MTH-632 PDEs
Assignment (2): Classification and Canonical Forms

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2.2.2

a.

$$\begin{aligned}\Delta &= B^2 - 4AC\\ &= 5^2 - 4(4 * 1)\\ &= 25 - 16\\ &= 9 > 0\\ \Longrightarrow &\text{ The problem is hyperbolic.}\end{aligned}$$

b.

$$\begin{aligned}\Delta &= B^2 - 4AC\\ &= 1^2 - 4(1 * 1)\\ &= 1 - 4\\ &= -3 < 0\\ \Longrightarrow &\text{ The problem is elliptic.}\end{aligned}$$

c.

$$\begin{aligned}\Delta &= B^2 - 4AC\\ &= 10^2 - 4(3 * 3)\\ &= 100 - 36\\ &= 64 > 0\\ \Longrightarrow &\text{ The problem is hyperbolic.}\end{aligned}$$

d.

$$\begin{aligned}\Delta &= B^2 - 4AC\\ &= 2^2 - 4(1 * 3)\\ &= 4 - 12\\ &= -8 < 0\\ \Longrightarrow &\text{ The problem is elliptic.}\end{aligned}$$

e.

$$\begin{aligned}\Delta &= B^2 - 4AC \\ &= (-4)^2 - 4(2 * 2) \\ &= 16 - 16 \\ &= 0 \\ \implies &\text{The problem is parabolic.}\end{aligned}$$

f.

$$\begin{aligned}\Delta &= B^2 - 4AC \\ &= 5^2 - 4(1 * 4) \\ &= 25 - 16 \\ &= 9 > 0 \\ \implies &\text{The problem is hyperbolic.}\end{aligned}$$

2.3.1 & 2.3.2

a.

Find the characteristic equation, characteristic curves and obtain a canonical form for:

$$xu_{xx} + u_{yy} = x^2$$

Solution:

$A = x \qquad B = 0 \qquad C = 1 \qquad D = 0 \qquad E = 0 \qquad F = 0 \qquad G = x^2$

$$\begin{aligned}\Delta &= B^2 - 4AC \\ &= 0 - 4(x * 1) \\ &= -4x.\end{aligned}$$

$$\begin{aligned}\implies x &< 0 \text{ then hyperbolic} \\ x &= 0 \text{ then parabolic} \\ x &> 0 \text{ then elliptic.}\end{aligned}$$

1. $x < 0$ (*hyperbolic*):

Step 1: Characteristic Equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{B \pm \sqrt{B - 4AC}}{2A} \\ &= \frac{0 \pm \sqrt{0 - 4x}}{2x} \\ &= \pm \frac{2\sqrt{-x}}{2x} \\ &= \pm(-x)^{-\frac{1}{2}}.\end{aligned}$$

Step 2: Solve for the Characteristic Curves:

$$\xi = \phi_1(x, y) = y + 2x^{\frac{1}{2}} = C_1$$
$$\eta = \phi_2(x, y) = y - 2x^{\frac{1}{2}} = C_2$$

Step 3: Compute the Canonical Coefficients:

$$\begin{aligned}
\xi_x &= x^{-\frac{1}{2}} & \eta_x &= -x^{-\frac{1}{2}} \\
\xi_{xx} &= -\frac{1}{2}x^{-\frac{3}{2}} & \eta_{xx} &= \frac{1}{2}x^{-\frac{3}{2}} \\
\xi_y &= 1 & \eta_y &= 1 \\
\xi_{yy} &= 0 & \eta_{yy} &= 0.
\end{aligned}$$

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 4 \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = \frac{1}{2}x^{-\frac{1}{2}} \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = -\frac{1}{2}x^{-\frac{1}{2}} \\
F^* &= F = 0 \\
G^* &= G = x^2.
\end{aligned}$$

Step 4: Transform to the Canonical Form:

$$4u_\xi\eta + \frac{1}{2}x^{-\frac{1}{2}}u_\xi - \frac{1}{2}x^{-\frac{1}{2}}u_\eta = x^2.$$

Step 5: Compute x and y in terms of ξ and η:

$$\begin{aligned}
y &= \frac{1}{2}(\xi + \eta) \\
x &= [\frac{1}{4}(\xi - \eta)]^2
\end{aligned}$$

Finally, substitute x and y in terms of ξ and η in the Canonical Form.

Same steps should be repeated for the parabolic and elliptic cases.

b,c,d,e,f

Can be solved in the same way as part (a) solved above.

2.4.1 & 2.4.2

Can be solved in the same way as problem 2.3.1&2.3.2 above.

2.5.2