Linear Algebra Assignments

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1.2.13

No. Because the inverse is undefined for some elements in V.

To show this, let's begin by finding the identity element. Let $a = (a_1, a_2) \in V$, we want to find the element $id = (id_1, id_2)$ such that:

$$\begin{aligned} a + id &= a \\ (a_1, a_2) + (id_1, id_2) &= (a_1, a_2) \\ (a_1 + id_1, a_2 id_2) &= (a1, a_2) \end{aligned} \tag{1}$$

(1) implies that $id = (0, 1)$.

Next, let's find the inverse element a^{-1} for any $a \in V$:

$$\begin{aligned} a + a^{-1} &= id \\ (a_1, a_2) + (a_1^{-1}, a_2^{-1}) &= (0, 1) \\ (a_1 + a_1^{-1}, a_2 * a_2^{-1}) &= (0, 1) \end{aligned} \tag{2}$$

(2) implies that $a_1^{-1} = -a_1$ and $a_2^{-1} = \frac{1}{a_2}$. But whenever $a_2 = 0$, a_2^{-1} is undefined.

Since the inverse is not defined for all $a \in V$, then V is not a vector space.

1.2.14

Yes.

Since the set of elements in V over $\mathbb{R}$ is a subset of V over $\mathbb{C}$, and V over $\mathbb{C}$ is a vector space, this problem can be reduced to checking whether V over $\mathbb{R}$ is a subspace of V over $\mathbb{C}$. We only need to check 2 conditions: closure under vector addition and closure and scalar multiplication.

Closure under Vector Addition

Let a and $b \in V$, then:

$$a + b = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

Since a and $b \in \mathbb{R} \Rightarrow a_1$, a_2, b_1, and $b_2 \in \mathbb{R} \Rightarrow a_1 + b_1$ and $a_2 + b_2 \in \mathbb{R} \Rightarrow a + b \in V$. Therefore V is closed under vector addition.

Closure under Scalar Multiplication

Let $a \in V$ and $c \in \mathbb{R}$, then:

$$ca = c(a_1, a_2) = (c * a_1, c * a_2)$$

Since $a \in V \Rightarrow a_1, a_2 \in \mathbb{R} \Rightarrow ca_1$ and $ca_2 \in \mathbb{R} \Rightarrow ca \in V$. Therefore V is closed under scalar multiplication.

1.2.19

No. Because it fails axiom 8: $(k_1 + k_2)u = k_1u + k_2u$.

Counter Example

Let $u = (1, 1)$, $k_1 = 1$, and $k_2 = 1$:

$$\begin{aligned}(k_1 + k_2)u &= (1 + 1)(1, 1) \\ &= (2)(1, 1) \\ &= (2, \frac{1}{2}) \qquad (3)\end{aligned}$$

But,

$$\begin{aligned}k_1u + k_2u &= 1(1, 1) + 1(1, 1) \\ &= (1, 1) + (1, 1) \\ &= (2, 2) \qquad (4)\end{aligned}$$

Since (3) $\neq$ (4), V is not a vector space.