Quiz 1: Design and Analysis of Algorithms
23rd September 2021

Maximum Marks: 100, Total time: 2 hours

1. Solve the following recurrences. Note the the functions below are all defined on the domain $\mathbb{N}$:

   (a) $T(n) = T(n-1) + n, T(1) = 1$.

   (b) $T(n) = T(\frac{n}{2}) + O(n), T(1) = c$, where $c > 0$ is some constant.

   (c) $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$, $T(1) = c$, where $c > 0$ is some constant.

2. The *Lucas Sequence* $1, 3, 4, 7, 11, 18, 29, \ldots$ is defined by $a_1 = 1, a_2 = 3, a_n = a_{n-1} + a_{n-2}$. Give a closed form expression for a_n. Also prove that $a_n = O(1.75^n)$.

3. Prove that if G is a graph such that every vertex has even degree, then the graph G can be written as union of edge$-$disjoint cycles.

4. Is it true that the number of people currently living on our planet and having an odd number of siblings is even?

5. Suppose $G = (V, E)$ is a simple connected graph. Let S be the set of all spanning trees of G. We say that spanning trees T and $T^{'}$ are neighbours of each other if and only if T contains exactly one edge, which is not in $T^{'}$ and $T^{'}$ contains exactly one edge, which is not in T.
   Now imagine constructing a simple graph $G^{'} = (V^{'}, E^{'})$. $G^{'}$ has a vertex for each spanning tree in S. Further, any two vertices in $G^{'}$ are adjacent if the corresponding spanning trees are neighbours of each other. Is $G^{'}$ connected?

6. Let G be a bipartite graph on $2n$ vertices and degree of every vertex is 10. Prove that G has a perfect matching.

7. Given a simple connected graph $G = (V, E)$, an **independent set** is defined as a subset $S \subseteq V$, such that no two vertices $u, v \in S$, are adjacent to each other.
   Additionally, we also say that G is $t-$ **colourable** if its vertices can be coloured using at most t colours, such that any two adjacent vertices do not receive the same colour. Show that if the degree of each vertex in G is at most d, then G contains an independent set of size at least $\frac{n}{d+1}$, where $n = |V|$.