Functional Analysis Assignment (Chapter 1)

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1.1.1

The distance on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$. We must check that the 4 axioms (M1 to M4) are satisfied.

M1 holds, since the absolute value of the difference between 2 real points is real, finite, and non-negative.

M2 holds, since $d(x, y) = |x - y| = 0 \iff x = y$.

M3 holds, since $d(x, y) = |x - y| = |y - x| = d(y, x)$.

M4 holds, since the triangle inequality holds for the absolute value.

1.1.2

$d(x, y) = (x - y)^2$ is not a valid metric since it does not satisfy the triangle inequality.

Proof. (By Counterexample)
Let $a, b, c \in R$, where $a = 0, b = 1, c = 5$, then:

$d(a, c) = (a - c)^2 = (0 - 5)^2 = 25.$
$d(a, b) = (a - b)^2 = (0 - 1)^2 = 1.$
$d(b, c) = (b - c)^2 = (1 - 5)^2 = 16.$
$d(a, b) + d(b, c) = 1 + 16 = 17.$

Therefore, $d(a, c) \not\leq d(a, b) + d(b, c)$.

$\square$

1.1.3

Since the distance function $d(x, y) = |x - y|$ defines a metric on $\mathbb{R}$, as shown in 1.1.1, then it is clear that the square root of that metric is also real, finite, and non-negative (i.e. M1 holds); definite (i.e. M2 holds); and symmetric (i.e. M3 holds).

The triangle inequality can be shown to hold by noting that the square root function is an increasing function with a negative second derivative in the interval $(0, \infty)$.

This shows that $d(x, y) = \sqrt{|x - y|}$ is a metric on $\mathbb{R}$.

1.1.4

i. $|X| = 2$

Let $X = \{a, b\}$, then d must satisfy:

$d(a, a) = d(b, b) = 0$, and
$d(a, b) = d(b, a) = c$, where c is any non-negative real number.

ii. $|X| = 1$

In this case the only valid metric is $d(a, a) = 0$.

1.1.5

i. Conditions for kd to be a metric

If d is a metric, then kd automatically satisfies axioms M2-M4.

For axiom M1 to hold, k must be a non-negative real number.

ii. Conditions for $k + d$ to be a metric

To satisfy axiom M2, k must be zero.

1.1.6

Proof. (By Induction on the Length of the Sequence)
Let $X = (x_j),\ Y = (y_j),\ Z = (z_j)$ be 3 bounded sequences.

Base case
Consider the subsequence of X, Y, Z consisting of just their first element. Then by the triangle inequality for numbers:

$$\begin{aligned} & |x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \\ \implies & \sup |x_1 - z_1| \leq \sup(|x_1 - y_1| + |y_1 - z_1|) \\ \implies & \sup |x_1 - z_1| \leq \sup |x_1 - y_1| + \sup |y_1 - z_1|. \end{aligned}$$

Therefore $d(x, z) \leq d(x, y) + d(y, z)$ holds for sequences of length 1.

Inductive step
Next, we'll consider the sub-sequences of X, Y, Z consisting of the first $n+1$ elements. Suppose that the induction hypothesis holds for sequences of length n, i.e.:

$$\sup_{j \in \{1..n\}} |x_j - z_j| \leq \sup_{j \in \{1..n\}} |x_j - y_j| + \sup_{j \in \{1..n\}} |y_j - z_j|.$$

Then we can partition each sequence of length $n+1$ into 2 sub-sequences: the first sequence contains the first n elements and the second contains the last element.

The distance between any 2 sequences of length $n+1$ then becomes:
$\max(\sup_{j\in 1..n} |x_j - z_j|, \ \sup |x_{n+1} - z_{n+1}|)$

Finally, applying the induction hypothesis we get:
$\max(\sup_{j\in 1..n} |x_j - z_j|, \ \sup |x_{n+1} - z_{n+1}|) \leq \max(\sup_{j\in 1..n} |x_j - y_j| + \sup_{j\in 1..n} |y_j - z_j|, \ \sup |x_{n+1} - y_{n+1}| + \sup |y_{n+1} - z_{n+1}|)$

$\implies \sup_{j\in 1..n+1} |x_j - z_j| \leq \sup_{j\in 1..n+1} |x_j - y_j| + \sup_{j\in 1..n+1} |y_j - z_j|$

Therefore $d(x,z) \leq d(x,y) + d(y,z)$ holds for sequences of any length.

$\square$

1.1.7

$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

Which is the discrete metric.

1.1.8

M1 holds because $|x(t) - y(t)|$ is a positive function, and the integral of a positive function is positive.

M2 holds because $|x(t) - x(t)| = 0$, and the integral of the zero function is 0.

M3 holds because $d(x,y) = \int_a^b |x(t) - y(t)| dt = \int_a^b |y(t) - x(t)| dt = d(y,x)$.

To show that M4 holds, let $x, y, z \in X$, then:

$$\begin{aligned}
d(x,z) &= \int_a^b |x(t) - z(t)|\ dt \\
&= \int_a^b |(x(t) - y(t)) - (z(t) - y(t))|\ dt \\
&\geq \int_a^b (|x(t) - y(t)| - |z(t) - y(t)|)\ dt \qquad \text{(By the triangle inequality of absolute values)} \\
&= \int_a^b |x(t) - y(t)| dt - \int_a^b |z(t) - y(t)| dt \\
&= d(x,y) - d(z,y)
\end{aligned}$$

$$\implies d(x,y) \leq d(x,z) + d(z,y).$$

1.2.3

Proof.

Let (ζ_j), (η_j), and $(\theta_j) \in l^p$, where (ζ_j) is any point, and define (η_j) and (θ_j) as follows:

$$\eta_j := \begin{cases} 1 & j \leq n \\ 0 & j > n \end{cases}$$

$$\theta_j := \begin{cases} \zeta_j & j \leq n \\ 0 & j > n \end{cases}$$

Applying the Cauchy-Schwarz inequality to (θ_j) and (η_j), we get:

$$\sum_{j=1}^{\infty} |\theta_j \eta_j| \leq \sqrt{\sum_{k=1}^{\infty} |\theta_j|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_j|^2}$$

$$\sum_{j=1}^{n} |\theta_j \eta_j| \leq \sqrt{\sum_{k=1}^{n} |\theta_j|^2} \sqrt{\sum_{m=1}^{n} |\eta_j|^2}$$

$$\sum_{j=1}^{n} |\theta_j| \leq \sqrt{\sum_{k=1}^{n} |\theta_j|^2} \sqrt{\sum_{m=1}^{n} 1}$$

$$\sum_{j=1}^{n} |\zeta_j| \leq \sqrt{\sum_{k=1}^{n} |\zeta_j|^2} \sqrt{n}$$

$$\left(\sum_{j=1}^{n} |\zeta_j| \right)^2 \leq n \sum_{k=1}^{n} |\zeta_j^2|$$

□

1.2.4

1.2.5

Proof.

The sequence $\zeta_j = \frac{1}{j}$ is divergent for $p = 1$ but convergent for all $p > 1$.

□

1.2.11

Proof.

M1 holds because:

$$\begin{aligned}
&d(x,y) \geq 0 \\
\implies &\frac{d(x,y)}{1+d(x,y)} \geq 0 \\
\implies &\tilde{d}(x,y) \geq 0.
\end{aligned}$$

M2 holds because:
$\underline{\implies}$:

$$\begin{aligned}
&\tilde{d}(x,y) = 0 \\
\implies &\frac{d(x,y)}{1+d(x,y)} = 0 \\
\implies &d(x,y) = 0 \\
\implies &x = y.
\end{aligned}$$

$\underline{\impliedby}$:

$$\begin{aligned}
&x = y \\
\implies &d(x,y) = 0 \\
\implies &\frac{d(x,y)}{1+d(x,y)} = 0 \\
\implies &\tilde{d}(x,y) = 0.
\end{aligned}$$

M3 holds because:

$$\begin{aligned}
&d(x,y) = d(y,x) \\
\implies &\frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} \\
\implies &\tilde{d}(x,y) = \tilde{d}(y,x).
\end{aligned}$$

M4 holds because:

$$\begin{aligned}
&d(x,z) &&\leq d(x,y) + d(y,z) \\
\implies &\frac{d(x,z)}{1+d(x,z)} &&\leq \frac{d(x,y)}{1+d(x,z)} + \frac{d(y,z)}{1+d(x,z)} \\
\implies &\frac{d(x,z)}{1+d(x,z)} &&\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\
\implies &\tilde{d}(x,z) &&\leq \tilde{d}(x,y) + \tilde{d}(y,z).
\end{aligned}$$

$\square$

1.3.4

Proof.
$\implies$ **(By Construction)**:

Let A be a non-empty open subset, and $\delta(A)$ be the diameter of A.
Since A is non-empty and open, then $|A| > 1$.
Let a_1, a_2 be 2 different points in A.
Consider the 2 open balls $B(a_1, \delta(A))$ and $B(a_2, \delta(A))$.
We have $B(a_1, \delta(A)) = B(a_2, \delta(A)) = A$.
Thus, $A = B(a_1, \delta(A)) \cup B(a_2, \delta(A))$.

$\impliedby$ **(By Strong Induction)**:

We will prove a **stronger** result: The union of non-empty open subsets is a non-empty open subset.
Let $B = (B_1, B_2, \ldots, B_n)$ be a list of open subsets (balls **or not**) in X.

Base case: $|B| = 2$

$$\begin{aligned} & A = B_1 \cup B_2 \\ \implies & \forall a \in A\ ,\ a \in B_1 \lor a \in B_2 \\ \implies & \forall a \in A,\ \exists \text{ a ball about } a \\ \implies & A \text{ is a non-empty open subset of } X. \end{aligned}$$

Inductive step: $|B| = k > 2$

$$\begin{aligned} A &= \bigcup_{j=1}^{k} B_j \\ &= \bigcup_{j=1}^{k-1} B_j + B_n \\ &= C + B_k \end{aligned} \tag{1}$$

Where C is a non-empty open subset by the induction hypothesis for $n = k - 1$.

Applying the induction hypothesis to (1) for $n = 2$, we conclude that A is a non-empty open subset in X.

□

1.3.6

Proof. (By Construction)

We will prove this statement by recursively (and infinitely) applying the definition of an accumulation point on increasingly smaller ϵ-neighborhoods.

Let the sequence of (ϵ_j) be such that $\forall j,\ \epsilon_j > 0$.

Since x_0 is an accumulation point, then each ϵ_j-neighborhood contains at least one point $y \in A$.

Let y_j to be any point in the ϵ_j-neighborhood of x_0 that satisfies the following condition:

$$d(y_j, x_0) = \max_{\forall y_k \in \tilde{B}(x_0;\epsilon_j)} d(y_k, x_0)$$

Define ϵ_j recursively as follows:

$$\epsilon_j := \begin{cases} \epsilon_1 & j = 1 \\ d(x_0, y_{j-1})/2 & j > 1 \end{cases}$$

Then the sequence (y_j) is an infinite sequence of distinct points in A, all contained inside the ϵ_1-neighborhood of x_0.

Since ϵ_1 was arbitrarily chosen, this completes the proof.

$\square$

1.3.13

Proof.
This follows directly from the definition that a separable space X contains a subset Y that is countable and dense in X.

$\square$

1.3.14

Proof.
$\underline{\Longrightarrow}$:
Let $T : X \longrightarrow Y$ be a continuous map and M be a closed subset in Y.

$\Longrightarrow M^c$ is open	(Because the complement of a closed set is an open set by definition 1.3-2)
$\Longrightarrow T^{-1}(M^c)$ is open	(By continuity of T)
$\Longrightarrow (T^{-1}(M^c))^c$ is closed	(Because the complement of an open set is a closed set by definition 1.3-2)
$\Longrightarrow T^{-1}((M^c)^c)$ is closed	(By the identity: $f^{-1}(A^c) = [f^{-1}(A)]^c$)
$\Longrightarrow T^{-1}(M)$ is open.	(By the identity: $A = [A^c]^c$)

$\underline{\Longleftarrow}$:

Let $T : X \longrightarrow Y$ be a map such that the inverse image of any closed set in Y is a closed set in X.

Let A be an open set in Y.

$\Longrightarrow A^c$ is closed	(Because the complement of an open set is a closed set by definition 1.3-2)
$\Longrightarrow T(A^c)$ is closed	(By our assumption the inverse image of any closed set is a closed set)
$\Longrightarrow [T(A^c)]^c$ is open	(Because the complement of a closed set is an open set by definition 1.3-2)
$\Longrightarrow T([A^c]^c)$ is open	(By the identity: $f^{-1}(A^c) = [f^{-1}(A)]^c$)
$\Longrightarrow T(A)$ is open	(By the identity: $A = [A^c]^c$)
$\Longrightarrow T$ is continuous.	

$\square$

1.4.2

Proof.

$$\begin{aligned}
&(x_{n_k}) \text{ converges to x}\\
\Longrightarrow\ &\forall\epsilon > 0,\ \exists K(\epsilon) \ni \forall k > K,\ d(x_{n_k}, x) < \epsilon\\
\Longrightarrow\ &\forall\epsilon > 0,\ \exists N(\epsilon) = K(\epsilon) \ni \forall n > N,\ d(x_n, x) < \epsilon \quad \text{(Because } (x_n) \text{ is Cauchy)}\\
\Longrightarrow\ &(x_n) \text{ converges to } x.
\end{aligned}$$

□

1.4.4

Proof.
Take ϵ with any concrete value, say $\epsilon = 1$.

Because (x_n) is Cauchy, then:

$$\exists N \ni \forall n, m > N, d(x_n, x_m) < \epsilon = 1.$$

Define $a = \max\limits_{\forall i,j\in\{1,\ldots,N\}} d(x_i, x_j)$.

Therefore we have $\forall n$:

$$d(x_n, x_N) \leq \max(1, a) \leq 1 + a$$

By the triangle inequality we have $\forall n, m$:

$$\begin{aligned}
d(x_n, d_m) &\leq d(x_n, x_N) + d(x_N, x_m)\\
&\leq (1 + a) + (1 + a) = 2(1 + a) = u.
\end{aligned}$$

This shows that u is an upperbound for the Cauchy sequence.

□

1.4.5

Proof. (By Counterexample)
i. Boundedness does **not** imply Cauchiness:
Consider the sequence $x_n = (-1)^n$.
(x_n) is bounded by 2, but the sequence is not Cauchy, because for any $0 < \epsilon < 2$:

$$\nexists N \ni \forall n, m > N,\ d(x_n, x_m) < \epsilon.$$

ii. Boundedness does **not** imply Convergence:
Consider the sequence $x_n = sin(n)$.
(x_n) is bounded by 2, but the sequence oscillates and does not converge.

□

1.4.6

Proof.

Take any $\epsilon > 0$, then since (x_n) and (y_n) are Cauchy:

$$\exists N_x \ni \forall n, m > N,\ d(x_n, x_m) < \epsilon$$

and,

$$\exists N_y \ni \forall n, m > N,\ d(y_n, y_m) < \epsilon.$$

Let $N = \max\{N_x, N_y\}$, then $\forall n > N$:

$$d(x_n, x_N) < \epsilon\ \wedge\ d(y_n, y_N) < \epsilon.$$

By the triangle inequality we have $\forall n > N$:

$$\begin{aligned} a_n = d(x_n, y_n) &\leq d(x_n, x_N) + d(x_N, y_N) + d(y_N, y_n) \\ &\leq \epsilon + c + \epsilon \\ &= 2\epsilon + c. \end{aligned}$$

Finally, $\forall n, m > N$:

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, a_N) + d(a_N, a_m) \\ &= |d(x_n, y_n) - d(x_N, y_N)| + |d(x_m, y_m) - d(x_N, y_N)| \\ &\leq |2\epsilon + c - c| + |2\epsilon + c - c| \\ &= |2\epsilon| + |2\epsilon| \\ &\leq 4\epsilon. \end{aligned}$$

Therefore (a_n) is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is complete, then (a_n) is convergent.

$\square$

1.5.2

Proof.

Assume (x_i) is a Cauchy sequence in X

$$\Longrightarrow \forall \epsilon > 0,\ \exists N \ni \forall j, k \geq N:\ d(x_j, x_k) = \max_{i=1}^{n} |x_i^{(j)} - x_i^{(k)}| < \epsilon \tag{1}$$

$\Longrightarrow \forall \epsilon > 0,\ \exists N \ni \forall j, k \geq N,\ \forall i \in \{1, \ldots, n\}:\ |x_i^{(j)} - x_i^{(k)}| < \epsilon$

$\Longrightarrow \forall i \in \{1, \ldots, n\}:\ (x_i^{(1)}, x_i^{(2)}, \ldots)$ is a Cauchy sequence of real numbers

$\Longrightarrow \forall i \in \{1, \ldots, n\}:\ (x_i^{(1)}, x_i^{(2)}, \ldots)$ converges to a limit point x_i because $\mathbb{R}$ is complete.

Next, we define a candidate limit for (x_i):

$$x = (x_1, \ldots, x_n).$$

Clearly, $x \in X$, and by (1) we have:

$$\forall j \geq N : d(x_j, x) = \epsilon.$$

This shows that x is the limit of (x_i) and proves completeness of X.

□

1.5.5

Proof.
Assume (x_n) is a Cauchy sequence in X
$\Longrightarrow\ \forall \epsilon > 0,\ \exists N \ni \forall n, m \geq N:\ d(x_n, x_m) = |x_n - x_m| < \epsilon.$

Take $\epsilon = 0.5$
$\Longrightarrow\ \exists N \ni \forall n, m \geq N : |x_n - x_m| < 0.5$
$\Longrightarrow\ \forall n, m \geq N : x_n = x_m$
$\Longrightarrow\ (x_n) \longrightarrow x_N.$

□

1.5.6

Proof. (By Counterexample)
Take the sequence $x_n = n$.

First we show that it is Cauchy.

Given any $\epsilon > 0$, we can take $N(\epsilon) = \tan(\frac{\pi}{2} - \epsilon)$ so that $\forall n, m \geq N$:

$$\arctan(x_m), \arctan(x_n) \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$$

$$
\begin{aligned}
\implies d(x_n, x_m) &= |\arctan(x_n) - \arctan(x_m)| \\
&\leq (\frac{\pi}{2} - \epsilon) - \frac{\pi}{2} \\
&= \epsilon.
\end{aligned}
$$

Therefore (x_n) is Cauchy as desired.

Now we show that (x_n) does not converge.
We observe that the sequence wants to converge to $\frac{\pi}{2}$, since:

$$\lim_{n\to\infty} d(x_n, \frac{\pi}{2}) = 0$$

But there is no element $x \in \mathbb{R}$, such that $\arctan(x) = \frac{\pi}{2}$.

Thus our metric space is **incomplete.**

□

1.5.8

Proof.

Let (x_n) be a Cauchy sequence in $Y \subseteq [a, b]$, and let J = [a,b]

$$\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N : d(x_n, x_m) = \max_{t\in J} |x_n(t) - x_m(t)| < \epsilon \tag{1}$$

$\implies \forall \epsilon > 0, \exists N \ni \forall n, m \geq N,\ \forall t \in J : |x_n(t) - x_m(t)| < \epsilon$

$\implies \forall t_0 \in J :\ (x_1(t_0), x_2(t_0), \ldots)$ is a Cauchy sequence of real numbers

$\implies \forall t_0 \in J :\ (x_1(t_0), x_2(t_0), \ldots)$ converges to a limit point, say $x_{lim}(t_0)$, because $\mathbb{R}$ is complete

Now, we define a candidate limit x for (x_n).

Define x pointwise so that $\forall t_0 \in J :\ x(t_0) = x_{lim}(t_0)$.

From the defintion above, we obviously have: $x(a) = x(b)$.

From (1) with $m \to \infty$ we have $\forall n > N$:

$$
\begin{aligned}
d(x_n, x) = \max_{t\in J} |x_n(t) - x(t)| < \epsilon \\
\implies \forall t_0 \in J : |x_n(t_0) - x(t_0) < \epsilon
\end{aligned}
$$

This shows that x_m converges to x **uniformly** on J.

Since the x_m's are continuous on J and the convergence is uniform, the limit function x is also continuous on J.

Because $x \in C[a,b]$ and $x(a) = x(b)$, then $x \in Y$, and thus Y is complete.

$\square$

2.1.5

Proof.

$$
\begin{aligned}
&\sum_{i=1}^{n} \alpha_i x_i = 0_v \\
\implies &\sum_{i=1}^{n} \alpha_i t^i = 0_v \\
\implies &\forall i,\ \alpha_i = 0 \\
\implies &(x_1, \ldots, x_n) \text{ is linearly independent.}
\end{aligned}
$$

□

2.1.6

Proof. (By Contradiction)

Let $x \in X$.

Suppose x has 2 different representations, x_1 and x_2, in the basis $(e_1, \ldots, e_n)$, where:

$$x_1 = \sum_{j=1}^{n} \alpha_j e_j \qquad \text{and} \qquad x_2 = \sum_{k=1}^{n} \beta_k e_k$$

Then we have:

$$
\begin{aligned}
&x_1 = x_2 \\
\implies &\sum_{j=1}^{n} \alpha_j e_j = \sum_{k=1}^{n} \beta e_k \\
\implies &\forall i,\ \alpha_i = \beta_i \qquad \text{(Because } (e_1, \ldots, e_n) \text{ is linearly independent.)}
\end{aligned}
$$

This is a contradiction to our assumption that x_1 and x_2 have different representations.

Thus, we conclude that every non-zero vector must have a unique representation in a given basis.

□

2.1.10

Proof.

i. $V = Y \cap Z$ is a subspace:

We must check 3 conditions:

1. $0_v \in V$

$$\begin{aligned} & Y \text{ and } Z \text{ are vector spaces} \\ \implies & 0_v \in Y \wedge 0_v \in Z \\ \implies & 0_v \in V = Y \cap Z. \end{aligned}$$

2. $v_1, v_2 \in V \implies v_1 + v_2 \in V$

$$\begin{aligned} & v1, v_2 \in V \\ \implies & v_1, v_2 \in Y \wedge v_1, v_2 \in Z \\ \implies & v_1 + v_2 \in Y \wedge v_1 + v_2 \in Z \\ \implies & v_1 + v_2 \in V = Y \cup Z. \end{aligned}$$

3. $k \in K$ and $v \in V \implies kv \in V$

$$\begin{aligned} & v \in V \\ \implies & v \in Y \wedge v \in Z \\ \implies & kv \in Y \wedge kv \in Z \\ \implies & kv \in V = Y \cup Z. \end{aligned}$$

ii. $V = Y \cup Z$ is not a subspace:

Consider the following counterexample:
Let $V = R^2$, $Y = \{(x, 0) : x \in R\}$, $Z = \{(0, y) : y \in R\}$, and $V = Y \cup Z$.
Clearly, Y and Z are subspaces of R^2.
Take $v_1 = (1, 0)$ and $v_2 = (0, 1)$.
We have $v_1, v_2 \in V$, but $v_1 + v_2 = (1, 1) \notin V$.
This shows that the union of subspaces fails to be closed under vector addition.

□

2.1.11

Proof.
We must check 3 conditions:
1. $0_v \in M$

$$\begin{aligned} & \text{Let } v \in M \\ \implies & 0v = 0_v \in M. \end{aligned}$$

2. $v_1, v_2 \in M \implies v_1 + v_2 \in M$

$$\begin{aligned}
& v1, v_2 \in M \\
\Longrightarrow & \, v_1 = \sum_{\forall m_i \in M} \alpha_i m_i \quad \text{and} \quad v_2 = \sum_{\forall m_j \in M} \beta_j m_j \\
\Longrightarrow & \, v_1 + v_2 = \sum_{\forall m_k \in M} (\alpha_k + \beta_k) m_k = \sum_{\forall m_k \in M} \gamma_k m_k \\
\Longrightarrow & \, v_1 + v_2 \in M.
\end{aligned}$$

3. $k \in K$ and $v \in M \implies kv \in M$

$$\begin{aligned}
& v \in V \\
\Longrightarrow & \, v = \sum_{\forall m_i \in M} \alpha_i m_i \\
\Longrightarrow & \, kv = \sum_{\forall m_i \in M} (k\alpha_i) m_i \\
\Longrightarrow & \, kv = \sum_{\forall m_i \in M} \gamma_i m_i \\
\Longrightarrow & \, kv \in M.
\end{aligned}$$

$\square$

2.2.6

Proof.

Let $x = (\zeta_1, \zeta_2)$, and $y = (\eta_1, \eta_2)$.

i. $||x||_1 = |\zeta_1| + |\zeta_2|$

N1 holds because:

$$||x||_1 = |\zeta_1| + |\zeta_2| \geq 0.$$

N2 holds because:

$\implies$:

$$\begin{aligned}
&||x||_1 = 0 \\
\implies &|\zeta_1| + |\zeta_2| = 0 \\
\implies &|\zeta_1|, |\zeta_2| = 0 \\
\implies &\zeta_1, \zeta_2 = 0 \\
\implies &x = (0, 0) = 0_v.
\end{aligned}$$

$\impliedby$:

$$\begin{aligned}
&x = 0 \\
\implies &\zeta_1, \zeta_2 = 0 \\
\implies &|\zeta_1|, |\zeta_2| = 0 \\
\implies &|\zeta_1| + |\zeta_2| = 0 \\
\implies &||x||_1 = 0
\end{aligned}$$

N3 holds because:

$$\begin{aligned}
||\alpha x||_1 &= |\alpha\zeta_1| + |\alpha\zeta_2| \\
&= |\alpha||\zeta_1| + |\alpha||\zeta_2| \\
&= |\alpha|(|\zeta_1| + |\zeta_2|) \\
&= |\alpha| \; ||x||_1
\end{aligned}$$

N4 holds because:

$$\begin{aligned}
||x + y||_1 &= |\zeta_1 + \eta_1| + |\zeta_2 + \eta_2| \\
&\leq |\zeta_1| + |\eta_1| + |\zeta_2| + |\eta_2| \\
&= (|\zeta_1| + |\zeta_2|) + (|\eta_1| + |\eta_2|) \\
&= ||x||_1 + ||y||_1.
\end{aligned}$$

ii. $||x||_2 = (\zeta_1^2 + \zeta_2^2)^{\frac{1}{2}}$

N1 holds because:

$$||x||_2 = (|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \geq 0.$$

N2 holds because:
$\implies$:

$$\begin{aligned}
&||x||_2 = 0 \\
\implies &(|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} = 0 \\
\implies &|\zeta_1|^2 + |\zeta_2|^2 = 0 \\
\implies &|\zeta_1|^2, |\zeta_2|^2 = 0 \\
\implies &|\zeta_1|, |\zeta_2| = 0 \\
\implies &\zeta_1, \zeta_2 = 0 \\
\implies &x = (0,0) = 0_v.
\end{aligned}$$

$\impliedby$:

$$\begin{aligned}
&x = 0 \\
\implies &\zeta_1, \zeta_2 = 0 \\
\implies &|\zeta_1|, |\zeta_2| = 0 \\
\implies &|\zeta_1|^2, |\zeta_2|^2 = 0 \\
\implies &|\zeta_1|^2 + |\zeta_2|^2 = 0 \\
\implies &(|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} = 0 \\
\implies &||x||_2 = 0
\end{aligned}$$

N3 holds because:

$$\begin{aligned}
||\alpha x||_2 &= (|\alpha\zeta_1|^2 + |\alpha\zeta_2|^2)^{\frac{1}{2}} \\
&= (|\alpha|^2|\zeta_1|^2 + |\alpha|^2|\zeta_2|^2)^{\frac{1}{2}} \\
&= |\alpha|(|\zeta_1|^2 + |\zeta_2|^2)^{\frac{1}{2}} \\
&= |\alpha|\ ||x||_2
\end{aligned}$$

N4 holds because:

$$\begin{aligned}
||x+y||_2^2 &= (\zeta_1+\eta_1)^2+(\zeta_2+\eta_2)^2\\
&= \zeta_1^2+2\zeta_1\eta_1+\eta_1^2+\zeta_2^2+2\zeta_2\eta_2+\eta_2^2\\
&= (\zeta_1^2+\zeta_2^2)+(\eta_1^2+\eta_2^2)+2(\zeta_1\eta_1+\zeta_2\eta_2)\\
&\leq (\zeta_1^2+\zeta_2^2)+(\eta_1^2+\eta_2^2)+2\sqrt{\zeta_1^2+\zeta_2^2}\sqrt{\eta_1^2+\eta_2^2} \qquad \text{(By the Cauchy-Schwartz inequality)}\\
&= ||x||_2^2+||y||_2^2+2||x||_2\,||y||_2\\
&= (||x||_2+||y||_2)^2
\end{aligned}$$

$\implies ||x+y||_2 \leq ||x||_2+||y||_2.$

iii. $||x||_\infty = \max\{|\zeta_1|,|\zeta_2|\}$

N1 holds because:

$$||x||_\infty = \max\{|\zeta_1|,|\zeta_2|\} \geq 0.$$

N2 holds because:
$\implies$:

$$\begin{aligned}
&||x||_\infty = 0\\
\implies &\max\{|\zeta_1|,|\zeta_2|\} = 0\\
\implies &|\zeta_1|,|\zeta_2| = 0\\
\implies &\zeta_1,\zeta_2 = 0\\
\implies &x = (\zeta_1,\zeta_2) = (0,0) = 0_v.
\end{aligned}$$

$\impliedby$:

$$\begin{aligned}
&x = 0\\
\implies &\zeta_1,\zeta_2 = 0\\
\implies &|\zeta_1|,|\zeta_2| = 0\\
\implies &\max\{|\zeta_1|,|\zeta_2|\} = 0\\
\implies &||x||_\infty = 0
\end{aligned}$$

N3 holds because:

$$\begin{aligned}
||\alpha x||_\infty &= \max\{|\alpha\zeta_1|,|\alpha\zeta_2|\}\\
&= |\alpha|\ \max\{|\zeta_1|,|\zeta_2|\}\\
&= |\alpha|\ ||x||_\infty
\end{aligned}$$

N4 holds because:

$$
\begin{aligned}
||x+y||_\infty &= \max\{|\zeta_1+\eta_2|, |\zeta_2+\eta_2|\} \\
&\leq \max\{|\zeta_1|+|\eta_2|, |\zeta_2|+|\eta_2|\} \\
&\leq \max\{|\zeta_1|, |\zeta_2|\} + \max\{|\eta_1|, |\eta_2|\} \\
&= ||x||_\infty + ||y||_\infty.
\end{aligned}
$$

$\square$

2.2.8

2.6.2

Proof.

Let $v_1, v_2 \in R^2$, where $v_1 = (\zeta_1, \zeta_2)$, $v_2 = (\eta_1, \eta_2)$, and $\alpha \in R$.

i. $\underline{T_1 : (\zeta_1, \zeta_2) \longmapsto (\zeta_1, 0)}$

Additivity:

$$\begin{aligned} T_1(v_1 + v_2) &= T_1(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_1 + \eta_1, 0) \\ &= (\zeta_1, 0) + (\eta_1, 0) \\ &= T_1v_1 + T_1v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_1(\alpha v_1) &= T_1(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\alpha\zeta_1, 0) \\ &= \alpha(\zeta_1, 0) \\ &= \alpha T_1 v_1. \end{aligned}$$

Geometric interpretation: Projection onto the x-axis.

ii. $\underline{T_2 : (\zeta_1, \zeta_2) \longmapsto (0, \zeta_2)}$

Proof is similar to part (i).

Geometric interpretation: Projection onto the y-axis.

iii. $\underline{T_3 : (\zeta_1, \zeta_2) \longmapsto (\zeta_2, \zeta_1)}$

Additivity:

$$\begin{aligned} T_3(v_1 + v_2) &= T_3(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\zeta_2 + \eta_2, \zeta_1 + \eta_1) \\ &= (\zeta_2, \zeta_1) + (\eta_2, \eta_1) \\ &= T_3v_1 + T_3v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_3(\alpha v_1) &= T_3(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\alpha\zeta_2, \alpha\zeta_1) \\ &= \alpha(\zeta_2, \zeta_1) \\ &= \alpha T_3 v_1 \end{aligned}$$

Geometric interpretation: Reflection across the line $y = x$.

iv. $T_4 : (\zeta_1, \zeta_2) \longmapsto (\gamma\zeta_1, \gamma\zeta_2)$

Additivity:

$$\begin{aligned} T_4(v_1 + v_2) &= T_4(\zeta_1 + \eta_1, \zeta_2 + \eta_2) \\ &= (\gamma(\zeta_1 + \eta_1), \gamma(\zeta_2 + \eta_2)) \\ &= (\gamma\zeta_1, \gamma\zeta_2) + (\gamma\eta_1, \gamma\eta_2) \\ &= T_4 v_1 + T_4 v_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} T_4(\alpha v_1) &= T_4(\alpha\zeta_1, \alpha\zeta_2) \\ &= (\gamma\alpha\zeta_1, \gamma\alpha\zeta_2) \\ &= \alpha(\gamma\zeta_1, \gamma\zeta_2) \\ &= \alpha T_4 v_1 \end{aligned}$$

□

2.6.6

Proof.
Let X, Y, Z be vector spaces over the same field K.
Let $T : X \longrightarrow Y$, $S : Y \longrightarrow Z$.
Then, the composite operator ST exists and $ST : X \longrightarrow Z$.

Let $x_1, x_2 \in X$, and $\alpha \in K$.

Additivity:

$$\begin{aligned} (ST)(x_1 + x_2) &= S(T(x_1 + x_2)) \\ &= S(Tx_1 + Tx_2) \\ &= (ST)x_1 + (ST)x_2. \end{aligned}$$

Homogeneity:

$$\begin{aligned} (ST)(\alpha x_1) &= S(T(\alpha x_1)) \\ &= S(\alpha T x_1) \\ &= \alpha(ST)x_1. \end{aligned}$$

□

2.7.2

Proof.

$\underline{\Longrightarrow:}$

Let $T : X \longrightarrow Y$ be a bounded linear operator, and B_x be a bounded set in X.

Since B_x is bounded, then $\forall x \in B_x$:

$$||x|| \leq c_x.$$

Let B_y be the image of B_x under T: $B_y = T(B_x)$.

Since T is bounded, then $\forall y \in B_y$:

$$\begin{aligned} ||y|| &= ||Tx|| \\ &\leq ||T||\,||x|| \\ &\leq ||T||c_x = c_y. \end{aligned}$$

Thus, B_y is bounded.

$\underline{\Longleftarrow:}$

Suppose T maps bounded sets in X into bounded sets Y.

Let $B_x = \{x \in X : ||x|| = 1\}$.
$\Longrightarrow$ B_x is bounded.

Let B_y be the image of B_x under T: $B_y = T(B_x)$.
$\Longrightarrow$ B_y is bounded
$\Longrightarrow$ $\forall y \in B_y :\ ||y|| \leq c$
$\Longrightarrow$ $\forall x \in B_x : ||Tx|| \leq c$
$\Longrightarrow$ T is bounded by *Lemma* $2.7-2$.

□

2.7.3

Proof.

Let $x \in X$ s.t. $||x|| = \alpha < 1$, then:

$$\begin{aligned}
&y = \frac{1}{\alpha}x \\
&\implies ||y|| = \frac{1}{\alpha}||x|| = 1 \\
&\implies ||Ty|| \leq ||T|| \\
&\implies ||T(\frac{1}{\alpha}x)|| \leq ||T|| \\
&\implies \frac{1}{\alpha}||Tx|| \leq ||T|| \\
&\implies ||Tx|| \leq \alpha||T|| < ||T||.
\end{aligned}$$

□

3.1.4

Proof.

$$\begin{aligned}\langle x+y, x-y\rangle &= \langle x, x-y\rangle + \langle y, x-y\rangle \\ &= \langle x,x\rangle - \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle \\ &= 0 - \langle x,y\rangle + \langle x,y\rangle - 0 = 0 \\ &= 0.\end{aligned}$$

Geometric interpretation for $X = \mathbb{R}^2$:

Since $||x|| = ||y||$, then the vectors x and y form a rhombus. The vectors $x+y$ and $x-y$ are the diagonals of that rhombus.

Therefore, this statement simply states that the diagonals of a rhombus are perpendicular/orthogonal.

Geometric interpretation for $X = \mathbb{C}^1$:

Same as $X = \mathbb{R}^2$, since $\mathbb{R}^2$ and $\mathbb{C}^1$ have the same geometry.

$\square$

3.1.7

Proof. (By Contradiction)

Let $u, v \in V$ s.t. $u \neq 0_v$ and $v := au$, where $a \neq 1$. Then $u \neq v$, and:

$$\begin{aligned}&\forall x \in X : \langle x, u\rangle = \langle x, v\rangle \\ \implies &\forall x \in X : \langle x, u\rangle = \langle x, au\rangle \\ \implies &\forall x \in X : \langle x, u\rangle = \bar{a}\langle x, u\rangle\end{aligned}$$

But this is impossible since $a \neq 1$. Thus, we must have $u = v$.

$\square$

3.1.11

Proof.

To check if the given norm is induced by some inner-product, we could check whether the parallelogram equality holds, but there's an even simpler proof.

First, we note that this space is an l^p space with $p = 1$.

Finally, we note that it was shown in $3.1-7$ that the space l^p with $p \neq 2$ is not an inner-product space.

Therefore, there is no inner-product that induces the given norm.

$\square$

3.2.1

The Euclidean inner product on $\mathbb{R}^2$ and $\mathbb{R}^3$ reduces to the dot product.

The dot product of 2 vectors is the product of their length times cosine the angle between them:

$$\begin{aligned}\langle x, y\rangle &= x \cdot y \\ &= ||x||\,||y||\cos(\theta)\end{aligned}$$

Taking the absolute value of both sides we get:

$$\begin{aligned}|\langle x, y\rangle| &= ||x||\,||y||\,|\cos(\theta)| \\ &\leq ||x||\,||y||\end{aligned}$$

Which is the same result given by the Schwarz inequality.

3.2.2

i. The trivial subspace containing just the zero sequence.
ii. Subspaces whose components are all 0 after some index n.

3.2.4

Proof.
By the continuity of the inner product (Lemma 3.2-2), we have:

$$\langle x_n, y\rangle \to \langle x, y\rangle \tag{1}$$

Next, we define the sequence (a_n) by: $a_n := \langle x_n, y\rangle$. Then:

$$\begin{aligned}&\forall n \in \mathbb{N}:\ a_n = 0 \\ \Longrightarrow\ & a_n \to 0 \\ \Longrightarrow\ & \langle x_n, y\rangle \to 0 \qquad (2)\end{aligned}$$

Finally:

$$\begin{aligned}(1) \text{ and } (2) &\Longrightarrow \langle x, y\rangle \to 0 \\ &\Longrightarrow x \perp y.\end{aligned}$$

□

3.2.5

Proof.
To prove that $x_n \to x$, we will show that $||x_n - x|| \to 0$:

$$\begin{aligned} ||x_n - x|| &= \langle x_n - x, x_n - x\rangle \\ &= \langle x_n, x_n\rangle - \langle x_n, x\rangle - \langle x, x_n\rangle + \langle x, x\rangle \\ &= ||x_n||^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + ||x_n||^2 \end{aligned}$$

Taking the limit as $n \to \infty$:

$$||x_n - x|| \to 2||x||^2 - 2\langle x, x\rangle = 2||x||^2 - 2||x||^2 = 0.$$

□

3.2.8

Proof.
$\underline{\Longrightarrow}$:

$$\begin{aligned} &\langle x, y\rangle = 0 \\ \Longrightarrow\ &\forall \bar{\alpha} :\ \bar{\alpha}\langle x, y\rangle = 0 \\ \Longrightarrow\ &\forall \alpha :\ \langle x, \alpha y\rangle = 0 \\ \Longrightarrow\ &\forall \alpha :\ \langle x, \alpha y\rangle = 0\ \wedge\ \langle \alpha y, x\rangle = 0 \end{aligned}$$

Next, we have $\forall \alpha$:

$$\begin{aligned} ||x + \alpha y||^2 &= \langle x + \alpha y, x + \alpha y\rangle \\ &= \langle x, x\rangle + \langle x, \alpha y\rangle + \langle \alpha y, x\rangle + \langle y, y\rangle \\ &= ||x||^2 + 0 + 0 + ||y||^2 \\ &\geq ||x||^2 \end{aligned}$$

Finally, taking the square root, we arrive at our desired result:

$$||x + \alpha y|| \geq ||x||.$$

$\underline{\Longleftarrow}$:

$$
\begin{aligned}
&\forall\alpha:\ ||x|| &&\leq ||x+\alpha y||\\
\Longrightarrow &\forall\alpha:\ ||x||^2 &&\leq ||x+\alpha y||^2\\
& &&= \langle x+\alpha y, x+\alpha y\rangle\\
& &&= \langle x,x\rangle + \langle x,\alpha y\rangle + \langle \alpha y, x\rangle + \langle y,y\rangle\\
& &&= ||x||^2 + \overline{\langle \alpha y, x\rangle} + \langle \alpha y, x\rangle + ||y||^2\\
& &&= ||x||^2 + 2Re\langle \alpha y, x\rangle + ||y||^2\\
& &&= ||x||^2 + 2Re(\alpha\langle y, x\rangle) + ||y||^2\\
\Longrightarrow &\forall\alpha:\ 2Re\langle \alpha y, x\rangle \leq 0\\
\Longrightarrow &\langle x,y\rangle = 0.
\end{aligned}
$$

$\square$