $\frac{\partial trace(X^T A X)}{\partial X} = X^T (A + A^T)$

Eigen value and rank

1. Prove that if P is a full rank matrix, matrices M and $P^{-1}MP$ have the same set of eigen values
2. Prove that sum of dimensions of eigenspaces of an $n * n$ matrix M can't exceed n. (M has at most n different eigenvalues)

Symmetric positive definite Prove that every symmetric positive definite matrix A has a unique factorization of the form $A = LL^T$, where L is a lower triangular matrix with positive diagonal entries.

Basic Concepts

1. Random variable X has PMF $f(x) = \begin{cases} a * (1/3)^{x-2} & x = 2, 3, ... \\ 0 & \text{o.w} \end{cases}$.
   (a) Find a
   (b) Find CDF $F_X(x)$
   (c) Find $P(X > 5)$
   (d) Find $P(X > 5 | X < 7)$
2. X, Y are R.Vs:
   (a) Prove $E[E[X|Y]] = E[X]$
   (b) Prove $var(X) = E[var(X|Y)] + var(E[X|Y])$
3. Assume $X_1, X_2, ...$ are i.i.d. Now we define another R.V as N such that :
   $$X_1 \geq X_2 \geq ... \geq X_{N-1} < X_N$$
   for $N \geq 2$ find $E[N]$

Dirichlet We recorded the attendance of students at week 1 and week 2. Let the probability that a student attends in both weeks be θ_{11}, the probability that a student attends in week 1 but not Week 2 be θ_{10} and so on. The data are as follows.

Attendance	Probability	Observed frequency
Week1 & Week2	θ_{11}	$n_{11} = 25$
Week1 but not Week2	θ_{10}	$n_{10} = 7$
Week2 but not Week1	θ_{01}	$n_{01} = 6$
Neither weeks	θ_{00}	$n_{00} = 13$

Suppose that the prior distribution for $(\theta_{11}, \theta_{10}, \theta_{01}, \theta_{00})$ is a Dirichlet distribution with density proportional to

$$\theta_{11}^3 \theta_{10} \theta_{01} \theta_{00}^2$$

1. Find the prior means and prior variances of $\theta_{11}, \theta_{10}, \theta_{01}, \theta_{00}$
2. Find the posterior distribution.
3. Find the posterior means and posterior variances of $\theta_{11}, \theta_{10}, \theta_{01}, \theta_{00}$

Multivariate Gaussian Suppose $X \sim N_3(\mu, \sum)$ where $\mu^T = (2, -3, 1)$ and $\sum = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

1. Find the distribution of $3X_1 - 2X_2 + X_3$
2. Relabel the variables if necessary, and find a 2×1 vector a such that X_2 and $X_2 - a^T \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ are independent.

ML & MAP Suppose X, Y are independent Normal R.Vs with mean μ and variance 1, where $\mu \sim Uni(0, 1)$. $f_\mu(t) = \begin{cases} 1 & t \in [0, 1] \\ 0 & o.w \end{cases}$

1. Find joint distribution of μ, X, Y.(Find $f_{\mu,X,Y}(t, x, y)$)
2. Find the MAP estimate of μ.
3. Evaluate the estimator you found in part (2) if the data is as given below.
   (a) x= 3/4 , y = 1
   (b) x= 1/2 , y = 2