MTH-632 PDEs
Assignment (4):
Chapter 4: Separation of Variables
& Chapter 6: Strum-Liouville Eigenvalue Problem

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i.

1.

We are required to find the eigenvalue of the following PDE subject to different boundary conditions:

$$X''(x) + \lambda X(x) = 0$$

We begin by finding the general solution for $X(x)$:

$$\begin{aligned} &X''(x) + \lambda X(x) = 0 \\ \implies &X''(x) = -\lambda X(x) \end{aligned}$$

Depending on the value of λ, the solution can have one of the following 3 forms:
<u>case 1: $\lambda < 0$</u>

$$\begin{aligned} X(x) &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ X'(x) &= \sqrt{\lambda}Ae^{\sqrt{\lambda}x} - \sqrt{\lambda}Be^{-\sqrt{\lambda}x} \end{aligned}$$

<u>case 2: $\lambda = 0$</u>

$$\begin{aligned} X(x) &= Ax + B \\ X'(x) &= A \end{aligned}$$

<u>case 3: $\lambda > 0$</u>

$$\begin{aligned} X(x) &= A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \\ X'(x) &= -\sqrt{\lambda}A\sin(\sqrt{\lambda}x) + \sqrt{\lambda}B\cos(\sqrt{\lambda}x) \end{aligned}$$

Now, we determine the values of λ for the following boundary conditions in each of the 3 cases.

a. <u>$X(0) = X(\pi) = 0$</u>

Assume case 1, then:

$$\begin{aligned} &X(0) = 0 \\ \implies &Ae^0 + Be^0 = 0 \\ \implies &A + B = 0 \\ \implies &B = -A \\ \implies &X(x) = Ae^{\sqrt{\lambda}x} - Ae^{-\sqrt{\lambda}x}. \end{aligned}$$

and,

$$\begin{aligned}
&X(\pi) = 0 \\
\implies &Ae^{\sqrt{\lambda}\pi} - Ae^{-\sqrt{\lambda}\pi} = 0 \\
\implies &A[e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}] = 0 \\
\implies &A = 0 \qquad \text{or} \qquad e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi} = 0 \\
\implies &A = 0 \qquad \text{or} \qquad \sqrt{\lambda}\pi + \sqrt{\lambda}\pi = 0 \\
\implies &A = 0 \qquad \text{or} \qquad \lambda = 0.
\end{aligned}$$

But $\lambda < 0$, therefore we must have $A = B = 0$ which is a trivial solution.

Assume case 2, then:

$$\begin{aligned}
&X(0) = 0 \\
\implies &A * 0 + B = 0 \\
\implies &B = 0 \\
\implies &X(x) = Ax.
\end{aligned}$$

and,

$$\begin{aligned}
&X(\pi) = 0 \\
\implies &A * \pi = 0 \\
\implies &A = 0.
\end{aligned}$$

The solution $A = B = 0$ is trivial, so we ignore it.
Assume case 3, then:

$$\begin{aligned}
&X(0) = 0 \\
\implies &A\cos(0) + B\sin(0) = 0 \\
\implies &A = 0 \\
\implies &X(0) = B\sin(\sqrt{\lambda}x).
\end{aligned}$$

and,

$$\begin{aligned}
&X(\pi) = 0 \\
\implies &B\sin(\sqrt{\lambda}\pi) = 0 \\
\implies &\sin(\sqrt{\lambda}\pi) = 0 \\
\implies &\lambda_n = n^2, \quad n \in I.
\end{aligned}$$

b. $\underline{X'(0) = X'(L) = 0}$

case 1: $\lambda < 0$

$$\begin{aligned}
& X'(0) = 0 \\
\implies & \sqrt{\lambda}Ae^0 - \sqrt{\lambda}Be^0 = 0 \\
\implies & \sqrt{\lambda}A - \sqrt{\lambda}B = 0 \\
\implies & \sqrt{\lambda}A - \sqrt{\lambda}B = 0 \\
\implies & \sqrt{\lambda}[A - B] = 0 \\
\implies & A - B = 0 \\
\implies & A = B \\
\implies & X'(x) = \sqrt{\lambda}Ae^{\sqrt{\lambda}x} - \sqrt{\lambda}Ae^{-\sqrt{\lambda}x} \\
\implies & X'(x) = \sqrt{\lambda}A[e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}]
\end{aligned}$$

and,

$$\begin{aligned}
& X'(L) = 0 \\
\implies & \sqrt{\lambda}A[e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}] = 0 \\
\implies & \sqrt{\lambda}A = 0 \quad \text{or} \quad e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L} = 0 \\
\implies & \sqrt{\lambda}A = 0 \quad \text{or} \quad \sqrt{\lambda}L + \sqrt{\lambda}L = 0 \\
\implies & A = 0 \quad \text{or} \quad \lambda = 0.
\end{aligned}$$

But $\lambda < 0$, therefore we must have $A = B = 0$ which is a trivial solution.

case 2: $\lambda = 0$

$$\begin{aligned}
& X'(0) = 0 \\
\implies & A = 0.
\end{aligned}$$

and,

$$\begin{aligned}
& X'(L) = 0 \\
\implies & A = 0 \\
\implies & X(x) = B.
\end{aligned}$$

Therefore, $\lambda = 0$ is an eigenvalue.

case 3: $\lambda > 0$

$$\begin{aligned} & X'(0) = 0 \\ \Longrightarrow & -\sqrt{\lambda} A \sin(\sqrt{\lambda} * 0) + \sqrt{\lambda} B \cos(\sqrt{\lambda} * 0) = 0 \\ \Longrightarrow & \sqrt{\lambda} B = 0 \\ \Longrightarrow & B = 0 \\ \Longrightarrow & X'(x) = -\sqrt{\lambda} A \sin(\sqrt{\lambda} x). \end{aligned}$$

and,

$$\begin{aligned} & X'(L) = 0 \\ \Longrightarrow & -\sqrt{\lambda} A \sin(\sqrt{\lambda} * L) = 0 \\ \Longrightarrow & \sin(\sqrt{\lambda} * L) = 0 \\ \Longrightarrow & \lambda_n = (\frac{n\pi}{L})^2, \quad n \in I. \end{aligned}$$

Therefore, the eigenvalues are:

$$\lambda = 0, \qquad\qquad \lambda_n = (\frac{n\pi}{L})^2, \quad n \in I.$$

c. $X(0) = X'(L) = 0$

case 1: $\lambda < 0$

$$\begin{aligned} & X(0) = 0 \\ \Longrightarrow & B = -A \qquad \text{(from 1.a)} \end{aligned}$$

and,

$$\begin{aligned} & X'(L) = 0 \\ \Longrightarrow & \sqrt{\lambda} A e^{\sqrt{\lambda} L} + \sqrt{\lambda} A e^{-\sqrt{\lambda} L} = 0 \\ \Longrightarrow & \sqrt{\lambda} A [e^{\sqrt{\lambda} L} + e^{-\sqrt{\lambda} L}] = 0 \\ \Longrightarrow & A = 0 \quad \text{or} \quad e^{\sqrt{\lambda} L} + e^{-\sqrt{\lambda} L} = 0 \\ \Longrightarrow & A = 0 \quad \text{or} \quad \sqrt{\lambda} L - \sqrt{\lambda} L = 0 \\ \Longrightarrow & A = 0 \quad \text{or} \quad \lambda = 0 \\ \Longrightarrow & A = 0. \end{aligned}$$

This is a trivial solution.

case 2: $\lambda = 0$

$$\begin{aligned} & X(0) = 0 \\ \Longrightarrow\ & Ax + B = 0 \\ \Longrightarrow\ & B = 0. \end{aligned}$$

and,

$$\begin{aligned} & X'(L) = 0 \\ \Longrightarrow\ & A = 0 \\ \Longrightarrow\ & X(x) = 0. \end{aligned}$$

This is a trivial solution.

case 3: $\lambda > 0$

$$\begin{aligned} & X(x) = 0 \\ \Longrightarrow\ & A\cos(\sqrt{\lambda} * 0) + B\sin(\sqrt{\lambda} * 0) = 0 \\ \Longrightarrow\ & A = 0. \end{aligned}$$

and,

$$\begin{aligned} & X'(L) = 0 \\ \Longrightarrow\ & \sqrt{\lambda} B\cos(\sqrt{\lambda} L) = 0 \\ \Longrightarrow\ & \cos(\sqrt{\lambda} L) = 0 \\ \Longrightarrow\ & \lambda = (\frac{(n+\frac{1}{2})\pi}{2})^2, \quad n \in I. \end{aligned}$$

d. $X'(0) = X(L) = 0$

case 1: $\lambda < 0$

$$\begin{aligned} & X'(0) \\ \Longrightarrow\ & A = B \qquad \text{(from 1.b)} \end{aligned}$$

and,

$$\begin{aligned}
&X(L) = 0 \\
\Longrightarrow\ & Ae^{\sqrt{\lambda}L} + Ae^{-\sqrt{\lambda}L} = 0 \\
\Longrightarrow\ & A[e^{\sqrt{\lambda}L} + e^{-\sqrt{\lambda}L}] = 0 \\
\Longrightarrow\ & A = 0 \qquad \text{or} \qquad e^{\sqrt{\lambda}L} + e^{-\sqrt{\lambda}L} = 0 \\
\Longrightarrow\ & A = 0 \qquad \text{or} \qquad \sqrt{\lambda}L - \sqrt{\lambda}L = 0 \\
\Longrightarrow\ & A = 0.
\end{aligned}$$

This is a trivial solution.

case 2: $\lambda = 0$

$$\begin{aligned}
&X'(0) = 0 \\
\Longrightarrow\ & A = 0.
\end{aligned}$$

and,

$$\begin{aligned}
&X(L) = 0 \\
\Longrightarrow\ & B = 0.
\end{aligned}$$

This is a trivial solution.

case 3: $\lambda > 0$

$$\begin{aligned}
&X'(0) = 0 \\
\Longrightarrow\ & B = 0 \qquad \text{(from 1.b)}
\end{aligned}$$

and,

$$\begin{aligned}
&X(L) = 0 \\
\Longrightarrow\ & A\cos(\sqrt{\lambda}L) = 0 \\
\Longrightarrow\ & \cos(\sqrt{\lambda}L) = 0 \\
\Longrightarrow\ & \lambda = (\frac{(n+\frac{1}{2})\pi}{2})^2, \quad n \in I.
\end{aligned}$$

e. $X(0) = 0$ and $X'(L) + X(L) = 0$

case 1: $\lambda < 0$

$$\begin{aligned}
&X(0) = 0 \\
\Longrightarrow\ & B = -A \qquad \text{(from 1.a)}
\end{aligned}$$

and,

$$
\begin{aligned}
& X'(L) + X(L) = 0 \\
\Longrightarrow\ & [\sqrt{\lambda} A e^{\sqrt{\lambda} L} + \sqrt{\lambda} A e^{-\sqrt{\lambda} L}] + [A e^{\sqrt{\lambda} L} - A e^{-\sqrt{\lambda} L}] = 0 \\
\Longrightarrow\ & A[\sqrt{\lambda} e^{\sqrt{\lambda} L} + \sqrt{\lambda} e^{-\sqrt{\lambda} L} + e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L}] \\
\Longrightarrow\ & A = 0 \qquad \text{or} \qquad \sqrt{\lambda} e^{\sqrt{\lambda} L} + \sqrt{\lambda} e^{-\sqrt{\lambda} L} + e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L} = 0 \\
\Longrightarrow\ & A = 0 \qquad \text{or} \qquad \lambda L - \lambda L + \sqrt{\lambda} L + \sqrt{\lambda} L = 0 \\
\Longrightarrow\ & A = 0 \qquad \text{or} \qquad 2\sqrt{\lambda} L = 0 \\
\Longrightarrow\ & A = 0 \qquad \text{or} \qquad \lambda = 0 \\
\Longrightarrow\ & A = 0.
\end{aligned}
$$

This is a trivial solution.

case 2: $\lambda = 0$

$$
\begin{aligned}
& X(0) = 0 \\
\Longrightarrow\ & B = 0.
\end{aligned}
$$

and,

$$
\begin{aligned}
& X'(L) + X(L) = 0 \\
\Longrightarrow\ & AL = 0 \\
\Longrightarrow\ & A = 0.
\end{aligned}
$$

This is a trivial solution.

case 3: $\lambda > 0$

$$
\begin{aligned}
& X(0) = 0 \\
\Longrightarrow\ & A\cos(\sqrt{\lambda} * 0) + B\sin(\sqrt{\lambda} * 0) = 0 \\
\Longrightarrow\ & A = 0.
\end{aligned}
$$

and,

$$
\begin{aligned}
& X'(L) + X(L) = 0 \\
\Longrightarrow\ & \sqrt{\lambda} B\cos(\sqrt{\lambda} L) + B\sin(\sqrt{\lambda} L) = 0 \\
\Longrightarrow\ & B[\sqrt{\lambda}\cos(\sqrt{\lambda} L) + \sin(\sqrt{\lambda} L)] = 0 \\
\Longrightarrow\ & \sqrt{\lambda}\cos(\sqrt{\lambda} L) + \sin(\sqrt{\lambda} L) = 0 \\
\Longrightarrow\ & \tan(\sqrt{\lambda} L) = -\sqrt{\lambda} \\
\Longrightarrow\ & \sqrt{\lambda} L = \arctan(-\sqrt{\lambda})
\end{aligned}
$$

This can be solved numerically (with MATLAB for instance).

ii.

1.

We use the separation of variables technique to solve the heat equation.

Assume the solution has the form:

$$u(x,t) = X(x)T(t)$$

Next, we compute u_t and $u_x x$:

$$\begin{aligned} u_t &= X(x)T\dot(t) \\ u_x &= X'(x)T(t) \\ u_{xx} &= X''(x)T(t). \end{aligned}$$

Substitute into the heat equation:

$$\begin{aligned} & u_t = ku_{xx} \\ \implies & X(x)\dot{T}(t) = kX''(x)T(t) \\ \implies & \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{kT(t)} \\ \implies & \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{kT(t)} = -\lambda \\ \implies & \frac{X''(x)}{X(x)} = -\lambda \qquad \text{and} \qquad \frac{\dot{T}(t)}{kT(t)} = -\lambda. \end{aligned}$$

First, we solve the spatial ODE along with the boundary conditions to obtain $X(x)$ as follows:

$$\begin{aligned} & \frac{X''(x)}{X(x)} = -\lambda \\ \implies & X''(x) + \lambda X(x) = 0. \end{aligned}$$

and,

$$\begin{aligned} & u(0,t) = 0 \\ \implies & X(0)T(t) = 0 \\ \implies & X(0) = 0 \end{aligned}$$

and,

$$\begin{aligned} & u(L,t) = 0 \\ \implies & X(L)T(t) = 0 \\ \implies & X(L) = 0 \end{aligned}$$

This matches the ODE and boundary conditions of problem 1.a above. Therefore the eigenvalues and th corresponding spatial eigenvectors are given by:

$$\begin{aligned}
\lambda_n &= \left(\frac{n\pi}{L}\right)^2, && n \in I \\
\implies X_n(x) &= \sin\left(\frac{n\pi}{L}x\right), && n \in I.
\end{aligned}$$

Having found the eignevalues λ_n, we now solve for the temporal eigenvectors $T(t)$:

$$\begin{aligned}
\frac{\dot{T_n}(t)}{kT_n(t)} &= -\lambda_n, && n \in I \\
\implies \dot{T_n}(t) &= -k\lambda_n T_n(t), && n \in I \\
\implies \dot{T_n}(t) &= -k\left(\frac{n\pi}{L}\right)^2 T_n(t), && n \in I \\
\implies T_n(t) &= e^{-k\left(\frac{n\pi}{L}\right)^2 t}, && n \in I.
\end{aligned}$$

Therefore, $u_n(x,t)$ is given by:

$$\begin{aligned}
u_n(x,t) &= b_n X_n(x) T_n(t) \\
\implies u_n(x,t) &= b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.
\end{aligned}$$

The complete solution $u(x,t)$ is the superposition of $u_n(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

The constants b_n can be found by applying the initial condition.

a.

$$\begin{aligned}
u(x,0) &= 6\sin\left(\frac{9\pi}{L}x\right) \\
\implies \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) &= 6\sin\left(\frac{9\pi}{L}x\right) \\
\implies b_9 = 6 \qquad &\text{and} \qquad b_n = 0 \quad \forall n : n \neq 6 \\
\implies u(x,t) &= 6\sin\left(\frac{9\pi}{L}x\right) e^{-k\left(\frac{9\pi}{L}\right)^2 t}.
\end{aligned}$$

b.

$$\begin{aligned}
u(x,0) &= 2\cos\left(\frac{3\pi}{L}x\right) \\
\implies b_n &= \frac{\int_{x=0}^{L} 2\cos\left(\frac{3\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx}{\int_{x=0}^{L} \sin^2\left(\frac{n\pi}{L}x\right)}.
\end{aligned}$$

3.

Note to self: This problem explores periodic boundary conditions.

case 1: $\lambda < 0$

$$\begin{aligned}
&\phi(0) = \phi(2\pi) \\
\implies &Ae^{\sqrt{-\lambda}*0} + Be^{-\sqrt{-\lambda}*0} = Ae^{\sqrt{-\lambda}2\pi} + Be^{-\sqrt{-\lambda}2\pi} \\
\implies &A + B = Ae^{\sqrt{-\lambda}2\pi} + Be^{-\sqrt{-\lambda}2\pi} \\
\implies &A[1 - e^{\sqrt{-\lambda}2\pi}] + B[1 - e^{-\sqrt{-\lambda}2\pi}] = 0 \qquad (1)
\end{aligned}$$

and,

$$\begin{aligned}
&\phi'(0) = \phi'(2\pi) \\
\implies &\sqrt{-\lambda}Ae^{\sqrt{-\lambda}*0} - \sqrt{-\lambda}Be^{-\sqrt{-\lambda}*0} = \sqrt{-\lambda}Ae^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda}Be^{-\sqrt{-\lambda}2\pi} \\
\implies &\sqrt{-\lambda}A - \sqrt{-\lambda}B = \sqrt{-\lambda}Ae^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda}Be^{-\sqrt{-\lambda}2\pi} \\
\implies &\sqrt{-\lambda}A[1 - e^{\sqrt{-\lambda}2\pi}] - \sqrt{-\lambda}B[1 - e^{-\sqrt{-\lambda}2\pi}] = 0 \qquad (2)
\end{aligned}$$

(1) and (2) is a $2x2$ system of linear equation. It can be put in the form $Mx = 0$.

$$M = \begin{pmatrix} 1 - e^{\sqrt{-\lambda}2\pi} & 1 - e^{-\sqrt{-\lambda}2\pi} \\ \sqrt{-\lambda}[1 - e^{\sqrt{-\lambda}2\pi}] & \sqrt{-\lambda}[1 - e^{-\sqrt{-\lambda}2\pi}] \end{pmatrix}$$

The system has a non-trivial solution only if the M is non-injective (i.e. has a zero determinant). Thus:

$$\begin{aligned}
&\det(M) = 0 \\
\implies &(1 - e^{\sqrt{-\lambda}2\pi}) * (\sqrt{-\lambda}[1 - e^{-\sqrt{-\lambda}2\pi}]) - (1 - e^{-\sqrt{-\lambda}2\pi}) * (\sqrt{-\lambda}[1 - e^{\sqrt{-\lambda}2\pi}]) = 0 \\
\implies &-2\sqrt{\lambda}(1 - e^{\sqrt{-\lambda}2\pi}) * (1 - e^{-\sqrt{-\lambda}2\pi}) = 0 \\
\implies &1 - e^{\sqrt{-\lambda}2\pi} = 0 \quad \text{or} \quad 1 - e^{-\sqrt{-\lambda}2\pi} = 0 \\
\implies &e^{\sqrt{-\lambda}2\pi} = 1 \quad \text{or} \quad e^{-\sqrt{-\lambda}2\pi} = 1 \\
\implies &\sqrt{-\lambda}2\pi = \ln(1) = 0 \quad \text{or} \quad -\sqrt{-\lambda}2\pi = \ln(1) = 0 \\
\implies &\sqrt{-\lambda} = 0 \quad \text{or} \quad \sqrt{-\lambda} = 0
\end{aligned}$$

This is impossible, because $\lambda < 0$.
Thus, the solution in this case is the trivial solution with $A = B = 0$.

case 2: $\lambda = 0$

$$\begin{aligned}
&\phi(0) = \phi(2\pi) \\
\implies &B = 2\pi A + B \\
\implies &A = 0.
\end{aligned}$$

and,

$$\begin{aligned}
&\phi'(0) = \phi'(2\pi) \\
\implies &A = A \\
\implies &\phi(x) = B.
\end{aligned}$$

<u>case 3: $\lambda > 0$</u>

$$\begin{aligned}
&\phi(0) = \phi(2\pi) \\
\implies &A\cos(\sqrt{\lambda} * 0) + B\sin(\sqrt{\lambda} * 0) = A\cos(\sqrt{\lambda}2\pi) + B\sin(\sqrt{\lambda}2\pi) \\
\implies &A = A\cos(\sqrt{\lambda}2\pi) + B\sin(\sqrt{\lambda}2\pi) \\
\implies &A[1 - \cos(\sqrt{\lambda}2\pi)] - B\sin(\sqrt{\lambda}2\pi) = 0 \qquad (3)
\end{aligned}$$

and,

$$\begin{aligned}
&\phi'(0) = \phi'(2\pi) \\
\implies &-\sqrt{\lambda}A\sin(\sqrt{\lambda} * 0) + \sqrt{\lambda}B\cos(\sqrt{\lambda} * 0) = -\sqrt{\lambda}A\sin(\sqrt{\lambda}2\pi) + \sqrt{\lambda}B\cos(\sqrt{\lambda}2\pi) \\
\implies &B\sqrt{\lambda} = -\sqrt{\lambda}A\sin(\sqrt{\lambda}2\pi) + \sqrt{\lambda}B\cos(\sqrt{\lambda}2\pi) \\
\implies &B\sqrt{\lambda} + \sqrt{\lambda}A\sin(\sqrt{\lambda}2\pi) - \sqrt{\lambda}B\cos(\sqrt{\lambda}2\pi) = 0 \\
\implies &\sqrt{\lambda}A\sin(\sqrt{\lambda}2\pi) + \sqrt{\lambda}B[1 - \cos(\sqrt{\lambda}2\pi)] = 0 \qquad (4)
\end{aligned}$$

(3) and (4) is a $2x2$ system of linear equation. It can be put in the form $Mx = 0$.

$$M = \begin{pmatrix} 1 - \cos(\sqrt{\lambda}2\pi) & -\sin(\sqrt{\lambda}2\pi) \\ \sqrt{\lambda}\sin(\sqrt{\lambda}2\pi) & \sqrt{\lambda}[1 - \cos(\sqrt{\lambda}2\pi)] \end{pmatrix}$$

The system has a non-trivial solution only if the M is non-injective (i.e. has a zero determinant). Thus:

$$
\begin{aligned}
&\det(M) = 0 \\
\implies &(1-\cos(\sqrt{\lambda}2\pi)) * \sqrt{\lambda}[1-\cos(\sqrt{\lambda}2\pi)] + \sin(\sqrt{\lambda}2\pi) * \sqrt{\lambda}\sin(\sqrt{\lambda}2\pi) = 0 \\
\implies &\sqrt{\lambda}(1-\cos(\sqrt{\lambda}2\pi))^2 + \sqrt{\lambda}\sin^2(\sqrt{\lambda}2\pi) = 0 \\
\implies &\sqrt{\lambda}(1-2\cos(\sqrt{\lambda}2\pi) + \cos^2(\sqrt{\lambda}2\pi)) + \sqrt{\lambda}\sin^2(\sqrt{\lambda}2\pi) = 0 \\
\implies &\sqrt{\lambda}(1-2\cos(\sqrt{\lambda}2\pi) + \cos^2(\sqrt{\lambda}2\pi) + \sin^2(\sqrt{\lambda}2\pi)) = 0 \\
\implies &\sqrt{\lambda}(1-2\cos(\sqrt{\lambda}2\pi) + 1) = 0 \\
\implies &\sqrt{\lambda}(2-2\cos(\sqrt{\lambda}2\pi)) = 0 \\
\implies &2\sqrt{\lambda}(1-\cos(\sqrt{\lambda}2\pi)) = 0 \\
\implies &1-\cos(\sqrt{\lambda}2\pi) = 0 \\
\implies &\cos(\sqrt{\lambda}2\pi) = 1 \\
\implies &\cos(\sqrt{\lambda}2\pi) = 1 \\
\implies &\sqrt{\lambda}2\pi = 2\pi n \\
\implies &\sqrt{\lambda} = n \\
\implies &\lambda_n = n^2, \quad n \in I.
\end{aligned}
$$

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iii.

1.

a.

Boundary Condition Check:

$$\begin{aligned} &X(0) = 0 \\ \implies &\beta_1 X(0) + \beta_2 X'(0) = 0 \qquad \text{for } \beta_1 = 1 \text{ and } \beta_2 = 0. \end{aligned}$$

and,

$$\begin{aligned} &X'(L) = 0 \\ \implies &\beta_3 X(L) + \beta_4 X'(L) = 0 \qquad \text{for } \beta_3 = 0 \text{ and } \beta_4 = 1. \end{aligned} \tag{5}$$

$\implies$ The boundary conditions are satisfied.

PDE form Check:

$$\begin{aligned} &X''(x) + \lambda X(x) = 0 \\ \implies &p(x) = 1,\ q(x) = 0,\ \sigma(x) = 1 \text{ in the Sturm-Liouville Equation:} \\ &\frac{d}{dx}\left(p(x)\frac{dX(x)}{dx}\right) + q(x)X(x) + \lambda\sigma(x)X(x) = 0 \\ \implies &p,\ q,\ \sigma \text{ are real and continuous, and } p,\ \sigma \text{ are positive on } [0, L]. \end{aligned} \tag{6}$$

(5) and (6) $\implies$ The problem is a **regular** Sturm-Liouville problem.

b.

$$\begin{aligned} &X_n = \sin((n - \frac{1}{2})x) \qquad\qquad n \in \{1, 2, \dots\} \\ &\lambda_n = (n - \frac{1}{2})^2. \end{aligned}$$

3.

a.

Boundary Condition Check:

$$
\begin{aligned}
& X(0)=0 \quad X'(0)=0 \\
\implies & \beta_1 X(0)+\beta_2 X'(0)=0 \qquad \text{for } \beta_1=1 \text{ and } \beta_2=-1.
\end{aligned}
$$

and,

$$
\begin{aligned}
& X(L)=0 \quad X'(L)=0 \\
\implies & \beta_3 X(L)+\beta_4 X'(L)=0 \qquad \text{for } \beta_3=1 \text{ and } \beta_4=-1.
\end{aligned} \tag{7}
$$

$\implies$ The boundary conditions are satisfied.

PDE form Check:

$$
\text{This is equivalent to problem 1 above which checks out.} \tag{8}
$$

(7) and (8) $\implies$ The problem is a **regular** Sturm-Liouville problem.

b.

The same problem was solved before, and the solution was:

$$
\begin{aligned}
X_n &= A_n \cos\left(\frac{2n\pi}{L}x\right)+B_n \sin\left(\frac{2n\pi}{L}x\right) \qquad n\in\{1,2,\dots\} \\
\lambda_n &= \left(\frac{2n\pi}{L}\right)^2.
\end{aligned}
$$