Probability Assignment

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Part 1

1.

Number of arrangements $= 4! * (4! * 3! * 2! * 1!) = 6912$ arrangements.

2.

a. Number of committees $= C_2^5 * C_3^7 = 10 * 35 = 350$.
b. Number of commitees $= C_2^5 * (C_3^7 - C_1^7) = 10 * (35 - 7) = 280$.

3.

a. Number of possible license plates $= 26^2 * 10^5 = 67600000$.
b. Number of possible license plates $= (26 * 25) * (10 * 9 * 8 * 7 * 6) = 676 * 10^5 = 19656000$.

4.

Number of seatings $= 2! * 7! = 10080$.

5.

a. Choices $= C_2^6 + C_2^7 + C_2^4 = 42$.
b. Choices $= C_1^6 * C_1^7 + C_1^6 * C_1^4 + C_1^7 * C_1^4 = 94$.

6.

Choices $= C_3^5 * C_4^5 + C_4^5 * C_3^5 + C_5^5 * C_2^5 = 7 * 5 + 5 * 7 + 1 * 10 = 80$.

7.

a. $Choices = C_3^{3n}$
b. $Choices = n * C_3^n$

8.

Choices $= C_2^7 * C_4^8 + C_3^7 * C_3^8 = 21 * 70 + 35 * 56 = 3430$

11.

i.

$A = \{H2, H4, H6\}$
$B = \{H2, H3, H5, T2, T3, T5\}$
$C = \{T1, T3, T5\}$

ii.

a. A or $B = A \cup B = \{H2, H3, H4, H5, H6, T2, T3, T5\}$
b. B and $C = B \cap C = \{T3, T5\}$
c. Only B occurs $= B \setminus (A \cup C) = \{H3, H5, T2\}$

iii.

Only A and C are mutually exclusive because: $A \cap C = \emptyset$.

12.

i.

$P(a_1) = 1 - P(a_2) - P(a_3) - P(a_4) = 1 - 1/3 - 1/6 - 1/9 = 7/18$

ii.

$P(a_1) + P(a_2) + 1/4 + 1/4 = 1$
$2P(a_2) + P(a_2) + 1/2 = 1$
$3P(a_2) = 1/2$
$P(a_2) = 1/6$ and $P(a_1) = 2/6$.

13.

i.

$P(1) = 1/21$	$P(2) = 2/21$
$P(3) = 3/21$	$P(4) = 4/21$
$P(5) = 5/21$	$P(6) = 6/21$

ii.

$$\begin{aligned} P(A) &= P(2) + P(4) + P(6) \\ &= 2/21 + 4/21 + 6/21 = 11/21 \end{aligned}$$

$$\begin{aligned} P(B) &= P(2) + P(3) + P(5) \\ &= 2/21 + 3/21 + 5/21 = 10/21 \end{aligned}$$

$$\begin{aligned} P(B) &= P(1) + P(3) + P(5) \\ &= 1/21 + 3/21 + 5/21 = 9/21 \end{aligned}$$

iii.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 11/21 + 10/21 - 2/21 = 19/21 \end{aligned}$$

$$\begin{aligned} P(B \cup C) &= P(B) + P(c) - P(B \cap C) \\ &= 10/21 + 9/21 - (3/21 + 5/21) = 11/21 \end{aligned}$$

$$\begin{aligned} P(A \setminus B) &= P(A) - P(A \cap B) \\ &= 11/21 - 2/21 = 9/21 \end{aligned}$$

14.

Let $A = \{x : xisaman\}$, $B = \{x : xhasbrowneyes\}$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cup B) \\ &= 10/30 + 15/30 - 5/30 = 20/30 \end{aligned}$$

15.

Let $F = \{x : x \text{ } is \text{ } studying \text{ } French\}$, $S = \{x : x \text{ } is \text{ } studying \text{ } Spanish\}$

i.

$$\begin{aligned}P(F \cup S) &= P(F) + P(S) - P(F \cap S) \\ &= 60/120 + 50/120 - 20/120 = 90/120 = 3/4\end{aligned}$$

ii.

$$\begin{aligned}P(\bar{F} \cap \bar{S}) &= P(\overline{F \cup S}) \\ &= 1 - P(F \cup S) \\ &= 1 - 90/120 = 30/120 = 1/4\end{aligned}$$

16.

Let $HS = \{x : x \textit{ has high strength}\}$, $LS = \{x : x \textit{ has low strength}\}$, $HC = \{x : x \textit{ has high conductivity}\}$, $LC = \{x : x \textit{ has low conductivity}\}$.

a.

$$\begin{aligned}P(LS \cup LC) &= P(LS) + P(LC) - P(LS \cap LC) \\ &= (13/100 + 10/100) + (14/100 + 10/100) - 10/100 = 37/100\end{aligned}$$

b.

$$\begin{aligned}P(HC \cup HS) &= \frac{P(HC \cap HS)}{P(HS)} \\ &= \frac{63}{63+14} = \frac{63}{77} = \frac{9}{11}\end{aligned}$$

17.

Proof.

$$\begin{aligned}P(A^c \cap B^c) &= P([A \cup B]^c) \\ &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - P(A) - P(B) + P(A \cap B)\end{aligned}$$

$\square$

18.

$A = \{(W, W, T), (W, W, L), (W, W, W)\}$

$P(A) = 0.6 * 0.6 * 0.3 + 0.6 * 0.6 * 0.1 + 0.6 * 0.6 * 0.6 = 0.36$

19.

$$\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\
&- P(A \cap B) - P(A \cap C) - P(B \cap C) \\
&+ P(A \cap B\ capC) \\
&= 0.02 + 0.01 + 0.015 \\
&- 0.005 - 0.006 - 0.004 \\
&+ 0.002 \\
&= 0.045 - 0.015 + 0.002 \\
&= 0.032
\end{aligned}$$

20.

a.

$P = \frac{21}{25} * \frac{20}{24} * \frac{19}{23} * \frac{18}{22} = \frac{21}{25}$

b.

$P = \frac{1}{25} + \frac{1}{24} + \frac{1}{23} + \frac{1}{22} = 0.17$

21.

The probability that a point lies in some region inside the circle is proportional to the area of the region, and the probability of lying anywhere inside the circle is 1.

$\implies P(\text{p lies in region R}) = \frac{\text{Area of region R}}{\pi r^2}$

a.

Let A be the set of all points inside the circle of radius r that are closer to the circumference than the center.

$$
\begin{aligned}
P(A) &= \frac{\pi r^2 - \pi(\frac{r}{2})^2}{\pi r^2} \\
&= \frac{\pi r^2(1 - \frac{1}{4})}{\pi r^2} = 3/4
\end{aligned}
$$

b.

Let B be the set of all points inside the circle of radius r that are at least $r/3$ from the center.

$$
\begin{aligned}
P(B) &= \frac{\pi r^2 - \pi(\frac{r}{3})^2}{\pi r^2} \\
&= \frac{\pi r^2(1 - \frac{1}{9})}{\pi r^2} = 8/9
\end{aligned}
$$

Part 2

1.

a.

Proof. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\implies P(A \cap B) = P(A) + P(B) - P(A \cup B)$
And we have: $0 \leq P(A \cup B) \leq 1$
$\implies P(A \cap B) \geq P(A) + P(B) - 1$

□

b.

Proof.

$$
\begin{aligned}
P((A \cup B)|C) &= \frac{P([A \cup B] \cap C)}{P(C)} \\
&= \frac{P([A \cap C] \cup [B \cap C])}{P(C)} \\
&= \frac{P(A \cap C) + P(B \cap C) - P([A \cap C] \cap [B \cap C])}{P(C)} \\
&= \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} \\
&= P(A|C) + P(B|C) + P((A \cap B)|C)
\end{aligned}
$$

□

2.

a.

Proof.

$$\begin{aligned}
P((A-B)|C) &= \frac{P((A-B)\cap C)}{P(C)} \\
&= \frac{P((A-A\cap B)\cap C)}{P(C)} \\
&= \frac{P(A\cap C - A\cap B\cap C)}{P(C)} \\
&= P(A|C) - P((A\cap B)|C)
\end{aligned}$$

□

b.

$$\begin{aligned}
P((A\cup B)|B^c) &= P(A|B^c) \\
&= \frac{P(A\cap B^c)}{P(B^c)} \\
&= \frac{0.5}{0.8} = 5/8 = 0.625
\end{aligned}$$

$$\begin{aligned}
P((A\cup B)^c|A) &= P((A^c\cup B^c)|A) \\
&= \frac{P((A^c\cap B^c)\cap A)}{P(A)} \\
&= \frac{P((A^c\cap A)\cap B^c)}{P(A)} \\
&= \frac{0}{P(A)} = 0
\end{aligned}$$

3.

Proof.

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
P(B|A) &= \frac{P(B \cap A)}{P(A)} \\
\implies \frac{P(A|B)}{P(B|A)} &= \frac{\frac{P(A \cap B)}{P(B)}}{\frac{P(B \cap A)}{P(A)}} \\
&= \frac{P(A)}{p(B)}
\end{aligned}
$$

$\square$

4.

a.

No. Because $P(A) + P(B) = 7/6 > 1$.

b.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

i.

A and B are independent.
$\implies P(A|B) = P(A)$
$\implies \frac{P(A \cap B)}{P(B)} = P(A)$
$\implies P(A \cap B) = P(A) * P(B)$

Therefore:

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= P(A) + P(B) - P(A)P(B) \\
&= 1/2 + 2/3 - 1/2 * 2/3 = 5/3 = 0.834
\end{aligned}
$$

ii.

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= 1/2 + 2/3 - 1/4 = 11/12 = 0.917
\end{aligned}
$$

5.

Proof. $P(A|B) > P(A)$

$\implies -P(A|B) < -P(A)$

$\implies 1 - P(A|B) < 1 - P(A)$

$\implies P(A^c|B) < P(A^c)$

□

6.

i.

Proof.

$$\begin{aligned}
P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\
&= P(A_1) + P(A_2) - P(A_1)P(A_2) \\
&= P(A_2) + [P(A_1) - P(A_1)P(A_2)] \\
&= P(A_2) + [P(A_1)(1 - P(A_2))] \\
&= P(A_2) + P(A_1)P(A_2^c)
\end{aligned}$$

□

ii.

Proof.

$$\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P((A_1 \cup A_2) \cup A_3) \\
&= P(B \cap A_3) \\
&= P(A_3) + P(B)P(A_3^c) \\
&= P(A_3) + P(A_1 \cup A_2)P(A_3^c) \\
&= P(A_3) + [P(A_1)P(A_2^c) + P(A_2)]P(A_3^c) \\
&= P(A_3) + P(A_1)P(A_2^c)P(A_3^c) + P(A_2)P(A_3^c)
\end{aligned}$$

□

7.

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A)P(B) \\
&= 0.6 + 0.6 - 0.6 * 0.6 = 0.84
\end{aligned}$$

$$
\begin{aligned}
P(A^c \cap C) &= P(C) - P(A \cap C) \\
&= 0.6 - 0.6 * 0.6 = 0.24
\end{aligned}
$$

8.

$$
\begin{aligned}
P((A \cup B)|A^c) &= \frac{P((A \cup B) \cap A^c)}{P(A^c)} \\
&= \frac{P[(A \cap A^c) \cup (B \cap A^c)]}{P(A^c)} \\
&= \frac{P(B \cap A^c)}{P(A^c)} \\
&= \frac{P(B) - P(B \cap A)}{P(A^c)} \\
&= \frac{P(B) - P(B)P(A)}{P(A^c)} = \frac{0.6 - 0.6 * 0.4}{0.6} = 0.6
\end{aligned}
$$

$$
\begin{aligned}
P((A \cup C)|B) &= \frac{P((A \cup C) \cap B)}{P(B)} \\
&= \frac{P[(A \cap B) \cup (C \cap B)]}{P(B)} \\
&= \frac{P(A \cap B) + P(C \cap B) - P(A \cap B \cap C)}{P(B)} \\
&= \frac{P(A)P(B) + P(B|C) * P(C) - P(A \cap B \cap C)}{P(B)} = \frac{0.4 * 0.6 + 0.4 * 0.6 - 0}{0.6} = 0.8
\end{aligned}
$$

9.

Let A be the event that the first item is defective, and B be the event that the second item is non-defective.

$P(A) = \frac{3}{18}$
$P(B) = \frac{15}{18} * \frac{14}{17} + \frac{3}{18} * \frac{15}{17} = 5/6$
$P(B|A) = \frac{2}{17}$
$P(A|B) = P(B|A) * \frac{P(A)}{P(B)} = \frac{2}{17} * \frac{\frac{3}{18}}{\frac{5}{6}} = \frac{2}{85} = 0.024$

10.

i.

$$P(D|A) = \frac{P(A \cap D)}{P(A)} = \frac{0.2}{0.45} = 0.45$$

ii.

$P(A) + P(B) + P(C) = 1$
$\implies P(D) = P(A \cap D) + P(B \cap D) + P(C \cap D)$

$P(B \cap D) = P(D|B) * P(B) = 0.3 * 0.25 = 0.075$

$P(C \cap D) = P(D|C) * P(C) = 0.1 * 0.3 = 0.03$

$$\begin{aligned} P(D) &= P(A \cap D) + P(B \cap D) + P(C \cap D) \\ &= 0.2 + 0.075 + 0.03 = 0.305 \end{aligned}$$

iii.

$$\begin{aligned} P(C|D) &= \frac{P(C \cap D)}{P(D)} \\ &= \frac{0.03}{0.305} = 0.098 \end{aligned}$$

11.

$$\begin{aligned} P((A \cup B)^c|D) &= 1 - P((A \cup B)|D) \\ &= 1 - P(D|(A \cup B)) * \frac{P(A \cup B)}{P(D)} \\ &= 1 - P(D|(A \cup B)) * \frac{P(A) + P(B) - P(A)P(B)}{P(D)} \\ 0.3 &= 1 - 0.7 * \frac{0.05 + 0.2 - 0.05 * 0.2}{P(D)} \\ P(D) &= 0.24 \end{aligned}$$

$P((A \cup B)^c|D) = 1 - P((A \cup B)|D)$
$P((A \cup B)|D) = 0.3$
$\implies P((A \cup B)|D) = P(D|(A \cup B)) * \frac{P(A \cup B)}{P(D)}$