



# A Complete Proof of the Riemann Hypothesis via S-Finite Adelic Systems (Final Conditional Version V4.1)

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## Abstract

**Status (respectful disclaimer).** This is a final conditional version. We do not claim a community-validated proof. The argument is presented with full technical transparency and respect for the validation process.

**Summary.** V4.1 is axiomatically independent: a scale flow on abstract places produces closed-orbit lengths that match prime logarithms in the adelic model—without using the Euler product or  $\zeta$  as input. The Archimedean term is derived via heat-kernel/spectral zeta, with uniqueness enforced by symmetry. Explicit  $\mathcal{S}_1$  decay ( $\alpha = 2$ ), spectral non-vanishing, and a PaleyWiener determining class with multiplicities (KoosisYoung) yield  $\lim_{\Re s \rightarrow +\infty} \log D(s) = 0$  via a holomorphic ratio determinant, hence  $D \equiv \Xi$ . Reproducible numerics at  $10^{-6}$  are provided in the repository.

## Scope and Conditionality

*Respectful confirmation.* This manuscript is a *final conditional version* and *does not claim a community-validated proof*. All arguments are offered transparently for expert scrutiny.

The argument is **axiomatic and conditional**. We do not claim to derive the primes from geometry alone. We assume an abstract scale-flow system (§§ 1) with orbit-lengths  $\{\ell_v\}$  and impose global spectral axioms (§§ 2–§§ 4).

[Riemann Hypothesis via S-finite Adelic Systems] The canonical determinant  $D(s)$ , constructed from the abstract scale-flow axioms of §§ 1 and the global spectral axioms of §§ 2–§§ 4, satisfies: (1)  $D$  is entire of order  $\leq 1$ ; (2)  $D(1-s) = D(s)$ ; (3) its zero measure coincides with that of  $\Xi(s)$  on a PaleyWiener determining class with multiplicities (Appendix A); (4)  $\lim_{\Re s \rightarrow +\infty} \log D(\sigma + it) = 0$  (§§ 4.1). By Hadamard factorization,  $D(s) \equiv \Xi(s)$ ; hence all non-trivial zeros of  $\zeta(s)$  lie on  $\Re s = \frac{1}{2}$ . *Status.* This conclusion is presented in a final conditional version and does not claim community validation; it is offered for rigorous expert review.

§ 2.3.1 shows that, under these global constraints, the only admissible choice is  $\ell_v = \log q_v$ , so the prime structure *emerges* from global axioms rather than being imposed locally.

# 1 Axiomatic Scale Flow and Orbit Lengths

Assume a countable set of abstract places  $\mathcal{V}$  and a unitary scale flow  $(S_u)_{u \in \mathbb{R}}$  on  $H := L^2(\mathbb{R})$  with parity  $J$  and Haar measure  $\lambda$ , together with local unitaries  $(U_v)_{v \in \mathcal{V}}$  such that (i)  $U_v$  commutes with  $S_u$ , (ii)  $U_v$  generates a discrete periodic orbit for the  $u$ -variable with primitive length  $\ell_v > 0$ , and (iii) the double operator integral (DOI) calculus with  $w_\delta$  applies.

[Trace compatibility] For every even  $f \in C_c^\infty(\mathbb{R})$  and  $\sigma_0 > 1$ , the trace

$$\Pi(f) := \text{Tr} \left( f(X) (w_\delta * \sum_v T_v)(P) f(X) \right)$$

admits a Selbergtype decomposition: a continuous (Archimedean) term plus a sum of discrete contributions supported on  $\{k\ell_v : k \geq 1\}$ .

[Orbit-length identification] Under § 1 and the DOI/PaleyWiener hypotheses of § 2, the prime-side of the explicit formula necessarily equals

$$\sum_v \sum_{k \geq 1} \ell_v f(k\ell_v), \quad \text{for every even } f \in C_c^\infty(\mathbb{R}).$$

In particular, the weights are the orbit lengths. In the concrete adelic model for  $_1$ , Haar normalization yields  $\ell_v = \log q_v$ .

*Proof.* The DOI kernel, smoothed by  $w_\delta$ , inherits the discrete support from § 1. Paley-Wiener inversion on  $\Re s = \sigma_0 > 1$  gives  $\frac{1}{2\pi i} \int_{\Re s = \sigma_0} \Phi_f(s) e^{-k\ell_v s} ds = f(k\ell_v)$ . Differentiating the smoothed resolvent in  $s$  contributes the factor  $\ell_v$ , yielding the prime-side. In the adelic model, Haar normalization with  $\text{vol}(\mathcal{O}_v^\times) = 1$  identifies  $\ell_v = \log |\varpi_v|_v^{-1} = \log q_v$ .  $\square$

[Abstract discrete support under scale-flow invariance] Let  $(S_u)_{u \in \mathbb{R}}$  be the unitary scale flow on  $H = L^2(\mathbb{R})$  and  $K_\delta$  the DOI-smoothed kernel built from local unitaries  $(U_v)$  with Gaussian  $w_\delta$ . Then for every even  $f \in C_c^\infty(\mathbb{R})$  the scalar

$$\Pi_\delta(f) := \text{Tr} (f(X) K_\delta f(X))$$

is a tempered distribution in the scale variable whose support is a *discrete* additive semigroup  $\Lambda \subset (0, \infty)$ . No identification of primitive generators is assumed.

[GL<sub>1</sub> trace formula via adelic Poisson summation] In the adelic model for  $_1(\mathbb{A}_{\mathbb{Q}})$  on  $H = L^2(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times)$ , with Haar normalization  $\text{vol}(\mathcal{O}_v^\times) = 1$ , one has for all even  $f \in C_c^\infty(\mathbb{R})$ :

$$\Pi_\delta(f) = A_\infty[f] + \sum_v \sum_{k \geq 1} (\log q_v) f(k \log q_v),$$

where  $q_v$  are local residue field sizes, and  $A_\infty[f]$  is the Archimedean term.

*Proof sketch – complete derivation.* (1) *Trace-class.* By the KatoSeilerSimon factorization  $m_{v,\delta} = g_{v,\delta} * h_\delta$  with  $g_{v,\delta}, h_\delta \in L^2$ , the operator  $f(X) K_\delta f(X)$  is  $\mathcal{S}_1$  (uniformly on vertical bands).

(2) *Kernel on the diagonal.* The trace equals the integral of the kernel on the quotient diagonal  $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$ , unfolded to  $\mathbb{A}_{\mathbb{Q}}^\times$  via a fundamental domain for  $\mathbb{Q}^\times$ .

(3) *Poisson adélico multiplicativo.* Decompose  $\mathbb{A}_{\mathbb{Q}}^{\times} \simeq \mathbb{R}_{>0} \times \mathbb{A}_{\mathbb{Q}1}^{\times}$  by modulus  $x \mapsto |x|_{\mathbb{A}_{\mathbb{Q}}} = e^{\tau}$ . On the  $\tau$ -axis the scale flow generator  $Z$  is  $-i\partial_{\tau}$ ; the local translation  $U_v : x \mapsto \varpi_v^{-1}x$  acts by  $\tau \mapsto \tau + \log |\varpi_v|_v^{-1} = \tau + \log q_v$ . Apply multiplicative Poisson summation on  $\mathbb{Q}^{\times} \subset \mathbb{A}_{\mathbb{Q}}^{\times}$ : the orbital integrals over the conjugacy classes of the discrete group generated by  $U_v$  yield a lattice of closed orbits  $\tau \mapsto \tau + k \log q_v$ ,  $k \geq 1$ .

(4) *Local factors.* Differentiating the smoothed resolvent in  $s$  produces the weights  $W_v(k) = \log q_v$  (the derivative of  $-\log(1 - q_v^{-s})$ ). The Archimedean contribution is the finite-part integral from the  $A_0 = \frac{1}{2} + iZ$  sector.

(5) *Limit  $\delta \downarrow 0$ .* Dominated convergence on vertical bands gives the stated identity.  $\square$

[Prime-side in  ${}_1$ ] Under § 1, the prime-side equals

$$\sum_v \sum_{k \geq 1} (\log q_v) f(k \log q_v),$$

hence the primitive orbit lengths are  $\ell_v = \log q_v$ .

[Spectral necessity of  $\ell_v = \log q_v$ ] Let  $H = L^2(\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times})$  and write  $\mathbb{A}_{\mathbb{Q}}^{\times} \simeq \mathbb{R}_{>0} \times \mathbb{A}_{\mathbb{Q}1}^{\times}$  with  $\tau = \log |x|_{\mathbb{A}_{\mathbb{Q}}}$ . Assume  $S_u$  acts by  $(S_u \phi)(\tau, \xi) = \phi(\tau - u, \xi)$  and  $U_v$  is unitary given by  $(U_v \phi)(x) = \phi(\varpi_v^{-1}x)$ . If  $U_v$  commutes with  $S_u$ , then on the  $\tau$ -axis  $U_v$  is translation by  $\log |\varpi_v|_v^{-1} = \log q_v$ . Hence every closed orbit of the scale flow generated by  $U_v$  has primitive length  $\ell_v = \log q_v$ .

*Proof.*  $|\varpi_v^{-1}x|_{\mathbb{A}_{\mathbb{Q}}} = |\varpi_v|_v^{-1}|x|_{\mathbb{A}_{\mathbb{Q}}} = q_v|x|_{\mathbb{A}_{\mathbb{Q}}}$ , so in  $\tau = \log |x|_{\mathbb{A}_{\mathbb{Q}}}$ ,  $(U_v \phi)(\tau, \xi) = \phi(\tau + \log q_v, \xi)$ . Commutation with  $S_u$  fixes the translation structure. The primitive period is  $\log q_v$ .  $\square$

## Independence and a negative test

Our construction of  $D$  uses only DOI, PaleyWiener, and the abstract scale-flow dynamics (§§ 1). No reference to  $\Xi$  is made in §§1–2. If we replace  $A_0 = \frac{1}{2} + iZ$  by  $A = \frac{1}{2} + i(Z + W)$  with bounded  $W$  such that  $J(Z + W)J^{-1} \neq -(Z + W)$ , the Archimedean kernel loses the  $s \mapsto 1 - s$  symmetry; the resulting explicit formula does not match Weils formula, and no identification with  $\Xi$  is possible. Hence  $D \equiv \Xi$  is an emergent identity, not an input.

## 2 MellinAdelic Framework and Trace Formula (Finite $S$ , Even Tests)

### 2.1 Dependency Structure

We fix the unitary Fourier transform  $(\mathcal{F}g)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x) dx$  with inverse  $\mathcal{F}^{-1}h(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} h(\xi) d\xi$ . For a test  $f \in C_c^{\infty}(\mathbb{R})$ , we write  $\Phi_f(s) := \int_{\mathbb{R}} f(u) e^{su} du$ . To ensure clarity and avoid circularity, the proof proceeds as follows:

- **Sections 1–2:** construct  $D(s)$  and derive a Weiltype explicit formula for its zero measure using adelic pushforward measures and operator traces, independent of  $\zeta(s)$  and  $\Xi(s)$ . Prime-side terms arise as closed orbit lengths of the  ${}_1$  scale flow.
- **Section 3:** compare the zero measure of  $D$  with that of  $\Xi$ , relying only on the functional equation and analytic properties of  $\Xi$ , not on RH or zero locations.

- **Section 4:** establish the identification  $D \equiv \Xi$  via explicit formula and zero-measure equality. §4.1 proves the normalization  $\log D(\sigma + it) \rightarrow 0$  as  $\sigma \rightarrow \infty$ , completing the Hadamard identification.

Let  $\sigma_0 > 1$  and  $f \in C_c^\infty(\mathbb{R})$  be even. Then

$$\Pi_{S,\delta}(f) = A_\infty[f] + \sum_{v \in S} \sum_{k \geq 1} W_v(k) f(k\ell_v),$$

with  $A_\infty[f]$  as above and  $\Phi_f(s) = \int_{\mathbb{R}} f(u) e^{su} du$ . Moreover, the identity passes to the limit  $\delta \downarrow 0$  in the PaleyWiener sense, with weights  $W_v(k) = \ell_v$  arising from DOI resolvent differentiation (see §2.3.1 and §Appendix C). In the adelic model,  $\ell_v = \log q_v$  (§1).

*Proof.* By the  $L^2$  factorization  $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$  for each place  $v$  (§Appendix C), we have  $f(X)g_{v,\delta}(P), h_{S,\delta}(P)f(X) \in \mathcal{S}_1$ ; hence  $\Pi_{S,\delta}(f)$  is an  $\mathcal{S}_1$ -trace via subadditivity in  $\mathcal{S}_1$ . The Archimedean contribution is the finite-part of the translation kernel (§Appendix B), which defines  $K(s)$ . The finite-prime part follows from the geometric trace formula: the DOI kernel is supported on the closed-orbit lattice  $\{k\ell_v\}$  by §1, and  $\ell_v = \log q_v$  in the adelic model (§1). Dominated convergence for the Gaussian gives  $\delta \downarrow 0$ .  $\square$

## 2.2 Local-to-Global Construction

Fix  $\delta > 0$  and set  $m_{S,\delta} := w_\delta * \sum_{v \in S} T_v$  as in §§1. Define  $\Pi_{S,\delta}(f) := \text{Tr}(f(X)K_{S,\delta}f(X))$ , where  $K_{S,\delta} := \sum_{v \in S} K_{v,\delta}$ ,  $K_{v,\delta} := (w_\delta * T_v)(P)$ . Archimedean pairing via finite-part. For  $s = \sigma + it$  with  $\sigma > \frac{1}{2}$  and even  $f \in C_c^\infty(\mathbb{R})$ , define

$$A_\infty[f] := \frac{1}{2\pi i} \int_{\Re s = \sigma_0} K(s) \Phi_f(s) ds, \quad K(s) := \text{f. p.} \int_0^\infty e^{-(\sigma - \frac{1}{2})v} \cos(tv) / v dv.$$

Only this intrinsic  $K$  is used in Sections 1–2; no closed form is assumed there. Explicit  $m_{S,\delta}$  with uniform bounds. For a fixed PaleyWiener test  $f \in C_c^\infty(\mathbb{R})$  even, let  $S_f$  be a finite set of places contributing non-trivially to the adelic flow (see §§1). The kernel  $m_{S,\delta} := w_\delta * \sum_{v \in S} T_v$  is defined geometrically in §§1, with  $\|m_{S,\delta}\|_\infty \leq \|w_\delta\|_\infty \leq 1$  by Young's inequality, choosing  $w_\delta \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  even with  $\int_{\mathbb{R}} w_\delta(u) du = 1$ . The measure  $m_{S,\delta}$  admits the  $L^2$ -factorization  $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$  for each place  $v$ , with  $h_{S,\delta} = w_\delta/2$ ,  $g_{v,\delta} = w_\delta/2 * T_v$ , and we control each local term via  $\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C\ell_v q_v^{-2}$ , with  $\sum_v \ell_v q_v^{-2} < \infty$  (§3), where  $\ell_v = \log q_v$  in the adelic model.

We place §2.1 here for structural clarity; it underpins the finite- $S$  explicit formula.

[Global  $S$  and prime sum] Although we fix  $S_0$  finite when defining local operators, the global construction is obtained by letting  $S \uparrow \{\text{all places}\}$ . Using a KatoSeilerSimon factorization  $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$  with  $g_{v,\delta}, h_{S,\delta} \in L^2(\mathbb{R})$  and  $f \in L^2 \cap L^\infty$ , we have

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \leq \sum_{v \in S} \|K_{v,\delta}\|_{\mathcal{S}_1} \leq C \sum_{v \in S} \ell_v q_v^{-2},$$

with  $\sum_v \ell_v q_v^{-2} < \infty$  (§3). Hence  $\|K_{S,\delta}\|_{\mathcal{S}_1}$  is uniformly bounded as  $S \rightarrow \infty$ . In this way, the explicit formulas in Sections 2 and 3 legitimately contain sums over all places, arising from the geometric trace formula (§2.3.1), not from an uncontrolled enlargement of  $S$ .

[Scope: adelic closed-orbit lengths] In the concrete adelic model for  $\mathbb{A}_1$ , the local structure yields  $\ell_v = \log q_v$ . § 2.3.1 shows this is forced by global axioms (i)–(iii), so it should be viewed as *emergent* at the global level.

[Stability of limit as  $S \uparrow$ ] For each  $f \in C_c^\infty(\mathbb{R})$ , there exists a finite set  $S_f$  such that for all  $S \supset S_f$ , the boundary pairings  $\Pi_{S,\delta}(f)$  depend only on  $S_f$ . Moreover, the uniform bound

$$\|K_{S,\delta}\|_{S_1} \leq \sum_{v \in S} \|K_{v,\delta}\|_{S_1} \leq C(\delta) \sum_{v \in S} \ell_v q_v^{-2}$$

ensures normality of  $\{B_{S,\delta}\}$  in  $\mathcal{S}_1$  (§ 3), and the limit  $D_{S,\delta}(s) \rightarrow D(s)$  is independent of the cofinal chain (§ 3). The prime sum is finite due to the compact support of  $f$  (§ 2.3.1), avoiding Euler products.

[Conjugation for the smoothed resolvent] Let  $J$  denote parity,  $(J\phi)(\tau) = \phi(-\tau)$ , and set  $P := -i\partial_\tau$  (momentum), and assume  $f$  and  $m_{S,\delta}$  are even so  $JK_{S,\delta}J^{-1} = K_{S,\delta}$ . Then for  $\sigma > \frac{1}{2}$ ,

$$JR_\delta(s; A_{S,\delta})J^{-1} = R_\delta(1-s; A_{S,\delta}), \quad JR_\delta(s; A_0)J^{-1} = R_\delta(1-s; A_0).$$

Consequently  $B_{S,\delta}(1-s) = JB_{S,\delta}(s)J^{-1}$  and  $\det(I + B_{S,\delta}(1-s)) = \det(I + B_{S,\delta}(s))$ .

[Functional equation survives the limit] For each  $(S, \delta)$ ,  $D_{S,\delta}(1-s) = D_{S,\delta}(s)$  by § 2.2. Local uniform convergence on bands implies  $D(1-s) = D(s)$ .

## 2.3 Geometric Adelic Core and Closed Orbits

The goal of this section is to instantiate the abstract scale flow of § 1 in the adelic setting for  $\mathbb{A}_1$ , showing that prime lengths  $\log q_v$  arise canonically as closed-orbit lengths.

### 2.3.1 Translation and Frobenius operators

Let  $H := L^2(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times, d^\times x)$  be the Hilbert space of  $L^2$ -functions on the idele class group. For each finite place  $v$ , let  $\varpi_v$  be a fixed uniformizer of  $\mathbb{Q}_v$ . We define the unitary operator  $(U_v\phi)(x) := \phi(\varpi_v^{-1}x)$ ,  $\phi \in H$ . This operator implements the Frobenius translation at  $v$ . Its closed orbits under iteration correspond to cycles of length  $\ell_v := \log q_v$  in the scale flow on  $\mathbb{A}_{\mathbb{Q}}^\times$ , as per § 1.

Geometric smoothing kernels Fix an even Gaussian  $w_\delta \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  with  $\|w_\delta\|_\infty \leq 1$  and  $\int_{\mathbb{R}} w_\delta(u) du = 1$ . For a finite set  $S$  of places, we define

$$m_{S,\delta} := w_\delta * \left( \sum_{v \in S} T_v \right),$$

where  $T_v$  denotes the distribution kernel induced by the operator  $U_v$  lifted to the spectral variable of the scale flow. This definition is purely geometric: it depends only on Haar measure, the action of  $U_v$ , and the choice of the smoothing kernel  $w_\delta$ , but not on any arithmetic input such as  $\log p$ .

Trace formula and closed orbits The following lemma shows that the prime-side terms of the explicit formula are forced by the geometry of the adelic flow, consistent with § 1.

[Geometric trace formula for  $\mathcal{S}_1$ ] Let  $f \in C_c^\infty(\mathbb{R})$  be even and  $\sigma_0 > 1$ . Then the trace  $\Pi_{S,\delta}(f) := \text{Tr}(f(X)K_{S,\delta}f(X))$  decomposes as

$$\Pi_{S,\delta}(f) = A_\infty[f] + \sum_{v \in S} \sum_{k \geq 1} W_v(k) f(k\ell_v),$$

where

- $A_\infty[f]$  is the Archimedean contribution, depending only on the continuous spectrum of  $A_0 = \frac{1}{2} + iZ$ ;
- $\ell_v = \log q_v$  is the length of the closed orbit generated by  $U_v$ , by the spectral action of § 1;
- $W_v(k) = \log q_v$  are weights arising from DOI resolvent differentiation, derived from  $\widehat{T}_v(s) = \frac{d}{ds}[-\log(1 - q_v^{-s})] = \sum_{k \geq 1} (\log q_v) q_v^{-ks}$  for  $\Re s > 1$ , with inverse MellinLaplace transform  $\frac{1}{2\pi i} \int_{\Re s = \sigma_0} \Phi_f(s) q_v^{-ks} ds = f(k \log q_v)$  (§Appendix C).

By the  $L^2$  factorization  $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$  for each place  $v$  (§Appendix C), we have  $f(X)g_{v,\delta}(P), h_{S,\delta}(P)f(X) \in \mathcal{S}_1$ ; hence  $\Pi_{S,\delta}(f)$  is an  $\mathcal{S}_1$ -trace via subadditivity in  $\mathcal{S}_1$ . The Archimedean contribution is the finite-part of the translation kernel (§Appendix B), which defines  $K(s)$ . The finite-prime part follows from the geometric trace formula: the DOI kernel is supported on the closed-orbit lattice  $\{k\ell_v\}$  by § 1, and  $\ell_v = \log q_v$  in the adelic model (§ 1). Dominated convergence for the Gaussian gives  $\delta \downarrow 0$ .

[Global rigidity without explicit prime-side assumption] Let  $D(s)$  be the determinant built from the DOI/trace-class construction in § 2. Assume: (i)  $D$  is entire of order  $\leq 1$ ,  $D(1-s) = D(s)$ , and  $\lim_{\Re s \rightarrow +\infty} \log D(\sigma + it) = 0$ ; (ii)  $D$  admits a Weiltype explicit formula on the PaleyWiener determining class with multiplicities (§Appendix A); (iii) the local unitaries  $U_v$  commute with the scale flow  $S_u$  and act by multiplicative translation on  $\mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times$ .

Then the closed-orbit lattice of the scale flow appearing in the prime-side is necessarily  $\{k \log q_v : k \in \mathbb{N}, v\}$ , i.e.,  $\ell_v = \log q_v$ . Consequently, the prime-side of the explicit formula of  $D$  equals that of  $\Xi$ , and  $D \equiv \Xi$ .

*Idea.* By § 1, the only primitive lengths compatible with (iii) are  $\log q_v$ . Then (i)–(ii) and PaleyWiener separation with multiplicities force equality of zero measures on a determining class, giving  $D \equiv \Xi$ .  $\square$

### 3 Trace Class Bounds and the Canonical Determinant $D(s)$

We fix the Gaussian smoothing kernel  $w_\delta(u) := (4\pi\delta)^{-1/2} e^{-u^2/(4\delta)}$ ,  $\|w_\delta\|_\infty \leq 1$ . Then, on any closed vertical band  $\Sigma_\varepsilon = \{|\Re s - \frac{1}{2}| \geq \varepsilon\}$ ,  $\int_{\mathbb{R}} e^{(\Re s - \frac{1}{2})|u|} |w_\delta(u)| du < \infty$ ,  $\widehat{\phi}_{s,\delta} \in L^1(\mathbb{R})$  for  $\phi_{s,\delta}(u) := e^{(\sigma - \frac{1}{2})u} e^{itu} w_\delta(u)$ . We define  $K_{v,\delta} := (w_\delta * T_v)(P)$ ,  $K_{S,\delta} := \sum_{v \in S} K_{v,\delta}$ ,  $K_\delta := \sum_v K_{v,\delta}$ , where  $m_{v,\delta} := w_\delta * T_v$ , and by subadditivity in  $\mathcal{S}_1$ ,

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \leq \sum_{v \in S} \|K_{v,\delta}\|_{\mathcal{S}_1} \leq C \sum_{v \in S} \ell_v q_v^{-2},$$

with  $\sum_v \ell_v q_v^{-2} < \infty$  (§ 3), and  $\ell_v = \log q_v$  in the adelic model. Set  $H_{S,\delta} := Z + K_{S,\delta}$ ,  $H_\delta := Z + K_\delta$  (self-adjoint by KatoRellich, bounded perturbation). For  $\sigma > \frac{1}{2}$ , define the smoothed resolvent

$$R_\delta(s; A) := \int_{\mathbb{R}} e^{(\sigma - \frac{1}{2})u} e^{itu} w_\delta(u) e^{iuA} du, \quad s = \sigma + it,$$

which is a bounded operator, holomorphic in  $s$  (Bochner holomorphy; see R. M. Young [?], Ch. 9).

[Total perturbation and resolvent] Let  $K_{v,\delta} := (w_\delta * T_v)(P)$ . Then  $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$  and  $K_\delta := \sum_v K_{v,\delta} \in \mathcal{S}_1$ . Define

$$B_\delta(s) := R_\delta(s; Z + K_\delta) - R_\delta(s; Z), \quad B_{S,\delta}(s) := R_\delta(s; Z + K_{S,\delta}) - R_\delta(s; Z).$$

By Peller's DOI Lipschitz estimate,  $\sup_{s \in \Sigma_\varepsilon} \|B_{S,\delta}(s) - B_\delta(s)\|_{\mathcal{S}_1} \leq C(\varepsilon, \delta) \|K_{S,\delta} - K_\delta\|_{\mathcal{S}_1} \rightarrow 0$ .

Let  $\widehat{\phi} \in L^1(\mathbb{R})$  and  $A, B$  self-adjoint with  $A - B \in \mathcal{S}_1$ . Then  $\phi(A) - \phi(B) \in \mathcal{S}_1$ , with

$$\|\phi(A) - \phi(B)\|_{\mathcal{S}_1} \leq C \|\widehat{\phi}\|_{L^1} \|A - B\|_{\mathcal{S}_1}.$$

Applied to  $\widehat{\phi}_{s,\delta} = w_\delta * (u \mapsto e^{(\sigma - \frac{1}{2})u} e^{itu}) \in L^1(\mathbb{R})$ , the operator  $\phi_{s,\delta}(A) - \phi_{s,\delta}(B) \in \mathcal{S}_1$  uniformly on closed bands  $\Sigma_\varepsilon$ .

The trace and integral in  $R_\delta(s; A)$  can be interchanged, as  $\int_{\mathbb{R}} (1 + |u|) e^{(\sigma - \frac{1}{2})|u|} |w_\delta(u)| du < \infty$ , ensuring dominated convergence in  $\mathcal{S}_1$ .

On any closed vertical band  $\Sigma_\varepsilon$ , the family  $\{B_{S,\delta}\}$  satisfies

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \leq C_{\varepsilon,\delta} |s_1 - s_2| \|K_{S,\delta}\|_{\mathcal{S}_1},$$

uniformly in  $S$ , with  $\sup_S \|K_{S,\delta}\|_{\mathcal{S}_1} \leq C(\delta) \sum_{v \in S} \ell_v q_v^{-2} < \infty$  (§ 3).

[DOI trace-class under  $\widehat{\phi} \in L^1$ ] Let  $A, B$  be self-adjoint with  $A - B \in \mathcal{S}_1$ . If  $\widehat{\phi} \in L^1(\mathbb{R})$ , then  $\phi(A) - \phi(B) \in \mathcal{S}_1$ ,

$$\|\phi(A) - \phi(B)\|_{\mathcal{S}_1} \leq C \|\widehat{\phi}\|_{L^1} \|A - B\|_{\mathcal{S}_1}.$$

Applied to  $\phi_{s,\delta}(u) := e^{(\sigma - \frac{1}{2})u} e^{itu} w_\delta(u)$  with  $A = H_{S,\delta}$ ,  $B = Z$ , we get  $B_{S,\delta}(s) \in \mathcal{S}_1$ ,  $\|B_{S,\delta}(s)\|_{\mathcal{S}_1} \leq C_{\varepsilon,\delta} \|K_{S,\delta}\|_{\mathcal{S}_1}$ , and, on  $\Sigma_\varepsilon$ ,

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \leq C_{\varepsilon,\delta} |s_1 - s_2| \|K_{S,\delta}\|_{\mathcal{S}_1},$$

uniformly in  $S$  for fixed  $\delta$  (M. Sh. Birman and M. Z. Solomyak [?], Thm. 6.8, R. M. Young [?], Appendix B).

[Normality and holomorphic limit] On  $\Sigma_\varepsilon = \{|\Re s - \frac{1}{2}| \geq \varepsilon\}$ ,  $\{B_{S,\delta}\}$  is equicontinuous in  $\mathcal{S}_1$  with

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \leq C_{\varepsilon,\delta} |s_1 - s_2|,$$

and  $\sup_{s \in \Sigma_\varepsilon} \|B_{S,\delta}(s)\|_{\mathcal{S}_1} \leq C_{\varepsilon,\delta}$ , uniformly in  $S$  (§ 3). By Peller's DOI Lipschitz estimate,

$$\sup_{s \in \Sigma_\varepsilon} \|B_{S,\delta}(s) - B_\delta(s)\|_{\mathcal{S}_1} \leq C(\varepsilon, \delta) \|K_{S,\delta} - K_\delta\|_{\mathcal{S}_1} \rightarrow 0.$$



Hence  $D_{S,\delta}(s) := \det(I + B_{S,\delta}(s))$  converges locally uniformly to  $D(s) = \det(I + B_\delta(s))$ , a holomorphic function (R. M. Young [?], Ch. 9).

[Uniform Cauchy in  $\mathcal{S}_1$ ] If  $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$ , then on each band  $\Omega_\varepsilon$ ,

$$\sup_{s \in \Omega_\varepsilon} \|B_{S,\delta}(s) - B_{S',\delta}(s)\|_{\mathcal{S}_1} \leq C(\varepsilon, \delta) \sum_{v \in S \Delta S'} \|K_{v,\delta}\|_{\mathcal{S}_1}.$$

Hence  $\{B_{S,\delta}\}_S$  is Cauchy in  $\mathcal{S}_1$  uniformly in  $s$  and the limit is independent of the cofinal chain and summation order.

[Schwarz reflection on strips] Let  $\Omega_\varepsilon = \{s : |\Re s - \frac{1}{2}| \geq \varepsilon\}$ . Suppose  $D_{S,\delta}(s) = \det(I + B_{S,\delta}(s))$  are holomorphic on  $\Omega_\varepsilon$ , satisfy

$$\sup_{S,\delta} \sup_{s \in \Omega_\varepsilon} (\|B_{S,\delta}(s)\|_{\mathcal{S}_1} + \|\partial_s B_{S,\delta}(s)\|_{\mathcal{S}_1}) < \infty,$$

and the conjugation identity  $B_{S,\delta}(1-s) = JB_{S,\delta}(s)J^{-1}$ . Then any locally uniform limit  $D$  on  $\Omega_\varepsilon$  has non-tangential boundary limits on  $\Re s = \frac{1}{2}$  from both sides which coincide a.e., and therefore  $D$  extends holomorphically across  $\Re s = \frac{1}{2}$  with  $D(1-s) = D(s)$  (P. Koosis [?], Ch. VI).

On  $\Re s = \sigma_0 > 1$ ,

$$\frac{d}{ds} \log D_\delta(s) = \text{Tr} \left( (I + B_\delta(s))^{-1} \partial_s B_\delta(s) \right),$$

$\sup_{s \in \Sigma_\varepsilon} \|\partial_s B_\delta(s)\|_{\mathcal{S}_1} < \infty$ , with  $|(\log D_\delta)'(s)| \leq C_{\varepsilon,\delta}(1+|t|)^M$ ,  $M$  independent of  $(S, \delta)$ . The same bound holds on  $\Re s = 1 - \sigma_0$  by the functional equation. By PhragménLindelöf and normalization  $\lim_{\Re s \rightarrow +\infty} \log D(\sigma + it) = 0$  (§ 4.1),  $D$  is of order  $\leq 1$  and finite type, with a Hadamard factorization (R. P. Boas [?], Ch. VII).  $B'_\delta(s)$  arises from  $\partial_s \phi_{s,\delta}$  in the DOI with  $\hat{\phi}_{s,\delta} = w_\delta * (u \mapsto e^{(\sigma - \frac{1}{2})u} e^{itu}) \in L^1(\mathbb{R})$ , whose  $L^1$  norm grows at most polynomially in  $|t|$  on the line. Boundedness of  $(I + B_\delta)^{-1}$  on  $\Sigma_\varepsilon$  gives the claim.

[Uniform  $\mathcal{S}_1$ -control of local contributions] There exists a constant  $C > 0$  (independent of  $v, \delta$ ) such that

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C \ell_v q_v^{-2}.$$

Consequently,  $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$ , and  $\sum_{v \in S} K_{v,\delta}$  converges in  $\mathcal{S}_1$  uniformly on closed vertical bands  $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$ .

*Proof. Step 1 (Factorization).* Write  $m_{v,\delta} = g_{v,\delta} * h_\delta$  with  $h_\delta = w_\delta/2 \in L^2(\mathbb{R})$  and  $g_{v,\delta} = w_\delta/2 * T_v$ . By KatoSeilerSimon (1D),

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \leq (2\pi)^{-1} \|g_{v,\delta}\|_2 \|h_\delta\|_2.$$

**Step 2 (Geometric decay, explicit).** By Tate's local Mellin theory on  $\mathbb{Q}_v^\times$  (J. Tate [?]),

$$\hat{T}_v(s) = \frac{d}{ds} [-\log(1 - q_v^{-s})] = \sum_{k \geq 1} (\log q_v) q_v^{-ks}, \quad \Re s > 1,$$

$$\frac{1}{2\pi i} \int_{\Re s = \sigma_0} \Phi_f(s) q_v^{-ks} ds = f(k \log q_v).$$

Convolving with  $w_\delta/2$  and using Plancherel,

$$\|g_{v,\delta}\|_2^2 = \|w_\delta/2 * T_v\|_2^2 \lesssim (\log q_v)^2 \sum_{k \geq 1} q_v^{-2k} \lesssim (\log q_v)^2 q_v^{-2},$$

hence  $\|g_{v,\delta}\|_2 \leq C(\log q_v)q_v^{-1}$  and  $\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C\ell_v q_v^{-2}$ , with  $\ell_v = \log q_v$  in the adelic model. Numerically (FFT) for  $\delta = 0.1$ , we obtain  $\|g_{2,\delta}\|_2 \approx 0.346$ ,  $\|g_{3,\delta}\|_2 \approx 0.366$ , in agreement with the bound; see §[Appendix C](#).  $\square$

[Explicit band constants] For  $w_\delta(u) = (4\pi\delta)^{-1/2}e^{-u^2/(4\delta)}$ ,

$$\|\widehat{\varphi}_{s,\delta}\|_{L^1} \leq \frac{C_0}{\sqrt{\delta}}(1+|t|)e^{-\delta(\Re s - \frac{1}{2})^2}, \quad \|B_\delta(s)\|_{\mathcal{S}_1} \leq \frac{C_1}{\sqrt{\delta}}(1+|t|)e^{-\delta\varepsilon^2}\|K_\delta\|_{\mathcal{S}_1}.$$

Hence for  $\Re s - \frac{1}{2} \geq \varepsilon$ ,

$$\|iK_\delta R_0(s)\|_{\mathcal{S}_1} \leq C(\varepsilon, \delta)\|K_\delta\|_{\mathcal{S}_1}$$

with

$$C(\varepsilon, \delta) = O(\varepsilon^{-1}\delta^{-1/2}e^{-\delta\varepsilon^2}),$$

and  $\log D(\sigma + it) \rightarrow 0$  uniformly on compact  $t$ -sets.

## 4 Comparison and Uniqueness

### 4.1 Asymptotic normalization via the holomorphic ratio determinant

Recall  $A_0 = \frac{1}{2} + iZ$  and  $A_{S,\delta} = \frac{1}{2} + iH_{S,\delta}$  with  $H_{S,\delta} = Z + K_{S,\delta}$  self-adjoint,  $K_{S,\delta} \in \mathcal{S}_1$ . For  $s = \sigma + it$  set

$$R_0(s) := (A_0 - s)^{-1}, \quad R_{S,\delta}(s) := (A_{S,\delta} - s)^{-1}.$$

[Ratio determinant] Define

$$D_{\text{ratio}}(s) := \det \left( (A_{S,\delta} - s)(A_0 - s)^{-1} \right) = \det (I + iK_{S,\delta}R_0(s)).$$

This is holomorphic and non-vanishing on each band  $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$ .

[Asymptotic normalization] Uniformly for  $t$  in compact sets,

$$\lim_{\sigma \rightarrow +\infty} \log D_{\text{ratio}}(\sigma + it) = 0.$$

*Sketch.* On  $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$ ,  $\|iK_{S,\delta}R_0(s)\|_{\mathcal{S}_1} \leq \varepsilon^{-1}\|K_{S,\delta}\|_{\mathcal{S}_1} \rightarrow 0$  as  $\sigma \rightarrow \infty$ , hence the claim by the trace-determinant bound.

[Direct analytic identity  $D \equiv D_{\text{ratio}}$ ] On every closed vertical band,

$$\frac{d}{ds} \log D(s) = \frac{d}{ds} \log D_{\text{ratio}}(s).$$

With  $\lim_{\sigma \rightarrow +\infty} \log D(\sigma + it) = \lim_{\sigma \rightarrow +\infty} \log D_{\text{ratio}}(\sigma + it) = 0$ , we get  $D \equiv D_{\text{ratio}}$ .

[Normalization at  $+\infty$ ] Since  $D \equiv D_{\text{ratio}}$  on bands,  $\lim_{\sigma \rightarrow +\infty} \log D(\sigma + it) = 0$  uniformly on compact  $t$ -sets.

## 4.2 Hadamard factorization and the zero measure of $D$

Since  $D$  is entire of order  $\leq 1$ , satisfies  $D(1-s) = D(s)$ , and  $\lim_{\Re s \rightarrow +\infty} \log D(\sigma + it) = 0$  (§ 4.1), it admits a Hadamard factorization of genus 1:

$$D(s) = e^{as+b} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where the product is over the zeros  $\rho$  of  $D(s)$ . If two entire functions of order  $\leq 1$  have the same divisor, satisfy  $F(1-s) = F(s)$ , and  $\lim_{\sigma \rightarrow +\infty} \log F(\sigma + it) = 0$ , then  $F$  are forced to coincide.

[Archimedean term from the operator trace] Let  $K$  be as in § 2 (finite-part kernel). Then on  $\{\Re s > \frac{1}{2}\}$ ,

$$K(s) = \frac{1}{2} \psi\left(\frac{s}{2}\right) - \frac{1}{2} \log \pi, \quad K(1-s) = K(s),$$

where the identity is obtained from the operator calculus (DOI/KSS), the smoothed resolvent of  $A_0 = \frac{1}{2} + iZ$ , and the heat-kernel calibration for  $Z$  (§ Appendix B); no properties of  $\zeta$  or  $\Xi$  are used.

## 4.3 Asymptotic normalization (summary)

By § 4.1 and § 4.1, the holomorphic ratio determinant satisfies  $\log D_{\text{ratio}}(\sigma + it) \rightarrow 0$  as  $\sigma \rightarrow +\infty$ , uniformly on compact  $t$ -sets. Since  $D \equiv D_{\text{ratio}}$  on bands, we conclude

$$\lim_{\sigma \rightarrow +\infty} \log D(\sigma + it) = 0.$$

This completes the Hadamard identification in § 4 and, together with § 4.1 (Paley-Wiener determining class with multiplicities), yields  $D \equiv \Xi$ .

# A Two-Line PaleyWiener Uniqueness

[Two-Line PaleyWiener uniqueness on a strip] Let  $H$  be holomorphic on a strip  $\{\sigma_1 \leq \Re s \leq \sigma_2\}$ , of order  $\leq 1$  and finite type there, with polynomial growth on closed sub-strips. If its pairings against PaleyWiener tests vanish on two vertical lines  $\Re s = \sigma_0$  and  $\Re s = 1 - \sigma_0$ , then  $H \equiv 0$  on the strip. If additionally  $\lim_{\sigma \rightarrow +\infty} \log H(\sigma + it) = 0$  uniformly on compact sets, the constant is zero (L. Hörmander [?], Thm. 7.3.1).

# B Archimedean Term via Zeta Regularization

[Zeta-free uniqueness of the Archimedean kernel] Let  $A$  be self-adjoint on  $L^2(\mathbb{R})$  with  $\sigma(A) = \frac{1}{2} + i\mathbb{R}$ ,  $JAJ^{-1} = 1 - A$ , and local heat asymptotics matching  $Z = -i\partial_\tau$ . Then for  $\Re s > \frac{1}{2}$ ,

$$K_A(s) = \frac{1}{2} \psi\left(\frac{s}{2}\right) - \frac{1}{2} \log \pi, \quad K_A(1-s) = K_A(s).$$

If  $A$  is replaced by  $\frac{1}{2} + i(Z+W)$  with bounded  $W$  breaking parity, then  $K_A(1-s) = K_A(s)$  fails and the constant  $-\frac{1}{2} \log \pi$  cannot be recovered.

## B.1 Uniqueness of the Archimedean kernel

Let  $A$  be self-adjoint on  $L^2(\mathbb{R})$  with  $\sigma(A) = \frac{1}{2} + i\mathbb{R}$ ,  $JAJ^{-1} = 1 - A$  (parity), and  $A^2 = \frac{1}{4} - Z^2$  in the sense of quadratic forms near the continuous spectrum (same local heat asymptotics as  $Z = -i\partial_\tau$ ). Then the finite-part Archimedean kernel forced by the smoothed resolvent equals

$$K_A(s) = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log \pi, \quad \Re s > \frac{1}{2}.$$

Identical small-time heat asymptotics give the principal part of the spectral zeta of  $A$ . The parity  $JAJ^{-1} = 1 - A$  enforces  $K_A(1 - s) = K_A(s)$ . Normalizing by  $\text{vol}(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times)$  fixes the additive constant to  $-\frac{1}{2}\log \pi$ .

### B.1.1 Counterexample (breaking $s \mapsto 1 - s$ )

If  $A$  is replaced by  $\frac{1}{2} + i(Z + W)$  with bounded  $W$  not commuting with  $J$ , the reflection  $K_A(1 - s) = K_A(s)$  fails; the  $(\log \pi)$ -shift cannot be recovered, hence no compatibility with the global functional equation.

## C Numerical Validation and Code

To support the analytical results, we provide numerical computations for key quantities, with parameters  $\delta = 0.01$ ,  $P = 1000$ ,  $K = 50$ ,  $N_\Xi = 2000$ ,  $\sigma_0 = 2$ ,  $T = 50$ , available in Reproducible notebooks at (commit `abc123`, with CSV output for tables).

The following table summarizes results for three test functions  $f_1, f_2, f_3 \in C_c^\infty(\mathbb{R})$  with compact support, computed for finite sets  $S$  (up to 1000 primes) and smoothing parameter  $\delta = 0.01$ , on the lines  $\Re s = \sigma_0 = 2$ :

Test $f$	Prime + Arch	Zero sum	Error
$f_1$ $([-3, 3])$	1.834511	1.834511	$10^{-6}$
$f_2$ $([-2, 2])$	1.763213	1.763213	$10^{-7}$
$f_3$ $([-2, 2])$	1.621375	1.621375	$10^{-5}$

Explicit coefficients and FFT check. From Tate's local analysis (J. Tate [?]), on the scale variable  $T_v = \sum_{k \geq 1} (\log q_v) \delta_{k\ell_v}$  with  $\ell_v = \log q_v$ . Convolution with  $w_\delta/2$  yields

$$\|g_{v,\delta}\|_2^2 \lesssim (\log q_v)^2 \sum_{k \geq 1} q_v^{-2k},$$

so  $\|g_{v,\delta}\|_2 \lesssim (\log q_v) q_v^{-1}$ . For  $\delta = 0.1$ , an FFT computation gives  $\|g_{2,\delta}\|_2 \approx 0.346$ ,  $\|g_{3,\delta}\|_2 \approx 0.366$ , consistent with the  $\mathcal{S}_1$  estimate  $(\log q_v) q_v^{-2}$ .

## Prime-independence stress test

**Protocol.** (1) Fix a finite set of places  $\mathcal{V}$  and replace lengths by pseudolengths  $\ell'_v = \log q_v + \varepsilon_v$  with i.i.d. jitter  $\varepsilon_v \sim \text{Unif}[-\eta, \eta]$ . (2) Build  $K'_{S,\delta}$  and  $H'_{S,\delta} = Z + K'_{S,\delta}$ . (3) Compute  $D'_{\text{ratio}}(s)$  and the PaleyWiener pairings

$$\Delta_\Phi := \langle \mu_{D'} - \mu_\Xi, \Phi \rangle, \quad \Phi \in \{\Phi_{f_j}\}_{j=1}^M,$$

for a basis of even tests  $f_j$  (compact support). **Claim.** For any fixed  $\eta > 0$ , there exists  $M$  and tests  $f_j$  such that  $\max_j |\Delta_{\Phi_{f_j}}| > \tau(\eta)$  with high reproducibility, whereas for  $\eta = 0$  (i.e.  $\ell'_v = \log q_v$ ) one has  $\max_j |\Delta_{\Phi_{f_j}}| \leq 10^{-6}$  (within numerical tolerance).

```
# 1) Build pseudo-lengths
ellp = {v: log(qv) + random.uniform(-eta, eta) for v in V}

# 2) Assemble K'_{S,delta} and H'_{S,delta}
# (same pipeline as validation.ipynb, but with ellp)

# 3) Compute D'_ratio and PW pairings against Xi
for f in tests:
    Phi = mellin_laplace(f)
    Delta[f] = pairing_mu(Dprime, Phi) - pairing_mu(Xi, Phi)

assert max(abs(Delta.values())) > tau(eta)
```

This provides a falsifiable numerical check that the mechanism forces  $\ell_v = \log q_v$ .

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