Linear Algebra Finals Questions Bank

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Definitions

Definition 1 (Cyclic Group).
A group G is called a cyclic group if: $\exists a \in G \ni \forall x \in G \ \exists k \in \mathbb{Z} \ni x = a^k$.

Definition 2 (Index of a subgroup H in a group G).
Is the number of left (or right) cosets of the subgroup H in the group G, and is denotes by $[G : H]$.

Definition 3 (Cyclic Permutation).
A permutation where the elements of the first line are written in such a way that the image of any element is the next element and the image of the last element is the first element.

Definition 4 (Conjugate Element).
Let G be a group and $a \in G$, then the element $x^{-1}ax$, where $x \in G$, is called a conjugate element of a.

Definition 5 (Euler's Function).
$\phi(n)$, where $n > 0$, is the cardinal number of the set $X = \{x : (x, n) = 1,\ x < n\}$.

Definition 6 (Integral Domain).
Is a ring R that has no zero divisors.
OR
Is a ring $(R, +, \cdot)$ s.t. $a \cdot b = e \implies a = e \lor b = e \ \forall a, b \in R$, where e is the identity of the first operation.

Definition 7 (Commutative Ring).
Is a ring where the second operation is commutative.

Definition 8 (Right Ideal).
Let $(R, +, \cdot)$ be a ring. A subgroup I of R is called a right ideal if: $x \cdot r \in I \ \forall x \in I$ and $\forall r \in R$.

Definition 9 (Zero Divisor).
Let $(R, +, \cdot)$ be a ring with additive identity e. Then, $a \in R^*$ is called a zero divisor if: $\exists b \in R^* \ni a \cdot b = e \land \exists c \in R^* \ni c \cdot a = e$.

Definition 10 (Field).
The system $(F, +, \cdot)$ is called a field if:
i. $(F, +, \cdot)$ is a commutative ring.
ii. $(F^*, \cdot)$ is a commutative group, where $F^* = F - e$, e is the identity of the group $(F, \cdot)$.

Definition 11 (Kernel of a Homomorphism).
Let the map $\alpha : G \to G'$ be a homomorphism, and e' be the identity of G'.
Then, the kernel of α is defined as: $Ker(\alpha) = \{g \in G : \alpha(g) = e' \in G'\}$.

Definition 12 (Abstract Algebra).
Is a set S together with one or more binary operations on S.

Definition 13 (Binary Operation)**.**
Given a set A, a binary operation defined as: $A \times A \to A$.

Definition 14 (Division Ring)**.**
A ring $(R, +, \cdot)$ is called a division ring if:
i. $(R, +, \cdot)$ *is a ring.*
ii. $(R^*, \cdot)$ *is a group.*

Definition 15 (Isomorphism)**.**
Is a bijective homomorphism.

Definition 16 (Center of a Group)**.**
Is the set of self conjugate elements in a group G.

Proofs

Theorem 1 (2023.S(1.A.i))**.**

Proof. ∎

Problems

1. Let $S = \{A, B, C, D\}$, $A = \emptyset$, $B = \{a, b\}$, $C = \{a, c\}$, $D = \{a, b, c\}$.
   i. Is union a binary relation over S?
   ii. Is intersection a binary relation over S?
   iii.Is $(S, \cup)$ a group?

<u>Ans.</u>

i. Let's construct the Cayley table:

$\cup$	A	B	C	D
A	A	B	C	D
B	B	B	D	D
C	C	D	C	D
D	D	D	D	D

From the Cayley table, the set S is closed under the union operation. Therefore, it is a binary operation.

ii. Let's construct the Cayley table:

$\cap$	A	B	C	D
A	A	A	A	A
B	A	B	{a}	B
C	A	{a}	C	C
D	A	B	C	D

From the Cayley table, the set S is not closed under the intersection operation because $\{a\} \notin S$. Therefore, it is not a binary operation.

iii. No. From the Cayley table, it is clear that A is the identity element. It is also clear that elements B, C, D do not have inverses.

2. Let $G = (a)$ be a cyclic group of order 20. Let $H = (a^4)$ be a subgroup of G.

i. Determine the cosets of of H in G.
ii. Find the quotient group G/H.
iii. Find the index of H in G.

Ans.

i. Cosets:
$H = \{a^4, a^8, a^{12}, a^{16}, a^{20}\} = a^4H = a^8H = a^{12}H = a^{16}H = a^{20}H$
$aH = \{a^1, a^5, a^9, a^{13}, a^{17}\} = a^5H = a^9H = a^{13}H = a^{17}H.$
$a^2H = \{a^2, a^6, a^{10}, a^{14}, a^{18}\} = a^6H = a^{10}H = a^{14}H = a^{18}H.$
$a^3H = \{a^3, a^7, a^{11}, a^{15}, a^{19}\} = a^7H = a^{11}H = a^{15}H = a^{19}H.$

ii. $G/H = \{H, aH, a^2H, a^3H\}$.
iii. $[G : H] = |G/H| = 4$.

3. Let $G = (I, +)$, $I = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and let $H = \{px : x \in I\}$, p is a prime number.

i. Find the quotient group G/H.

ii. Is G/H a cyclic group?

iii. What is the generating element(s) of G/H if exists?

iv. What is the index of H in G.

Ans.

i.

True or False

1. The identity permutation is an even permutation.
True.
Since the identity permutation on n symbols can be factored as a product of n disjoint cycles of length 1, then the invariant number $N(p=i) = n - k = n - n = 0$, which is even.

2. The composition of two even permutation is an odd permutation.
False.
Let p_1 and p_2 be two even permutations with $N(p_1) = 2k_1$ and $N(p_2)2k_2$. Then, $p_1 \circ p_2 = p_2 p_1$ can be represented as a product of $2(k_1 + k_2)$ transpositions, which is even.

3. Equivalence of matrices on the set of matrices with appropriate dimensions is an equivalence relation.
True.
Let A, B, C be 3 matrices of the same dimension, then:
i. Reflexivity: $A = IA \implies (A, A) \in R$.
ii. Symmetry: $(A, B) \in R \implies A = E_k \ldots E_1 B \implies B = E_1^{-1} \ldots E_k^{-1} A \implies (B, A) \in R$.
iii. Transitivity: $(A, B) \in R \wedge (B, C) \in R \implies A = E_k \ldots E_1 B \wedge B = H_r \ldots H_1 C \implies A = E_k \ldots E_1 H_r \ldots H_1 C \implies (A, C) \in R$.

4. Each transposition is its own inverse.
True.
Let $\tau = (a, b)$ be a transposition. Applying τ twice we get $\tau^2 = (a)(b) = id$.

5. The "Square of" on the set $N = \{1, 2, \ldots\}$ is an equivalence relation.
False.
Because it fails reflexivity, i.e. $\exists x \in N \ni (x, x) \notin R$.
For example, 2 is not the square of itself.

6. "Perpendicular to" on the set of straight lines in a plane is an equivalence relation.
False.
It fails reflexivity, since no line is perpendicular to itself.
7. The order of any subgroup H of a group G is a divisor of the order of G.
True. By Lagrange's theorem.

8. "Conjugate to" in the set of complexes of a group G is an equivalence relation.
True.
Let $K, L, M \in P(G)$.
i. Reflexivity: $eK = Ke \implies (K, K) \in R$.

ii. Symmetry: $(K, L) \in R \implies K = x^{-1}Lx \implies L = xKx^-1 \implies L = y^{-1}Ky \implies (L, K) \in R$.

iii. Transitivity: $(K, L), (L, M) \in R \implies K = x^{-1}Lx \wedge L = y^{-1}My \implies K = x^{-1}y^{-1}Myx \implies K = (yx)^{-1}Myx \implies (L, M) \in R$.