



# Probability Review

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## Part I

Design and Analysis  
of Algorithms I

# Topics Covered

- Sample spaces
- Events
- Random variables
- Expectation
- Linearity of Expectation

See also:

- Lehman-Leighton notes (free PDF)
- Wikibook on Discrete Probability

# Concept #1 – Sample Spaces

Sample Space  $\Omega$  : “all possible outcomes”  
[ in algorithms,  $\Omega$  is usually finite ]

Also : each outcome  $i \in \Omega$  has a probability  $p(i) \geq 0$

Constraint :  $\sum_{i \in \Omega} p(i) = 1$

36 ordered pairs

Example #1 : Rolling 2 dice.  $\Omega = \{(1,1), (2,1), (3,1), \dots, (5,6), (6,6)\}$

Example #2 : Choosing a random pivot in outer QuickSort call.

$\Omega = \{1, 2, 3, \dots, n\}$  (index of pivot) and  $p(i) = 1/n$  for all  $i \in \Omega$

# Concept #2 – Events

An event is a subset  $S \subseteq \Omega$

The probability of an event  $S$  is  $\sum_{i \in S} p(i)$

Consider the event (i.e., the subset of outcomes for which) “the sum of the two dice is 7”. What is the probability of this event?

$$S = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$\Pr[S] = 6/36 = 1/6$$

☐  $1/36$

☐  $1/12$

☒  $1/6$

☐  $1/2$

Consider the event (i.e., the subset of outcomes for which) “the chosen pivot gives a 25-75 split of better”. What is the probability of this event?

☐  $1/n$

☐  $1/4$

☒  $1/2$

☐  $3/4$

$S = \{(n/4+1)^{\text{th}} \text{ smallest element}, \dots, (3n/4)^{\text{th}} \text{ smallest element}\}$

$$\Pr[S] = (n/2)/n = 1/2$$

# Concept #2 – Events

An event is a subset

The probability of an event  $S$  is

Ex#1 : sum of dice = 7.  $S = \{(1,1),(2,1),(3,1),\dots,(5,6),(6,6)\}$   
 $\Pr[S] = 6/36 = 1/6$

Ex#2 : pivot gives 25-75 split or better.  
 $S = \{(n/4+1)^{\text{th}} \text{ smallest element}, \dots, (3n/4)^{\text{th}} \text{ smallest element}\}$   
 $\Pr[S] = (n/2)/n = 1/2$

# Concept #3 - Random Variables

A Random Variable  $X$  is a real-valued function

$$X : \Omega \rightarrow \mathbb{R}$$

Ex#1 : Sum of the two dice

Min -> 2

Max -> 12

Ex#2 : Size of subarray passed to 1<sup>st</sup> recursive call.

Min -> 0

Max ->  $n - 1$



# Concept #4 - Expectation

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable.

The expectation  $E[X]$  of  $X$  = average value of  $X$

$$= \sum_{i \in \Omega} X(i) \cdot p(i)$$

What is the expectation of the sum of two dice?

☐ 6.5

☒ 7

☐ 7.5

☐ 8

Which of the following is closest to the expectation of the size of the subarray passed to the first recursive call in QuickSort?

Let  $X$  = subarray size

$$\begin{aligned}\text{Then } E[X] &= (1/n)*0 + (1/n)*2 + \dots + (1/n)*(n-1) \\ &= (n-1)/2\end{aligned}$$

- ☐  $n/4$
- ☐  $n/3$
- ☒  $n/2$
- ☐  $3n/4$

# Concept #4 - Expectation

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable.

The expectation  $E[X]$  of  $X$  = average value of  $X$

$$= \sum_{i \in \Omega} X(i) \cdot p(i)$$

Ex#1 : Sum of the two dice,  $E[X] = 7$

Ex#2 : Size of subarray passed to 1<sup>st</sup> recursive call.

$$E[X] = (n-1)/2$$

# Concept #5 – Linearity of Expectation

Claim [LIN EXP] : Let  $X_1, \dots, X_n$  be random variables defined on  $\Omega$ . Then :

$$E\left[\sum_{j=1}^n X_j\right] = \sum_{j=1}^n E[X_j]$$

Ex#1 : if  $X_1, X_2$  = the two dice, then  
 $E[X_j] = (1/6)(1+2+3+4+5+6) = 3.5$

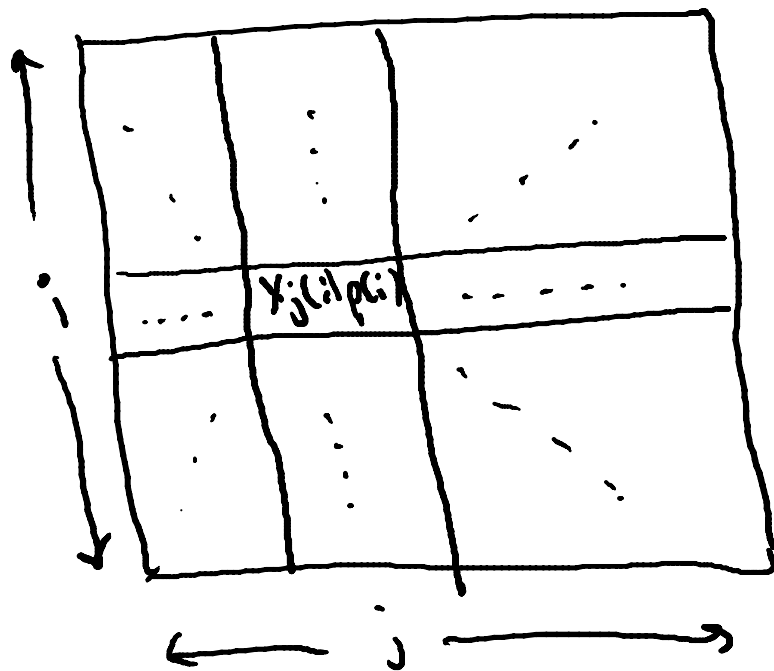
By LIN EXP :  $E[X_1 + X_2] = E[X_1] + E[X_2] = 3.5 + 3.5 = 7$

CRUCIALLY:  
HOLDS EVEN WHEN  
 $X_j$ 's ARE NOT  
INDEPENDENT!  
[WOULD FAIL IF  
REPLACE SUMS WITH  
PRODUCTS]

# Linearity of Expectation (Proof)

$$\begin{aligned}\sum_{j=1}^n E[X_j] &= \sum_{j=1}^n \sum_{i \in \Omega} X_j(i) p(i) \\ &= \sum_{i \in \Omega} \sum_{j=1}^n X_j(i) p(i) \\ &= \sum_{i \in \Omega} p(i) \sum_{j=1}^n X_j(i) \\ &= E\left[\sum_{j=1}^n X_j\right]\end{aligned}$$

Q.E.D.



# Example: Load Balancing

Problem : need to assign  $n$  processes to  $n$  servers.

Proposed Solution : assign each process to a random server

Question : what is the expected number of processes assigned to a server ?

# Load Balancing Solution

Sample Space  $\Omega$  = all  $n^n$  assignments of processes to servers, each equally likely.

Let  $Y$  = total number of processes assigned to the first server.

Goal : compute  $E[Y]$

Let  $X_j = \begin{cases} 1 & \text{if } j\text{th process assigned to first server} \\ 0 & \text{otherwise} \end{cases}$

“indicator random variable”



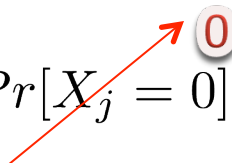
Note  $Y = \sum_{j=1}^n X_j$

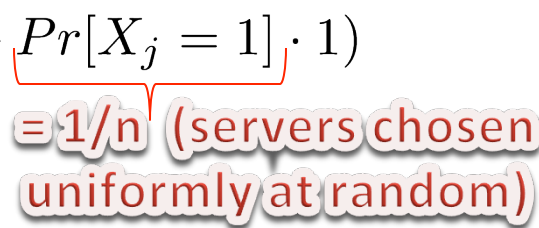


# Load Balancing Solution (con'd)

We have

$$\begin{aligned} E[Y] &= E\left[\sum_{j=1}^n X_j\right] \\ &= \sum_{j=1}^n E[X_j] \\ &= \sum_{j=1}^n (Pr[X_j = 0] \cdot 0 + Pr[X_j = 1] \cdot 1) \\ &= \sum_{j=1}^n \frac{1}{n} = 1 \end{aligned}$$

 **0**

 **= 1/n (servers chosen uniformly at random)**



# Probability Review

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## Part II

Design and Analysis  
of Algorithms I

# Topics Covered

- Conditional probability
- Independence of events and random variables

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# Concept #1 – Sample Spaces

Sample Space  $\Omega$  : “all possible outcomes”  
[ in algorithms,  $\Omega$  is usually finite ]

Also : each outcome  $i \in \Omega$  has a probability  $p(i) \geq 0$

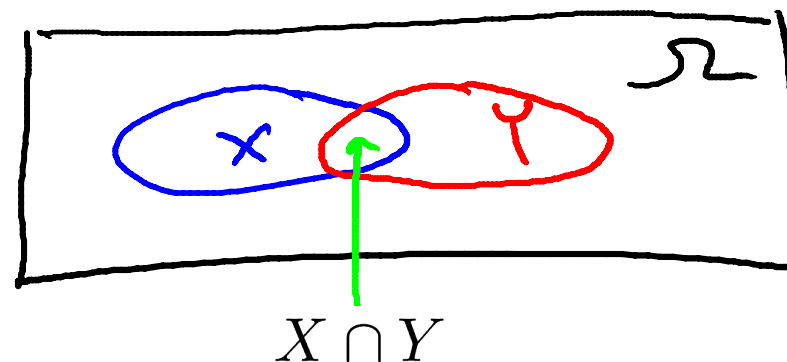
Constraint :  $\sum_{i \in \Omega} p(i) = 1$

An event is a subset  $S \subseteq \Omega$

The probability of an event  $S$  is  $\sum_{i \in S} p(i)$

## Concept #6 – Conditional Probability

Let  $X, Y \subseteq \Omega$  be events.



$$\text{Then } Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$$

("X given Y")

Suppose you roll two fair dice. What is the probability that at least one die is a 1, given that the sum of the two dice is 7?

$X$  = at least one die is a 1

$Y$  = sum of two dice = 7

$$= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$\Rightarrow X \cap Y = \{(1,6), (6,1)\}$$

$$Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]} = \frac{(2/36)}{(6/36)} = \frac{1}{3}$$

☐  $1/36$

☐  $1/6$

☒  $1/3$

☐  $1/2$

# Concept #7 – Independence (of Events)

Definition : Events  $X, Y \subseteq \Omega$  are independent  
if (and only if)  $Pr[X \cap Y] = Pr[X] \cdot Pr[Y]$

You check : this holds if and only if  $Pr[X | Y] = Pr[X]$   
 $\iff Pr[Y | X] = Pr\{Y\}$

WARNING : can be a very subtle concept.  
(intuition is often incorrect!)

# Independence (of Random Variables)

Definition : random variables  $A, B$  (both defined on  $\Omega$ ) are independent if and only if the events  $\Pr[A=a], \Pr[B=b]$  are independent for all  $a, b$ . [ $\Leftrightarrow \Pr[A = a \text{ and } B = b] = \Pr[A=a] \cdot \Pr[B=b]$  ]

Claim : if  $A, B$  are independent, then  $E[AB] = E[A] \cdot E[B]$

Proof : 
$$\begin{aligned} E[AB] &= \sum_{a,b} (a \cdot b) \cdot \Pr[A = a \text{ and } B = b] \\ &= \sum_{a,b} (a \cdot b) \cdot \Pr[A = a] \cdot \Pr[B = b] \quad (\text{Since } A, B \text{ independent}) \\ &= \left( \sum_a a \cdot \Pr[A = a] \right) \left( \sum_b b \cdot \Pr[B = b] \right) \end{aligned}$$

$E[A]$   $\leftarrow$   $\rightarrow$   $E[B]$

**Q.E.D.**



# Example

Let  $X_1, X_2 \in \{0, 1\}$  be random, and  $X_3 = X_1 \oplus X_2$  XOR

formally :  $\Omega = \{000, 101, 011, 110\}$ , each equally likely.

Claim :  $X_1$  and  $X_3$  are independent random variables (you check)

Claim :  $X_1X_3$  and  $X_2$  are not independent random variables.

Proof : suffices to show that

$$E[X_1X_2X_3] \neq E[X_1X_3]E[X_2]$$

= 0 = E[X1]E[X3] = 1/4 = 1/2

Since  $X_1$  and  $X_3$  independent