# Functional Analysis Assignment (Chapter 1)

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## 1.1.1

The distance on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$. We must check that the 4 axioms (M1 to M4) are satisfied.

M1 holds, since the absolute value of the difference between 2 real points is real, finite, and non-negative.

M2 holds, since $d(x, y) = |x - y| = 0 \iff x = y$.

M3 holds, since $d(x, y) = |x - y| = |y - x| = d(y, x)$.

M4 holds, since the triangle inequality holds for the absolute value.

## 1.1.2

$d(x, y) = (x - y)^2$ is not a valid metric since it does not satisfy the triangle inequality.

*Proof.* (By Counterexample)
Let $a, b, c \in R$, where $a = 0, b = 1, c = 5$, then:

$d(a, c) = (a - c)^2 = (0 - 5)^2 = 25.$
$d(a, b) = (a - b)^2 = (0 - 1)^2 = 1.$
$d(b, c) = (b - c)^2 = (1 - 5)^2 = 16.$
$d(a, b) + d(b, c) = 1 + 16 = 17.$

Therefore, $d(a, c) \nleq d(a, b) + d(b, c)$.

□

## 1.1.3

Since the distance function $d(x, y) = |x - y|$ defines a metric on $\mathbb{R}$, as shown in 1.1.1, then it is clear that the square root of that metric is also real, finite, and non-negative (i.e. M1 holds); definite (i.e. M2 holds); and symmetric (i.e. M3 holds).

The triangle inequality can be shown to hold by noting that the square root function is an increasing function with a negative second derivative in the interval $(0, \infty)$.

This shows that $d(x, y) = \sqrt{|x - y|}$ is a metric on $\mathbb{R}$.

## 1.1.4

### i. $|X| = 2$

Let $X = \{a, b\}$, then $d$ must satisfy:

$d(a, a) = d(b, b) = 0$, and
$d(a, b) = d(b, a) = c$, where $c$ is any non-negative real number.

### ii. $|X| = 1$

In this case the only valid metric is $d(a, a) = 0$.

## 1.1.5

### i. Conditions for $kd$ to be a metric

If $d$ is a metric, then $kd$ automatically satisfies axioms M2-M4.

For axiom M1 to hold, $k$ must be a non-negative real number.

### ii. Conditions for $k + d$ to be a metric

To satisfy axiom M2, $k$ must be zero.

## 1.1.6

*Proof.* (By Induction on the Length of the Sequence)
Let $X = (x_j),\ Y = (y_j),\ Z = (z_j)$ be 3 bounded sequences.

**Base case**
Consider the subsequence of $X, Y, Z$ consisting of just their first element. Then by the triangle inequality for numbers:

$$\begin{aligned} & |x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \\ \implies & \sup |x_1 - z_1| \leq \sup(|x_1 - y_1| + |y_1 - z_1|) \\ \implies & \sup |x_1 - z_1| \leq \sup |x_1 - y_1| + \sup |y_1 - z_1|. \end{aligned}$$

Therefore $d(x, z) \leq d(x, y) + d(y, z)$ holds for sequences of length 1.

**Inductive step**
Next, we'll consider the sub-sequences of $X, Y, Z$ consisting of the first $n + 1$ elements. Suppose that the induction hypothesis holds for sequences of length $n$, i.e.:

$$\sup_{j \in \{1..n\}} |x_j - z_j| \leq \sup_{j \in \{1..n\}} |x_j - y_j| + \sup_{j \in \{1..n\}} |y_j - z_j|.$$

Then we can partition each sequence of length $n+1$ into 2 sub-sequences: the first sequence contains the first $n$ elements and the second contains the last element.

The distance between any 2 sequences of length $n+1$ then becomes:
$\max(\sup_{j\in 1..n} |x_j - z_j|,\ \sup |x_{n+1} - z_{n+1}|)$

Finally, applying the induction hypothesis we get:
$\max(\sup_{j\in 1..n} |x_j - z_j|,\ \sup |x_{n+1} - z_{n+1}|) \leq \max(\sup_{j\in 1..n} |x_j - y_j| + \sup_{j\in 1..n} |y_j - z_j|,\ \sup |x_{n+1} - y_{n+1}| + \sup |y_{n+1} - z_{n+1}|)$

$$\implies \sup_{j\in 1..n+1} |x_j - z_j| \leq \sup_{j\in 1..n+1} |x_j - y_j| + \sup_{j\in 1..n+1} |y_j - z_j|$$

Therefore $d(x,z) \leq d(x,y) + d(y,z)$ holds for sequences of any length.

$\square$

## 1.1.7

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Which is the discrete metric.

## 1.1.8

M1 holds because $|x(t) - y(t)|$ is a positive function, and the integral of a positive function is positive.

M2 holds because $|x(t) - x(t)| = 0$, and the integral of the zero function is 0.

M3 holds because $d(x,y) = \int_a^b |x(t) - y(t)| dt = \int_a^b |y(t) - x(t)| dt = d(y,x)$.

To show that M4 holds, let $x, y, z \in X$, then:

$$
\begin{aligned}
d(x,z) &= \int_a^b |x(t) - z(t)| \; dt \\
&= \int_a^b |(x(t) - y(t)) - (z(t) - y(t))| \; dt \\
&\geq \int_a^b (|x(t) - y(t)| - |z(t) - y(t)|) \; dt \qquad \text{(By the triangle inequality of absolute values)} \\
&= \int_a^b |x(t) - y(t)| dt - \int_a^b |z(t) - y(t)| dt \\
&= d(x,y) - d(z,y)
\end{aligned}
$$

$\implies d(x,y) \leq d(x,z) + d(z,y).$

## 1.2.3

*Proof.*

Let $(\zeta_j)$, $(\eta_j)$, and $(\theta_j) \in l^p$, where $(\zeta_j)$ is any point, and define $(\eta_j)$ and $(\theta_j)$ as follows:

$$\eta_j := \begin{cases} 1 & j \leq n \\ 0 & j > n \end{cases}$$

$$\theta_j := \begin{cases} \zeta_j & j \leq n \\ 0 & j > n \end{cases}$$

Applying the Cauchy-Schwarz inequality to $(\theta_j)$ and $(\eta_j)$, we get:

$$\begin{aligned}
\sum_{j=1}^{\infty} |\theta_j \eta_j| &\leq \sqrt{\sum_{k=1}^{\infty} |\theta_j|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_j|^2} \\
\sum_{j=1}^{n} |\theta_j \eta_j| &\leq \sqrt{\sum_{k=1}^{n} |\theta_j|^2} \sqrt{\sum_{m=1}^{n} |\eta_j|^2} \\
\sum_{j=1}^{n} |\theta_j| &\leq \sqrt{\sum_{k=1}^{n} |\theta_j|^2} \sqrt{\sum_{m=1}^{n} 1} \\
\sum_{j=1}^{n} |\zeta_j| &\leq \sqrt{\sum_{k=1}^{n} |\zeta_j|^2} \sqrt{n} \\
\left( \sum_{j=1}^{n} |\zeta_j| \right)^2 &\leq n \sum_{k=1}^{n} |\zeta_j^2|
\end{aligned}$$

□

## 1.2.4

## 1.2.5

*Proof.*

The sequence $\zeta_j = \frac{1}{j}$ is divergent for $p = 1$ but convergent for all $p > 1$.

□

## 1.2.11

*Proof.*

M1 holds because:

$$\begin{aligned}
&d(x,y) \geq 0 \\
\implies &\frac{d(x,y)}{1+d(x,y)} \geq 0 \\
\implies &\tilde{d}(x,y) \geq 0.
\end{aligned}$$

M2 holds because:
$\underline{\implies}$:

$$\begin{aligned}
&\tilde{d}(x,y) = 0 \\
\implies &\frac{d(x,y)}{1+d(x,y)} = 0 \\
\implies &d(x,y) = 0 \\
\implies &x = y.
\end{aligned}$$

$\underline{\impliedby}$:

$$\begin{aligned}
&x = y \\
\implies &d(x,y) = 0 \\
\implies &\frac{d(x,y)}{1+d(x,y)} = 0 \\
\implies &\tilde{d}(x,y) = 0.
\end{aligned}$$

M3 holds because:

$$\begin{aligned}
&d(x,y) = d(y,x) \\
\implies &\frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} \\
\implies &\tilde{d}(x,y) = \tilde{d}(y,x).
\end{aligned}$$

M4 holds because:

$$\begin{aligned}
&d(x,z) &&\leq d(x,y) + d(y,z) \\
\implies &\frac{d(x,z)}{1+d(x,z)} &&\leq \frac{d(x,y)}{1+d(x,z)} + \frac{d(y,z)}{1+d(x,z)} \\
\implies &\frac{d(x,z)}{1+d(x,z)} &&\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\
\implies &\tilde{d}(x,z) &&\leq \tilde{d}(x,y) + \tilde{d}(y,z).
\end{aligned}$$

□

### 1.3.4

*Proof.*
$\implies$ **(By Construction)**:

Let $A$ be a non-empty open subset, and $\delta(A)$ be the diameter of A.
Since $A$ is non-empty and open, then $|A| > 1$.
Let $a_1$, $a_2$ be 2 different points in $A$.
Consider the 2 open balls $B(a_1, \delta(A))$ and $B(a_2, \delta(A))$.
We have $B(a_1, \delta(A)) = B(a_2, \delta(A)) = A$.
Thus, $A = B(a_1, \delta(A)) \cup B(a_2, \delta(A))$.

$\impliedby$ **(By Strong Induction)**:

We will prove a **stronger** result: The union of non-empty open subsets is a non-empty open subset.
Let $B = (B_1, B_2, \ldots, B_n)$ be a list of open subsets (balls **or not**) in $X$.

Base case: $|B| = 2$

$$\begin{aligned} &A = B_1 \cup B_2 \\ \implies &\forall a \in A\,,\ a \in B_1 \lor a \in B_2 \\ \implies &\forall a \in A,\ \exists \text{ a ball about } a \\ \implies &A \text{ is a non-empty open subset of } X. \end{aligned}$$

Inductive step: $|B| = k > 2$

$$\begin{aligned} A &= \bigcup_{j=1}^{k} B_j \\ &= \bigcup_{j=1}^{k-1} B_j + B_n \\ &= C + B_k \end{aligned} \tag{1}$$

Where $C$ is a non-empty open subset by the induction hypothesis for $n = k - 1$.

Applying the induction hypothesis to (1) for $n = 2$, we conclude that $A$ is a non-empty open subset in $X$.

□

### 1.3.6

*Proof.* (By Construction)

We will prove this statement by recursively (and infinitely) applying the definition of an accumulation point on increasingly smaller $\epsilon$-neighborhoods.

Let the sequence of $(\epsilon_j)$ be such that $\forall j,\ \epsilon_j > 0$.

Since $x_0$ is an accumulation point, then each $\epsilon_j$-neighborhood contains at least one point $y \in A$.

Let $y_j$ to be any point in the $\epsilon_j$-neighborhood of $x_0$ that satisfies the following condition:

$$d(y_j, x_0) = \max_{\forall y_k \in \tilde{B}(x_0;\epsilon_j)} d(y_k, x_0)$$

Define $\epsilon_j$ recursively as follows:

$$\epsilon_j := \begin{cases} \epsilon_1 & j = 1 \\ d(x_0, y_{j-1})/2 & j > 1 \end{cases}$$

Then the sequence $(y_j)$ is an infinite sequence of distinct points in A, all contained inside the $\epsilon_1$-neighborhood of $x_0$.

Since $\epsilon_1$ was arbitrarily chosen, this completes the proof.

□

## 1.3.13

*Proof.*
This follows directly from the definition that a separable space $X$ contains a subset $Y$ that is countable and dense in $X$.

□

## 1.3.14

*Proof.*
$\underline{\Longrightarrow}$:
Let $T : X \longrightarrow Y$ be a continuous map and $M$ be a closed subset in $Y$.

| | |
|---|---|
| $\Longrightarrow M^c$ is open | (Because the complement of a closed set is an open set by definition 1.3-2) |
| $\Longrightarrow T^{-1}(M^c)$ is open | (By continuity of T) |
| $\Longrightarrow (T^{-1}(M^c))^c$ is closed | (Because the complement of an open set is a closed set by definition 1.3-2) |
| $\Longrightarrow T^{-1}((M^c)^c)$ is closed | (By the identity: $f^{-1}(A^c) = [f^{-1}(A)]^c$) |
| $\Longrightarrow T^{-1}(M)$ is open. | (By the identity: $A = [A^c]^c$) |

$\underline{\Longleftarrow}$:

Let $T : X \longrightarrow Y$ be a map such that the inverse image of any closed set in $Y$ is a closed set in $X$.

Let $A$ be an open set in $Y$.

| | |
|---|---|
| $\Longrightarrow A^c$ is closed | (Because the complement of an open set is a closed set by definition 1.3-2) |
| $\Longrightarrow T(A^c)$ is closed | (By our assumption the inverse image of any closed set is a closed set) |
| $\Longrightarrow [T(A^c)]^c$ is open | (Because the complement of a closed set is an open set by definition 1.3-2) |
| $\Longrightarrow T([A^c]^c)$ is open | (By the identity: $f^{-1}(A^c) = [f^{-1}(A)]^c$) |
| $\Longrightarrow T(A)$ is open | (By the identity: $A = [A^c]^c$) |
| $\Longrightarrow T$ is continuous. | |

$\square$

### 1.4.2

*Proof.*

$$
\begin{aligned}
&(x_{n_k}) \text{ converges to x} \\
\implies &\forall \epsilon > 0,\ \exists K(\epsilon) \ni \forall k > K,\ d(x_{n_k}, x) < \epsilon \\
\implies &\forall \epsilon > 0,\ \exists N(\epsilon) = K(\epsilon) \ni \forall n > N,\ d(x_n, x) < \epsilon \quad \text{(Because } (x_n) \text{ is Cauchy)} \\
\implies &(x_n) \text{ converges to } x.
\end{aligned}
$$

□

### 1.4.4

*Proof.*
Take $\epsilon$ with any concrete value, say $\epsilon = 1$.

Because $(x_n)$ is Cauchy, then:

$$\exists N \ni \forall n, m > N, d(x_n, x_m) < \epsilon = 1.$$

Define $a = \max\limits_{\forall i,j \in \{1,\ldots,N\}} d(x_i, x_j)$.

Therefore we have $\forall n$:

$$d(x_n, x_N) \leq \max(1, a) \leq 1 + a$$

By the triangle inequality we have $\forall n, m$:

$$
\begin{aligned}
d(x_n, d_m) &\leq d(x_n, x_N) + d(x_N, x_m) \\
&\leq (1 + a) + (1 + a) = 2(1 + a) = u.
\end{aligned}
$$

This shows that $u$ is an upperbound for the Cauchy sequence.

□

### 1.4.5

*Proof.* (By Counterexample)
i. Boundedness does **not** imply Cauchiness:
Consider the sequence $x_n = (-1)^n$.
$(x_n)$ is bounded by 2, but the sequence is not Cauchy, because for any $0 < \epsilon < 2$:

$$\nexists N \ni \forall n, m > N,\ d(x_n, x_m) < \epsilon.$$

ii. Boundedness does **not** imply Convergence:
Consider the sequence $x_n = sin(n)$.
$(x_n)$ is bounded by 2, but the sequence oscillates and does not converge.

□

### 1.4.6

*Proof.*
Take any $\epsilon > 0$, then since $(x_n)$ and $(y_n)$ are Cauchy:

$$\exists N_x \ni \forall n, m > N,\ d(x_n, x_m) < \epsilon$$

and,

$$\exists N_y \ni \forall n, m > N,\ d(y_n, y_m) < \epsilon.$$

Let $N = \max\{N_x, N_y\}$, then $\forall n > N$:

$$d(x_n, x_N) < \epsilon \ \wedge\ d(y_n, y_N) < \epsilon.$$

By the triangle inequality we have $\forall n > N$:

$$\begin{aligned} a_n = d(x_n, y_n) &\leq d(x_n, x_N) + d(x_N, y_N) + d(y_N, y_n) \\ &\leq \epsilon + c + \epsilon \\ &= 2\epsilon + c. \end{aligned}$$

Finally, $\forall n, m > N$:

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, a_N) + d(a_N, a_m) \\ &= |d(x_n, y_n) - d(x_N, y_N)| + |d(x_m, y_m) - d(x_N, y_N)| \\ &\leq |2\epsilon + c - c| + |2\epsilon + c - c| \\ &= |2\epsilon| + |2\epsilon| \\ &\leq 4\epsilon. \end{aligned}$$

Therefore $(a_n)$ is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is complete, then $(a_n)$ is convergent.

□

### 1.5.2

*Proof.*

Assume $(x_i)$ is a Cauchy sequence in $X$

$$\implies \forall \epsilon > 0,\ \exists N \ni \forall j, k \geq N:\ d(x_j, x_k) = \max_{i=1}^{n} |x_i^{(j)} - x_i^{(k)}| < \epsilon \tag{1}$$

$\implies \forall \epsilon > 0,\ \exists N \ni \forall j, k \geq N,\ \forall i \in \{1, \ldots, n\}:\ |x_i^{(j)} - x_i^{(k)}| < \epsilon$

$\implies \forall i \in \{1, \ldots, n\}:\ (x_i^{(1)}, x_i^{(2)}, \ldots)$ is a Cauchy sequence of real numbers

$\implies \forall i \in \{1, \ldots, n\}:\ (x_i^{(1)}, x_i^{(2)}, \ldots)$ converges to a limit point $x_i$ because $\mathbb{R}$ is complete.

Next, we define a candidate limit for $(x_i)$:

$$x = (x_1, \ldots, x_n).$$

Clearly, $x \in X$, and by (1) we have:

$$\forall j \geq N : d(x_j, x) = \epsilon.$$

This shows that $x$ is the limit of $(x_i)$ and proves completeness of $X$.

□

### 1.5.5

*Proof.*

Assume $(x_n)$ is a Cauchy sequence in X

$\implies\ \forall \epsilon > 0,\ \exists N \ni \forall n, m \geq N:\ d(x_n, x_m) = |x_n - x_m| < \epsilon.$

Take $\epsilon = 0.5$

$\implies\ \exists N \ni \forall n, m \geq N : |x_n - x_m| < 0.5$

$\implies\ \forall n, m \geq N : x_n = x_m$

$\implies\ (x_n) \longrightarrow x_N.$

□

### 1.5.6

*Proof.* (By Counterexample)

Take the sequence $x_n = n$.

First we show that it is Cauchy.

Given any $\epsilon > 0$, we can take $N(\epsilon) = \tan(\frac{\pi}{2} - \epsilon)$ so that $\forall n, m \geq N$:

$$\arctan(x_m), \arctan(x_n) \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$$

$$\begin{aligned}\implies d(x_n, x_m) &= |\arctan(x_n) - \arctan(x_m)| \\ &\le (\frac{\pi}{2} - \epsilon) - \frac{\pi}{2} \\ &= \epsilon.\end{aligned}$$

Therefore $(x_n)$ is Cauchy as desired.

Now we show that $(x_n)$ does not converge.
We observe that the sequence wants to converge to $\frac{\pi}{2}$, since:

$$\lim_{n\to\infty} d(x_n, \frac{\pi}{2}) = 0$$

But there is no element $x \in \mathbb{R}$, such that $\arctan(x) = \frac{\pi}{2}$.

Thus our metric space is **incomplete.**

□

## 1.5.8

*Proof.*

Let $(x_n)$ be a Cauchy sequence in $Y \subseteq [a, b]$, and let J = [a,b]

$$\implies \forall \epsilon > 0, \exists N \ni \forall n, m \ge N : d(x_n, x_m) = \max_{t\in J} |x_n(t) - x_m(t)| < \epsilon \quad (1)$$

$$\implies \forall \epsilon > 0, \exists N \ni \forall n, m \ge N,\ \forall t \in J : |x_n(t) - x_m(t)| < \epsilon$$

$$\implies \forall t_0 \in J :\ (x_1(t_0), x_2(t_0), \ldots) \text{ is a Cauchy sequence of real numbers}$$

$$\implies \forall t_0 \in J :\ (x_1(t_0), x_2(t_0), \ldots) \text{ converges to a limit point, say } x_{lim}(t_0)\text{, because } \mathbb{R} \text{ is complete}$$

Now, we define a candidate limit $x$ for $(x_n)$.

Define $x$ pointwise so that $\forall t_0 \in J :\ x(t_0) = x_{lim}(t_0)$.

From the defintion above, we obviously have: $x(a) = x(b)$.

From (1) with $m \to \infty$ we have $\forall n > N$:

$$\begin{aligned} d(x_n, x) = \max_{t\in J} |x_n(t) - x(t)| < \epsilon \\ \implies \forall t_0 \in J : |x_n(t_0) - x(t_0) < \epsilon \end{aligned}$$

This shows that $x_m$ converges to $x$ **uniformly** on $J$.

Since the $x_m$'s are continuous on $J$ and the convergence is uniform, the limit function $x$ is also continuous on $J$.

Because $x \in C[a,b]$ and $x(a) = x(b)$, then $x \in Y$, and thus $Y$ is complete.

$\square$

## 2.1.5

*Proof.*

$$
\begin{aligned}
& \sum_{i=1}^{n} \alpha_i x_i = 0_v \\
\implies & \sum_{i=1}^{n} \alpha_i t^i = 0_v \\
\implies & \forall i,\ \alpha_i = 0 \\
\implies & (x_1, \ldots, x_n) \text{ is linearly independent.}
\end{aligned}
$$

□

## 2.1.6

*Proof.* (By Contradiction)

Let $x \in X$.

Suppose $x$ has 2 different representations, $x_1$ and $x_2$, in the basis $(e_1, \ldots, e_n)$, where:

$$x_1 = \sum_{j=1}^{n} \alpha_j e_j \qquad \text{and} \qquad x_2 = \sum_{k=1}^{n} \beta_k e_k$$

Then we have:

$$
\begin{aligned}
& x_1 = x_2 \\
\implies & \sum_{j=1}^{n} \alpha_j e_j = \sum_{k=1}^{n} \beta e_k \\
\implies & \forall i,\ \alpha_i = \beta_i \qquad \text{(Because } (e_1, \ldots, e_n) \text{ is linearly independent.)}
\end{aligned}
$$

This is a contradiction to our assumption that $x_1$ and $x_2$ have different representations.

Thus, we conclude that every non-zero vector must have a unique representation in a given basis.

□

## 2.1.10

*Proof.*

i. $V = Y \cap Z$ is a subspace:

We must check 3 conditions:

1. $0_v \in V$

$$\begin{aligned} & Y \text{ and } Z \text{ are vector spaces} \\ \implies & 0_v \in Y \wedge 0_v \in Z \\ \implies & 0_v \in V = Y \cap Z. \end{aligned}$$

2. $v_1, v_2 \in V \implies v_1 + v_2 \in V$

$$\begin{aligned} & v1, v_2 \in V \\ \implies & v_1, v_2 \in Y \wedge v_1, v_2 \in Z \\ \implies & v_1 + v_2 \in Y \wedge v_1 + v_2 \in Z \\ \implies & v_1 + v_2 \in V = Y \cup Z. \end{aligned}$$

3. $k \in K$ and $v \in V \implies kv \in V$

$$\begin{aligned} & v \in V \\ \implies & v \in Y \wedge v \in Z \\ \implies & kv \in Y \wedge kv \in Z \\ \implies & kv \in V = Y \cup Z. \end{aligned}$$

ii. $V = Y \cup Z$ is not a subspace:

Consider the following counterexample:
Let $V = R^2$, $Y = \{(x, 0) : x \in R\}$, $Z = \{(0, y) : y \in R\}$, and $V = Y \cup Z$.
Clearly, $Y$ and $Z$ are subspaces of $R^2$.
Take $v_1 = (1, 0)$ and $v_2 = (0, 1)$.
We have $v_1, v_2 \in V$, but $v_1 + v_2 = (1, 1) \notin V$.
This shows that the union of subspaces fails to be closed under vector addition.

□

## 2.1.11

*Proof.*
We must check 3 conditions:
1. $0_v \in M$

$$\begin{aligned} & \text{Let } v \in M \\ \implies & 0v = 0_v \in M. \end{aligned}$$

2. $v_1, v_2 \in M \implies v_1 + v_2 \in M$

$$\begin{aligned} & v1, v_2 \in M \\ \implies & v_1 = \sum_{\forall m_i \in M} \alpha_i m_i \quad \text{and} \quad v_2 = \sum_{\forall m_j \in M} \beta_j m_j \\ \implies & v_1 + v_2 = \sum_{\forall m_k \in M} (\alpha_k + \beta_k) m_k = \sum_{\forall m_k \in M} \gamma_k m_k \\ \implies & v_1 + v_2 \in M. \end{aligned}$$

3. $k \in K$ and $v \in M \implies kv \in M$

$$\begin{aligned} & v \in V \\ \implies & v = \sum_{\forall m_i \in M} \alpha_i m_i \\ \implies & kv = \sum_{\forall m_i \in M} (k\alpha_i) m_i \\ \implies & kv = \sum_{\forall m_i \in M} \gamma_i m_i \\ \implies & kv \in M. \end{aligned}$$

$\square$

## 2.2.6

*Proof.*

Let $x = (\zeta_1, \zeta_2)$, and $y = (\eta_1, \eta_2)$.

<u>i. $||x||_1 = |\zeta_1| + |\zeta_2|$</u>

N1 holds because:

$$||x||_1 = |\zeta_1| + |\zeta_2| \geq 0.$$

N2 holds because:

<u>$\implies$ :</u>

$$\begin{aligned} &||x||_1 = 0 \\ \implies &|\zeta_1| + |\zeta_2| = 0 \\ \implies &|\zeta_1|, |\zeta_2| = 0 \\ \implies &\zeta_1, \zeta_2 = 0 \\ \implies &x = (0, 0) = 0_v. \end{aligned}$$

<u>$\impliedby$ :</u>

$$\begin{aligned} &x = 0 \\ \implies &\zeta_1, \zeta_2 = 0 \\ \implies &|\zeta_1|, |\zeta_2| = 0 \\ \implies &|\zeta_1| + |\zeta_2| = 0 \\ \implies &||x||_1 = 0 \end{aligned}$$

N3 holds because:

$$\begin{aligned} ||\alpha x||_1 &= |\alpha\zeta_1| + |\alpha\zeta_2| \\ &= |\alpha||\zeta_1| + |\alpha||\zeta_2| \\ &= |\alpha|(|\zeta_1| + |\zeta_2|) \\ &= |\alpha| \, ||x||_1 \end{aligned}$$

N4 holds because:

$$\begin{aligned} ||x + y||_1 &= |\zeta_1 + \eta_1| + |\zeta_2 + \eta_2| \\ &\leq |\zeta_1| + |\eta_1| + |\zeta_2| + |\eta_2| \\ &= (|\zeta_1| + |\zeta_2|) + (|\eta_1| + |\eta_2|) \\ &= ||x||_1 + ||y||_1. \end{aligned}$$

ii. $||x||_2 = (\zeta_1^2 + \zeta_2^2)^{\frac{1}{2}}$

N1 holds because:

$$||x||_2 = (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \geq 0.$$

N2 holds because:
$\implies$ :

$$\begin{aligned}
&||x||_2 = 0 \\
\implies &(|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} = 0 \\
\implies &|\zeta_1|^2 + |\zeta_2|^2 = 0 \\
\implies &|\zeta_1|^2, |\zeta_2|^2 = 0 \\
\implies &|\zeta_1|, |\zeta_2| = 0 \\
\implies &\zeta_1, \zeta_2 = 0 \\
\implies &x = (0,0) = 0_v.
\end{aligned}$$

$\impliedby$ :

$$\begin{aligned}
&x = 0 \\
\implies &\zeta_1, \zeta_2 = 0 \\
\implies &|\zeta_1|, |\zeta_2| = 0 \\
\implies &|\zeta_1|^2, |\zeta_2|^2 = 0 \\
\implies &|\zeta_1|^2 + |\zeta_2|^2 = 0 \\
\implies &(|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} = 0 \\
\implies &||x||_2 = 0
\end{aligned}$$

N3 holds because:

$$\begin{aligned}
||\alpha x||_2 &= (|\alpha\zeta_1|^2 + |\alpha\zeta_2|^2)^{\frac{1}{2}} \\
&= (|\alpha|^2|\zeta_1|^2 + |\alpha|^2|\zeta_2|^2)^{\frac{1}{2}} \\
&= |\alpha|(|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \\
&= |\alpha| \; ||x||_2
\end{aligned}$$

N4 holds because:

$$\begin{aligned}
||x+y||_2^2 &= (\zeta_1+\eta_1)^2+(\zeta_2+\eta_2)^2 \\
&= \zeta_1^2+2\zeta_1\eta_1+\eta_1^2+\zeta_2^2+2\zeta_2\eta_2+\eta_2^2 \\
&= (\zeta_1^2+\zeta_2^2)+(\eta_1^2+\eta_2^2)+2(\zeta_1\eta_1+\zeta_2\eta_2) \\
&\leq (\zeta_1^2+\zeta_2^2)+(\eta_1^2+\eta_2^2)+2\sqrt{\zeta_1^2+\zeta_2^2}\sqrt{\eta_1^2+\eta_2^2} \qquad \text{(By the Cauchy-Schwartz inequality)} \\
&= ||x||_2^2+||y||_2^2+2||x||_2\,||y||_2 \\
&= (||x||_2+||y||_2)^2
\end{aligned}$$

$$\implies ||x+y||_2 \leq ||x||_2+||y||_2.$$

iii. $||x||_\infty = \max\{|\zeta_1|,|\zeta_2|\}$

N1 holds because:

$$||x||_\infty = \max\{|\zeta_1|,|\zeta_2|\} \geq 0.$$

N2 holds because:
$\implies$ :

$$\begin{aligned}
&||x||_\infty = 0 \\
\implies &\max\{|\zeta_1|,|\zeta_2|\} = 0 \\
\implies &|\zeta_1|,|\zeta_2| = 0 \\
\implies &\zeta_1,\zeta_2 = 0 \\
\implies &x = (\zeta_1,\zeta_2) = (0,0) = 0_v.
\end{aligned}$$

$\impliedby$ :

$$\begin{aligned}
&x = 0 \\
\implies &\zeta_1,\zeta_2 = 0 \\
\implies &|\zeta_1|,|\zeta_2| = 0 \\
\implies &\max\{|\zeta_1|,|\zeta_2|\} = 0 \\
\implies &||x||_\infty = 0
\end{aligned}$$

N3 holds because:

$$\begin{aligned}
||\alpha x||_\infty &= \max\{|\alpha\zeta_1|,|\alpha\zeta_2|\} \\
&= |\alpha|\ \max\{|\zeta_1|,|\zeta_2|\} \\
&= |\alpha|\ ||x||_\infty
\end{aligned}$$

N4 holds because:

$$\begin{aligned}||x+y||_\infty &= \max\{|\zeta_1+\eta_2|, |\zeta_2+\eta_2|\}\\ &\leq \max\{|\zeta_1|+|\eta_2|, |\zeta_2|+|\eta_2|\}\\ &\leq \max\{|\zeta_1|, |\zeta_2|\}+\max\{|\eta_1|, |\eta_2|\}\\ &= ||x||_\infty+||y||_\infty.\end{aligned}$$

□

## 2.2.8

## 2.6.2

*Proof.*

Let $v_1, v_2 \in R^2$, where $v_1 = (\zeta_1, \zeta_2)$, $v_2 = (\eta_1, \eta_2)$, and $\alpha \in R$.

i. $\underline{T_1 : (\zeta_1, \zeta_2) \longmapsto (\zeta_1, 0)}$

Additivity:

$$\begin{aligned} T_1(v_1 + v_2) &= T_1(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_1 + \eta_1, 0) \\ &= (\zeta_1, 0) + (\eta_1, 0) \\ &= T_1 v_1 + T_1 v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_1(\alpha v_1) &= T_1(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\alpha\zeta_1, 0) \\ &= \alpha(\zeta_1, 0) \\ &= \alpha T_1 v_1. \end{aligned}$$

Geometric interpretation: Projection onto the x-axis.

ii. $\underline{T_2 : (\zeta_1, \zeta_2) \longmapsto (0, \zeta_2)}$

Proof is similar to part (i).

Geometric interpretation: Projection onto the y-axis.

iii. $\underline{T_3 : (\zeta_1, \zeta_2) \longmapsto (\zeta_2, \zeta_1)}$

Additivity:

$$\begin{aligned} T_3(v_1 + v_2) &= T_3(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_2 + \eta_2, \zeta_1 + \eta_1) \\ &= (\zeta_2, \zeta_1) + (\eta_2, \eta_1) \\ &= T_3 v_1 + T_3 v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_3(\alpha v_1) &= T_3(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\alpha\zeta_2, \alpha\zeta_1) \\ &= \alpha(\zeta_2, \zeta_1) \\ &= \alpha T_3 v_1 \end{aligned}$$

Geometric interpretation: Reflection across the line $y = x$.

iv. $T_4 : (\zeta_1, \zeta_2) \longmapsto (\gamma\zeta_1, \gamma\zeta_2)$

Additivity:

$$\begin{aligned} T_4(v_1 + v_2) &= T_4(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\gamma(\zeta_1 + \eta_1), \gamma(\zeta_2 + \eta_2)) \\ &= (\gamma\zeta_1, \gamma\zeta_2) + (\gamma\eta_1, \gamma\eta_2) \\ &= T_4 v_1 + T_4 v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_4(\alpha v_1) &= T_4(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\gamma\alpha\zeta_1, \gamma\alpha\zeta_2) \\ &= \alpha(\gamma\zeta_1, \gamma\zeta_2) \\ &= \alpha T_4 v_1 \end{aligned}$$

□

## 2.6.6

*Proof.*

Let $X, Y, Z$ be vector spaces over the same field $K$.
Let $T : X \longrightarrow Y$, $S : Y \longrightarrow Z$.
Then, the composite operator $ST$ exists and $ST : X \longrightarrow Z$.

Let $x_1, x_2 \in X$, and $\alpha \in K$.

Additivity:

$$\begin{aligned} (ST)(x_1 + x_2) &= S(T(x_1 + x_2)) \\ &= S(Tx_1 + Tx_2) \\ &= (ST)x_1 + (ST)x_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} (ST)(\alpha x_1) &= S(T(\alpha x_1)) \\ &= S(\alpha T x_1) \\ &= \alpha(ST)x_1. \end{aligned}$$

□

### 2.7.2

*Proof.*

$\underline{\Longrightarrow:}$

Let $T : X \longrightarrow Y$ be a bounded linear operator, and $B_x$ be a bounded set in $X$.

Since $B_x$ is bounded, then $\forall x \in B_x$:

$$||x|| \leq c_x.$$

Let $B_y$ be the image of $B_x$ under $T$: $B_y = T(B_x)$.

Since $T$ is bounded, then $\forall y \in B_y$:

$$\begin{aligned} ||y|| &= ||Tx|| \\ &\leq ||T||\,||x|| \\ &\leq ||T||c_x = c_y. \end{aligned}$$

Thus, $B_y$ is bounded.

$\underline{\Longleftarrow:}$

Suppose $T$ maps bounded sets in $X$ into bounded sets $Y$.

Let $B_x = \{x \in X : ||x|| = 1\}$.
$\Longrightarrow$ $B_x$ is bounded.

Let $B_y$ be the image of $B_x$ under $T$: $B_y = T(B_x)$.
$\Longrightarrow$ $B_y$ is bounded
$\Longrightarrow$ $\forall y \in B_y :\ ||y|| \leq c$
$\Longrightarrow$ $\forall x \in B_x : ||Tx|| \leq c$
$\Longrightarrow$ $T$ is bounded by *Lemma* $2.7-2$.

□

### 2.7.3

*Proof.*

Let $x \in X$ s.t. $||x|| = \alpha < 1$, then:

$$\begin{aligned}
&y = \frac{1}{\alpha}x \\
&\implies ||y|| = \frac{1}{\alpha}||x|| = 1 \\
&\implies ||Ty|| \leq ||T|| \\
&\implies ||T(\frac{1}{\alpha}x)|| \leq ||T|| \\
&\implies \frac{1}{\alpha}||Tx|| \leq ||T|| \\
&\implies ||Tx|| \leq \alpha||T|| < ||T||.
\end{aligned}$$

$\square$

## 3.1.4

*Proof.*

$$\begin{aligned}\langle x+y, x-y\rangle &= \langle x, x-y\rangle + \langle y, x-y\rangle \\ &= \langle x,x\rangle - \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle \\ &= 0 - \langle x,y\rangle + \langle x,y\rangle - 0 = 0 \\ &= 0.\end{aligned}$$

Geometric interpretation for $X = \mathbb{R}^2$:

Since $||x|| = ||y||$, then the vectors $x$ and $y$ form a rhombus. The vectors $x+y$ and $x-y$ are the diagonals of that rhombus.

Therefore, this statement simply states that the diagonals of a rhombus are perpendicular/orthogonal.

Geometric interpretation for $X = \mathbb{C}^1$:

Same as $X = \mathbb{R}^2$, since $\mathbb{R}^2$ and $\mathbb{C}^1$ have the same geometry.

□

## 3.1.7

*Proof.* (By Contradiction)

Let $u, v \in V$ s.t. $u \neq 0_v$ and $v := au$, where $a \neq 1$. Then $u \neq v$, and:

$$\begin{aligned} &\forall x \in X : \langle x, u\rangle = \langle x, v\rangle \\ \Longrightarrow &\forall x \in X : \langle x, u\rangle = \langle x, au\rangle \\ \Longrightarrow &\forall x \in X : \langle x, u\rangle = \bar{a}\langle x, u\rangle \end{aligned}$$

But this is impossible since $a \neq 1$. Thus, we must have $u = v$.

□

## 3.1.11

*Proof.*

To check if the given norm is induced by some inner-product, we could check whether the parallelogram equality holds, but there's an even simpler proof.

First, we note that this space is an $l^p$ space with $p = 1$.

Finally, we note that it was shown in $3.1 - 7$ that the space $l^p$ with $p \neq 2$ is not an inner-product space.

Therefore, there is no inner-product that induces the given norm.

□

## 3.2.1

The Euclidean inner product on $\mathbb{R}^2$ and $\mathbb{R}^3$ reduces to the dot product.

The dot product of 2 vectors is the product of their length times cosine the angle between them:

$$\begin{aligned}\langle x, y\rangle &= x \cdot y \\ &= ||x||\,||y|| \cos(\theta)\end{aligned}$$

Taking the absolute value of both sides we get:

$$\begin{aligned}|\langle x, y\rangle| &= ||x||\,||y||\,|\cos(\theta)| \\ &\leq ||x||\,||y||\end{aligned}$$

Which is the same result given by the Schwarz inequality.

## 3.2.2

i. The trivial subspace containing just the zero sequence.
ii. Subspaces whose components are all 0 after some index $n$.

## 3.2.4

*Proof.*
By the continuity of the inner product (Lemma 3.2-2), we have:

$$\langle x_n, y\rangle \to \langle x, y\rangle \tag{1}$$

Next, we define the sequence $(a_n)$ by: $a_n := \langle x_n, y\rangle$. Then:

$$\begin{aligned}&\forall n \in \mathbb{N}:\ a_n = 0 \\ \implies &a_n \to 0 \\ \implies &\langle x_n, y\rangle \to 0 \qquad (2)\end{aligned}$$

Finally:

$$\begin{aligned}(1) \text{ and } (2) &\implies \langle x, y\rangle \to 0 \\ &\implies x \perp y.\end{aligned}$$

□

## 3.2.5

*Proof.*
To prove that $x_n \to x$, we will show that $||x_n - x|| \to 0$:

$$\begin{aligned} ||x_n - x|| &= \langle x_n - x, x_n - x \rangle \\ &= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle \\ &= ||x_n||^2 - \langle x_n, x \rangle - \langle x, x_n \rangle + ||x_n||^2 \end{aligned}$$

Taking the limit as $n \to \infty$:

$$||x_n - x|| \to 2||x||^2 - 2\langle x, x \rangle = 2||x||^2 - 2||x||^2 = 0.$$

□

## 3.2.8

*Proof.*
$\underline{\implies}$:

$$\begin{aligned} &\langle x, y \rangle = 0 \\ \implies &\forall \bar{\alpha} : \ \bar{\alpha}\langle x, y \rangle = 0 \\ \implies &\forall \alpha : \ \langle x, \alpha y \rangle = 0 \\ \implies &\forall \alpha : \ \langle x, \alpha y \rangle = 0 \ \wedge \ \langle \alpha y, x \rangle = 0 \end{aligned}$$

Next, we have $\forall \alpha$:

$$\begin{aligned} ||x + \alpha y||^2 &= \langle x + \alpha y, x + \alpha y \rangle \\ &= \langle x, x \rangle + \langle x, \alpha y \rangle + \langle \alpha y, x \rangle + \langle y, y \rangle \\ &= ||x||^2 + 0 + 0 + ||y||^2 \\ &\geq ||x||^2 \end{aligned}$$

Finally, taking the square root, we arrive at our desired result:

$$||x + \alpha y|| \geq ||x||.$$

$\underline{\Longleftarrow}$:

$$\begin{aligned}
&\forall\alpha:\ ||x|| &&\leq ||x+\alpha y||\\
\Longrightarrow\ &\forall\alpha:\ ||x||^2 &&\leq ||x+\alpha y||^2\\
& &&= \langle x+\alpha y, x+\alpha y\rangle\\
& &&= \langle x,x\rangle + \langle x,\alpha y\rangle + \langle \alpha y, x\rangle + \langle y,y\rangle\\
& &&= ||x||^2 + \overline{\langle \alpha y, x\rangle} + \langle \alpha y, x\rangle + ||y||^2\\
& &&= ||x||^2 + 2Re\langle \alpha y, x\rangle + ||y||^2\\
& &&= ||x||^2 + 2Re(\alpha\langle y, x\rangle) + ||y||^2\\
\Longrightarrow\ &\forall\alpha:\ 2Re\langle\alpha y, x\rangle \leq 0\\
\Longrightarrow\ &\langle x, y\rangle = 0.
\end{aligned}$$

$\square$

### 4.8.1

*Proof.* ∎

## 5.1.2

*Proof.*
Let $x, y \in X$

$$\begin{aligned}
d(Tx, Ty) &= |Tx - Ty| \\
&= |\frac{x}{2} + x^{-1} - \frac{y}{2} - y^{-1}| \\
&= |\frac{x-y}{2} + \frac{y-x}{xy}| \\
&= |x-y|\,|\frac{1}{2} - \frac{1}{xy}| \\
&\leq \frac{1}{2}\,|x-y| \\
&= \frac{1}{2}d(x, y)
\end{aligned}$$

Therefore, $T$ is a contraction and the smallest $\alpha$ is $\frac{1}{2}$.

□

## 5.1.5

*Proof.* (By Contradiction)
Suppose $T$ has 2 different fixed points $x$ and $y$, then:

$$\begin{aligned}
&x = Tx \wedge y = Ty \\
\implies &d(x, y) = d(Tx, Ty) \qquad (1)
\end{aligned}$$

But we are given that:

$$d(Tx, Ty) < d(x, y) \qquad (2)$$

(1) and (2) $\implies d(x, y) < d(x, y)$.

This is a contradiction. Thus, $T$ has a unique fixed point.

□

## 5.1.6

<u>*i.* $T$ is a contraction $\implies T^n$ is a contraction:</u>

*Proof.* (By Induction)
<u>Base case ($n = 1$):</u>
$T^1 = T$ which is a contraction by assumption.

<u>Inductive step ($n = k + 1$):</u>

Suppose the induction hypothesis holds for $n = k$, and let $x, y \in X$, then:

$$d(T^k x, T^k y) \leq \beta\, d(x, y) \qquad \text{(where } 0 < \beta < 1\text{)} \tag{1}$$

Next, let $T^k x = u$ and $T^k y = v$. Because $T$ is a contraction then:

$$d(Tu, Tv) \leq \alpha\, d(u, v) \qquad \text{(where } 0 < \alpha < 1\text{)} \tag{2}$$

Putting (1) and (2) together, we get:

$$\begin{aligned} d(T^{k+1}x, T^{k+1}y) &= d(Tu, Tv) \\ &\leq \alpha\, d(u, v) \\ &\leq \alpha\, \beta\, d(x, y) \\ &= \gamma\, d(x, y) \qquad \text{(where } 0 < \gamma < 1\text{)} \end{aligned}$$

Therefore, $T^{k+1}$ is a contraction.

□

*ii.* $T^n$ is a contraction $\not\Longrightarrow$ $T$ is a contraction:

*Proof.* (By Counterexample)

Let the mapping $T^n : R \longrightarrow R$ be defined by:

$$Tx := \begin{cases} -x, & \text{if } x \geq 0 \\ \frac{x}{2}, & \text{if } x < 0 \end{cases}$$

Then, $T$ does not contract the distance between positive numbers.
However, $T^2$ is a contraction.

□