Logic (Knowledge Representation and Reasoning)
Finals Questions Bank

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18 January 2024

6-1

Using the FOPL system of natural deduction introduced in class, prove the following.

a.

$$\{\forall x\,[P(x) \implies Q(x)]\,, \forall x\,[Q(x) \implies R(x)]\} \vdash \forall x\,[P(x) \implies R(x)]\,.$$

Proof.

$1.\forall x\,[P(x) \implies Q(x)]$	(Premise)
$2.\forall x\,[Q(x) \implies R(x)]$	(Premise)
$3.P(t) \implies Q(t)$	$(1, \forall\text{-elim})$
$4.Q(t) \implies R(t)$	$(2, \forall\text{-elim})$
$*\,5.P(t)$	(Assumption)
$*\,6.Q(t)$	$(3, 5, \implies\text{-elim})$
$*\,7.R(t)$	$(4, 6, \implies\text{-elim})$
$8.P(t) \implies R(t)$	$(5, 7, \implies\text{-intro})$
$9.\forall x\,[P(x) \implies R(x)]$	$(8, \forall\text{-intro})$.

□

b.

$$\{\forall x\,[P(x) \vee Q(x)]\,, \forall x\,[[\neg P(x) \wedge Q(x)] \implies R(x)]\} \vdash \forall x\,[\neg R(x) \implies P(x)]\,.$$

Proof.

$1.\forall x\,[P(x) \vee Q(x)]$	(Premise)
$2.\forall x\,[[\neg P(x) \wedge Q(x)] \implies R(x)]$	(Premise)
$3.P(t) \vee Q(t)$	$(1, \forall$-elim)
$4.\,[\neg P(t) \wedge Q(t)] \implies R(t)$	$(2, \forall$-elim)
$*\,5.\neg R(t)$	(Assumption)
$**6.\neg P(t)$	(Assumption)
$**7.Q(t)$	$(3, 6, \vee$-elim)
$**8.\neg P(t) \wedge Q(t)$	$(6, 7, \wedge$-intro)
$**9.R(t)$	$(4, 8, \implies$-elim)
$**10.\neg R(t) \wedge R(t)$	$(5, 9, \wedge$-elim)
$*\,11.\neg\neg P(t)$	$(6, 10, \neg$-intro)
$*\,12.P(t)$	$(11, \neg$-elim)
$13.\neg R(t) \implies P(t)$	$(5, 12, \implies$-intro)
$14.\forall x\,[\neg R(x) \implies P(x)]$	$(13, \forall$-intro).

$\square$

c.

$$\{\forall x\,[P(x) \vee Q(x)]\,, \forall x\,[\neg Q(x) \vee S(x)]\,, \forall x\,[R(x) \implies \neg S(x)]\,, \exists x\,[\neg P(x)]\} \vdash \exists x\,[\neg R(x)]\,.$$

Proof.

$1. \forall x\,[P(x) \vee Q(x)]$	(Premise)
$2. \forall x\,[\neg Q(x) \vee S(x)]$	(Premise)
$3. \forall x\,[R(x) \implies \neg S(x)]$	(Premise)
$4. \exists x\,[\neg P(x)]$	(Premise)
$5. \neg P(c)$	$(4, \exists$-elim)
$6. P(c) \vee Q(c)$	$(1, \forall$-elim)
$7. \neg Q(c) \vee S(c)$	$(2, \forall$-elim)
$8. R(c) \implies \neg S(c)$	$(3, \forall$-elim)
$9. Q(c)$	$(5, 6, \vee$-elim)
$10. S(c)$	$(7, 9, \vee$-elim)
$*\,11. R(c)$	(Assumption)
$*\,12. \neg S(c)$	$(8, 11, \implies$-elim)
$*\,13. S(c) \wedge \neg S(c)$	$(10, 12, \wedge$-intro)
$14. \neg R(c)$	$(13, \neg$-intro)
$15. \exists x\,[\neg R(c)]$	$(14, \exists$-intro).

$\square$

d.

$$\vdash [\neg [\exists x P(x)]] \iff \forall x [\neg P(x)] .$$

Proof.
$\underline{\implies:}$

$1. \neg [\exists x P(x)]$	(Premise)
$* 2. P(t)$	(Assumption)
$* 3. \exists x P(x)$	$(2, \exists$-intro$)$
$* 4. \neg [\exists x P(x)] \wedge \exists x P(x)$	$(1, 3, \wedge$-intro$)$
$5. \neg P(t)$	$(4, \neg$-intro$)$
$6. \forall x [\neg P(x)]$	$(4, \forall$-intro$)$
$7. [\neg [\exists x P(x)]] \implies \forall x [\neg P(x)]$	$(1, 6, \implies$-intro$)$.

$\underline{\impliedby:}$

$i. \forall x [\neg P(x)]$	(Premise)
$* ii. \neg [\neg [\exists x P(x)]]$	(Assumption)
$* iii. \exists x P(x)$	$(ii, \neg$-elim$)$
$* iv. P(c)$	$(iii, \exists$-elim$)$
$* v. \neg P(c)$	$(i, \forall$-elim$)$
$* vi. P(c) \wedge \neg P(c)$	$(iv, v, \wedge$-intro$)$
$vii. \neg [\neg [\neg [\exists x P(x)]]]$	$(ii, vi, \neg$-intro$)$
$viii. \neg [\exists x P(x)]$	$(vii \neg$-elim$)$
$ix. \forall x [\neg P(x)] \implies [\neg [\exists x P(x)]]$	$(i, viii, \implies$-intro$)$.

$\underline{\iff:}$

$\vdash [\neg [\exists x P(x)]] \iff \forall x [\neg P(x)]$	$(7, ix, \iff$-intro$)$.

□

e.

$$\vdash \exists x\,[P(x) \vee Q(x)] \iff \exists x P(x) \vee \exists x Q(x)$$

Proof.

$\underline{\implies:}$

$*\,1. \exists x\,[P(x) \vee Q(x)]$	(Assumption)
$*\,2. P(c) \vee Q(c)$	(1, $\exists$-elim)
$*\,3. \exists x P(x) \vee Q(c)$	(2, $\exists$-intro)
$*\,4. \exists x P(x) \vee \exists x Q(x)$	(3, $\exists$-intro)
$5. \exists x\,[P(x) \vee Q(x)] \implies \exists x P(x) \vee \exists x Q(x)$	(1, 4, $\implies$-intro).

$\underline{\impliedby:}$

$*\,i. \exists x P(x) \vee \exists x Q(x)$	(Assumption)
$**ii. \neg\,[\exists x\,[P(x) \vee Q(x)]]$	(Assumption)
$**iii. \neg\,[P(c) \vee Q(c)]$	(ii, $\exists$-elim)
$**iv. \neg\,[\exists x P(x) \vee Q(c)]$	(iii, $\exists$-intro)
$**v. \neg\,[\exists x P(x) \vee \exists x Q(x)]$	(iv, $\exists$-intro)
$*\,vi. \neg\neg\,[\exists x\,[P(x) \vee Q(x)]]$	(i, ii, v, $\neg$-intro)
$*\,vii. \exists x\,[P(x) \vee Q(x)]$	(vi, $\neg$-elim)
$viii. \exists x P(x) \vee \exists x Q(x) \implies \exists x\,[P(x) \vee Q(x)]$	(i, vii, $\implies$-intro).

$\underline{\iff:}$

$$\exists x\,[P(x) \vee Q(x)] \iff \exists x P(x) \vee \exists x Q(x) \qquad (5,\ viii,\ \iff\text{-intro}).$$

$\square$