A COMPLETE CONDITIONAL RESOLUTION OF THE RIEMANN HYPOTHESIS VIA S-FINITE ADELIC SPECTRAL SYSTEMS

JOSÉ MANUEL MOTA BURRUEZO
INSTITUTOCONSCIENCIA@PROTON.ME
INSTITUTO CONCIENCIA CUÁNTICA (ICQ)
PALMA DE MALLORCA, SPAIN
GITHUB: HTTPS://GITHUB.COM/MOTANOVA84/-JMMOTABURR-RIEMANN-ADELIC
ZENODO DOI: 10.5281/ZENODO.17116291

Final Conditional Version V4.1 (September 2025)

Status (respectful disclaimer). This is a final conditional version. It does not claim community validation but is offered with complete transparency and reproducibility for expert review.

Abstract. This paper presents a complete conditional resolution of the Riemann Hypothesis, based on a spectral framework built from S-finite adelic systems. We define a canonical determinant $D(s)$, constructed from operator-theoretic principles alone, without using the Euler product or the Riemann zeta function $\zeta(s)$ as input. The determinant $D(s)$ arises from a scale-invariant flow over abstract places, smoothed via double operator integrals (DOI), and satisfies:

- $D(s)$ is entire of order ≤ 1,
- $D(1-s) = D(s)$ by spectral symmetry,
- $\lim_{\Re(s)\to+\infty} \log D(s) = 0$ (normalization),
- $D(s) \equiv \Xi(s)$, where $\Xi(s)$ is the completed Riemann xi-function.

The trace formula derived from this system recovers the logarithmic prime structure $\ell_v = \log q_v$ as a geometric consequence of closed spectral orbits, not as an assumption. The zero measure of $D(s)$ coincides with that of $\Xi(s)$ on a PaleyWiener determining class with multiplicities. This yields a conditional identification $D(s) = \Xi(s)$, and thus a conditional proof of the Riemann Hypothesis:

$$\zeta(s) = 0 \Rightarrow \Re(s) = \frac{1}{2}.$$

All results are presented with full transparency, including detailed appendices on trace-class convergence, uniqueness theorems, and numerical validation. The code and data are openly provided at the GitHub repository above.

This construction is offered as a rigorous, conditional framework for expert scrutiny. The core claim is that under the S-finite axioms and spectral regularity conditions detailed herein, the Riemann Hypothesis holds.

1. Axiomatic Scale Flow and Spectral System

1.1. **Abstract Framework.** Let V be a countable set of abstract places (both Archimedean and non-Archimedean), and let $H := L^2(\mathbb{R})$ be the Hilbert space of square-integrable functions. We consider a unitary scale-flow group $(S_u)_{u\in\mathbb{R}} \subset \mathcal{U}(H)$, acting by dilations along a spectral axis $\tau \in \mathbb{R}$, with generator $Z = -i\frac{d}{d\tau}$.

Each place $v \in V$ is associated with a local unitary operator $U_v \in \mathcal{U}(H)$, satisfying a discrete orbit condition and compatibility with the global scale flow.

We define the axiomatic system as follows.[1]

1.2. S-Finite Axioms.

Assumption 1.1 (Scale Commutativity (A1))**.** *Each local unitary U_v commutes with the scale-flow:*

$$U_v S_u = S_u U_v \quad \textit{for all } u \in \mathbb{R}$$

1.3. **Trace Structure and Discrete Support.** We define the smoothed trace functional:

$$\Pi_{S,\delta}(f) := \mathrm{Tr}\left(f(X) K_{S,\delta} f(X)\right),$$

for all even test functions $f \in C_c^\infty(\mathbb{R})$. The operator $f(X)$ denotes multiplication by f, acting on the scale variable.

Assumption 1.4 (Trace Decomposition — Selberg Type)**.** *For all even test functions $f \in C_c^\infty(\mathbb{R})$, the trace admits a decomposition of the form:*

$$\Pi_{S,\delta}(f) = A_\infty[f] + \sum_{v \in S} \sum_{k \geq 1} W_v(k) f(k\ell_v),$$

where $A_\infty[f]$ is a continuous (Archimedean) contribution, and the second term is a discrete sum over the closed orbit lengths ℓ_v.

1.4. **Length Identification.** We define the system to be *spectrally geometrized* if the orbit lengths ℓ_v match logarithmic lengths $\log q_v$, where q_v is the local norm at place v. In the adelic model for GL_1, we will later show that:

$$\ell_v = \log q_v.$$

This identification will emerge as a *consequence* of the global spectral axioms, not as an assumption.

Remark 1.5 (Role of ℓ_v)**.** *The values ℓ_v are not inserted by hand; they are the* primitive orbit lengths *arising from the periodic action of U_v on the spectral coordinate τ. The eventual identification $\ell_v = \log q_v$ will follow from operator symmetries and explicit formula inversion, as shown in Section 3.*

2. Construction of the Canonical Determinant $D(s)$

2.1. **Smoothing and Operator Perturbation.** Let $Z = -i\frac{d}{d\tau}$ be the generator of the scale-flow (S_u), acting on the Hilbert space $H = L^2(\mathbb{R})$. Let $P = Z$ by notation. Consider the total perturbation kernel:

$$K_{S,\delta} := \sum_{v \in S} K_{v,\delta}, \quad \text{where} \quad K_{v,\delta} := \left(w_\delta * T_v\right)(P),$$

with $w_\delta \in \mathcal{S}(\mathbb{R})$ an even Gaussian smoothing kernel.

We define the perturbed (self-adjoint) operator:

$$A_{S,\delta} := Z + K_{S,\delta}.$$

This defines a family of trace-class perturbations of the unperturbed operator $A_0 := Z$, indexed by finite sets $S \subset V$.

2.2. **Smoothed Resolvent and Trace Perturbation.** Let $s = \sigma + it \in \mathbb{C}$, with $\sigma > \frac{1}{2}$. Define the smoothed resolvent kernel:

$$R_\delta(s; A) := \int_{\mathbb{R}} e^{(\sigma - \frac{1}{2})u} e^{itu} w_\delta(u) e^{iuA}\, du.$$

Then we define the difference operator:

$$B_{S,\delta}(s) := R_\delta(s; A_{S,\delta}) - R_\delta(s; A_0),$$

and the canonical determinant:

$$D_{S,\delta}(s) := \det\left(I + B_{S,\delta}(s)\right).$$

2.3. **Holomorphy and Schatten Control.**

Proposition 2.1. *For each fixed $\delta > 0$, and on every vertical strip $\Omega_\varepsilon = \{s : |\Re(s) - \frac{1}{2}| \geq \varepsilon\}$, the operator $B_{S,\delta}(s) \in \mathcal{S}_1$ (trace-class), and the map $s \mapsto D_{S,\delta}(s)$ is holomorphic on Ω_ε.*

Sketch. Since $w_\delta \in \mathcal{S}(\mathbb{R})$, the smoothed resolvent is an operator-valued Bochner integral. The boundedness and trace-class property follow from KatoSeilerSimon estimates on convolutions and perturbation theory. Holomorphy follows from standard results on trace-class valued holomorphic families (Simon, 2005). □

2.4. **Limit and Canonical Determinant** $D(s)$**.**

Taking the limit $S \uparrow V$, we define the full kernel:

$$K_\delta := \sum_{v \in V} K_{v,\delta}, \quad A_\delta := Z + K_\delta.$$

By uniform convergence in $\mathcal{S}_1$, the family $B_{S,\delta}(s) \to B_\delta(s) := R_\delta(s; A_\delta) - R_\delta(s; A_0)$ uniformly on Ω_ε, and we define the canonical determinant:

$$D(s) := \det\left(I + B_\delta(s)\right).$$

2.5. **Functional Equation.**

Let J be the parity operator on H, defined by $(J\varphi)(\tau) := \varphi(-\tau)$. Then $JZJ^{-1} = -Z$, and $JA_\delta J^{-1} = 1 - A_\delta$. This yields the symmetry:

$$B_\delta(1-s) = JB_\delta(s)J^{-1} \quad \Rightarrow \quad D(1-s) = D(s).$$

2.6. **Remarks.**

Remark 2.2 (Zeta-Free Construction)**.** *At no point is $\zeta(s)$, $\Xi(s)$, or the Euler product used in the definition of $D(s)$. The entire construction arises from operator theory, smoothing, and spectral perturbations of a scale-invariant system.*

Remark 2.3 (Order and Growth)**.** *The determinant $D(s)$ is entire of order ≤ 1, as shown in Section 4, by Hadamard theory and uniform norm control on $B_\delta(s)$. Its zero set and asymptotics will be analyzed via explicit formulas and spectral analysis in the next sections.*

3. Trace Formula and Geometric Emergence of Logarithmic Lengths

3.1. **Adelic Model for** GL_1**.**

We now interpret the scale-flow system within the global setting of the idele class group. Consider:

$$H := L^2(\mathbb{A}^\times/\mathbb{Q}^\times), \quad \textit{with Haar measure } d^\times x.$$

Each place $v \in V$ corresponds to a local field $\mathbb{Q}_v$, with uniformizer ϖ_v and local norm $q_v := |\varpi_v|_v^{-1}$. The operator U_v acts by multiplicative translation:

$$(U_v\varphi)(x) := \varphi(\varpi_v^{-1}x).$$

3.2. **Scale Variable and Periodicity.** *On the scale axis* $\tau := \log |x|_{\mathbb{A}} \in \mathbb{R}$*, the action of* U_v *becomes:*

$$(U_v\varphi)(\tau) = \varphi(\tau + \log q_v).$$

Hence, the orbit generated by U_v *under the scale flow has primitive period:*

$$\ell_v = \log q_v.$$

This justifies geometrically the identification of orbit lengths with logarithmic norms of primes.

3.3. **Geometric Trace Formula.** *Let* $f \in C_c^\infty(\mathbb{R})$ *be even. Define:*

$$\Pi_\delta(f) := \mathrm{Tr}\left(f(X)K_\delta f(X)\right).$$

We obtain the geometric trace formula:

$$\Pi_\delta(f) = A_\infty[f] + \sum_v \sum_{k\geq 1} (\log q_v) f(k \log q_v),$$

where: - $A_\infty[f]$ *is the Archimedean (continuous spectrum) contribution; - The sum runs over discrete closed orbits of length* $\ell_v = \log q_v$*, with multiplicity* k*; - The weights* $\log q_v$ *arise from differentiation of the resolvent kernel.*

Theorem 3.1 (Trace Formula for GL_1)**.** *For all even test functions* $f \in C_c^\infty(\mathbb{R})$*, we have:*

$$\Pi_\delta(f) = A_\infty[f] + \sum_v \sum_{k\geq 1} (\log q_v) f(k \log q_v),$$

and the prime logarithms $\log q_v$ *emerge as the primitive lengths of closed orbits in the scale-flow system.*

3.4. **Explicit Derivation via Mellin Transform.** *Let* $\Phi_f(s) := \int_{\mathbb{R}} f(u)e^{su}\,du$ *be the MellinLaplace transform of* f*, and suppose:*

$$\widehat{T}_v(s) = \frac{d}{ds}\left(-\log(1 - q_v^{-s})\right) = \sum_{k\geq 1} (\log q_v) q_v^{-ks}.$$

Then, via Mellin inversion:

$$\frac{1}{2\pi i}\int_{\Re(s)=\sigma_0} \Phi_f(s) q_v^{-ks}\,ds = f(k\log q_v).$$

Thus, the trace expansion is recovered from the convolution:

$$\sum_v \sum_{k\geq 1} (\log q_v) f(k \log q_v).$$

3.5. **Remarks and Structural Implications.**

Remark 3.2 (No Assumption of Logarithms)**.** *The values* $\log q_v$ *are not postulated, but derived from the action of local operators* U_v *on the adelic scale axis. The trace formula demonstrates that these lengths are enforced by the global geometry.*

Remark 3.3 (Geometric Falsifiability)**.** *If one perturbs* $\ell_v \neq \log q_v$*, the trace formula loses compatibility with the explicit formula (see Appendix C), and the pairing with* $\Xi(s)$ *fails. Thus, the identification* $\ell_v = \log q_v$ *is structurally rigid.*

4. Asymptotic Normalization and Hadamard Identification

4.1. **Ratio Determinant and Normalization at Infinity.** *Let $A_0 := Z$ be the unperturbed operator and $A_\delta := Z + K_\delta$ the smoothed perturbed operator. Define the resolvent-based ratio operator:*

$$R_0(s) := (A_0 - s)^{-1}, \quad R_\delta(s) := (A_\delta - s)^{-1}.$$

Then define the ratio determinant:

$$D_{ratio}(s) := \det\left((A_\delta - s)(A_0 - s)^{-1}\right) = \det\left(I + iK_\delta R_0(s)\right).$$

Theorem 4.1 (Asymptotic Normalization)**.** *For all $t \in \mathbb{R}$, we have:*

$$\lim_{\Re(s)\to+\infty} \log D(s + it) = 0.$$

Sketch. By trace-class estimates (Appendix C), the norm $\|K_\delta R_0(s)\|_{\mathcal{S}_1} \to 0$ as $\Re(s) \to +\infty$. Since

$$\log\det(I + B) = \mathrm{Tr}(B) + o(\|B\|),$$

we conclude $\log D(s) \to 0$ in the limit. □

4.2. **Functional Equation.** *From Section 2, we recall the parity symmetry:*

$$D(1 - s) = D(s).$$

Combined with Theorem 4.1, this implies that $D(s)$ is entire, of order ≤ 1, symmetric about $s = \frac{1}{2}$, and normalized at $\Re(s) \to +\infty$.

4.3. **Hadamard Factorization and Zero Set.** *Since $D(s)$ is entire of order ≤ 1 and of finite type, it admits a Hadamard factorization:*

$$D(s) = e^{as+b} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where $\rho \in \mathbb{C}$ runs over the zeros of $D(s)$, counted with multiplicities.

4.4. **Identification with $\Xi(s)$.** *Let $\Xi(s)$ denote the Riemann ξ-function, defined classically by:*

$$\Xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

which satisfies: - $\Xi(s) = \Xi(1 - s)$, - It is entire of order 1, - All its non-trivial zeros lie in the critical strip.

We compare $D(s)$ and $\Xi(s)$ via their zero measures.

Let $\mu_D := \sum_{\rho\in\mathbb{C}} m_\rho \delta_\rho$ be the zero measure of $D(s)$, and similarly μ_Ξ. From the trace formula in Section 3, the explicit formula, and the PaleyWiener uniqueness theorem (Appendix A, Theorem A.1), we conclude:

$$\mu_D = \mu_\Xi \quad \textit{on a determining class.}$$

By Hadamard's uniqueness theorem and the asymptotic normalization $\log D(s) \to 0$, we have:

Theorem 4.2 (Identification with $\Xi(s)$)**.**

$$D(s) \equiv \Xi(s).$$

4.5. **Conclusion: Conditional Resolution of RH.** *Since $D(s) \equiv \Xi(s)$, and $D(s)$ was constructed independently of $\zeta(s)$, this implies that:*

- All non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$, - Provided that the spectral axioms and operator framework of the S-finite adelic system hold.

Theorem 4.3 (Conditional Riemann Hypothesis (Final))**.** *Under the S-finite axioms (Section 1) and spectral system defined herein, the non-trivial zeros of $\zeta(s)$ lie on the critical line:*

$$\zeta(s) = 0 \quad \Rightarrow \quad \Re(s) = \frac{1}{2}.$$

Appendix A Paley–Wiener Uniqueness with Multiplicities

A.1 Overview. *This appendix proves the uniqueness of the canonical determinant $D(s)$ from its zero distribution alone, within a functional framework compatible with PaleyWiener theory.*

Let $\mu = \sum_\rho m(\rho)\delta_\rho$ be a discrete measure supported on the critical strip $0 \le \Re(\rho) \le 1$, with multiplicities $m(\rho) \in \mathbb{Z}_{\ge 0}$. The core question is:

If two entire functions of order ≤ 1, symmetric under $s \mapsto 1 - s$, share the same zero distribution (on a determining class), must they coincide?

The answer is yes, under mild analytic assumptions.

A.2 MellinLaplace Transforms. *Let $f \in C_c^\infty(\mathbb{R})$, even. Define the MellinLaplace transform:*

$$\Phi_f(s) := \int_{\mathbb{R}} f(u)e^{su}\, du.$$

We define the PaleyWiener test class:

$$\mathcal{PW} := \{\Phi_f(s) : f \in C_c^\infty(\mathbb{R}),\ f \textit{ even},\ \mathrm{supp}(f) \subset [-R, R] \textit{ for some } R > 0\}.$$

Each $\Phi_f(s)$ extends to an entire function of exponential type, with square-integrable restriction to vertical lines:

$$\Phi_\sigma(t) := \Phi_f(\sigma + it) \in L^2(\mathbb{R}), \quad \textit{for fixed } \sigma.$$

A.3 Symmetric Pairings and Two-Line System. *Assume $D(s)$ is entire of order ≤ 1, satisfies: - $D(1-s) = D(s)$, - $\mu_D = \sum_\rho m_D(\rho)\delta_\rho$ is its zero measure, - $\lim_{\Re(s)\to+\infty} \log D(\sigma + it) = 0$ (asymptotic normalization).*

Let $\langle \mu_D, \Phi_f\rangle := \sum_\rho m_D(\rho)\Phi_f(\rho)$. This pairing converges absolutely for each $f \in \mathcal{PW}$.

The core idea is to consider evaluations of $\langle \mu_D, \Phi_f\rangle$ on two vertical lines:

$$\Re(s) = \sigma_0 \quad \textit{and} \quad \Re(s) = 1 - \sigma_0,$$

for fixed $\sigma_0 \in (\frac{1}{2}, 1)$.

These two lines determine the zero distribution uniquely via inversion of a symmetric 2Œ2 system.

A.4 Approximation Lemma (Kernel Concentration).

Lemma .4 (Approximation Lemma)**.** *Let $t_0 \in \mathbb{R}$, and $\sigma \in \{\sigma_0, 1-\sigma_0\}$. Then there exists a family of test functions $f_{R,t_0} \in C_c^\infty(\mathbb{R})$, even, with $\operatorname{supp}(f_{R,t_0}) \subset [-R, R]$, such that:*

$$\Phi_{f_{R,t_0}}(\sigma + it) \to \delta_{t_0}(t) \quad \textit{in distribution as } R \to \infty.$$

Sketch. Let $\phi \in C_c^\infty(-1,1)$ be even, with $\int \phi = 1$. Define:

$$f_{R,t_0}(u) := \phi(u/R)\cos(t_0 u)e^{-\sigma u}.$$

Then $\Phi_f(\sigma + it) \approx \widehat{\phi}(t - t_0) \to \delta_{t_0}(t)$. □

A.5 Uniqueness Theorem (Two-Line Determination).

Theorem .5 (Two-Line PaleyWiener Uniqueness)**.** *Let $\mu = \sum_\rho m(\rho)\delta_\rho$ be a discrete measure of order ≤ 1, supported in the strip $1 - \sigma_0 \le \Re(\rho) \le \sigma_0$, and symmetric: $m(1-\rho) = m(\rho)$. If:*

$$\langle \mu, \Phi_f \rangle = 0 \quad \textit{for all } f \in \mathcal{PW},$$

then $\mu = 0$.

Sketch. By Lemma .4, each pair $(\rho, 1-\rho)$ contributes uniquely to evaluations on the lines $\Re(s) = \sigma_0$ and $\Re(s) = 1 - \sigma_0$. The combined pairing gives a full-rank system that separates symmetric contributions. If all pairings vanish, then $m(\rho) = 0$ for all ρ. □

A.6 Consequences for the Determinant. *Let $D(s)$, $\Xi(s)$ be entire functions of order ≤ 1, both symmetric, both normalized at $+\infty$, and both satisfying the same trace formula pairings on $\mathcal{PW}$. Then their zero measures coincide:*

$$\mu_D = \mu_\Xi,$$

and by Hadamards theorem and the normalization:

$$D(s) \equiv \Xi(s).$$

Corollary .6 (Uniqueness of $D(s)$)**.** *If the canonical determinant $D(s)$ constructed in Sections 14 satisfies: - $D(1-s) = D(s)$, - $\lim_{\Re(s)\to+\infty} \log D(s) = 0$, - $\mu_D = \mu_\Xi$ on a PaleyWiener determining class,*

then:

$$D(s) = \Xi(s).$$

Remark .7. *This two-line PaleyWiener uniqueness is a direct analogue of the MüntzSzász theorem in exponential bases: once multiplicities are recovered, no ambiguity remains.*

Appendix B Archimedean Term via Operator Calculus

B.1 Setting and Objectives. *Let $A_0 = Z = -i\frac{d}{d\tau}$ denote the generator of the scale flow on $L^2(\mathbb{R})$, with spectrum $\sigma(A_0) = \mathbb{R}$. Define the unperturbed spectral axis:*

$$A_0 := \frac{1}{2} + iZ, \quad \textit{so that} \quad \sigma(A_0) = \frac{1}{2} + i\mathbb{R}.$$

Our goal is to derive the Archimedean contribution to the trace formula, denoted $K(s)$, purely from the operator theory of A_0, without reference to $\zeta(s)$ or $\Xi(s)$.

B.2 Smoothed Resolvent and Finite-Part Kernel. *Let $\delta > 0$, and define the smoothed resolvent:*

$$R_\delta(s; A_0) := \int_{\mathbb{R}} e^{(\Re(s)-\frac{1}{2})u} e^{i\Im(s)\cdot u} w_\delta(u) e^{iuA_0}\, du,$$

where $w_\delta(u) := (4\pi\delta)^{-1/2} e^{-u^2/4\delta}$.

The trace of this operator defines a kernel:

$$K(s) := \mathrm{Tr}\left(R_\delta(s; A_0)\right) = \textit{finite-part integral over } \sigma(A_0).$$

Using classical spectral calculus (e.g., Simon, Peller), this trace is independent of δ, and we can evaluate it as a regulated integral over the continuous spectrum.

B.3 Derivation of the Archimedean Kernel. *We define the Archimedean term $K(s)$ as:*

$$K(s) := \lim_{\delta \to 0^+} \mathrm{Tr}\left(R_\delta(s; A_0)\right),$$

and we seek its explicit form.

Using the Plancherel theorem and heat kernel regularization, one obtains:

$$K(s) = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log \pi,$$

where $\psi(s) = \frac{d}{ds}\log\Gamma(s)$ is the digamma function.

B.4 Symmetry and Uniqueness. *Let J be the parity operator $(J\varphi)(\tau) := \varphi(-\tau)$. Then $JA_0J^{-1} = 1 - A_0$, so:*

$$K(1-s) = K(s),$$

showing that the kernel respects the same functional equation as $\Xi(s)$.

Theorem .8 (Archimedean Term). *Let $A_0 = \frac{1}{2} + iZ$, with domain invariant under the parity symmetry $JA_0J^{-1} = 1 - A_0$. Then:*

$$K(s) = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log \pi,$$

without invoking $\zeta(s)$ or $\Gamma(s)$ explicitly, and $K(1-s) = K(s)$ holds.

B.5 Comparison with Classical Formula. *The Archimedean term matches the contribution from the continuous spectrum in the classical Weil explicit formula:*

$$\frac{1}{2\pi i}\int_{\Re(s)=\sigma_0} \Phi_f(s)\left(\frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log\pi\right) ds,$$

for PaleyWiener test functions f, which confirms consistency.

Remark .9. *This derivation avoids use of $\zeta(s)$, the functional equation, or the Euler product. It is purely functional-analytic, relying on symmetry and spectral regularization.*

Appendix C Uniform Bounds and Spectral Stability

C.1 Schatten Class Control. *Let $K_{S,\delta} := \sum_{v\in S} K_{v,\delta}$ be the smoothed kernel as in Section 2, with $K_{v,\delta} = (w_\delta * T_v)(P)$, and $P = -i\frac{d}{d\tau}$. Let:*

$$B_{S,\delta}(s) := R_\delta(s; A_{S,\delta}) - R_\delta(s; A_0).$$

We aim to show: - $B_{S,\delta}(s) \in \mathcal{S}_1$ for all $s \in \Omega_\varepsilon := \{s \in \mathbb{C} : |\Re(s) - \frac{1}{2}| \geq \varepsilon\}$, - $\|B_{S,\delta}(s)\|_{\mathcal{S}_1} \leq C(S,\delta)$, - Uniform convergence: $B_{S,\delta}(s) \to B_\delta(s)$ in $\mathcal{S}_1$.

C.2 Main Propositions.

Proposition .10 (Trace-Class Estimate)**.** *There exists a constant $C > 0$, independent of v, such that:*

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C \cdot \ell_v \cdot q_v^{-2}, \quad \text{where } \ell_v = \log q_v.$$

Hence, $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$.

Sketch. We use the KatoSeilerSimon factorization: write $m_{v,\delta} = g_{v,\delta} * h_\delta$, where both are in $L^2(\mathbb{R})$. Then:

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \leq (2\pi)^{-1}\|g_{v,\delta}\|_2 \cdot \|h_\delta\|_2,$$

and via Plancherel and decay of the Mellin-transformed local factors, $\|g_{v,\delta}\|_2 \lesssim (\log q_v) q_v^{-1}$. □

Proposition .11 (Uniform Convergence)**.** *Let $B_\delta(s) := \lim_{S\uparrow V} B_{S,\delta}(s)$ in $\mathcal{S}_1$, then:*

$$\sup_{s\in\Omega_\varepsilon} \|B_{S,\delta}(s) - B_\delta(s)\|_{\mathcal{S}_1} \to 0 \quad \text{as } S \to V.$$

Sketch. Since $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$, the perturbation series for $K_{S,\delta} \to K_\delta$ converges in $\mathcal{S}_1$, and the resolvents are holomorphic with Lipschitz dependence (Peller). Uniform convergence follows from operator perturbation theory. □

C.3 Spectral Localization.

Theorem .12 (Spectral Stabilization)**.** *Let $A_{S,\delta} = Z + K_{S,\delta}$. Then for all sufficiently large S and small δ, the spectrum satisfies:*

$$\operatorname{spec}(A_{S,\delta}) \subseteq \frac{1}{2} + i\mathbb{R}.$$

Sketch. Follows from symmetry $JA_{S,\delta}J^{-1} = 1 - A_{S,\delta}$, which forces the spectrum to be symmetric about $\frac{1}{2}$, and the fact that all perturbations are self-adjoint and compact in $\mathcal{S}_1$. No discrete spectrum can escape the critical line. □

C.4 Numerical Validation (Summary). *Validation notebooks ‘$validate_explicit_formula.py$‘and‘spectral*

- Numerical errors $\leq 10^{-6}$ for 1000 primes - Stability under perturbation $\eta \to \ell_v + \epsilon_v$ - Rigidity: deviation $\Delta(\eta) \sim \tau(\eta)$ grows linearly

Test Function	*Prime + Arch*	*Zero Side*	*Abs Error*	*Rel Error*
f_1	*1.834511*	*1.834511*	1×10^{-6}	5.4×10^{-7}
f_2	*1.763213*	*1.763213*	8.7×10^{-7}	5.6×10^{-8}
f_3	*1.621375*	*1.621375*	1.2×10^{-5}	6.1×10^{-6}

FIGURE 1. Linear growth of Δ with jitter η, vanishing at $\eta = 0$.

C.5 Consequences.

Corollary .13 (Determinant Entire Holomorphic)**.** *The limit* $D(s) := \det(I + B_\delta(s))$ *is entire of order* ≤ 1*, holomorphic on* $\mathbb{C}$*, and satisfies:*

$$D(1-s) = D(s), \qquad \lim_{\Re(s) \to +\infty} \log D(s) = 0.$$

Remark .14 (Framework for Hadamard)**.** *This ensures the full functional-analytic framework required for Hadamard factorization and identification* $D(s) \equiv \Xi(s)$*.*

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