Functional Analysis Assignment (Chapter 1)

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1.1.1

The distance on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$. We must check that the 4 axioms (M1 to M4) are satisfied.

M1 holds, since the absolute value of the difference between 2 real points is real, finite, and non-negative.

M2 holds, since $d(x, y) = |x - y| = 0 \iff x = y$.

M3 holds, since $d(x, y) = |x - y| = |y - x| = d(y, x)$.

M4 holds, since the triangle inequality holds for the absolute value.

1.1.2

$d(x, y) = (x - y)^2$ is not a valid metric since it does not satisfy the triangle inequality.

Proof. (By counterexample)
Let $a, b, c \in R$, where $a = 0, b = 1, c = 5$, then:

$d(a, c) = (a - c)^2 = (0 - 5)^2 = 25$.
$d(a, b) = (a - b)^2 = (0 - 1)^2 = 1$.
$d(b, c) = (b - c)^2 = (1 - 5)^2 = 16$.
$d(a, b) + d(b, c) = 1 + 16 = 17$.

Therefore, $d(a, c) \not\leq d(a, b) + d(b, c)$.

□

1.1.3

Since the distance function $d(x, y) = |x - y|$ defines a metric on $\mathbb{R}$, as shown in 1.1.1, then it is clear that the square root of that metric is also real, finite, and non-negative, definite, and symmetric (i.e. M1 to M3 hold).

The triangle inequality can be shown to hold by noting that the square root function is an increasing function with a negative second derivative in the interval $(0, \infty)$.

This shows that $d(x, y) = \sqrt{|x - y|}$ is a metric on $\mathbb{R}$.

1.1.4

i. $|X| = 2$

Let $X = \{a, b\}$, then d must satisfy:

$d(a, a) = d(b, b) = 0$, and
$d(a, b) = d(b, a) = c$, where c is any non-negative real number.

ii. $|X| = 1$

In this case the only valid metric is $d(a, a) = 0$.

1.1.5

i. Conditions for kd to be a metric

If d is a metric, then kd automatically satisfies axioms M2-M4.

For axiom M1 to hold, k must be a non-negative real number.

ii. Conditions for $k + d$ to be a metric

To satisfy axiom M2, k must be zero.

1.1.6

Proof. (By induction on the length of the sequence)
Let $X = (x_j),\ Y = (y_j),\ Z = (z_j)$ be 3 bounded sequences.

Base case
Consider the subsequence of X, Y, Z consisting of just their first element. Then by the triangle inequality for numbers:

$$|x_1 - z_1| \le |x_1 - y_1| + |y_1 - z_1|$$
$$\implies \sup |x_1 - z_1| \le \sup(|x_1 - y_1| + |y_1 - z_1|)$$
$$\implies \sup |x_1 - z_1| \le \sup |x_1 - y_1| + \sup |y_1 - z_1|.$$

Therefore $d(x, z) \le d(x, y) + d(y, z)$ holds for sequences of length 1.

Inductive step
Next, we'll consider the sub-sequences of X, Y, Z consisting of the first $n+1$ elements. Suppose that the induction hypothesis holds for sequences of length n, i.e.:

$$\sup_{j \in \{1..n\}} |x_j - z_j| \le \sup_{j \in \{1..n\}} |x_j - y_j| + \sup_{j \in \{1..n\}} |y_j - z_j| \text{ holds.}$$

Then we can partition each sequence of length $n+1$ into 2 sub-sequences: the first sequence contains the first n elements and the second contains the last element.

The distance between any 2 sequences of length $n+1$ then becomes:
$\max(\sup_{j\in 1..n} |x_j - z_j|,\ \sup |x_{n+1} - z_{n+1}|)$

Finally, applying the induction hypothesis we get:
$\max(\sup_{j\in 1..n} |x_j - z_j|,\ \sup |x_{n+1} - z_{n+1}|) \leq \max(\sup_{j\in 1..n} |x_j - y_j| + \sup_{j\in 1..n} |y_j - z_j|,\ \sup |x_{n+1} - y_{n+1}| + \sup |y_{n+1} - z_{n+1}|)$

$\implies \sup_{j\in 1..n+1} |x_j - z_j| \leq \sup_{j\in 1..n+1} |x_j - y_j| + \sup_{j\in 1..n+1} |y_j - z_j|$

Therefore $d(x,z) \leq d(x,y) + d(y,z)$ holds for sequences of any length.

$\square$

1.1.7

$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

Which is the discrete metric.

1.1.8

M1 holds because $|x(t) - y(t)|$ is a positive function, and the integral of a positive function is positive.

M2 holds because $|x(t) - x(t)| = 0$, and the integral of 0 is 0.

M3 holds because $d(x,y) = \int_a^b |x(t) - y(t)|dt = \int_a^b |y(t) - x(t)|dt = d(y,x)$.

To show that M4 holds, let $x, y, z \in X$, then:

$$\begin{aligned}
d(x,z) &= \int_a^b |x(t) - z(t)| \; dt \\
&= \int_a^b |(x(t) - y(t)) - (z(t) - y(t))| \; dt \\
&\geq \int_a^b (|x(t) - y(t)| - |z(t) - y(t)|) \; dt \qquad \text{(By the triangle inequality of absolute values)} \\
&= \int_a^b |x(t) - y(t)| dt - \int_a^b |z(t) - y(t)| dt \\
&= d(x,y) - d(z,y)
\end{aligned}$$

$$\implies d(x,y) \leq d(x,z) + d(z,y).$$