Real Analysis Assignments

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1.1.7

A_n is the set of all positive multiples of $(n+1)$.

a.

$$\begin{aligned} A_1 \cap A_2 &= \{x : \text{x is a positive multiple of both 2 and 3}\} \\ &= \{x : \text{x is a positive multiple of 6}\} \\ &= A_5 \\ &= \{6, 12, 18, \ldots\}. \end{aligned}$$

b.

i.

$$\begin{aligned} \cup\{A_n : n \in \mathbf{N}\} &= \{x : \text{x is a positive multiple of 2 or 3 or 4 or} \ldots\} \\ &= \mathbf{N} - \{1\}. \end{aligned}$$

ii.

$$\begin{aligned} \cap\{A_n : n \in \mathbf{N}\} &= \{x : \text{x is a positive multiple of 2 and 3 and 4 and} \ldots\} \\ &= \emptyset \text{ (because } \forall n \in N \; n \notin A_n). \end{aligned}$$

1.1.22

a.

Proof.
Let $x_1, x_2 \in A$, and assume $f(x_1) = f(x_2)$
$\implies g(f(x_1)) = g(f(x_2))$
$\implies (g \circ f)(x_1) = (g \circ f)(x_2)$
$\implies x_1 = x_2$, by injectivity of $g \circ f$
$\implies$ f is injective.

□

b.

Proof.
Let $c \in C$
$\implies \exists a \in A : (g \circ f)(a) = c$
$\implies \exists a \in A : g(f(a)) = c$

$\implies \exists b \in B : g(b) = c$
$\implies$ g is surjective.

$\square$

1.3.7

Proof:

Forward direction: If a set T_1 is denumerable, then there exists a bijection from $T1$ onto a denumerable set T_2.

Since T_1 is denumerable, then there exists a bijection f_1 from N onto T_1.
And since T_2 is denumerable, then there exists a bijection f_2 from N onto T_2.
Also, since f_1 is a bijection, then f_1^{-1} exists and is a bijection from T_1 onto N.
Then, the function defined by $f_2 \circ f_1^{-1}$ is a bijection from $T1$ onto T_2, since the compositiomn of bijective functions is bijective.

Reverse direction: If there exists a bijection from a set $T1$ onto a denumerable set T_2, then T_1 is denumerable.

Let g be the bijection from $T1$ to $T2$.
Since T_2 is denumerable, then there exists a bijection f_2 from N onto T_2.
Also, since g is a bijection, then g^{-1} exists and is a bijection from T_2 onto T_1.
Then, the function defined as $f_1 := g^{-1} \circ f_2$ is a bijection from N onto T_1, since the compositiomn of bijective functions is bijective.
This implies that the set T_1 is denumerable.

This completes the proof. $\square$

1.3.8

Let's define the set S_i as $S_i := \{i\}$, then for each i, S_i is a finite set (of cardinality $= 1$).
But the union $\cup_{i=1}^{\infty} S_i = N$ is infinite, because N is infinite.

2.1.4

By the trichotomy of a, we have three cases: (i) $a < 0$, (ii) $a = 0$, and (iii) $a > 0$.

(i): $a < 0 \Rightarrow a \cdot a > 0 \Rightarrow a \cdot a > a \Rightarrow a \cdot a \neq a$. Therefore, a cannot be less than 0.

(ii): $a = 0 \Rightarrow a \cdot a = a$, because $0 \cdot 0 = 0$.

(iii): $a > 0$ and $a \cdot a = a \Rightarrow a^{-1} \cdot a \cdot a = a^{-1} \cdot a \Rightarrow a = 1$.

Therefore, $a = 0$ or $a = 1$. $\square$

2.1.23

Proof:

Forward direction: For $a > 0$, $b > 0$, and $n \in N$: If $a < b$, then $a^n < b^n$.

We use induction.

Base case $(n = 1)$: This case is trivially true because it is given by the hypothesis: $a < b \iff a^1 < b^1$.

Inductive step $(n > 1)$: By the induction hypothesis, we have:

$$a < b \Rightarrow a^n < b^n$$

Since $b > 0$, multiplying $a^n < b^n$ by b, we get:

$$ba^n < b^{n+1} \tag{1}$$

Since $a < b$, then $b - a > 0$; and since $a > 0$, then $a^n > 0$.
Also, since $(b - a) > 0$ and $a^n > 0$, we have:

$$(b - a)a^n > 0 \Rightarrow ba^n - a^{n+1} > 0 \Rightarrow ba^n > a^{n+1} \tag{2}$$

Combining (1) and (2) together, we get:

$$a^{n+1} < ba^n < b^{n+1}$$

Therefore, $a < b \Rightarrow a^{n+1} < b^{n+1}$, thus closing the induction.

Reverse direction: For $a > 0$, $b > 0$, and $n \in N$: If $a^n < b^n$, then $a < b$.

We use induction.

Base case $(n = 1)$: This case is trivially true because it is given by the hypothesis: $a^1 < b^1 \iff a < b$.

Inductive step $(n > 1)$: By the induction hypothesis, we have:

$$a^n < b^n \Rightarrow a < b \tag{3}$$

The contrapositive of (3) is:

$$a \geq b \Rightarrow a^n \geq b^n \tag{4}$$

Since $b > 0$, multiplying $a^n \geq b^n$ by b, we get:

$$ba^n \geq b^{n+1} \tag{5}$$

Since $a \geq b$, then $a - b \geq 0$; and since $a > 0$, then $a^n > 0$.
Also, since $(a - b) \geq 0$ and $a^n > 0$, we have:

$$(a-b)a^n \geq 0 \Rightarrow a^{n+1} - ba^n \geq 0 \Rightarrow a^{n+1} \geq ba^n \tag{6}$$

Combining (5) and (6) together, we get:

$$a^{n+1} \geq ba^n \geq b^{n+1} \tag{7}$$

Putting (4), (5), (6), and (7) together, we get:

$$a \geq b \Rightarrow a^{n+1} \geq b^{n+1} \tag{8}$$

Taking the contrapositive of (8), we get:

$$a^{n+1} < b^{n+1} \Rightarrow a < b \tag{9}$$

This closes the induction and completes the proof. □

2.2.16

$V_\epsilon(a) = \{x \in R : |x - a| < \epsilon\}$
$V_\delta(a) = \{x \in R : |x - a| < \delta\}$

(i) $V_\epsilon(a) \cup V_\delta(a) = \{x \in R : |x - a| < \epsilon$ and $|x - a| < \delta\}$

Let $\gamma = \min(\epsilon, \delta)$.

Lower Bound: $|x-a| < \epsilon$ and $|x-a| < \delta \Rightarrow x > a-\epsilon$ and $x > a-\delta \Rightarrow x > a-\gamma$

Upper Bound: $|x-a| < \epsilon$ and $|x-a| < \delta \Rightarrow x < a+\epsilon$ and $x < a+\delta \Rightarrow x < a+\gamma$

Therefore, $V_\epsilon(a) \cup V_\delta(a)$ is in the γ-neighbourhood of a.

(ii) $V_\epsilon(a) \cap V_\delta(a) = \{x \in R : |x - a| < \epsilon$ or $|x - a| < \delta\}$

Let $\gamma = \max(\epsilon, \delta)$.

Lower Bound: $|x - a| < \epsilon$ or $|x - a| < \delta \Rightarrow x > a - \epsilon$ or $x > a - \delta \Rightarrow x > a - \gamma$

Upper Bound: $|x-a| < \epsilon$ and $|x-a| < \delta \Rightarrow x < a+\epsilon$ and $x < a+\delta \Rightarrow x < a+\gamma$

Therefore, $V_\epsilon(a) \cap V_\delta(a)$ is in the γ-neighbourhood of a.

2.2.17

Without loss of generality, assume that $b > a$. Let $\epsilon = \frac{b-a}{2}$

Then, $U_\epsilon(a) = \{x \in R : |x - a| < \epsilon\}$, and $V_\epsilon(b) = \{x \in R : |x - b| < \epsilon\}$

$U_\epsilon(a) \cap V_\epsilon(b) = \{x \in R : |x - a| < \epsilon \text{ and } |x - b| < \epsilon\}$

Lower Bound:

$$\begin{aligned}
&|x - a| < \epsilon \text{ and } |x - b| < \epsilon \\
&\implies x > a - \epsilon \text{ and } x > b - \epsilon \\
&\implies x > a - \tfrac{b-a}{2} \text{ and } x > b - \tfrac{b-a}{2} \\
&\implies x > \tfrac{3a-b}{2} \text{ and } x > \tfrac{b+a}{2} \\
&\implies x > \tfrac{b+a}{2}
\end{aligned}$$

Upper Bound:

$$\begin{aligned}
&|x - a| < \epsilon \text{ and } |x - b| < \epsilon \\
&\implies x < a + \epsilon \text{ and } x < b + \epsilon \\
&\implies x < a + \tfrac{b-a}{2} \text{ and } x < b + \tfrac{b-a}{2} \\
&\implies x < \tfrac{a+b}{2} \text{ and } x < b + \tfrac{b-a}{2} \\
&\implies x < \tfrac{a+b}{2}
\end{aligned}$$

Since the lower bound and the upper bound do not intersect, then $U_\epsilon(a) \cap V_\epsilon(b) = \emptyset$. $\square$

2.3.4

$S_4 = \{1 - (-1)^n/n : n \in N\}$

Let $S' = \{-(-1)^n/n : n \in N\}$. Then: $S_4 = 1 + S'$.

$inf(S') = -(-1)^2/2 = -1/2$

$sup(S') = -(-1)^1/1 = 1$

$inf(S_4)$:

$$inf(S_4) = 1 + inf(S') = 1 + (-1/2) = 1/2$$

$sup(S_4)$:

$$sup(S_4) = 1 + sup(S') = 1 + 1 = 2$$

2.3.11

Proof. By the definition of the infimum and supremum we have $inf(S_0) \leq sup(S_0)$. So we need only show that (i)$inf(S) \leq inf(S_0)$, and (ii)$sup(S_0) \leq sup(S)$.

i. $\underline{inf(S) \leq inf(S_0)}$**:** We prove this by contradiction.
Suppose that $inf(S_0) < inf(S)$.
Since $inf(S_0)$ is a infimum for S_0, then $\forall \epsilon > 0 \; \exists s \in S_0 : s < inf(S_0) + \epsilon$.
Taking $\epsilon = [inf(S) - inf(S_0)]/2$
$\Rightarrow \exists s \in S_0$ and $s < inf(S)$
$\Rightarrow s \in S_0 \; and \; s \notin S$
$\Rightarrow S_0 \not\subset S$.
This is a contradiction. Therefore, we must conclude that $inf(S) \leq inf(S_0)$.

ii. $\underline{sup(S_0) \leq sup(S)}$**:** We prove this by contradiction.
Suppose that $sup(S_0) > sup(S)$.
Since $sup(S_0)$ is a supremum for S_0, then $\forall \epsilon > 0 \; \exists s \in S_0 : s > sup(S_0) + \epsilon$.
Taking $\epsilon = [sup(S_0) - sup(S)]/2$
$\Rightarrow \exists s \in S_0$ and $s > sup(S)$
$\Rightarrow s \in S_0 \; and \; s \notin S$
$\Rightarrow S_0 \not\subset S$.
This is a contradiction. Therefore, we must conclude that $sup(S_0) \leq sup(S)$. □

3.1.1.b

$x_n := (-1)^n/n = (-1, 1/2, -1/3, 1/4, -1/5, ...)$

3.1.5.d

Required to show: $\lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$

Proof. Given any $\epsilon > 0$, we need to find $k(\epsilon)$ such that for all $n \geq k$: $|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| < \epsilon$:

$$
\begin{aligned}
|\frac{n^2-1}{2n^2+3}-\frac{1}{2}| &< \epsilon \\
|\frac{2n^2-2-2n^2-3}{4n^2+6}| &< \epsilon \\
|\frac{-5}{4n^2+6}| &< \epsilon \\
\frac{5}{4n^2+6} &< \epsilon \\
\frac{5}{4n^2+6} &\leq \frac{5}{n^2} \leq \frac{5}{n} < \epsilon
\end{aligned}
$$

Taking $k(\epsilon) = 5/\epsilon$ satisfies the required conditions.

□

3.1.7

a

Required to show: $\lim(\frac{1}{ln(n+1)}) = 0$

Proof. Given any $\epsilon > 0$, we need to find $k(\epsilon)$ such that for all $n \geq k$: $|\frac{1}{ln(n+1)} - 0| < \epsilon$:

$$
\begin{aligned}
|\frac{1}{ln(n+1)} - 0| &< \epsilon \\
\frac{1}{ln(n+1)} &< \epsilon \\
ln(n+1) &> \frac{1}{\epsilon} \\
n+1 &> e^{\frac{1}{\epsilon}} \\
n &> e^{\frac{1}{\epsilon}} - 1
\end{aligned}
$$

Taking $k(\epsilon) = e^{\frac{1}{\epsilon}} - 1$ satisfies the required conditions.

□

b

i. $k(1/2) = e^2 - 1 = 7$
ii. $k(1/10) = e^{10} - 1 = 22026$

3.1.9

Proof. $\lim(x_n) = 0 \Rightarrow \forall\epsilon > 0$ there exists $k(\epsilon)$ such that for all $n \geq k : \; |x_n - 0| < \epsilon$.
$\Rightarrow \forall\epsilon > 0$ there exists $k(\epsilon)$ such that for all $n \geq k : x_n < \epsilon$ (because $x_n > 0$).
$\Rightarrow \forall\epsilon > 0$ there exists $k(\epsilon)$ such that for all $n \geq k : \sqrt{x_n} - 0 < \sqrt{\epsilon} = \epsilon^{'}$.
Since $\epsilon^{'}$ can take on any value greater than zero, then by the definition of the limit of a sequence this shows that $\lim(\sqrt{x_n}) = 0$.

□

3.1.12

Required to show: $\lim(\sqrt{n^2+1} - n) = 0$

Proof. Given any $\epsilon > 0$, we need to find $k(\epsilon)$ such that for all $n \geq k$: $|(\sqrt{n^2+1} - n) - 0| < \epsilon$:

$$\begin{aligned}
|(\sqrt{n^2+1} - n) - 0| &< \epsilon \\
|\sqrt{n^2+1} - n| &< \epsilon \\
\sqrt{n^2+1} - n \leq \sqrt{(n+1/n)^2} - n &< \epsilon \\
\sqrt{n^2+1} - n \leq (n+1/n) - n &< \epsilon \\
\sqrt{n^2+1} - n \leq 1/n &< \epsilon
\end{aligned}$$

Taking $k(\epsilon) = \frac{1}{\epsilon}$ satisfies the required conditions.

□

3.2.2

a.

$X = (sin^2(n))$, $Y = (cos^2(n)) \implies X + Y = (1)$.

b.

$X = ((-1)^n)$, $Y = ((-1)^{n+1}) \implies X * Y = (-1)$.

3.2.13

Multiply and divide by the complex conjugate to put the sequence into a more favorable form:

$$
\begin{aligned}
(\sqrt{(n+a)(n+b)}-n) * \frac{\sqrt{(n+a)(n+b)}+n}{\sqrt{(n+a)(n+b)}+n} &= \frac{(n+a)(n+b)-n^2}{\sqrt{(n+a)(n+b)}+n} \\
&= \frac{n^2+an+bn-ab-n^2}{\sqrt{(n+a)(n+b)}+n} \\
&= \frac{an+bn-ab}{\sqrt{(n+a)(n+b)}+n} \\
&= \frac{an+bn-ab}{\sqrt{(n+a)(n+b)}+n} \\
&= \frac{a+b-\frac{ab}{n}}{\sqrt{(1+\frac{a}{n})(1+\frac{b}{n})}+1}
\end{aligned}
$$

Now taking the limit:

$$
\begin{aligned}
\lim \frac{a+b-\frac{ab}{n}}{\sqrt{(1+\frac{a}{n})(1+\frac{b}{n})}+1} &= \frac{\lim a+\lim b-\lim\frac{ab}{n}}{\lim\sqrt{1+\frac{a}{n}} * \lim\sqrt{1+\frac{b}{n}}+\lim 1} \\
&= \frac{a+b-0}{1*1+1} = \frac{a+b}{2}
\end{aligned}
$$

3.2.14

a.

We have:
$\quad 0 \le 1/n^2 \le 1/n$
$\implies n^0 \le n^{1/n^2} \le n^{1/n}$
And we know the following limits:
$\quad \lim(n^0) = 1 \ \wedge \ \lim(n^{1/n}) = \lim(n^{\lim 1/n}) = \lim(n^0) = 1$
$\implies \lim(n^{1/n^2}) = 1$ (By the squeeze theorem).

b.

We have:
$\quad 1 \le n! \le n^n$
$\implies 1^{1/n^2} \le (n!)^{1/n^2} \le (n^n)^{1/n^2}$
And we know the following limits:
$\quad \lim(1^{1/n^2}) = 1^0 = 1 \ \wedge \ \lim((n^n)^{1/n^2}) = \lim(n^{1/n}) = n^0 = 1$
$\implies \lim((n!)^{1/n^2}) = 1$ (By the squeeze theorem).

3.2.22

The definition given for (y_n) is exactly the definition of the limit.
Therefore, $\lim(y_n) = (x_n)$.
And, because (x_n) is convergent, then (y_n) must also be convergent.

3.3.3

We first show that (x_n) is bounded below by 2 using induction.

Proof.
Base case $(n = 1)$: This case is trivially true since $x_1 \geq 2$.

Inductive step $(n > 1)$: Assume $x_n \geq 2$
$\implies x_{n+1} = 1 + \sqrt{x_n - 1} \geq 1 + \sqrt{2 - 1} = 2.$

□

Next, we show that (x_n) is decreasing using induction.

Proof.
Base case $(n = 1)$: $x_2 = 1 + \sqrt{x_1 - 1} \leq x_1 f$ for $x \geq 2$.

Inductive step $(n > 1)$: Assume $x_{n+1} \leq x_n$
$\implies x_{n+2} = 1 + \sqrt{x_{n+1} - 1} \leq 1 + \sqrt{x_n - 1} = x_{n+1}.$

□

Finally, to find the limit we note that at the limit we have:
$x_{n+1} = x_n$
$\implies lim(x_n) = x = 1 + \sqrt{x - 1}$
$\implies x = 2.$

3.4.9

Proof. (By contradiction)
Suppose $\lim X = a \neq 0$.
$\implies$ All **subsequences** of X must converge to a (By theorem 3.4.2).
$\implies$ All **subsequences of subsequences** of X must converge to a (Again by theorem 3.4.2).

But this is a contradiction to our initial assumption that all subsequences of X contain a subsequence that converges to zero. Therefore, we conclude that $limX = 0$.

□

3.5.9

Proof.

$$\begin{aligned} m > n \implies |x_m - x_n| &< r^n + r^{n+1} + \ldots + r^{m-1} \\ &\leq \frac{r^n}{1-r} \\ &< \epsilon \ \ \forall \epsilon > 0 \ and \ n \geq H(\epsilon) \ \text{(Because} \lim \frac{r^n}{1-r} = 0\text{)}. \end{aligned}$$

$\implies (x_n)$ is a Cauchy sequence.

□

3.7.9

a.

Proof.
The sequence (cos(n)) does not converge to zero.
$\implies \sum_{n=1}^{\infty} cos(n)$ is divergent.

□

b.

Proof.
Let $X := (\frac{1}{n^2})$ and $Y = (\frac{cos(n)}{n^2})$.
$\implies X$ is convergent and $x_n < y_n$
$\implies Y$ is convergent (By the comparison test).

□

4.2.14

Proof.
$\lim_{x \to c} f = L \implies |f(x) - L| < \epsilon$

And we have: $|f(x) - L| \geq ||f(x)| - |L||$

Therefore: $||f(x)| - |L|| < \epsilon$

This implies that $\lim_{x \to c} |f| = |L|$

□

5.4.2

<u>On A = [1, ∞):</u>

Proof.
Let $x, u \geq 1$.

$$\begin{aligned}
\implies |f(x) - f(u)| &= |\frac{1}{x^2} - \frac{1}{u^2}| \\
&= |\frac{u^2 - x^2}{u^2x^2}| \\
&= |\frac{(u+x)(u-x)}{u^2x^2}| \\
&= |\frac{(u+x)(u-x)}{u^2x^2}| \\
&= \frac{(u+x)}{u^2x^2}|u-x| \\
&= (\frac{1}{ux^2} + \frac{1}{u^2x})|u-x| \\
&\leq 2|u-x|
\end{aligned}$$

$\implies \forall \epsilon > 0 \ \exists \delta(\epsilon) = \frac{\epsilon}{2} : (\forall x, u \in A : |x - u| < \delta \implies |f(x) - f(u)| < \epsilon)$
$\implies f$ is uniformly continuous on A.

□

<u>On B = [0, ∞):</u>

Proof.
Take $x = \frac{1}{n}$ and $u = \frac{1}{n+1}$.
$\implies |x - u| = |\frac{1}{n} - \frac{1}{n+1}| = |\frac{1}{n^2+n}|$
$\implies lim(|x - u|) = 0.$

But we also have:
$|f(x) - f(u)| = |n^2 - (n+1)^2| = |n^2 - n^2 - 2n - 1| = 2n + 1 \geq 1.$

This shows that if we take ϵ to be any value < 1, then there is no corresponding δ that can satisfy the condition $|x - u| < \delta \implies |f(x) - f(u)|$.
$\implies f$ is not uniformly continuous on B.

□

6.2.14

Proof. (By Contradiction)
Suppose there exists $a, b \in I$ such that $f'(a) > 0$ and $f'(b) < 0$.

$$\implies \exists c \in I : f'(c) = 0 \qquad \text{(By Darboux's theorem)}$$

This is a contradiction. Therefore, we must conclude that either $f'(x) > 0 \ \forall x \in I$ or $f'(x) < 0 \ \forall x \in I$.

□

6.2.15

Proof.
Let $f'(x) \leq C$, and $x_1, x_2 \in I$.

$$\begin{aligned} \implies \exists z \in I : |f(x_1) - f(x_2)| &= |f'(z)(x_1 - x_2)| \qquad \text{(By the Mean Value Theorem)} \\ &\leq C|x_1 - x_2| \end{aligned}$$

$\implies f$ satisfies the Lipschitz condition.

□

7.2.18

Proof.
Let $M := sup(f)$ and $p \in [a, b]$ such that $f(p) = M$.

Since f is continuous at p, then given an $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\begin{aligned} |x - p| \leq \delta &\implies |f(x) - f(p)| \leq \epsilon \\ &\implies f(p) - \epsilon \leq f(x) \leq f(p) \\ &\implies M - \epsilon \leq f(x) \leq M \\ &\implies (M - \epsilon)^n \leq f(x)^n \leq M^n \\ &\implies 2\delta(M - \epsilon)^n \leq \int_{p-\delta}^{p+\delta} f(x)^n \leq (b - a)M^n \\ &\implies 2\delta(M - \epsilon)^n \leq \int_{p-\delta}^{p+\delta} f(x)^n \leq \int_a^b f(x)^n \leq (b - a)M^n \\ &\implies 2\delta(M - \epsilon)^n \leq \int_a^b f(x)^n \leq (b - a)M^n \\ &\implies (2\delta)^{\frac{1}{n}}(M - \epsilon) \leq \left(\int_a^b f(x)^n\right)^{\frac{1}{n}} \leq (b - a)^{\frac{1}{n}} M \\ &\implies (M - \epsilon) \leq M_n \leq M. \end{aligned}$$

$\implies \lim M_n = M.$

□

7.4.7

a.

Proof.
Let $P = [0, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, 1]$

$$\implies L(g;P) = 0 * (\frac{1}{2} - \epsilon) + 0 * (2\epsilon) + 1 * (\frac{1}{2} - \epsilon) = \frac{1}{2} - \epsilon$$
$$U(g;P) = 0 * (\frac{1}{2} - \epsilon) + 1 * (2\epsilon) + 1 * (\frac{1}{2} - \epsilon) = \frac{1}{2} + \epsilon$$

$\implies U(g;P) - L(g;P) = 2\epsilon$
$\implies$ The Darboux integral of g on $[0,1]$ is $\frac{1}{2}$.

□

b.

Let $P = [0, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, 1]$

$$\implies L(g;P) = 0 * (\frac{1}{2} - \epsilon) + 0 * (2\epsilon) + 1 * (\frac{1}{2} - \epsilon) = \frac{1}{2} - \epsilon$$
$$U(g;P) = 0 * (\frac{1}{2} - \epsilon) + 13 * (2\epsilon) + 1 * (\frac{1}{2} - \epsilon) = \frac{1}{2} + 25\epsilon$$

$\implies U(g;P) - L(g;P) = 26\epsilon$
$\implies$ g is Darboux integrable on $[0,1]$ with an integral value of $\frac{1}{2}$.

8.1.23

Proof.
Let M be the larger of the upper bounds on f_n and g_n.

$$\begin{aligned}
\implies |f_n g_n - fg| &= |f_n g_n + f_n g - f_n g - fg| \\
&\leq |f_n g_n - f_n g| + |f_n g - fg| \\
&= |f_n||g_n - g| + |g||f_n - f| \\
&\leq M|g_n - g| + M|f_n - f| \\
&< \epsilon.
\end{aligned}$$

$\implies lim(f_n g_n) = fg.$

□

Extras

Theorem 3.1.4 (Uniqueness of Limits)**.**
A sequence in $\mathbb{R}$ can have at most one limit.

Proof. (By contradiction)
Suppose for the sake of contradiction that x' and x'' are two limits of the sequence (x_n) and $x' \neq x''$.
$\implies \forall \epsilon > 0 : \; (\exists K' : \forall n \geq K' : |x_n - x'| < \epsilon) \wedge \; (\exists K'' : \forall n \geq K'' : |x_n - x''| < \epsilon)$

$$\begin{aligned} \implies \forall n \geq K = max(K', K'') : |x' - x''| &= |x' - x_n + x_n - x''| \\ &\leq |x' - x_n| + |x_n - x''| \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

$\implies \; x' = x''$, since we can make ϵ as small as we wish.

But this is a contradiction to our initial assumption that $x' \neq x''$. Therefore, we conclude that a sequence in $\mathbb{R}$ can have at most one limit.

□

Theorem 3.2.2.
A convergent sequence of real numbers is bounded.

Proof.
Let $\lim (x_n) = x$ and $\epsilon := 1$.
$\implies \exists K = K(1) : \forall n \geq K : |x_n - x| < \epsilon = 1$

$$\begin{aligned} \implies \; |x_n| &= |x_n - x + x| \\ &\leq |x_n - x| + |x| \\ &< 1 + |x|. \end{aligned}$$

Define $M := sup\{|x_1|, |x_2|, \ldots, |x_{K-1}|, 1 + |x|\}$.
$\implies \forall n \in \mathbf{N} : |x_n| \leq M$.
$\implies$ The sequence (x_n) is bounded.

□

Theorem 3.4.8 (Bolzano-Weierstrass Theorem)**.**
A bounded sequence of real numbers has a convergent subsequence.

Proof.
Let X be our bounded sequence.
By the Monotone Subsequence Theorem, X has a subsequence X' that is monotone.
Since X is bounded, then so is X'.
Since X' is monotone and bounded, then, by the Monotone Convergence Theorem, it is convergent.

□

Theorem 5.3.7 (Bolzano's Intermediate Value Theorem)**.**

Let I be an interval and let $f : I \to \boldsymbol{R}$ be continuous on I. If $a, b \in I$ and if $k \in \boldsymbol{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

Proof. (By Cases)

Case 1: $a < b$:

Define $g(x) := f(x) - k$.

$\implies g(a) < 0 < g(b)$

$\implies \exists c$ where $a < c < b : g(c) = 0$ (By the Location of Roots Theorem)

$\implies \exists c : f(c) = g(c) + k = k$.

Case 2: $b < a$:

Define $h(x) := k - f(x)$.

$\implies h(b) < 0 < h(a)$

$\implies \exists c$ where $b < c < a : g(c) = 0$ (By the Location of Roots Theorem)

$\implies \exists c : f(c) = k - h(c) = k$.

□

Theorem 6.2.12 (Darboux's Theorem)**.**

If f is differentiable on $I = [a, b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point c in (a, b) such that $f'(c) = k$.

Proof.

Suppose that $f'(a) < k < f'(b)$.

Define $g(x) := kx - f(x)$ for $x \in I$.

Since g is continuous, it attains a maximum value on I.

Since $g'(a) = k - f'(a) > 0$, then the maximum of g does not occur at $x = a$.

Similarly, since $g'(b) = k - f'(b) < 0$, then the maximum of g does not occur at $x = b$.

Therefore, by the Interior Extremum theorem, g attains its maximum at some point c in (a, b) where $g'(c) = k - f'(c) = 0$. Hence $f'(c) = k$.

□