Linear Algebra Finals Questions Bank

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Proofs

Theorem 1 (2023.S(1.A.i))**.**
i. Disprove: W is a subspace of the vector space V and $v \in V$. Then, the set defined by $v + W = \{v + w : w \in W\}$, is a subspace of V.

ii. Under what condition is $v + W$ a subspace of V?

Proof.
i. By counterexample:
Let $V = R^2$, $W = \{w \in V : w = (x, 0)\}$, and $v = (0, 1)$.
$\implies v + W = \{u \in V : u = (x, 1)\}$.
$\implies 0_v = (0, 0) \notin v + W$
$\implies$ v+W is not a subspace.

ii. It's clear that if $v \in W$, then $v + W = W$, which is a subspace of V as desired.
If, however, $v \notin W$, then, by the counterexample provided, $v + W$ is not a subspace of V.
Therefore, the sufficient and necessary condition is: $v \in W$.

$\square$

Note: The transformation $v+W = \{v+w : w \in W\}$ is an **affine transformation**, which is not—necessarily—linear.

Theorem 2 (2023.S(1.A.ii))**.**

Let $x_1, \dots, x_{n+1}$ be distinct elements of F. Then, the functions $f_i(x) = \Pi_{k=1,k\neq i}^{n+1} \frac{(x-x_k)}{(x_i-x_k)}$ for $i = 1, \dots, n+1$ form a basis for $P_n(F)$.

Note to self: A possible source of confusion here is that F is not an infinite field, rather it's a finite field given by $F = \{x_1, \dots, x_n\}$.

Proof.

Since $dim(P_n) = n+1$ and we have $n+1$ functions/vectors, then it's sufficient to check either one of the following conditions:

i. The $n+1$ functions/vectors span P_n

ii. The $n+1$ functions/vectors are linearly independent.

We check (ii). Let $x_j \in \{x_1, \dots, x_{n+1}\}$, then:

$$\begin{aligned} 0_v &= \sum_{i=1}^{n+1} a_i f_i(x_j) \\ &= \sum_{i=1}^{n+1} a_i \Pi_{k=1,k\neq i}^{n+1} \frac{x_j - x_k}{x_i - x_k} \\ &= \sum_{i=1}^{n+1} a_i \delta_{ij} \\ &= a_j. \end{aligned}$$

$\implies \forall j \in \{1, \dots, n+1\} : \; a_j = 0.$

$\implies (f_1, \dots, f_n)$ are linearly independent.

$\implies (f_1, \dots, f_n)$ is a basis for P_n.

$\square$

Notes:

- The tricky part in this question is to realize that the complicated form of f_i reduces to δ_{ij}.

- This result can be generalized to all function spaces (not just polynomials) over a finite field. The generalization can be stated as follows: Any function space over a finite field of cardinality n has a basis given by the functions f_i where $f_i(x_j) := \delta_{ij}$ for $i, j \in \{1, \dots, n\}$.

Theorem 3 (2023.S(1.A.iii))**.**

If $T \in Hom(V)$, W is a T-invariant subspace of V, and $V = R(T) \oplus W$, then $W \subseteq N(T)$.

Proof. (By contradiction)

Suppose that $W \not\subseteq N(T)$
$\implies \exists w \in W$ s.t. $w \notin N(T)$
$\implies Tw \in R(T)$
$\implies Tw \in R(T)$ and $Tw \in W$ (Because W is T-invariant)
$\implies \exists v = Tw \in V$ s.t. v is not uniquely represented as a sum from $R(T)$ and W.

This is a contradiction. Therefore, we must conclude that $W \subseteq N(T)$. $\square$

Theorem 4 (2023.S(1.A.iv)).

Disprove: A linear operator on an infinite-dimensional vector space has no eigenvectors.

Proof. (By counterexample)

Consider the vector space $V = \mathbb{R}^\infty$ over $\mathbb{R}$.

Let T be a linear operator on V defined by $T := \lambda I_\infty$, where $\lambda \in \mathbb{R}$.

Then, all vectors in V are eigenvectors with eigenvalue $= \lambda$.

$\square$

Theorem 5 (2023.S(1.A.v))**.**

*If S is a **subset** of an inner product space V, then $Span(S)$ is a **subspace** of $(S^{\perp})^{\perp}$.*

Proof.

We will show that $(S^{\perp})^{\perp} = Span(S)$, thus, concluding that $Span(S)$ is a subspace (an improper subspace) of $(S^{\perp})^{\perp}$.

We start from the fact (given by another theorem) that if U is a subspace, then $U = (U^{\perp})^{\perp}$. So, now we need only show that $Span(S)^{\perp} = S^{\perp}$. This amounts to showing that $(i) Span(S)^{\perp} \subseteq S^{\perp}$ and $(ii) S^{\perp} \subseteq Span(S)^{\perp}$.

(i) Is true since any vector that is orthgonal to all vectors in $Span(S)$ must also be orthogonal to all vectors in S.

(ii) Let $s_o \in S^{\perp}$ and $s \in Span(S)$.

$\Longrightarrow\ s = \sum a_i s_i,\ \forall s_i \in S$

$\Longrightarrow < \sum a_i s_i, s_o >= \sum < a_i s_i, s_o >= \sum a_i < s_i, s_o >= \sum a_i * 0 = 0.$

$\Longrightarrow\ s_o \in Span(S)^{\perp}$

$\Longrightarrow\ S^{\perp} \subseteq Span(S)^{\perp}.$

□

Theorem 6 (2023.S(1.B), 2022.S(1.A.ii))**.**

Let V be an n-dimensional vector space and $T \in Hom(V, W)$. Prove that:

i. $nullity(T) + rank(T) = n$.

ii. T is injective iff T carries linearly independent subsets of V onto linearly independent subsets of W. In other words:

T is injective $\iff$ If $(v_1, \ldots, v_k)$ are linearly independent, then $(Tv_1, \ldots, Tv_k)$ are linearly independent.

Proof. Part i

Let $nullity(T) = m$, $0 \le m \le n$, and $B_N = (u_1, \ldots, u_m)$ be a basis for N(T).

Extend B_N to a basis for V: $B_V = (u_1, \ldots, u_m, u_{m+1}, \ldots, u_n)$.

Let $v \in V$, then $v = \sum_{i=1}^{n} a_i u_i$.
Apply T to both sides:

$$\begin{aligned} T(v) &= T(\sum_{i=1}^{n} a_i u_i) \\ &= \sum_{i=1}^{n} a_i T(u_i) \\ &= \sum_{i=m+1}^{n} a_i T(u_i) \end{aligned}$$

This shows that $(Tv_{m+1}, \ldots, Tv_n)$ spans $R(T)$.
Next, we show that it is also linearly independent:

$$\begin{aligned} 0_v &= \sum_{i=m+1}^{n} a_i T(u_i) \\ &= T(\sum_{i=m+1}^{n} a_i u_i) \end{aligned}$$

$\implies \sum_{i=m+1}^{n} a_i u_i \in N(T)$
$\implies \sum_{i=1}^{m} a_i u_i = \sum_{i=m+1}^{n} a_i u_i$
$\implies a_i = 0$, for $i = 1, ..., n$. (Because $(u_1, \ldots, u_n)$ is linearly independent)
$\implies (Tu_{m+1}, \ldots, Tu_n)$ is linearly independent and hence is a bsis for range T.

□

Proof. Part ii

<u>Forward direction:</u> Assume T is injective and $(v_1, \ldots, v_n)$ is linearly independent, then:

$$
\begin{aligned}
0_v &= a_1Tv_1 + \ldots + a_nTv_n \\
&= T(a_1v_1 + \ldots + a_nv_n)
\end{aligned}
$$

$\implies a1, \ldots, a_n = 0$ (Because T is injective)
$\implies (Tv_1, ..., Tv_n)$ is linearly independent.

Converse direction: Assume $(v_1, \ldots, v_n)$ and $(Tv_1, \ldots, Tv_n)$ are both linearly independent.
$\implies$ If $a_1Tv_1 + \ldots + a_nTv_n = 0_v$, then $a_1, \ldots, a_n = 0$.
$\implies$ If $T(a_1v_1 + \ldots + a_nv_n) = 0$, then $a_1, \ldots, a_n = 0$.
$\implies N(T) = \{0_v\}$.
$\implies T$ is injective.

$\square$

Theorem 7 (2023.S(2.A)).

Let $T : P_n(R) \to R^{n+1}$ *be such that:*

$T(\sum_{i=0}^{n} c_i t^i) = (x_0, x_1, \ldots, x_n)$

where: $x_k = \int_0^1 f(t)dt$ *for* $k = 0, \ldots, n$.

Show that T *is invertible.*

Question to self: What is $f(t)$*? Also, is it "for some* $f(t)$*" or "for all* $f(t)$*"?*
It cannot be "for all $f(t)$*, because it's obviously false for* $f(t) = 0$.
So it has to be for some $f(t)$.

Proof. ∎

Theorem 8 (2023.S(2.B), 2021.F(2.A))**.**

Let V and W be n-dimensional vector spaces with order bases α and β respectively. If T is an isomorphism from V onto W with $[T]_\alpha^\beta = A$, show that $[T^{-1}]_\beta^\alpha = A^{-1}$

Proof.

$\square$

Theorem 9 (2023.S(2.C)).

Let $V = M_{2x2}(R)$ and $T(A) = A^t + 2tr(A)I_2$, where $A \in V$ and A^t is the transpose of A.

Find an ordered Basis β for V so that $[T]\beta$ is a diagonal matrix.

Theorem 10 (2023.S(3.B)).

Let $V = W \oplus W^{\perp}$ and T be the projection on W along $W^{\perp}$.

Show that $T^* = T$.

Theorem 11 (2023.S(3.C)).

Let T be a linear operator on the inner product space V. Show that:

$< T(u), T(v) >=< u, v > \quad \forall u, v \in V \iff ||T(u)|| = ||u|| \quad \forall u \in V$.

Theorem 12 (2022.S(1.A.i)).

If V is a vector space and $S_1, S_2 \subseteq V$ with $S_1 \subseteq S_2$, then $S_2^{\perp}$ is a ***subspace*** *of $S_1^{\perp}$.*

Theorem 13 (2022.S(1.B)).

Let $V = M_{2x2}(R)$.

i. Show that V has a basis that contains bases for its subspaces U and W, where:

$U = \{A \in V : A^T = A\}$ *and* $W = \{A \in V : A^T = -A\}$.

ii. Show that $V = U \oplus W$.

Theorem 14 (2022.S(1.D)).

Let V be the vector space of complex numbers over the field $\mathbb{R}$, i.e. C^1 over $\mathbb{R}$.

Let $T : V \to V$ be defined by $T(z) = \bar{z}$, the complex conjugate of z.

i. Show that T is linear.

ii. Show that T is not linear if V is redefined to be over the complex field $\mathbb{C}$.

Theorem 15 (2022.S(2.A)).

Let V be an n-dimensional vector space with bases $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$.

If $P \in Hom(V)$, such that $P(\alpha_i) = \beta_i \quad \forall i$, derive the relation between $[V]_\alpha$ and $[V]_\beta$ for $v \in V$.

Theorem 16 (2022.S(2.B)).

TODO: Reformulate from a problem to a statement.

Let $T : P_2(R) \to P_2(R)$ be such that:

$T(a + bt + ct^2) = -2b - 3c + (a + 3b + 3c)t + ct^2$.

i. Find a basis for for the eigenspace E_1.

ii. Is T diagonalizable?

iii. Is there an operator on $P_2(R)$ whose null space is E_1?

Theorem 17 (2022.S(3.B), 2021.F(3.D)).

Let V be an inner product space, and $T \in Hom(V)$.

i. Show that $N(T^*T) = N(T)$.

ii. [Prove or Disprove] $rank(T^*T) = rank(T)$.

Theorem 18 (2021.F(1.A.i)).
V is an inner product vector space and $S \subseteq V \ \wedge\ S \neq \emptyset$
$\implies S^{\perp}$ is a subspace of V.

Theorem 19 (2021.F(1.A.ii)).
[Disprove] If $T \in Hom(V, W)$, $dim(V) = dim(W) = 2$, and $\{v_1, v_2\}$ is a basis for V, then $\{T(v_1 - v_2), T(v_1)\}$ is a basis for W.

Theorem 20 (2021.F(3.B)).
Let V be an inner product space, and W be a finite-dimensional subspace of V.
Show that: $w \in W \implies \exists v \in V \ni v \in W^{\perp} \wedge < v, u > \neq 0$.

Theorem 21 (2021.F(3.B)).
Let V be an inner product space, and W be a finite-dimensional subspace of V.
Show that: $w \in W \implies \exists v \in V \ni v \in W^{\perp} \wedge < v, u > \neq 0$.

Theorem 22 (2012.F(1.B)).
Let $V = M_{2x2}(R)$,
$B \in V$ such that $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$,
$W_1 = \{A \in V : AB = BA\}$,
$W_2 = \{A \in V : A^T = A\}$.
i. Show that W_1 is a subspace of of V.
ii. Find $dim(W_1)$.
iii. [Prove or Disprove] $V = W_1 \oplus W_2$.

Theorem 23 (2012.F(1.D)).
Let $T_1, T_2 \in Hom(V, W)$.
Show that: $rank(T_1 + T_2) \leq rank(T_1) + rank(T_2)$.

Problems

1.

[2023.S(3.A)]
Let $V = C([-1, 1])$ with the inner product $< f, g >= \int_{-1}^{1} f(t)g(t)dt \qquad \forall f, g \in V$.
i. Find an orthonormal basis for $P_2(R)$ as a subspace of V and use it to compute the best quadratic approximation of $f(t) = e^t$ on $[-1, 1]$.
ii. For $T \in Hom(P_1(R))$ with $P_1(R)$ as a subspace of V and $T(f) = f' + 3f$, evaluate $T^* = 1 + 3t$.

2.

[2022.S(1.C), 2021.F(1.B)]

Let $V = R^3$.

i. Suggest two 2-dimensional subspaces W_1 and W_2 of V such that $V = W_1 + W_2$.

ii. With $W_1 = \{v \in V : v = (x, y, 0)\}$, define two projections **on** W_1 **along** two distinct subspaces W_2 amd W_3 of V.

3.

[2022.S(3.A), 2021.F(3.A)]

Let $V = P_1(R) = Span(\{1, t\})$ with the inner product $< f, g >= \int_0^1 f(t)g(t)dt$, $\forall f, g \in V$.

i. Find an orthonormal basis for V.

ii. Find the orthogonal projection of $f(t) = t^2$ on V.

iii. Let $T(f) = f'(t) + 3f(t)$ be a linear operator on V. Is $(t - 2)$ an eigenvector for T?

iv. Evaluate $T^*(2t - 1)$.

4.

[2021.F(1.C)]

Let $T : P_2(R) \to R^2$ be such that $T(a_0 + a_1t + a_2t^2) = (a_1 + a_2, a_0 - a_1)$.

i. Find a basis for $ker(T)$. ii. Is T surjective? iii. Find a two-dimensional subspace of of $P_2(R)$ such that its image under T is a one-dimensional subspace of R^3. iv. Find the matrix representation of T relative to $\{1, t - 1, t^2 + 1\}$ as a basis for $P_2(R)$ and $\{(0, 1), (1, 1)\}$ as a basis for R^2.

5.

[2021.F(2.B)]

T is a linear operator on $P_2(R)$ defined by: $T(f(x) = xf'(x) + f(2) + f(3))$.

i. Is T diagonalizable?

ii. Find an eigenpair for T.

6.

[2021.F(3.C)]

Let $V = \mathbb{C}^2$ and $T(z_1, z_2) = (2z_1 + iz_2, (1 - i)z_1)$.

Evaluate $T^*(3 - i, 1 + 2i)$.

6.

[2012.F(4.B)]

Find the minimal l_2-norm solution to the system:

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 3y + z &= 2 \\ 4x + 7y - z &= 4. \end{aligned}$$