MTH-684 Logic
Assignment (1): Propositional Logic

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1-3

Proof. (By Strong Induction)

We will show that this inequality holds using proof by strong induction on the number of connectives n in a WFF ϕ:

<u>Base case $(n = 0)$:</u>

This case represents **atomic expressions**.

From the definition of sub on **atomic expressions**, it is clear that $sub(\phi) = \{A\}$, where A is an atomic expression.

$\implies |sub(\phi)| = 1$

$\implies |sub(\phi)| \leq 2n + 1 = 1$ holds.

<u>Inductive step $(n = k + 1)$:</u>

This case represents **composite expressions**.

Suppose the induction hypothesis holds for $n \leq k$, we will do a case analysis on all connective types to show that it also holds for $n = k + 1$.

<u>**(i):** $\neg$</u>

$$\begin{aligned} &\phi = \neg\psi \\ \implies &sub(\phi) = sub(\psi) \cup \{\neg\psi\} \\ \implies &|sub(\phi)| \leq |sub(\psi)| + 1 \\ \implies &|sub(\phi)| \leq (2k + 1) + 1 = 2(k + 1) \leq 2(k + 1) + 1. \end{aligned}$$

<u>**(ii):** $\wedge$</u>

$$\begin{aligned} &\phi = \psi \wedge \omega \\ \implies &sub(\phi) = sub(\psi) \cup sub(\omega) \cup \{(\psi \wedge \omega)\} \\ \implies &|sub(\phi)| \leq |sub(\psi)| + |sub(\omega)| + 1 \\ \implies &|sub(\phi)| \leq [2r + 1] + [2(k - r) + 1] + 1 \\ \implies &|sub(\phi)| \leq 2k + 3 = 2(k + 1) + 1. \end{aligned}$$

<u>**(iii - v):** $\vee$, $\implies$, $\iff$</u>

These cases all have similar proof to case ii.

This closes the induction, and thus $|sub(\phi)| \leq 2n + 1$ is true.

$\square$