Operations Research and Optimization Assignments

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1.

In general, given an optimization problem in the standard form $Ax = b$ and $x_i \geq 0$, where A is an mxn matrix and $rank(A) = m$; then the basic solutions are constructed by solving the system of equations using any m linearly independent columns at a time (the basic variables), and setting the remaining variables (the non-basic variables) to zero.

i.

a.

$dim(A) = 2x4 \Rightarrow$ There are ${}^4C_2 = 6$ possible basic solutions, corresponding to the following bases:

$$\begin{array}{ll} B_1 = (A_1, A_2) & B_1 = (A_1, A_3) \\ B_3 = (A_1, A_4) & B_4 = (A_2, A_3) \\ B_5 = (A_2, A_4) & B_6 = (A_3, A_4) \end{array}$$

Basic solution corresponding to B_1:

$$\begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 35/2 \\ -12 \end{bmatrix}$$

Therefore, $bs_1 = (35/2, -12, 0, 0)$.

Basic solution corresponding to B_2:

$$\begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7/2 \\ 3 \end{bmatrix}$$

Therefore, $bs_2 = (-7/2, 0, 3, 0)$.

Basic solution corresponding to B_3:

$$\begin{bmatrix} 4 & 7 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 13/22 \\ 12/11 \end{bmatrix}$$

Therefore, $bs_3 = (13/22, 0, 0, 12/11)$.

Basic solution corresponding to B_4:

$$\begin{bmatrix} 5 & 8 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5/2 \end{bmatrix}$$

Therefore, $bs_4 = (0, -2, 5/2, 0)$.

Basic solution corresponding to B_5:

$$\begin{bmatrix}5 & 7\\2 & 9\end{bmatrix}\begin{bmatrix}x_2\\x_4\end{bmatrix} = \begin{bmatrix}10\\11\end{bmatrix} \Rightarrow \begin{bmatrix}x_2\\x_4\end{bmatrix} = \begin{bmatrix}13/31\\35/31\end{bmatrix}$$

Therefore, $bs_5 = (0, 13/31, 0, 35/31)$.

<u>**Basic solution corresponding to B_6:**</u>

$$\begin{bmatrix}8 & 7\\6 & 9\end{bmatrix}\begin{bmatrix}x_3\\x_4\end{bmatrix} = \begin{bmatrix}10\\11\end{bmatrix} \Rightarrow \begin{bmatrix}x_3\\x_4\end{bmatrix} = \begin{bmatrix}13/30\\14/15\end{bmatrix}$$

Therefore, $bs_6 = (0, 0, 13/30, 14/15)$.

b.

The basic <u>**feasible**</u> solutions are: $\{bs_3, bs_5, bs_6\}$.

c.

The maximum number of basic solutions is: ${}^4C_2 = 6$.
This occurs when $rank(A) = 2$.

ii.

a.

$dim(A) = 2x3 \Rightarrow$ There are ${}^3C_2 = 3$ possible basic solutions, corresponding to the following bases:

$$B_1 = (A_1, A_2) \qquad B_1 = (A_1, A_3) \qquad B_3 = (A_2, A_3)$$

<u>**Basic solution corresponding to B_1:**</u>

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

Therefore, $bs_1 = (2, 1, 0)$.

<u>**Basic solution corresponding to B_2:**</u>

$$\begin{bmatrix}1 & 1\\2 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_3\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix}$$

Therefore, $bs_2 = (5, 0, -1)$.

<u>**Basic solution corresponding to B_3:**</u>

$$\begin{bmatrix}2 & 1\\1 & 5\end{bmatrix}\begin{bmatrix}x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} \Rightarrow \begin{bmatrix}x_2\\x_3\end{bmatrix} = \begin{bmatrix}5/3\\2/3\end{bmatrix}$$

Therefore, $bs_3 = (0, 5/3, 2/3)$.

b.

The basic **feasible** solutions are: $\{bs_1, bs_3\}$.

c.

The maximum number of basic solutions is: ${}^3C_2 = 3$.
This occurs when $rank(A) = 2$.

2.

a.

First, we transform the problem from a maximization problem to the equivalent minimization problem by multiplying the RHS of the cost function by -1.

Second, we put the problem in the standard form by introducing 3 slack variables x_3, x_4, x_5 to change the inequalities to equations.

The problem now becomes:

$$\text{Minimize } z = -5x_1 - 3x_2$$

Subject to:

$$\begin{gathered} 9x_1 + 3x_2 + x_3 = 27 \\ 2x_1 + x_2 + x_4 = 7 \\ 2x_1 + 2x_2 + x_5 = 12 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{gathered}$$

The tableau for this problem is:

0	-5	-3	0	0	0
27	9	3	1	0	0
7	2	1	0	1	0
12	2	2	0	0	1

We get the initial basic feasible solution $bfs_0 = (0, 0, 27, 7, 12)$ with a cost $z = 0$.

Next, we pivot on x_{11} because $\bar{c_1} < 0$ so it is profitable for column 1 to enter the basis, and $\theta_0 = 1/3$ at $l = 1$:

15	0	-4/3	5/9	0	0
3	1	1/3	1/9	0	0
1	0	1/3	-2/9	1	0
6	0	4/3	-2/9	0	1

We get the $bfs = (3, 0, 0, 1, 6)$ with a cost $z = -15$. Next, we pivot on x_{22}:

19	0	0	-1/3	4	0
2	1	0	1/3	-1	0
3	0	1	-2/3	3	0
2	0	0	2/3	-4	1

We get the $bfs = (2, 3, 0, 0, 2)$ with a cost $z = -19$. Next, we pivot on x_{33}:

20	0	0	0	2	1/2
1	1	0	0	1	-1/2
5	0	1	0	-1	1
3	0	0	1	-6	3/2

We get the $bfs = (1, 5, 3, 0, 0)$ with cost $z = -20$.
This solution is optimal because all $\bar{c_i} \geq 0$.

3.

First, we transform the problem from a maximization problem to the equivalent minimization problem by multiplying the RHS of the cost function by -1.

Second, we put the problem in the standard form by introducing 3 slack variables x_4, x_5, x_6 to change the inequalities to equations.

The problem now becomes:

$$\text{Minimize } z = -5x_1 - 3x_2 - 4x_3$$

Subject to:

$$\begin{aligned} 3x_1 + 6x_2 + 2x_3 + x_4 &= 12 \\ 1x_1 + 2x_2 + 2x_3 + x_5 &= 8 \\ 4x_1 + 2x_2 + 4x_3 + x_6 &= 17 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

The tableau for this problem is:

0	-5	-3	-4	0	0	0
12	3	6	2	1	0	0
8	1	2	2	0	1	0
17	4	2	4	0	0	1

We get the initial basic feasible solution $bfs_0 = (0, 0, 0, 12, 8, 17)$ with a cost $z = 0$.

Next, we pivot on x_{11} because $\bar{c_1} < 0$ so it is profitable for column 1 to enter the basis, and $\theta_0 = 12/3 = 4$ at $l = 1$:

20	0	7	-2/3	5/4	0	0
4	1	2	2/3	1/3	0	0
4	0	0	4/3	-1/3	1	0
1	0	-6	4/3	-4/3	0	1
41/2	0	4	0	7/12	0	1/2
7/2	1	5	0	1	0	-1/2
3	0	6	0	1	1	-1
3/4	0	-9/2	1	-1	0	3/4

We get the $bfs = (7/2, 0, 3/4, 0, 3, 0)$ with cost $z = -41/2$.
This solution is optimal because all $\bar{c_i} \geq 0$.

4.

First, we put the problem in the standard form by introducing 3 slack variables $x5, x_6, x_7$ to change the inequalities to equations.

The problem now becomes:

$$\text{Minimize } z = -x_1 - 2x_2 + 3x_3 - x_4$$

Subject to:

$$\begin{aligned} 1x_1 + 2x_2 + x_3 - x_4 + x_5 &= 1 \\ 2x_1 + 3x_2 + -1x_3 + 2x_4 + x_6 &= 2 \\ -1x_1 + 2x_2 + 3x_3 - 3x_4 + x_7 &= 3 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

The tableau for this problem is:

0	-1	-2	3	-1	0	0	0
1	1	2	1	-1	1	0	0
2	2	3	-1	2	0	1	0
3	-1	2	3	-3	0	0	1
1	0	0	4	-2	1	0	0
1/2	1/2	1	1/2	-1/2	1/2	0	0
1/2	1/2	0	-5/2	7/2	-3/2	1	0
2	-2	0	2	-2	-1	0	1
9/7	2/7	0	18/7	0	29/7	4/7	0
4/7	4/7	1	1/7	0	2/7	1/7	0
1/7	1/7	0	-5/7	1	-3/7	2/7	0
16/7	-12/7	0	4/7	0	-13/7	4/7	1

We get the $bfs = (0, 4/7, 0, 1/7, 0, 0, 16/7)$ with cost $z = -9/7$.
This solution is optimal because all $\bar{c_i} \geq 0$.

5.

First, we transform the problem from a maximization problem to the equivalent minimization problem by multiplying the RHS of the cost function by -1.

Second, we put the problem in the standard form by introducing a slack variable x_3 to the first constraint, and a surplus variable to the second constraint, to change the inequalities to equations.

The problem now becomes:

$$\text{Minimize } z = -2x_1 - 4x_2$$

Subject to:

$$\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ 6x_1 + 4x_2 - x_4 &= 12 \\ x_1 + 4x_2 &= 20 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

To find an initial feasible solution, we use the 2-phase method.

Phase I:

We introduce 2 artificial variables x_1^a, x_2^a to the second and third equations respectively, and minimize the cost function $w = x_1^a + x_2^a$. The problem becomes:

$$\text{Minimize } w = x_1^a + x_2^a = 32 - 7x_1 - 8x_2 + x_4$$

Subject to:

$$\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ 6x_1 + 4x_2 - x_4 + x_1^a &= 12 \\ x_1 + 4x_2 + x_2^a &= 20 \\ x_1, x_2, x_3, x_4, x_1^a, x_2^a &\geq 0 \end{aligned}$$

0	-2	-4	0	0	0	0
-32	-7	-8	0	1	0	0
8	1	1	1	0	0	0
12	6	4	0	-1	1	0
20	1	4	0	0	0	1
12	4	0	0	-1	1	0
-8	5	0	0	-1	2	0
5	-1/2	0	1	1/4	-1/4	0
2	3/2	1	0	-1/4	1/4	0
8	-5	0	0	1	-1	1
20	-1	0	0	0	0	1
0	0	0	0	0	1	1
3	3/4	0	1	0	0	-1/4
5	1/4	1	0	0	0	1/4
8	-5	0	0	1	-1	1

Phase II:

20	-1	0	0	0
3	3/4	0	1	0
5	1/4	1	0	0
8	-5	0	0	1
24	0	0	4/3	0
4	1	0	4/3	0
4	0	1	-1/3	0
28	0	0	20/3	1

We get the $bfs = (4, 4, 0, 28)$ with cost $z = -24$.
This solution is optimal because all $\bar{c}_i \geq 0$.

6.

First, we put the problem in the standard form by introducing 2 slack variables $x5, x_6$ and 2 surplus variables x_7, x_8 to change the inequalities to equations.

The problem now becomes:

$$\text{Minimize } z = 20x_1 + 15x_2 + 10x_3 + 12x_4$$

Subject to:

$$\begin{aligned} 2x_1 + x_2 + 1.5x_3 - 0.5x_4 + x_5 &= 250 \\ x_1 + 0.5x_2 + 2x_3 + 1.5x_4 + x_6 &= 200 \\ x_1 + x_2 - x_7 &= 100 \\ x_3 - x_8 &= 20 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 &\geq 0 \end{aligned}$$

To find an initial feasible solution, we use the 2-phase method.

Phase I:

We introduce 2 artificial variables x_1^a, x_2^a to the third and fourth equations respectively, and minimize the cost function $w = x_1^a + x_2^a$. The problem becomes:

$$\text{Minimize } w = x_1^a + x_2^a = 120 - x_1 - x_2 - x_3 + x_7 + x_8$$

Subject to:

$$\begin{aligned} 2x_1 + x_2 + 1.5x_3 - 0.5x_4 + x_5 &= 250 \\ x_1 + 0.5x_2 + 2x_3 + 1.5x_4 + x_6 &= 200 \\ x_1 + x_2 - x_7 + x_1^a &= 100 \\ x_3 - x_8 + x_2^a &= 20 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_1^a, x_2^a &\geq 0 \end{aligned}$$

0	20	15	10	12	0	0	0	0	0	0
-120	-1	-1	-1	0	0	0	1	1	0	0
250	2	1	1.5	-0.5	1	0	0	0	0	0
200	1	0.5	2	1.5	0	1	0	0	0	0
100	1	1	0	0	0	0	-1	0	1	0
20	0	0	1	0	0	0	0	-1	0	1
-2000	0	-5	10	12	0	0	20	0	-20	0
-20	0	0	-1	0	0	0	0	1	1	0
50	0	-1	1.5	0.5	1	0	2	0	-2	0
100	0	-0.5	2	1.5	0	1	1	0	-1	0
100	1	1	0	0	0	0	-1	0	1	0
20	0	0	1	0	0	0	0	-1	0	1
-2200	0	-5	0	12	0	0	20	10	-20	-10
0	0	0	-0	0	0	0	0	0	1	1
20	0	-1	0	0.5	1	0	2	1.5	-2	-1.5
60	0	-0.5	0	1.5	0	1	1	2	-1	-2
100	1	1	0	0	0	0	-1	0	1	0
20	0	0	1	0	0	0	0	-1	0	1

Phase II:

-2200	0	-5	0	12	0	0	20	10
20	0	-1	0	0.5	1	0	2	1.5
60	0	-0.5	0	1.5	0	1	1	2
100	1	1	0	0	0	0	-1	0
20	0	0	1	0	0	0	0	-1
-1700	5	0	0	12	0	0	15	10
120	1	0	0	0.5	1	0	1	1.5
110	0.5	0	0	1.5	0	1	0.5	2
100	1	1	0	0	0	0	-1	0
20	0	0	1	0	0	0	0	-1

We get the $bfs = (0, 100, 20, 0, 0, 0, 0, 0)$ with cost $z = 1700$.
This solution is optimal because all $\bar{c}_i \geq 0$.