

# Functional Analysis Assignment (Chapter 1)

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### 1.1.1

The distance on  $\mathbb{R}$  is defined by  $d(x, y) = |x - y|$ . We must check that the 4 axioms (M1 to M4) are satisfied.

M1 holds, since the absolute value of the difference between 2 real points is real, finite, and non-negative.

M2 holds, since  $d(x, y) = |x - y| = 0 \iff x = y$ .

M3 holds, since  $d(x, y) = |x - y| = |y - x| = d(y, x)$ .

M4 holds, since the triangle inequality holds for the absolute value.

### 1.1.2

$d(x, y) = (x - y)^2$  is not a valid metric since it does not satisfy the triangle inequality.

*Proof.* (By Counterexample)

Let  $a, b, c \in \mathbb{R}$ , where  $a = 0, b = 1, c = 5$ , then:

$$d(a, c) = (a - c)^2 = (0 - 5)^2 = 25.$$

$$d(a, b) = (a - b)^2 = (0 - 1)^2 = 1.$$

$$d(b, c) = (b - c)^2 = (1 - 5)^2 = 16.$$

$$d(a, b) + d(b, c) = 1 + 16 = 17.$$

Therefore,  $d(a, c) \not\leq d(a, b) + d(b, c)$ .

□

### 1.1.3

Since the distance function  $d(x, y) = |x - y|$  defines a metric on  $\mathbb{R}$ , as shown in 1.1.1, then it is clear that the square root of that metric is also real, finite, and non-negative (i.e. M1 holds); definite (i.e. M2 holds); and symmetric (i.e. M3 holds).

The triangle inequality can be shown to hold by noting that the square root function is an increasing function with a negative second derivative in the interval  $(0, \infty)$ .

This shows that  $d(x, y) = \sqrt{|x - y|}$  is a metric on  $\mathbb{R}$ .

### 1.1.4

i.  $|X| = 2$

Let  $X = \{a, b\}$ , then  $d$  must satisfy:

$$\begin{aligned} d(a, a) &= d(b, b) = 0, \text{ and} \\ d(a, b) &= d(b, a) = c, \text{ where } c \text{ is any non-negative real number.} \end{aligned}$$

ii.  $|X| = 1$

In this case the only valid metric is  $d(a, a) = 0$ .

### 1.1.5

i. **Conditions for  $kd$  to be a metric**

If  $d$  is a metric, then  $kd$  automatically satisfies axioms M2-M4.

For axiom M1 to hold,  $k$  must be a non-negative real number.

ii. **Conditions for  $k + d$  to be a metric**

To satisfy axiom M2,  $k$  must be zero.

### 1.1.6

*Proof.* (By Induction on the Length of the Sequence)

Let  $X = (x_j)$ ,  $Y = (y_j)$ ,  $Z = (z_j)$  be 3 bounded sequences.

#### Base case

Consider the subsequence of  $X, Y, Z$  consisting of just their first element.

Then by the triangle inequality for numbers:

$$\begin{aligned} |x_1 - z_1| &\leq |x_1 - y_1| + |y_1 - z_1| \\ \implies \sup |x_1 - z_1| &\leq \sup(|x_1 - y_1| + |y_1 - z_1|) \\ \implies \sup |x_1 - z_1| &\leq \sup |x_1 - y_1| + \sup |y_1 - z_1|. \end{aligned}$$

Therefore  $d(x, z) \leq d(x, y) + d(y, z)$  holds for sequences of length 1.

#### Inductive step

Next, we'll consider the sub-sequences of  $X, Y, Z$  consisting of the first  $n + 1$  elements. Suppose that the induction hypothesis holds for sequences of length  $n$ , i.e.:

$$\sup_{j \in \{1..n\}} |x_j - z_j| \leq \sup_{j \in \{1..n\}} |x_j - y_j| + \sup_{j \in \{1..n\}} |y_j - z_j|.$$

Then we can partition each sequence of length  $n + 1$  into 2 sub-sequences: the first sequence contains the first  $n$  elements and the second contains the last element.

The distance between any 2 sequences of length  $n + 1$  then becomes:

$$\max(\sup_{j \in 1..n} |x_j - z_j|, \sup |x_{n+1} - z_{n+1}|)$$

Finally, applying the induction hypothesis we get:

$$\max(\sup_{j \in 1..n} |x_j - z_j|, \sup |x_{n+1} - z_{n+1}|) \leq \max(\sup_{j \in 1..n} |x_j - y_j| + \sup_{j \in 1..n} |y_j - z_j|, \sup |x_{n+1} - y_{n+1}| + \sup |y_{n+1} - z_{n+1}|)$$

$$\implies \sup_{j \in 1..n+1} |x_j - z_j| \leq \sup_{j \in 1..n+1} |x_j - y_j| + \sup_{j \in 1..n+1} |y_j - z_j|$$

Therefore  $d(x, z) \leq d(x, y) + d(y, z)$  holds for sequences of any length.  $\square$

### 1.1.7

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Which is the discrete metric.

### 1.1.8

M1 holds because  $|x(t) - y(t)|$  is a positive function, and the integral of a positive function is positive.

M2 holds because  $|x(t) - x(t)| = 0$ , and the integral of the zero function is 0.

M3 holds because  $d(x, y) = \int_a^b |x(t) - y(t)| dt = \int_a^b |y(t) - x(t)| dt = d(y, x)$ .

To show that M4 holds, let  $x, y, z \in X$ , then:

$$\begin{aligned}
 d(x, z) &= \int_a^b |x(t) - z(t)| \, dt \\
 &= \int_a^b |(x(t) - y(t)) - (z(t) - y(t))| \, dt \\
 &\geq \int_a^b (|x(t) - y(t)| - |z(t) - y(t)|) \, dt \quad (\text{By the triangle inequality of absolute values}) \\
 &= \int_a^b |x(t) - y(t)| \, dt - \int_a^b |z(t) - y(t)| \, dt \\
 &= d(x, y) - d(z, y)
 \end{aligned}$$

$$\implies d(x, y) \leq d(x, z) + d(z, y).$$

### 1.2.3

*Proof.*

Let  $(\zeta_j)$ ,  $(\eta_j)$ , and  $(\theta_j) \in l^p$ , where  $(\zeta_j)$  is any point, and define  $(\eta_j)$  and  $(\theta_j)$  as follows:

$$\eta_j := \begin{cases} 1 & j \leq n \\ 0 & j > n \end{cases}$$

$$\theta_j := \begin{cases} \zeta_j & j \leq n \\ 0 & j > n \end{cases}$$

Applying the Cauchy-Schwarz inequality to  $(\theta_j)$  and  $(\eta_j)$ , we get:

$$\begin{aligned} \sum_{j=1}^{\infty} |\theta_j \eta_j| &\leq \sqrt{\sum_{k=1}^{\infty} |\theta_j|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_j|^2} \\ \sum_{j=1}^n |\theta_j \eta_j| &\leq \sqrt{\sum_{k=1}^n |\theta_j|^2} \sqrt{\sum_{m=1}^n |\eta_j|^2} \\ \sum_{j=1}^n |\theta_j| &\leq \sqrt{\sum_{k=1}^n |\theta_j|^2} \sqrt{\sum_{m=1}^n 1} \\ \sum_{j=1}^n |\zeta_j| &\leq \sqrt{\sum_{k=1}^n |\zeta_j|^2} \sqrt{n} \\ \left( \sum_{j=1}^n |\zeta_j| \right)^2 &\leq n \sum_{k=1}^n |\zeta_j|^2 \end{aligned}$$

□

### 1.2.4

### 1.2.5

*Proof.*

The sequence  $\zeta_j = \frac{1}{j}$  is divergent for  $p = 1$  but convergent for all  $p > 1$ .

□

### 1.2.11

*Proof.*

M1 holds because:

$$\begin{aligned}
& d(x, y) \geq 0 \\
\implies & \frac{d(x, y)}{1 + d(x, y)} \geq 0 \\
\implies & \tilde{d}(x, y) \geq 0.
\end{aligned}$$

M2 holds because:

$\implies$ :

$$\begin{aligned}
& \tilde{d}(x, y) = 0 \\
\implies & \frac{d(x, y)}{1 + d(x, y)} = 0 \\
\implies & d(x, y) = 0 \\
\implies & x = y.
\end{aligned}$$

$\Leftarrow$ :

$$\begin{aligned}
& x = y \\
\implies & d(x, y) = 0 \\
\implies & \frac{d(x, y)}{1 + d(x, y)} = 0 \\
\implies & \tilde{d}(x, y) = 0.
\end{aligned}$$

M3 holds because:

$$\begin{aligned}
& d(x, y) = d(y, x) \\
\implies & \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} \\
\implies & \tilde{d}(x, y) = \tilde{d}(y, x).
\end{aligned}$$

M4 holds because:

$$\begin{aligned}
& d(x, z) \leq d(x, y) + d(y, z) \\
\implies & \frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y)}{1 + d(x, z)} + \frac{d(y, z)}{1 + d(x, z)} \\
\implies & \frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\
\implies & \tilde{d}(x, z) \leq \tilde{d}(x, y) + \tilde{d}(y, z).
\end{aligned}$$

□

### 1.3.4

*Proof.*

$\Rightarrow$  (By Construction):

Let  $A$  be a non-empty open subset, and  $\delta(A)$  be the diameter of  $A$ .

Since  $A$  is non-empty and open, then  $|A| > 1$ .

Let  $a_1, a_2$  be 2 different points in  $A$ .

Consider the 2 open balls  $B(a_1, \delta(A))$  and  $B(a_2, \delta(A))$ .

We have  $B(a_1, \delta(A)) = B(a_2, \delta(A)) = A$ .

Thus,  $A = B(a_1, \delta(A)) \cup B(a_2, \delta(A))$ .

$\Leftarrow$  (By Strong Induction):

We will prove a **stronger** result: The union of non-empty open subsets is a non-empty open subset.

Let  $B = (B_1, B_2, \dots, B_n)$  be a list of open subsets (balls **or not**) in  $X$ .

Base case:  $|B| = 2$

$$A = B_1 \cup B_2$$

$$\Rightarrow \forall a \in A, a \in B_1 \vee a \in B_2$$

$$\Rightarrow \forall a \in A, \exists \text{ a ball about } a$$

$$\Rightarrow A \text{ is a non-empty open subset of } X.$$

Inductive step:  $|B| = k > 2$

$$\begin{aligned} A &= \bigcup_{j=1}^k B_j \\ &= \bigcup_{j=1}^{k-1} B_j + B_k \\ &= C + B_k \end{aligned} \tag{1}$$

Where  $C$  is a non-empty open subset by the induction hypothesis for  $n = k - 1$ .

Applying the induction hypothesis to (1) for  $n = 2$ , we conclude that  $A$  is a non-empty open subset in  $X$ .

□

### 1.3.6

*Proof.* (By Construction)



We will prove this statement by recursively (and infinitely) applying the definition of an accumulation point on increasingly smaller  $\epsilon$ -neighborhoods.

Let the sequence of  $(\epsilon_j)$  be such that  $\forall j, \epsilon_j > 0$ .

Since  $x_0$  is an accumulation point, then each  $\epsilon_j$ -neighborhood contains at least one point  $y \in A$ .

Let  $y_j$  to be any point in the  $\epsilon_j$ -neighborhood of  $x_0$  that satisfies the following condition:

$$d(y_j, x_0) = \max_{\forall y_k \in \tilde{B}(x_0; \epsilon_j)} d(y_k, x_0)$$

Define  $\epsilon_j$  recursively as follows:

$$\epsilon_j := \begin{cases} \epsilon_1 & j = 1 \\ d(x_0, y_{j-1})/2 & j > 1 \end{cases}$$

Then the sequence  $(y_j)$  is an infinite sequence of distinct points in  $A$ , all contained inside the  $\epsilon_1$ -neighborhood of  $x_0$ .

Since  $\epsilon_1$  was arbitrarily chosen, this completes the proof. □

### 1.3.13

*Proof.*

This follows directly from the definition that a separable space  $X$  contains a subset  $Y$  that is countable and dense in  $X$ . □

### 1.3.14

*Proof.*

$\implies$ :

Let  $T : X \longrightarrow Y$  be a continuous map and  $M$  be a closed subset in  $Y$ .

- |                                      |   |
|--------------------------------------|---|
| $\implies M^c$ is open               | (Because the complement of a closed set is an open set by definition 1.3-2) |
| $\implies T^{-1}(M^c)$ is open       | (By continuity of $T$ )   |
| $\implies (T^{-1}(M^c))^c$ is closed | (Because the complement of an open set is a closed set by definition 1.3-2) |
| $\implies T^{-1}((M^c)^c)$ is closed | (By the identity: $f^{-1}(A^c) = [f^{-1}(A)]^c$ )                           |
| $\implies T^{-1}(M)$ is open.        | (By the identity: $A = [A^c]^c$ )   |

$\Leftarrow$ :

Let  $T : X \rightarrow Y$  be a map such that the inverse image of any closed set in  $Y$  is a closed set in  $X$ .

Let  $A$  be an open set in  $Y$ .

- $\Rightarrow A^c$  is closed (Because the complement of an open set is a closed set by definition 1.3-2)
- $\Rightarrow T(A^c)$  is closed (By our assumption the inverse image of any closed set is a closed set)
- $\Rightarrow [T(A^c)]^c$  is open (Because the complement of a closed set is an open set by definition 1.3-2)
- $\Rightarrow T([A^c]^c)$  is open (By the identity:  $f^{-1}(A^c) = [f^{-1}(A)]^c$ )
- $\Rightarrow T(A)$  is open (By the identity:  $A = [A^c]^c$ )
- $\Rightarrow T$  is continuous.

□

### 1.4.2

*Proof.*

$(x_{n_k})$  converges to  $x$   
 $\implies \forall \epsilon > 0, \exists K(\epsilon) \ni \forall k > K, d(x_{n_k}, x) < \epsilon$   
 $\implies \forall \epsilon > 0, \exists N(\epsilon) = K(\epsilon) \ni \forall n > N, d(x_n, x) < \epsilon$  (Because  $(x_n)$  is Cauchy)  
 $\implies (x_n)$  converges to  $x$ .

□

### 1.4.4

*Proof.*

Take  $\epsilon$  with any concrete value, say  $\epsilon = 1$ .

Because  $(x_n)$  is Cauchy, then:

$$\exists N \ni \forall n, m > N, d(x_n, x_m) < \epsilon = 1.$$

Define  $a = \max_{i,j \in \{1, \dots, N\}} d(x_i, x_j)$ .

Therefore we have  $\forall n$ :

$$d(x_n, x_N) \leq \max(1, a) \leq 1 + a$$

By the triangle inequality we have  $\forall n, m$ :

$$\begin{aligned}
 d(x_n, x_m) &\leq d(x_n, x_N) + d(x_N, x_m) \\
 &\leq (1 + a) + (1 + a) = 2(1 + a) = u.
 \end{aligned}$$

This shows that  $u$  is an upperbound for the Cauchy sequence.

□

### 1.4.5

*Proof.* (By Counterexample)

i. Boundedness does **not** imply Cauchiness:

Consider the sequence  $x_n = (-1)^n$ .

$(x_n)$  is bounded by 2, but the sequence is not Cauchy, because for any  $0 < \epsilon < 2$ :

$$\nexists N \ni \forall n, m > N, d(x_n, x_m) < \epsilon.$$

ii. Boundedness does **not** imply Convergence:

Consider the sequence  $x_n = \sin(n)$ .

$(x_n)$  is bounded by 2, but the sequence oscillates and does not converge.

□

### 1.4.6

*Proof.*

Take any  $\epsilon > 0$ , then since  $(x_n)$  and  $(y_n)$  are Cauchy:

$$\exists N_x \ni \forall n, m > N, d(x_n, x_m) < \epsilon$$

and,

$$\exists N_y \ni \forall n, m > N, d(y_n, y_m) < \epsilon.$$

Let  $N = \max\{N_x, N_y\}$ , then  $\forall n > N$ :

$$d(x_n, x_N) < \epsilon \wedge d(y_n, y_N) < \epsilon.$$

By the triangle inequality we have  $\forall n > N$ :

$$\begin{aligned} a_n = d(x_n, y_n) &\leq d(x_n, x_N) + d(x_N, y_N) + d(y_N, y_n) \\ &\leq \epsilon + c + \epsilon \\ &= 2\epsilon + c. \end{aligned}$$

Finally,  $\forall n, m > N$ :

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, a_N) + d(a_N, a_m) \\ &= |d(x_n, y_n) - d(x_N, y_N)| + |d(x_m, y_m) - d(x_N, y_N)| \\ &\leq |2\epsilon + c - c| + |2\epsilon + c - c| \\ &= |2\epsilon| + |2\epsilon| \\ &\leq 4\epsilon. \end{aligned}$$

Therefore  $(a_n)$  is a Cauchy sequence in  $\mathbb{R}$ , and since  $\mathbb{R}$  is complete, then  $(a_n)$  is convergent.

□

## 1.5.2

*Proof.*

Assume  $(x_i)$  is a Cauchy sequence in  $X$

$$\implies \forall \epsilon > 0, \exists N \ni \forall j, k \geq N : d(x_j, x_k) = \max_{i=1}^n |x_i^{(j)} - x_i^{(k)}| < \epsilon \quad (1)$$

$$\implies \forall \epsilon > 0, \exists N \ni \forall j, k \geq N, \forall i \in \{1, \dots, n\} : |x_i^{(j)} - x_i^{(k)}| < \epsilon$$

$$\implies \forall i \in \{1, \dots, n\} : (x_i^{(1)}, x_i^{(2)}, \dots) \text{ is a Cauchy sequence of real numbers}$$

$$\implies \forall i \in \{1, \dots, n\} : (x_i^{(1)}, x_i^{(2)}, \dots) \text{ converges to a limit point } x_i \text{ because } \mathbb{R} \text{ is complete.}$$

Next, we define a candidate limit for  $(x_i)$ :

$$x = (x_1, \dots, x_n).$$

Clearly,  $x \in X$ , and by (1) we have:

$$\forall j \geq N : d(x_j, x) = \epsilon.$$

This shows that  $x$  is the limit of  $(x_i)$  and proves completeness of  $X$ . □

## 1.5.5

*Proof.*

Assume  $(x_n)$  is a Cauchy sequence in  $X$

$$\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N : d(x_n, x_m) = |x_n - x_m| < \epsilon.$$

Take  $\epsilon = 0.5$

$$\implies \exists N \ni \forall n, m \geq N : |x_n - x_m| < 0.5$$

$$\implies \forall n, m \geq N : x_n = x_m$$

$$\implies (x_n) \longrightarrow x_N.$$
□

## 1.5.6

*Proof.* (By Counterexample)

Take the sequence  $x_n = n$ .

First we show that it is Cauchy.

Given any  $\epsilon > 0$ , we can take  $N(\epsilon) = \tan(\frac{\pi}{2} - \epsilon)$  so that  $\forall n, m \geq N$ :

$$\arctan(x_m), \arctan(x_n) \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$$

$$\begin{aligned}
\implies d(x_n, x_m) &= |\arctan(x_n) - \arctan(x_m)| \\
&\leq \left(\frac{\pi}{2} - \epsilon\right) - \frac{\pi}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore  $(x_n)$  is Cauchy as desired.

Now we show that  $(x_n)$  does not converge.

We observe that the sequence wants to converge to  $\frac{\pi}{2}$ , since:

$$\lim_{n \rightarrow \infty} d(x_n, \frac{\pi}{2}) = 0$$

But there is no element  $x \in \mathbb{R}$ , such that  $\arctan(x) = \frac{\pi}{2}$ .

Thus our metric space is **incomplete**.

□

## 1.5.8

*Proof.*

$$\begin{aligned}
&\text{Let } (x_n) \text{ be a Cauchy sequence in } Y \subseteq [a, b], \text{ and let } J = [a, b] \\
\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N : d(x_n, x_m) &= \max_{t \in J} |x_n(t) - x_m(t)| < \epsilon \quad (1) \\
\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N, \forall t \in J : |x_n(t) - x_m(t)| < \epsilon \\
\implies \forall t_0 \in J : (x_1(t_0), x_2(t_0), \dots) &\text{ is a Cauchy sequence of real numbers} \\
\implies \forall t_0 \in J : (x_1(t_0), x_2(t_0), \dots) &\text{ converges to a limit point, say } x_{lim}(t_0), \text{ because } \mathbb{R} \text{ is complete}
\end{aligned}$$

Now, we define a candidate limit  $x$  for  $(x_n)$ .

Define  $x$  pointwise so that  $\forall t_0 \in J : x(t_0) = x_{lim}(t_0)$ .

From the definition above, we obviously have:  $x(a) = x(b)$ .

From (1) with  $m \rightarrow \infty$  we have  $\forall n > N$ :

$$\begin{aligned}
d(x_n, x) &= \max_{t \in J} |x_n(t) - x(t)| < \epsilon \\
\implies \forall t_0 \in J : |x_n(t_0) - x(t_0)| &< \epsilon
\end{aligned}$$

This shows that  $x_n$  converges to  $x$  **uniformly** on  $J$ .

Since the  $x_n$ 's are continuous on  $J$  and the convergence is uniform, the limit function  $x$  is also continuous on  $J$ .

Because  $x \in C[a, b]$  and  $x(a) = x(b)$ , then  $x \in Y$ , and thus  $Y$  is complete.  $\square$

### 2.1.5

*Proof.*

$$\begin{aligned}
& \sum_{i=1}^n \alpha_i x_i = 0_v \\
\implies & \sum_{i=1}^n \alpha_i t^i = 0_v \\
\implies & \forall i, \alpha_i = 0 \\
\implies & (x_1, \dots, x_n) \text{ is linearly independent.}
\end{aligned}$$

□

### 2.1.6

*Proof.* (By Contradiction)

Let  $x \in X$ .

Suppose  $x$  has 2 different representations,  $x_1$  and  $x_2$ , in the basis  $(e_1, \dots, e_n)$ , where:

$$x_1 = \sum_{j=1}^n \alpha_j e_j \quad \text{and} \quad x_2 = \sum_{k=1}^n \beta_k e_k$$

Then we have:

$$\begin{aligned}
& x_1 = x_2 \\
\implies & \sum_{j=1}^n \alpha_j e_j = \sum_{k=1}^n \beta_k e_k \\
\implies & \forall i, \alpha_i = \beta_i \quad (\text{Because } (e_1, \dots, e_n) \text{ is linearly independent.})
\end{aligned}$$

This is a contradiction to our assumption that  $x_1$  and  $x_2$  have different representations.

Thus, we conclude that every non-zero vector must have a unique representation in a given basis.

□

### 2.1.10

*Proof.*

i.  $V = Y \cap Z$  is a subspace:

We must check 3 conditions:



$$1. 0_v \in V$$

$Y$  and  $Z$  are vector spaces

$$\implies 0_v \in Y \wedge 0_v \in Z$$

$$\implies 0_v \in V = Y \cap Z.$$

$$2. v_1, v_2 \in V \implies v_1 + v_2 \in V$$

$$v_1, v_2 \in V$$

$$\implies v_1, v_2 \in Y \wedge v_1, v_2 \in Z$$

$$\implies v_1 + v_2 \in Y \wedge v_1 + v_2 \in Z$$

$$\implies v_1 + v_2 \in V = Y \cup Z.$$

$$3. k \in K \text{ and } v \in V \implies kv \in V$$

$$v \in V$$

$$\implies v \in Y \wedge v \in Z$$

$$\implies kv \in Y \wedge kv \in Z$$

$$\implies kv \in V = Y \cup Z.$$

ii.  $V = Y \cup Z$  is not a subspace:

Consider the following counterexample:

Let  $V = \mathbb{R}^2$ ,  $Y = \{(x, 0) : x \in \mathbb{R}\}$ ,  $Z = \{(0, y) : y \in \mathbb{R}\}$ , and  $V = Y \cup Z$ .

Clearly,  $Y$  and  $Z$  are subspaces of  $\mathbb{R}^2$ .

Take  $v_1 = (1, 0)$  and  $v_2 = (0, 1)$ .

We have  $v_1, v_2 \in V$ , but  $v_1 + v_2 = (1, 1) \notin V$ .

This shows that the union of subspaces fails to be closed under vector addition.

□

## 2.1.11

*Proof.*

We must check 3 conditions:

$$1. 0_v \in M$$

$$\text{Let } v \in M$$

$$\implies 0v = 0_v \in M.$$

$$2. v_1, v_2 \in M \implies v_1 + v_2 \in M$$

$$\begin{aligned} & v_1, v_2 \in M \\ \implies v_1 &= \sum_{\forall m_i \in M} \alpha_i m_i \quad \text{and} \quad v_2 = \sum_{\forall m_j \in M} \beta_j m_j \\ \implies v_1 + v_2 &= \sum_{\forall m_k \in M} (\alpha_k + \beta_k) m_k = \sum_{\forall m_k \in M} \gamma_k m_k \\ \implies v_1 + v_2 &\in M. \end{aligned}$$

$$3. k \in K \text{ and } v \in M \implies kv \in M$$

$$\begin{aligned} & v \in V \\ \implies v &= \sum_{\forall m_i \in M} \alpha_i m_i \\ \implies kv &= \sum_{\forall m_i \in M} (k\alpha_i) m_i \\ \implies kv &= \sum_{\forall m_i \in M} \gamma_i m_i \\ \implies kv &\in M. \end{aligned}$$

□

## 2.2.6

*Proof.*

Let  $x = (\zeta_1, \zeta_2)$ , and  $y = (\eta_1, \eta_2)$ .

i.  $\|x\|_1 = |\zeta_1| + |\zeta_2|$

N1 holds because:

$$\|x\|_1 = |\zeta_1| + |\zeta_2| \geq 0.$$

N2 holds because:

$\implies$  :

$$\begin{aligned} \|x\|_1 &= 0 \\ \implies |\zeta_1| + |\zeta_2| &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies x &= (0, 0) = 0_v. \end{aligned}$$

$\Leftarrow$  :

$$\begin{aligned} x &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies |\zeta_1| + |\zeta_2| &= 0 \\ \implies \|x\|_1 &= 0 \end{aligned}$$

N3 holds because:

$$\begin{aligned} \|\alpha x\|_1 &= |\alpha \zeta_1| + |\alpha \zeta_2| \\ &= |\alpha| |\zeta_1| + |\alpha| |\zeta_2| \\ &= |\alpha| (|\zeta_1| + |\zeta_2|) \\ &= |\alpha| \|x\|_1 \end{aligned}$$

N4 holds because:

$$\begin{aligned} \|x + y\|_1 &= |\zeta_1 + \eta_1| + |\zeta_2 + \eta_2| \\ &\leq |\zeta_1| + |\eta_1| + |\zeta_2| + |\eta_2| \\ &= (|\zeta_1| + |\zeta_2|) + (|\eta_1| + |\eta_2|) \\ &= \|x\|_1 + \|y\|_1. \end{aligned}$$

ii.  $\|x\|_2 = (\zeta_1^2 + \zeta_2^2)^{\frac{1}{2}}$

N1 holds because:

$$\|x\|_2 = (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \geq 0.$$

N2 holds because:

$\implies :$

$$\begin{aligned} \|x\|_2 &= 0 \\ \implies (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} &= 0 \\ \implies |\zeta_1|^2 + |\zeta_2|^2 &= 0 \\ \implies |\zeta_1|^2, |\zeta_2|^2 &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies x &= (0, 0) = 0_v. \end{aligned}$$

$\Leftarrow :$

$$\begin{aligned} x &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies |\zeta_1|^2, |\zeta_2|^2 &= 0 \\ \implies |\zeta_1|^2 + |\zeta_2|^2 &= 0 \\ \implies (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} &= 0 \\ \implies \|x\|_2 &= 0 \end{aligned}$$

N3 holds because:

$$\begin{aligned} \|\alpha x\|_2 &= (|\alpha \zeta_1|^2 + |\alpha \zeta_2|^2)^{\frac{1}{2}} \\ &= (|\alpha|^2 |\zeta_1|^2 + |\alpha|^2 |\zeta_2|^2)^{\frac{1}{2}} \\ &= |\alpha| (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \\ &= |\alpha| \|x\|_2 \end{aligned}$$

N4 holds because:

$$\begin{aligned}
\|x + y\|_2^2 &= (\zeta_1 + \eta_1)^2 + (\zeta_2 + \eta_2)^2 \\
&= \zeta_1^2 + 2\zeta_1\eta_1 + \eta_1^2 + \zeta_2^2 + 2\zeta_2\eta_2 + \eta_2^2 \\
&= (\zeta_1^2 + \zeta_2^2) + (\eta_1^2 + \eta_2^2) + 2(\zeta_1\eta_1 + \zeta_2\eta_2) \\
&\leq (\zeta_1^2 + \zeta_2^2) + (\eta_1^2 + \eta_2^2) + 2\sqrt{\zeta_1^2 + \zeta_2^2}\sqrt{\eta_1^2 + \eta_2^2} \quad (\text{By the Cauchy-Schwartz inequality}) \\
&= \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 \\
&= (\|x\|_2 + \|y\|_2)^2
\end{aligned}$$

$$\implies \|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

$$\text{iii. } \|x\|_\infty = \max\{|\zeta_1|, |\zeta_2|\}$$

N1 holds because:

$$\|x\|_\infty = \max\{|\zeta_1|, |\zeta_2|\} \geq 0.$$

N2 holds because:

$\implies$  :

$$\begin{aligned}
&\|x\|_\infty = 0 \\
&\implies \max\{|\zeta_1|, |\zeta_2|\} = 0 \\
&\implies |\zeta_1|, |\zeta_2| = 0 \\
&\implies \zeta_1, \zeta_2 = 0 \\
&\implies x = (\zeta_1, \zeta_2) = (0, 0) = 0_v.
\end{aligned}$$

$\Leftarrow$  :

$$\begin{aligned}
&x = 0 \\
&\implies \zeta_1, \zeta_2 = 0 \\
&\implies |\zeta_1|, |\zeta_2| = 0 \\
&\implies \max\{|\zeta_1|, |\zeta_2|\} = 0 \\
&\implies \|x\|_\infty = 0
\end{aligned}$$

N3 holds because:

$$\begin{aligned}
\|\alpha x\|_\infty &= \max\{|\alpha\zeta_1|, |\alpha\zeta_2|\} \\
&= |\alpha| \max\{|\zeta_1|, |\zeta_2|\} \\
&= |\alpha| \|x\|_\infty
\end{aligned}$$

N4 holds because:

$$\begin{aligned} \|x + y\|_\infty &= \max\{|\zeta_1 + \eta_2|, |\zeta_2 + \eta_2|\} \\ &\leq \max\{|\zeta_1| + |\eta_2|, |\zeta_2| + |\eta_2|\} \\ &\leq \max\{|\zeta_1|, |\zeta_2|\} + \max\{|\eta_1|, |\eta_2|\} \\ &= \|x\|_\infty + \|y\|_\infty. \end{aligned}$$

□

## 2.2.8

## 2.6.2

*Proof.*

Let  $v_1, v_2 \in R^2$ , where  $v_1 = (\zeta_1, \zeta_2)$ ,  $v_2 = (\eta_1, \eta_2)$ , and  $\alpha \in R$ .

i.  $T_1 : (\zeta_1, \zeta_2) \mapsto (\zeta_1, 0)$

Additivity:

$$\begin{aligned} T_1(v_1 + v_2) &= T_1(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_1 + \eta_1, 0) \\ &= (\zeta_1, 0) + (\eta_1, 0) \\ &= T_1v_1 + T_1v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_1(\alpha v_1) &= T_1(\alpha \zeta_1, \alpha \zeta_2) \\ &= (\alpha \zeta_1, 0) \\ &= \alpha(\zeta_1, 0) \\ &= \alpha T_1v_1. \end{aligned}$$

Geometric interpretation: Projection onto the x-axis.

ii.  $T_2 : (\zeta_1, \zeta_2) \mapsto (0, \zeta_2)$

Proof is similar to part (i).

Geometric interpretation: Projection onto the y-axis.

iii.  $T_3 : (\zeta_1, \zeta_2) \mapsto (\zeta_2, \zeta_1)$

Additivity:

$$\begin{aligned} T_3(v_1 + v_2) &= T_3(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_2 + \eta_2, \zeta_1 + \eta_1) \\ &= (\zeta_2, \zeta_1) + (\eta_2, \eta_1) \\ &= T_3v_1 + T_3v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_3(\alpha v_1) &= T_3(\alpha \zeta_1, \alpha \zeta_2) \\ &= (\alpha \zeta_2, \alpha \zeta_1) \\ &= \alpha(\zeta_2, \zeta_1) \\ &= \alpha T_3v_1 \end{aligned}$$

Geometric interpretation: Reflection across the line  $y = x$ .

$$\text{iv. } T_4 : (\zeta_1, \zeta_2) \mapsto (\gamma\zeta_1, \gamma\zeta_2)$$

Additivity:

$$\begin{aligned} T_4(v_1 + v_2) &= T_4(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\gamma(\zeta_1 + \eta_1), \gamma(\zeta_2 + \eta_2)) \\ &= (\gamma\zeta_1, \gamma\zeta_2) + (\gamma\eta_1, \gamma\eta_2) \\ &= T_4v_1 + T_4v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_4(\alpha v_1) &= T_4(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\gamma\alpha\zeta_1, \gamma\alpha\zeta_2) \\ &= \alpha(\gamma\zeta_1, \gamma\zeta_2) \\ &= \alpha T_4v_1 \end{aligned}$$

□

## 2.6.6

*Proof.*

Let  $X, Y, Z$  be vector spaces over the same field  $K$ .

Let  $T : X \longrightarrow Y$ ,  $S : Y \longrightarrow Z$ .

Then, the composite operator  $ST$  exists and  $ST : X \longrightarrow Z$ .

Let  $x_1, x_2 \in X$ , and  $\alpha \in K$ .

Additivity:

$$\begin{aligned} (ST)(x_1 + x_2) &= S(T(x_1 + x_2)) \\ &= S(Tx_1 + Tx_2) \\ &= (ST)x_1 + (ST)x_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} (ST)(\alpha x_1) &= S(T(\alpha x_1)) \\ &= S(\alpha Tx_1) \\ &= \alpha(ST)x_1. \end{aligned}$$

□



### 2.7.2

*Proof.*

$\implies :$

Let  $T : X \longrightarrow Y$  be a bounded linear operator, and  $B_x$  be a bounded set in  $X$ .

Since  $B_x$  is bounded, then  $\forall x \in B_x :$

$$\|x\| \leq c_x.$$

Let  $B_y$  be the image of  $B_x$  under  $T$ :  $B_y = T(B_x)$ .

Since  $T$  is bounded, then  $\forall y \in B_y :$

$$\begin{aligned} \|y\| &= \|Tx\| \\ &\leq \|T\| \|x\| \\ &\leq \|T\| c_x = c_y. \end{aligned}$$

Thus,  $B_y$  is bounded.

$\Longleftarrow :$

Suppose  $T$  maps bounded sets in  $X$  into bounded sets  $Y$ .

Let  $B_x = \{x \in X : \|x\| = 1\}$ .

$\implies B_x$  is bounded.

Let  $B_y$  be the image of  $B_x$  under  $T$ :  $B_y = T(B_x)$ .

$\implies B_y$  is bounded

$\implies \forall y \in B_y : \|y\| \leq c$

$\implies \forall x \in B_x : \|Tx\| \leq c$

$\implies T$  is bounded by *Lemma 2.7 – 2*.

□

### 2.7.3

*Proof.*

Let  $x \in X$  s.t.  $\|x\| = \alpha < 1$ , then:

$$\begin{aligned}y &= \frac{1}{\alpha}x \\ \Rightarrow \|y\| &= \frac{1}{\alpha}\|x\| = 1 \\ \Rightarrow \|Ty\| &\leq \|T\| \\ \Rightarrow \|T(\frac{1}{\alpha}x)\| &\leq \|T\| \\ \Rightarrow \frac{1}{\alpha}\|Tx\| &\leq \|T\| \\ \Rightarrow \|Tx\| &\leq \alpha\|T\| < \|T\|.\end{aligned}$$

□

### 3.1.4

*Proof.*

$$\begin{aligned}
\langle x + y, x - y \rangle &= \langle x, x - y \rangle + \langle y, x - y \rangle \\
&= \langle x, x \rangle - \langle x, y \rangle + \langle y, x \rangle - \langle y, y \rangle \\
&= 0 - \langle x, y \rangle + \langle x, y \rangle - 0 = 0 \\
&= 0.
\end{aligned}$$

Geometric interpretation for  $X = \mathbb{R}^2$ :

Since  $\|x\| = \|y\|$ , then the vectors  $x$  and  $y$  form a rhombus. The vectors  $x + y$  and  $x - y$  are the diagonals of that rhombus.

Therefore, this statement simply states that the diagonals of a rhombus are perpendicular/orthogonal.

Geometric interpretation for  $X = \mathbb{C}^1$ :

Same as  $X = \mathbb{R}^2$ , since  $\mathbb{R}^2$  and  $\mathbb{C}^1$  have the same geometry. □

### 3.1.7

*Proof.* (By Contradiction)

Let  $u, v \in V$  s.t.  $u \neq 0_v$  and  $v := au$ , where  $a \neq 1$ . Then  $u \neq v$ , and:

$$\begin{aligned}
&\forall x \in X : \langle x, u \rangle = \langle x, v \rangle \\
&\implies \forall x \in X : \langle x, u \rangle = \langle x, au \rangle \\
&\implies \forall x \in X : \langle x, u \rangle = a \langle x, u \rangle
\end{aligned}$$

But this is impossible since  $a \neq 1$ . Thus, we must have  $u = v$ . □

### 3.1.11

*Proof.*

To check if the given norm is induced by some inner-product, we could check whether the parallelogram equality holds, but there's an even simpler proof.

First, we note that this space is an  $l^p$  space with  $p = 1$ .

Finally, we note that it was shown in 3.1 – 7 that the space  $l^p$  with  $p \neq 2$  is not an inner-product space.

Therefore, there is no inner-product that induces the given norm. □

### 3.2.1

The Euclidean inner product on  $\mathbb{R}^2$  and  $\mathbb{R}^3$  reduces to the dot product.

The dot product of 2 vectors is the product of their length times cosine the angle between them:

$$\begin{aligned}\langle x, y \rangle &= x \cdot y \\ &= \|x\| \|y\| \cos(\theta)\end{aligned}$$

Taking the absolute value of both sides we get:

$$\begin{aligned}|\langle x, y \rangle| &= \|x\| \|y\| |\cos(\theta)| \\ &\leq \|x\| \|y\|\end{aligned}$$

Which is the same result given by the Schwarz inequality.

### 3.2.2

- i. The trivial subspace containing just the zero sequence.
- ii. Subspaces whose components are all 0 after some index  $n$ .

### 3.2.4

*Proof.*

By the continuity of the inner product (Lemma 3.2-2), we have:

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle \tag{1}$$

Next, we define the sequence  $(a_n)$  by:  $a_n := \langle x_n, y \rangle$ . Then:

$$\begin{aligned}\forall n \in \mathbb{N} : a_n &= 0 \\ \implies a_n &\rightarrow 0 \\ \implies \langle x_n, y \rangle &\rightarrow 0\end{aligned} \tag{2}$$

Finally:

$$\begin{aligned}(1) \text{ and } (2) &\implies \langle x, y \rangle \rightarrow 0 \\ &\implies x \perp y.\end{aligned}$$

□

### 3.2.5

*Proof.*

To prove that  $x_n \rightarrow x$ , we will show that  $\|x_n - x\| \rightarrow 0$ :

$$\begin{aligned}\|x_n - x\| &= \langle x_n - x, x_n - x \rangle \\ &= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle \\ &= \|x_n\|^2 - \langle x_n, x \rangle - \langle x, x_n \rangle + \|x\|^2\end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ :

$$\|x_n - x\| \rightarrow 2\|x\|^2 - 2\langle x, x \rangle = 2\|x\|^2 - 2\|x\|^2 = 0.$$

□

### 3.2.8

*Proof.*

$\implies$ :

$$\begin{aligned}\langle x, y \rangle &= 0 \\ \implies \forall \bar{\alpha} : \bar{\alpha} \langle x, y \rangle &= 0 \\ \implies \forall \alpha : \langle x, \alpha y \rangle &= 0 \\ \implies \forall \alpha : \langle x, \alpha y \rangle = 0 \wedge \langle \alpha y, x \rangle &= 0\end{aligned}$$

Next, we have  $\forall \alpha$ :

$$\begin{aligned}\|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y \rangle \\ &= \langle x, x \rangle + \langle x, \alpha y \rangle + \langle \alpha y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 0 + 0 + \|y\|^2 \\ &\geq \|x\|^2\end{aligned}$$

Finally, taking the square root, we arrive at our desired result:

$$\|x + \alpha y\| \geq \|x\|.$$

$\Leftarrow$ :

$$\begin{aligned}
& \forall \alpha : \|x\| \leq \|x + \alpha y\| \\
\implies \forall \alpha : \|x\|^2 & \leq \|x + \alpha y\|^2 \\
& = \langle x + \alpha y, x + \alpha y \rangle \\
& = \langle x, x \rangle + \langle x, \alpha y \rangle + \langle \alpha y, x \rangle + \langle y, y \rangle \\
& = \|x\|^2 + \overline{\langle \alpha y, x \rangle} + \langle \alpha y, x \rangle + \|y\|^2 \\
& = \|x\|^2 + 2\operatorname{Re}\langle \alpha y, x \rangle + \|y\|^2 \\
& = \|x\|^2 + 2\operatorname{Re}(\alpha \langle y, x \rangle) + \|y\|^2 \\
\implies \forall \alpha : 2\operatorname{Re}\langle \alpha y, x \rangle & \leq 0 \\
\implies \langle x, y \rangle & = 0.
\end{aligned}$$

□

### 5.1.2

*Proof.*

Let  $x, y \in X$

$$\begin{aligned}
 d(Tx, Ty) &= |Tx - Ty| \\
 &= \left| \frac{x}{2} + x^{-1} - \frac{y}{2} - y^{-1} \right| \\
 &= \left| \frac{x-y}{2} + \frac{y-x}{xy} \right| \\
 &= |x-y| \left| \frac{1}{2} - \frac{1}{xy} \right| \\
 &\leq \frac{1}{2} |x-y| \\
 &= \frac{1}{2} d(x, y)
 \end{aligned}$$

Therefore,  $T$  is a contraction and the smallest  $\alpha$  is  $\frac{1}{2}$ .

□

### 5.1.5

*Proof.* (By Contradiction)

Suppose  $T$  has 2 different fixed points  $x$  and  $y$ , then:

$$\begin{aligned}
 x &= Tx \wedge y = Ty \\
 \implies d(x, y) &= d(Tx, Ty)
 \end{aligned} \tag{1}$$

But we are given that:

$$d(Tx, Ty) < d(x, y) \tag{2}$$

(1) and (2)  $\implies d(x, y) < d(x, y)$ .

This is a contradiction. Thus,  $T$  has a unique fixed point.

□

### 5.1.6

$i. T$  is a contraction  $\implies T^n$  is a contraction:

*Proof.* (By Induction)

Base case ( $n = 1$ ):

$T^1 = T$  which is a contraction by assumption.

Inductive step ( $n = k + 1$ ):

Suppose the induction hypothesis holds for  $n = k$ , and let  $x, y \in X$ , then:

$$d(T^k x, T^k y) \leq \beta d(x, y) \quad (\text{where } 0 < \beta < 1) \quad (1)$$

Next, let  $T^k x = u$  and  $T^k y = v$ . Because  $T$  is a contraction then:

$$d(Tu, Tv) \leq \alpha d(u, v) \quad (\text{where } 0 < \alpha < 1) \quad (2)$$

Putting (1) and (2) together, we get:

$$\begin{aligned} d(T^{k+1}x, T^{k+1}y) &= d(Tu, Tv) \\ &\leq \alpha d(u, v) \\ &\leq \alpha \beta d(x, y) \\ &= \gamma d(x, y) \quad (\text{where } 0 < \gamma < 1) \end{aligned}$$

Therefore,  $T^{k+1}$  is a contraction.

□

ii.  $T^n$  is a contraction  $\not\Rightarrow T$  is a contraction:

*Proof.* (By Counterexample)

Let the mapping  $T^n : R \rightarrow R$  be defined by:

$$Tx := \begin{cases} -x, & \text{if } x \geq 0 \\ \frac{x}{2}, & \text{if } x < 0 \end{cases}$$

Then,  $T$  does not contract the distance between positive numbers.  
However,  $T^2$  is a contraction.

□