MTH-684 Logic
Assignment (5): First-Order Predicate Logic

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5-3

b.

Given:

$$
\begin{aligned}
&\mathcal{U} = \{switch1, switch2, bulb\} \\
&\mathcal{F} = \{s1^0, s2^0, b^0\} \\
&\mathcal{I}_{\mathcal{F}} = \{(s1, switch1), (s2, switch2), (b, bulb)\} \\
&\mathcal{P} = \{Switch^1, Bulb^1, Up^1, Down^1, On^1, Off^1\} \\
&\mathcal{I}_{\mathcal{P}}(Switch) = \{switch1, switch2\} \\
&\mathcal{I}_{\mathcal{P}}(Bulb) = \{bulb\} \\
&\mathcal{I}_{\mathcal{P}}(Up) = \{switch1\} \\
&\mathcal{I}_{\mathcal{P}}(Down) = \{switch2\} \\
&\mathcal{I}_{\mathcal{P}}(On) = \{bulb\} \\
&\mathcal{I}_{\mathcal{P}}(Off) = \{\} \\
&M = (\mathcal{U}, \mathcal{I}_{\mathcal{F}}, \mathcal{I}_{\mathcal{P}}) \\
&\mathcal{V} = \{x, y, z_1, z_2, ...\} \\
&s: \\
&\quad s(x) = switch1 \\
&\quad s(y) = switch2 \\
&\quad s(z_i) = bulb
\end{aligned}
$$

Let:

$$P(x, y) := Switch(x) \land Switch(y) \land Down(x) \land Down(y)$$
$$Q(z_1) := Bulb(z_1) \implies Off(z_1)$$

Determine the truth value of the formula:

$$\forall x(\forall y(P(x, y) \implies \forall z_1 Q(z_1)))$$

<u>Solution:</u>

$$
\begin{aligned}
& [[\forall x(\forall y(P(x, y) \implies \forall z_1 Q(z_1)))]]^{M,s} = \top & \\
\iff & [[\forall y(P(x, y) \implies \forall z_1 Q(z_1))]]^{M,s[a/x]} = \top & \text{for every } a \in \mathcal{U} \\
\iff & [[P(x, y) \implies \forall z_1 Q(z_1)]]^{M,s[a/x][b/y]} = \top & \text{for every } a, b \in \mathcal{U} \\
\iff & [[\neg P(x, y)) \lor \forall z_1 Q(z_1)]]^{M,s[a/x][b/y]} = \top & \text{for every } a, b \in \mathcal{U} \\
\iff & ([[\neg P(x, y)]]^{M,s[a/x][b/y]} = \top & \\
& \quad \text{or} & \\
& [[\forall z_1 Q(z_1)]]^{M,s[a/x][b/y]} = \top) & \text{for every } a, b \in \mathcal{U} \\
\iff & ([[P(x, y)]]^{M,s[a/x][b/y]} = \bot & \\
& \quad \text{or} & \\
& [[Q(z_1)]]^{M,s[a/x][b/y][c/z_1]} = \top) & \text{for every } a, b, c \in \mathcal{U} \\
\iff & ([[Switch(x) \land Switch(y) \land Down(x) \land Down(y)]]^{M,s[a/x][b/y]} = \bot &
\end{aligned}
$$

$$
\begin{aligned}
& \quad \text{or} \\
& [[Bulb(z_1) \implies Off(z_1)]]^{M,s[a/x][b/y][c/z_1]} = \top) && \text{for every } a,b,c \in \mathcal{U} \\
\iff & ([[Switch(x) \land Switch(y) \land Down(x) \land Down(y)]]^{M,s[a/x][b/y]} = \bot \\
& \quad \text{or} \\
& [[\neg Bulb(z_1) \lor Off(z_1)]]^{M,s[a/x][b/y][c/z_1]} = \top) && \text{for every } a,b,c \in \mathcal{U} \\
\iff & ([[Switch(x)]]^{M,s[a/x][b/y]} = [[Switch(y)]]^{M,s[a/x][b/y]} = [[Down(x)]]^{M,s[a/x][b/y]} = [[Down(y)]]^{M,s[a/x][b/y]} = \bot \\
& \quad \text{or} \\
& [[\neg Bulb(z_1)]]^{M,s[a/x][b/y][c/z_1]} = \top \text{ or } [[Off(z_1)]]^{M,s[a/x][b/y][c/z_1]} = \top) && \text{for every } a,b,c \in \mathcal{U} \\
\iff & ([[Switch(x)]]^{M,s[a/x][b/y]} = [[Switch(y)]]^{M,s[a/x][b/y]} = [[Down(x)]]^{M,s[a/x][b/y]} = [[Down(y)]]^{M,s[a/x][b/y]} = \bot \\
& \quad \text{or} \\
& ([[Bulb(z_1)]]^{M,s[a/x][b/y][c/z_1]} = \bot \text{ or } [[Off(z_1)]]^{M,s[a/x][b/y][c/z_1]} = \top)) && \text{for every } a,b,c \in \mathcal{U} \\
\iff & (([[x]]^{M,s[a/x][b/y]} \notin \{switch1, switch2\} \text{ and } [[y]]^{M,s[a/x][b/y]} \notin \{switch1, switch2\} \text{ and } [[x]]^{M,s[a/x][b/y]} \notin \{switch2\} \text{ and } [[y]]^{M,s[a/x][b/y]} \notin \{switch2\}) \\
& \quad \text{or} \\
& ([[z_1]]^{M,s[a/x][b/y][c/z_1]} \notin \{bulb\} \text{ or } [[z_1]]^{M,s[a/x][b/y][c/z_1]} \in \emptyset)) && \text{for every } a,b,c \in \mathcal{U} \\
\iff & ((a \notin \{switch1, switch2\} \text{ and } b \notin \{switch1, switch2\} \text{ and } a \notin \{switch2\} \text{ and } b \notin \{switch2\}) \\
& \quad \text{or} \\
& (c \notin \{bulb\} \text{ or } c \in \emptyset)) && \text{for every } a,b,c \in \mathcal{U}
\end{aligned}
$$

Take $a = switch2, b = switch2, c = bulb$, then:

$$
\begin{aligned}
&((switch2 \notin \{switch1, switch2\} \text{ and } switch2 \notin \{switch1, switch2\} \text{ and } switch2 \notin \{switch2\} \text{ and } switch2 \notin \{switch2\}) \\
&\quad \text{or} \\
&(bulb \notin \{bulb\} \text{ or } bulb \in \emptyset)) \\
\iff &((\bot \text{ and } \bot \text{ and } \bot \text{ and } \bot) \\
&\quad \text{or} \\
&(\bot \text{ or } \bot)) \\
\iff &(\bot \\
&\quad \text{or} \\
&\bot) \\
\iff &\bot
\end{aligned}
$$

$\iff$ The formula is false under the given interpretation structure.