

Linear Algebra (MTH401) Finals Questions Bank

Mostafa Hassanein

14 January 2024

Proofs

Theorem 1 (2023.S(1.A.i)).

i. *Disprove: W is a subspace of the vector space V and $v \in V$. Then, the set defined by $v + W = \{v + w : w \in W\}$, is a subspace of V .*

ii. *Under what condition is $v + W$ a subspace of V ?*

Proof.

i. By counterexample:

Let $V = \mathbb{R}^2$, $W = \{w \in V : w = (x, 0)\}$, and $v = (0, 1)$.

$\implies v + W = \{u \in V : u = (x, 1)\}$.

$\implies 0_v = (0, 0) \notin v + W$

$\implies v + W$ is not a subspace.

ii. It's clear that if $v \in W$, then $v + W = W$, which is a subspace of V as desired.

If, however, $v \notin W$, then, by the counterexample provided, $v + W$ is not a subspace of V .

Therefore, the sufficient and necessary condition is: $v \in W$.

□

Note: The transformation $v + W = \{v + w : w \in W\}$ is an **affine transformation**, which is not—necessarily—linear.

Theorem 2 (2023.S(1.A.ii)).

Let x_1, \dots, x_{n+1} be distinct elements of F . Then, the functions $f_i(x) = \prod_{k=1, k \neq i}^{n+1} \frac{(x-x_k)}{(x_i-x_k)}$ for $i = 1, \dots, n+1$ form a basis for $P_n(F)$.

Note to self: A possible source of confusion here is that F is not an infinite field, rather it's a finite field given by $F = \{x_1, \dots, x_n\}$.

Proof.

Since $\dim(P_n) = n+1$ and we have $n+1$ functions/vectors, then it's sufficient to check either one of the following conditions:

- i. The $n+1$ functions/vectors span P_n
- ii. The $n+1$ functions/vectors are linearly independent.

We check (ii). Let $x_j, a_i \in \{x_1, \dots, x_{n+1}\}$, then:

$$\begin{aligned} 0_v &= \sum_{i=1}^{n+1} a_i f_i(x_j) \\ &= \sum_{i=1}^{n+1} a_i \prod_{k=1, k \neq i}^{n+1} \frac{x_j - x_k}{x_i - x_k} \\ &= \sum_{i=1}^{n+1} a_i \delta_{ij} \\ &= a_j. \end{aligned}$$

$\implies \forall j \in \{1, \dots, n+1\} : a_j = 0$.

$\implies (f_1, \dots, f_n)$ are linearly independent.

$\implies (f_1, \dots, f_n)$ is a basis for P_n .

□

Notes:

- The tricky part in this question is to realize that the complicated form of f_i reduces to δ_{ij} when applying f_i to x_j .

- Intuitively, this is because the output of applying $f_i(x)$ to some $x_j \in F$ must be one of the finite scalars in F .

- I suppose this result can be generalized to all function spaces (not just polynomials) over a finite field. The generalization can be stated as follows: Any function space over a finite field of cardinality n has a basis given by the functions f_i where $f_i(x_j) := \delta_{ij}$ for $i, j \in \{1, \dots, n\}$.

Theorem 3 (2023.S(1.A.iii)).

If $T \in \text{Hom}(V)$, W is a T -invariant subspace of V , and $V = R(T) \oplus W$, then $W \subseteq N(T)$.

Proof. (By contradiction)

Suppose that $W \not\subseteq N(T)$

$\implies \exists w \in W$ s.t. $w \notin N(T)$

$\implies Tw \in R(T)$

$\implies Tw \in R(T)$ and $Tw \in W$ (Because W is T -invariant)

$\implies \exists v = Tw \in V$ s.t. v is not uniquely represented as a sum from $R(T)$ and W .

$\implies V \neq R(T) \oplus W$.

This is a contradiction. Therefore, we must conclude that $W \subseteq N(T)$.

□

Theorem 4 (2023.S(1.A.iv)).

Disprove: A linear operator on an infinite-dimensional vector space has no eigenvectors.

Proof. (By counterexample)

Consider the vector space $V = \mathbb{R}^\infty$ over \mathbb{R} .

Let T be a linear operator on V defined by $T := \lambda I_\infty$, where $\lambda \in \mathbb{R}$.

Then, all vectors in V are eigenvectors with eigenvalue $= \lambda$.

□

Theorem 5 (2023.S(1.A.v)).

If S is a subset of an inner product space V , then $\text{Span}(S)$ is a subspace of $(S^\perp)^\perp$.

Proof.

We prove a stronger result: $(S^\perp)^\perp = \text{Span}(S)$.

We start from the fact (given by another theorem) that if U is a subspace, then $U = (U^\perp)^\perp$. So, now we need only show that $\text{Span}(S)^\perp = S^\perp$. This amounts to showing that (i) $\text{Span}(S)^\perp \subseteq S^\perp$ and (ii) $S^\perp \subseteq \text{Span}(S)^\perp$.

(i) Is true since any vector that is orthogonal to all vectors in $\text{Span}(S)$ must also be orthogonal to all vectors in S .

(ii) Let $s_o \in S^\perp$ and $s \in \text{Span}(S)$.

$$\implies s = \sum a_i s_i, \forall s_i \in S$$

$$\implies \langle \sum a_i s_i, s_o \rangle = \sum \langle a_i s_i, s_o \rangle = \sum a_i \langle s_i, s_o \rangle = \sum a_i * 0 = 0.$$

$$\implies s_o \in \text{Span}(S)^\perp$$

$$\implies S^\perp \subseteq \text{Span}(S)^\perp.$$

□

Theorem 6 (2023.S(1.B), 2022.S(1.A.ii)).

Let V be an n -dimensional vector space and $T \in \text{Hom}(V, W)$. Prove that:

i. $\text{nullity}(T) + \text{rank}(T) = n$.

ii. T is injective iff T carries linearly independent subsets of V onto linearly independent subsets of W . In other words:

T is injective \iff If (v_1, \dots, v_k) are linearly independent, then (Tv_1, \dots, Tv_k) are linearly independent.

Proof. Part i

Let $\text{nullity}(T) = m$, $0 \leq m \leq n$, and $B_N = (u_1, \dots, u_m)$ be a basis for $N(T)$.

Extend B_N to a basis for V : $B_V = (u_1, \dots, u_m, u_{m+1}, \dots, u_n)$.

Let $v \in V$, then $v = \sum_{i=1}^n a_i u_i$.

Apply T to both sides:

$$\begin{aligned} T(v) &= T\left(\sum_{i=1}^n a_i u_i\right) \\ &= \sum_{i=1}^n a_i T(u_i) \\ &= \sum_{i=m+1}^n a_i T(u_i) \end{aligned}$$

This shows that (Tv_{m+1}, \dots, Tv_n) spans $R(T)$.

Next, we show that it is also linearly independent:

$$\begin{aligned} 0_v &= \sum_{i=m+1}^n a_i T(u_i) \\ &= T\left(\sum_{i=m+1}^n a_i u_i\right) \end{aligned}$$

$$\implies \sum_{i=m+1}^n a_i u_i \in N(T)$$

$$\implies \sum_{i=1}^n a_i u_i = \sum_{i=m+1}^n a_i u_i$$

$$\implies a_i = 0, \text{ for } i = 1, \dots, m. \quad (\text{Because } (u_1, \dots, u_m) \text{ is linearly independent})$$

$$\implies (Tu_{m+1}, \dots, Tu_n) \text{ is linearly independent and hence is a basis for range } T.$$

□

Proof. Part ii

Forward direction: Assume T is injective and (v_1, \dots, v_n) is linearly independent, then:

$$\begin{aligned} 0_w &= a_1 T v_1 + \dots + a_n T v_n \\ &= T(a_1 v_1 + \dots + a_n v_n) \end{aligned}$$

$\implies a_1 v_1 + \dots + a_n v_n = 0_v$ (Because T is injective)

$\implies a_1, \dots, a_n = 0$ (Because (v_1, \dots, v_n) is linearly independent)

$\implies (T v_1, \dots, T v_n)$ is linearly independent.

Converse direction: Assume (v_1, \dots, v_n) and $(T v_1, \dots, T v_n)$ are both linearly independent.

\implies If $a_1 T v_1 + \dots + a_n T v_n = 0_w$, then $a_1, \dots, a_n = 0$

\implies If $T(a_1 v_1 + \dots + a_n v_n) = 0_w$, then $a_1 v_1 + \dots + a_n v_n = 0_v$

$\implies N(T) = \{0_v\}$

$\implies T$ is injective.

□

Theorem 7 (2023.S(2.A)).

Let $T : P_n(R) \rightarrow R^{n+1}$ be such that:

$$T(\sum_{i=0}^n c_i t^i) = (x_0, x_1, \dots, x_n)$$

where: $x_k = \int_0^1 f(t) dt$ for $k = 0, \dots, n$.

Show that T is invertible.

Question to self: What is $f(t)$? Also, is it "for some $f(t)$ " or "for all $f(t)$ "?

It cannot be "for all $f(t)$ ", because it's obviously false for $f(t) = 0$.

So it has to be for some $f(t)$.

Proof.

□

Theorem 8 (2023.S(2.B), 2021.F(2.A)).

Let V and W be n -dimensional vector spaces with order bases α and β respectively. If T is an isomorphism from V onto W with $[T]_{\alpha}^{\beta} = A$, show that $[T^{-1}]_{\beta}^{\alpha} = A^{-1}$

Proof.

□

Theorem 9 (2023.S(2.C)).

Let $V = M_{2 \times 2}(R)$ and $T(A) = A^t + 2\text{tr}(A)I_2$, where $A \in V$ and A^t is the transpose of A .

Find an ordered Basis β for V so that $[T]\beta$ is a diagonal matrix.

Proof.

□

Theorem 10 (2023.S(3.B)).

Let $V = W \oplus W^\perp$ and T be the projection on W along W^\perp .

Show that $T^* = T$.

Proof.

Let $u = w_1 + w_2$, $v = w'_1 + w'_2 \in V$ where $w_1, w'_1 \in W$ and $w_2, w'_2 \in W^\perp$.

$$\begin{aligned}\langle Tu, v \rangle &= \langle T(w_1 + w_2), w'_1 + w'_2 \rangle \\ &= \langle w_1, w'_1 + w'_2 \rangle \\ &= \langle w_1, w'_1 \rangle + \langle w_1, w'_2 \rangle \\ &= \langle w_1, w'_1 \rangle + 0 \\ &= \langle w_1, w'_1 \rangle + \langle w_2, w'_1 \rangle \\ &= \langle w_1 + w_2, w'_1 \rangle \\ &= \langle w_1 + w_2, T(w'_1 + w'_2) \rangle \\ &= \langle u, Tv \rangle\end{aligned}$$

$$\implies T^* = T.$$

□

Theorem 11 (2023.S(3.C)).

Let T be a linear operator on the inner product space V . Show that:

$$\langle Tu, Tv \rangle = \langle u, v \rangle \quad \forall u, v \in V \iff \|Tu\| = \|u\| \quad \forall u \in V.$$

Proof.

$\implies :$

$$\begin{aligned} & \langle Tu, Tv \rangle = \langle u, v \rangle \\ \implies & \langle u, T^*Tv \rangle = \langle u, v \rangle \\ \implies & T^*T = I \\ \implies & \|Tu\|^2 = \langle Tu, Tu \rangle \\ \implies & \|Tu\|^2 = \langle u, T^*Tu \rangle \\ \implies & \|Tu\|^2 = \langle u, Iu \rangle \\ \implies & \|Tu\|^2 = \langle u, u \rangle \\ \implies & \|Tu\| = \sqrt{\langle u, u \rangle} = \|u\|. \end{aligned}$$

$\impliedby :$

$$\begin{aligned} & \|Tu\| = \|u\| \\ \implies & \|Tu\|^2 = \|u\|^2 \\ \implies & \langle Tu, Tu \rangle = \langle u, u \rangle \\ \implies & \langle u, T^*Tu \rangle = \langle u, u \rangle \\ \implies & T^*T = I \\ \implies & \langle Tu, Tv \rangle = \langle u, T^*Tv \rangle \\ \implies & \langle Tu, Tv \rangle = \langle u, Iv \rangle = \langle u, v \rangle. \end{aligned}$$

□

Theorem 12 (2022.S(1.A.i)).

If V is a vector space and $S_1, S_2 \subseteq V$ with $S_1 \subseteq S_2$, then S_2^\perp is a subspace of S_1^\perp .

Proof.

Since for any $A \subseteq V$, A^\perp is a subspace of V , we need only show that: S_2^\perp is a subset of S_1^\perp .

$$\begin{aligned} & \text{Let } s_2^\perp \in S_2^\perp \\ \implies & \forall s_2 \in S_2, \langle s_2^\perp, s_2 \rangle = 0 \\ \implies & \forall s_1 \in (S_2 \cap S_1 = S_1), \langle s_2^\perp, s_1 \rangle = 0 \\ \implies & S_2^\perp \subseteq S_1^\perp. \end{aligned}$$

□

Theorem 13 (2022.S(1.B)).

Let $V = M_{2 \times 2}(R)$.

i. Show that V has a basis that contains bases for its subspaces U and W , where:

$$U = \{A \in V : A^T = A\} \text{ and } W = \{A \in V : A^T = -A\}.$$

ii. Show that $V = U \oplus W$.

Proof.

□

Theorem 14 (2022.S(1.D)).

Let V be the vector space of complex numbers over the field \mathbb{R} , i.e. C^1 over \mathbb{R} .

Let $T : V \rightarrow V$ be defined by $T(z) = \bar{z}$, the complex conjugate of z .

i. Show that T is linear.

ii. Show that T is not linear if V is redefined to be over the complex field \mathbb{C} .

Proof.

□

Theorem 15 (2022.S(2.A)).

Let V be an n -dimensional vector space with bases $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$.

If $P \in \text{Hom}(V)$, such that $P(\alpha_i) = \beta_i \quad \forall i$, derive the relation between $[V]_\alpha$ and $[V]_\beta$ for $v \in V$.

Proof.

□

Theorem 16 (2022.S(2.B)).

TODO: Reformulate from a problem to a statement.

Let $T : P_2(R) \rightarrow P_2(R)$ be such that:

$$T(a + bt + ct^2) = -2b - 3c + (a + 3b + 3c)t + ct^2.$$

i. Find a basis for the eigenspace E_1 .

ii. Is T diagonalizable?

iii. Is there an operator on $P_2(R)$ whose null space is E_1 ?

Proof.

□

Theorem 17 (2022.S(3.B), 2021.F(3.D)).

Let V be an inner product space, and $T \in \text{Hom}(V)$.

*i. Show that $N(T^*T) = N(T)$.*

*ii. [Prove or Disprove] $\text{rank}(T^*T) = \text{rank}(T)$.*

Proof.

□

Theorem 18 (2021.F(1.A.i)).

*V is an inner product vector space and $S \subseteq V \wedge S \neq \emptyset$
 $\implies S^\perp$ is a subspace of V .*

Proof.

i. Inclusion of the Zero Vector:

$$\begin{aligned} \forall s \in S, \langle 0_v, s \rangle &= 0 \\ \implies 0_v &\in S^\perp. \end{aligned}$$

ii. Closure under Vector Addition:

$$\begin{aligned} \text{Let } s_1^\perp, s_2^\perp &\in S^\perp, s \in S \\ \implies \langle s_1^\perp, s \rangle &= 0 \wedge \langle s_2^\perp, s \rangle = 0 \\ \implies \langle s_1^\perp, s \rangle + \langle s_2^\perp, s \rangle &= 0 \\ \implies \langle s_1^\perp + s_2^\perp, s \rangle &= 0 \\ \implies s_1^\perp + s_2^\perp &\in S^\perp. \end{aligned}$$

iii. Closure under Scalar Multiplication:

$$\begin{aligned} \text{Let } s^\perp &\in S^\perp, s \in S, k \in F \\ \implies \langle s^\perp, s \rangle &= 0 \\ \implies k \langle s^\perp, s \rangle &= 0 \\ \implies \langle ks^\perp, s \rangle &= 0 \\ \implies ks^\perp &\in S^\perp. \end{aligned}$$

□

Theorem 19 (2021.F(1.A.ii)).

[Disprove] If $T \in \text{Hom}(V, W)$, $\dim(V) = \dim(W) = 2$, and $\{v_1, v_2\}$ is a basis for V , then $\{T(v_1 - v_2), T(v_1)\}$ is a basis for W .

Proof.

□

Theorem 20 (2021.F(3.B)).

Let V be an inner product space, and W be a finite-dimensional subspace of V .

Show that: $w \in W \implies \exists v \in V \ni v \notin W^\perp \wedge \langle w, v \rangle \neq 0$.

Notes to self:

1. I don't think this theorem holds when $W = V$. So I'm going to assume $W \neq V$ in the proof.

2. I don't think this theorem holds when $w = 0_v$. So I'm going to assume $w \neq 0_v$ for the proof.

Proof.

Let W be a **proper** subspace of V , then $\dim(V) = n > \dim(W) = m$.

Let $\{v_i\}_{i=1}^m$ be an ordered orthonormal basis for W , and extend it to an orthonormal basis for V : $\{v_i\}_{i=1}^n$. Then, $\{v_i\}_{i=m+1}^n$ is an orthonormal basis for W^\perp .

Let $w = \sum_{i=1}^m a_i v_i$. Since $w \neq 0_v$, then at least one a_i is non-zero. Let k be the index of the first non-zero a_i .

Next, let $v = b_k v_k + b_n v_n$, where $a_1, a_n \neq 0$.

Then, $v \notin W^\perp$, and:

$$\begin{aligned} \langle w, v \rangle &= \langle w, b_k v_k + b_n v_n \rangle \\ &= \langle w, b_k v_k \rangle + \langle w, b_n v_n \rangle \\ &= \overline{b_k} \langle w, v_k \rangle + \overline{b_n} \langle w, v_n \rangle \\ &= \overline{b_k} \langle w, v_k \rangle + 0 \\ &= \overline{b_k} \left\langle \sum_{i=k}^m a_i v_i, v_k \right\rangle \\ &= \overline{b_k} \langle a_k v_k, v_k \rangle + \overline{b_k} \left\langle \sum_{i=k+1}^m a_i v_i, v_k \right\rangle \\ &= \overline{b_k} a_k \langle v_k, v_k \rangle + 0 \\ &= \overline{a_k} a_k \neq 0 \end{aligned}$$

□

Theorem 21 (2012.F(1.B)).

Let $V = M_{2 \times 2}(R)$,

$B \in V$ such that $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$,

$W_1 = \{A \in V : AB = BA\}$,

$W_2 = \{A \in V : A^T = A\}$.

i. Show that W_1 is a subspace of V .

ii. Find $\dim(W_1)$.

iii. [Prove or Disprove] $V = W_1 \oplus W_2$.

Proof.

□

Theorem 22 (2012.F(1.D)).

Let $T_1, T_2 \in \text{Hom}(V, W)$.

Show that: $\text{rank}(T_1 + T_2) \leq \text{rank}(T_1) + \text{rank}(T_2)$.

Proof.

□

Problems

1.

[2023.S(3.A)]

Let $V = C([-1, 1])$ with the inner product $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt \quad \forall f, g \in V$.

i. Find an orthonormal basis for $P_2(R)$ as a subspace of V and use it to compute the best quadratic approximation of $f(t) = e^t$ on $[-1, 1]$.

ii. For $T \in \text{Hom}(P_1(R))$ with $P_1(R)$ as a subspace of V and $T(f) = f' + 3f$, evaluate $T^*(1 + 3t)$.

Solution:

i. The ordered set $\alpha = \{a_i\}_{i=0}^2 = \{t^i\}_{i=0}^2$ is a basis for $P_2(R)$. We use the Gram-Schmidt process to transform α to an orthonormal basis $\beta = \{b_i\}_{i=0}^2$.

$\underline{b_1}$:

$$\|a_1\|^2 = \langle a_1, a_1 \rangle = \langle 1, 1 \rangle = \int_{-1}^1 1 * 1 dt = [t]_{-1}^1 = 2.$$

$$b_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}}.$$

$\underline{b_2}$:

$$a_2 = t.$$

$$\langle a_2, b_1 \rangle = \int_{-1}^1 t * \frac{1}{\sqrt{2}} dt = \left[\frac{t^2}{2\sqrt{2}} \right]_{-1}^1 = 0.$$

$$a_2 - \langle a_2, b_1 \rangle b_1 = t.$$

$$\|a_2 - \langle a_2, b_1 \rangle b_1\|^2 = \|t\|^2 = \langle t, t \rangle = \int_{-1}^1 t^2 dt = \left[\frac{t^3}{3} \right]_{-1}^1 = \frac{2}{3}.$$

$$b_2 = \frac{a_2 - \langle a_2, b_1 \rangle b_1}{\|a_2 - \langle a_2, b_1 \rangle b_1\|} = \sqrt{\frac{3}{2}} t$$

$\underline{b_3}$:

$$a_3 = t^2.$$

$$\langle a_3, b_1 \rangle = \int_{-1}^1 t^2 * \frac{1}{\sqrt{2}} dt = \left[\frac{t^3}{3\sqrt{2}} \right]_{-1}^1 = \frac{\sqrt{2}}{3}.$$

$$\langle a_3, b_2 \rangle = \int_{-1}^1 t^2 * \sqrt{\frac{3}{2}} t dt = \left[\frac{t^4}{4\sqrt{2}} \right]_{-1}^1 = 0.$$

$$a_3 - \langle a_3, b_1 \rangle b_1 - \langle a_3, b_2 \rangle b_2 = t^2 - \frac{\sqrt{2}}{3} * \frac{1}{\sqrt{2}} = t^2 - \frac{1}{3}.$$

$$\|a_3 - \langle a_3, b_1 \rangle b_1 - \langle a_3, b_2 \rangle b_2\|^2 = \|t^2 - \frac{1}{3}\|^2 = \langle t^2 - \frac{1}{3}, t^2 - \frac{1}{3} \rangle = \int_{-1}^1 (t^2 - \frac{1}{3}) * (t^2 - \frac{1}{3}) dt = \int_{-1}^1 (t^4 - \frac{2}{3}t^2 + \frac{1}{9}) dt = [\frac{t^5}{5} - \frac{2}{9}t^3 + \frac{1}{9}t]_{-1}^1 = \frac{8}{45}.$$

$$b_3 = \frac{a_3 - \langle a_3, b_1 \rangle b_1 - \langle a_3, b_2 \rangle b_2}{\|a_3 - \langle a_3, b_1 \rangle b_1 - \langle a_3, b_2 \rangle b_2\|} = \frac{t^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}}.$$

Best quadratic approximation of $f(t) = e^t$:

The best approximation is given by: $P_{P_2}(f(t)) = \langle f(t), b_1 \rangle b_1 + \langle f(t), b_2 \rangle b_2 + \langle f(t), b_3 \rangle b_3$.

$$\langle f(t), b_1 \rangle = \langle e^t, \frac{1}{\sqrt{2}} \rangle = \int_{-1}^1 \frac{1}{\sqrt{2}} e^t dt = [\frac{1}{\sqrt{2}} e^t]_{-1}^1 = 1.662.$$

$$\langle f(t), b_2 \rangle = \langle e^t, \sqrt{\frac{3}{2}}t \rangle = \int_{-1}^1 e^t * \sqrt{\frac{3}{2}}t dt = 0.901.$$

$$\langle f(t), b_3 \rangle = \langle e^t, \sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}} \rangle = \int_{-1}^1 e^t * (\sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}}) dt = 0.226.$$

$$\implies f(t) \approx 0.226(\sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}}) + 0.901 * \sqrt{\frac{3}{2}}t + \frac{1.662}{\sqrt{2}} = 0.536t^2 + 1.103t + 0.997$$

ii. Let $v = a_0b_1 + a_1b_2 = (a_0, a_1), w = a'_0b_1 + a'_1b_2 = (a'_0, a'_1) \in P_1(R)$.

$$\begin{aligned} \langle v, T^*w \rangle &= \langle Tv, w \rangle \\ &= \langle T(a_0, a_1), (a'_0, a'_1) \rangle \\ &= \langle (3a_0 + a_1, 3a_1), (a'_0, a'_1) \rangle \\ &= \langle (3a_0 + a_1, 0), (a'_0, 0) \rangle + \langle (0, 3a_1), (0, a'_1) \rangle \\ &= \langle (3a_0, 0), (a'_0, 0) \rangle + \langle (a_1, 0), (a'_0, 0) \rangle + \langle (0, 3a_1), (0, a'_1) \rangle \\ &= \langle (a_0, a_1), (3a'_0, a'_0 + 3a'_1) \rangle \end{aligned}$$

$$\implies T^*(a_0, a_1) = (3a_0, a_0 + 3a_1).$$

Let $v = 1 + 3t$, then:

$$\langle v, b_1 \rangle = \sqrt{2}.$$

$$\langle v, b_2 \rangle = 2.45.$$

Therefore, $v = \sqrt{2}b_1 + 2.45b_2 = (\sqrt{2}, 2.45)$.

$$\implies T^*v = (3\sqrt{2}, \sqrt{2} + 3 * 2.45) = (4.24, 8.76).$$

2.

[2022.S(1.C), 2021.F(1.B)]

Let $V = R^3$.

- i. Suggest two 2-dimensional subspaces W_1 and W_2 of V such that $V = W_1 + W_2$.
- ii. With $W_1 = \{v \in V : v = (x, y, 0)\}$, define two projections on W_1 along two distinct subspaces W_2 and W_3 of V .

3.

[2022.S(3.A), 2021.F(3.A)]

Let $V = P_1(R) = \text{Span}(\{1, t\})$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt, \forall f, g \in V$.

- i. Find an orthonormal basis for V .
- ii. Find the orthogonal projection of $f(t) = t^2$ on V .
- iii. Let $T(f) = f'(t) + 3f(t)$ be a linear operator on V . Is $(t - 2)$ an eigenvector for T ?
- iv. Evaluate $T^*(2t - 1)$.

Solution:

i. Let $\alpha = \{a_i\} = \{t^i\}$ be the standard basis. We use the Gram-Schmidt process to orthonormalize this basis to obtain $E = \{e_i\}$.

$$\underline{e_1:}$$

$$\|a_1\|^2 = \langle 1, 1 \rangle = 1.$$

$$e_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{1}} = 1.$$

$$\underline{e_2:}$$

$$\langle a_2, e_1 \rangle = \langle t, 1 \rangle = \frac{1}{2}.$$

$$a_2 - \langle a_2, e_1 \rangle e_1 = t - \frac{1}{2}.$$

$$\|a_2 - \langle a_2, e_1 \rangle e_1\|^2 = \|t - \frac{1}{2}\|^2 = \langle t - \frac{1}{2}, t - \frac{1}{2} \rangle = \frac{1}{12}.$$

$$e_2 = \frac{a_2 - \langle a_2, e_1 \rangle e_1}{\|a_2 - \langle a_2, e_1 \rangle e_1\|} = \frac{t - \frac{1}{2}}{\sqrt{\frac{1}{12}}} = \sqrt{12}(t - \frac{1}{2}).$$

$$\text{ii.}$$

$$\langle f(t), e_1 \rangle = \langle t^2, 1 \rangle = 0.333.$$

$$\langle f(t), e_2 \rangle = \langle t^2, \sqrt{12}(t - \frac{1}{2}) \rangle = 0.289.$$

$$\implies f(t) = e^t \approx 0.333 + 0.289 * \sqrt{12}(t - \frac{1}{2}) = -0.168 + 1.001t.$$

$$\text{iii. } T(t - 2) = 1 + 3(t - 2) = -5 + 3t \neq \lambda(t - 2).$$

$$\implies (t - 2) \text{ is not an eigenvector of } T.$$

$$\text{iv. Let } v = a_1 e_1 + a_2 e_2 = (a_1, a_2), w = b_1 e_1 + b_2 e_2 = (b_1, b_2) \in P_1(R)$$

$$\begin{aligned}
\langle v, T^*w \rangle &= \langle Tv, w \rangle \\
&= \langle T(a_1, a_2), (b_1, b_2) \rangle \\
&= \langle (3a_1 + a_2, 3a_2), (b_1, b_2) \rangle \\
&= \langle (3a_1 + a_2, 0), (b_1, 0) \rangle + \langle (0, 3a_2), (0, b_2) \rangle \\
&= \langle (3a_1, 0), (b_1, 0) \rangle + \langle (a_1, 0), (b_1, 0) \rangle + \langle (0, 3a_2), (0, b_2) \rangle \\
&= 3a_1b_1 + a_1b_1 + 3a_2b_2 \\
&= \langle (a_1, a_2), (3b_1, b_1 + 3b_2) \rangle.
\end{aligned}$$

$$\implies T^*(c_1, c_2) = (3c_1, c_1 + 3c_2).$$

Let $u = 2t - 1$.

$$u = \langle u, e_1 \rangle e_1 + \langle u, e_2 \rangle e_2 = 0.577e_2.$$

$$T^*u = (0, 1.731) .$$

4.

[2021.F(1.C)]

Let $T : P_2(R) \rightarrow R^2$ be such that $T(a_0 + a_1t + a_2t^2) = (a_1 + a_2, a_0 - a_1)$.

i. Find a basis for $\ker(T)$. ii. Is T surjective? iii. Find a two-dimensional subspace of $P_2(R)$ such that its image under T is a one-dimensional subspace of R^2 . iv. Find the matrix representation of T relative to $\{1, t - 1, t^2 + 1\}$ as a basis for $P_2(R)$ and $\{(0, 1), (1, 1)\}$ as a basis for R^2 .

5.

[2021.F(2.B)]

T is a linear operator on $P_2(R)$ defined by: $T(f(x)) = xf'(x) + f(2) + f(3)$.

- i. Is T diagonalizable?
- ii. Find an eigenpair for T .

6.

[2021.F(3.C)]

Let $V = \mathbb{C}^2$ and $T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1)$.
Evaluate $T^*(3-i, 1+2i)$.

Solution:

Let $w = (w_1, w_2), z = (z_1, z_2) \in C^2$.

$$\begin{aligned}\langle w, T^*z \rangle &= \langle Tw, z \rangle \\ &= \langle T(w_1, w_2), (z_1, z_2) \rangle \\ &= \langle (2w_1 + iw_2, (1-i)w_1), (z_1, z_2) \rangle \\ &= (2w_1 + iw_2)z_1 + (1-i)w_1z_2 \\ &= w_1(2z_1 + (1-i)z_2) + w_2(iz_1) \\ &= \langle (w_1, w_2), (2z_1 + (1-i)z_2, iz_1) \rangle\end{aligned}$$

$\implies T^*(z_1, z_2) = (2z_1 + (1-i)z_2, iz_1)$.
Let $v = (3-i, 1+2i)$, then $T^*v = (2(3-i) + (1-i)(1+2i), i(3-i)) = (\dots, \dots)$.

7.

[2012.F(4.B)]

Find the minimal l_2 -norm solution to the system:

$$x + 2y - z = 1$$

$$2x + 3y + z = 2$$

$$4x + 7y - z = 4.$$