# MTH-684 Logic
# Assignment (2): Propositional Logic

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2 October 2024

## 2-3

To prove an implication we assume the antecedent and show that the consequent holds.

### a.

*Proof.*
Let $T_3 = T_1 \cup T_2$.
Suppose all WFFs in $T_3$ are true, then all WFFs in $T_1$, and $T_2$ are also true.
From $T_1 \models \phi$, we conclude that $\phi$ is true.
From $T_2 \models \psi$, we conclude that $\psi$ is true.
Putting it all together, we get: $T_3 \models \phi \wedge \psi$.

$\square$

### b.

*Proof.*
Let $T_4 = T_1 \cup T_2 \cup T_3$.
Suppose all WFFs in $T_4$ are true, then all WFFs in $T_1$, $T_2$, and $T_3$ are true.
From $T_1 \models \phi$, we conclude that $\phi \vee \psi$ is true. We have 3 cases:

1. $\phi = \top$ and $\psi = \bot$:
In this case, all WFFs in $T_2 \cup \{\phi\}$ are true.
From $T_2 \cup \{\phi\} \models \zeta$, we conclude that $\zeta$ is true.

2. $\phi = \bot$ and $\psi = \top$:
In this case, all WFFs in $T_3 \cup \{\psi\}$ are true.
From $T_3 \cup \{\psi\} \models \zeta$, we conclude that $\zeta$ is true.

3. $\phi = \top$ and $\psi = \top$:
Using either case 1 or 2, it is clear that $\zeta$ is true.

Therefore, we conclude that $T_4 = T_1 \cup T_2 \cup T_3 \models \zeta$.

$\square$

### c.

*Proof.*
We want to show that $T_1 \models (\phi \implies \psi)$ is true, so we assume $T_1$ and attempt to show that $\phi \implies \psi$ holds.

Next, to show that $\phi \implies \psi$, we assume $\phi$ and attempt to show that $\psi$ holds.

Putting our assumptions together: $T_1$ and $\phi$ are true. Therefore, all WFFs in $T_1 \cup \{\phi\}$ are true.

From $T_1 \cup \{\phi\} \models \psi$, we conclude that $\psi$ is true.

Therefore, $T_1 \models (\phi \implies \psi)$ as desired.

$\square$

## d.

*Proof.* (By Contradiction)

Suppose $T_1 \cup T_2$ is true.
For the sake of contradiction, suppose that $\phi$ is true.
From $T_1 \cup \{\phi\} \models \psi$, we conclude that $\psi$ is true.
From $T_2 \cup \{\phi\} \models \neg\psi$, we conclude that $\neg\psi$ is true.
But then we have: $\psi \wedge \neg\psi$ is true. This is a contradiction.
Therefore, we must conclude that: $T_1 \cup T_2 \models \neg\phi$.

$\square$