MTH-632 PDEs
Assignment (6):
Chapter 9: Numerical Analysis of PDEs
Finite Difference (FD) Technique

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1.

The governing algebraic (finite difference) equation for the steady state temperature is given by:

$$T_{i,j} = \frac{T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1}}{4}$$

We start with an initial guess for the temperature at the interior nodes to be zero.

<u>iteration 1:</u>

$$\begin{aligned}
T_{1,1} &= \frac{T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2}}{4} = \frac{75 + 0 + 0 + 0}{4} = 18.75 \\
T_{2,1} &= \frac{T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2}}{4} = \frac{0 + 50 + 0 + 0}{4} = 12.5 \\
T_{1,2} &= \frac{T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3}}{4} = \frac{75 + 0 + 0 + 100}{4} = 43.75 \\
T_{2,2} &= \frac{T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3}}{4} = \frac{0 + 50 + 0 + 100}{4} = 37.5 \\
|\epsilon_{a_{11}}| &= 100\%.
\end{aligned}$$

<u>iteration 2:</u>

$$\begin{aligned}
T_{1,1} &= \frac{T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2}}{4} = \frac{75 + 12.5 + 0 + 43.75}{4} = 32.81 \\
T_{2,1} &= \frac{T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2}}{4} = \frac{18.75 + 50 + 0 + 37.5}{4} = 26.56 \\
T_{1,2} &= \frac{T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3}}{4} = \frac{75 + 37.5 + 18.75 + 100}{4} = 57.81 \\
T_{2,2} &= \frac{T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3}}{4} = \frac{43.75 + 50 + 12.5 + 100}{4} = 51.56 \\
|\epsilon_{a_{11}}| &= \left| \frac{32.81 - 18.75}{32.81} \right| = 43\%.
\end{aligned}$$

<u>iteration 3:</u>

$$\begin{aligned}
T_{1,1} &= \frac{T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2}}{4} = \frac{75 + 26.56 + 0 + 57.81}{4} = 39.84 \\
T_{2,1} &= \frac{T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2}}{4} = \frac{32.81 + 50 + 0 + 51.56}{4} = 33.59 \\
T_{1,2} &= \frac{T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3}}{4} = \frac{75 + 51.56 + 32.81 + 100}{4} = 64.84 \\
T_{2,2} &= \frac{T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3}}{4} = \frac{57.81 + 50 + 26.56 + 100}{4} = 58.59 \\
|\epsilon_{a_{11}}| &= \left| \frac{39.84 - 32.81}{39.84} \right| = 17.6\%.
\end{aligned}$$

iteration 4:

$$T_{1,1} = \frac{T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2}}{4} = \frac{75 + 33.59 + 0 + 64.84}{4} = 43.36$$
$$T_{2,1} = \frac{T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2}}{4} = \frac{39.84 + 50 + 0 + 58.59}{4} = 37.11$$
$$T_{1,2} = \frac{T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3}}{4} = \frac{75 + 58.59 + 39.84 + 100}{4} = 68.36$$
$$T_{2,2} = \frac{T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3}}{4} = \frac{64.84 + 50 + 33.59 + 100}{4} = 62.11$$
$$|\epsilon_{a_{11}}| = \left|\frac{43.36 - 39.84}{43.36}\right| = 8.1\%.$$

iteration 5:

$$T_{1,1} = \frac{T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2}}{4} = \frac{75 + 37.11 + 0 + 68.36}{4} = 45.12$$
$$T_{2,1} = \frac{T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2}}{4} = \frac{43.36 + 50 + 0 + 62.11}{4} = 38.87$$
$$T_{1,2} = \frac{T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3}}{4} = \frac{75 + 62.11 + 43.36 + 100}{4} = 70.12$$
$$T_{2,2} = \frac{T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3}}{4} = \frac{68.36 + 50 + 37.11 + 100}{4} = 63.87$$
$$|\epsilon_{a_{11}}| = \left|\frac{45.12 - 43.36}{45.12}\right| = 3.9\%.$$

iteration 6:

$$T_{1,1} = \frac{T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2}}{4} = \frac{75 + 38.87 + 0 + 70.12}{4} = 45.98$$
$$T_{2,1} = \frac{T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2}}{4} = \frac{45.12 + 50 + 0 + 63.87}{4} = 39.75$$
$$T_{1,2} = \frac{T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3}}{4} = \frac{75 + 63.87 + 45.12 + 100}{4} = 71$$
$$T_{2,2} = \frac{T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3}}{4} = \frac{70.12 + 50 + 38.87 + 100}{4} = 64.75$$
$$|\epsilon_{a_{11}}| = \left|\frac{45.98 - 45.12}{45.98}\right| = 1.8\%.$$

iteration 7:

$$T_{1,1} = \frac{T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2}}{4} = \frac{75 + 39.75 + 0 + 71}{4} = 46.44$$
$$T_{2,1} = \frac{T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2}}{4} = \frac{45.98 + 50 + 0 + 64.75}{4} = 40.18$$
$$T_{1,2} = \frac{T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3}}{4} = \frac{75 + 64.75 + 45.98 + 100}{4} = 71.43$$
$$T_{2,2} = \frac{T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3}}{4} = \frac{71 + 50 + 39.75 + 100}{4} = 65.19$$
$$|\epsilon_{a_{11}}| = \left|\frac{46.44 - 45.98}{46.44}\right| = 0.99\%$$
$$|\epsilon_{a_{21}}| = \left|\frac{40.18 - 39.75}{40.18}\right| = 1\%$$
$$|\epsilon_{a_{12}}| = \left|\frac{71.43 - 71}{71.43}\right| = 0.6\%$$
$$|\epsilon_{a_{22}}| = \left|\frac{65.19 - 64.75}{65.19}\right| = 0.6\%.$$

2.

We use an explicit scheme.

The finite difference equation for Poisson's equation in 3D is given by:

$$T_{i,j,k} = \frac{T_{i-1,j,k} + T_{i+1,j,k} + T_{i,j-1,k} + T_{i,j+1,k} + T_{i,j,k-1} + T_{i,j,k-1} - h^2 f_{i,j,k}}{6}$$

We start with an initial guess for the temperature at the interior nodes to be zero.

iteration 1:

$$\begin{aligned} T_{1,1,1} &= \frac{T_{0,1,1} + T_{2,1,1} + T_{1,0,1} + T_{1,2,1} + T_{1,1,0} + T_{1,1,2} - h^2 f_{1,1,1}}{6} \\ &= \frac{0+0+0+0+0+0-\frac{1}{36}(-10)}{6} = \frac{10}{216} \\ &\vdots \end{aligned}$$

There are 25 points interior points, the other points can be computed in a similar manner.

On each iteration we should compute the maximum error, and if the error is low enough (say less than 1%), we stop, otherwise we continue onto the next iteration.

5.

To solve the PDE:

$$c_t = Dc_{xx} - Uc_x - kc$$

using finite difference, we compute the finite difference approximation to the derivatives as follows:

$$\begin{aligned}
c_t &= \frac{c_i^{l+1} - c_i^l}{\Delta t} \\
c_x &= \frac{c_{i+1}^l - c_i^l}{\Delta x} \\
c_{xx} &= \frac{c_{i+1}^l - 2c_i^l + c_{i-1}^l}{\Delta x}
\end{aligned}$$

Next, we plug these approximations to the PDE:

$$\frac{c_i^{l+1} - c_i^l}{\Delta t} = D\frac{c_{i+1}^l - 2c_i^l + c_{i-1}^l}{\Delta x^2} - U\frac{c_{i+1}^l - c_i^l}{\Delta x} - kc_i^l$$

$$\implies c_i^l[k - \frac{1}{\Delta t} + \frac{2D}{\Delta x^2} - \frac{U}{\Delta x}] + c_{i-1}^l[-\frac{D}{\Delta x^2}] + c_{i+1}^l[-\frac{D}{\Delta x^2} + \frac{U}{\Delta x}] + c_i^{l+1}[\frac{1}{\Delta t}] = 0$$

Plugging in the following values for the constants:

$$\begin{aligned}
k &= 0.15 & D &= 100 & U &= 1 \\
\Delta x &= 1 & \Delta t &= 0.005
\end{aligned}$$

we get:

$$-\frac{17}{20}c_i^l - 100c_{i-1}^l - 99c_{i+1}^l + 200c_i^{l+1} = 0$$

Solving for c_i^{l+1}:

$$c_i^{l+1} = \frac{1}{2}c_{i-1}^l + \frac{17}{4000}c_i^l + \frac{99}{200}c_{i+1}^l$$

We can nnow use the Liebmann's (Gauss-Seidel) method to solve for the concentrations iteratively.

iteration 1, timestep 1:

$$\begin{aligned}
c_1^1 &= \frac{1}{2}c_0^0 + \frac{17}{4000}c_1^0 + \frac{99}{200}c_2^0 = -\frac{1}{2}100 + \frac{17}{4000}0 + \frac{99}{200}0 = 50 \\
c_2^1 &= \frac{1}{2}c_1^0 + \frac{17}{4000}c_2^0 + \frac{99}{200}c_3^0 = \frac{1}{2}0 + \frac{17}{4000}0 + \frac{99}{200}0 = 0 \\
&\vdots \\
c_9^1 &= \frac{1}{2}c_8^0 + \frac{17}{4000}c_9^0 + \frac{99}{200}c_{10}^0 = \frac{1}{2}0 + \frac{17}{4000}0 + \frac{99}{200}0 = 0
\end{aligned}$$

iteration 1, timestep 2:

$$c_1^2 = \frac{1}{2}c_0^1 + \frac{17}{4000}c_1^1 + \frac{99}{200}c_2^1 = \frac{1}{2}100 + \frac{17}{4000}50 + \frac{99}{200}0 = 50.21$$
$$c_2^2 = \frac{1}{2}c_1^1 + \frac{17}{4000}c_2^1 + \frac{99}{200}c_3^1 = \frac{1}{2}50 + \frac{17}{4000}0 + \frac{99}{200}0 = 25$$
$$\vdots$$

This should be continued until the final timestep (timestep 9). Then, the same steps should be iterated until the maximum error falls below the required threshold.