# Operations Research and Optimization Questions Bank

Mostafa Hassanein

21 January 2024

# Proofs

**Theorem 1.**

*Given the following LP:*

*Minimize* $C^T X$

*Subject to,*

$x \in F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}.$

*Show that:*

*Corresponding to* $F$ *there exists a polytope* $P \subseteq R^{n-m}$.

*Proof.*

Assume $rank(A) = m$.

Then $A$ can be arranged into $m$ basis columns and $(n-m)$ non-basis columns, thus the constraint $AX = b$ becomes:

$$\begin{bmatrix} n-m & & m \\ N & | & B \\ & | & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-m} \\ x_{n-m+1} \\ x_b \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$\implies NX_N + BX_B = b$

$\implies X_B = B^{-1}b - B^{-1}NX_N$

$\implies x_i = b'_i - \sum_{j=1}^{n-m} a'_{ij}x_j \qquad \text{for } i = n-m+1 \ldots n. \qquad (1)$

Substituting (1) into the non-negativity constraints we get:

$$\left\{ \begin{matrix} x_j > 0, & j = 1 \ldots n-m \\ b'_i - \sum_{j=1}^{n-m} a'_{ij}x_j > 0 & i = n-m+1 \ldots n \end{matrix} \right\} \qquad (2)$$

But (2) describes the intersection of $n$ halfspaces in $R^{n-m}$.

Thus, it describes a convex polytope in $R^{n-m}$.

Conversely, ...

$\square$

**Theorem 2.**

*Given the following LP:*

*Minimize* $C^T X$

*Subject to,*

$x \in F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}.$

*Show that:*

*If the given LP has an optimal solution, then at least one bfs is optimal. Furthermore, if q bfs's are optimal then their convex combinations are also optimal.*

*Proof.*

We know that a BFS in $F$ corresponds to a vertex in $P$.

So it suffices to show that an optimal solution occurs at one of the vertices of $P$.

Since $P$ is closed and bounded, then the cost function $c$ attains its min in $P$.

Let $x_o$ be an optimal solution, then:

$x_o = \sum_{i=1}^{N} \alpha_i v_i, \quad 0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{N} \alpha_i = 1.$

Let $v_j$ be the vertex with the lowest cost, then:

$c^T x_o = \sum_{i=1}^{N} \alpha_i c^T v_i \geq \sum_{i=1}^{N} \alpha_i c^T v_j = c^T v_j.$

$\Longrightarrow$ $v_j$ is optimal.

Next, suppose the vertices $v_{j1}, \ldots, v_{ja}$ are optimal, then:

$c^T \sum_{i=1}^{a} \alpha_i v_i = \sum_{i=1}^{a} \alpha_i c^T v_i = c^T v_{j1}.$

Therefore, their convex combination is optimal.

□

**Theorem 3.**

*Given the following LP:*

*Minimize* $C^T X$

*Subject to,*

$x \in F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}$.

*Starting from a bfs* $x_o$ *with basis* $B = \{A_{B(i)}, i = 1, \ldots, m\}$*, show how to obtain an adjacent bfs* $x'_0$ *with basic* $B'$ *containing* $A_j \notin B$*.*

*Proof.*

Let $x_o$ be an optimal solution corresponding to the basis $B = \{A_{B(i)}\}$, then:

$\sum_{i=1}^{m} x_{oi} A_{B(i)} = b. \qquad (1)$

Let $A_j \notin B$, then:

$A_j - \sum_{i=1}^{m} x_{ij} A(B(i)) = 0. \qquad (2)$

Multiply (2) by $\theta$ and subtract from (1):

$\sum_{i=1}^{m} (x_{io} - x_{ij}) A_{B(i)} - A_j = b.$

Assume $x_o$ is not degenerate, then $x_{io} > 0$.

As $\theta$ increases we move from the current bfs to a feasible solution with $(m+1)$ positive components.

How far can we increase $\theta$ and remain feasible?

Until $\theta = \min \frac{x_{io}}{x_{ij}}$. At this point we reach an adjacent bfs with $m$ strictly positive components and with $A_j$ in the basis.

□

**Theorem 4.**

*Given the following LP:*

*Minimize* $C^T X$

*Subject to,*

$x \in F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}$.

*Derive the effect of the step from* $x_o$ *to* $x'_o$ *on the cost* $C^T X$*, Hence deduce the optimality criterion.*

*Proof.*

The cost of the bfs with basis $B$ is: $\sum_i^m x_{io} C_{B(i)}$.

Define $z_j = \sum_{i=1}^m x_{ij} C_{B(i)}$ using the tablea after diagonalization.
Therefore: $Z^T = C_B^T B^{-1} A$.

A pivot step in which $x_j$ enters the basis changes the cost by the amount:

$\theta_o \bar{c_j} = \theta_o (c_j - z_j)$.

Therefore, it is profitable to bring $A_j$ into the basis exactly when $\bar{c_j} < 0$.

Furthermore, when $\forall j \ \bar{c_j} \geq 0$, then we've arrived at an optimal solution.

$\square$

**Theorem 5 & 6.**

*Given the following LP:*

*Minimize* $C^T X$

*Subject to,*

$x \in F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}$.

*Show that:*

*The same pivoting rules in the Simplex Table can be applied on the characteristic row (row of relative costs).*

*OR*

*In the simplex tableau the same pivoting rules can be applied to the zeroth row:* $-Z_o = -Z + \sum_{j=1\ A_j \notin B}^{n} \bar{c}_j x_j$.

*Proof.*

We add to the tableau Row 0 containing $\bar{c}_j$. To do this we have:

$0 = -Z + \sum_{i=1}^{n} c_i x_i$. (*)

Then, multiplying each row of the table by $-C_{B(i)}$ and adding to (*) we get Row 0 as:

$-Z_o = -Z + \sum_{j=1\ A_j \notin B}^{n} \bar{c}_j x_j$.

□

**Theorem 8.a.**

*Show that if 2 distinct bases correspond to the same bfs, then it is degenrate.*

*Proof.*

Suppose that $B$ and $B'$ both determine the same bfs $x$.

Then $x$ has zeros in $(n-m)$ columns not in $B$.

But that implies that it must also have zeros in at least one column in $B - B' \neq \emptyset$, hence it is degenerate.

□

**Theorem 8.b.**

*Given the following LP:*

*Minimize* $C^T X$

*Subject to,*

$x \in F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}$.

*Show that:*

*For a degenerate bfs with* $P < m$ *positive component it may correspond up to* $C_{n-m}^{n-p} = \frac{(n-p)!}{(n-m)!(m-p)!}$ *different bases.*

*Proof.*

The number of zeros in this degenerate bfs $= (n-p)$.

To determine a bfs we must choose $(n-m)$ non-basic variables (the zero variables).

Therefore, the possible number of ways we can choose the non-basic variables is given by:

$$C_{n-m}^{n-p} = \frac{(n-p)!}{(n-m)!(m-p)!}.$$

$\square$

**Theorem 7.**

*Given the following LP:*

*Minimize* $C^T X$

*Subject to,*

$x \in F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}$.

*Show that:*

*A vector* $X$ *is an optimal solution of the problem if there exist vectors* $r$ *and* $w$ *such that:*

*a.* $AX = b, x \geq 0$.

*b.* $A^T w + r = c, r \geq 0$.

*c.* $r^T x = 0$.

*and, in this case* $w$ *is an optimal solution of the dual problem.*

**Theorem 9.a.**

*For a primal-dual pair show that:*

*a. If the primal has an optimal solution, so does its dual and at optimality their costs are equal.*

**Theorem 9.b.**

*For a primal-dual pair show that:*

*b. If either problem has unbounded objective value, then the other has no feasible solution.*

**Theorem 9.c.**

*For a primal-dual pair show that:*

*c. The dual of the dual is the primal.*

**Theorem 10..**

*Show the possible categories of primal-dual pair.*

**Theorem 11..**

*Show that* $F = \{x \in R^n : AX = b, x \geq 0, b \in R^m\}$ *is a convex polyhedron.*

*Proof.*

Same as Theorem(1). $\square$

**Theorem 12.a.**

*Suppose* $C_1, \ldots, C_p$ *are convex in* $R^n$.

*Prove that:* *a.*$\bigcap_{i=1}^{P}$ *is convex.*

**Theorem 12.b.**

*Suppose* $C_1, \ldots, C_p$ *are convex in* $R^n$.

*Prove that:* $\quad b.\bigcup_{i=1}^{P}$ *is not convex.*

**Theorem 13.**

*Show that a feasible pair* $x, \pi$ *in a primal-dual pair is optimal iff:*

$\forall i, j: \ \pi_i(a_i^T - b) = 0$ *and* $(C_j - \pi^T A_j)x_j = 0.$

**Theorem 14.**

*Show that the linear system $AX = b,\ x \geq 0$ has no solution* ***iff*** *the system $A^T u \leq 0,\ b^T u > 0$ has a solution.*

**Theorem 15.a.** *(Necessary Condition for Multivariable Optimal Solution)*
*Show that if $f(x)$ has a local min (max) at $x = x^*$ and if the first partial derivatives of $f(x)$* ***exist*** *at $x^*$, then: $\nabla f(x^*) = 0$ is a necessary condition.*

*Proof.*
Perform a taylor expansion around $x^*$:
$f(x^* + h) = f(x^*) + h^T \nabla f(x^*) + R_1(x^*, h)$.
$\implies f(x^* + h) - f(x^*) = h^T \nabla f(x^*) + R_1(x^*, h)$.

For small $h$, the first order term dominates the higher order terms. Therefore, the sign of the LHS depends only on the sign of the first term (component-wise).

But the sign of the first term depends on the sign of $h$, so the only way to remove that dependency on the sign of $h$ is to have $\nabla f(x^*) = 0$.

$\square$

**Theorem 15.b.** *(Sufficient Conditions for Multivariable Optimal Solution)*

*Show that the sufficient condition for a stationary point $x^*$ to be a local minimum is the second partial derivatives (Hessian) of $f$ at $x^*$ is a positive definite matrix: $H(x^*) = \nabla^2 f(x^*) > 0$.*

*Proof.*

Perform a Taylor's expansion around $x^*$:

$f(x^* + h) = f(x^*) + h^T \nabla f(x^*) + \frac{1}{2} h^T \nabla^2 f(x^* + \theta h) h \quad$ for $0 < \theta < 1$.

Then at the stationary point $x^*$:

$$
\begin{aligned}
f(x^* + h) - f(x^*) &= h^T \nabla f(x^*) + \frac{1}{2} h^T \nabla^2 f(x^* + \theta h) h \\
&= 0 + \frac{1}{2} h^T \nabla^2 f(x^* + \theta h) h \\
&= \frac{1}{2} h^T \nabla^2 f(x^* + \theta h) h
\end{aligned}
$$

To be a local min, the RHS should be $> 0$, then:

$H(x^*) = \nabla^2 f(x^*) > 0$

$\implies$ $H$ is a positive definite matrix.

$\square$

**Theorem 16.**

*Show that the linear system $Ax = b$, $x \geq 0$ has no solution iff the system $A^T u \leq 0$, $b^T u > 0$ has a solution.*

*Proof.* **Forward direction:** (By Contradiction)

Given that $Ax = b$, $x \geq 0$ has a solution, suppose for the sake of contradiction that $A^T u \leq 0$, $b^T u > 0$ also has a solution.

$Ax = b$
$\implies x^T A^T = b^T$
$\implies x^T A^T u = b^T u$
$\implies x^T < 0$

This is a contradiction. Therefore the system $A^T u \leq 0$, $b^T u > 0$ has no solution.

**Reverse direction:** (By Contradiction)

Given that $A^T u \leq 0$, $b^T u > 0$ has a solution, suppose for the sake of contradiction that $Ax = b$, $x \geq 0$ also has a solution.

$A^T u \leq 0$
$\implies u^T A \leq 0$
$\implies u^T Ax \leq 0$
$\implies u^T b \leq 0$
$\implies b^T u \leq 0$

This is a contradiction. Therefore the system $Ax = b$, $x \geq 0$ has no solution.

□

**Theorem 17.**

*KKT Condition.*

*Given a linear programming problem in its standard form, a vector $x$ is an optimal solution to the problem iff $\exists$ vectors $r, \pi$ such that:*

1. $Ax = b$, $x \geq 0$ *(Primal feasibility).*
2. $A^T\pi + r = c$, $r \geq 0$ *(Dual feasibility).*
3. $r^T x = 0$ *(Complementary slackness).*

*In this case $\pi$ is an optimal solution to the dual problem.*

**Theorem 17.**

*Degenerecy:*

*Definition An LP is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value.*

# Problems

## 1.

Minimize: $f(x) = 0.65 - \frac{0.75}{x^2+1} - 0.65 tan^{-1}(\frac{1}{x})$, for $x \in (0, 3)$, using the **golden search method**. (Take n = 6).

## 2.

Minimize: $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$ along the direction $s_1 = [4\ 0]^T$, using the **quadratic interpolation method**.

(Start with $x_1 = [-1\ 1]^T$ and use a maximum of 2 refits.)