Mathematical Tripos

Part II

Graph Theory

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0 Preface

The course is self-contained and has almost no prerequisites (the only ones are mainly IA material). The book recommended for a more in-depth study of the material is I. B. Bollobas's "Modern Graph Theory" (published by Springer).

1 Introduction

A preliminary definition of graphs:

Definition. A graph consists of some 'vertices' with soem pairs of vertices joined by 'edges'

1.1 Types of graph problems:

1.1.1 Bridges of Konigsberg

Question. Is it possible to walk round the city crossing each bridge precisely once and returning to the starting point?

Question. Is it possible to walk round the multigraph traversing each edge precisely once and finishing at the starting vertex?

1.1.2 Four Colour Problem

Question. How many colours are needed to colour a map?

1.1.3 Simultaneous coset representation

Let G be a finite group and $H \leq G$. Let $n = |G : H|$. Then we know $\exists a_i, b_i \in G$ s.t. $a_i H$ are the left cosets and Hb_i are the right cosets of H in G.

Question. $\exists ? c_i$ s.t. $c_i H$ are the left cosets and Hc_i are the right cosets of H in G

By considering X the set of left cosets and Y the set of right cosets of H in G, let $E = X \cup Y$ and $V = \{(gH, Hg) : g \in G\}$.

Question. $\exists ? \epsilon \subset E$ s.t. $\forall v \in V$, $\exists$ unique $e \in \epsilon$ s.t. e has v as an endpoint

1.1.4 Fermat equation mod p

$x^n + y^n = z^n$ has no non-trivial solutions in $\mathbb{Z}$ if $n \geq 3$.

Question. Does $x^n + y^n = z^n$ have any non-trivial solutions in $\mathbb{Z}_p$?

Theorem 1.1. *Let $n \in \mathbb{N}$. Then for $\forall$ sufficiently large p, there are $x, y, z \neq 0 \pmod{p}$ with $x^n + y^n \equiv z^n \pmod{p}$.*

Proof. Let $G = \mathbb{Z}_p^\times$, the multiplicative group of residues modulo p. Let $H = \{g^n : g \in G\} \leq G$. We want $x, y, z \in H$ s.t. $x + y = z$.
Looking at the cosets of H, we have $|H| \geq \frac{|G|}{n}$. We also note that $\forall g \in G$, if $\exists u, v, w \in gH$, then $g^{-1}u + g^{-1}v = g^{-1}w$ and $g^{-1}u, g^{-1}, g^{-1}w \in H$. We have reduced the problem to the following combinatorial statement: ☐

Theorem 1.2 (Schur's theorem)**.** *Let n be a positive integer. Then for $\forall$ sufficiently large k, if $[k] = \{1, 2, ..., k\}$ is partitioned in n parts, $\exists u, v, w$ in the same part s.t. $u + v = w$.*