

Functional Analysis Assignment (Chapter 1)

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1.1.1

The distance on \mathbb{R} is defined by $d(x, y) = |x - y|$. We must check that the 4 axioms (M1 to M4) are satisfied.

M1 holds, since the absolute value of the difference between 2 real points is real, finite, and non-negative.

M2 holds, since $d(x, y) = |x - y| = 0 \iff x = y$.

M3 holds, since $d(x, y) = |x - y| = |y - x| = d(y, x)$.

M4 holds, since the triangle inequality holds for the absolute value.

1.1.2

$d(x, y) = (x - y)^2$ is not a valid metric since it does not satisfy the triangle inequality.

Proof. (By Counterexample)

Let $a, b, c \in \mathbb{R}$, where $a = 0, b = 1, c = 5$, then:

$$d(a, c) = (a - c)^2 = (0 - 5)^2 = 25.$$

$$d(a, b) = (a - b)^2 = (0 - 1)^2 = 1.$$

$$d(b, c) = (b - c)^2 = (1 - 5)^2 = 16.$$

$$d(a, b) + d(b, c) = 1 + 16 = 17.$$

Therefore, $d(a, c) \not\leq d(a, b) + d(b, c)$.

□

1.1.3

Since the distance function $d(x, y) = |x - y|$ defines a metric on \mathbb{R} , as shown in 1.1.1, then it is clear that the square root of that metric is also real, finite, and non-negative (i.e. M1 holds); definite (i.e. M2 holds); and symmetric (i.e. M3 holds).

The triangle inequality can be shown to hold by noting that the square root function is an increasing function with a negative second derivative in the interval $(0, \infty)$.

This shows that $d(x, y) = \sqrt{|x - y|}$ is a metric on \mathbb{R} .

1.1.4

i. $|X| = 2$

Let $X = \{a, b\}$, then d must satisfy:

$$\begin{aligned} d(a, a) &= d(b, b) = 0, \text{ and} \\ d(a, b) &= d(b, a) = c, \text{ where } c \text{ is any non-negative real number.} \end{aligned}$$

ii. $|X| = 1$

In this case the only valid metric is $d(a, a) = 0$.

1.1.5

i. **Conditions for kd to be a metric**

If d is a metric, then kd automatically satisfies axioms M2-M4.

For axiom M1 to hold, k must be a non-negative real number.

ii. **Conditions for $k + d$ to be a metric**

To satisfy axiom M2, k must be zero.

1.1.6

Proof. (By Induction on the Length of the Sequence)

Let $X = (x_j)$, $Y = (y_j)$, $Z = (z_j)$ be 3 bounded sequences.

Base case

Consider the subsequence of X, Y, Z consisting of just their first element.

Then by the triangle inequality for numbers:

$$\begin{aligned} |x_1 - z_1| &\leq |x_1 - y_1| + |y_1 - z_1| \\ \implies \sup |x_1 - z_1| &\leq \sup(|x_1 - y_1| + |y_1 - z_1|) \\ \implies \sup |x_1 - z_1| &\leq \sup |x_1 - y_1| + \sup |y_1 - z_1|. \end{aligned}$$

Therefore $d(x, z) \leq d(x, y) + d(y, z)$ holds for sequences of length 1.

Inductive step

Next, we'll consider the sub-sequences of X, Y, Z consisting of the first $n + 1$ elements. Suppose that the induction hypothesis holds for sequences of length n , i.e.:

$$\sup_{j \in \{1..n\}} |x_j - z_j| \leq \sup_{j \in \{1..n\}} |x_j - y_j| + \sup_{j \in \{1..n\}} |y_j - z_j|.$$

Then we can partition each sequence of length $n + 1$ into 2 sub-sequences: the first sequence contains the first n elements and the second contains the last element.

The distance between any 2 sequences of length $n + 1$ then becomes:

$$\max(\sup_{j \in 1..n} |x_j - z_j|, \sup |x_{n+1} - z_{n+1}|)$$

Finally, applying the induction hypothesis we get:

$$\max(\sup_{j \in 1..n} |x_j - z_j|, \sup |x_{n+1} - z_{n+1}|) \leq \max(\sup_{j \in 1..n} |x_j - y_j| + \sup_{j \in 1..n} |y_j - z_j|, \sup |x_{n+1} - y_{n+1}| + \sup |y_{n+1} - z_{n+1}|)$$

$$\implies \sup_{j \in 1..n+1} |x_j - z_j| \leq \sup_{j \in 1..n+1} |x_j - y_j| + \sup_{j \in 1..n+1} |y_j - z_j|$$

Therefore $d(x, z) \leq d(x, y) + d(y, z)$ holds for sequences of any length. \square

1.1.7

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Which is the discrete metric.

1.1.8

M1 holds because $|x(t) - y(t)|$ is a positive function, and the integral of a positive function is positive.

M2 holds because $|x(t) - x(t)| = 0$, and the integral of the zero function is 0.

M3 holds because $d(x, y) = \int_a^b |x(t) - y(t)| dt = \int_a^b |y(t) - x(t)| dt = d(y, x)$.

To show that M4 holds, let $x, y, z \in X$, then:

$$\begin{aligned}
d(x, z) &= \int_a^b |x(t) - z(t)| \, dt \\
&= \int_a^b |(x(t) - y(t)) - (z(t) - y(t))| \, dt \\
&\geq \int_a^b (|x(t) - y(t)| - |z(t) - y(t)|) \, dt && \text{(By the triangle inequality of absolute values)} \\
&= \int_a^b |x(t) - y(t)| \, dt - \int_a^b |z(t) - y(t)| \, dt \\
&= d(x, y) - d(z, y)
\end{aligned}$$

$$\implies d(x, y) \leq d(x, z) + d(z, y).$$

1.2.3

Proof.

Let (ζ_j) , (η_j) , and $(\theta_j) \in l^p$, where (ζ_j) is any point, and define (η_j) and (θ_j) as follows:

$$\eta_j := \begin{cases} 1 & j \leq n \\ 0 & j > n \end{cases}$$

$$\theta_j := \begin{cases} \zeta_j & j \leq n \\ 0 & j > n \end{cases}$$

Applying the Cauchy-Schwarz inequality to (θ_j) and (η_j) , we get:

$$\begin{aligned} \sum_{j=1}^{\infty} |\theta_j \eta_j| &\leq \sqrt{\sum_{k=1}^{\infty} |\theta_j|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_j|^2} \\ \sum_{j=1}^n |\theta_j \eta_j| &\leq \sqrt{\sum_{k=1}^n |\theta_j|^2} \sqrt{\sum_{m=1}^n |\eta_j|^2} \\ \sum_{j=1}^n |\theta_j| &\leq \sqrt{\sum_{k=1}^n |\theta_j|^2} \sqrt{\sum_{m=1}^n 1} \\ \sum_{j=1}^n |\zeta_j| &\leq \sqrt{\sum_{k=1}^n |\zeta_j|^2} \sqrt{n} \\ \left(\sum_{j=1}^n |\zeta_j| \right)^2 &\leq n \sum_{k=1}^n |\zeta_j|^2 \end{aligned}$$

□

1.2.4

1.2.5

Proof.

The sequence $\zeta_j = \frac{1}{j}$ is divergent for $p = 1$ but convergent for all $p > 1$.

□

1.2.11

Proof.

M1 holds because:

$$\begin{aligned}
& d(x, y) \geq 0 \\
\implies & \frac{d(x, y)}{1 + d(x, y)} \geq 0 \\
\implies & \tilde{d}(x, y) \geq 0.
\end{aligned}$$

M2 holds because:

\implies :

$$\begin{aligned}
& \tilde{d}(x, y) = 0 \\
\implies & \frac{d(x, y)}{1 + d(x, y)} = 0 \\
\implies & d(x, y) = 0 \\
\implies & x = y.
\end{aligned}$$

\Leftarrow :

$$\begin{aligned}
& x = y \\
\implies & d(x, y) = 0 \\
\implies & \frac{d(x, y)}{1 + d(x, y)} = 0 \\
\implies & \tilde{d}(x, y) = 0.
\end{aligned}$$

M3 holds because:

$$\begin{aligned}
& d(x, y) = d(y, x) \\
\implies & \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} \\
\implies & \tilde{d}(x, y) = \tilde{d}(y, x).
\end{aligned}$$

M4 holds because:

$$\begin{aligned}
& d(x, z) \leq d(x, y) + d(y, z) \\
\implies & \frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y)}{1 + d(x, z)} + \frac{d(y, z)}{1 + d(x, z)} \\
\implies & \frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\
\implies & \tilde{d}(x, z) \leq \tilde{d}(x, y) + \tilde{d}(y, z).
\end{aligned}$$

□

1.3.4

Proof.

\Rightarrow (By Construction):

Let A be a non-empty open subset, and $\delta(A)$ be the diameter of A .

Since A is non-empty and open, then $|A| > 1$.

Let a_1, a_2 be 2 different points in A .

Consider the 2 open balls $B(a_1, \delta(A))$ and $B(a_2, \delta(A))$.

We have $B(a_1, \delta(A)) = B(a_2, \delta(A)) = A$.

Thus, $A = B(a_1, \delta(A)) \cup B(a_2, \delta(A))$.

\Leftarrow (By Strong Induction):

We will prove a **stronger** result: The union of non-empty open subsets is a non-empty open subset.

Let $B = (B_1, B_2, \dots, B_n)$ be a list of open subsets (balls **or not**) in X .

Base case: $|B| = 2$

$$A = B_1 \cup B_2$$

$$\Rightarrow \forall a \in A, a \in B_1 \vee a \in B_2$$

$$\Rightarrow \forall a \in A, \exists \text{ a ball about } a$$

$$\Rightarrow A \text{ is a non-empty open subset of } X.$$

Inductive step: $|B| = k > 2$

$$\begin{aligned} A &= \bigcup_{j=1}^k B_j \\ &= \bigcup_{j=1}^{k-1} B_j + B_k \\ &= C + B_k \end{aligned} \tag{1}$$

Where C is a non-empty open subset by the induction hypothesis for $n = k - 1$.

Applying the induction hypothesis to (1) for $n = 2$, we conclude that A is a non-empty open subset in X .

□

1.3.6

Proof. (By Construction)

We will prove this statement by recursively (and infinitely) applying the definition of an accumulation point on increasingly smaller ϵ -neighborhoods.

Let the sequence of (ϵ_j) be such that $\forall j, \epsilon_j > 0$.

Since x_0 is an accumulation point, then each ϵ_j -neighborhood contains at least one point $y \in A$.

Let y_j to be any point in the ϵ_j -neighborhood of x_0 that satisfies the following condition:

$$d(y_j, x_0) = \max_{\forall y_k \in \tilde{B}(x_0; \epsilon_j)} d(y_k, x_0)$$

Define ϵ_j recursively as follows:

$$\epsilon_j := \begin{cases} \epsilon_1 & j = 1 \\ d(x_0, y_{j-1})/2 & j > 1 \end{cases}$$

Then the sequence (y_j) is an infinite sequence of distinct points in A , all contained inside the ϵ_1 -neighborhood of x_0 .

Since ϵ_1 was arbitrarily chosen, this completes the proof. □

1.3.13

Proof.

This follows directly from the definition that a separable space X contains a subset Y that is countable and dense in X . □

1.3.14

Proof.

\implies :

Let $T : X \longrightarrow Y$ be a continuous map and M be a closed subset in Y .

- | | |
|--------------------------------------|---|
| $\implies M^c$ is open | (Because the complement of a closed set is an open set by definition 1.3-2) |
| $\implies T^{-1}(M^c)$ is open | (By continuity of T) |
| $\implies (T^{-1}(M^c))^c$ is closed | (Because the complement of an open set is a closed set by definition 1.3-2) |
| $\implies T^{-1}((M^c)^c)$ is closed | (By the identity: $f^{-1}(A^c) = [f^{-1}(A)]^c$) |
| $\implies T^{-1}(M)$ is closed. | (By the identity: $A = [A^c]^c$) |

\Leftarrow :

Let $T : X \rightarrow Y$ be a map such that the inverse image of any closed set in Y is a closed set in X .

Let A be an open set in Y .

- $\Rightarrow A^c$ is closed (Because the complement of an open set is a closed set by definition 1.3-2)
- $\Rightarrow T(A^c)$ is closed (By our assumption the inverse image of any closed set is a closed set)
- $\Rightarrow [T(A^c)]^c$ is open (Because the complement of a closed set is an open set by definition 1.3-2)
- $\Rightarrow T([A^c]^c)$ is open (By the identity: $f^{-1}(A^c) = [f^{-1}(A)]^c$)
- $\Rightarrow T(A)$ is open (By the identity: $A = [A^c]^c$)
- $\Rightarrow T$ is continuous.

□

1.4.2

Proof.

(x_{n_k}) converges to x
 $\implies \forall \epsilon > 0, \exists K(\epsilon) \ni \forall k > K, d(x_{n_k}, x) < \epsilon$
 $\implies \forall \epsilon > 0, \exists N(\epsilon) = K(\epsilon) \ni \forall n > N, d(x_n, x) < \epsilon$ (Because (x_n) is Cauchy)
 $\implies (x_n)$ converges to x .

□

1.4.4

Proof.

Take ϵ with any concrete value, say $\epsilon = 1$.

Because (x_n) is Cauchy, then:

$$\exists N \ni \forall n, m > N, d(x_n, x_m) < \epsilon = 1.$$

Define $a = \max_{i,j \in \{1, \dots, N\}} d(x_i, x_j)$.

Therefore we have $\forall n$:

$$d(x_n, x_N) \leq \max(1, a) \leq 1 + a$$

By the triangle inequality we have $\forall n, m$:

$$\begin{aligned}
 d(x_n, x_m) &\leq d(x_n, x_N) + d(x_N, x_m) \\
 &\leq (1 + a) + (1 + a) = 2(1 + a) = u.
 \end{aligned}$$

This shows that u is an upperbound for the Cauchy sequence.

□

1.4.5

Proof. (By Counterexample)

i. Boundedness does **not** imply Cauchiness:

Consider the sequence $x_n = (-1)^n$.

(x_n) is bounded by 2, but the sequence is not Cauchy, because for any $0 < \epsilon < 2$:

$$\nexists N \ni \forall n, m > N, d(x_n, x_m) < \epsilon.$$

ii. Boundedness does **not** imply Convergence:

Consider the sequence $x_n = \sin(n)$.

(x_n) is bounded by 2, but the sequence oscillates and does not converge.

□

1.4.6

Proof.

Take any $\epsilon > 0$, then since (x_n) and (y_n) are Cauchy:

$$\exists N_x \ni \forall n, m > N, d(x_n, x_m) < \epsilon$$

and,

$$\exists N_y \ni \forall n, m > N, d(y_n, y_m) < \epsilon.$$

Let $N = \max\{N_x, N_y\}$, then $\forall n > N$:

$$d(x_n, x_N) < \epsilon \wedge d(y_n, y_N) < \epsilon.$$

By the triangle inequality we have $\forall n > N$:

$$\begin{aligned} a_n = d(x_n, y_n) &\leq d(x_n, x_N) + d(x_N, y_N) + d(y_N, y_n) \\ &\leq \epsilon + c + \epsilon \\ &= 2\epsilon + c. \end{aligned}$$

Finally, $\forall n, m > N$:

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, a_N) + d(a_N, a_m) \\ &= |d(x_n, y_n) - d(x_N, y_N)| + |d(x_m, y_m) - d(x_N, y_N)| \\ &\leq |2\epsilon + c - c| + |2\epsilon + c - c| \\ &= |2\epsilon| + |2\epsilon| \\ &\leq 4\epsilon. \end{aligned}$$

Therefore (a_n) is a Cauchy sequence in \mathbb{R} , and since \mathbb{R} is complete, then (a_n) is convergent.

□

1.5.2

Proof.

Assume (x_i) is a Cauchy sequence in X

$$\implies \forall \epsilon > 0, \exists N \ni \forall j, k \geq N : d(x_j, x_k) = \max_{i=1}^n |x_i^{(j)} - x_i^{(k)}| < \epsilon \quad (1)$$

$$\implies \forall \epsilon > 0, \exists N \ni \forall j, k \geq N, \forall i \in \{1, \dots, n\} : |x_i^{(j)} - x_i^{(k)}| < \epsilon$$

$$\implies \forall i \in \{1, \dots, n\} : (x_i^{(1)}, x_i^{(2)}, \dots) \text{ is a Cauchy sequence of real numbers}$$

$$\implies \forall i \in \{1, \dots, n\} : (x_i^{(1)}, x_i^{(2)}, \dots) \text{ converges to a limit point } x_i \text{ because } \mathbb{R} \text{ is complete.}$$

Next, we define a candidate limit for (x_i) :

$$x = (x_1, \dots, x_n).$$

Clearly, $x \in X$, and by (1) we have:

$$\forall j \geq N : d(x_j, x) = \epsilon.$$

This shows that x is the limit of (x_i) and proves completeness of X . □

1.5.5

Proof.

Assume (x_n) is a Cauchy sequence in X

$$\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N : d(x_n, x_m) = |x_n - x_m| < \epsilon.$$

Take $\epsilon = 0.5$

$$\implies \exists N \ni \forall n, m \geq N : |x_n - x_m| < 0.5$$

$$\implies \forall n, m \geq N : x_n = x_m$$

$$\implies (x_n) \longrightarrow x_N.$$
□

1.5.6

Proof. (By Counterexample)

Take the sequence $x_n = n$.

First we show that it is Cauchy.

Given any $\epsilon > 0$, we can take $N(\epsilon) = \tan(\frac{\pi}{2} - \epsilon)$ so that $\forall n, m \geq N$:

$$\arctan(x_m), \arctan(x_n) \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$$

$$\begin{aligned}
\implies d(x_n, x_m) &= |\arctan(x_n) - \arctan(x_m)| \\
&\leq \left(\frac{\pi}{2} - \epsilon\right) - \frac{\pi}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore (x_n) is Cauchy as desired.

Now we show that (x_n) does not converge.

We observe that the sequence wants to converge to $\frac{\pi}{2}$, since:

$$\lim_{n \rightarrow \infty} d(x_n, \frac{\pi}{2}) = 0$$

But there is no element $x \in \mathbb{R}$, such that $\arctan(x) = \frac{\pi}{2}$.

Thus our metric space is **incomplete**.

□

1.5.8

Proof.

$$\begin{aligned}
&\text{Let } (x_n) \text{ be a Cauchy sequence in } Y \subseteq [a, b], \text{ and let } J = [a, b] \\
\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N : d(x_n, x_m) &= \max_{t \in J} |x_n(t) - x_m(t)| < \epsilon \quad (1) \\
\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N, \forall t \in J : |x_n(t) - x_m(t)| < \epsilon \\
\implies \forall t_0 \in J : (x_1(t_0), x_2(t_0), \dots) &\text{ is a Cauchy sequence of real numbers} \\
\implies \forall t_0 \in J : (x_1(t_0), x_2(t_0), \dots) &\text{ converges to a limit point, say } x_{lim}(t_0), \text{ because } \mathbb{R} \text{ is complete}
\end{aligned}$$

Now, we define a candidate limit x for (x_n) .

Define x pointwise so that $\forall t_0 \in J : x(t_0) = x_{lim}(t_0)$.

From the definition above, we obviously have: $x(a) = x(b)$.

From (1) with $m \rightarrow \infty$ we have $\forall n > N$:

$$\begin{aligned}
d(x_n, x) &= \max_{t \in J} |x_n(t) - x(t)| < \epsilon \\
\implies \forall t_0 \in J : |x_n(t_0) - x(t_0)| &< \epsilon
\end{aligned}$$

This shows that x_n converges to x **uniformly** on J .

Since the x_n 's are continuous on J and the convergence is uniform, the limit function x is also continuous on J .

Because $x \in C[a, b]$ and $x(a) = x(b)$, then $x \in Y$, and thus Y is complete. \square

2.1.5

Proof.

$$\begin{aligned}
& \sum_{i=1}^n \alpha_i x_i = 0_v \\
\implies & \sum_{i=1}^n \alpha_i t^i = 0_v \\
\implies & \forall i, \alpha_i = 0 \\
\implies & (x_1, \dots, x_n) \text{ is linearly independent.}
\end{aligned}$$

□

2.1.6

Proof. (By Contradiction)

Let $x \in X$.

Suppose x has 2 different representations, x_1 and x_2 , in the basis (e_1, \dots, e_n) , where:

$$x_1 = \sum_{j=1}^n \alpha_j e_j \quad \text{and} \quad x_2 = \sum_{k=1}^n \beta_k e_k$$

Then we have:

$$\begin{aligned}
& x_1 = x_2 \\
\implies & \sum_{j=1}^n \alpha_j e_j = \sum_{k=1}^n \beta_k e_k \\
\implies & \forall i, \alpha_i = \beta_i \quad (\text{Because } (e_1, \dots, e_n) \text{ is linearly independent.})
\end{aligned}$$

This is a contradiction to our assumption that x_1 and x_2 have different representations.

Thus, we conclude that every non-zero vector must have a unique representation in a given basis.

□

2.1.10

Proof.

i. $V = Y \cap Z$ is a subspace:

We must check 3 conditions:

$$1. 0_v \in V$$

$$\begin{aligned} & Y \text{ and } Z \text{ are vector spaces} \\ \implies 0_v \in Y \wedge 0_v \in Z \\ \implies 0_v \in V = Y \cap Z. \end{aligned}$$

$$2. v_1, v_2 \in V \implies v_1 + v_2 \in V$$

$$\begin{aligned} & v_1, v_2 \in V \\ \implies v_1, v_2 \in Y \wedge v_1, v_2 \in Z \\ \implies v_1 + v_2 \in Y \wedge v_1 + v_2 \in Z \\ \implies v_1 + v_2 \in V = Y \cup Z. \end{aligned}$$

$$3. k \in K \text{ and } v \in V \implies kv \in V$$

$$\begin{aligned} & v \in V \\ \implies v \in Y \wedge v \in Z \\ \implies kv \in Y \wedge kv \in Z \\ \implies kv \in V = Y \cup Z. \end{aligned}$$

ii. $V = Y \cup Z$ is not a subspace:

Consider the following counterexample:

Let $V = \mathbb{R}^2$, $Y = \{(x, 0) : x \in \mathbb{R}\}$, $Z = \{(0, y) : y \in \mathbb{R}\}$, and $V = Y \cup Z$.

Clearly, Y and Z are subspaces of \mathbb{R}^2 .

Take $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

We have $v_1, v_2 \in V$, but $v_1 + v_2 = (1, 1) \notin V$.

This shows that the union of subspaces fails to be closed under vector addition.

□

2.1.11

Proof.

We must check 3 conditions:

$$1. 0_v \in M$$

$$\begin{aligned} & \text{Let } v \in M \\ \implies 0v = 0_v \in M. \end{aligned}$$

$$2. v_1, v_2 \in M \implies v_1 + v_2 \in M$$

$$\begin{aligned} & v_1, v_2 \in M \\ \implies v_1 &= \sum_{\forall m_i \in M} \alpha_i m_i \quad \text{and} \quad v_2 = \sum_{\forall m_j \in M} \beta_j m_j \\ \implies v_1 + v_2 &= \sum_{\forall m_k \in M} (\alpha_k + \beta_k) m_k = \sum_{\forall m_k \in M} \gamma_k m_k \\ \implies v_1 + v_2 &\in M. \end{aligned}$$

$$3. k \in K \text{ and } v \in M \implies kv \in M$$

$$\begin{aligned} & v \in V \\ \implies v &= \sum_{\forall m_i \in M} \alpha_i m_i \\ \implies kv &= \sum_{\forall m_i \in M} (k\alpha_i) m_i \\ \implies kv &= \sum_{\forall m_i \in M} \gamma_i m_i \\ \implies kv &\in M. \end{aligned}$$

□

2.2.6

Proof.

Let $x = (\zeta_1, \zeta_2)$, and $y = (\eta_1, \eta_2)$.

i. $\|x\|_1 = |\zeta_1| + |\zeta_2|$

N1 holds because:

$$\|x\|_1 = |\zeta_1| + |\zeta_2| \geq 0.$$

N2 holds because:

\implies :

$$\begin{aligned} \|x\|_1 &= 0 \\ \implies |\zeta_1| + |\zeta_2| &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies x &= (0, 0) = 0_v. \end{aligned}$$

\Leftarrow :

$$\begin{aligned} x &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies |\zeta_1| + |\zeta_2| &= 0 \\ \implies \|x\|_1 &= 0 \end{aligned}$$

N3 holds because:

$$\begin{aligned} \|\alpha x\|_1 &= |\alpha \zeta_1| + |\alpha \zeta_2| \\ &= |\alpha| |\zeta_1| + |\alpha| |\zeta_2| \\ &= |\alpha| (|\zeta_1| + |\zeta_2|) \\ &= |\alpha| \|x\|_1 \end{aligned}$$

N4 holds because:

$$\begin{aligned} \|x + y\|_1 &= |\zeta_1 + \eta_1| + |\zeta_2 + \eta_2| \\ &\leq |\zeta_1| + |\eta_1| + |\zeta_2| + |\eta_2| \\ &= (|\zeta_1| + |\zeta_2|) + (|\eta_1| + |\eta_2|) \\ &= \|x\|_1 + \|y\|_1. \end{aligned}$$

ii. $\|x\|_2 = (\zeta_1^2 + \zeta_2^2)^{\frac{1}{2}}$

N1 holds because:

$$\|x\|_2 = (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \geq 0.$$

N2 holds because:

$\implies :$

$$\begin{aligned} \|x\|_2 &= 0 \\ \implies (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} &= 0 \\ \implies |\zeta_1|^2 + |\zeta_2|^2 &= 0 \\ \implies |\zeta_1|^2, |\zeta_2|^2 &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies x &= (0, 0) = 0_v. \end{aligned}$$

$\Leftarrow :$

$$\begin{aligned} x &= 0 \\ \implies \zeta_1, \zeta_2 &= 0 \\ \implies |\zeta_1|, |\zeta_2| &= 0 \\ \implies |\zeta_1|^2, |\zeta_2|^2 &= 0 \\ \implies |\zeta_1|^2 + |\zeta_2|^2 &= 0 \\ \implies (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} &= 0 \\ \implies \|x\|_2 &= 0 \end{aligned}$$

N3 holds because:

$$\begin{aligned} \|\alpha x\|_2 &= (|\alpha \zeta_1|^2 + |\alpha \zeta_2|^2)^{\frac{1}{2}} \\ &= (|\alpha|^2 |\zeta_1|^2 + |\alpha|^2 |\zeta_2|^2)^{\frac{1}{2}} \\ &= |\alpha| (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \\ &= |\alpha| \|x\|_2 \end{aligned}$$

N4 holds because:

$$\begin{aligned}
\|x + y\|_2^2 &= (\zeta_1 + \eta_1)^2 + (\zeta_2 + \eta_2)^2 \\
&= \zeta_1^2 + 2\zeta_1\eta_1 + \eta_1^2 + \zeta_2^2 + 2\zeta_2\eta_2 + \eta_2^2 \\
&= (\zeta_1^2 + \zeta_2^2) + (\eta_1^2 + \eta_2^2) + 2(\zeta_1\eta_1 + \zeta_2\eta_2) \\
&\leq (\zeta_1^2 + \zeta_2^2) + (\eta_1^2 + \eta_2^2) + 2\sqrt{\zeta_1^2 + \zeta_2^2}\sqrt{\eta_1^2 + \eta_2^2} \quad (\text{By the Cauchy-Schwartz inequality}) \\
&= \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 \\
&= (\|x\|_2 + \|y\|_2)^2
\end{aligned}$$

$$\implies \|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

$$\text{iii. } \|x\|_\infty = \max\{|\zeta_1|, |\zeta_2|\}$$

N1 holds because:

$$\|x\|_\infty = \max\{|\zeta_1|, |\zeta_2|\} \geq 0.$$

N2 holds because:

\implies :

$$\begin{aligned}
&\|x\|_\infty = 0 \\
&\implies \max\{|\zeta_1|, |\zeta_2|\} = 0 \\
&\implies |\zeta_1|, |\zeta_2| = 0 \\
&\implies \zeta_1, \zeta_2 = 0 \\
&\implies x = (\zeta_1, \zeta_2) = (0, 0) = 0_v.
\end{aligned}$$

\Leftarrow :

$$\begin{aligned}
&x = 0 \\
&\implies \zeta_1, \zeta_2 = 0 \\
&\implies |\zeta_1|, |\zeta_2| = 0 \\
&\implies \max\{|\zeta_1|, |\zeta_2|\} = 0 \\
&\implies \|x\|_\infty = 0
\end{aligned}$$

N3 holds because:

$$\begin{aligned}
\|\alpha x\|_\infty &= \max\{|\alpha\zeta_1|, |\alpha\zeta_2|\} \\
&= |\alpha| \max\{|\zeta_1|, |\zeta_2|\} \\
&= |\alpha| \|x\|_\infty
\end{aligned}$$

N4 holds because:

$$\begin{aligned} \|x + y\|_\infty &= \max\{|\zeta_1 + \eta_2|, |\zeta_2 + \eta_2|\} \\ &\leq \max\{|\zeta_1| + |\eta_2|, |\zeta_2| + |\eta_2|\} \\ &\leq \max\{|\zeta_1|, |\zeta_2|\} + \max\{|\eta_1|, |\eta_2|\} \\ &= \|x\|_\infty + \|y\|_\infty. \end{aligned}$$

□

2.2.8

2.6.2

Proof.

Let $v_1, v_2 \in R^2$, where $v_1 = (\zeta_1, \zeta_2)$, $v_2 = (\eta_1, \eta_2)$, and $\alpha \in R$.

i. $T_1 : (\zeta_1, \zeta_2) \mapsto (\zeta_1, 0)$

Additivity:

$$\begin{aligned} T_1(v_1 + v_2) &= T_1(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_1 + \eta_1, 0) \\ &= (\zeta_1, 0) + (\eta_1, 0) \\ &= T_1v_1 + T_1v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_1(\alpha v_1) &= T_1(\alpha \zeta_1, \alpha \zeta_2) \\ &= (\alpha \zeta_1, 0) \\ &= \alpha(\zeta_1, 0) \\ &= \alpha T_1v_1. \end{aligned}$$

Geometric interpretation: Projection onto the x-axis.

ii. $T_2 : (\zeta_1, \zeta_2) \mapsto (0, \zeta_2)$

Proof is similar to part (i).

Geometric interpretation: Projection onto the y-axis.

iii. $T_3 : (\zeta_1, \zeta_2) \mapsto (\zeta_2, \zeta_1)$

Additivity:

$$\begin{aligned} T_3(v_1 + v_2) &= T_3(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_2 + \eta_2, \zeta_1 + \eta_1) \\ &= (\zeta_2, \zeta_1) + (\eta_2, \eta_1) \\ &= T_3v_1 + T_3v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_3(\alpha v_1) &= T_3(\alpha \zeta_1, \alpha \zeta_2) \\ &= (\alpha \zeta_2, \alpha \zeta_1) \\ &= \alpha(\zeta_2, \zeta_1) \\ &= \alpha T_3v_1 \end{aligned}$$

Geometric interpretation: Reflection across the line $y = x$.

$$\text{iv. } T_4 : (\zeta_1, \zeta_2) \mapsto (\gamma\zeta_1, \gamma\zeta_2)$$

Additivity:

$$\begin{aligned} T_4(v_1 + v_2) &= T_4(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\gamma(\zeta_1 + \eta_1), \gamma(\zeta_2 + \eta_2)) \\ &= (\gamma\zeta_1, \gamma\zeta_2) + (\gamma\eta_1, \gamma\eta_2) \\ &= T_4v_1 + T_4v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_4(\alpha v_1) &= T_4(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\gamma\alpha\zeta_1, \gamma\alpha\zeta_2) \\ &= \alpha(\gamma\zeta_1, \gamma\zeta_2) \\ &= \alpha T_4v_1 \end{aligned}$$

□

2.6.6

Proof.

Let X, Y, Z be vector spaces over the same field K .

Let $T : X \longrightarrow Y$, $S : Y \longrightarrow Z$.

Then, the composite operator ST exists and $ST : X \longrightarrow Z$.

Let $x_1, x_2 \in X$, and $\alpha \in K$.

Additivity:

$$\begin{aligned} (ST)(x_1 + x_2) &= S(T(x_1 + x_2)) \\ &= S(Tx_1 + Tx_2) \\ &= (ST)x_1 + (ST)x_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} (ST)(\alpha x_1) &= S(T(\alpha x_1)) \\ &= S(\alpha Tx_1) \\ &= \alpha(ST)x_1. \end{aligned}$$

□

2.7.2

Proof.

$\implies :$

Let $T : X \longrightarrow Y$ be a bounded linear operator, and B_x be a bounded set in X .

Since B_x is bounded, then $\forall x \in B_x :$

$$\|x\| \leq c_x.$$

Let B_y be the image of B_x under T : $B_y = T(B_x)$.

Since T is bounded, then $\forall y \in B_y :$

$$\begin{aligned} \|y\| &= \|Tx\| \\ &\leq \|T\| \|x\| \\ &\leq \|T\| c_x = c_y. \end{aligned}$$

Thus, B_y is bounded.

$\impliedby :$

Suppose T maps bounded sets in X into bounded sets Y .

Let $B_x = \{x \in X : \|x\| = 1\}$.

$\implies B_x$ is bounded.

Let B_y be the image of B_x under T : $B_y = T(B_x)$.

$\implies B_y$ is bounded

$\implies \forall y \in B_y : \|y\| \leq c$

$\implies \forall x \in B_x : \|Tx\| \leq c$

$\implies T$ is bounded by *Lemma 2.7 – 2*.

□

2.7.3

Proof.

Let $x \in X$ s.t. $\|x\| = \alpha < 1$, then:

$$\begin{aligned}y &= \frac{1}{\alpha}x \\ \Rightarrow \|y\| &= \frac{1}{\alpha}\|x\| = 1 \\ \Rightarrow \|Ty\| &\leq \|T\| \\ \Rightarrow \|T(\frac{1}{\alpha}x)\| &\leq \|T\| \\ \Rightarrow \frac{1}{\alpha}\|Tx\| &\leq \|T\| \\ \Rightarrow \|Tx\| &\leq \alpha\|T\| < \|T\|.\end{aligned}$$

□

3.1.4

Proof.

$$\begin{aligned}
 \langle x + y, x - y \rangle &= \langle x, x - y \rangle + \langle y, x - y \rangle \\
 &= \langle x, x \rangle - \langle x, y \rangle + \langle y, x \rangle - \langle y, y \rangle \\
 &= 0 - \langle x, y \rangle + \langle x, y \rangle - 0 = 0 \\
 &= 0.
 \end{aligned}$$

Geometric interpretation for $X = \mathbb{R}^2$:

Since $\|x\| = \|y\|$, then the vectors x and y form a rhombus. The vectors $x + y$ and $x - y$ are the diagonals of that rhombus.

Therefore, this statement simply states that the diagonals of a rhombus are perpendicular/orthogonal.

Geometric interpretation for $X = \mathbb{C}^1$:

Same as $X = \mathbb{R}^2$, since \mathbb{R}^2 and \mathbb{C}^1 have the same geometry. □

3.1.7

Proof. (By Contradiction)

Let $u, v \in V$ s.t. $u \neq 0_v$ and $v := au$, where $a \neq 1$. Then $u \neq v$, and:

$$\begin{aligned}
 &\forall x \in X : \langle x, u \rangle = \langle x, v \rangle \\
 \implies &\forall x \in X : \langle x, u \rangle = \langle x, au \rangle \\
 \implies &\forall x \in X : \langle x, u \rangle = a \langle x, u \rangle
 \end{aligned}$$

But this is impossible since $a \neq 1$. Thus, we must have $u = v$. □

3.1.11

Proof.

To check if the given norm is induced by some inner-product, we could check whether the parallelogram equality holds, but there's an even simpler proof.

First, we note that this space is an l^p space with $p = 1$.

Finally, we note that it was shown in 3.1 – 7 that the space l^p with $p \neq 2$ is not an inner-product space.

Therefore, there is no inner-product that induces the given norm. □

3.2.1

The Euclidean inner product on \mathbb{R}^2 and \mathbb{R}^3 reduces to the dot product.

The dot product of 2 vectors is the product of their length times cosine the angle between them:

$$\begin{aligned}\langle x, y \rangle &= x \cdot y \\ &= \|x\| \|y\| \cos(\theta)\end{aligned}$$

Taking the absolute value of both sides we get:

$$\begin{aligned}|\langle x, y \rangle| &= \|x\| \|y\| |\cos(\theta)| \\ &\leq \|x\| \|y\|\end{aligned}$$

Which is the same result given by the Schwarz inequality.

3.2.2

- i. The trivial subspace containing just the zero sequence.
- ii. Subspaces whose components are all 0 after some index n .

3.2.4

Proof.

By the continuity of the inner product (Lemma 3.2-2), we have:

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle \tag{1}$$

Next, we define the sequence (a_n) by: $a_n := \langle x_n, y \rangle$. Then:

$$\begin{aligned}\forall n \in \mathbb{N} : a_n &= 0 \\ \implies a_n &\rightarrow 0 \\ \implies \langle x_n, y \rangle &\rightarrow 0\end{aligned} \tag{2}$$

Finally:

$$\begin{aligned}(1) \text{ and } (2) &\implies \langle x, y \rangle \rightarrow 0 \\ &\implies x \perp y.\end{aligned}$$

□

3.2.5

Proof.

To prove that $x_n \rightarrow x$, we will show that $\|x_n - x\| \rightarrow 0$:

$$\begin{aligned}\|x_n - x\| &= \langle x_n - x, x_n - x \rangle \\ &= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle \\ &= \|x_n\|^2 - \langle x_n, x \rangle - \langle x, x_n \rangle + \|x\|^2\end{aligned}$$

Taking the limit as $n \rightarrow \infty$:

$$\|x_n - x\| \rightarrow 2\|x\|^2 - 2\langle x, x \rangle = 2\|x\|^2 - 2\|x\|^2 = 0.$$

□

3.2.8

Proof.

\implies :

$$\begin{aligned}\langle x, y \rangle &= 0 \\ \implies \forall \bar{\alpha} : \bar{\alpha} \langle x, y \rangle &= 0 \\ \implies \forall \alpha : \langle x, \alpha y \rangle &= 0 \\ \implies \forall \alpha : \langle x, \alpha y \rangle = 0 \wedge \langle \alpha y, x \rangle &= 0\end{aligned}$$

Next, we have $\forall \alpha$:

$$\begin{aligned}\|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y \rangle \\ &= \langle x, x \rangle + \langle x, \alpha y \rangle + \langle \alpha y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 0 + 0 + \|y\|^2 \\ &\geq \|x\|^2\end{aligned}$$

Finally, taking the square root, we arrive at our desired result:

$$\|x + \alpha y\| \geq \|x\|.$$

\Leftarrow :

$$\begin{aligned} & \forall \alpha : \|x\| \leq \|x + \alpha y\| \\ \implies \forall \alpha : \|x\|^2 & \leq \|x + \alpha y\|^2 \\ & = \langle x + \alpha y, x + \alpha y \rangle \\ & = \langle x, x \rangle + \langle x, \alpha y \rangle + \langle \alpha y, x \rangle + \langle y, y \rangle \\ & = \|x\|^2 + \overline{\langle \alpha y, x \rangle} + \langle \alpha y, x \rangle + \|y\|^2 \\ & = \|x\|^2 + 2\operatorname{Re}\langle \alpha y, x \rangle + \|y\|^2 \\ & = \|x\|^2 + 2\operatorname{Re}(\alpha \langle y, x \rangle) + \|y\|^2 \\ \implies \forall \alpha : 2\operatorname{Re}\langle \alpha y, x \rangle & \leq 0 \\ \implies \langle x, y \rangle & = 0. \end{aligned}$$

□

4.8.1

Proof.

□

5.1.2

Proof.

Let $x, y \in X$

$$\begin{aligned}
 d(Tx, Ty) &= |Tx - Ty| \\
 &= \left| \frac{x}{2} + x^{-1} - \frac{y}{2} - y^{-1} \right| \\
 &= \left| \frac{x-y}{2} + \frac{y-x}{xy} \right| \\
 &= |x-y| \left| \frac{1}{2} - \frac{1}{xy} \right| \\
 &\leq \frac{1}{2} |x-y| \\
 &= \frac{1}{2} d(x, y)
 \end{aligned}$$

Therefore, T is a contraction and the smallest α is $\frac{1}{2}$.

□

5.1.5

Proof. (By Contradiction)

Suppose T has 2 different fixed points x and y , then:

$$\begin{aligned}
 x &= Tx \wedge y = Ty \\
 \implies d(x, y) &= d(Tx, Ty)
 \end{aligned} \tag{1}$$

But we are given that:

$$d(Tx, Ty) < d(x, y) \tag{2}$$

(1) and (2) $\implies d(x, y) < d(x, y)$.

This is a contradiction. Thus, T has a unique fixed point.

□

5.1.6

$i. T$ is a contraction $\implies T^n$ is a contraction:

Proof. (By Induction)

Base case ($n = 1$):

$T^1 = T$ which is a contraction by assumption.

Inductive step ($n = k + 1$):

Suppose the induction hypothesis holds for $n = k$, and let $x, y \in X$, then:

$$d(T^k x, T^k y) \leq \beta d(x, y) \quad (\text{where } 0 < \beta < 1) \quad (1)$$

Next, let $T^k x = u$ and $T^k y = v$. Because T is a contraction then:

$$d(Tu, Tv) \leq \alpha d(u, v) \quad (\text{where } 0 < \alpha < 1) \quad (2)$$

Putting (1) and (2) together, we get:

$$\begin{aligned} d(T^{k+1}x, T^{k+1}y) &= d(Tu, Tv) \\ &\leq \alpha d(u, v) \\ &\leq \alpha \beta d(x, y) \\ &= \gamma d(x, y) \quad (\text{where } 0 < \gamma < 1) \end{aligned}$$

Therefore, T^{k+1} is a contraction.

□

ii. T^n is a contraction $\not\Rightarrow T$ is a contraction:

Proof. (By Counterexample)

Let the mapping $T^n : R \rightarrow R$ be defined by:

$$Tx := \begin{cases} -x, & \text{if } x \geq 0 \\ \frac{x}{2}, & \text{if } x < 0 \end{cases}$$

Then, T does not contract the distance between positive numbers.
However, T^2 is a contraction.

□