A generalized Llewellyn Thomas algorithm for solving banded cyclic/periodic matrices:

Banded matrices can arise from finite difference methods, eg. :

$a_j z_{j-1}+b_j z_j+c_j z_{j+1}=d_j$

for a 3-point difference equation or

$a_j z_{j-2}+b_j z_{j-1}+c_j z_j+d_j z_{j+1}+e_j z_{j+2}=f_j$

for a 5-point formula.

A periodic version of a banded matrix can come from periodic boundary conditions. In these cases, none of the coefficients above are zero, whereas in a “regular” banded matrix, the boundary conditions make the off diagonal terms 0.

For the first example, the three (m=3) point equation $a_j z_{j-1}+b_j z_j+c_j z_{j+1}=d_j$ can be written as a cyclic or periodic version of the tridiagonal matrix with non-zero out of band elements:

$$\begin{bmatrix} b_1 & c_1 & . & .. & . & 0 & a_1 \\ a_2 & b_2 & c_2 & .. & . & . & 0 \\ . & . & . & .. & a_{n-1} & b_{n-1} & c_{n-1} \\ c_n & 0 & . & .. & 0 & a_n & b_n \end{bmatrix} \begin{bmatrix} z_1 \\ . \\ . \\ z_n \end{bmatrix} = \begin{bmatrix} d_1 \\ . \\ . \\ d_n \end{bmatrix}$$ i.e. tridiagonal except for two off-diagonal terms.

The system can be solved in O[n] with the following algorithm. Starting with the 1st and nth equations and working inwards to the 2nd and n-1st an so on, for the jth pair of equations we can write:
for j:{2,3,4…} 1 < j < n/2

$$\begin{bmatrix} b_j & 0 \\ 0 & b_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix} + \begin{bmatrix} a_j & 0 \\ 0 & c_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j-1} \\ z_{n-j+2} \end{bmatrix} = \begin{bmatrix} d_j \\ d_{n-j+1} \end{bmatrix} - \begin{bmatrix} c_j & 0 \\ 0 & a_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j+1} \\ z_{n-j} \end{bmatrix}$$

which we rewrite as

$$\begin{bmatrix} b_j & 0 \\ 0 & b_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix} + \begin{bmatrix} a_j & 0 \\ 0 & c_{n-j+1} \end{bmatrix} \left\{ \tilde{A}_{j-1} \begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix} + \vec{v}_{j-1} \right\} = \begin{bmatrix} d_j \\ d_{n-j+1} \end{bmatrix} - \begin{bmatrix} c_j & 0 \\ 0 & a_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j+1} \\ z_{n-j} \end{bmatrix}$$

for j:{1,2,3,4…} 0 < j < n/2

defining $\begin{bmatrix} z_{j-1} \\ z_{n-j+2} \end{bmatrix} = \left\{ \tilde{A}_{j-1} \begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix} + \vec{v}_{j-1} \right\}$ for j:{2,3,4…} 1 < j < n/2

then we have

$$\begin{bmatrix} b_j & 0 \\ 0 & b_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix} + \begin{bmatrix} a_j & 0 \\ 0 & c_{n-j+1} \end{bmatrix} \left\{ \tilde{A}_{j-1} \begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix} + \vec{v}_{j-1} \right\} = \begin{bmatrix} d_j \\ d_{n-j+1} \end{bmatrix} - \begin{bmatrix} c_j & 0 \\ 0 & a_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j+1} \\ z_{n-j} \end{bmatrix}$$

and $\tilde{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \vec{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for j=1 because at j=1:

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_n \end{bmatrix}+\begin{bmatrix} a_1 & 0 \\ 0 & c_n \end{bmatrix}\begin{bmatrix} z_n \\ z_1 \end{bmatrix}=\begin{bmatrix} d_1 \\ d_n \end{bmatrix}-\begin{bmatrix} c_1 & 0 \\ 0 & a_n \end{bmatrix}\begin{bmatrix} z_2 \\ z_{n-1} \end{bmatrix}$$

Each pair is solved for in terms of the inner next pair of values. Using the first equation with j=1 yields the result for the second equation for j=2 by solving a 2x2 system of equations. Using the matrix A_1 and vector v_1, the process is iterated repeatedly almost n/2 times, solving for all the A_j and v_j.

At the center of the matrix we have to solve either a 2x2 system of equations if n is even or a 3x3 system if n is odd. If n is even the final equation becomes, with j=n/2:

$$\begin{bmatrix} b_{n/2} & 0 \\ 0 & b_{n/2+1} \end{bmatrix}\begin{bmatrix} z_{n/2} \\ z_{n/2+1} \end{bmatrix}+\begin{bmatrix} a_{n/2} & 0 \\ 0 & c_{n/2+1} \end{bmatrix}\left\{\widetilde{A}_{n/2-1}\begin{bmatrix} z_{n/2} \\ z_{n/2+1} \end{bmatrix}+\vec{v}_{n/2-1}\right\}=\begin{bmatrix} d_{n/2} \\ d_{n/2+1} \end{bmatrix}-\begin{bmatrix} c_{n/2} & 0 \\ 0 & a_{n/2+1} \end{bmatrix}\begin{bmatrix} z_{n/2+1} \\ z_{n/2} \end{bmatrix}$$

which is a 2x2 system solvable for the middle two second derivatives.

If n is odd, the middle term is included:

$$\begin{bmatrix} b_j & c_j & 0 \\ a_{j+1} & b_{j+1} & c_{j+1} \\ 0 & a_{j+2} & b_{j+2} \end{bmatrix}\begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \end{bmatrix}=\begin{bmatrix} d_j \\ d_{j+1} \\ d_{j+2} \end{bmatrix}-\begin{bmatrix} a_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{j+2} \end{bmatrix}\begin{bmatrix} z_{j-1} \\ 0 \\ z_{j+3} \end{bmatrix}$$

$$\begin{bmatrix} z_{j-1} \\ z_{j+3} \end{bmatrix}=\widetilde{A}_{j-1}\begin{bmatrix} z_j \\ z_{j+2} \end{bmatrix}+\vec{v}_{j-1}, j=\frac{(n-1)}{2}$$

With

$$\begin{bmatrix} z_{j-1} \\ 0 \\ z_{j+3} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_{j-1} \\ z_{j+3} \end{bmatrix}=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\widetilde{A}_{j-1}\begin{bmatrix} z_j \\ z_{j+2} \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\vec{v}_{j-1}\right\} \quad \text{then}$$

$$\begin{bmatrix} b_j & c_j & 0 \\ a_{j+1} & b_{j+1} & c_{j+1} \\ 0 & a_{j+2} & b_{j+2} \end{bmatrix}\begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \end{bmatrix}=\begin{bmatrix} d_j \\ d_{j+1} \\ d_{j+2} \end{bmatrix}-\begin{bmatrix} a_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{j+2} \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\widetilde{A}_{j-1}\begin{bmatrix} z_j \\ z_{j+2} \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\vec{v}_{j-1}\right\}$$

or, writing the components of $\widetilde{A}_{j-1}$ explicitly:

$$\begin{bmatrix} b_j+a_j A^{11}_{j-1} & c_j & a_j A^{12}_{j-1} \\ a_{j+1} & b_{j+1} & c_{j+1} \\ c_{j+2} A^{21}_{j-1} & a_{j+2} & b_{j+2}+c_{j+2} A^{22}_{j-1} \end{bmatrix}\begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \end{bmatrix}=\begin{bmatrix} d_j-a_j v^1_{j-1} \\ d_{j+1} \\ d_{j+2}-c_{j+2} v^2_{j-1} \end{bmatrix}$$

We can then solve for all of the z_j by back substitution using $\begin{bmatrix} z_{j-1} \\ z_{n-j+2} \end{bmatrix}=\left\{\widetilde{A}_{j-1}\begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix}+\vec{v}_{j-1}\right\}$

Similarly, the m=5 point scheme $a_j z_{j-2}+b_j z_{j-1}+c_j z_j+d_j z_{j+1}+e_j z_{j+2}=f_j$ results in

$$\begin{bmatrix} c_1 & d_1 & e_1 & . & . & . & a_1 & b_1 \\ b_2 & c_2 & d_2 & e_2 & . & . & . & a_2 \\ . & . & . & . & . & . & . & . \\ e_{n-1} & . & . & . & a_{n-1} & b_{n-1} & c_{n-1} & d_{n-1} \\ d_n & e_n & . & . & . & a_n & b_n & c_n \end{bmatrix} \begin{bmatrix} z_1 \\ . \\ . \\ . \\ z_n \end{bmatrix} = \begin{bmatrix} f_1 \\ . \\ . \\ . \\ f_n \end{bmatrix}$$ with 6 off-diagonal non-zero elements.

We can then write, for any set of 4 equations j:

$$\begin{bmatrix} c_j & d_j & 0 & 0 \\ b_{j+1} & c_{j+1} & 0 & 0 \\ 0 & 0 & c_{n-j} & d_{n-j} \\ 0 & 0 & b_{n-j+1} & c_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{j+1} \\ z_{n-j} \\ z_{n-j+1} \end{bmatrix} + \begin{bmatrix} a_j & b_j & 0 & 0 \\ 0 & a_{j+1} & 0 & 0 \\ 0 & 0 & e_{n-j} & 0 \\ 0 & 0 & d_{n-j} & e_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j-2} \\ z_{j-1} \\ z_{n-j+2} \\ z_{n-j+3} \end{bmatrix} = \begin{bmatrix} f_j \\ f_{j+1} \\ f_{n-j} \\ f_{n-j+1} \end{bmatrix} - \begin{bmatrix} e_j & 0 & 0 & 0 \\ d_{j+1} & e_{j+1} & 0 & 0 \\ 0 & 0 & a_{n-j} & b_{n-j} \\ 0 & 0 & 0 & a_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j+2} \\ z_{j+3} \\ z_{n-j-2} \\ z_{n-j-1} \end{bmatrix}$$

where we can define

$$\begin{bmatrix} z_{j-2} \\ z_{j-1} \\ z_{n-j+2} \\ z_{n-j+3} \end{bmatrix} = \tilde{A}_{j-1} \begin{bmatrix} z_j \\ z_{j+1} \\ z_{n-j} \\ z_{n-j+1} \end{bmatrix} + \vec{v}_{j-1}$$ as before for j:{3,5,7…} j>1 and $\tilde{A}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \vec{v}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ for j=1.

and then

$$\begin{bmatrix} c_j & d_j & 0 & 0 \\ b_{j+1} & c_{j+1} & 0 & 0 \\ 0 & 0 & c_{n-j} & d_{n-j} \\ 0 & 0 & b_{n-j+1} & c_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{j+1} \\ z_{n-j} \\ z_{n-j+1} \end{bmatrix} + \begin{bmatrix} a_j & b_j & 0 & 0 \\ 0 & a_{j+1} & 0 & 0 \\ 0 & 0 & e_{n-j} & 0 \\ 0 & 0 & d_{n-j} & e_{n-j+1} \end{bmatrix} \left(\tilde{A}_j \begin{bmatrix} z_j \\ z_{j+1} \\ z_{n-j} \\ z_{n-j+1} \end{bmatrix} + \vec{v}_j \right)$$
$$= \begin{bmatrix} f_j \\ f_{j+1} \\ f_{n-j} \\ f_{n-j+1} \end{bmatrix} - \begin{bmatrix} e_j & 0 & 0 & 0 \\ d_{j+1} & e_{j+1} & 0 & 0 \\ 0 & 0 & a_{n-j} & b_{n-j} \\ 0 & 0 & 0 & a_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j+2} \\ z_{j+3} \\ z_{n-j-2} \\ z_{n-j-1} \end{bmatrix}$$

We then have either 5,6 or 7 rows left after iterating the recurrence relations: for example with 6 rows with j=n/2-2 ? (n-1)/2 we have:

$$\begin{bmatrix} c_j & d_j & e_j & 0 & 0 & 0 \\ b_{j+1} & c_{j+1} & d_{j+1} & e_{j+1} & 0 & 0 \\ a_{j+2} & b_{j+2} & c_{j+2} & d_{j+2} & e_{j+2} & 0 \\ 0 & a_{n-j-1} & b_{n-j-1} & c_{n-j-1} & d_{n-j-1} & e_{n-j-1} \\ 0 & 0 & a_{n-j} & b_{n-j} & c_{n-j} & d_{n-j} \\ 0 & 0 & 0 & a_{n-j+1} & b_{n-j+1} & c_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \\ z_{j+3} \\ z_{j+4} \\ z_{j+5} \end{bmatrix} = \begin{bmatrix} f_j \\ f_{j+1} \\ f_{j+2} \\ f_{j+3} \\ f_{j+4} \\ f_{j+5} \end{bmatrix} - \begin{bmatrix} a_j & b_j & 0 & 0 & 0 & 0 \\ 0 & a_{j+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{n-j} & 0 \\ 0 & 0 & 0 & 0 & d_{n-j} & e_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j-2} \\ z_{j-1} \\ 0 \\ 0 \\ z_{n-j+2} \\ z_{n-j+3} \end{bmatrix}$$

but

$$\begin{bmatrix} z_{j-2} \\ z_{j-1} \\ 0 \\ 0 \\ z_{n-j+2} \\ z_{n-j+3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{j-2} \\ z_{j-1} \\ z_{n-j+2} \\ z_{n-j+3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left\{ \widetilde{A}_{j-1} \begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \\ z_{j+3} \end{bmatrix} + \vec{v}_{j-1} \right\}$$

and so

$$\begin{bmatrix} c_j & d_j & e_j & 0 & 0 & 0 \\ b_{j+1} & c_{j+1} & d_{j+1} & e_{j+1} & 0 & 0 \\ a_{j+2} & b_{j+2} & c_{j+2} & d_{j+2} & e_{j+2} & 0 \\ 0 & a_{n-j-1} & b_{n-j-1} & c_{n-j-1} & d_{n-j-1} & e_{n-j-1} \\ 0 & 0 & a_{n-j} & b_{n-j} & c_{n-j} & d_{n-j} \\ 0 & 0 & 0 & a_{n-j+1} & b_{n-j+1} & c_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \\ z_{j+3} \\ z_{j+4} \\ z_{j+5} \end{bmatrix}$$

$$= \begin{bmatrix} f_j \\ f_{j+1} \\ f_{j+2} \\ f_{j+3} \\ f_{j+4} \\ f_{j+5} \end{bmatrix} - \begin{bmatrix} a_j & b_j & 0 & 0 & 0 & 0 \\ 0 & a_{j+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{n-j} & 0 \\ 0 & 0 & 0 & 0 & d_{n-j} & e_{n-j+1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left\{ \widetilde{A}_{j-1} \begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \\ z_{j+3} \end{bmatrix} + \vec{v}_{j-1} \right\}$$

and
$$\begin{bmatrix} a_j & b_j & 0 & 0 & 0 & 0 \\ 0 & a_{j+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{n-j} & 0 \\ 0 & 0 & 0 & 0 & d_{n-j} & e_{n-j+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_j & b_j & 0 & 0 \\ 0 & a_{j+1} & 0 & 0 \\ 0 & 0 & e_{n-j} & 0 \\ 0 & 0 & d_{n-j} & e_{n-j+1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can workout similar formulae for 5 and 7 rows left, with:

$$\begin{bmatrix} c_j & d_j & e_j & 0 & 0 \\ b_{j+1} & c_{j+1} & d_{j+1} & e_{j+1} & 0 \\ a_{(n+1)/2} & b_{(n+1)/2} & c_{(n+1)/2} & d_{(n+1)/2} & e_{(n+1)/2} \\ 0 & a_{n-j} & b_{n-j} & c_{n-j} & d_{n-j} \\ 0 & 0 & a_{n-j+1} & b_{n-j+1} & c_{n-j+1} \end{bmatrix} \quad \text{with 5 rows}$$

and

$$\begin{bmatrix} c_j & d_j & e_j & 0 & 0 & 0 & 0 \\ b_{j+1} & c_{j+1} & d_{j+1} & e_{j+1} & 0 & 0 & 0 \\ a_{j+2} & b_{j+2} & c_{j+2} & d_{j+2} & e_{j+2} & 0 & 0 \\ 0 & a_{(n+1)/2} & b_{(n+1)/2} & c_{(n+1)/2} & d_{(n+1)/2} & e_{(n+1)/2} & 0 \\ 0 & 0 & a_{n-j-1} & b_{n-j-1} & c_{n-j-1} & d_{n-j-1} & e_{n-j-1} \\ 0 & 0 & 0 & a_{n-j} & b_{n-j} & c_{n-j} & d_{n-j} \\ 0 & 0 & 0 & 0 & a_{n-j+1} & b_{n-j+1} & c_{n-j+1} \end{bmatrix} \quad \text{with 7 rows}$$

Writing more generally, if we write the first equation corresponding to the 3 point formula

$$\begin{bmatrix} b_j & 0 \\ 0 & b_{n-j+1} \end{bmatrix} \begin{bmatrix} z_j \\ z_{n-j+1} \end{bmatrix} + \begin{bmatrix} a_j & 0 \\ 0 & c_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j-1} \\ z_{n-j+2} \end{bmatrix} = \begin{bmatrix} d_j \\ d_{n-j+1} \end{bmatrix} - \begin{bmatrix} c_j & 0 \\ 0 & a_{n-j+1} \end{bmatrix} \begin{bmatrix} z_{j+1} \\ z_{n-j} \end{bmatrix}$$

as

$$\widetilde{C}_{jj} \vec{z}_{jj} + \widetilde{S}_{jj} \vec{z}_{jj-1} = \vec{b}_{jj} - \widetilde{P}_{jj} \vec{z}_{jj+1} \quad \text{and defining} \quad KU = \frac{(m-1)}{2}$$

where $\vec{z}_{jj}$ is understood to be the corresponding vector with length 2 KU (or m-1) defined as

$$(z_j, z_{j+1} \ldots. z_{j+(m-3)/2}, z_{n-j+1-(m-3)/2} \ldots z_{n-j}, z_{n-j+1}) = (z_j, z_{j+1} \ldots. z_{j+KU-1}, z_{n-j+2-KU} \ldots z_{n-j}, z_{n-j+1})$$

with $\vec{b}_{jj}$ defined similarly so that $\widetilde{C}_{jj}, \widetilde{S}_{jj}, \widetilde{P}_{jj}, \vec{b}_{jj} \vec{z}_{jj}$ are defined for jj every j KU apart.

Substituting $\widetilde{A}_{jj}$ and v_{jj} defined as:

$$\vec{z}_{jj-1} = \widetilde{A}_{jj-1} \vec{z}_{jj} + \vec{v}_{jj-1}$$

results in the second equation as

$$\{\widetilde{C}_{jj}+\widetilde{S}\,\widetilde{A_{jj-1}}\}\vec{z}_{jj}+\widetilde{S}_{jj}\,v_{\vec{jj-1}}=\vec{b}_{jj}-\widetilde{P}_{jj}\,z_{\vec{jj+1}}$$

Using the definition of $\widetilde{A}_{jj}$ and v_{jj} and the second equation we can recognize the recurrence relations:

$$\widetilde{A}_{jj}=-\{\widetilde{C}_{jj}+\widetilde{S}_{jj}\,\widetilde{A_{jj-1}}\}^{-1}\widetilde{P}_{jj}$$

$$\vec{v}_{jj}=\{\widetilde{C}_{jj}+\widetilde{S}_{jj}\,\widetilde{A_{jj-1}}\}^{-1}\{\vec{b}_{jj}-\widetilde{S}_{jj}\,v_{\vec{jj-1}}\}$$

which are solved every jj, reducing the number of equations by (m-1) each iteration with $\widetilde{A}_0=\begin{bmatrix} 0 & I_{\widetilde{(m-1)/2}} \\ I_{\widetilde{(m-1)/2}} & 0 \end{bmatrix}$ an (m-1)x(m-1) circular permutation matrix and $I_{\widetilde{(m-1)/2}}$ the (m-1)/2x(m-1)/2 identity matrix.

At the center of the system matrix, if modulo(n,m-1) =0, then $z_{\vec{jj+1}}=\widetilde{A}_0\vec{z}_{jj}$ (recognizing that the final $\widetilde{A_{j-1}}=\{\widetilde{A}_0\}^{-1}$)

so that the final inner values are given by

$\vec{z}_{jj}=\{\widetilde{C}_{jj}+\widetilde{P}_{jj}\widetilde{A}_0+\widetilde{S}_{jj}\,\widetilde{A_{jj-1}}\}^{-1}\{\vec{b}_{jj}-\widetilde{S}_j\,v_{\vec{jj-1}}\}$ for the last 2*(m-1) values. More generally, if modulo(n,m-1) = p /= 0, we have m-1+p rows left. The central rows can be written

$$\vec{z}_L=\{\ z_j, z_{j+1}, \ldots. z_{m-1+p}\}\quad \text{with } j=(n-m-p+2)/2$$

We can then state that these central values are given by:

$$\vec{z}_L=[\widetilde{C}_L+\widetilde{P}_L\,\widetilde{A}_0+\widetilde{I}_0^*S_{jj}\,\widetilde{A_{jj-1}}\,\widetilde{I}_0^{*T}]^{-1}\{\vec{b}_L-\widetilde{I}_0^*\widetilde{S}_{jj}\,v_{\vec{jj-1}}\}\quad\text{or}$$

$$\vec{z}_L=[\widetilde{C_{jjL}}+\widetilde{I}_0^*S_{jj}\,\widetilde{A_{jj-1}}\,\widetilde{I}_0^{*T}]^{-1}\{\vec{b}_L-\widetilde{I}_0^*\widetilde{S}_{jj}\,v_{\vec{jj-1}}\}$$

where $\widetilde{C_{jjL}}=\widetilde{C}_L+\widetilde{P}_L\,\widetilde{A}_0$ and $\vec{b}_L$ consists of the last 2*KU+p inner values.

$\widetilde{I}_0^*$ is an (m-1) x (m-1+p) matrix composed of the (m-1) x (m-1) identity matrix with p=modulo(n,m-1) extra zero rows . Note we use the fact that $\widetilde{I}_0^*\widetilde{I}_0^{*\mathrm{T}}=I_{\widetilde{m-1+p}}$ that for the central rows where

j=(n-m-p+2)/2: $\widetilde{I}_0^*\vec{z}_L=\widetilde{I}_0^*\{\ z_j, z_{j+1}, \ldots. z_{j+m-1+p}\}^T=\widetilde{I}_0^*\vec{z}_j=$
$\widetilde{I}_0^*\{z_j, z_{j+1}\ldots. z_{j+(m-3)/2}, 0,0,..0, z_{n-j+1-(m-3)/2}\ldots z_{n-j}, z_{n-j+1}\}^T$

In general, then, the algorithm consists of computing the matrix and vector every (m-1) sets of rows with

$$\widetilde{A}_{jj}=-\{\widetilde{C}_{jj}+\widetilde{S}_{jj}\,\widetilde{A_{jj-1}}\}^{-1}\widetilde{P}_{jj}$$

$$\vec{v}_{jj}=\left(\widetilde{C}_{jj}+\widetilde{S}_{jj}\widetilde{A_{jj-1}}\right)^{-1}\left(\vec{b}_{jj}-\widetilde{S}_{jj}\vec{v_{jj-1}}\right)$$

for n/(m-1) number of times. (integer division)

then computing the m-1+p central values needing only $\widetilde{A_{jj-1}}$ and v_{jj-1}

$$\vec{z}_L=[\widetilde{C}_L+\widetilde{P}_L\widetilde{A}_0+\widetilde{I}_0^*S_{jj}\widetilde{A_{jj-1}}\widetilde{I}_0^{*T}]^{-1}\left(\vec{b}_L-\widetilde{I}_0^*\widetilde{S}_{jj}\vec{v_{jj-1}}\right)$$

or

$$\vec{z}_L=[\widetilde{C_{jjL}}+\widetilde{I}_0^*S_{jj}\widetilde{A_{jj-1}}\widetilde{I}_0^{*T}]^{-1}\left(\vec{b}_L-\widetilde{I}_0^*\widetilde{S}_{jj}\vec{v_{jj-1}}\right)$$

where $\widetilde{C_{jjL}}=\widetilde{C}_L+\widetilde{P}_L\widetilde{A}_0$ and $\vec{b}_L$ consists of the last 2*KU+p inner values.

where $\widetilde{I}_0^*$ is an (m-1) x (m-1+p) matrix composed of the (m-1) x (m-1) identity matrix with p=modulo(n,m-1) extra central zero rows , and $\widetilde{A}_0$ is the (m-1) x (m-1) circular permutation matrix composed of the identity matrix from above.

Once the center values are known the stored matrices $\widetilde{A}_{jj}$ and vectors $\vec{v}_{jj}$ are used for back substitution using $\vec{z_{jj-1}}=\widetilde{A}_{jj-1}\vec{z_{jj}}+\vec{v}_{jj-1}$.

In general, an m-point equation, with m-odd, can be written:

$$\sum_{i=1}^{i=m}AB(i,j)*z_{i+j-(m+1)/2}=b_j \quad \text{in contrast with} \quad \sum_{i=1}^{m}a_{ij}z_{i+j}=b_j$$

or

$$\sum_{i=-(m-1)/2}^{i=(m-1)/2}AB((m+1)/2+i,j)*z_{i+j}=b_j$$

where the AB matrix is m x n, with n the number of points, similar to the general LAPACK band matrix (https://www.netlib.org/lapack/lug/node124.html) notation:

$$AB(KU+1+i-j,j)=a_{ij}\,with\,KU=(m-1)/2$$

or, generalizing when we allow wrapping the periodic coefficients

$$AB(2*KU+2-mod(N+KU+1+i-j,N),i)=a_{ij}\,with\,KU=(m-1)/2$$

In the nonperiodic banded matrix, the coefficients AB(i,j)=0 when i+j-(m+1)/2 is < 1 or > n.
With the above definition of AB[i,j] we have the square (i x k) for each j matrices:

$$\widetilde{C}_{jj}=\begin{cases} AB[KU+k-i+1, j+i-1] & \forall i,k\{1,2...KU\} \\ AB[KU+k-i+1, n-2*KU+i-j+1] & \forall i,k\{KU+1...2*KU\} \\ else\ 0 \end{cases}$$

$$\widetilde{S}_{jj}=\begin{cases} AB[1+k-i, j+i-1] & \forall i,k\{1,2...KU\} \\ AB[2*KU+1+k-i, n-2*KU+i-j] & \forall i,k\{KU+1...2*KU\} \\ else\ 0 \end{cases}$$

$$\widetilde{P}_{jj}=\begin{cases} AB[2*KU+1+k-i, j+i-1] & \forall i,k\{1,2...KU\} \\ AB[1+k-i, n-2*KU+i-j] & \forall i,k\{KU+1...2*KU\} \\ else\ 0 \end{cases}$$

The last Cjj and Pjj are (m-1+p) x (m-1+p), which means the last $\widetilde{A}_0$ is also (m-1+p) x (m-1+p). In combination we define CjL

$$\widetilde{C}_{jjL}=\widetilde{C}_L+\widetilde{P}_L\widetilde{A}_0=\begin{cases} AB[KU+1+k-i, j+i-1] & \forall i,k\{1,2...2*KU+p\} \\ else\ 0 \end{cases}$$