MTH-684 Logic
Assignment (6): Inference in First-Order Predicate Logic (FOPL)

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Using the FOPL system of natural deduction introduced in class, prove the following.

$$\vdash \exists x\,[P(x) \vee Q(x)] \iff \exists x P(x) \vee \exists x Q(x)$$

Proof.

$\underline{\implies:}$

$* \, 1.\exists x\,[P(x) \vee Q(x)]$	(Assumption)
$* \, 2.P(c) \vee Q(c)$	(1, $\exists$-elim)
$* \, 3.\exists x P(x) \vee Q(c)$	(2, $\exists$-intro)
$* \, 4.\exists x P(x) \vee \exists x Q(x)$	(3, $\exists$-intro)
$5.\exists x\,[P(x) \vee Q(x)] \implies \exists x P(x) \vee \exists x Q(x)$	(1, 4, $\implies$-intro).

$\underline{\impliedby:}$

$* \, i.\exists x P(x) \vee \exists x Q(x)$	(Assumption)
$* * ii.\neg\,[\exists x\,[P(x) \vee Q(x)]]$	(Assumption)
$* * iii.\neg\,[P(c) \vee Q(c)]$	(ii, $\exists$-elim)
$* * iv.\neg\,[\exists x P(x) \vee Q(c)]$	(iii, $\exists$-intro)
$* * v.\neg\,[\exists x P(x) \vee \exists x Q(x)]$	(iv, $\exists$-intro)
$* \, vi.\neg\neg\,[\exists x\,[P(x) \vee Q(x)]]$	(i, ii, v, $\neg$-intro)
$* \, vii.\exists x\,[P(x) \vee Q(x)]$	(vi, $\neg$-elim)
$viii.\exists x P(x) \vee \exists x Q(x) \implies \exists x\,[P(x) \vee Q(x)]$	(i, vii, $\implies$-intro).

$\underline{\iff:}$

$$\exists x\,[P(x) \vee Q(x)] \iff \exists x P(x) \vee \exists x Q(x) \qquad (5,\ viii,\ \iff\text{-intro}).$$

$\square$