Abstract Algebra Assignment (4): Application on Groups of Permutations

Mostafa Hassanein

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1.

$|G = (a)| = 5 \;\Rightarrow\; |G_A| = \phi(5) = |\{1,2,3,4\}| = 4.$

$G_A = \{f_t(a^j) = a^{tj} : t \in \{1,2,3,4\},\; j \in \{1..5\}\}.$

$f_1(a^j) = a^j; \quad j \in \{1..5\}$
$$f_1 = \begin{pmatrix} a & a^2 & a^3 & a^4 & a^5 \\ a & a^2 & a^3 & a^4 & a^5 \end{pmatrix} = id$$

$f_2(a^j) = a^{2j}, \quad j \in \{1..5\}$
$$f_2 = \begin{pmatrix} a & a^2 & a^3 & a^4 & a^5 \\ a^2 & a^4 & a & a^3 & a^5 \end{pmatrix} = (a\ a^2\ a^4\ a^3)$$

$f_3(a^j) = a^{3j}, \quad j \in \{1..5\}$
$$f_3 = \begin{pmatrix} a & a^2 & a^3 & a^4 & a^5 \\ a^3 & a & a^4 & a^2 & a^5 \end{pmatrix} = (a\ a^3\ a^4\ a^2)$$

$f_4(a^j) = a^{4j}, \quad j \in \{1..5\}$
$$f_4 = \begin{pmatrix} a & a^2 & a^3 & a^4 & a^5 \\ a^4 & a^3 & a^2 & a^1 & a^5 \end{pmatrix} = (a\ a^4)(a^2\ a^3)$$

G_A is a cyclic group.
Its generating elements are f_2 and f_3 since $|f_2| = |f_3| = 4 = |G_A|$.

2.

We know that:

i. The cosets of a subgroup partition the group.
ii. The size of any coset is equal to the size of the subgroup.
iii. The size of $A_n = S_n/2$.

(i), (ii), and (iii) $\Rightarrow H$ has 2 cosets.

Since H is one of the cosets (because $id \circ H = H$ and $H \circ id = H$), then the other coset must be $S_n \setminus H = S_n \setminus A_n = O_n$, which are the odd permutations.

$S_n/H = \{H, O_n\}$
$[S_n : H] = |S_n/H| = 2$

3.

Let's first construct the Cayley table:

$\circ$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
p_1	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
p_2	p_2	p_3	p_4	p_1	p_8	p_7	p_5	p_6
p_3	p_3	p_4	p_1	p_2	p_6	p_5	p_3	p_7
p_4	p_4	p_1	p_2	p_3	p_7	p_8	p_6	p_5
p_5	p_5	p_7	p_6	p_8	p_1	p_3	p_2	p_4
p_6	p_6	p_8	p_5	p_7	p_3	p_1	p_4	p_2
p_7	p_7	p_6	p_8	p_5	p_4	p_2	p_1	p_3
p_8	p_8	p_5	p_7	p_6	p_2	p_4	p_3	p_1

To prove that $H = \{p_1, p_3, p_7, p_8\}$ is self conjugate, we have to show that all elements in H commute with all elements in G i.e. $\forall h \in H \; and \; \forall p \in G : p^{-1} \circ h \circ p = h$:

$$
\begin{aligned}
&p_1^{-1} \circ p_1 \circ p_1 = p_1\\
&p_2^{-1} \circ p_1 \circ p_2 = p_1\\
&p_3^{-1} \circ p_1 \circ p_3 = p_1\\
&p_4^{-1} \circ p_1 \circ p_4 = p_1\\
&p_5^{-1} \circ p_1 \circ p_5 = p_1\\
&p_6^{-1} \circ p_1 \circ p_6 = p_1\\
&p_7^{-1} \circ p_1 \circ p_7 = p_1\\
&p_8^{-1} \circ p_1 \circ p_8 = p_1
\end{aligned}
$$

$$
\begin{aligned}
&p_1^{-1} \circ p_3 \circ p_1 = p_3\\
&p_2^{-1} \circ p_3 \circ p_2 = p_3\\
&p_3^{-1} \circ p_3 \circ p_3 = p_3\\
&p_4^{-1} \circ p_3 \circ p_4 = p_3\\
&p_5^{-1} \circ p_3 \circ p_5 = p_3\\
&p_6^{-1} \circ p_3 \circ p_6 = p_3\\
&p_7^{-1} \circ p_3 \circ p_7 = p_3\\
&p_8^{-1} \circ p_3 \circ p_8 = p_3
\end{aligned}
$$

$$
\begin{aligned}
&p_1^{-1} \circ p_7 \circ p_1 = p_7\\
&p_2^{-1} \circ p_7 \circ p_2 = p_7\\
&p_3^{-1} \circ p_7 \circ p_3 = p_7\\
&p_4^{-1} \circ p_7 \circ p_4 = p_7\\
&p_5^{-1} \circ p_7 \circ p_5 = p_7\\
&p_6^{-1} \circ p_7 \circ p_6 = p_7\\
&p_7^{-1} \circ p_7 \circ p_7 = p_7\\
&p_8^{-1} \circ p_7 \circ p_8 = p_7
\end{aligned}
$$

$$
\begin{aligned}
&p_1^{-1} \circ p_8 \circ p_1 = p_8\\
&p_2^{-1} \circ p_8 \circ p_2 = p_8\\
&p_3^{-1} \circ p_8 \circ p_3 = p_8\\
&p_4^{-1} \circ p_8 \circ p_4 = p_8
\end{aligned}
$$

$p_5^{-1} \circ p_8 \circ p_5 = p_8$
$p_6^{-1} \circ p_8 \circ p_6 = p_8$
$p_7^{-1} \circ p_8 \circ p_7 = p_8$
$p_8^{-1} \circ p_8 \circ p_8 = p_8$

The right cosets of H are $H \circ b$, where $b \in \{p_1 ... p_8\}$:

$H \circ p_1 = H = \{p_1, p_3, p_7, p_8\} = H \circ p_3 = H \circ p_7 = H \circ p_8$

$H \circ p_2 = \{p_1, p_3, p_7, p_8\} \circ p_2 = \{p_2, p_4, p_6, p_5\} = H \circ p_4 = H \circ p_6 = H \circ p_6$

$S/H = \{H \circ p_1, H \circ p_2\}$
$[S : H] = |S/H| = 2$

4.

$G_A = \{f_a : f_a = \begin{pmatrix} x \\ a^{-1} \circ x \circ a \end{pmatrix} \quad \forall x \in G, \ \forall a \in G\}$

We start by constructing the Cayley table:

$\circ$	p_1	p_2	p_3	p_4
p_1	p_1	p_2	p_3	p_4
p_2	p_2	p_1	p_4	p_3
p_3	p_3	p_4	p_1	p_2
p_4	p_4	p_3	p_2	p_1

Element	p_1	p_2	p_3	p_4
Order	1	2	2	2

$f_1(p_1) = p_1^{-1} \circ (p_1 \circ p_1) = p_1^{-1} \circ p_1 = p_1 \circ p_1 = p_1$
$f_1(p_2) = p_2$
$f_1(p_3) = p_3$
$f_1(p_4) = p_4$

$f_2(p_1) = p_2^{-1} \circ (p_1 \circ p_2) = p_2^{-1} \circ p_2 = p_2 \circ p_2 = p_1$
$f_2(p_2) = p_2^{-1} \circ (p_2 \circ p_2) = p_2^{-1} \circ p_1 = p_2 \circ p_1 = p_2$
$f_2(p_3) = p_2^{-1} \circ (p_3 \circ p_2) = p_2^{-1} \circ p_4 = p_2 \circ p_4 = p_3$
$f_2(p_4) = p_2^{-1} \circ (p_4 \circ p_2) = p_2^{-1} \circ p_3 = p_2 \circ p_3 = p_4$

$f_3(p_1) = p_3^{-1} \circ (p_1 \circ p_3) = p_3^{-1} \circ p_3 = p_3 \circ p_3 = p_1$
$f_3(p_2) = p_3^{-1} \circ (p_2 \circ p_3) = p_3^{-1} \circ p_4 = p_3 \circ p_4 = p_2$
$f_3(p_3) = p_3^{-1} \circ (p_3 \circ p_3) = p_3^{-1} \circ p_1 = p_3 \circ p_1 = p_3$
$f_3(p_4) = p_3^{-1} \circ (p_4 \circ p_3) = p_3^{-1} \circ p_2 = p_3 \circ p_2 = p_4$

$f_4(p_1) = p_4^{-1} \circ (p_1 \circ p_4) = p_4^{-1} \circ p_4 = p_4 \circ p_4 = p_1$
$f_4(p_2) = p_4^{-1} \circ (p_2 \circ p_4) = p_4^{-1} \circ p_3 = p_4 \circ p_4 = p_2$
$f_4(p_3) = p_4^{-1} \circ (p_3 \circ p_4) = p_4^{-1} \circ p_2 = p_4 \circ p_2 = p_3$

$f_4(p_4) = p_4^{-1} \circ (p_4 \circ p_4) = p_4^{-1} \circ p_1 = p_4 \circ p_1 = p_4$

We have $f_1 = f_2 = f_3 = f_4 = p_1$ (i.e. the identity permutation).
$\Rightarrow \; G_A = \{p_1\}$
$\Rightarrow |G_A| = 1 = |p_1|$.
$\Rightarrow G_A$ is a cyclic group whose generating element is p_1.