Functional Analysis (MTH414) Finals Questions Bank

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<u>Definitions</u>

Definition 1 (Metric space)**.** A metric space is an ordered pair (X, d), where X is a set, and $d : X \times X \longrightarrow R$ is a distance function such that $\forall x, y, z \in X$:

$M1$ (Non-negativity) :	$d(x, y) \geq 0$
$M2$ (Definiteness) :	$d(x, x) = 0 \iff x = 0$
$M3$ (Symmetry) :	$d(x, y) = d(y, x)$
$M4$ (Triangle inequality) :	$d(x, z) \leq d(x, y) + d(y, z)$

Note: To save time, you could just mention that d satisfies the axioms of non-negativity, definiteness, symmetry, triangle inequality.

Definition 2 (Open set)**.** An open set is a set X where $\forall x \in X$:

$$\exists r : \; B(x, r) \text{ is wholly contained inside } X.$$

Alternatively:

A subset M of a metric space X is said to be open if it contains a ball about each of its points.

Definition 3 (Closed set)**.** A closed set is a set that is <u>not</u> open.
Alternatively:

A subset K of X is said to be closed if its complement (in X) is open.

Definition 4 (Interior point)**.** A point x is an interior point of a set $M \subseteq X$ if M is a neighborhood of x.

Definition 5 (Dense set)**.** A subset M of a metric space X is dense in X if $\overline{M} = X$.

Definition 6 (Separable space)**.** A space X is separable if it contains a <u>countable subset</u> M which is <u>dense</u> in X

Definition 7 (Complete space)**.** A space X is complete if every Cauchy sequence in X converges.

Definition 8 (Finite dimensional vector space)**.** A vector space V is finite dimensional if $\exists n \in N$, such that X contains a set of n linearly independet vectors, whereas any set of $n + 1$ vectors is linearly dependent.

Definition 9 (Infinite dimensional vector space)**.** A vector space V is infinite dimensional if it is <u>not</u> finite dimensional.

Definition 10 (Normed space)**.** Is an ordered pair $(X, ||)$, where X is a vector space, and $|| : X \longrightarrow R$ is a norm function such that $\forall x, y \in X$:

$N1$ (Non-negativity) :	$\|\|x\|\| \geq 0$
$N2$ (Definiteness) :	$\|\|x\|\| = 0 \iff x = 0$
$N3$ (Homogeneity) :	$\|\|\alpha\, x\|\| = \alpha\, \|\|x\|\|$
$N4$ (Triangle inequality) :	$\|\|x + y\|\| \leq \|\|x\|\| + \|\|y\|\|$

Definition 11 (Banach space)**.** Is a complete normed space.

Definition 12 (Hilbert space)**.** Is a complete inner product space.

Definition 13 (Linear operator)**.** A linear operator $T : X \longrightarrow Y$ is a mapping between vector spaces such that $\forall x, y \in X$ and $\forall \alpha \in K$:

$$T(\alpha\, x + y) = \alpha\, Tx + Ty$$

Definition 14 (Linear functional)**.** A linear functional $f : X \longrightarrow K$ is a linear operator whose domain is a vector space X and whose range is the scalar field K of X.

Definition 15 (Bounded linear operator)**.** A linear operator $T : X \longrightarrow Y$ between normed vector spaces is bounded if $\exists c \in R$ such that:

$$\forall x \in X :\ ||Tx|| \leq c||x||$$

Definition 16 (Inner product space)**.** Is an ordered pair $(X, \langle\rangle)$, where X is a vector space, and $\langle\rangle : X \times X \longrightarrow R$ is an inner-product function such that $\forall x, y, z \in X$:

$$\begin{array}{ll} IP1 \text{ (Non-negativity) :} & \langle x, x\rangle \geq 0 \\ IP2 \text{ (Definiteness) :} & \langle x, x\rangle = 0 \iff x = 0 \\ IP3.1 \text{ (Additivity) :} & \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \\ IP3.2 \text{ (Homogeneity) :} & \langle \alpha\, x, x\rangle = \alpha\, \langle x, x\rangle \\ IP4 \text{ (Conjugate symmetry) :} & \langle x, y\rangle = \overline{\langle y, x\rangle} \end{array}$$

Definition 17 (Strong convergence)**.** A sequence (x_n) in a normed space X converges strongly if $\exists x \in X$ such that:

$$\lim_{n\to\infty} ||x_n - x|| = 0.$$

Definition 18 (Weak convergence)**.** A sequence (x_n) in a normed space X converges weakly if $\exists x \in X$ such that $\forall f \in X'$:

$$\lim_{n\to\infty} f(x_n) = f(x).$$

True or false (without proof)

1. The sequence space l^∞ is a metric space. True
2. The complex plane $\mathbb{C}$ is separable. True
3. Every Cauchy sequence in a metric space is convergent sequence. False
4. The function sapce $C[a, b]$ is both a Banach space and a Hilbert space. False
5. The integral operator is a bounded linear operator. True
6. All normed spaces are inner product spaces. False
7. The space l^p with $p \neq 2$ is a Hilbert space. False

Proofs

Theorem 1 (Completeness of $\mathbb{R}^n$ and $\mathbb{C}^n$)**.**

Proof.
Let F denote either R or C. Then F is a complete field.

Let (x_m) be a Cauchy sequence in F^n, where $x_m = (x_1^{(m)}, \ldots, x_n^{(m)})$.

Since (x_m) is Cauchy, then $\forall \epsilon > 0$, $\exists N$ such that $\forall m, r \geq N$:

$$d(x_m, x_r) = \left(\sum_{j=1}^{n} (x_j^{(m)} - x_j^{(r)})^2 \right)^{1/2} < \epsilon$$

$\implies \forall j \in 1, \ldots, n: \ (x_j^{(m)} - x_j^{(r)})^2 < \epsilon^2$

$\implies \forall j \in 1, \ldots, n: \ |x_j^{(m)} - x_j^{(r)}| < \epsilon$

$\implies \forall j \in 1, \ldots, n:$ The sequence $(x_j^{(1)}, x_j^{(2)}, \ldots)$ is Cauchy

$\implies \forall j \in 1, \ldots, n:$ The sequence $(x_j^{(1)}, x_j^{(2)}, \ldots)$ converges (by the completeness of F).

Denote the limit of the above sequences by x_j.

Next, we define our candidate limit as follows:

$$x = (x_1, \ldots, x_n)$$

Finally, it is clear that $x \in F^n$, and $\forall m \geq N$:

$$d(x_m, x) < \epsilon$$

This shows that x is indeed the limit of (x_n), and proves completeness of F^n.

□

Theorem 2 (Completeness of the function space $C[a, b]$)**.**

Proof.

Let (x_m) be a Cauchy sequence in $C[a, b]$, and $J = [a, b]$.

Then, $\forall \epsilon > 0$, $\exists N$ such that $\forall m, n > N$:

$$d(x_m, x_n) = \max_{t \in J} |x_m(t) - x_n(t)| < \epsilon \tag{1}$$

$\implies \forall t_0 \in J :\ |x_m(t_0) - x_n(t_0)| < \epsilon$

$\implies \forall t_0 \in J :$ The sequence $(x_1(t_0), x_2(t_0), \ldots)$ is a Cauchy sequence of real numbers

$\implies \forall t_0 \in J :$ The sequence $(x_1(t_0), x_2(t_0), \ldots)$ converges (by completeness of R).

Denote the limit of the above sequences by $x_{lim}(t_0)$.

Next, we define our candidate limit x. We define x pointwise as follows:

$$\forall t \in J :\ x(t) = x_{lim}(t)$$

Now we show that $(x_m(t))$ converges uniformly on J.
To do this we take the limit of (1) as $n \to \infty$:

$$\begin{aligned} d(x_m, x) &= \max_{t \in J} |x_m(t) - x(t)| < \epsilon \\ &\implies \forall t_0 \in J :\ |x_m(t_0) - x(t_0)| < \epsilon \\ &\implies (x_m(t)) \text{ converges uniformly on } J. \end{aligned}$$

Finally, since the x_m's are continuous on J and the convergence is uniform, then the limit function x is continuous on J, and hence $x \in C[a, b]$.

Therefore, the space $C[a, b]$ is complete.

□

Theorem 3 (Compactness).

Proof.

□

Theorem 4 (Banach fixed point theorem (Contraction Theorem)).

Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \longrightarrow X$ be a contraction on X. Then T has precisely one fixed point.

Proof.

Proof sketch:

First, we construct a sequence (x_n) and show that it is Cauchy, so that it converges in the complete space X.

Second, we prove that its limit, x, is a fixed point of T.

Finally, we show that T has no other fixed points.

Choose any $x_0 \in X$, and define the sequence (x_n) recursively as follows:

$$x_n = \begin{cases} x_0, & n = 0 \\ Tx_{n-1}, & n > 0 \end{cases}$$

We now show that (x_n) is Cauchy.

Because T is a contraction we have:

$$\begin{aligned} d(x_{m+1}, x_m) &= d(Tx_m, Tx_{m-1}) \\ &\leq \alpha\, d(x_m, x_{m-1}) \\ &= \alpha\, d(Tx_{m-1}, Tx_{m-2}) \\ &\leq \alpha^2\, d(x_{m-1}, x_{m-2}) \\ &\vdots \\ &= \alpha^m\, d(x_1, x_0) \end{aligned} \tag{1}$$

Applying the triangle inequality to (1), we get $\forall n > m$:

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-1}, x_n) \\ &\leq (\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-1})\, d(x_0, x_1) \end{aligned} \tag{2}$$

Applying the geometric series formula to (2), we get:

$$\begin{aligned} d(x_m, x_n) &= \alpha^m \frac{1 - \alpha^{n-m}}{1 - \alpha}\, d(x_0, x_1) \\ &= \beta^m\, d(x_0, x_1) && \text{for some } 0 < \beta < 1 \\ &< \epsilon && \forall \epsilon, \text{ and } \forall m, n > N(\epsilon) \end{aligned}$$

$\Longrightarrow$ (x_n) is Cauchy.

$\Longrightarrow$ (x_n) converges by the completeness of X to a limit point, say x.

Next, we show that the point x is a fixed point of T.

By the triangle inequality, we have:

$$\begin{aligned}
d(x, Tx) &\leq d(x, x_m) + d(x_m, Tx) \\
&\leq d(x, x_m) + \alpha\, d(x_{m-1}, x) \\
&< \epsilon && \forall \epsilon > 0 \text{ and } \forall m \geq N(\epsilon)
\end{aligned}$$

$\implies d(x, Tx) = 0$
$\implies x = Tx$
$\implies x$ is a fixed point of T.

Finally, we show that x is the only fixed point of T.
Suppose x and x' are fixed points of T, then:

$$\begin{aligned}
& d(x, x') = d(Tx, Tx') \leq \alpha\, d(x, x') \\
\implies & d(x, x') = 0 \\
\implies & x = x'.
\end{aligned}$$

$\square$

Theorem 5 (Fredholm Integral Equation)**.**

Proof.

A Fredhom integral equation of the second kind has the form:

$$x(t) - \mu \int_a^b k(t,\tau)x(\tau)d\tau = v(t) \tag{1}$$

where:

$[a, b]$: is a given interval.

x: is an unknown function on $[a, b]$.

μ: is a parameter/constant.

k: The kernel of the equation, is a function defined on the square $G = [a, b]x[a, b]$.

v: is a given function on $[a, b]$.

Let's restrict this equation to the function space $C[a, b]$, with the metric:

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|$$

We assume that $v \in C[a, b]$ and k is continuous on G. Then, k is bounded on G:

$$\forall t, \tau \in G : \ |k(t, \tau) \leq c$$

Next, we we assume that the soluton to (1) is a fixed point of some operator T, therefore we can replace x with Tx in (1) to get:

$$Tx(t) = v(t) + \mu \int_a^b k(t,\tau)x(\tau)d\tau \tag{2}$$

Since v and k are continuous, then (2) defines an operator $T : C[a, b] \longrightarrow C[a, b]$.

Next, we derive the condition for T to be a contraction:

$$\begin{aligned}
d(Tx, Ty) &= \max_{t \in J} |Tx(t) - Ty(t)| \\
&= |\mu| \max_{t \in J} | \int_a^b k(t,\tau)(x(\tau) - y(\tau))| \, d\tau \\
&\leq |\mu| \max_{t \in J} \int_a^b |k(t,\tau)| \, |x(\tau) - y(\tau)| \, d\tau \\
&\leq |\mu| \, c \, \max_{\sigma \in J} |x(\sigma) - y(\sigma)| \int_a^b d\tau \\
&= |\mu| \, c \, d(x, y) \, (b - a) \\
&= \alpha \, d(x, y) \qquad\qquad \text{where } \alpha = |\mu| \, c \, (b - a)
\end{aligned}$$

$\Longrightarrow T$ is a contraction when $|\mu| < \frac{1}{c(b-a)}$.

We now state the existence and uniqueness of a solution to (1): Given the restrictions stated above, (1) has a solution $x \in J$. x is the limit of the iterative sequence $(x_0, x_1, \ldots)$, where x_0 is any point in $C[a, b]$, and:

$$x_{n+1}(t) = v(t) + \mu \int_a^b k(t, \tau) x_n(\tau) d\tau.$$

$\square$

Theorem 6 (Volterra Integral Equation)**.**

Proof.

We assume that the solution is a fixed point of some operator T, and replace x with Tx:

$$Tx(t) = v(t) + \mu \int_a^\tau k(t,\tau)x(\tau)d\tau$$

Since k is continuous on R, and R is closed and bounded, then k is bounded:

$$\forall t, \tau \in R: \; |k(t,\tau)| \leq c$$

And, $\forall x, y \in C[a,b]$ we have:

$$\begin{aligned} d(Tx, Ty) &= |Tx(t) - Ty(t)| \\ &= |\mu| \, |\int_a^t k(t,\tau)[x(\tau) - y(\tau)] \, d\tau| \\ &\leq |\mu| \int_a^t |k(t,\tau)| \, |x(\tau) - y(\tau)| \, d\tau \\ &\leq |\mu| \, c \, d(x,y) \int_a^t d\tau \\ &\leq |\mu| \, c \, (t-a) \, d(x,y) \end{aligned} \tag{1}$$

Next, we show by induction that:

$$|T^m x(t) - T^m y(t)| \leq |\mu|^m c^m \frac{(t-a)^m}{m!} \tag{2}$$

<u>Base case (1)</u>: Holds by (1).

<u>Inductive step $(m+1)$</u>:

Suppose the I.H. holds for m, then:

$$\begin{aligned} |T^{k+1}x(t) - T^{k+1}y(t)| &= |\mu| \, |\int_a^t k(t,\tau) \, [T^m x(t) - T^m y(t)] \, d\tau| \\ &\leq |\mu| \, c \int_a^t \; |\mu|^m c^m \frac{(t-a)^m}{m!} d\tau \\ &= |\mu|^{m+1} \; c^{m+1} \; \frac{(t-a)^{m+1}}{(m+1)!} d(x,y). \end{aligned}$$

Using $t - a \leq b - a$ on the RHS of (2), and taking the max over $t \in J$ on the LHS, we get:

$$d(T^m x, T^m y) \leq \alpha_m d(x,y) \qquad \text{where } \alpha_m = |\mu|^m c^m \frac{(b-a)^2}{m!}$$

This implies that for any fixed μ and sufficiently large m, $\alpha < 1$.
Hence T^m is a contraction on $C[a,b]$ and has a unique fixed point.

□