Design and Analysis of Algorithms I

Introduction

Why Study Algorithms?

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- important for all other branches of computer science
- plays a key role in modern technological innovation
  - "Everyone knows Moore's Law – a prediction made in 1965 by Intel co-founder Gordon Moore that the density of transistors in integrated circuits would continue to double every 1 to 2 years….in many areas, performance gains due to improvements in algorithms have vastly exceeded even the dramatic performance gains due to increased processor speed."
    - Excerpt from *Report to the President and Congress: Designing a Digital Future,* December 2010 (page 71).

Why Study Algorithms?

- important for all other branches of computer science
- plays a key role in modern technological innovation
- provides novel “lens” on processes outside of computer science and technology
  - quantum mechanics, economic markets, evolution

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- challenging (i.e., good for the brain!)
- fun

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Introduction

Integer Multiplication

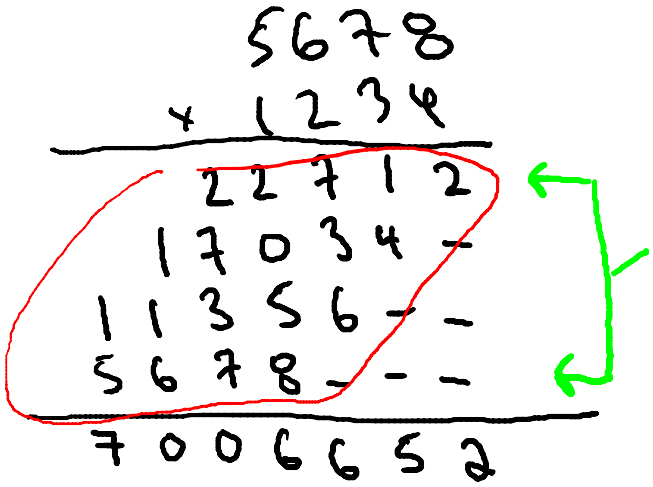
Integer Multiplication

Input : 2 n-digit numbers x and y

Output : product x*y

“Primitive Operation” - add or multiply 2 single-digit numbers

The Grade-School Algorithm



Roughly n operations per row up to a constant

# of operations overall ~ constant* n^2

The Algorithm Designer's Mantra

"Perhaps the most important principle for the good algorithm designer is to refuse to be content."

-Aho, Hopcroft, and Ullman, *The Design and Analysis of Computer Algorithms*, 1974

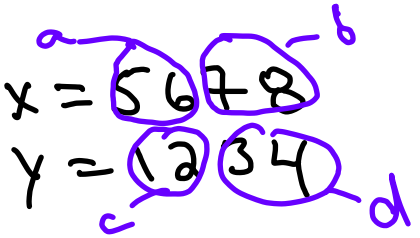
CAN WE DO BETTER ?
[than the "obvious" method]

Design and Analysis of Algorithms I

Introduction

Karatsuba Multiplication

Example



Step 1: compute $a \cdot c = 672$

Step 2: compute $b \cdot d = 2652$

Step 3: compute $(a+b)(c+d) = 134 \cdot 46 = 6164$

Step 4: compute ③ − ② − ① $= 2840$

Step 5:

$$\begin{array}{r} 6720000 \\ 2652 \\ 284000 \\ \hline 7006652 \end{array} = (1234)(5678)$$

A Recursive Algorithm

Write $x = 10^{n/2}a + b$ and $y = 10^{n/2}c + d$

Where a,b,c,d are n/2-digit numbers.

[example: a=56, b=78, c=12, d=34]

Then $x.y = (10^{n/2}a + b)(10^{n/2}c + d)$

$$= (10^n ac + 10^{n/2}(ad + bc) + bd \qquad (*)$$

Idea : recursively compute ac, ad, bc, bd, then compute (*) in the obvious way

Simple Base Case Omitted

Karatsuba Multiplication

$$x.y = (10^n ac + 10^{n/2}(ad + bc) + bd$$

1. Recursively compute ac
2. Recursively compute bd
3. Recursively compute (a+b)(c+d) = ac+bd+ad+bc

Gauss' Trick : (3) – (1) – (2) = ad + bc

Upshot : Only need 3 recursive multiplications (and some additions)

Q : which is the fastest algorithm ?

Introduction

About The Course

Design and Analysis
of Algorithms I

Course Topics

- Vocabulary for design and analysis of algorithms
- Divide and conquer algorithm design paradigm
- Randomization in algorithm design
- Primitives for reasoning about graphs
- Use and implementation of data structures

Course Topics

- Vocabulary for design and analysis of algorithms
  - E.g., “Big-Oh” notation
  - “sweet spot” for high-level reasoning about algorithms

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- Vocabulary for design and analysis of algorithms
- Divide and conquer algorithm design paradigm
  - Will apply to: Integer multiplication, sorting, matrix multiplication, closest pair
  - General analysis methods ("Master Method/Theorem")

Course Topics

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- Randomization in algorithm design
  - Will apply to: QuickSort, primality testing, graph partitioning, hashing.

Course Topics

- Vocabulary for design and analysis of algorithms
- Divide and conquer algorithm design paradigm
- Randomization in algorithm design
- Primitives for reasoning about graphs
  - Connectivity information, shortest paths, structure of information and social networks.

Course Topics

- Vocabulary for design and analysis of algorithms
- Divide and conquer algorithm design paradigm
- Randomization in algorithm design
- Primitives for reasoning about graphs
- Use and implementation of data structures
  - Heaps, balanced binary search trees, hashing and some variants (e.g., bloom filters)

Topics in Sequel Course

- Greedy algorithm design paradigm

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- Dynamic programming algorithm design paradigm

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Topics in Sequel Course

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- NP-complete problems and what to do about them

Topics in Sequel Course

- Greedy algorithm design paradigm
- Dynamic programming algorithm design paradigm
- NP-complete problems and what to do about them
- Fast heuristics with provable guarantees
- Fast exact algorithms for special cases
- Exact algorithms that beat brute-force search

Skills You'll Learn

- Become a better programmer

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- Sharpen your mathematical and analytical skills

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- Become a better programmer
- Sharpen your mathematical and analytical skills
- Start "thinking algorithmically"
- Literacy with computer science's "greatest hits"
- Ace your technical interviews

Who Are You?

- It doesn't really matter. (It's a free course, after all.)

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- Ideally, you know some programming.
- Doesn't matter which language(s) you know.
  - But you should be capable of translating high-level algorithm descriptions into working programs in *some* programming language.

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- Doesn't matter which language(s) you know.
- Some (perhaps rusty) mathematical experience.
  - Basic discrete math, proofs by induction, etc.

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- *Excellent free reference: "*Mathematics for Computer Science", by Eric Lehman and Tom Leighton. (Easy to find on the Web.)

Supporting Materials

- All (annotated) slides available from course site.

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  - Kleinberg/Tardos, *Algorithm Design*, 2005.
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  - Cormen/Leiserson/Rivest/Stein, *Introduction to Algorithms*, 2009 (3rd edition).
  - Mehlhorn/Sanders, *Data Structures and Algorithms: The Basic Toolbox*, 2008.

Biggest influence on instructor

Freely available online

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  - Mehlhorn/Sanders, *Data Structures and Algorithms: The Basic Toolbox*, 2008.
- No specific development environment required.
  - But you should be able to write and execute programs.

Assessment

- No grades per se. (Details on a certificate of accomplishment TBA.)
- Weekly homeworks.
  - Test understand of material
  - Synchronize students, greatly helps discussion forum
  - Intellectual challenge

Assessment

- No grades per se. (Details on a certificate of accomplishment TBA.)
- Weekly homeworks.
- Assessment tools currently just a “1.0” technology.
  - We’ll do our best!
- Will sometimes propose harder “challenge problems”
  - Will not be graded; discuss solutions via course forum

Design and Analysis
of Algorithms I

Introduction

Guiding Principles

Guiding Principle #1

"worst – case analysis" : our running time bound holds for every input of length n.
-Particularly appropriate for "general-purpose" routines

As Opposed to
--"average-case" analysis
--benchmarks

REQUIRES DOMAIN KNOWLEDGE

BONUS : worst case usually easier to analyze.

Guiding Principle #2

Won't pay much attention to constant factors, lower-order terms

<u>Justifications</u>

1. Way easier
2. Constants depend on architecture / compiler / programmer anyways
3. Lose very little predictive power (as we'll see)

Guiding Principle #3

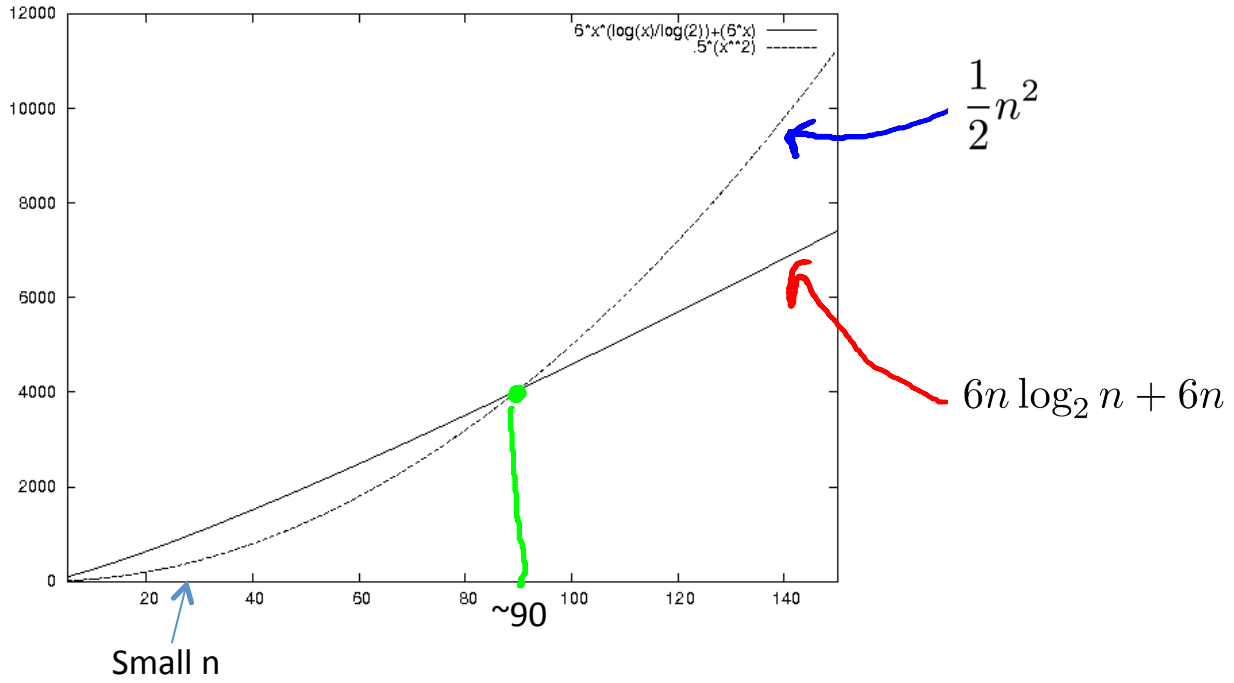
Asymptotic Analysis : focus on running time for large input sizes n

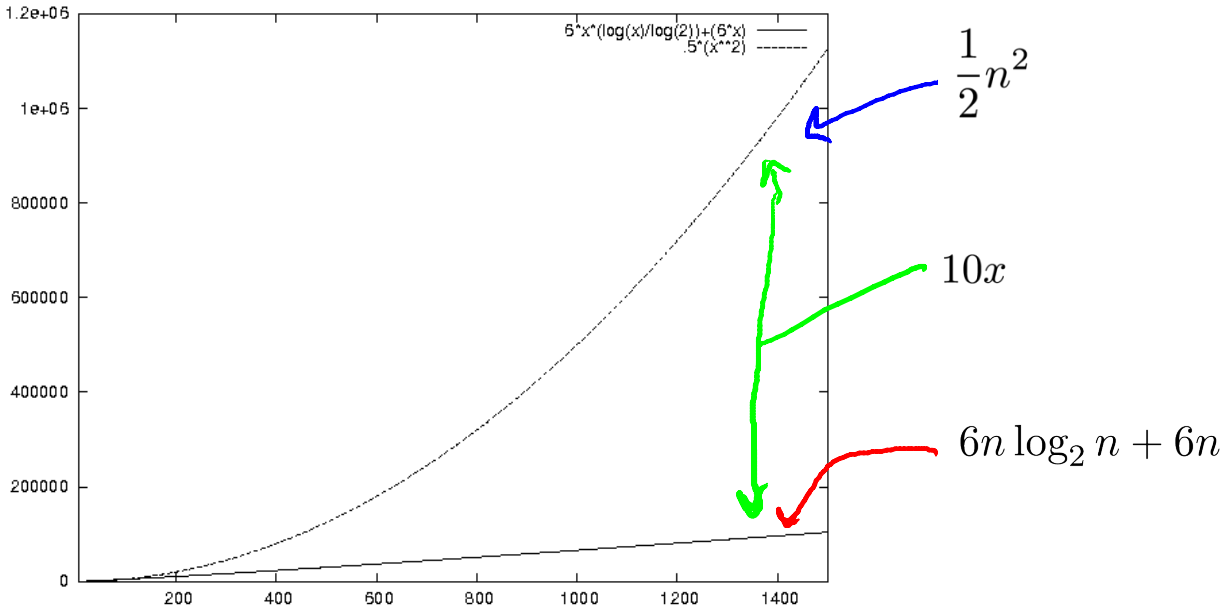
Eg : $6n \log_2 + 6n$ "better than" $\frac{1}{2}n^2$

MERGE SORT

INSERTION SORT

Justification: Only big problems are interesting!


6*x*(log(x)/log(2))+(6*x)
.5*(x**2)
12000
10000
8000
6000
4000
2000
0
20
40
60
80
100
120
140
~90
Small n
$\frac{1}{2}n^2$
$6n \log_2 n + 6n$



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What Is a "Fast" Algorithm?

This Course : adopt these three biases as guiding principles

fast algorithm $\approx$ worst-case running time grows slowly with input size

Usually : want as close to linear (O(n)) as possible

Design and Analysis
of Algorithms I

Introduction

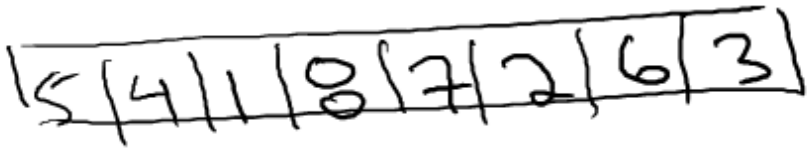
Merge Sort (Overview)

Why Study Merge Sort?

- Good introduction to divide & conquer
  - Improves over Selection, Insertion, Bubble sorts
- Calibrate your preparation
- Motivates guiding principles for algorithm analysis (worst-case and asymptotic analysis)
- Analysis generalizes to "Master Method"

The Sorting Problem

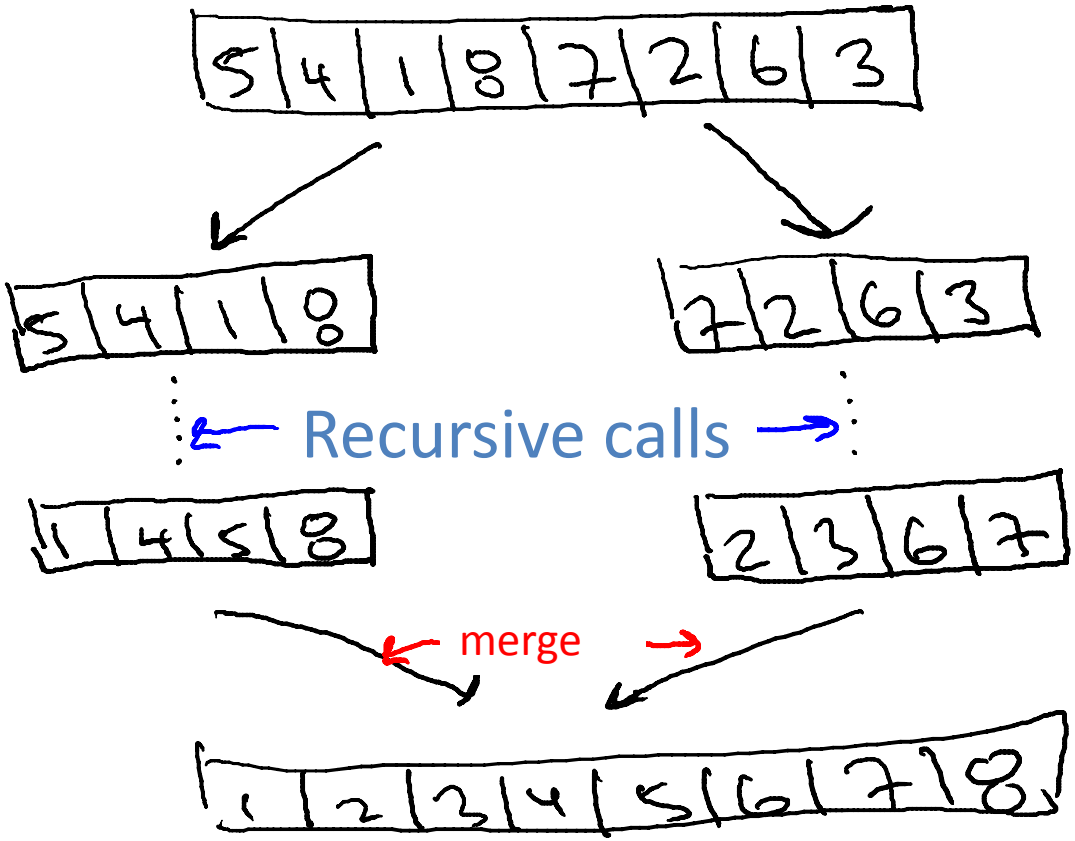
Input : array of n numbers, unsorted.



Assume Distinct

Output : Same numbers, sorted in increasing order

Merge Sort: Example



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Design and Analysis of Algorithms I

Introduction

Merge Sort (Pseudocode)

Tim

Merge Sort: Pseudocode

-- recursively sort 1st half of the input array

-- recursively sort 2nd half of the input array

-- merge two sorted sublists into one

[ignores base cases]

Pseudocode for Merge:

```
C = output [length = n]
A = 1st sorted array [n/2]
B = 2nd sorted array [n/2]
i = 1
j = 1
```

```
for k = 1 to n
        if A(i) < B(j)
                C(k) = A(i)
                i++
        else [B(j) < A(i)]
                C(k) = B(j)
                j++
end
```

(ignores end cases)

Merge Sort Running Time?

Key Question : running time of Merge Sort on array of n numbers ?

[running time ~ # of lines of code executed]

Pseudocode for Merge:

C = output [length = n]

A = 1st sorted array [n/2]

B = 2nd sorted array [n/2]

i = 1

j = 1

} 2 operations

```
for k = 1 to n
        if A(i) < B(j)
                C(k) = A(i)
                i++
        else [B(j) < A(i)]
                C(k) = B(j)
                j++
end
```

(ignores end cases)

Running Time of Merge

Upshot : running time of Merge on array of m numbers is $\leq 4m + 2$

$\leq 6m$ (Since $m \geq 1$)

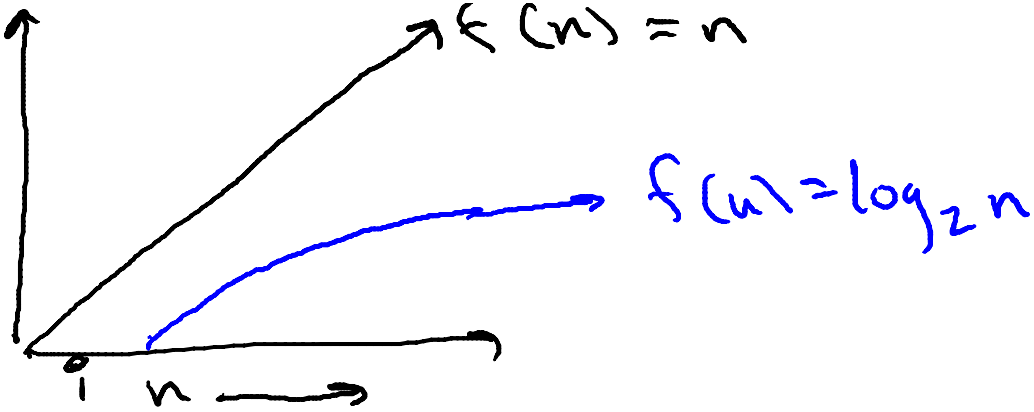
Running Time of Merge Sort

Claim : Merge Sort requires

$\leq 6n \log_2 n + 6n$ operations

to sort n numbers.

Recall : $= \log_2 n$ is the # of times you divide by 2 until you get down to 1



Design and Analysis of Algorithms I

Introduction

Merge Sort (Analysis)

Tim

Running Time of Merge Sort

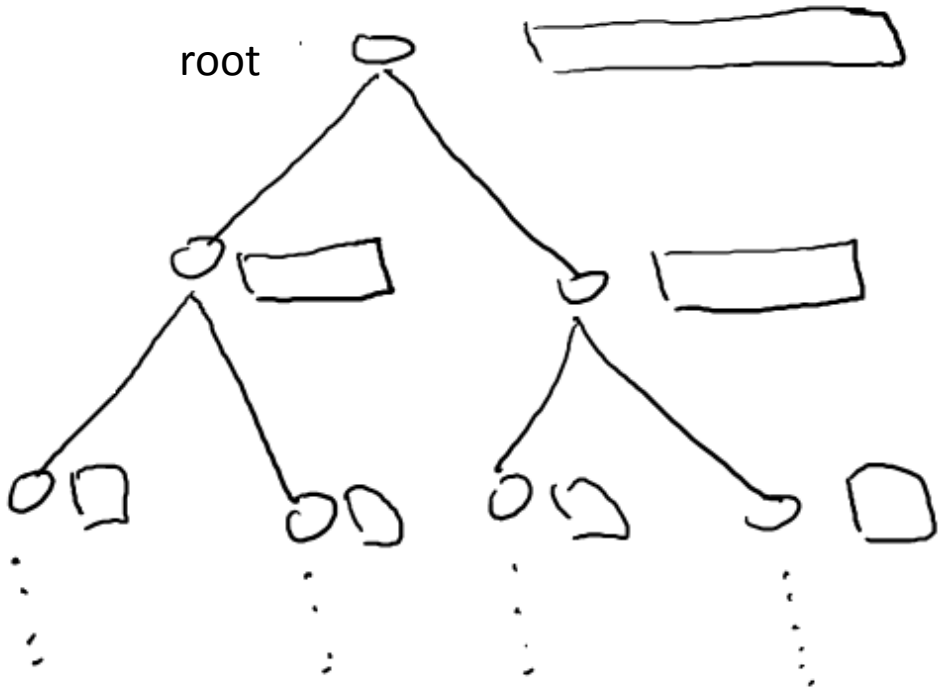
Claim: For every input array of n numbers, Merge Sort produces a sorted output array and uses at most $6n \log_2 n + 6n$ operations.

Proof of claim (assuming n = power of 2):

Level 0
[outer call to Merge Sort]

Level 1
(1st recursive calls)

Level 2



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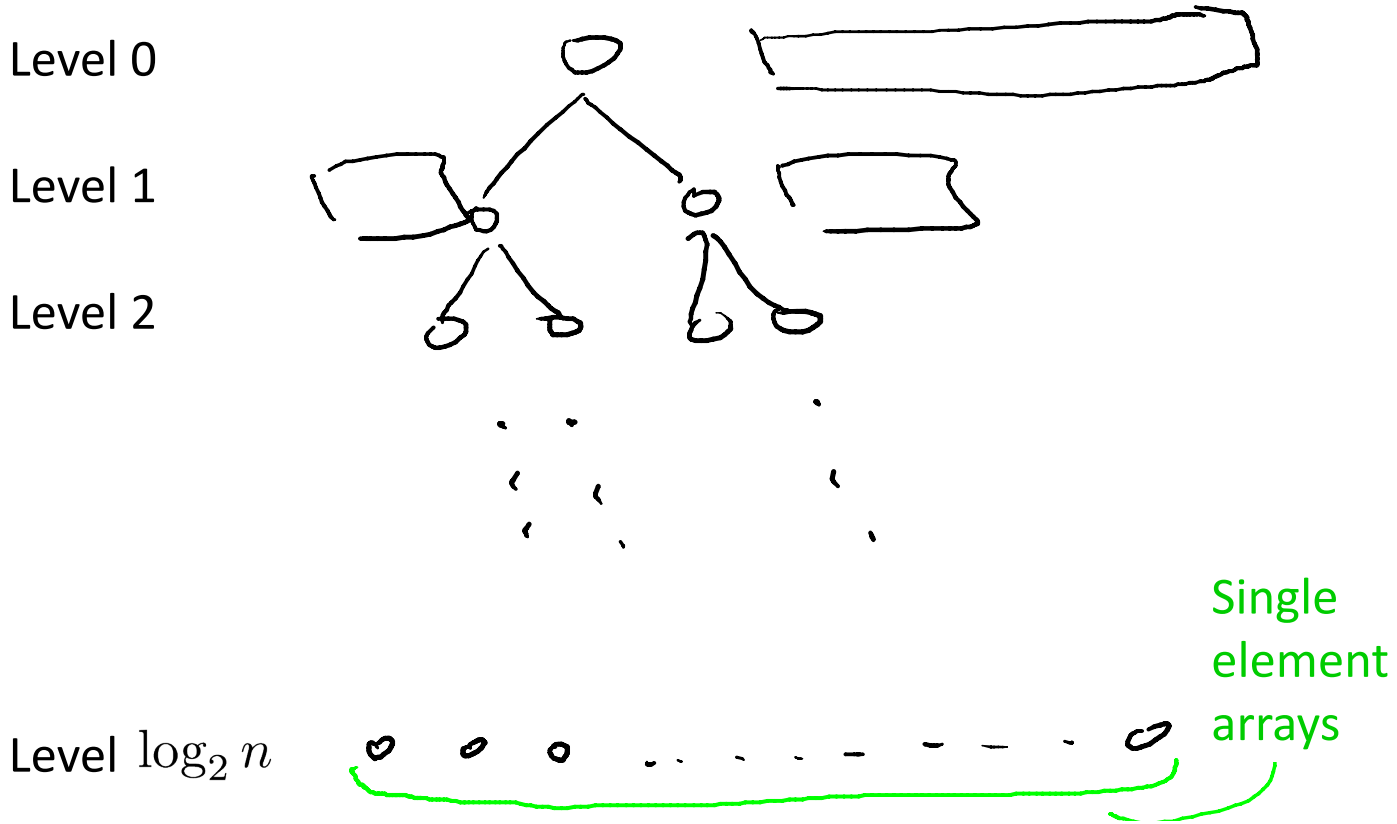
Roughly how many levels does this recursion tree have (as a function of n, the length of the input array)?

- A constant number (independent of n).
- $\log_2 n$ $(\log_2 n + 1)$ to be exact!
- $\sqrt{n}$
- n

Proof of claim (assuming n = power of 2):



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What is the pattern ? Fill in the blanks in the following statement: at each level j = 0,1,2,.., $\log_2 n$, there are <blank> subproblems, each of size <blank>.

- 2^j *and* 2^j, *respectively*
- $n/2^j$ *and* $n/2^j$, *respectively*
- 2^j *and* $n/2^j$, *respectively*
- $n/2^j$ *and* 2^j, *respectively*

Proof of claim (assuming n = power of 2) :

At each level j=0,1,2,.., $\log_2 n$,

Total # of operations at level j = 0,1,2,…, $\log_2 n$

$$\leq 2^j * 6(\frac{n}{2^j}) = 6n$$

# of level-j subproblems

Size of level-j subproblem

Work per level – j subproblem

Total

$$6n(\log_2 n + 1)$$

Work per level

# of levels

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Running Time of Merge Sort

Claim: For every input array of n numbers, Merge Sort produces a sorted output array and uses at most $6n \log_2 n + 6n$ operations.

QED!