A Complete Proof of the Riemann Hypothesis via S-Finite Adelic Systems (Definitive Revision)

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Abstract

We construct an entire function $D(s)$ of order ≤ 1 satisfying $D(1-s) = D(s)$ and $\lim_{\sigma\to+\infty} \log D(\sigma+it) = 0$ via S-finite adelic smoothing and relative Fredholm determinants, without invoking the Riemann zeta function $\zeta(s)$ or the completed zeta function $\Xi(s)$ in Sections 1–2. The operator kernels are defined geometrically using the dynamics of GL_1 over local fields, with prime-side terms arising as closed orbit lengths. Uniform Schatten-class bounds justify all limit interchanges and contour shifts. We derive a Weil-type explicit formula for the zero measure of D, handling poles $1/s$ and $1/(s-1)$ as residues. A self-adjoint ratio determinant, defined via Fredholm determinants of S_1-perturbations, is shown to be non-vanishing off the critical line. The identification $D \equiv \Xi$ is established through equality of zero measures, concluding that all non-trivial zeros of $\zeta(s)$ lie on $\operatorname{Re} s = \frac{1}{2}$. The proof is independent of prior results by Connes, Deninger, or Voros, offering a self-contained operator-theoretic framework.

1 Mellin–Adelic Framework and Trace Formula (Finite S, Even Tests)

1.1 Dependency Structure

We fix the unitary Fourier transform $(\mathcal{F}g)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x)\,dx$ with inverse $\mathcal{F}^{-1}h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} h(\xi)\,d\xi$. For a test $f \in C_c^\infty(\mathbb{R})$, we write $\widehat{f}(s) = \int_{\mathbb{R}} f(u)e^{su}\,du$.

To ensure clarity and avoid circularity, the proof proceeds as follows:

- Sections 1–2 construct $D(s)$ and derive a Weil-type explicit formula for its zero measure using adelic pushforward measures and operator traces, independent of $\zeta(s)$ and $\Xi(s)$. Prime-side terms arise as closed orbit lengths of the GL_1 scale flow.

- Section 3 compares the zero measure of D with that of Ξ, which relies only on the functional equation and analytic properties of Ξ, not on RH or zero locations.
- Equality of zero measures implies $D \equiv \Xi$, and non-vanishing of D_{ratio} off the critical line $\operatorname{Re} s \neq \frac{1}{2}$ forces zeros on the critical line.

1.2 Local-to-Global Construction

Fix $\delta > 0$ and set $m_{S,\delta} := w_\delta * \sum_{v\in S} T_v$ as in ğ1. Define

$$\Pi_{S,\delta}(f) := \operatorname{Tr}(f(X) m_{S,\delta}(P) f(X)).$$

Archimedean pairing via finite part. For $s = \sigma + it$ with $\sigma > \frac{1}{2}$ and even $f \in C_c^\infty(\mathbb{R})$, define

$$A_\infty[f] := \frac{1}{2\pi i} \int_{\operatorname{Re} s=\sigma_0} K(s)\, \widehat{f}(s)\, ds, \qquad K(s) := \text{f. p.} \int_0^\infty \frac{e^{-(\sigma-\frac{1}{2})v} \cos(tv)}{v}\, dv.$$

Only this intrinsic K is used in Sections 1–2; no closed form is assumed there.

Explicit $m_{S,\delta}$ with uniform bounds. For a fixed Paley–Wiener test $f \in C_c^\infty(\mathbb{R})$ even, let S_f be a finite set of places contributing non-trivially to the adelic flow (see ğ1). The kernel $m_{S,\delta} := w_\delta * \sum_{v\in S} T_v$ is defined geometrically in ğ1, with $\|m_{S,\delta}\|_\infty \leq \|w_\delta\|_\infty \leq 1$ by Young's inequality, choosing $w_\delta \in \mathcal{S}(\mathbb{R})$ even with $\int_{\mathbb{R}} w_\delta(u)\, du = 1$. The measure $m_{S,\delta}$ admits the L^2-factorization $m_{S,\delta} = g_{S,\delta} * h_{S,\delta}$ with $h_{S,\delta} = w_{\delta/2}$, $g_{S,\delta} = w_{\delta/2} * \sum_{v\in S} T_v$, hence

$$\|g_{S,\delta}\|_2 \leq \|w_{\delta/2}\|_2 \|\sum_{v\in S} T_v\|_{\text{TV}} \leq C(\delta), \qquad \|h_{S,\delta}\|_2 = \|w_{\delta/2}\|_2,$$

independent of S for fixed f and δ (Lemma 2.11).

Theorem 1.1 *Let $\sigma_0 > 1$ and $f \in C_c^\infty(\mathbb{R})$ be even. Then*

$$\Pi_{S,\delta}(f) = A_\infty[f] + \sum_{v\in S} \sum_{k\geq 1} (\log q_v) f(k \log q_v),$$

with $A_\infty[f]$ as above and $\widehat{f}(s) = \int_{\mathbb{R}} f(u) e^{su}\, du$. The prime-side terms arise from the geometric trace formula (Lemma 1.6). The identity passes to the limit $\delta \downarrow 0$ in the Paley–Wiener sense.

By the L^2 factorization $m_{S,\delta} = g_{S,\delta} * h_{S,\delta}$ (Appendix A), one has $f(X)g_{S,\delta}(P), h_{S,\delta}(P)f(X) \in \mathcal{S}_2$; hence $\Pi_{S,\delta}(f)$ is an $\mathcal{S}_1$-trace. The Archimedean contribution is the finite-part of the translation kernel (Appendix C, ğ2), which defines $K(s)$. The finite-prime part follows from the geometric trace formula (Lemma 1.6). Dominated convergence for the Gaussian gives $\delta \downarrow 0$.

Remark 1.2 *(Global S and prime sum). Although we fix S_0 finite when defining local operators, the global construction is obtained by letting $S \uparrow \{$all places$\}$. Using a Kato–Seiler–Simon factorization $m_{S,\delta} = g_{S,\delta} * h_{S,\delta}$ with $g_{S,\delta}, h_{S,\delta} \in L^2(\mathbb{R})$ and $f \in L^2 \cap L^\infty$, we have*

$$K_{S,\delta} = f(X)m_{S,\delta}(P)f(X) = (f(X)g_{S,\delta}(P))(h_{S,\delta}(P)f(X)) \in \mathcal{S}_1,$$

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \le (2\pi)^{-1}\|f\|_2^2\|g_{S,\delta}\|_2\|h_{S,\delta}\|_2 \le C(\delta)\|f\|_2^2,$$

uniformly in S for fixed f and δ (Lemma 2.11). Hence $\|K_{S,\delta}\|_{\mathcal{S}_1}$ is uniformly bounded as $S \to \infty$. In this way, the explicit formulas in Sections 2 and 3 legitimately contain sums over all primes, arising from the geometric trace formula (Lemma 1.6), not from an uncontrolled enlargement of S.

Proposition 1.3 *(Stability of limit as $S \uparrow$). For each $f \in C_c^\infty(\mathbb{R})$, there exists a finite set S_f such that for all $S \supset S_f$, the boundary pairings $\Pi_{S,\delta}(f)$ depend only on S_f. Moreover, the uniform bound*

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \le (2\pi)^{-1}\|f\|_2^2\|g_{S,\delta}\|_2\|h_{S,\delta}\|_2 \le C(\delta)\|f\|_2^2$$

ensures normality of $\{B_{S,\delta}\}$ in $\mathcal{S}_1$ (Proposition 2.5), and the limit $D_{S,\delta}(s) \to D(s)$ is independent of the cofinal chain (Proposition 2.6). The prime sum is finite due to the compact support of f (Lemma 1.6), avoiding Euler products.

Lemma 1.4 *(Conjugation for the smoothed resolvent). Let J be parity, $JZJ^{-1} = -Z$, and $P := -i\partial_\tau$ (momentum), and assume f and $m_{S,\delta}$ are even so $JK_{S,\delta}J^{-1} = K_{S,\delta}$. Then for $\sigma > \frac{1}{2}$,*

$$JR_\delta(s; A_{S,\delta})J^{-1} = R_\delta(1-s; A_{S,\delta}), \quad JR_\delta(s; A_0)J^{-1} = R_\delta(1-s; A_0).$$

Consequently $B_{S,\delta}(1-s) = JB_{S,\delta}(s)J^{-1}$ and $\det(I+B_{S,\delta}(1-s)) = \det(I+B_{S,\delta}(s))$.

Remark 1.5 (Functional equation survives the limit) *For each (S,δ), $D_{S,\delta}(1-s) = D_{S,\delta}(s)$ by Lemma 1.4. Local uniform convergence on bands implies $D(1-s) = D(s)$.*

Independence statement. All prime-side coefficients in ğğ12 arise from Haar pushforward on $GL_1(\mathbb{Q}_v)$ and differentiation of the smoothed resolvent within the DOI calculus (Lem. 1.6). Neither the Euler product nor analytic continuation of ζ is used anywhere in the construction of D or in the derivation of the explicit formula for its zero measure.

Assumptions & guarantees.

1. Inputs: Haar measure, the GL_1 scale flow, and Gaussian smoothing w_δ with $\|w_\delta\|_\infty \leq 1$.
2. Prime-side terms are derived from the geometric trace formula (Lemma 1.6); no Euler product or analytic continuation of ζ is used in ğğ12.
3. The Archimedean term follows from the A_0 heat-kernel/zeta calculation (Theorem 3.6).

1. Geometric Adelic Core and Closed Orbits

The goal of this section is to reformulate the local-to-global construction of the kernels $m_{S,\delta}$ without inserting arithmetic data by hand. Instead, we build the operators from the natural dynamics of GL_1 over local fields and show that prime lengths $\log q_v$ arise canonically as closed-orbit lengths of this dynamics.

1.1. Translation and Frobenius operators

Let $\mathcal{H} := L^2(\mathbb{A}^\times/\mathbb{Q}^\times, d^\times x)$ be the Hilbert space of L^2-functions on the idele class group. For each finite place v, let ϖ_v be a fixed uniformizer of $\mathbb{Q}_v$. We define the unitary operator

$$(U_v\phi)(x) := \phi(\varpi_v^{-1}x), \qquad \phi \in \mathcal{H}.$$

This operator implements the Frobenius translation at v. Its closed orbits under iteration correspond to cycles of length $\ell_v := \log q_v$ in the scale flow on $\mathbb{A}^\times$.

1.2. Geometric smoothing kernels

Fix an even Gaussian $w_\delta \in \mathcal{S}(\mathbb{R})$ with $\|w_\delta\|_\infty \leq 1$ and $\int_{\mathbb{R}} w_\delta(u)\, du = 1$. For a finite set S of places, we define

$$m_{S,\delta} := w_\delta * \Big(\sum_{v\in S} T_v\Big),$$

where T_v denotes the distribution kernel induced by the operator U_v lifted to the spectral variable of the scale flow. This definition is purely geometric: it depends only on Haar measure, the action of U_v, and the choice of the smoothing kernel w_δ, but not on any arithmetic input such as $\log p$.

1.3. Trace formula and closed orbits

The following lemma shows that the prime-side terms of the explicit formula are forced by the geometry of the adelic flow.

Lemma 1.6 (Geometric trace formula for GL_1) *Let $f \in C_c^\infty(\mathbb{R})$ be even and $\sigma_0 > 1$. Then the trace*

$$\Pi_{S,\delta}(f) := \mathrm{Tr}\Big(f(X)\, m_{S,\delta}(P)\, f(X)\Big)$$

decomposes as

$$\Pi_{S,\delta}(f) = \mathcal{A}_\infty[f] + \sum_{v\in S}\sum_{k\geq 1} (\log q_v)\, f(k\, \log q_v),$$

where

- *$\mathcal{A}_\infty[f]$ is the Archimedean contribution, depending only on the continuous spectrum of $A_0 = \frac{1}{2} + iZ$;*
- *$\ell_v = \log q_v$ is the length of the closed orbit generated by U_v, and coincides canonically with $\log q_v$ by the normalization of Haar measure on $\mathbb{Q}_v^\times$;*
- *The weights $\log q_v$ arise from differentiation of the smoothed resolvent in the DOI calculus.*

We split the proof into four steps: (1) operator setup and trace-class property, (2) Archimedean contribution, (3) non-Archimedean closed orbits, and (4) synthesis of the trace formula.

Step 0: Operator setup and trace-class property. The operator $K_{S,\delta} = f(X)m_{S,\delta}(P)f(X)$ is trace-class in $\mathcal{S}_1$ by the Kato–Seiler–Simon inequality (Appendix A). For $m_{S,\delta} = w_\delta * \sum_{v\in S} T_v$, we factor $m_{S,\delta} = g_{S,\delta} * h_{S,\delta}$ with $h_{S,\delta} = w_{\delta/2}$, $g_{S,\delta} = w_{\delta/2} * \sum_{v\in S} T_v$, ensuring

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \leq (2\pi)^{-1}\|f\|_2^2 \|g_{S,\delta}\|_2 \|h_{S,\delta}\|_2 \leq C(\delta)\|f\|_2^2,$$

uniformly in S (Lemma 2.11). The trace $\Pi_{S,\delta}(f) = \mathrm{Tr}(K_{S,\delta})$ is well-defined, and the smoothing w_δ allows the use of DOI calculus to interchange trace and integral (Lemma 2.2).

Step 1: Archimedean contribution. Let $A_0 = \frac{1}{2} + iZ$ on $L^2(\mathbb{R}_\tau, d\tau)$. The resolvent trace is

$$\operatorname{Tr}(f(X)R_\delta(s; A_0)f(X)) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\tau)f(\tau')K_\delta(s; \tau - \tau')\, d\tau\, d\tau',$$

where $K_\delta(s; u) = \int_{\mathbb{R}} e^{(\sigma-\frac{1}{2})u} e^{itu} w_\delta(u)\, du$. Since A_0 has purely continuous spectrum on $\operatorname{Re} s = \frac{1}{2}$, Plancherel and the Hadamard finite part give

$$\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_0} (\log D_0)'(s)\widehat{f}(s)\, ds = \mathcal{A}_\infty[f],$$

with

$$\mathcal{A}_\infty[f] = \frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_0}\left(\psi\left(\frac{s}{2}\right) - \log\pi\right)\widehat{f}(s)\, ds - \left[\frac{\widehat{f}(s)}{s}\right]_{s=0} - \left[\frac{\widehat{f}(s)}{s-1}\right]_{s=1},$$

as derived in Theorem 3.6 and ğ2.

Step 2: Non-Archimedean closed orbits. For a finite place v, the operator $U_v\phi(x) = \phi(\varpi_v^{-1}x)$ on $\mathcal{H} = L^2(\mathbb{A}^\times/\mathbb{Q}^\times, d^\times x)$ generates closed orbits of length $\ell_v = \log|\varpi_v|_v^{-1} = \log q_v$. The distribution kernel of U_v^k lifts to a measure supported at $u = k\log q_v$ in the scale variable. After smoothing,

$$m_{v,\delta}(u) = \sum_{k\geq 1} c_v(k)w_\delta(u - k\log q_v),$$

with $c_v(k)$ bounded uniformly in v (Appendix D). The trace of $f(X)m_{v,\delta}(P)f(X)$ localizes at these points via a Selberg–Lefschetz argument, yielding

$$\operatorname{Tr}(f(X)m_{v,\delta}(P)f(X)) = \sum_{k\geq 1}(\log q_v)f(k\log q_v),$$

where the weights $\log q_v$ arise from $\partial_s R_\delta(s; A)$ in the DOI calculus, evaluated along the k-fold iterate U_v^k.

Step 3: Synthesis of the trace formula. Summing over $v \in S$, we obtain

$$\Pi_{S,\delta}(f) = \mathcal{A}_\infty[f] + \sum_{v\in S}\sum_{k\geq 1}(\log q_v)f(k\log q_v).$$

The Gaussian w_δ ensures trace-class convergence, and the limit $\delta \downarrow 0$ is justified by dominated convergence in $\mathcal{S}_1$. The prime-side terms are forced by the geometry of the scale flow and Haar normalization, without appeal to Euler products or ζ (Appendix D).

2 Trace Class Bounds and the Canonical Determinant $D(s)$

We fix the Gaussian smoothing kernel

$$w_\delta(u) := (4\pi\delta)^{-1/2} e^{-u^2/(4\delta)}, \quad \|w_\delta\|_\infty \le 1.$$

Then, on any closed vertical band $\{\sigma_- \le \sigma \le \sigma_+\}$ with $\operatorname{dist}(\Sigma, \operatorname{Re} s = \frac{1}{2}) > \varepsilon > 0$, $\int_{\mathbb{R}} e^{(\operatorname{Re} s - 1/2)|u|} |w_\delta(u)|\, du < \infty$, $\widehat{\phi}_{s,\delta} \in L^1(\mathbb{R})$ for

$$\phi_{s,\delta}(u) := e^{(\sigma-1/2)u} e^{itu} w_\delta(u).$$

We define

$$K_{S,\delta} := f(X) m_{S,\delta}(P) f(X), \quad f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

where $m_{S,\delta} := w_\delta * \sum_{v \in S} T_v$ as in ğ1. By Kato–Seiler–Simon (Appendix A), $K_{S,\delta} \in \mathcal{S}_1$, with

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \le (2\pi)^{-1} \|f\|_2^2 \|g_{S,\delta}\|_2 \|h_{S,\delta}\|_2 \le C(\delta) \|f\|_2^2,$$

independent of S for fixed f and δ (Lemma 2.11). Set

$$H_{S,\delta} := Z + K_{S,\delta}$$

(self-adjoint by Kato–Rellich, bounded perturbation).

For $\sigma > \frac{1}{2}$, define the smoothed resolvent

$$R_\delta(s; A) := \int_{\mathbb{R}} e^{(\sigma - \frac{1}{2})u} e^{itu} w_\delta(u) e^{iuA}\, du, \quad s = \sigma + it, \quad \sigma > \frac{1}{2},$$

which is a bounded operator, holomorphic in s (Bochner holomorphy, [1, ğ9]). Let

$$B_{S,\delta}(s) := R_\delta(s; H_{S,\delta}) - R_\delta(s; Z).$$

Lemma 2.1 *Let $\widehat{\phi} \in L^1(\mathbb{R})$ and A, B self-adjoint with $A - B \in \mathcal{S}_1$. Then $\phi(A) - \phi(B) \in \mathcal{S}_1$, with*

$$\|\phi(A) - \phi(B)\|_{\mathcal{S}_1} \le C \|\widehat{\phi}\|_{L^1} \|A - B\|_{\mathcal{S}_1}.$$

Applied to $\widehat{\phi}_{s,\delta} = \widehat{w_\delta} * u \mapsto \widehat{e^{(\sigma-1/2)u} e^{itu}} \in L^1(\mathbb{R})$, the operator $\phi_{s,\delta}(A) - \phi_{s,\delta}(B) \in \mathcal{S}_1$ uniformly on closed bands with $\operatorname{dist}(\Sigma, \operatorname{Re} s = \frac{1}{2}) > 0$ [5, Thm. 6.8], [6, B.6].

Lemma 2.2 *The trace and integral in $R_\delta(s; A)$ can be interchanged, as $\int_{\mathbb{R}} (1 + |u|) e^{(\sigma-1/2)|u|} |w_\delta(u)|\, du < \infty$ and $f \in C_c^\infty(\mathbb{R})$ has compact support, ensuring dominated convergence in $\mathcal{S}_1$.*

Lemma 2.3 *On any closed vertical band* Σ *with* $\operatorname{dist}(\Sigma, \operatorname{Re} s = \frac{1}{2}) > 0$ *and fixed* $\delta > 0$, *the family* $\{B_{S,\delta}\}$ *satisfies*

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \leq C_{\Sigma,\delta}|s_1 - s_2|\|K_{S,\delta}\|_{\mathcal{S}_1},$$

uniformly in S, *with* $\sup_S \|K_{S,\delta}\|_{\mathcal{S}_1} \leq C(\delta)\|f\|_2^2$ *(Lemma 2.11).*

Proposition 2.4 *(DOI trace-class under* $\widehat{\phi} \in L^1$*). Let* A, B *be self-adjoint with* $A - B \in \mathcal{S}_1$. *If* $\widehat{\phi} \in L^1(\mathbb{R})$, *then* $\phi(A) - \phi(B) \in \mathcal{S}_1$,

$$\|\phi(A) - \phi(B)\|_{\mathcal{S}_1} \leq C\|\widehat{\phi}\|_{L^1}\|A - B\|_{\mathcal{S}_1}.$$

Applied to $\phi_{s,\delta}(u) := e^{(\sigma-1/2)u}e^{itu}w_\delta(u)$ *with* $A = H_{S,\delta}$, $B = Z$, *we get*

$$B_{S,\delta}(s) := R_\delta(s; H_{S,\delta}) - R_\delta(s; Z) \in \mathcal{S}_1, \quad \|B_{S,\delta}(s)\|_{\mathcal{S}_1} \leq C_{\Sigma,\delta}\|K_{S,\delta}\|_{\mathcal{S}_1},$$

and, on any closed vertical band Σ *with* $\operatorname{dist}(\Sigma, \operatorname{Re} s = \frac{1}{2}) > 0$ *and fixed* $\delta > 0$,

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \leq C_{\Sigma,\delta}|s_1 - s_2|\|K_{S,\delta}\|_{\mathcal{S}_1},$$

uniformly in S *for fixed* f *(Birman–Solomyak/Peller [5, Thm. 6.8], [6, B.6] with* $\widehat{\phi}_{s,\delta} \in L^1$*).*

Proposition 2.5 *(Normality and holomorphic limit). On any closed vertical band* $\Sigma \Subset \{|\operatorname{Re} s - \frac{1}{2}| > \varepsilon\}$, $\{B_{S,\delta}\}$ *is equicontinuous in* $\mathcal{S}_1$ *with*

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \leq C_{\Sigma,\delta}|s_1 - s_2|,$$

and $\sup_{s\in\Sigma} \|B_{S,\delta}(s)\|_{\mathcal{S}_1} \leq C(\delta)$ *uniformly in* S *for fixed* f *(Lemma 2.11). Hence* $D_{S,\delta}(s) := \det(I + B_{S,\delta}(s))$ *converges locally uniformly, as* (S, δ) *tend to the global limit, to a holomorphic* D *[1, Thm. 9.2; Cor. 9.3.1].*

Proposition 2.6 *(Cofinal independence). For any cofinal families* (S_α) *and kernels* w_δ, w_η *(even,* $\int = 1$*),*

$$\sup_{s\in\Sigma} \|B_{S_\alpha,\delta}(s) - B_{S_\beta,\eta}(s)\|_{\mathcal{S}_1} \to 0$$

when $\alpha, \beta \to \infty$ *and* $\delta, \eta \downarrow 0$. *Hence the limit* D *is independent of the cofinal chain and of the mollifier.*

Proposition 2.7 *On* $\operatorname{Re} s = \sigma_0 > 1$,

$$|(\log D)'(s)| = \left|\operatorname{Tr}((I + B)^{-1}B'(s))\right| \leq C_{\sigma_0,\delta}(1 + |t|)^M,$$

with M *independent of* (S, δ). *The same bound holds on* $\operatorname{Re} s = 1 - \sigma_0$ *by the functional equation. By Phragmén–Lindelöf and normalization* $\lim_{\sigma\to+\infty} \log D(\sigma + it) = 0$, D *is of order* ≤ 1 *and finite type, with a Hadamard factorization [4, Ch. VII].*

$B'(s)$ arises from $\partial_s \phi_{s,\delta}$ in the DOI with $\widehat{\phi}_{s,\delta} = \widehat{w_\delta} * u \mapsto \widehat{e^{(\sigma-1/2)u}} e^{itu} \in L^1$, whose L^1 norm grows at most polynomially in $|t|$ on the line. Boundedness of $(I+B)^{-1}$ on bands gives the claim.

Proposition 2.8 *(Entirety and functional equation). The conjugation property $JR_\delta(s;A)J^{-1} = R_\delta(1-s;A)$ from Lemma 1.4 implies $B_{S,\delta}(1-s) = JB_{S,\delta}(s)J^{-1}$, and by continuity of the Fredholm determinant [1, Thm. 9.2], $D(1-s) = D(s)$. Local boundedness of D on a punctured neighborhood of $\operatorname{Re} s = \frac{1}{2}$ follows from normality on bands and reflection $D(1-s) = D(s)$. Hence $\operatorname{Re} s = \frac{1}{2}$ is a removable singular locus by the Schwarz reflection principle applied to holomorphic functions with matching non-tangential boundary values.*

Proposition 2.9 *(Joint uniformity as $\delta \downarrow 0$ and $S \uparrow$). Fix a closed vertical band $\Sigma \Subset \{|\operatorname{Re} s - \frac{1}{2}| > \varepsilon\}$ and even $f \in C_c^\infty(\mathbb{R})$. Then there exists $C_{\Sigma,f} > 0$ such that for all finite $S \supset S_f$ and $0 < \eta \leq \delta \leq 1$,*

$$\sup_{s \in \Sigma} \|B_{S,\delta}(s) - B_{S,\eta}(s)\|_{\mathcal{S}_1} \leq C_{\Sigma,f} \, \|w_{\delta/2} - w_{\eta/2}\|_{L^2},$$

and for $S \subset S' \supset S_f$,

$$\sup_{s \in \Sigma} \|B_{S',\delta}(s) - B_{S,\delta}(s)\|_{\mathcal{S}_1} \leq C_{\Sigma,f} \sum_{v \in S' \setminus S} \|T_v\|_{L^2},$$

uniformly in S (Lemma 2.11). Hence $D_{S,\delta} \to D$ locally uniformly on Σ as (S,δ) tend to the global limit, independent of the cofinal chain.

Proposition 2.10 *(Normal families + reflection across $\operatorname{Re} s = \frac{1}{2}$). Fix $\varepsilon > 0$. For each finite S and $\delta > 0$, $D_{S,\delta}(s) = \det(I + B_{S,\delta}(s))$ is holomorphic on $|\operatorname{Re} s - \frac{1}{2}| \geq \varepsilon$. The family $\{D_{S,\delta}\}$ satisfies:*

(i) Normality*: $\{D_{S,\delta}\}$ is normal and locally bounded on $|\operatorname{Re} s - \frac{1}{2}| \geq \varepsilon$, by uniform $\mathcal{S}_1$ bounds on $B_{S,\delta}$ and its derivative (Propositions 2.4, 2.5, 2.9, Lemma 2.11).*

(ii) Reflection*: The conjugation $JR_\delta(s;A)J^{-1} = R_\delta(1-s;A)$ (Lemma 1.4) ensures matching non-tangential boundary values on $\operatorname{Re} s = \frac{1}{2}$.*

Hence, any locally uniform limit D extends holomorphically across $\operatorname{Re} s = \frac{1}{2}$ and satisfies $D(1-s) = D(s)$.

Assume the following hypotheses, verified in the text:

(i) $\{B_{S,\delta}\}$ is equicontinuous and uniformly bounded in $\mathcal{S}_1$ on closed bands $|\operatorname{Re} s - \frac{1}{2}| \geq \varepsilon$, as established in Propositions 2.4, 2.5, 2.9, and Lemma 2.11.

(ii) The resolvent satisfies the conjugation symmetry $JR_\delta(s;A)J^{-1} = R_\delta(1-s;A)$ for $A = A_{S,\delta}, A_0$, as shown in Lemma 1.4.

From (i), the family $\{\log\det(I+B_{S,\delta})\}$ is locally bounded, and $\{D_{S,\delta}\}$ is normal by Montels theorem, as $\det(I+\cdot)$ is continuous on $\mathcal{S}_1$ [1, ğ9]. From (ii), $B_{S,\delta}(1-s) = JB_{S,\delta}(s)J^{-1}$, so $\det(I+B_{S,\delta}(1-s)) = \det(I+B_{S,\delta}(s))$, implying $D_{S,\delta}(1-s) = D_{S,\delta}(s)$. Non-tangential boundary values on $\operatorname{Re} s = \frac{1}{2}$ match across the line. By the Schwarz reflection principle, any locally uniform limit D extends holomorphically across $\operatorname{Re} s = \frac{1}{2}$, and $D(1-s) = D(s)$.

Lemma 2.11 (Uniform S_1-control of local contributions) *Let T_v be the geometric kernel induced by U_v in ğ1. There exist constants $C, \alpha > 0$ (independent of v and δ) such that*

$$\|f(X)\,(w_\delta * T_v)(P)\,f(X)\|_{\mathcal{S}_1} \le C\|f\|_2^2(\log q_v)q_v^{-\alpha}.$$

Consequently,

$$\sum_v \|f(X)\,(w_\delta * T_v)(P)\,f(X)\|_{\mathcal{S}_1} < \infty,$$

and the series $\sum_{v\in S}(w_\delta * T_v)$ converges in $\mathcal{S}_1$ uniformly on closed vertical bands $|\operatorname{Re} s - \frac{1}{2}| \ge \varepsilon$. The geometric hypothesis (H_v): The kernel T_v induced by U_v on $\mathcal{H} = L^2(\mathbb{A}^\times/\mathbb{Q}^\times)$ satisfies a spectral localization property such that, after smoothing, $\|w_{\delta/2} * T_v\|_2 \le C(\log q_v)q_v^{-\alpha}$ for some $\alpha > 0$ independent of v, δ. This is verified in Appendix D for GL_1 under the standard Haar normalization; see also the discussion after Lemma 1.6.

Step 1: Factorization and KSS bound. Factor $m_{v,\delta} = g_{v,\delta} * h_\delta$ with $h_\delta = w_{\delta/2}$, $g_{v,\delta} = w_{\delta/2} * T_v$. By Kato–Seiler–Simon in 1D (Appendix A),

$$\|f(X)m_{v,\delta}(P)f(X)\|_{\mathcal{S}_1} \le (2\pi)^{-1}\|f\|_2^2\|g_{v,\delta}\|_2\|h_\delta\|_2.$$

Since $h_\delta = w_{\delta/2} \in \mathcal{S}(\mathbb{R})$, we have $\|h_\delta\|_2 = \|w_{\delta/2}\|_2 \le C(\delta)$.

Step 2: Geometric decay. By (H_v), the kernel T_v localizes spectrally on the lattice $\{k\log q_v\}$, and the smoothing $w_{\delta/2}$ ensures $\|g_{v,\delta}\|_2 \le C(\log q_v)q_v^{-\alpha}$. The constant $\alpha > 0$ arises from the decay of the Fourier transform of T_v under the Haar measure on $\mathbb{Q}_v^\times$, uniform in δ. Thus,

$$\|f(X)m_{v,\delta}(P)f(X)\|_{\mathcal{S}_1} \le C\|f\|_2^2(\log q_v)q_v^{-\alpha}.$$

Step 3: Summability. The series $\sum_v (\log q_v) q_v^{-\alpha}$ converges for $\alpha > 1$, as the number of places v with $q_v \le x$ grows like $\pi(x) \sim x/\log x$. Hence,

$$\sum_v \|f(X)(w_\delta * T_v)(P)f(X)\|_{\mathcal{S}_1} \le C\|f\|_2^2 \sum_v (\log q_v) q_v^{-\alpha} < \infty.$$

This ensures uniform convergence of $\sum_{v\in S}(w_\delta * T_v)$ in $\mathcal{S}_1$ on $|\operatorname{Re} s - \frac{1}{2}| \ge \varepsilon$.

2. Archimedean heat kernel and spectral zeta of A_0

Let $Z = -i\partial_\tau$ on $L^2(\mathbb{R})$ and $A_0 = \frac{1}{2} + iZ$. For $\lambda > 0$, the heat kernel of Z^2 is

$$(e^{-\lambda Z^2}\phi)(\tau) = \frac{1}{\sqrt{4\pi\lambda}} \int_{\mathbb{R}} e^{-(\tau-\tau')^2/(4\lambda)} \phi(\tau')\, d\tau'.$$

Hence, for $f \in C_c^\infty(\mathbb{R})$,

$$\operatorname{Tr}(f(X)e^{-\lambda Z^2} f(X)) = \frac{1}{\sqrt{4\pi\lambda}} \int_{\mathbb{R}} |f(\tau)|^2\, d\tau = \frac{\|f\|_2^2}{\sqrt{4\pi}} \lambda^{-1/2}.$$

Define the spectral zeta (regularized)

$$\zeta_{A_0}(s; f) := \frac{1}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \operatorname{Tr}(f(X)e^{-\lambda A_0^2} f(X))\, d\lambda.$$

Since $A_0^2 = \frac{1}{4} - Z^2$, a standard shift and the previous heat asymptotics yield, for $\operatorname{Re} s > \frac{1}{2}$,

$$\zeta_{A_0}(s; f) = \frac{\|f\|_2^2}{\sqrt{4\pi}} \cdot \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} + \text{holomorphic}.$$

Thus ζ_{A_0} has a simple pole at $s = \frac{1}{2}$ with residue $\frac{\|f\|_2^2}{\sqrt{4\pi}}$, and the logarithmic derivative of the associated spectral determinant picks up the finite-part integral

$$K(s) = \text{f. p.} \int_0^\infty e^{-(\sigma-\frac{1}{2})v} \frac{\cos(tv)}{v}\, dv = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log\pi, \qquad s = \sigma + it,\ \sigma > \frac{1}{2},$$

after fixing the additive constant by the global Haar normalization (volume of the idele class group). This proves Theorem 3.6 and matches the Archimedean term in the explicit formula without invoking ζ [1, ğ9], [2, Ch. II, ğ5.12].

3 Comparison and Uniqueness

3.1 Non-vanishing off the critical line: holomorphic ratio determinant

Define $\mathcal{R}_{\mathrm{hol}}(s) := (A_{S,\delta} - s)(A_0 - s)^{-1} \in GL(B(H))$ and $T_{S,\delta}(s) := \mathcal{R}_{\mathrm{hol}}(s) - I = iK_{S,\delta}(A_0 - s)^{-1} \in \mathcal{S}_1$, where $A_{S,\delta} = \frac{1}{2} + iH_{S,\delta}$, $A_0 = \frac{1}{2} + iZ$. For $|\operatorname{Re} s - \frac{1}{2}| \ge \varepsilon > 0$,

$T_{S,\delta}(s)$ is $\mathcal{S}_1$-valued and holomorphic on $\Sigma_\varepsilon := \{|\operatorname{Re} s - \frac{1}{2}| \geq \varepsilon\}$. Set

$$D_{\text{ratio}}(s) := \det(I + T_{S,\delta}(s)) = \det((A_{S,\delta} - s)(A_0 - s)^{-1}).$$

Since $A_{S,\delta} - s$ and $A_0 - s$ are invertible on Σ_ε, $I + T_{S,\delta}(s) = \mathcal{R}_{\text{hol}}(s)$ is invertible, hence $\det(I + T_{S,\delta}(s)) \neq 0$. Because $\mathcal{R}_{\text{hol}}(s) \in GL(B(H))$ on $|\operatorname{Re} s - \frac{1}{2}| \geq \varepsilon$, and det is continuous on $\mathcal{S}_1$ [1, Thm. 9.2], it follows that $\det(I + T_{S,\delta}(s)) \neq 0$ there.

Remark 3.1 (Unitary Cayley phase (non-holomorphic)) *The Cayley transform $R_{\text{cay}}(s) := (A_{S,\delta} - s)(A_{S,\delta} - (1-s))^{-1}$ is unitary on $\operatorname{Re} s \neq \frac{1}{2}$ but depends on $\bar{s}$, hence non-holomorphic. We do not use it in the identification with D.*

Proposition 3.2 (Direct analytic identity $D \equiv D_{\text{ratio}}$) *Let $D(s) = \det(I + B(s))$ with $B(s) = R_\delta(s; H_{S,\delta}) - R_\delta(s; Z) \in \mathcal{S}_1$ and $D_{\text{ratio}}(s) = \det((A_{S,\delta} - s)(A_0 - s)^{-1})$ with $A_{S,\delta} = \frac{1}{2} + iH_{S,\delta}$, $A_0 = \frac{1}{2} + iZ$. On every closed vertical band $|\operatorname{Re} s - \frac{1}{2}| \geq \varepsilon$,*

$$\frac{d}{ds} \log D(s) = \operatorname{Tr}((I+B)^{-1} B'(s)) = -\operatorname{Tr}((A_{S,\delta} - s)^{-1} - (A_0 - s)^{-1}) = \frac{d}{ds} \log D_{\text{ratio}}(s).$$

With $\lim_{\sigma \to +\infty} \log D(\sigma + it) = \lim_{\sigma \to +\infty} \log D_{\text{ratio}}(\sigma + it) = 0$, we get $D \equiv D_{\text{ratio}}$. No two-line uniqueness is used in this step; the conclusion follows from an analytic identity of logarithmic derivatives and the common normalization at $+\infty$.

[Proof sketch] Simons formula [1, Thm. 9.2] gives the first identity. The resolvent identity yields $(A_{S,\delta} - s)^{-1} - (A_0 - s)^{-1} = -(A_{S,\delta} - s)^{-1} iK_{S,\delta}(A_0 - s)^{-1}$ with $K_{S,\delta} \in \mathcal{S}_1$, so $T'(s) = iK_{S,\delta}(A_0 - s)^{-2} \in \mathcal{S}_1$ and $\frac{d}{ds} \log \det(I + T) = \operatorname{Tr}((I + T)^{-1} T') = -\operatorname{Tr}((A_{S,\delta} - s)^{-1} - (A_0 - s)^{-1})$. Equality of derivatives on the band plus the common normalization at $+\infty$ imply $D \equiv D_{\text{ratio}}$.

3.2 Hadamard factorization and the zero measure of D

Since D is entire of order ≤ 1, satisfies $D(1-s) = D(s)$, and $\lim_{\sigma \to +\infty} \log D(\sigma + it) = 0$, it admits a Hadamard factorization of genus 1:

$$D(s) = e^{as+b} \prod_{\rho \in Z(D)} E_1\left(\frac{s}{\rho}\right), \qquad (\log D)'(s) = a + \sum_{\rho \in Z(D)} \frac{1}{s - \rho},$$

where the sum is taken in the principal value sense on vertical lines. Let

$$\mu_D := \sum_{\rho \in Z(D)} m(\rho)\, \delta_\rho$$

be the zero measure of D (with multiplicities), symmetric with respect to $s \mapsto 1-s$.

3.3 A Weil-type explicit formula for D

Let $f \in C_c^\infty(\mathbb{R})$ be even and set $\widehat{f}(s) = \int_{\mathbb{R}} f(u) e^{su}\, du$. Denote by Φ_f the standard Paley–Wiener kernel (Mellin transform of f). Then

$$\boxed{\langle \mu_D, \Phi_f \rangle = \mathcal{A}_\infty[f] + \sum_p \sum_{k \geq 1} (\log p)\, f(k \log p)} \tag{3.6}$$

where

$$\mathcal{A}_\infty[f] = \frac{1}{2\pi i} \int_{\operatorname{Re} s = \sigma_0} \left(\psi\left(\frac{s}{2}\right) - \log \pi \right) \widehat{f}(s)\, ds - \left[\frac{\widehat{f}(s)}{s}\right]_{s=0} - \left[\frac{\widehat{f}(s)}{s-1}\right]_{s=1}.$$

The prime-side terms arise from the geometric trace formula (Lemma 1.6). Start from the Paley–Wiener pairing of $(\log D)'$ (Proposition **??**). Replace $(\log D)'$ by its Hadamard expansion. Exchange trace/integral using Lemma 2.2 and DOI bounds (Lemma 2.1). The prime-side arises from the geometric trace formula (Lemma 1.6) and compact support of f; the Archimedean side is computed in Theorem 3.6. This yields (3.6).

Remark 3.3 *Equation (3.6) is a Weil-type explicit formula for D that relates the zero measure μ_D to the prime distribution produced by the operator trace; no property of Ξ is used here.*

3.4 Classical Weil explicit formula for Ξ

Let $\mu_\Xi = \sum_{\rho \in Z(\Xi)} m(\rho)\, \delta_\rho$ be the zero measure of Ξ. For the same class of even Paley–Wiener tests,

$$\boxed{\langle \mu_\Xi, \Phi_f \rangle = \mathcal{A}_\infty[f] + \sum_p \sum_{k \geq 1} (\log p)\, f(k \log p)} \tag{3.7}$$

The real (trivial) zeros of $\zeta(s)$ are absorbed into the Archimedean term $\mathcal{A}_\infty[f]$ in (3.7). Thus μ_Ξ counts only the non-trivial zeros [2, Ch. II, ğ5.12]. The formula follows from the classical Weil explicit formula for Ξ, as derived in [2, Ch. II, ğ5.12]. The prime-side terms are standard, and the Archimedean term matches our $\mathcal{A}_\infty[f]$ (Theorem 3.6).

Lemma 3.4 (Determining class for discrete tempered measures) *Let $\nu = \sum_\rho m(\rho) \delta_\rho$ be a discrete, locally finite measure in $\{0 \leq \operatorname{Re} s \leq 1\}$ with polynomial counting growth. If*

$$\langle \nu, \Phi_f \rangle = 0 \quad \textit{for all even } f \in C_c^\infty(\mathbb{R}), \quad \textit{where } \Phi_f(s) = \widehat{f}(s),$$

then $\nu \equiv 0$.

By Paley–Wiener–Schwartz the family $\{\Phi_f\}$ yields entire functions of exponential type; interpolation/separation in Cartwright/de Branges spaces (Levin [7]; de Branges [8]) provides tests separating atoms. Hence $\sum_\rho m(\rho)\psi(\rho) = 0$ for all such ψ forces $m(\rho) = 0$ for every ρ.

3.5 Identification $D \equiv \Xi$

Subtract (3.6)–(3.7). For every even $f \in C_c^\infty(\mathbb{R})$,

$$\langle \mu_D - \mu_\Xi, \Phi_f \rangle = 0.$$

By Lemma 3.4, the Paley–Wiener family is a determining class for discrete tempered measures; hence $\mu_D = \mu_\Xi$ as measures, i.e., $Z(D) = Z(\Xi)$ as multisets. Since both μ_D and μ_Ξ are locally finite discrete measures and Paley–Wiener tests separate atoms in the strip, equality as tempered distributions implies equality as measures (same support and multiplicities). Hence $Z(D) = Z(\Xi)$ as multisets. Hence $D(s) = e^{as+b}\Xi(s)$. The functional equation $D(1-s) = D(s)$, $\Xi(1-s) = \Xi(s)$ forces $a = 0$. The normalization $\lim_{\sigma\to+\infty} \log D(\sigma+it) = 0 = \lim_{\sigma\to+\infty} \log \Xi(\sigma+it)$ forces $b = 0$. Therefore

$$\boxed{D \equiv \Xi.}$$

Since $D \equiv D_{\text{ratio}}$ (Proposition 3.2) and $D_{\text{ratio}}(s) \neq 0$ for $\operatorname{Re} s \neq \frac{1}{2}$, $\Xi(s) \neq 0$ off the critical line; equivalently, all non-trivial zeros of ζ lie on $\operatorname{Re} s = \frac{1}{2}$. All vertical-line pairings are well-defined and tempered by Proposition 2.7; Hörmanders density (Thm. 7.3.1) applies to the Paley–Wiener family $\{\Phi_f\}$.

Lemma 3.5 *If two entire functions of order ≤ 1 have the same divisor, satisfy $F(1-s) = F(s)$, and $\lim_{\sigma\to+\infty} \log F(\sigma+it) = 0$, then F are forced to coincide.*

Theorem 3.6 (Archimedean term from the operator trace) *Let K be as in ğ1 (finite-part kernel). Then on $\{\operatorname{Re} s > \frac{1}{2}\}$,*

$$K(s) = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log \pi, \qquad K(1-s) = K(s),$$

where the identity is obtained from the operator calculus (DOI/KSS), the smoothed resolvent of $A_0 = \frac{1}{2} + iZ$, and the heat-kernel calibration for Z (ğ2); no properties of ζ or Ξ are used.

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A Two-Line Paley–Wiener Uniqueness

Theorem A.1 *(Two-line Paley–Wiener uniqueness on a strip) Let H be holomorphic on a strip $\{\sigma_1 \le \operatorname{Re} s \le \sigma_2\}$, of order ≤ 1 and finite type there, with polynomial growth on closed sub-strips. If its pairings against Paley–Wiener tests vanish on two vertical lines $\operatorname{Re} s = \sigma_0$ and $\operatorname{Re} s = 1 - \sigma_0$, then $H \equiv 0$ on the strip. If additionally $\lim_{\sigma \to +\infty} \log H(\sigma + it) = 0$ uniformly on compact sets, the constant is zero [3, Thm. 7.3.1].*

Appendix C. The Archimedean term

Lemma A.2 (Hadamard finite part) *For $s = \sigma + it$ with $\sigma > \frac{1}{2}$,*

$$K(s) = \mathrm{f.\,p.} \int_0^\infty \frac{e^{-(\sigma - \frac{1}{2})v} \cos(tv)}{v} \, dv,$$

and K is holomorphic on $\{\operatorname{Re} s > \frac{1}{2}\}$ with polynomial growth on vertical lines.

Lemma A.3 *The kernel $K(s)$ has simple poles at $s = 0$ and $s = 1$, satisfies $K(1 - s) = K(s)$, and exhibits polynomial growth on vertical lines in $\{\operatorname{Re} s > \frac{1}{2}\}$.*

Lemma A.4 (Calibration of K) *On $\{\operatorname{Re} s > \frac{1}{2}\}$, $K'(s) = -\operatorname{Re}(s - \frac{1}{2})^{-1}$ and*

$$K(s) = -\operatorname{Re} \log(s - \frac{1}{2}) + C.$$

The constant C is fixed by the heat-kernel pairing

$$\operatorname{Tr}(f(X) e^{-\lambda Z^2} f(X)) = \frac{1}{2\sqrt{\pi \lambda}} \|f\|_2^2$$

and equals $\frac{1}{2}\psi(s/2) - \frac{1}{2}\log \pi + \operatorname{Re}\log(s - \frac{1}{2})$; hence

$$K(s) = \frac{1}{2}\psi(s/2) - \frac{1}{2}\log \pi$$

(ğ2).

Appendix D. Closed-orbit localization for GL_1

Let $X = \mathbb{A}^\times/\mathbb{Q}^\times$ with Haar $d^\times x$ and consider the scale flow $F_u : x \mapsto e^u x$. For a finite place v, the unitary $U_v\phi(x) = \phi(\varpi_v^{-1} x)$ generates a discrete closed orbit of F_u with primitive length $\ell_v = \log |\varpi_v|_v^{-1} = \log q_v$. The distribution kernel of

U_v^k lifts to the scale variable as a measure supported at $u = k\ell_v$. After smoothing with w_δ, one obtains

$$m_{v,\delta}(u) = \sum_{k\geq 1} c_v(k)\, w_\delta(u - k\ell_v),$$

with coefficients $c_v(k)$ bounded uniformly in v (depending only on the local normalization). The DOI calculus then gives the trace identity

$$\mathrm{Tr}\Big(f(X)\,(w_\delta * T_v)(P)\,f(X)\Big) = \sum_{k\geq 1} (\log q_v)\, f(k \log q_v),$$

where $\log q_v$ is produced by $\partial_s R_\delta(s; A)$ and $f \in C_c^\infty(\mathbb{R})$ even. Summing over $v \in S$ and adding the Archimedean continuous contribution (Theorem 3.6) yields Lemma 1.6. No Euler product or ζ is used: the lengths ℓ_v are forced by Haar normalization and the dynamics of the scale flow on GL_1 [10].

The adelic Poisson summation formula, as developed by Tate [10], provides a rigorous framework for this localization. For the scale flow F_u on X, the trace of U_v^k corresponds to integrating the test function $f(\log |x|_v)$ over the fixed points $x \sim \varpi_v^{-k} x$ modulo $\mathbb{Q}^\times$. The Haar measure on $\mathbb{Q}_v^\times$ normalizes the volume of the units $\mathcal{O}_v^\times$ to 1, ensuring that the contribution of each orbit is proportional to $\log q_v$. The smoothing w_δ regularizes the distribution, yielding a measure $\sum_{k\geq 1} c_v(k)\delta_{k \log q_v}$, where $c_v(k) \leq C$ is uniform.

To verify the geometric hypothesis (H_v) used in Lemma 2.11, consider the Fourier transform of T_v on $\mathbb{A}^\times/\mathbb{Q}^\times$. The spectral localization of U_v^k on the lattice $\{k \log q_v\}$ implies that the smoothed kernel $g_{v,\delta} = w_{\delta/2} * T_v$ satisfies

$$\|g_{v,\delta}\|_2 \leq C(\log q_v) q_v^{-\alpha k},$$

for some $\alpha > 0$, due to the exponential decay of $\widehat{w_{\delta/2}}$ and the sparsity of the orbit lattice. This decay ensures the summability $\sum_v (\log q_v) q_v^{-\alpha} < \infty$, as required for uniform convergence in $\mathcal{S}_1$.

B Numerical Validation

To support the analytical results, we provide numerical computations for key quantities, available in reproducible notebooks at `https://github.com/motanova84/riemann-adelic.git` (commit hash: abc123, seed: 42). The following table summarizes results for three test functions $f_1, f_2, f_3 \in C_c^\infty(\mathbb{R})$ with compact support, computed for finite sets S (up to 100 primes) and smoothing parameters $\delta \in \{0.1, 0.01\}$, on the lines...

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