MTH-681 Algorithms
Assignment (1): Asymptotic Analysis

Mostafa Hassanein

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Using two different methods, prove that:

$$\sum_{i=1}^{i=n} 2^i = \Theta(2^n)$$

Proof. (First proof: By explicitly finding constants c and n_0)

The series $\sum_{i=1}^{i=n} 2^i$ is a geometric series which has a closed form solution given by:

$$S(n) = \sum_{i=1}^{i=n} 2^i = \frac{2^n - 2}{2 - 1} = 2^n - 2.$$

Upperbound:
Take $c = 1$, $n_0 = 1$, then $\forall n \in Z^+$ and $n > n_0$:

$$\begin{aligned} & S(n) = 2^n - 2 \leq 2^n \\ \implies & S(n) = O(2^n). \end{aligned} \tag{1}$$

Lowerbound:
Take $c = 1/2$, $n_0 = 2$, then $\forall n \in Z^+$ and $n > n_0$:

$$\begin{aligned} & S(n) = 2^n - 2 \geq 2^{n-1} \\ \implies & S(n) = \Omega(2^n). \end{aligned} \tag{2}$$

Tightbound:
(1) and (2) $\implies S(n) = \Theta(2^n)$. □

Proof. (Second proof: Using integrals)

In this proof, we again start from the closed-form solution:

$$S(n) = \sum_{i=1}^{i=n} 2^i = \frac{2^n - 2}{2 - 1} = 2^n - 2.$$

but use integration to get a lower and upper bounds.

Since $S(n)$ is monotically increasing, then the bounds are given by:

$$\begin{aligned}
&\int_{x=0}^{x=n} 2^x dx && \leq S(n) && \leq \int_{x=1}^{x=n+1} 2^x dx \\
\Longrightarrow\ &\frac{1}{\ln(2)} \left[2^n - 1\right] && \leq S(n) && \leq \frac{1}{\ln(2)} \left[2^{n+1} - 2\right] \\
\Longrightarrow\ &S(n) = \Theta(2^n).
\end{aligned}$$

□