Quiz1

Rachit Parikh (CRS-2101) – rachitparikh2016@gmail.com

Design and Analysis of Algorithms

Disclaimer : I declare that all the work presented in this assignment is my own work and I have only consulted the internet when it was absolutely necessary. I have only used Vizing's theorem result directly, since we were taught that in our Discrete Mathematics course last semester.

Q-1

(a)

$$\begin{aligned} T(n) &= T(n-1) + n \\ &= T(n-2) + n - 1 + n \\ &= T(1) + 2 + 3 + \cdots + n \end{aligned}$$

$$\therefore T(n) = \frac{n(n-1)}{2} = O(n^2)$$

(b)

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + O(n) \\ &= T\left(\frac{n}{4}\right) + O\left(\frac{n}{2}\right) + O(n) \\ &= c + O(n) + O\left(\frac{n}{2}\right) + \cdots + O(1) - - \log_2 n \text{ terms} \end{aligned}$$

By definition of $O(n)$, if $h(n) \leq M.g(n) \implies h(n) = O(g(n))$
In our case $g(n) = n$ and $O(n) = h(n) \implies h(n) \leq M.n$

$$
\begin{aligned}
\therefore T(n) &= c + h(n) + h\left(\frac{n}{2}\right) + \cdots + h(1) \\
&\leq c + Mn + M\frac{n}{2} + \cdots + M \\
&\leq q + qn + q\frac{n}{2} + \cdots + q \\
&\leq q\left[1 + \left(\frac{\left(\frac{1}{2}\right)^{\log_2(n)-1} - 1}{\left(\frac{1}{2}\right) - 1}\right)\right] \\
&\leq q(3n-4) \\
&\leq dn, \text{ for some } d \in \mathbb{R}
\end{aligned}
$$

$$\implies T(n) = O(n)$$

(c)

We know that,

$$\left\lfloor \frac{x}{2} \right\rfloor = \begin{cases} \frac{x}{2} & \text{if } 2 \mid x \\ \frac{x-1}{2} & \text{otherwise} \end{cases}$$

Similarly,

$$\left\lfloor \frac{\left\lfloor \frac{x}{2} \right\rfloor}{2} \right\rfloor = \begin{cases} \frac{x}{4} & \text{if } 4 \mid x \\ \frac{x-1}{4} & \text{if } 2 | \frac{x-1}{2}, 2 \nmid \left\lfloor \frac{x}{2} \right\rfloor \\ \frac{x-2}{4} & \text{if } 2 | \left\lfloor \frac{x}{2} \right\rfloor, 2 \nmid \frac{x}{2} \\ \frac{x-3}{4} & \text{if } 2 \nmid \left\lfloor \frac{x}{2} \right\rfloor, 2 \nmid \frac{x-1}{2} \end{cases}$$

Let,

$$b_{n-1} = \left\lfloor \frac{b_n}{2} \right\rfloor$$

with,

$$b_n = n$$

Then we know that

$$b_{n-k} \leq \frac{n}{2^k}$$

Coming back to our recursion,

$$
\begin{aligned}
T(n) &= 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \\
&= 2T(b_{n-1}) + n \\
&= 2^2 T(b_{n-2}) + 2b_{n-1} + b_n \\
&= 2^k T(b_{n-k}) + 2^{k-1} b_{n-k+1} + \cdots + b_n
\end{aligned}
$$

At, $k = \log_2(n)$,

$$
\begin{aligned}
T(n) &= 2^{\log_2(n)}.T\left(b_{n-\log_2(n)}\right) + \sum_{i=0}^{k} 2^i b_{n-i} \\
&\leq nT\left(b_{n-\log_2(n)}\right) + \sum_{i=0}^{k} 2^i \frac{n}{2^i} \\
&\leq nT(1) + n\log_2 n \\
&\leq Mn\log_2 n
\end{aligned}
$$

Thus,

$$T(n) = O(n\log_2 n)$$

Q-2

Let, $a_n = x$, then for the equation $a_n = a_{n-1} + a_{n-2}$, we have

$x^2 = x + 1$

$x^2 - x - 1 = 0$

$\implies x = \left(\frac{1 \pm \sqrt{5}}{2}\right)$

Let,

$p = \frac{1+\sqrt{5}}{2}$ $\qquad$ $q = \frac{1-\sqrt{5}}{2}$

$\therefore q = 1 - p$

Now, our solution for a_n would be linear combination of both the solutions, Hence, $a_n = sp^n + tq^n$
By, the initial conditions, $a_1 = 1, a_2 = 3$,

$$1 = sp + qt$$

$$1 = s\left(\frac{1+\sqrt{5}}{2}\right) + t\left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = \frac{s+t}{2} + \sqrt{5}\left(\frac{s-t}{2}\right)$$

$$3 = sp^2 + tq^2$$

$$3 = s\left(\frac{1+\sqrt{5}}{2}\right)^2 + t\left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$3 = \frac{3}{2}(s+t) + \frac{\sqrt{5}}{2}(s-t)$$

By solving the above system of equations we get,

$$s = t = 1$$

Hence, the closed form for a_n is

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Now, we will try to find the $O(n)$ notation for a_n

$a_n = p^n + q^n$

$p = \dfrac{1+\sqrt{5}}{2} = 1.62 \implies q = 0.62$

$\therefore a_n = 1.62^n + (-0.62)^n$

For large n, we have $q^n \approx 0$, hence we can assume $a_n \approx 1.62^n$

$\exists N, \forall n \geq N,$

$a_n \approx 1.62^n \leq 1.75^n$

$\implies a_n = O(1.75^n)$

Q-3

Here, we will try to remove cycles from a graph G and prove that for a graph with even degree we can write the graph G as a union of disjoint edge cycles.

If there is no cycle in $G = (V, E)$, then there must be some vertex with degree 1, hence it is odd. Take a path $P = (ue_1v_1....e_kv_k)$, then the vertex u has degree 1, unless some $v_{k'}, k' > 2$ connects with u. If there exists a cycle C_1 in this graph then if we remove it from the graph, the degrees of all the vertices in C_1 would not change their parity. Let d be the degree of a vertex u in G, and d' be the degree of u in $G - C_1$, then $d' = d - 2k$ for some k. In other words, $2|d - d'$, thus we have parity of degree of vertices as an invariant property. If we keep continuing this process, ultimately we would have 0 edges in case of a graph where all the vertices have an even degree. This would lead to a graph without any edges which will make our statement vacuously true since there are no cycles here. Thus,

$$G = C_1 \cup C_2 \cup \cdots \cup C_t$$

for some t.

Q-4

First, let us define some properties of siblings. If A and B are siblings, B and C are siblings, then by definition A and C are also siblings. Hence it is a transitive property.

Construct a graph G such that people are vertices and there is an edge connecting them if and only if they are siblings. So if there are edges (u, v) and (v, w), then there is also an edge (u, w). Hence, if we construct a graph G, then all of its components are complete, since for any component every vertex will be connected to each other. So if there are n siblings in a component then they form K_n. Hence, in K_n for every vertex u we have $deg(u) = n - 1$. Then if $n - 1$ is odd then n is even. If u has odd number of siblings ($2 \nmid deg(u)$), then $2|n$. Thus, number of people with odd number of siblings is even.

Q-5

Claim : Any spanning tree T of graph G can be transformed into another spanning tree T' of graph G with finite number of operations.

Consider the case when there is 1 ucommon edge in both $T(V, E)$ and $T'(V, E')$. Let $e \notin E', e' \notin E$, if we want to transform T to T', we can do so by adding the edge $e' \in E'$ to E. Now, we would have a cycle as a spanning tree is maximally acyclic. So T will now become $T(V, E \cup \{e'\})$. Now if we remove e from E, we will again get a spanning tree which is none other than T'. This is easy to show as $E' = (E \cup \{e'\}) - \{e\}$. Hence, we can transform one tree to another by doing this operation of adding and removing edge $e \in (E - E') \cup (E' - E)$. Here we can see that in G' there is an edge between T and T', if we can use this operation.

Now, assume that there are k edges in $T(V, E)$, which are not present in $T'(V, E')$. Let, $D = E - E'$ and $D' = E' - E$. If we remove one edge from D it will make our tree disconnected and we would have two components, since a spanning tree is maximally acyclical. There has to be at least one edge in D' such that on adding it to the currently disconnected graph, it won't create any cycle. To show this, consider W and U be the vertices set of these components. If there does not exist any edge which can connect the components between U and W in D', then T' would be disconnected, hence a contradiction. Thus, there exists at least one edge in D' such that on adding it to the disconnected graph we can connect it and transform it into another tree T_1. Now we have $k - 1$ edges in $T_1(V, E_1)$, which are not present in $T'(V, E')$. By induction we can assume the claim to be true for $k - 1$ edges. Hence, we can see that any spanning tree T can be transformed into another tree T' with finite number of operations.

Since, each operation replace exactly one edge, the trees formed after immediate transformation are neighbors. Thus if we have k uncommon edges between T and T', then the transformation $T \to T_1 \cdots \to T'$, then $(T, T_1), (T_1, T_2), \ldots (T_q, T')$ are neighbors. Which means $(T, T_1), (T_1, T_2), \ldots (T_q, T')$ are edges in $G'(V', E')$. Since any tree can be transformed to another, there exists a path from one tree to another which means G' is connected.

Q-6

By handshake lemma, $\sum_{u \in V} deg(u) = 2|E| \implies 10(2n) = 2|E| \implies |E| = 10n$, thus number of edges $= 10n$

Let L and R be two set of vertices which form the bipartite graph G, such that $L \cup R = V$ and $L \cap R = \phi$ then then number of edges $= 10|L| = 10n \implies |L| = n \implies |R| = n$

Since we have $|L| = |R|$, we just need to prove that $|N(W)| \geq |W|$ where $W \subseteq L$, because it follows from Hall's marriage theorem that G would have a perfect matching if both of the above conditions hold. If $|W| < 10$, then it is already true, since each vertex has degree 10, $\implies |N(W)| > 10$. For $|W| \geq 10$, assume that $|N(W)| < 10$ then by pigeon hole principle, we would have at least one vertex in $N(W), v$ such that $|N(v)| > 10$, $\implies deg(v) > 10$, hence a contradiction. Thus, we have $|N(W)| \geq |W|$. This completes the proof.

Q-7

By Vizing's theorem, $\chi(G) = \Delta + 1$ or $\chi(G) = \Delta$, where $\Delta =$ maximum degree in the graph G, and $\chi(G) =$ minimum t for which the graph is $t-$colorable. Let, the independent sets for our graph G be $V_1, V_2, \ldots, V_k$ then we have k independent sets. Also, $|V_1| + \cdots + |V_k| = n$. Now, we know that no two vertices in an independent set are connected, which means that they can be colored using the same color. Let us color V_i with the color c_i then we have total of k colors used to color the graph. But by Vizing's theorem, we know that $k = \Delta + 1$ or $k = \Delta$. Thus, we have,

$$|V_1| + \cdots + |V_{\Delta+1}| = n$$

or, Hence by pigeonhole principle we have, at least one V_i such that $|V_i| \geq \frac{n}{\Delta}$

$$|V_1| + \cdots + |V_{\Delta}| = n$$

Here, we have at least one V_i such that $|V_i| \geq \left(\frac{n}{\Delta+1}\right)$
Also we know that,

$$\frac{n}{\Delta} > \frac{n}{\Delta+1},$$
$$d \geq \Delta$$

Hence, there exists and independent set of size at least $\frac{n}{d+1}$