Real Analysis Assignments

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1.3.7

Proof:

Forward direction: If a set T_1 is denumerable, then there exists a bijection from $T1$ onto a denumerable set T_2.

Since T_1 is denumerable, then there exists a bijection f_1 from N onto T_1.
And since T_2 is denumerable, then there exists a bijection f_2 from N onto T_2.
Also, since f_1 is a bijection, then f_1^{-1} exists and is a bijection from T_1 onto N.
Then, the function defined by $f_2 \circ f_1^{-1}$ is a bijection from $T1$ onto T_2, since the compositiomn of bijective functions is bijective.

Reverse direction: If there exists a bijection from a set $T1$ onto a denumerable set T_2, then T_1 is denumerable.

Let g be the bijection from $T1$ to $T2$.
Since T_2 is denumerable, then there exists a bijection f_2 from N onto T_2.
Also, since g is a bijection, then g^{-1} exists and is a bijection from T_2 onto T_1.
Then, the function defined as $f_1 := g^{-1} \circ f_2$ is a bijection from N onto T_1, since the compositiomn of bijective functions is bijective.
This implies that the set T_1 is denumerable.

This completes the proof. $\square$

1.3.8

Let's define the set S_i as $S_i := \{i\}$, then for each i, S_i is a finite set (of cardinality $= 1$).
But the union $\cup_{i=1}^{\infty} S_i = N$ is infinite, because N is infinite.

2.1.4

By the trichotomy of a, we have three cases: (i) $a < 0$, (ii) $a = 0$, and (iii) $a > 0$.

(i): $a < 0 \Rightarrow a \cdot a > 0 \Rightarrow a \cdot a > a \Rightarrow a \cdot a \neq a$. Therefore, a cannot be less than 0.

(ii): $a = 0 \Rightarrow a \cdot a = a$, because $0 \cdot 0 = 0$.

(iii): $a > 0$ and $a \cdot a = a \Rightarrow a^{-1} \cdot a \cdot a = a^{-1} \cdot a \Rightarrow a = 1$.

Therefore, $a = 0$ or $a = 1$. $\square$

2.1.23

Proof:

Forward direction: For $a > 0$, $b > 0$, and $n \in N$: If $a < b$, then $a^n < b^n$.

We use induction.

Base case $(n = 1)$: This case is trivially true because it is given by the hypothesis: $a < b \iff a^1 < b^1$.

Inductive step $(n > 1)$: By the induction hypothesis, we have:

$$a < b \Rightarrow a^n < b^n$$

Since $b > 0$, multiplying $a^n < b^n$ by b, we get:

$$ba^n < b^{n+1} \tag{1}$$

Since $a < b$, then $b - a > 0$; and since $a > 0$, then $a^n > 0$.
Also, since $(b - a) > 0$ and $a^n > 0$, we have:

$$(b - a)a^n > 0 \Rightarrow ba^n - a^{n+1} > 0 \Rightarrow ba^n > a^{n+1} \tag{2}$$

Combining (1) and (2) together, we get:

$$a^{n+1} < ba^n < b^{n+1}$$

Therefore, $a < b \Rightarrow a^{n+1} < b^{n+1}$, thus closing the induction.

Reverse direction: For $a > 0$, $b > 0$, and $n \in N$: If $a^n < b^n$, then $a < b$.

We use induction.

Base case $(n = 1)$: This case is trivially true because it is given by the hypothesis: $a^1 < b^1 \iff a < b$.

Inductive step $(n > 1)$: By the induction hypothesis, we have:

$$a^n < b^n \Rightarrow a < b \tag{3}$$

The contrapositive of (3) is:

$$a \geq b \Rightarrow a^n \geq b^n \tag{4}$$

Since $b > 0$, multiplying $a^n \geq b^n$ by b, we get:

$$ba^n \geq b^{n+1} \tag{5}$$

Since $a \geq b$, then $a - b \geq 0$; and since $a > 0$, then $a^n > 0$.
Also, since $(a - b) \geq 0$ and $a^n > 0$, we have:

$$(a - b)a^n \geq 0 \Rightarrow a^{n+1} - ba^n \geq 0 \Rightarrow a^{n+1} \geq ba^n \tag{6}$$

Combining (5) and (6) together, we get:

$$a^{n+1} \geq ba^n \geq b^{n+1} \tag{7}$$

Putting (4), (5), (6), and (7) together, we get:

$$a \geq b \Rightarrow a^{n+1} \geq b^{n+1} \tag{8}$$

Taking the contrapositive of (8), we get:

$$a^{n+1} < b^{n+1} \Rightarrow a < b \tag{9}$$

This closes the induction and completes the proof. □

2.2.16

$V_\epsilon(a) = \{x \in R : |x - a| < \epsilon\}$
$V_\delta(a) = \{x \in R : |x - a| < \delta\}$

(i) $V_\epsilon(a) \cup V_\delta(a) = \{x \in R : |x - a| < \epsilon$ and $|x - a| < \delta\}$

Let $\gamma = \min(\epsilon, \delta)$.

Lower Bound: $|x-a| < \epsilon$ and $|x-a| < \delta \Rightarrow x > a-\epsilon$ and $x > a-\delta \Rightarrow x > a-\gamma$

Upper Bound: $|x-a| < \epsilon$ and $|x-a| < \delta \Rightarrow x < a+\epsilon$ and $x < a+\delta \Rightarrow x < a+\gamma$

Therefore, $V_\epsilon(a) \cup V_\delta(a)$ is in the γ-neighbourhood of a.

(ii) $V_\epsilon(a) \cap V_\delta(a) = \{x \in R : |x - a| < \epsilon$ or $|x - a| < \delta\}$

Let $\gamma = \max(\epsilon, \delta)$.

Lower Bound: $|x - a| < \epsilon$ or $|x - a| < \delta \Rightarrow x > a - \epsilon$ or $x > a - \delta \Rightarrow x > a - \gamma$

Upper Bound: $|x-a| < \epsilon$ and $|x-a| < \delta \Rightarrow x < a+\epsilon$ and $x < a+\delta \Rightarrow x < a+\gamma$

Therefore, $V_\epsilon(a) \cap V_\delta(a)$ is in the γ-neighbourhood of a.

2.2.17

Without loss of generality, assume that $b > a$. Let $\epsilon = \frac{b-a}{2}$

Then, $U_\epsilon(a) = \{x \in R : |x-a| < \epsilon\}$, and $V_\epsilon(b) = \{x \in R : |x-b| < \epsilon\}$

$U_\epsilon(a) \cap V_\epsilon(b) = \{x \in R : |x-a| < \epsilon \text{ and } |x-b| < \epsilon\}$

Lower Bound: $|x-a| < \epsilon$ and $|x-b| < \epsilon \Rightarrow x > a - \epsilon$ and $x > b - \epsilon \Rightarrow x > a - \frac{b-a}{2}$ and $x > b - \frac{b-a}{2} \Rightarrow x > \frac{a-b}{2}$ and $x > \frac{b-a}{2} \Rightarrow x > \frac{b-a}{2}$

Upper Bound: $|x-a| < \epsilon$ and $|x-b| < \epsilon \Rightarrow x < a + \epsilon$ and $x < b + \epsilon \Rightarrow x < a + \frac{b-a}{2}$ and $x < b + \frac{b-a}{2} \Rightarrow x < \frac{a+b}{2}$ and $x < b + \frac{b-a}{2} \Rightarrow x < \frac{a+b}{2}$

But this is a contradiction, since the lower bound is greater than the upper bound. Therefore, we conclude that $U_\epsilon(a) \cap V_\epsilon(b) = \emptyset$. $\square$

2.3.4

$S_4 = \{1 - (-1)^n/n : n \in N\}$

Let $S' = \{-(-1)^n/n : n \in N\}$. Then: $S_4 = 1 + S'$.

$inf(S') = -(-1)^2/2 = -1/2$

$sup(S') = -(-1)^1/1 = 1$

$\underline{inf(S_4)}$**:**

$$inf(S_4) = 1 + inf(S') = 1 + (-1/2) = 1/2$$

$\underline{sup(S_4)}$**:**

$$sup(S_4) = 1 + sup(S') = 1 + 1 = 2$$

2.3.11

Proof. By the definition of the infimum and supremum we have $inf(S_0) \leq sup(S_0)$. So we need only show that (i)$inf(S) \leq inf(S_0)$, and (ii)$sup(S_0) \leq sup(S)$.

i. $\underline{inf(S) \leq inf(S_0)}$**:** We prove this by contradiction.
Suppose that $inf(S_0) < inf(S)$.
Since $inf(S_0)$ is a infimum for S_0, then $\forall \epsilon > 0\ \exists s \in S_0 : s < inf(S_0) + \epsilon$.
Taking $\epsilon = [inf(S) - inf(S_0)]/2$

$\Rightarrow \exists s \in S_0$ and $s < inf(S)$
$\Rightarrow s \in S_0$ *and* $s \notin S$
$\Rightarrow S_0 \not\subset S$.
This is a contradiction. Therefore, we must conclude that $inf(S) \leq inf(S_0)$.

ii. $sup(S_0) \leq sup(S)$**:** We prove this by contradiction.
Suppose that $sup(S_0) > sup(S)$.
Since $sup(S_0)$ is a supremum for S_0, then $\forall \epsilon > 0\ \exists s \in S_0 : s > sup(S_0) + \epsilon$.
Taking $\epsilon = [sup(S_0) - sup(S)]/2$
$\Rightarrow \exists s \in S_0$ and $s > sup(S)$
$\Rightarrow s \in S_0$ *and* $s \notin S$
$\Rightarrow S_0 \not\subset S$.
This is a contradiction. Therefore, we must conclude that $sup(S_0) \leq sup(S)$.
□

3.1.1.b

$x_n := (-1)^n/n = (-1, 1/2, -1/3, 1/4, -1/5, ...)$

3.1.5.d

Required to show: $\lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$

Proof. Given any $\epsilon > 0$, we need to find $k(\epsilon)$ such that for all $n \geq k$: $|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| < \epsilon$:

$$|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| < \epsilon$$
$$|\frac{2n^2-2-2n^2-3}{4n^2+6}| < \epsilon$$
$$|\frac{-5}{4n^2+6}| < \epsilon$$
$$\frac{5}{4n^2+6} < \epsilon$$
$$\frac{5}{4n^2+6} \leq \frac{5}{n^2} \leq \frac{5}{n} < \epsilon$$

Taking $k(\epsilon) = 5/\epsilon$ satisfies the required conditions.

□

3.1.7

a

Required to show: $\lim(\frac{1}{ln(n+1)}) = 0$

Proof. Given any $\epsilon > 0$, we need to find $k(\epsilon)$ such that for all $n \geq k$: $|\frac{1}{ln(n+1)} - 0| < \epsilon$:

$$\begin{aligned} |\frac{1}{ln(n+1)} - 0| &< \epsilon \\ \frac{1}{ln(n+1)} &< \epsilon \\ ln(n+1) &> \frac{1}{\epsilon} \\ n+1 &> e^{\frac{1}{\epsilon}} \\ n &> e^{\frac{1}{\epsilon}} - 1 \end{aligned}$$

Taking $k(\epsilon) = e^{\frac{1}{\epsilon}} - 1$ satisfies the required conditions.

□

b

i. $k(1/2) = e^2 - 1 = 7$
ii. $k(1/10) = e^{10} - 1 = 22026$

3.1.9

Proof. $\lim(x_n) = 0 \Rightarrow \forall\epsilon > 0$ there exists $k(\epsilon)$ such that for all $n \geq k : \ |x_n - 0| < \epsilon$.
$\Rightarrow \forall\epsilon > 0$ there exists $k(\epsilon)$ such that for all $n \geq k : x_n < \epsilon$ (because $x_n > 0$).
$\Rightarrow \forall\epsilon > 0$ there exists $k(\epsilon)$ such that for all $n \geq k : \sqrt{x_n} - 0 < \sqrt{\epsilon} = \epsilon^{'}$.
Since $\epsilon^{'}$ can take on any value greater than zero, then by the definition of the limit of a sequence this shows that $\lim(\sqrt{x_n}) = 0$.

□

3.1.12

Required to show: $\lim(\sqrt{n^2+1} - n) = 0$

Proof. Given any $\epsilon > 0$, we need to find $k(\epsilon)$ such that for all $n \geq k$: $|(\sqrt{n^2+1} - n) - 0| < \epsilon$:

$$\begin{aligned}
|(\sqrt{n^2+1} - n) - 0| &< \epsilon \\
|\sqrt{n^2+1} - n| &< \epsilon \\
\sqrt{n^2+1} - n \leq \sqrt{(n+1/n)^2} - n &< \epsilon \\
\sqrt{n^2+1} - n \leq (n+1/n) - n &< \epsilon \\
\sqrt{n^2+1} - n \leq 1/n &< \epsilon
\end{aligned}$$

Taking $k(\epsilon) = \frac{1}{\epsilon}$ satisfies the required conditions.

$\square$