MTH-632 PDEs
Assignment (5):
Chapter 8: Separation of Variables
(Non-Homogeneous)

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i.

2.

Step 1: Transform the problem to eliminate BC inhomogeneities

BCs are already homogeneous.

Step 2: Compute the eigenfunctions of the homogeneous PDE

From the previous chapter, we know that PDE/eigenvalue problem:

$$\begin{aligned}&\phi''(x) = -\lambda\phi(x)\\&\phi(0) = 0\\&\phi(\pi) = 0\end{aligned}$$

has solutions given by the following eigenfunctions:

$$\phi_n(x) = \cos(\sqrt{\lambda}x) = \cos(nx), \quad \lambda^2 = n^2, \quad n \in \{0, 1, \ldots\}.$$

Step 3: Expand the source function in the eigenfunction basis to obtain $s_n(t)$

$$\begin{aligned}s(x,t) &= \sum_{n=0}^{\infty} s_n(t)\phi_n(x)\\ \implies e^{-t} &= \sum_{n=0}^{\infty} s_n(t)\cos(nx)\\ \implies s_n(t) &= \frac{\int_{x=0}^{\pi} e^{-t}\cos(nx)dx}{\int_{x=0}^{\pi}\cos^2(nx)dx}\\ \implies s_n(t) &= \begin{cases} e^{-t}, & n = 0\\ 0, & n \neq 0\end{cases}\end{aligned}$$

Step 4: Expand the solution in the eigenfunction basis

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\phi_n(x).$$

Step 5: Compute u_t and u_{xx}

$$\begin{aligned}u_t &= \sum_{n=0}^{\infty} \dot{u}_n(t)\phi_n(x)\\ u_{xx} &= \sum_{n=0}^{\infty} u_n(t)\phi_n''(x) = -n^2\sum_{n=0}^{\infty} u_n(t)\phi_n(x).\end{aligned}$$

Step 6: Expand the initial condition in the eigenfunction basis to obtain $u_n(0)$

$$
\begin{aligned}
u(x,0) &= \sum_{n=0}^{\infty} u_n(0)\phi_n(x) \\
\implies \cos(2x) &= \sum_{n=0}^{\infty} u_n(0)\phi_n(x) \\
\implies u_n(0) &= \frac{\int_{x=0}^{\pi} \cos(2x)\cos(nx)dx}{\int_{x=0}^{\pi} \cos^2(nx)dx} \\
\implies u_n(0) &= \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}
\end{aligned}
$$

Step 7: Substitute back into the PDE

$$
\begin{aligned}
& u_t = u_{xx} + e^{-t} \\
\implies & \sum_{n=0}^{\infty} \dot{u}_n(t)\phi_n(x) = -\lambda^2 \sum_{n=0}^{\infty} u_n(t)\phi_n(x) + \sum_{n=0}^{\infty} s_n(t)\phi_n(x) \\
\implies & \dot{u}_n(t)\phi_n(x) = -n^2 u_n(t)\phi_n(x) + s_n(t)\phi_n(x) \\
\implies & \begin{cases} \dot{u}_0(t) = s_0(t) = e^{-t} \\ \dot{u}_n(t) = -n^2 u_n(t), \quad n \neq 0 \end{cases}
\end{aligned}
$$

Step 8: Solve the resulting ODEs

$$
u_n(x,t) = \begin{cases} A - e^{-t}, & n = 0 \\ C_n e^{-n^2 t}, & n \neq 0 \end{cases}
$$

Comparing with the initial conditions we determine the constants as:

$$
\begin{aligned}
A &= 1 \\
C_n &= \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}
\end{aligned}
$$

Step 8: Form the complete solution

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(t)\phi_n(x) \\
&= u_0(t)\phi_0(x) + u_2(t)\phi_2(x) \\
&= 1 - e^{-t} + e^{-4t}\cos(2x)
\end{aligned}
$$

ii.

2.

Step 1: Transform the problem to eliminate BC inhomogeneities

BCs are already homogeneous.

Step 2: Compute the eigenfunctions of the homogeneous PDE

From the previous chapter, we know that PDE/eigenvalue problem:

$$\begin{aligned} &\phi''(x) = -\lambda\phi(x) \\ &\phi(0) = 0 \\ &\phi(\pi) = 0 \end{aligned}$$

has solutions given by the following eigenfunctions:

$$\phi_n(x) = \sin(\sqrt{\lambda}x) = \sin(nx), \quad \lambda^2 = n^2, \quad n \in \{1, \ldots\}.$$

Step 3: Expand the source function in the eigenfunction basis to obtain $s_n(t)$

$$\begin{aligned} s(x,t) &= \sum_{n=1}^{\infty} s_n(t)\phi_n(x) \\ \implies cos(\omega t) &= \sum_{n=1}^{\infty} s_n(t)\sin(nx) \\ \implies s_n(t) &= \frac{\int_{x=0}^{\pi} \cos(\omega t)\sin(nx)dx}{\int_{x=0}^{\pi} \sin^2(nx)dx} \\ \implies s_n(t) &= \begin{cases} \frac{4}{n\pi}\cos(\omega t), & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases} \end{aligned}$$

Step 4: Expand the solution in the eigenfunction basis

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x).$$

Step 5: Compute u_t, u_{tt} and u_{xx}

$$
\begin{aligned}
u_t &= \sum_{n=1}^{\infty} \dot{u}_n(t)\phi_n(x) \\
u_{tt} &= \sum_{n=1}^{\infty} \ddot{u}_n(t)\phi_n(x) \\
u_{xx} &= \sum_{n=1}^{\infty} u_n(t)\phi_n''(x) = -n^2 \sum_{n=1}^{\infty} u_n(t)\phi_n(x).
\end{aligned}
$$

Step 6: Expand the initial condition in the eigenfunction basis to obtain $u_n(0)$ and $\dot{u}_n(0)$

$$
\begin{aligned}
u(x,0) &= \sum_{n=1}^{\infty} u_n(0)\phi_n(x) \\
\implies f(x) &= \sum_{n=1}^{\infty} u_n(0)\phi_n(x) \\
\implies u_n(0) &= \frac{\int_{x=0}^{\pi} f(x)\sin(nx)}{\int_{x=0}^{\pi} \sin^2(nx)} \\
\implies u_n(0) &= \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\sin(nx).
\end{aligned}
$$

$$
\begin{aligned}
\dot{u}(x,0) &= \sum_{n=1}^{\infty} \dot{u}_n(0)\phi_n(x) \\
\implies 0 &= \sum_{n=1}^{\infty} \dot{u}_n(0)\phi_n(x) \\
\implies \dot{u}_n(0) &= \frac{\int_{x=0}^{\pi} 0 * \sin(nx)}{\int_{x=0}^{\pi} 0 * \sin^2(nx)} = 0.
\end{aligned}
$$

Step 7: Substitute back into the PDE

$$
\begin{aligned}
& u_{tt} - c^2 u_{xx} + \beta u_t = \cos(\omega t) \\
\implies & \sum_{n=1}^{\infty} \ddot{u}_n(t)\phi_n(x) + c^2 n^2 \sum_{n=1}^{\infty} u_n(t)\phi_n(x) + \beta \sum_{n=1}^{\infty} \dot{u}_n(t)\phi_n(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\cos(\omega t)\phi_n(x) \\
\implies & \begin{cases} \ddot{u}_n(t) + c^2 n^2 u_n(t) + \beta \dot{u}_n(t) = \frac{4}{n\pi}\cos(\omega t), & n \text{ is odd} \\ \ddot{u}_n(t) + c^2 n^2 u_n(t) + \beta \dot{u}_n(t) = 0, & n \text{ is even.} \end{cases}
\end{aligned}
$$

Step 8: Solve the resulting ODEs

n is odd:

$$u_n(t) = u_n^h + u_n^p$$

Compue the homogeneous solution u_n^h:

We guess a solution of the form $u_n^h = Ae^{\mu_n t}$ and substitute into the ODE:

$$\begin{aligned}
&\mu_n^2 Ae^{\mu_n t} + \beta\mu_n Ae^{\mu_n t} + c^2n^2 Ae^{\mu_n t} = 0\\
\implies &\mu_n^2 + \beta\mu_n + c^2n^2 = 0\\
\implies &\mu_n = \frac{-\beta \pm \sqrt{\beta^2 - 4c^2n^2}}{2}
\end{aligned}$$

At this point, we assume the discriminant is negative, and thus we have 2 complex roots:

$$\mu_n = \frac{-\beta \pm i\sqrt{4c^2n^2 - \beta^2}}{2}$$

$$\begin{aligned}
u_n^h &= Re\left[A_1 e^{\frac{-\beta + i\sqrt{4c^2n^2-\beta^2}}{2}t} + A_2 e^{\frac{-\beta - i\sqrt{4c^2n^2-\beta^2}}{2}t}\right]\\
&= Re\left[e^{\frac{-\beta}{2}t}\left(A_1 e^{\frac{i\sqrt{4c^2n^2-\beta^2}}{2}t} + A_2 e^{\frac{-i\sqrt{4c^2n^2-\beta^2}}{2}t}\right)\right]\\
&= e^{\frac{-\beta}{2}t}\left(A_1\cos\frac{\sqrt{4c^2n^2-\beta^2}}{2}t + A_2\cos\frac{\sqrt{4c^2n^2-\beta^2}}{2}t\right)\\
&= Ae^{\frac{-\beta}{2}t}\cos(\frac{\sqrt{4c^2n^2-\beta^2}}{2}t + \phi)
\end{aligned}$$

Compue the particular solution u_n^p:

We guess a solution of the form $u_n^p = B_1\sin(\omega t) + B_2\cos(\omega t)$ and substitute into the ODE:

$$\begin{aligned}
&-\omega^2[B_1\sin(\omega t) + B_2\cos(\omega t)] + \beta\omega[B_1\cos(\omega t) - B_2\sin(\omega t)]\\
&+ c^2n^2[B_1\sin(\omega t) + B_2\cos(\omega t)] = \frac{4}{n\pi}\cos(\omega t)\\
\implies &\sin(\omega t)[-\omega^2B_1 - \beta\omega B_2 + c^2n^2B_1] + \cos(\omega t)[-\omega^2B_2 + \beta\omega B_1 + c^2n^2B_2] = \frac{4}{n\pi}\cos(\omega t)\\
\implies &-\omega^2B_1 - \beta\omega B_2 + c^2n^2B_1 = 0 \quad\text{and}\quad -\omega^2B_2 + \beta\omega B_1 + c^2n^2B_2 = \frac{4}{n\pi}\\
\implies &B_1[-\omega^2 + c^2n^2] + B_2[-\beta\omega] = 0 \quad\text{and}\quad B_1[\beta\omega] + B_2[-\omega^2 + c^2n^2] = \frac{4}{n\pi}
\end{aligned}$$

This is a $2x2$ system that can be solved for B_1 and B_2.

n is even:
This case has the same solution as u_n^h for the odd n case.

Step 8: Determine ϕ by comparing with the initial conditions

Step 9: Form the complete solution

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n^h(t)\phi_n(x) + \sum_{n=1}^{\infty} u_{2n-1}^p(t)\phi_{2n-1}(x) \\
&= \sum_{n=1}^{\infty} A_n e^{\frac{-\beta}{2}t} \cos\left(\frac{\sqrt{4c^2n^2-\beta^2}}{2}t+\phi\right)\sin(nx) + \sum_{n=1}^{\infty}[B_{1_{2n-1}}\sin(\omega t) + B_{2_{2n-1}}\cos(\omega t)]\sin((2n-1)x).
\end{aligned}
$$