MTH-632 PDEs
Assignment (3): Method of Characteristics

Mostafa Hassanein

2 Jan 2025

3.1.2

a.

Solve using the method of characteristics:

$$\begin{aligned} u_t + cu_x &= e^{2x} \\ \text{Subject to: } u(x,0) &= f(x). \end{aligned}$$

<u>Solution:</u>

$$\begin{aligned} &u_t + cu_x = e^{2x} \\ \Longrightarrow\ &\frac{dx}{dt} = c \\ \Longrightarrow\ &x = ct + x_0 \end{aligned}$$

and,

$$\begin{aligned} &u_t + cu_x = e^{2x} \\ \Longrightarrow\ &\frac{du}{dt} = e^{2x} \\ \Longrightarrow\ &\frac{du}{dt} = e^{2(ct+x_0)} \\ \Longrightarrow\ &u(x,t) = \frac{1}{2c}e^{2(ct+x_0)} + K \end{aligned}$$

Apply the initial condition to find K:

$$\begin{aligned} &u(x_0, 0) = f(x_0) \\ \Longrightarrow\ &\frac{1}{2c}e^{2x_0} + K = f(x_0) \\ \Longrightarrow\ &K = -\frac{1}{2c}e^{2x_0} + f(x_0) \end{aligned}$$

Substitute back in u:

$$\begin{aligned} u(x,t) &= \frac{1}{2c}e^{2(ct+x_0)} - \frac{1}{2c}e^{2x_0} + f(x_0) \\ &= \frac{1}{2c}e^{2(ct+x-ct)} - \frac{1}{2c}e^{2(x-ct)} + f(x-ct) \\ &= \frac{1}{2c}e^{2x} - \frac{1}{2c}e^{2(x-ct)} + f(x-ct). \end{aligned}$$

b.

Solve using the method of characteristics:

$$\begin{aligned} u_t + x u_x &= 1 \\ \text{Subject to: } u(x,0) &= f(x). \end{aligned}$$

<u>Solution:</u>

$$\begin{aligned} & u_t + x u_x = 1 \\ \implies & \frac{dx}{dt} = x \\ \implies & x = ce^t \\ \implies & x = x_0 e^t. \end{aligned}$$

and,

$$\begin{aligned} & u_t + x u_x = 1 \\ \implies & \frac{du}{dt} = 1 \\ \implies & u(x,t) = t + K \\ \implies & u(x_0, 0) = K = f(x_0) \\ \implies & u(x,t) = t + f(x_0) \\ \implies & u(x,t) = t + f(xe^{-t}). \end{aligned}$$

c.

Solve using the method of characteristics:

$$\begin{aligned} u_t + 3tu_x &= u \\ \text{Subject to: } u(x,0) &= f(x). \end{aligned}$$

<u>Solution:</u>

$$\begin{aligned} & u_t + 3tu_x = u \\ \implies & \frac{dx}{dt} = 3t \\ \implies & x = \frac{3}{2}t^2 + x_0. \end{aligned}$$

and,

$$\begin{aligned} & u_t + 3tu_x = u \\ \implies & \frac{du}{dt} = u \\ \implies & u(x,t) = Ke^t \\ \implies & u(x_0,0) = K = f(x_0) \\ \implies & u(x,t) = f(x_0)e^t \\ \implies & u(x,t) = f(x - \frac{3}{2}t^2)e^t. \end{aligned}$$

d.

Solve using the method of characteristics:

$$u_t - 2u_x = e^{2x}$$
$$\text{Subject to: } u(x,0) = \cos(x).$$

Solution:

This is the same as point $(a.)$ above, with the values $c = -2$ and $f(x) = \cos(x)$. Thus, the solution is given by:

$$\begin{aligned} u(x,t) &= \frac{1}{2c}e^{2x} - \frac{1}{2c}e^{2(x-ct)} + f(x-ct) \\ &= -\frac{1}{4}e^{2x} + \frac{1}{4}e^{2(x+2t)} + \cos(x+2t). \end{aligned}$$

e.

Solve using the method of characteristics:

$$u_t - t^2 u_x = -u$$
$$\text{Subject to: } u(x,0) = 3e^x.$$

Solution:

$$\begin{aligned} & u_t - t^2 u_x = -u \\ \implies & \frac{dx}{dt} = -t^2 \\ \implies & x = -\frac{1}{3}t^3 + x_0. \end{aligned}$$

and,

$$\begin{aligned} & u_t - t^2 u_x = -u \\ \implies & \frac{du}{dt} = -u \\ \implies & u(x,t) = Ke^{-t} \\ \implies & u(x_0,0) = K = 3e^{x_0} \\ \implies & u(x,t) = 3e^{x_0 - t} \\ \implies & u(x,t) = 3e^{x + \frac{1}{3}t^2 - t}. \end{aligned}$$

3.1.4

Solve:

$$\begin{aligned} u_t &= u \\ \text{Subject to: } u(x,0) &= 1 + \cos(x). \end{aligned}$$

Solution:

$$\begin{aligned} u_t &= u \\ \implies u(x,t) &= k(x)e^t. \end{aligned}$$

On the line $x = -2t$, we have:

$$\begin{aligned} u(x,t) &= 1 + \cos(x) \\ \implies k(x)e^t &= 1 + \cos(x) \\ \implies k(x)e^{-\frac{1}{2}x} &= 1 + \cos(x) \\ \implies k(x) &= e^{\frac{1}{2}x}\left[1 + \cos(x)\right]. \end{aligned}$$

Substituting back in u:

$$u(x,t) = e^{\frac{1}{2}x+t}\left[1 + \cos(x)\right].$$

3.1.6

a.

Solve the following first-order linear PDE:

$$\begin{aligned} u_t + cu_x &= e^{-3x} \\ \text{Subject to: } u(x,0) &= f(x). \end{aligned}$$

<u>Solution:</u>

$$\begin{aligned} u_t + cu_x &= e^{-3x} \\ \implies \frac{dx}{dt} &= c \\ \implies x &= ct + x_0. \end{aligned}$$

and,

$$\begin{aligned} & u_t + cu_x = e^{-3x} \\ \implies & \frac{du}{dt} = e^{-3x} \\ \implies & \frac{du}{dt} = e^{-3(ct+x_0)} \\ \implies & u(x,t) = k - \frac{1}{3c}e^{-3(ct+x_0)} \\ \implies & u(x_0,0) = k - \frac{1}{3c}e^{-3x_0} = f(x_0) \\ \implies & k = f(x_0) + \frac{1}{3c}e^{-3x_0}. \end{aligned}$$

Substituting back in u:

$$\begin{aligned} u(x,t) &= k - \frac{1}{3c}e^{-3(ct+x_0)} \\ &= f(x_0) + \frac{1}{3c}e^{-3x_0} - \frac{1}{3c}e^{-3(ct+x_0)} \\ &= f(x-ct) + \frac{1}{3c}e^{-3(x-ct)} - \frac{1}{3c}e^{-3x} \end{aligned}$$

b.

Solve the following first-order linear PDE:

$$\begin{aligned} u_t + tu_x &= 5 \\ \text{Subject to: } u(x,0) &= f(x). \end{aligned}$$

Solution:

$$\begin{aligned} & u_t + tu_x = 5 \\ \implies & \frac{dx}{dt} = t \\ \implies & x = \frac{1}{2}t^2 + x_0. \end{aligned}$$

and,

$$\begin{aligned} & u_t + tu_x = 5 \\ \implies & \frac{du}{dt} = 5 \\ \implies & u(x,t) = 5t + k = 5t + f(x_0) \\ \implies & u(x,t) = 5t + k = 5t + f(x - \frac{1}{2}t^2). \end{aligned}$$

c.

Solve the following first-order linear PDE:

$$\begin{aligned} u_t - t^2 u_x &= -u \\ \text{Subject to: } u(x,0) &= f(x). \end{aligned}$$

<u>Solution:</u>

$$\begin{aligned} & u_t - t^2 u_x = -u \\ \implies & \frac{dx}{dt} = -t^2 \\ \implies & x = -\frac{1}{3}t^3 + x_0. \end{aligned}$$

and,

$$\begin{aligned} & u_t - t^2 u_x = -u \\ \implies & \frac{du}{dt} = -u \\ \implies & u(x,t) = ke^{-t} \\ \implies & u(x_0,0) = k = f(x_0) \\ \implies & u(x,t) = f(x_0)e^{-t} \\ \implies & u(x,t) = f(x + \frac{1}{3}t^3)e^{-t}. \end{aligned}$$

d.

Solve the following first-order linear PDE:

$$\begin{aligned} & u_t + x u_x = t \\ \text{Subject to: } & u(x,0) = f(x). \end{aligned}$$

Solution:

$$\begin{aligned} & u_t + x u_x = t \\ \Longrightarrow & \frac{dx}{dt} = x \\ \Longrightarrow & x = ce^x = x_0 e^t. \end{aligned}$$

and,

$$\begin{aligned} & u_t + x u_x = t \\ \Longrightarrow & \frac{du}{dt} = t \\ \Longrightarrow & u(x,t) = \frac{1}{2}t^2 + k \\ \Longrightarrow & u(x_0, 0) = k = f(x_0) \\ \Longrightarrow & u(x,t) = \frac{1}{2}t^2 + f(x_0) \\ \Longrightarrow & u(x,t) = \frac{1}{2}t^2 + f(e^{-t}x) \end{aligned}$$

e.

Solve the following first-order linear PDE:

$$\begin{aligned} & u_t + xu_x = x \\ \text{Subject to: } & u(x,0) = f(x). \end{aligned}$$

Solution:

$$\begin{aligned} & u_t + xu_x = t \\ \implies & \frac{dx}{dt} = x \\ \implies & x = ce^x = x_0 e^t. \end{aligned}$$

and,

$$\begin{aligned} & u_t + xu_x = x \\ \implies & \frac{du}{dt} = x \\ \implies & \frac{du}{dt} = x_0 e^t \\ \implies & u(x,t) = x_0 e^t + k \\ \implies & u(x_0, 0) = x_0 + k = f(x_0) \\ \implies & k = f(x_0) - x_0 \\ \implies & u(x,t) = x_0 e^t + f(x_0) - x_0 \\ \implies & u(x,t) = x + f(xe^{-t}) - xe^{-t} \end{aligned}$$

3.2.2

Consider the problems:

$$u_t + 2uu_x = 0$$
$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & x > L \end{cases}$$

a. Determine the equation for the characteristics.
b. Determine the solution $u(x,t)$
c. Sketch the characteristic curves.
d. Sketch $u(x,t)$ for fixed t.

Solution:

a.

$$\frac{dx}{dt} = 2u$$
$$\frac{du}{dt} = 0.$$

b.

$$\frac{du}{dt} = 0$$
$$\implies u(x,t) = u(x_0,0) = f(x_0) = \begin{cases} 1 & x_0 < 0 \\ 1 + \frac{x_0}{L} & 0 < x_0 < L \\ 2 & x_0 > L \end{cases}$$

$$\frac{dx}{dt} = 2u$$
$$\implies x = 2ut + x_0$$
$$\implies x = \begin{cases} 2t + x_0 & x_0 < 0 \\ 2t + 2t\frac{x_0}{L} + x_0 & 0 < x_0 < L \\ 4t + x_0 & x_0 > L. \end{cases}$$

Substituting back in u:

$$\implies u(x,t) = u(x_0,0) = f(x_0) = \begin{cases} 1 & x-2t<0 \\ 1+\frac{x_0}{L} & 0<\frac{x-2t}{\frac{2t}{L}+1}<L \\ 2 & x-4t>L \end{cases}$$

3.2.4

Solve:

$$\begin{aligned} &u_t + t^2 u u_x = 5 \\ &u(x,0) = x. \end{aligned}$$

Solution:

$$\begin{aligned} &u_t + t^2 u u_x = 5 \\ \implies &\frac{du}{dt} = 5 \\ \implies &u(x,t) = 5t + k \\ \implies &u(x_0,0) = k = x_0 \\ \implies &u(x,t) = 5t + x_0 \end{aligned}$$

and,

$$\begin{aligned} &u_t + t^2 u u_x = 5 \\ \implies &\frac{dx}{dt} = t^2 u \\ \implies &\frac{dx}{dt} = t^2(5t + x_0) \\ \implies &\frac{dx}{dt} = 5t^3 + x_0 t^2 \\ \implies &x = \frac{5}{4}t^4 + \frac{1}{3}x_0 t^3 + x_0. \end{aligned}$$

Substitute back in u:

$$\begin{aligned} u(x,t) &= 5t + x_0 \\ &= 5t + \frac{x - \frac{5}{4}t^4}{\frac{1}{3}t^3 + 1}. \end{aligned}$$

3.2.2

Solve:

$$\rho_t + \rho^2\rho_x = 0$$
$$\rho(x,0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$$

<u>Solution:</u>

$$\begin{aligned} &\rho_t + \rho^2\rho_x = 0 \\ \Longrightarrow\ &\frac{d\rho}{dt} = 0 \\ \Longrightarrow\ &\rho(x,t) = k = \rho(x_0,0) = \begin{cases} 4 & x_0 < 0 \\ 3 & x_0 > 0 \end{cases} \end{aligned}$$

and,

$$\begin{aligned} &\frac{dx}{dt} = \rho^2 \\ &\frac{dx}{dt} = \rho^2(x_0,0) = \begin{cases} 16 & x_0 < 0 \\ 9 & x_0 > 0 \end{cases} \\ &x = \begin{cases} 16t + x_0 & x_0 < 0 \\ 9t + x_0 & x_0 > 0 \end{cases} \end{aligned}$$

Substitute back in ρ:

$$\rho(x,t) = \begin{cases} 4 & x - 16t < 0 \\ 3 & x - 9t > 0 \end{cases}$$

The discontinuity in the initial condition at $x_0 = 0$ will result in a shock. The shock characteristic is given by:

$$\begin{aligned} &\frac{dx_s}{dt} = \frac{[q]}{[u]} = \frac{\frac{1}{3}[3^3 - 4^3]}{3-4} = \frac{37}{3} \\ \Longrightarrow\ &x_s = \frac{37}{3}t + x_{s_0} = \frac{37}{3}t. \end{aligned}$$

The solution above the shock characteristic/line is $\rho = 4$, and below it is $\rho = 3$.

3.2.4

Solve:

$$u_t + 4uu_x = 0$$
$$u(x,0) = \begin{cases} 2 & x < -1 \\ 3 & x > -1 \end{cases}$$

Solution:

$$\begin{aligned} & u_t + 4uu_x = 0 \\ \implies & \frac{du}{dt} = 0 \\ \implies & u(x,t) = k = u(x_0,0) = \begin{cases} 2 & x_0 < -1 \\ 3 & x_0 > -1 \end{cases} \end{aligned}$$

and,

$$\begin{aligned} & u_t + 4uu_x = 0 \\ \implies & \frac{dx}{dt} = 4u \\ \implies & x = \begin{cases} 8t + x_0 & x_0 < -1 \\ 12t + x_0 & x_0 > -1 \end{cases} \end{aligned}$$

Substitute back in u:

$$\begin{aligned} u(x,t) &= \begin{cases} 2 & x - 8t < -1 \\ 3 & x - 12t > -1 \end{cases} \\ &= \begin{cases} 2 & x < 8t - 1 \\ 3 & x > 12t - 1 \end{cases} \end{aligned}$$

The discontinuity in the initial condition at $x_0 = -1$ will result in a fanning-out in the region $8t - 1 < x < 12 - 1$, where the solution is given by:

$$\begin{aligned} & \frac{dx}{dt} = 4u \\ \implies & x = 4ut + x_0 = 4ut - 1 \\ \implies & u = \frac{x+1}{4t}. \end{aligned}$$

3.2.6

Solve the quasilinear equation:

$$u_t + uu_x = 0$$
$$u(x,0) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

<u>Solution:</u>

$$\begin{aligned} & u_t + uu_x = 0 \\ \implies & \frac{du}{dt} = 0 \\ \implies & u(x,t) = k = u(x_0, 0) = \begin{cases} 0 & x_0 < 0 \\ x_0 & 0 \leq x_0 < 1 \\ 1 & x_0 \geq 1 \end{cases} \end{aligned}$$

and,

$$\begin{aligned} & u_t + uu_x = 0 \\ \implies & \frac{dx}{dt} = u \\ \implies & \frac{dx}{dt} = \begin{cases} 0 & x_0 < 0 \\ x_0 & 0 \leq x_0 < 1 \\ 1 & x_0 \geq 1 \end{cases} \\ \implies & x = \begin{cases} x_0 & x_0 < 0 \\ x_0 t + x_0 & 0 \leq x_0 < 1 \\ t + x_0 & x_0 \geq 1 \end{cases} \end{aligned}$$

Substitute back in u:

$$u(x,t) = \begin{cases} 0 & x < 0 \\ x & 0 \leq \frac{x}{1+t} < 1 \\ 1 & x - t \geq 1 \end{cases}$$

3.3.2

The general solution of the one dimensional wave equation:

$$u_{tt} - 4u_{xx} = 0$$

is given by:

$$u(x,t) = F(x-2t) + G(x+2t).$$

Find the solution subject to the initial conditions:

$$\begin{aligned} &u(x,0) = \cos(x) && -\infty < x < \infty \\ &u_t(x,0) = 0 && -\infty < x < \infty. \end{aligned}$$

Solution:

$$\begin{aligned} &u(x,0) = \cos(x) \\ \Longrightarrow\ &F(x) + G(x) = \cos(x). \end{aligned} \tag{1}$$

and,

$$\begin{aligned} &u_t(x,0) = 0 \\ \Longrightarrow\ &-2\dot{F}(x) + 2\dot{G}(x) = 0 \\ \Longrightarrow\ &-F(x) + G(x) = k. \end{aligned} \tag{2}$$

Solving (1) and (2) together, we get:

$$\begin{aligned} &G(x) = \frac{1}{2}\cos(x) + \frac{1}{2}k, && F(x) = \frac{1}{2}\cos(x) - \frac{1}{2}k \\ \Longrightarrow\ &G(x+2t) = \frac{1}{2}\cos(x+2t) + \frac{1}{2}k, && F(x-2t) = \frac{1}{2}\cos(x-2t) - \frac{1}{2}k \end{aligned}$$

Therefore:

$$\begin{aligned} u(x,t) &= F(x-2t) + G(x+2t) \\ &= \frac{1}{2}\cos(x+2t) + \frac{1}{2}k + F(x-2t) = \frac{1}{2}\cos(x-2t) - \frac{1}{2}k \\ &= \frac{1}{2}[\cos(x-2t) + \cos(x+2t)]. \end{aligned}$$

Problems

2

Solve:

$$
\begin{aligned}
& u_{tt} - c^2 u_{xx} = 0, && x < 0 \\
\text{Subject to:} & && \\
& u(x,0) = \sin(x), && x < 0 \\
& u_t(x,0) = 0, && x < 0 \\
& u(0,t) = e^{-t}, && t > 0.
\end{aligned}
$$

<u>Solution:</u>
By D'Alembert's formula for a semi-infinite domain:

$$
\begin{aligned}
u(x,t) &= \begin{cases} \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\eta)d\eta & x+ct<0 \\ h(t-\frac{x}{c}) + \frac{f(x-ct)-f(x+ct)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\eta)d\eta & x+ct>0 \end{cases} \\
&= \begin{cases} \frac{\sin(x-ct)-\sin(x+ct)}{2} & x+ct<0 \\ e^{-(t-\frac{x}{c})} + \frac{\sin(x-ct)-\sin(x+ct)}{2} & x+ct>0 \end{cases}
\end{aligned}
$$

4

Solve:

$$u_{tt} - c^2 u_{xx} = 0, \qquad x, t > 0$$

Subject to:

$$u(x,0) = 0$$
$$u_t(x,0) = 0$$
$$u_x(0,t) = h(t).$$

Solution:
For $x - ct > 0$, the solution is $u = 0$, because both $u(x,0) = 0$ and $u_t(x,0) = 0$, and reflections from the boundary have not reached yet.

For $x - ct < 0$, the solution will be the result of reflections at the boundary.