Partial Differential Equations (MTH-632) Finals Questions Bank
Final 2024

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2.

a.

Find the characteristic equation, characteristic curves and the canonical form of:

$$u_{xx} + u_{xy} + u_{yy} = 0$$

Solution:

b.

Solve using the method of characteristics the PDE:

$$u_t - 2u_x = e^{2x}$$

Subject to:

$$u(x,0) = \cos(x).$$

<u>Solution:</u>

$$\begin{aligned} & u_t - 2u_x = e^{2x} \\ \implies & \frac{dx}{dt} = -2 \\ \implies & x = -2t + x_0. \end{aligned}$$

and,

$$\begin{aligned} & u_t - 2u_x = e^{2x} \\ \implies & \frac{du}{dt} = e^{2x} \\ \implies & \frac{du}{dt} = e^{2(-2t+x_0)} \\ \implies & \frac{du}{dt} = e^{2x_0}e^{-4t} \\ \implies & u(x,t) = \frac{-e^{2x_0}e^{-4t}}{4} + K. \end{aligned}$$

To find K, we apply the initial conditions:

$$\begin{aligned} & u(x_0, 0) = \cos(x_0) \\ \implies & \frac{-e^{2x_0}e^0}{4} + K = \cos(x_0) \\ \implies & \frac{-e^{2x_0}}{4} + K = \cos(x_0) \\ \implies & K = \frac{e^{2x_0}}{4} + \cos(x_0) \end{aligned}$$

Finally, we get:

$$\begin{aligned} u(x,t) &= \frac{-e^{2x_0}e^{-4t}}{4} + \frac{e^{2x_0}}{4} + \cos(x_0) \\ &= \frac{1}{4}e^{2x_0}\left[-e^{-4t} + 1\right] + \cos(x_0) \\ &= \frac{1}{4}e^{2(x+2t)}\left[-e^{-4t} + 1\right] + \cos(x + 2t) \end{aligned}$$

c.

Solve the linear PDE:

$$u_t + tu_x = 5$$

Subject to:

$$u(x,0) = f(x).$$

<u>Solution:</u>

$$\begin{aligned} &u_t + tu_x = 5 \\ \implies &\frac{dx}{dt} = t \\ \implies &x = \frac{1}{2}t^2 + x_0 \\ \implies &x = \frac{1}{2}t^2 + x_0. \end{aligned}$$

and,

$$\begin{aligned} &u_t + tu_x = 5 \\ \implies &\frac{du}{dt} = 5 \\ \implies &u = 5t + K. \end{aligned}$$

To find K, we apply the initial conditions:

$$\begin{aligned} &u(x_0, 0) = f(x_0) \\ \implies &5 * 0 + K = f(x_0) \\ \implies &K = f(x_0). \end{aligned}$$

Finally, we get:

$$\begin{aligned} &u(x,t) = 5t + f(x_0) \\ \implies &u(x,t) = 5t + f(x - \frac{1}{2}t^2). \end{aligned}$$

3.

a.

Solve the equation:

$$u_t + 4uu_x = 0$$

Subject to:

$$u(x,0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$$

<u>Solution:</u>

$$\begin{aligned} & u_t + 4uu_x = 0 \\ \implies & \frac{du}{dt} = 0 \\ \implies & u(x,t) = K \end{aligned}$$

To find K, we apply the initial conditions:

$$\begin{aligned} u(x_0,0) &= \begin{cases} 3 & x_0 < 1 \\ 2 & x_0 > 1 \end{cases} \\ \implies K &= \begin{cases} 3 & x_0 < 1 \\ 2 & x_0 > 1 \end{cases} \end{aligned}$$

and,

$$\begin{aligned} & u_t + 4uu_x = 0 \\ \implies & \frac{dx}{dt} = \begin{cases} 12 & x_0 < 1 \\ 8 & x_0 > 1 \end{cases} \\ \implies & x = \begin{cases} 12t + x_0 & x_0 < 1 \\ 8t + x_0 & x_0 > 1 \end{cases} \end{aligned}$$

Therefore, away from the discontinuity we get :

$$u(x,t) = \begin{cases} 3 & x < 1 + 12t \\ 2 & x > 1 + 8t \end{cases}$$

To find the characteristic of the shock, we first rewrite the PDE in the conservation law form:

$$u_t + \frac{\partial(q(u))}{\partial x} = 0, \quad \text{where } q(u) = 2u^2.$$

then:

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[q]}{[u]} = \frac{2 * 2^2 - 2 * 3^2}{2 - 3} = 10 \\ \implies x_s &= 10t + x_{s_0} = 10t + 1. \end{aligned}$$

b.

i.

Assuming an infinite domain ($-\infty < x < \infty$), $u(x,0) = f(x)$ and $u_x(x,0) = g(x)$, then the solution is given by D'Alembert's formula:

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\eta)d\eta.$$

ii.

For $f(x) = \cos(x)$ and $g(x) = 0$:

$$u(x,t) = \frac{\cos(x-ct) + \cos(x+ct)}{2}$$

iii.

Transfer the PDE to a system of 2 ODEs as follows:

$$\begin{aligned}
&u_{tt} - c^2 u_x = 0\\
\implies &(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = 0\\
\implies &v = (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u \quad \text{and} \quad (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})v = 0.
\end{aligned}$$

c.

This is a semi-infinite domain ($0 \leq x < \infty$). The solution is given by the modified D'Alembert's formula:

$$u(x,t) = \begin{cases} \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\eta)d\eta & x-ct \geq 0 \\ h(t-\frac{x}{c}) + \frac{f(x-ct)-f(x+ct)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\eta)d\eta & x-ct < 0 \end{cases}.$$

For $f(x) = 0$ and $g(x) = 0$, and $c^2 = 4$, we have:

$$u(x,t) = \begin{cases} 0 & x-2t \geq 0 \\ h(t-\frac{x}{2}) & x-2t < 0 \end{cases}$$