# Linear Algebra (MTH401) Finals Questions Bank

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# Proofs

**Theorem 1** (2023.S(1.A.i)).
*i. Disprove: $W$ is a subspace of the vector space $V$ and $v \in V$. Then, the set defined by $v + W = \{v + w : w \in W\}$, is a subspace of $V$.*

*ii. Under what condition is $v + W$ a subspace of $V$?*

*Proof.*
i. By counterexample:
Let $V = R^2$, $W = \{w \in V : w = (x, 0)\}$, and $v = (0, 1)$.
$\implies v + W = \{u \in V : u = (x, 1)\}$.
$\implies 0_v = (0, 0) \notin v + W$
$\implies$ v+W is not a subspace.

ii. It's clear that if $v \in W$, then $v + W = W$, which is a subspace of $V$ as desired.
If, however, $v \notin W$, then, by the counterexample provided, $v + W$ is not a subspace of $V$.
Therefore, the sufficient and necessary condition is: $v \in W$.

□

Note: The transformation $v+W = \{v+w : w \in W\}$ is an **affine transformation**, which is not—necessarily—linear.

**Theorem 2** (2023.S(1.A.ii))**.**
*Let* $x_1, \ldots, x_{n+1}$ *be distinct elements of* $F$*. Then, the functions* $f_i(x) = \Pi_{k=1,k\neq i}^{n+1} \frac{(x-x_k)}{(x_i-x_k)}$ *for* $i = 1, \ldots, n+1$ *form a basis for* $P_n(F)$*.*

*Note to self: A possible source of confusion here is that* $F$ *is not an infinite field, rather it's a finite field given by* $F = \{x_1, \ldots, x_n\}$*.*

*Proof.*
Since $dim(P_n) = n+1$ and we have $n+1$ functions/vectors, then it's sufficient to check either one of the following conditions:
i. The $n+1$ functions/vectors span $P_n$
ii. The $n+1$ functions/vectors are linearly independent.

We check (ii). Let $x_j, a_i \in \{x_1, \ldots, x_{n+1}\}$, then:

$$\begin{aligned} 0_v &= \sum_{i=1}^{n+1} a_i f_i(x_j) \\ &= \sum_{i=1}^{n+1} a_i \Pi_{k=1,k\neq i}^{n+1} \frac{x_j - x_k}{x_i - x_k} \\ &= \sum_{i=1}^{n+1} a_i \delta_{ij} \\ &= a_j. \end{aligned}$$

$\implies \forall j \in \{1, \ldots, n+1\} : \ a_j = 0.$
$\implies (f_1, \ldots, f_n)$ are linearly independent.
$\implies (f_1, \ldots, f_n)$ is a basis for $P_n$.

□

Notes:
- The tricky part in this question is to realize that the complicated form of $f_i$ reduces to $\delta_{ij}$ when applying $f_i$ to $x_j$.
- Intuitively, this is because the output of applying $f_i(x)$ to some $x_j \in F$ must be one of the finite scalars in $F$.
- I suppose this result can be generalized to all function spaces (not just polynomials) over a finite field. The generalization can be stated as follows: Any function space over a finite field of cardinality $n$ has a basis given by the functions $f_i$ where $f_i(x_j) := \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$.

**Theorem 3** (2023.S(1.A.iii)).

*If $T \in Hom(V)$, $W$ is a $T$-invariant subspace of $V$, and $V = R(T) \oplus W$, then $W \subseteq N(T)$.*

*Proof.* (Direct Proof)

Let $w \in W$

$\implies Tw \in R(T)$ (by definition of the range) and $Tw \in W$ (because W is $T$-invariant)

$\implies Tw \in R(T) \cap W$.

But since $V = R(T) \oplus W$, then $R(T) \cap W = \{0_v\}$

$\implies Tw = 0_v$

$\implies w \in N(T)$

$\implies W \subseteq N(T)$.

$\square$

## Alternative Proof

*Proof.* (By contradiction)

Suppose that $W \not\subseteq N(T)$

$\implies \exists w \in W$ s.t. $w \notin N(T)$

$\implies Tw \in R(T)$ and $Tw \in W$ (Because $W$ is $T$-invariant)

$\implies R(T) \cap W \neq \{0_v\}$

$\implies V \neq R(T) \oplus W$.

This is a contradiction. Therefore, we must conclude that $W \subseteq N(T)$.

$\square$

**Theorem 4** (2023.S(1.A.iv))**.**

*Disprove: A linear operator on an infinite-dimensional vector space has no eigenvectors.*

*Proof.* (By counterexample)

Consider the vector space $V = \mathbb{R}^\infty$ over $\mathbb{R}$.

Let $T$ be a linear operator on $V$ defined by $T := \lambda I_\infty$, where $\lambda \in \mathbb{R}$.

Then, all vectors in $V$ are eigenvectors with eigenvalue $= \lambda$.

$\square$

**Theorem 5** (2023.S(1.A.v)).

*If $S$ is a* ***subset*** *of an inner product space $V$, then $Span(S)$ is a* ***subspace*** *of $(S^{\perp})^{\perp}$.*

*Proof.*

We prove a stronger result: $Span(S) = (S^{\perp})^{\perp}$.

We start from the fact (given by another theorem) that if $U$ is a subspace, then $U = (U^{\perp})^{\perp}$. So, now we need only show that $Span(S)^{\perp} = S^{\perp}$. This amounts to showing that $(i) Span(S)^{\perp} \subseteq S^{\perp}$ and $(ii) S^{\perp} \subseteq Span(S)^{\perp}$.

(i) Is true since any vector that is orthgonal to all vectors in $Span(S)$ must also be orthogonal to all vectors in $S$.

(ii) Let $s_o \in S^{\perp}$ and $s \in Span(S)$

$\implies s = \sum a_i s_i, \ \forall s_i \in S$

$\implies \langle s, s_o \rangle = \langle \sum a_i s_i, s_o \rangle = \sum \langle a_i s_i, s_o \rangle = \sum a_i \langle s_i, s_o \rangle = \sum a_i * 0 = 0$

$\implies s_o \in Span(S)^{\perp}$

$\implies S^{\perp} \subseteq Span(S)^{\perp}$.

$\square$

**Theorem 6** (2023.S(1.B), 2022.S(1.A.ii)).

*Let $V$ be an $n$-dimensional vector space and $T \in Hom(V, W)$. Prove that:*

*i. $nullity(T) + rank(T) = n$.*

*ii. $T$ is injective iff $T$ carries linearly independent subsets of $V$ onto linearly independent subsets of $W$. In other words:*

*$T$ is injective $\iff$ If $(v_1, \ldots, v_k)$ are linearly independent, then $(Tv_1, \ldots, Tv_k)$ are linearly independent.*

*Proof.* Part i

Let $nullity(T) = m$, $0 \leq m \leq n$, and $B_N = (u_1, \ldots, u_m)$ be a basis for N(T).

Extend $B_N$ to a basis for $V$: $B_V = (u_1, \ldots, u_m, u_{m+1}, \ldots, u_n)$.

Let $v \in V$, then $v = \sum_{i=1}^{n} a_i u_i$.
Apply $T$ to both sides:

$$\begin{aligned} T(v) &= T(\sum_{i=1}^{n} a_i u_i) \\ &= \sum_{i=1}^{n} a_i T(u_i) \\ &= \sum_{i=m+1}^{n} a_i T(u_i) \end{aligned}$$

This shows that $(Tv_{m+1}, \ldots, Tv_n)$ spans $R(T)$.
Next, we show that it is also linearly independent:

$$\begin{aligned} 0_v &= \sum_{i=m+1}^{n} a_i T(u_i) \\ &= T(\sum_{i=m+1}^{n} a_i u_i) \end{aligned}$$

$\implies \sum_{i=m+1}^{n} a_i u_i \in N(T)$
$\implies \sum_{i=1}^{m} a_i u_i = \sum_{i=m+1}^{n} a_i u_i$
$\implies a_i = 0$, for $i = 1, ..., n$. (Because $(u_1, \ldots, u_n)$ is linearly independent)
$\implies (Tu_{m+1}, \ldots, Tu_n)$ is linearly independent and hence is a bsis for range $T$.
□

*Proof.* Part ii

**Forward direction:**

Assume $T$ is injective and $(v_1, \ldots, v_n)$ is linearly independent, then:

$$
\begin{aligned}
0_w &= a_1Tv_1 + \ldots + a_nTv_n \\
&= T(a_1v_1 + \ldots + a_nv_n)
\end{aligned}
$$

$\implies a_1v_1 + \ldots + a_nv_n = 0_v$ (Because T is injective)
$\implies a1, \ldots, a_n = 0$ (Because $(v_1, \ldots, v_n)$ is linearly independent)
$\implies (Tv_1, ..., Tv_n)$ is linearly independent.

**Converse direction:**

$$
\begin{aligned}
&\text{Let } (v_1, \ldots, v_n) \text{ be a basis for } V \\
\implies &(v_1, \ldots, v_n) \text{ is linearly independent} \\
\implies &(Tv_1, \ldots, Tv_n) \text{ is linearly independent} \\
\implies &[a_1Tv_1 + \ldots + a_nTv_n = 0_w \implies a_1, \ldots, a_n = 0] \\
\implies &[T(a_1v_1 + \ldots + a_nv_n) = 0_w \implies a_1v_1 + \ldots + a_nv_n = 0_v] \\
\implies &N(T) = \{0_v\}
\end{aligned}
$$

$\implies T$ is injective.

$\square$

**Theorem 7** (2023.S(2.A)).

*Let $T : P_n(R) \to R^{n+1}$ be such that:*

$T(\sum_{i=0}^{n} c_i t^i) = (x_0, x_1, \dots, x_n)$

*where:* $x_k = \int_0^1 t^k f(t)dt$ *for* $k = 0, \dots, n$.

*Show that $T$ is invertible.*

*Proof.*

First, we note that the set $B_{P_n(R)} = \{t^0, \dots, t^n\}$ is a basis for $P_n(R)$, and $\langle f1, f2 \rangle = \int_0^1 f_1 * f_2 \, dt$ defines an inner-product on $P_n(R)$.

It follows that $T$ can be defined equivalently as follows:

$$Tf = (\langle t^0, f \rangle, \dots, \langle t^n, f \rangle)$$

Finally, we have the following series of implications:

$$\begin{aligned}
&Tf = 0_{R^n} \\
\implies &\forall i \in \{0, \dots, n\},\ \langle t^i, f \rangle = 0 \\
\implies &\sum_{i=0}^{n} a_i \langle t^i, f \rangle = 0 \\
\implies &\left\langle \sum_{i=0}^{n} a_i t^i, f \right\rangle = 0 \\
\implies &\forall f' \in P_n(R),\ \langle f', f \rangle = 0 \\
\implies &f = 0 = 0_{P_n(R)} \\
\implies &T \text{ is injective} \\
\implies &T \text{ is invertible.}
\end{aligned}$$

□

**Theorem 8** (2023.S(2.B), 2021.F(2.A)).

*Let $V$ and $W$ be $n$-dimensional vector spaces with order bases $\alpha$ and $\beta$ respectively. If $T$ is an isomorphism from $V$ onto $W$ with $[T]_\alpha^\beta = A$, show that $[T^{-1}]_\beta^\alpha = A^{-1}$*

*Proof.*

Let:

$$\begin{aligned}
[T]_\alpha^\beta &= A \\
[T^{-1}]_\beta^\alpha &= B \\
T\alpha_j &= \sum_{i=1}^n b_{ij}\beta_i \\
T^{-1}\beta_j &= \sum_{i=1}^n a_{ij}\alpha_i
\end{aligned}$$

Then, we have:

$$\begin{aligned}
\alpha_j &= (T^{-1}T)\alpha_j \\
&= T^{-1}(T\alpha_j) \\
&= T^{-1}\left(\sum_{i=1}^n b_{ij}\beta_i\right) \\
&= \sum_{i=1}^n b_{ij}T^{-1}\beta_i \\
&= \sum_{i=1}^n b_{ij}\sum_{k=1}^n a_{ki}\alpha_k \\
&= \sum_{k=1}^n \left(\sum_{i=1}^n b_{ij}a_{ki}\right)\alpha_k
\end{aligned}$$

$\implies \left(\sum_{i=1}^n a_{ij}b_{ki}\right) = \delta_{kj}$
$\iff BA = I$
$\iff B = A^{-1}.$

$\square$

**Theorem 9** (2023.S(2.C)).

*Let* $V = M_{2x2}(R)$ *and* $T(A) = A^t + 2tr(A)I_2$*, where* $A \in V$ *and* $A^t$ *is the transpose of* $A$.

*Find an ordered Basis* $\beta$ *for* $V$ *so that* $[T]_\beta$ *is a diagonal matrix.*

*Proof.*

Since $M_{2x2}(R)$ is isomorphic to $R^4$, we'll use $R^4$ in place of $M_{2x2}(R)$.

Let $\alpha$ be the standard basis for $R^4$:

$$\alpha = \{(1,0,0,0)^T, (0,1,0,0)^T, (0,0,1,0)^T, (0,0,0,1)^T\}$$

Then:

$$\begin{aligned}
&T(A) = A^t + 2tr(A)I \\
\implies T = &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.
\end{aligned}$$

Next, to find the eigenvalues of $T$, we solve the characteristic equation:

$$\begin{aligned}
&det(T - \lambda I) = 0 \\
\implies &det\begin{pmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{pmatrix} = 0 \\
\implies &(3-\lambda) * [-\lambda(-\lambda(3-\lambda)) - 1(3-\lambda)] = 0 \\
\implies &(3-\lambda) * [(3-\lambda)(\lambda^2 - 1)] = 0 \\
\implies &\lambda = 3,\ +1,\ -1.
\end{aligned}$$

□

**Theorem 10** (2023.S(3.B)).

*Let $V = W \oplus W^{\perp}$ and $T$ be the projection on $W$ along $W^{\perp}$.*

*Show that $T^* = T$.*

*Proof.*

Let $u = w_1 + w_2,\ v = w_1' + w_2' \ \in V$ where $w_1, w_1' \in W$ and $w_2, w_2' \in W^{\perp}$.

$$
\begin{aligned}
\langle Tu, v\rangle &= \langle T(w_1 + w_2), w_1' + w_2'\rangle \\
&= \langle w_1, w_1' + w_2'\rangle \\
&= \langle w_1, w_1'\rangle + \langle w_1, w_2'\rangle \\
&= \langle w_1, w_1'\rangle + 0 \\
&= \langle w_1, w_1'\rangle + \langle w_2, w_1'\rangle \\
&= \langle w_1 + w_2, w_1'\rangle \\
&= \langle w_1 + w_2, T(w_1' + w_2')\rangle \\
&= \langle u, Tv\rangle
\end{aligned}
$$

$\implies T^* = T$.

□

**Theorem 11** (2023.S(3.C)).

*Let $T$ be a linear operator on the inner product space $V$. Show that:*
$\langle Tu, Tv\rangle = \langle u, v\rangle \quad \forall u, v \in V \iff ||Tu|| = ||u|| \quad \forall u \in V$.

*Proof.*

$\underline{\implies}$:

Suppose $\langle Tu, Tv\rangle = \langle u, v\rangle$, then:

$$\begin{aligned} ||Tu|| &= \sqrt{\langle Tu, Tu\rangle} \\ &= \sqrt{\langle u, u\rangle} \\ &= ||u||. \end{aligned}$$

$\underline{\impliedby}$:

$$\begin{aligned} & ||Tu|| = ||u|| \\ \implies & ||Tu||^2 = ||u||^2 \\ \implies & \langle Tu, Tu\rangle = \langle u, u\rangle \\ \implies & \langle u, T^*Tu\rangle = \langle u, u\rangle \\ \implies & T^*T = I \\ \implies & \langle Tu, Tv\rangle = \langle u, T^*Tv\rangle \\ \implies & \langle Tu, Tv\rangle = \langle u, Iv\rangle = \langle u, v\rangle. \end{aligned}$$

□

**Theorem 12** (2022.S(1.A.i))**.**
*If $V$ is a vector space and $S_1, S_2 \subseteq V$ with $S_1 \subseteq S_2$, then $S_2^\perp$ is a* ***<u>subspace</u>*** *of $S_1^\perp$.*

*Proof.*
Since for any $A \subseteq V$, $A^\perp$ is a subspace of $V$, we need only show that: $S_2^\perp$ is a **<u>subset</u>** of $S_1^\perp$.

$$
\begin{aligned}
&\text{Let } s_2^\perp \in S_2^\perp \\
\implies &\forall s_2 \in S_2,\ \langle s_2^\perp, s_2 \rangle = 0 \\
\implies &\forall s_1 \in (S_2 \cap S_1 = S_1),\ \langle s_2^\perp, s_1 \rangle = 0 \\
\implies &S_2^\perp \subseteq S_1^\perp.
\end{aligned}
$$

□

**Theorem 13** (2022.S(1.B))**.**

*Let $V = M_{2x2}(R)$.*

*i. Show that $V$ has a basis that contains bases for its subspaces $U$ and $W$, where:*

$$U = \{A \in V : A^T = A\} \text{ and } W = \{A \in V : A^T = -A\}.$$

*ii. Show that $V = U \oplus W$.*

*Proof.*

$\square$

**Theorem 14** (2022.S(1.D)).

*Let $V$ be the vector space of complex numbers over the field $\mathbb{R}$, i.e. $C^1$ over $\mathbb{R}$.*

*Let $T : V \to V$ be defined by $T(z) = \bar{z}$, the complex conjugate of $z$.*

*i. Show that $T$ is linear.*

*ii. Show that $T$ is not linear if $V$ is redefined to be over the complex field $\mathbb{C}$.*

*Proof.*

$\square$

**Theorem 15** (2022.S(2.A)).

*Let $V$ be an $n$-dimensional vector space with bases $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$.*

*If $P \in Hom(V)$, such that $P(\alpha_i) = \beta_i \quad \forall i$, derive the relation between $[V]_\alpha$ and $[V]_\beta$ for $v \in V$.*

*Proof.*

□

**Theorem 16** (2022.S(2.B)).

*TODO: Reformulate from a problem to a statement.*

*Let* $T : P_2(R) \to P_2(R)$ *be such that:*

$$T(a + bt + ct^2) = -2b - 3c + (a + 3b + 3c)t + ct^2.$$

*i. Find a basis for for the eigenspace* $E_1$*.*

*ii. Is* $T$ *diagonalizable?*

*iii. Is there an operator on* $P_2(R)$ *whose null space is* $E_1$*?*

*Proof.*

$\square$

**Theorem 17** (2022.S(3.B), 2021.F(3.D)).
*Let $V$ be an inner product space, and $T \in Hom(V)$.*
*i. Show that $N(T^*T) = N(T)$.*
*ii. [Prove or Disprove] $rank(T^*T) = rank(T)$.*

*Proof.*

$\square$

**Theorem 18** (2021.F(1.A.i)).
*$V$ is an inner product vector space and $S \subseteq V \;\wedge\; S \neq \emptyset$*
*$\implies S^{\perp}$ is a subspace of $V$.*

*Proof.*
**<u>i. Inclusion of the Zero Vector</u>**:

$$\begin{aligned} &\forall s \in S,\ \langle 0_v, s\rangle = 0 \\ \implies &0_v \in S^{\perp}. \end{aligned}$$

**<u>ii. Closure under Vector Addition</u>**:

$$\begin{aligned} &\text{Let } s_1^{\perp}, s_2^{\perp} \in S^{\perp},\ s \in S \\ \implies &\langle s_1^{\perp}, s\rangle = 0 \wedge \langle s_2^{\perp}, s\rangle = 0 \\ \implies &\langle s_1^{\perp}, s\rangle + \langle s_2^{\perp}, s\rangle = 0 \\ \implies &\langle s_1^{\perp} + s_2^{\perp}, s\rangle = 0 \\ \implies &s_1^{\perp} + s_2^{\perp} \in S^{\perp}. \end{aligned}$$

**<u>iii. Closure under Scalar Multiplication</u>**:

$$\begin{aligned} &\text{Let } s^{\perp} \in S^{\perp},\ s \in S,\ k \in F \\ \implies &\langle s^{\perp}, s\rangle = 0 \\ \implies &k\langle s^{\perp}, s\rangle = 0 \\ \implies &\langle ks^{\perp}, s\rangle = 0 \\ \implies &ks^{\perp} \in S^{\perp}. \end{aligned}$$

□

**Theorem 19** (2021.F(1.A.ii))**.**

*[Disprove] If $T \in Hom(V, W)$, $dim(V) = dim(W) = 2$, and $\{v_1, v_2\}$ is a basis for $V$, then $\{T(v_1 - v_2), T(v_1)\}$ is a basis for $W$.*

*Proof.*

□

**Theorem 20** (2021.F(3.B)).

*Let $V$ be an inner product space, and $W$ be a finite-dimensional subspace of $V$.*

*Show that: $w \in W \implies \exists v \in V \ni v \notin W^{\perp} \wedge \langle w, v\rangle \neq 0$.*

***Notes to self:***

*1. I don't think this theorem holds when $W = V$. So I'm going to assume $W \neq V$ in the proof.*

*2. I don't think this theorem holds when $w = 0_v$. So I'm going to assume $w \neq 0_v$ for the proof.*

*Proof.*

Let $W$ be a **proper** subspace of $V$, then $dim(V) = n > dim(W) = m$.

Let $\{v_i\}_{i=1}^{m}$ be an ordered orthonormal basis for $W$, and extend it to an orthonormal basis for $V$: $\{v_i\}_{i=1}^{n}$. Then, $\{v_i\}_{i=m+1}^{n}$ is an orthonormal basis for $W^{\perp}$.

Let $w = \sum_{i=1}^{m} a_i v_i$. Since $w \neq 0_v$, then at least one $a_i$ is non-zero. Let $k$ be the index of the first non-zero $a_i$.

Next, let $v = b_k v_k + b_n v_n$, where $a_1, a_n \neq 0$.

Then, $v \notin W^{\perp}$, and:

$$\begin{aligned}
\langle w, v\rangle &= \langle w, b_k v_k + b_n v_n\rangle \\
&= \langle w, b_k v_k\rangle + \langle w, b_n v_n\rangle \\
&= \overline{b_k}\langle w, v_k\rangle + \overline{b_n}\langle w, v_n\rangle \\
&= \overline{b_k}\langle w, v_k\rangle + 0 \\
&= \overline{b_k}\langle \sum_{i=k}^{m} a_i v_i, v_k\rangle \\
&= \overline{b_k}\langle a_k v_k, v_k\rangle + \overline{b_k}\langle \sum_{i=k+1}^{m} a_i v_i, v_k\rangle \\
&= \overline{b_k} a_k \langle v_k, v_k\rangle + 0 \\
&= \overline{a_k} a_k \neq 0
\end{aligned}$$

□

**Theorem 21** (2012.F(1.B))**.**

*Let* $V = M_{2x2}(R)$*,*

$B \in V$ *such that* $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$*,*

$W_1 = \{A \in V : AB = BA\}$*,*

$W_2 = \{A \in V : A^T = A\}$*.*

*i. Show that* $W_1$ *is a subspace of of* $V$*.*

*ii. Find* $dim(W_1)$*.*

*iii. [Prove or Disprove]* $V = W_1 \oplus W_2$*.*

*Proof.*

□

**Theorem 22** (2012.F(1.D)).

*Let* $T_1, T_2 \in Hom(V, W)$.

*Show that:* $rank(T_1 + T_2) \leq rank(T_1) + rank(T_2)$.

*Proof.*

$\square$

# Problems

## 1.

[2023.S(3.A)]

Let $V = C([-1,1])$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt \;\; \forall f, g \in V$.

i. Find an orthonormal basis for $P_2(R)$ as a subspace of $V$ and use it to compute the best quadratic approximation of $f(t) = e^t$ on $[-1,1]$.

ii. For $T \in Hom(P_1(R))$ with $P_1(R)$ as a subspace of $V$ and $T(f) = f' + 3f$, evaluate $T^*(1+3t)$.

**Solution:**

i. The ordered set $\alpha = \{a_i\}_{i=0}^{2} = \{t^i\}_{i=0}^{2}$ is a basis for $P_2(R)$. We use the Gram-Schmidt process to transform $\alpha$ to an orthonormal basis $\beta = \{b_i\}_{i=0}^{2}$.

$\underline{b_1}$:
$||a_1||^2 = \langle a_1, a_1\rangle = \langle 1, 1\rangle = \int_{-1}^{1} 1 * 1dt = [t]_{-1}^{1} = 2.$

$b_1 = \frac{a_1}{||a_1||} = \frac{1}{\sqrt{2}}.$

$\underline{b_2}$:
$a_2 = t.$

$\langle a_2, b_1\rangle = \int_{-1}^{1} t * \frac{1}{\sqrt{2}}dt = [\frac{t^2}{2\sqrt{2}}]_{-1}^{1} = 0.$

$a_2 - \langle a_2, b_1\rangle b_1 = t.$

$||a_2 - \langle a_2, b_1\rangle b_1||^2 = ||t||^2 = \langle t, t\rangle = \int_{-1}^{1} t^2 dt = [\frac{t^3}{3}]_{-1}^{1} = \frac{2}{3}.$

$b_2 = \frac{a_2 - \langle a_2, b_1\rangle b_1}{||a_2 - \langle a_2, b_1\rangle b_1||} = \sqrt{\frac{3}{2}}t$

$\underline{b_3}$:
$a_3 = t^2.$

$\langle a_3, b_1\rangle = \int_{-1}^{1} t^2 * \frac{1}{\sqrt{2}}dt = [\frac{t^3}{3\sqrt{2}}]_{-1}^{1} = \frac{\sqrt{2}}{3}.$

$\langle a_3, b_2\rangle = \int_{-1}^{1} t^2 * \sqrt{\frac{3}{2}}tdt = [\frac{t^4}{4\sqrt{2}}]_{-1}^{1} = 0.$

$a_3 - \langle a_3, b_1\rangle b_1 - \langle a_3, b_2\rangle b_2 = t^2 - \frac{\sqrt{2}}{3} * \frac{1}{\sqrt{2}} = t^2 - \frac{1}{3}.$

$||a_3 - \langle a_3, b_1\rangle b_1 - \langle a_3, b_2\rangle b_2||^2 = ||t^2 - \frac{1}{3}||^2 = \langle t^2 - \frac{1}{3}, t^2 - \frac{1}{3}\rangle = \int_{-1}^{1}(t^2 - \frac{1}{3}) * (t^2 - \frac{1}{3})dt = \int_{-1}^{1}(t^4 - \frac{2}{3}t^2 + \frac{1}{9})dt = [\frac{t^5}{5} - \frac{2}{9}t^3 + \frac{1}{9}t]_{-1}^{1} = \frac{8}{45}$.

$b_3 = \frac{a_3 - \langle a_3, b_1\rangle b_1 - \langle a_3, b_2\rangle b_2}{||a_3 - \langle a_3, b_1\rangle b_1 - \langle a_3, b_2\rangle b_2||} = \frac{t^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}}$.

<u>Best quadratic approximation of $f(t) = e^t$:</u>
The best approximation is given by: $P_{P_2}(f(t)) = \langle f(t), b_1\rangle b_1 + \langle f(t), b_2\rangle b_2 + \langle f(t), b_3\rangle b_3$.

$\langle f(t), b_1\rangle = \langle e^t, \frac{1}{\sqrt{2}}\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}}e^t dt = [\frac{1}{\sqrt{2}}e^t]_{-1}^{1} = 1.662$.

$\langle f(t), b_2\rangle = \langle e^t, \sqrt{\frac{3}{2}}t\rangle = \int_{-1}^{1} e^t * \sqrt{\frac{3}{2}}t dt = 0.901$.

$\langle f(t), b_3\rangle = \langle e^t, \sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}}\rangle = \int_{-1}^{1} e^t * (\sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}})dt = 0.226$.

$\implies f(t) \approx 0.226(\sqrt{\frac{45}{8}}t^2 - \sqrt{\frac{5}{8}}) + 0.901 * \sqrt{\frac{3}{2}}t + \frac{1.662}{\sqrt{2}} = 0.536t^2 + 1.103t + 0.997$

ii. Let $v = a_0 b_1 + a_1 b_2 = (a_0, a_1), w = a'_0 b_1 + a'_1 b_2 = (a'_0, a'_1) \in P_1(R)$.

$$\begin{aligned}
\langle v, T^* w\rangle &= \langle Tv, w\rangle \\
&= \langle T(a_0, a_1), (a'_0, a'_1)\rangle \\
&= \langle (3a_0 + a_1, 3a_1), (a'_0, a'_1)\rangle \\
&= \langle (3a_0 + a_1, 0), (a'_0, 0)\rangle + \langle (0, 3a_1), (0, a'_1)\rangle \\
&= \langle (3a_0, 0), (a'_0, 0)\rangle + \langle (a_1, 0), (a'_0, 0)\rangle + \langle (0, 3a_1), (0, a'_1)\rangle \\
&= \langle (a_0, a_1), (3a'_0, a'_0 + 3a'_1)\rangle
\end{aligned}$$

$\implies T^*(a_0, a_1) = (3a_0, a_0 + 3a_1)$.

Let $v = 1 + 3t$, then:
$\langle v, b_1\rangle = \sqrt{2}$.
$\langle v, b_2\rangle = 2.45$.
Therefore, $v = \sqrt{2}b_1 + 2.45b_2 = (\sqrt{2}, 2.45)$.

$\implies T^* v = (3\sqrt{2}, \sqrt{2} + 3 * 2.45) = (4.24, 8.76)$.

## 2.

[2022.S(1.C), 2021.F(1.B)]

Let $V = R^3$.

i. Suggest two 2-dimensional subspaces $W_1$ and $W_2$ of $V$ such that $V = W_1 + W_2$.

ii. With $W_1 = \{v \in V : v = (x, y, 0)\}$, define two projections **on** $W_1$ **along** two distinct subspaces $W_2$ amd $W_3$ of $V$.

## 3.

[2022.S(3.A), 2021.F(3.A)]

Let $V = P_1(R) = Span(\{1, t\})$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt, \forall f, g \in V$.

i. Find an orthonormal basis for V.

ii. Find the orthogonal projection of $f(t) = t^2$ on $V$.

iii. Let $T(f) = f'(t) + 3f(t)$ be a linear operator on $V$. Is $(t - 2)$ an eigenvector for $T$?

iv. Evaluate $T^*(2t - 1)$.

**Solution:**

i. Let $\alpha = \{a_i\} = \{t^i\}$ be the standard basis. We use the Gram-Schmidt process to orthonormalize this basis to obtain $E = \{e_i\}$.

$\underline{e_1}$:
$||a_1||^2 = \langle 1, 1\rangle = 1.$

$e_1 = \frac{a1}{||a_1||} = \frac{1}{\sqrt{1}} = 1.$

$\underline{e_2}$:
$\langle a_2, e_1\rangle = \langle t, 1\rangle = \frac{1}{2}.$

$a_2 - \langle a_2, e_1\rangle e_1 = t - \frac{1}{2}.$

$||a_2 - \langle a_2, e_1\rangle e_1||^2 = ||t - \frac{1}{2}||^2 = \langle t - \frac{1}{2}, t - \frac{1}{2}\rangle = \frac{1}{12}.$

$e_2 = \frac{a_2 - \langle a_2, e_1\rangle e_1}{||a_2 - \langle a_2, e_1\rangle e_1||} = \frac{t - \frac{1}{2}}{\sqrt{\frac{1}{12}}} = \sqrt{12}(t - \frac{1}{2}).$

ii.
$\langle f(t), e_1\rangle = \langle t^2, 1\rangle = 0.333.$

$\langle f(t), e_2\rangle = \langle t^2, \sqrt{12}(t - \frac{1}{2})\rangle = 0.289.$

$\implies f(t) = e^t \approx 0.333 + 0.289 * \sqrt{12}(t - \frac{1}{2}) = -0.168 + 1.001t.$

iii. $T(t - 2) = 1 + 3(t - 2) = -5 + 3t \neq \lambda(t - 2).$
$\implies (t - 2)$ is not an eigenvector of $T$.

iv. Let $v = a_1e_1 + a_2e_2 = (a_1, a_2), w = b_1e_1 + b_2e_2 = (b_1, b_2) \in P_1(R)$

$$
\begin{aligned}
\langle v, T^*w\rangle &= \langle Tv, w\rangle \\
&= \langle T(a_1, a_2), (b_1, b_2)\rangle \\
&= \langle (3a_1 + a_2, 3a_2), (b_1, b_2)\rangle \\
&= \langle (3a_1 + a_2, 0), (b_1, 0)\rangle + \langle (0, 3a_2), (0, b_2)\rangle \\
&= \langle (3a_1, 0), (b_1, 0)\rangle + \langle (a_1, 0), (b_1, 0)\rangle + \langle (0, 3a_2), (0, b_2)\rangle \\
&= 3a_1b_1 + a2b_1 + 3a_2b_2 \\
&= \langle (a_1, a_2), (3b_1, b_1 + 3b_2)\rangle.
\end{aligned}
$$

$$\implies T^*(c_1, c_2) = (3c_1, c_1 + 3c_2).$$

Let $u = 2t - 1$.
$u = \langle u, e_1\rangle e_1 + \langle u, e_2\rangle e_2 = 0.577e_2$.
$T^*u = (0, 1.731)$ .

## 4.

[2021.F(1.C)]

Let $T : P_2(R) \to R^2$ be such that $T(a_0 + a_1t + a_2t^2) = (a_1 + a_2, a_0 - a_1)$.

i. Find a basis for $ker(T)$. ii. Is $T$ surjective? iii. Find a two-dimensional subspace of of $P_2(R)$ such that its image under $T$ is a one-dimensional subspace of $R^3$. iv. Find the matrix representation of $T$ relative to $\{1, t-1, t^2+1\}$ as a basis for $P_2(R)$ and $\{(0,1),(1,1)\}$ as a basis for $R^2$.

## 5.

[2021.F(2.B)]

$T$ is a linear operator on $P_2(R)$ defined by: $T(f(x) = xf'(x) + f(2) + f(3))$.

i. Is $T$ diagonalizable?

ii. Find an eigenpair for $T$.

## 6.

[2021.F(3.C)]

Let $V = \mathbb{C}^2$ and $T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1)$.
Evaluate $T^*(3-i, 1+2i)$.

**Solution:**

Let $w = (w_1, w_2), z = (z_1, z_2) \in C^2$.

$$\begin{aligned}
\langle w, T^* z\rangle &= \langle Tw, z\rangle \\
&= \langle T(w_1, w_2), (z_1, z_2)\rangle \\
&= \langle (2w_1 + iw_2, (1-i)w_1), (z_1, z_2)\rangle \\
&= (2w_1 + iw_2)z_1 + (1-i)w_1 z_2 \\
&= w_1(2z_1 + (1-i)z_2) + w_2(iz_1) \\
&= \langle (w_1, w_2), (2z_1 + (1-i)z_2, iz_1)\rangle
\end{aligned}$$

$\implies T^*(z_1, z_2) = (2z_1 + (1-i)z_2,\ iz_1)$.
Let $v = (3-i, 1+2i)$, then $T^* v = (2(3-i) + (1-i)(1+2i), i(3-i)) = (..., ...)$.

## 7.

[2012.F(4.B)]
Find the minimal $l_2$-norm solution to the system:

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 3y + z &= 2 \\ 4x + 7y - z &= 4. \end{aligned}$$

*Proof.*

$$\begin{aligned}
&\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 6 & 4 \end{pmatrix} \\
\to &\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{pmatrix} \\
\to &\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

□