# Assignment 4

1. Show that the following problems are in NP:
   (a) Does a graph $G$ have a simple path (i.e., with no vertex repeated) of length $k$?
   (b) Is an integer $n$ composite?
   (c) Does a graph $G$ have a vertex cover of size $k$?
   (d) Does an ILP have a solution at most $k$?

2. In the Hitting-Set problem, we are given a family of sets $\{S_1, S_2 \dots S_n\}$ and a budget $b$, and we wish to find a set $H$ of size $\leq b$ which intersects every $S_i$ , if such an $H$ exists. In other words, we want $H \cap S_i \neq \emptyset$ for all $i \in [n]$. Show that Hitting-Set is NP-complete.

3. Show that for any problem $\Pi$ in NP, there is an algorithm which solves $\Pi$ in time $O(2^{p(n)})$, where $n$ is the size of the input instance and $p(n)$ is a polynomial (which may depend on $\Pi$).

4. A vertex cover of an undirected graph $G = (V, E)$ is a subset $V_0 \subset V$ such that if $(u, v) \in E$ then either $u \in V_0$ or $v \in V_0$ or both (i.e. there is a vertex in $V_0$ "covering" every edge in $E$). The optimization version of the Vertex-Cover-Problem is to find a vertex cover of minimum size in $G$. It is known that Vertex-Cover-Problem is NP hard. However, while finding the problem exactly is hard; perhaps it is easy to find a good approximate solution. Consider the algorithm below:

   **Approximate Vertex Cover for G = (V, E)**

   1. $C = \emptyset$
   2. $E_0 = E$
   3. While $E_0 \neq \emptyset$:
      i. Let $(u, v) \in E_0$ be an arbitrary edge
      ii. $C = C \cup \{(u, v)\}$
      iii. Remove from $E_0$ every edge incident on either $u$ or $v$.

   Answer the following questions:

   (a) Show that the above algorithm outputs a vertex cover of $G$.

(b) Show that the algorithm runs in polynomial time.

(c) Show that the solution returned by the above algorithm is no more than twice the size of the optimal cover.

5. Consider the LONGEST PATH problem. Given a graph $G$ and an integer $g$, the goal is to find in $G$ a simple path of length $g$. Prove that LONGEST PATH is NP-complete.

6. A vertex cover of an undirected graph $G = (V, E)$ is a subset of the vertices which touches every edge—that is, a subset $S \subset V$ such that for each edge $\{u, v\} \in E$, one or both of $u, v$ are in $S$. Write down the ILP of this problem.