MTH-632 PDEs
Assignment (6):
Chapter 10: Green's Functions

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10.2.1

$$
\begin{aligned}
&u_t = u_{xx} + Q(x,t)0, \quad 0 \le x \le 1, \quad t > 0 \\
&u(x,0) = f(x) \\
&u(0,t) = A(t) \\
&u(1,t) = B(t).
\end{aligned}
$$

Obtain a solution of the form:

$$
u(x,t) = \int_0^1 f(s)G(x;s,t)ds + \int_0^1 \int_0^t Q(s,\tau)G(x;s,t-\tau)d\tau ds.
$$

Solution:

$$
\begin{aligned}
&w(x,t) = A(t) + x[B(t) - A(t)] \\
&v(x,t) = u(x,t) - w(x,t) \\
&y(x,t) = Q(x,t) - wt + w_{xx}.
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow\ &v_t = v_{xx} + y(x,t) \\
&v(x,0) = g(x) = f(x) - A(0) - x[B(0) - A(0)] \\
&v(0,t) = v(1,t) = 0.
\end{aligned}
$$

The eigenfunctions are given by:

$$
\begin{aligned}
&\phi_n(x) = sin(n\pi x) \\
&\lambda_n = (n\pi)^2 \\
&n = 1, 2, \ldots
\end{aligned}
$$

Expand $y(x,t)$ in the eigenfunction basis:

$$
\begin{aligned}
y(x,t) &= \sum_{n=1}^{\infty} y_n(t)\phi_n(x) \\
\Longrightarrow y_n(t) &= \frac{\int_0^1 y(x,y)\sin(n\pi x)dx}{\int_0^1 \sin^2(n\pi x)dx}.
\end{aligned}
$$

Expand $v(x,t)$ in the eigenfunction basis:

$$\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty} v_n(t)\phi(x) \\
\implies v(x,0) &= \sum_{n=1}^{\infty} v_n(0)\phi_n(x) = g(x) \\
\implies v_n(0) &= \frac{\int_0^1 g(x)\sin(n\pi x)dx}{\int_0^1 \sin^2(n\pi x)dx} = 2\int_0^1 g(x)\sin(n\pi x)dx.
\end{aligned}$$

Plugging into the PDE, we get:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[\dot{v}_n(t) + n^2\pi^2 v_n(t) - y_n(t)\right]\sin(n\pi x) = 0 \\
\implies &\forall n : \dot{v}_n(t) + n^2\pi^2 v_n(t) = y_n(t).
\end{aligned}$$

Using the method of variation of parameters, we can solve for $v_n(t)$ as follows:

$$\begin{aligned}
v_n(t) &= v_n(0)e^{-n^2\pi^2 t} + \int_0^t y_n(\tau)e^{-n^2\pi^2(t-\tau)}d\tau \\
&= e^{-n^2\pi^2 t}2\int_0^1 g(x)\sin(n\pi x)dx + \int_0^t y_n(\tau)e^{-n^2\pi^2(t-\tau)}d\tau.
\end{aligned}$$

Now, plugging into $v(x,t)$, we get:

$$\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty} v_n(t)\phi(x) \\
&= \sum_{n=1}^{\infty} \left[e^{-n^2\pi^2 t}2\int_0^1 g(x)\sin(n\pi x)dx + \int_0^t y_n(\tau)e^{-n^2\pi^2(t-\tau)}d\tau\right]\sin(n\pi x)
\end{aligned}$$