MTH-684 Logic
Assignment (2): Propositional Logic

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<u>**Note:**</u> I've only managed to edit part (a).

a.

Proof.
Let A be a truth assignment function and $[[.]]^A$ be its corresponding interpretation function under which $(\Gamma_1 \models \phi) = (\Gamma_2 \models \psi) = \top$.

We now want to show that $(\Gamma_1 \cup \Gamma_2 \models \phi \land \psi) = \top$ under the same interpretation function $[[.]]^A$.
So suppose that all formulas in $\Gamma_1 \cup \Gamma_2$ interpret to $\top$ under $[[.]]^A$.
Because all formulas in Γ_1 are $\top$ and $\Gamma_1 \models \phi$, we have $\phi = \top$.
And because all formulas in Γ_2 are $\top$ and $\Gamma_2 \models \psi$, we have $\psi = \top$.
Finally, from the definition of logical conjunction: $\phi \land \psi = \top \iff \phi = \psi = \top$, we conclude that $\phi \land \psi = \top$.

$\square$

b.

Proof.
Let $T_4 = T_1 \cup T_2 \cup T_3$.
Suppose all WFFs in T_4 are true, then all WFFs in T_1, T_2, and T_3 are true.
From $T_1 \models \phi$, we conclude that $\phi \lor \psi$ is true. We have 3 cases:

<u>1. $\phi = \top$ and $\psi = \bot$:</u>
In this case, all WFFs in $T_2 \cup \{\phi\}$ are true.
From $T_2 \cup \{\phi\} \models \zeta$, we conclude that ζ is true.

<u>2. $\phi = \bot$ and $\psi = \top$:</u>
In this case, all WFFs in $T_3 \cup \{\psi\}$ are true.
From $T_3 \cup \{\psi\} \models \zeta$, we conclude that ζ is true.

<u>3. $\phi = \top$ and $\psi = \top$:</u>
Using either case 1 or 2, it is clear that ζ is true.

Therefore, we conclude that $T_4 = T_1 \cup T_2 \cup T_3 \models \zeta$.

$\square$

c.

Proof.
We want to show that $T_1 \models (\phi \implies \psi)$ is true, so we assume T_1 and attempt to show that $\phi \implies \psi$ holds.

Next, to show that $\phi \implies \psi$, we assume ϕ and attempt to show that ψ holds.

Putting our assumptions together: T_1 and ϕ are true. Therefore, all WFFs in $T_1 \cup \{\phi\}$ are true.

From $T_1 \cup \{\phi\} \models \psi$, we conclude that ψ is true.

Therefore, $T_1 \models (\phi \implies \psi)$ as desired.

□

d.

Proof. (By Contradiction)

Suppose $T_1 \cup T_2$ is true.

For the sake of contradiction, suppose that ϕ is true.

From $T_1 \cup \{\phi\} \models \psi$, we conclude that ψ is true.

From $T_2 \cup \{\phi\} \models \neg\psi$, we conclude that $\neg\psi$ is true.

But then we have: $\psi \land \neg\psi$ is true. This is a contradiction.

Therefore, we must conclude that: $T_1 \cup T_2 \models \neg\phi$.

□