MTH-632 PDEs
Assignment (1): Introduction and Applications

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10 October 2024

1.

$$\frac{\partial z}{\partial x} = -12e^{-3x}\cos(3y) \qquad \frac{\partial z}{\partial y} = -12e^{-3x}sin(3y)$$

$$\frac{\partial^2 z}{\partial x^2} = 36e^{-3x}\cos(3y) \qquad \frac{\partial^2 z}{\partial y^2} = -36e^{-3x}cos(3y)$$

Then:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 36e^{-3x}\cos(3y) - 36e^{-3x}cos(3y) = 0$$
$$z(x, \pi/2) = 4e^{-3x}cos(\frac{3}{2}\pi) = 4e^{-3x} * 0 = 0$$
$$z(x, 0) = 4e^{-3x}cos(3 * 0) = 4e^{-3x} * 1 = 4e^{-3x}$$

Therefore $z(x, y) = 4e^{-3x}\cos(3y)$ is a solution to the given BVP.

2.

Let $u = 2x + y$.

a.

$$\frac{\partial u}{\partial x} = 2$$
$$\frac{\partial u}{\partial y} = 1$$
$$\frac{\partial v}{\partial x} = x\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + F(u) = 2x\frac{\partial F}{\partial u} + F(u)$$
$$\frac{\partial v}{\partial y} = x\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} = x\frac{\partial F}{\partial u}$$

Then:

$$x\frac{\partial v}{\partial x} - 2x\frac{\partial v}{\partial y} = x(2x\frac{\partial F}{\partial u} + F(u)) - 2x(x\frac{\partial F}{\partial u})$$
$$= xF(u) = v.$$

Therefore, $v(x, y) = xF(2x + y)$ is a general solution to the given PDE.

b.

$$
\begin{aligned}
&v(1,y) = y^2 \\
\implies &F(2+y) = y^2 \\
\implies &F = (u-2)^2 \\
\implies &v(x,y) = x*(2x+y-2)^2 \quad \text{is a particular solution.}
\end{aligned}
$$

3.

$$
\begin{aligned}
&\frac{\partial u}{\partial x} = F(x-3y) + 2G(2x+y) \\
&\frac{\partial u}{\partial y} = -3F(x-3y) + G(2x+y) \\
&7F(x-3y) = \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} \\
&7G(2x+y) = 3\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \\
\implies &7u = 4\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}.
\end{aligned}
$$

4.

a.

Let $u = 2y - 3x$.

$$
\begin{aligned}
&\frac{\partial z}{\partial x} = -3e^x\frac{\partial f}{\partial u} + e^x f \\
&\frac{\partial z}{\partial y} = 2e^x\frac{\partial f}{\partial u} \\
&2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 2e^x f \\
\implies &2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 2z \quad \text{is a PDE satisfying the general solution.}
\end{aligned}
$$

b.

Let $u = 2x + y$ and $v = x - 2y$.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2\frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} \\
\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u} - 2\frac{\partial g}{\partial v} \\
\frac{\partial^2 z}{\partial y\, \partial x} &= 2\frac{\partial^2 f}{\partial u^2} - 2\frac{\partial^2 g}{\partial v^2} \\
\frac{\partial^2 z}{\partial x\, \partial y} &= 2\frac{\partial^2 f}{\partial u^2} - 2\frac{\partial^2 g}{\partial v^2} \\
\implies \frac{\partial^2 z}{\partial x\, \partial y} &= \frac{\partial^2 z}{\partial y\, \partial x} \quad \text{is a PDE satisfying the general solution.}
\end{aligned}$$

5.

a.

$$\begin{aligned}
x\frac{\partial^2 z}{\partial x\, \partial y} + \frac{\partial z}{\partial y} &= 0 \\
\text{Let } u(x,y) = \frac{\partial z}{\partial y} \\
\implies x\frac{\partial}{\partial x}u + u &= 0 \\
\implies \frac{\partial}{\partial x}u &= (-\frac{1}{x})u \\
\implies u = F(y) * e^{-ln(x)} &= F(y) * e^{ln(x^{-1})} = F(Y) * x^{-1} \\
\implies z(x,y) = \int u\, dy &= x^{-1} \int F(Y)\, dy = x^{-1}H(y) + G(x)
\end{aligned}$$

b.

Boundary Condition 1:

$$\begin{aligned}
&z(x,0) = x^5 + x \\
\implies &x^{-1}H(y) + G(x) = x^5 + x \\
\implies &G(x) = x^5 + x \quad \text{and} \quad H(0) = 0
\end{aligned}$$

Boundary Condition 2:

$$\begin{aligned}
&z(2,y) = 3y^4 \\
\implies &2^{-1}H(y) + G(2) = 3y^4 \\
\implies &\frac{1}{2}H(y) + 2^5 + 2 = 3y^4 \\
\implies &H(y) = 6y^4 - 68
\end{aligned}$$

Therefore:

$$\begin{aligned}
z(x,y) &= x^{-1}H(y) + G(x) \\
&= x^{-1}(6y^4 - 68) + x^5 + x.
\end{aligned}$$

Verify:

$$\begin{aligned}
&\frac{\partial z}{\partial y} = 24x^{-1}y^3 \\
&\frac{\partial^2 z}{\partial x\,\partial y} = -24x^{-2}y^3 \\
&x\frac{\partial^2 z}{\partial x\,\partial y} = -24x^{-1}y^3 \\
\implies &x\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial z}{\partial y} = -24x^{-1}y^3 + 24x^{-1}y^3 = 0.
\end{aligned}$$

6.

$v = \frac{F(r-ct)+G(r+ct)}{r}$.
Let $u = r - ct$ and $w = r + ct$, then: $v = \frac{F(u)+G(w)}{r}$.

$$\frac{\partial v}{\partial r} = \frac{r\frac{\partial F}{\partial u} + r\frac{\partial G}{\partial w} - F - G}{r^2} \qquad \frac{\partial v}{\partial t} = \frac{-cr\frac{\partial F}{\partial u} + cr\frac{\partial G}{\partial w}}{r^2}$$

$$\begin{aligned}
\frac{\partial^2 v}{\partial r^2} &= \frac{r^2(r\frac{\partial^2 F}{\partial u^2} + \frac{\partial F}{\partial u}) + r^2(r\frac{\partial^2 G}{\partial w^2} + \frac{\partial G}{\partial w}) - \frac{\partial F}{\partial u} - \frac{\partial G}{\partial w} - 2r(r\frac{\partial F}{\partial u} + r\frac{\partial G}{\partial w} - F - G)}{r^4} \\
&= \frac{\frac{\partial^2 F}{\partial u^2}(r^3) + \frac{\partial^2 G}{\partial w^2}(r^3) + \frac{\partial F}{\partial u}(-r^2 - 1) + \frac{\partial G}{\partial w}(-r^2 - 1) + F(2r) + G(2r)}{r^4}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= \frac{r^2(c^2r\frac{\partial^2 F}{\partial u^2} + -c\frac{\partial F}{\partial u}) + r^2(c^2r\frac{\partial^2 G}{\partial w^2} + c\frac{\partial G}{\partial w}) + 2r(-cr\frac{\partial F}{\partial u} + cr\frac{\partial G}{\partial w})}{r^4} \\
&= \frac{\frac{\partial^2 F}{\partial u^2}(r^3c^2) + \frac{\partial^2 G}{\partial w^2}(r^3c^2) + \frac{\partial F}{\partial u}(-3cr^2) + \frac{\partial G}{\partial w}(3cr^2)}{r^4}
\end{aligned}$$

$$
\begin{aligned}
\implies \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} &= \frac{\frac{\partial^2 F}{\partial u^2}(r^3) + \frac{\partial^2 G}{\partial w^2}(r^3) + \frac{\partial F}{\partial u}(-r^2-1) + \frac{\partial G}{\partial w}(-r^2-1) + F(2r) + G(2r)}{r^4} \\
&+ \frac{2}{r}\left(\frac{r\frac{\partial F}{\partial u} + r\frac{\partial G}{\partial w} - F - G}{r^2}\right) \\
&= \frac{1}{c^2}\frac{\frac{\partial^2 F}{\partial u^2}(r^3c^2) + \frac{\partial^2 G}{\partial w^2}(r^3c^2) + \frac{\partial F}{\partial u}(-3cr^2) + \frac{\partial G}{\partial w}(3cr^2)}{r^4} \\
&= \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}.
\end{aligned}
$$

Problems:

1.

PDE	Order
(a)	2
(b)	3
(c)	4
(d)	2
(e)	1

3.

PDE	Linear	Non-Linear	Quasi-Linear	Homogenous
(a)	✓			
(b)	✓			✓
(c)			✓	
(d)		✓		
(e)	✓			✓
(f)			✓	
(g)		✓		
(h)	✓			✓
(i)			✓	

5.

Let $w = xy$ and $v = y/x$.

$$
\begin{aligned}
u_x &= yF' - x^{-1}G' + G \\
u_{xx} &= y^2F'' + yx^{-3}G'' + x^{-2}G' - x^{-2}G' = y^2F'' + yx^{-3}G'' \\
u_y &= xF' + G' \\
u_{yy} &= x^2F'' + x^{-1}G'' \\
x^2u_{xx} &= x^2y^2F'' + x^{-1}yG'' \\
y^2u_{yy} &= x^2y^2F'' + x^{-1}y^2G'' \\
\implies x^2u_{xx} - y^2u_{yy} &= [x^2y^2F'' + x^{-1}yG''] - [x^2y^2F'' + x^{-1}yG''] = 0.
\end{aligned}
$$