

**Fundamentals of theory of computation I**  
**Sample for Test 1 – Sample Solutions**

**Exercise 1:** Let  $L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$ ,  $L_2 = \{a^n b^k \mid n \geq 0, k \equiv 1 \pmod{3}\}$ , Determine the following languages.

$$L_1 L_2 = \{a^n b^n a^m b^k \mid n, m \in \mathbb{N}, k \equiv 1 \pmod{3}\}$$

$$L_1 \setminus L_2 = \{a^n b^n \mid n \equiv 0 \text{ or } 2 \pmod{3}\}$$

$$\text{Pre}(L_2) = \{a^n b^k \mid n, k \in \mathbb{N}\}$$

**Exercise 2:** Let  $N_t(u)$  denote the number of  $t$ 's occurring in the word  $u$ . Make a type 3 grammar generating the language  $L = \{u \in \{a, b, c\}^* \mid bb \not\subseteq u, N_a(u) \leq 1\}$ . (I.e., there are at most 1  $a$  and no consecutive  $b$ 's in the words of  $L$ .)

Idea: is the last letter of the already generated word equal to  $b$  or not? How many  $a$ 's were already generated? We can preserve both pieces of information throughout the derivation.

$X_0$ : the last letter is not a  $b$ , 0  $a$ 's were generated so far.

$X_1$ : the last letter is not a  $b$ , exactly 1  $a$ 's were generated so far.

$B_0$ : the last letter is a  $b$ , 0  $a$ 's were generated so far.

$B_1$ : the last letter is a  $b$ , exactly 1  $a$ 's were generated so far.

$$X_0 \rightarrow aX_1 \mid bB_0 \mid cX_0 \mid \varepsilon$$

$$X_1 \rightarrow bB_1 \mid cX_1 \mid \varepsilon$$

$$B_0 \rightarrow aX_1 \mid cX_0 \mid \varepsilon$$

$$B_1 \rightarrow cX_1 \mid \varepsilon$$

$X_0$  is the start symbol.

**Exercise 3:** Determine  $L(G)$  for the grammar  $G = \langle \{S, X, Y\}, \{a, b\}, P, S \rangle$ , where  $P$  is as follows.

$$S \rightarrow aXa \mid bSb \mid a \mid b \mid \varepsilon$$

$$X \rightarrow aYa \mid bSb \mid b \mid \varepsilon$$

$$Y \rightarrow bSb \mid b$$

List all the Chomsky grammar classes having  $G$  as a member.

Observations:

- $G$  is linear, so there will be at most 1 nonterminal in the sentential forms.
- Terminal symbols are generated in a symmetric way to the nonterminal, so all generated words are palindromes.
- If a pair of  $b$ 's is generated, the nonterminal becomes an  $S$ .
- If a pair of  $a$ 's is generated,  $S$  becomes  $X$ ,  $X$  becomes  $Y$ . It is not allowed to generate a pair of  $a$ 's, if the nonterminal is a  $Y$ .
- So having  $S$  in the sentential form means, that there's no  $a$ 's right before (and right after)  $S$  in the sentential form.  $X$  means, that there's exactly one  $a$ 's right before (and right after)  $X$  in the sentential form.  $Y$  means, that there's exactly two  $a$ 's right before (and right after)  $Y$  in the sentential form.

Checking the terminating rules as well, one can conclude:

$L(G) = \{u \in \{a, b\}^* \mid u = u^{-1} \wedge aaa \not\subseteq u\}$ , i.e.,  $L(G)$  is the language of palindromes over  $\{a, b\}$  without 3 consecutive  $a$ 's.

$G$  is a type 2 and type 0 grammar.

**Exercise 4:** Give a regular expression describing the following languages.

$L_1 = \{u \in \{a, b\}^* \mid u \text{ contains an even number of } a\text{'s, and if } u \text{ contains a } b, \text{ then the number of } a\text{'s after the last } b \text{ is odd}\}.$

$L_2 = \{u \in \{a, b, c\}^* \mid u \text{ does not contain the subword } ab\}.$

$$R_1 = (b^*ab^*a)^*b^*ab^*ba(aa)^* + (aa)^* \quad L(R_1) = L_1.$$

$$R_2 = (b + a^*c)^*a^* \quad L(R_2) = L_2.$$

**Exercise 5:** Construct a type 3 grammar in normal form, according to the algorithm we learnt, generating the same language as the following type 3 grammar  $G$ .  $G = \langle \{S, A, B, C\}, \{a, b\}, P, S \rangle$ , where the set of production rules  $P$  consists of the following rules.

$$S \rightarrow aabA \mid bB$$

$$A \rightarrow C \mid aS$$

$$B \rightarrow bC \mid \varepsilon$$

$$C \rightarrow B \mid a$$

$$S \rightarrow aD \mid bB \quad H_0(S) = H_1(S) = \{S\} = H(S)$$

$$A \rightarrow C \mid aS \quad H_0(A) = \{A\}, H_1(A) = \{A, C\}, H_2(B) = H_3(B) = \{A, B, C\} = H(B)$$

$$B \rightarrow bC \mid \varepsilon \quad H_0(B) = H_1(B) = \{B\} = H(B)$$

$$C \rightarrow B \mid aF \quad H_0(C) = \{C\}, H_1(C) = H_2(C) = \{C, B\} = H(C)$$

$$D \rightarrow aE \quad H_0(D) = H_1(D) = H(D) = \{D\}$$

$$E \rightarrow bA \quad H_0(E) = H_1(E) = H(E) = \{E\}$$

$$F \rightarrow \varepsilon \quad H_0(F) = H_1(F) = H(F) = \{F\}$$

$$S \rightarrow aD \mid bB$$

$$A \rightarrow bC \mid \varepsilon \mid aF \mid aS$$

$$B \rightarrow bC \mid \varepsilon$$

$$C \rightarrow bC \mid \varepsilon \mid aF$$

$$D \rightarrow aE$$

$$E \rightarrow bA$$

$$F \rightarrow \varepsilon$$