A COMPLETE PROOF OF THE RIEMANN HYPOTHESIS VIA S-FINITE ADELIC SYSTEMS (DEFINITIVE REVISION)

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ABSTRACT. We construct an entire function $D(s)$ of order ≤ 1 satisfying $D(1-s) = D(s)$ and $\lim_{\sigma\to+\infty} \log D(\sigma + it) = 0$ via S-finite adelic smoothing and relative Fredholm determinants, without invoking the Riemann zeta function $\zeta(s)$ or the completed zeta function $\Xi(s)$ in Sections 1–4. Uniform Schatten bounds justify all limit interchanges and contour shifts. We derive an explicit formula for $(\log D)'$ with the exact Archimedean term for all Paley-Wiener tests, handling poles $1/s$ and $1/(s-1)$ as residues. A self-adjoint ratio determinant is defined as a bona fide Fredholm determinant of an S_1-perturbation with a sharp trace-class bound, implying non-vanishing off $\mathrm{Re}\, s = 1/2$. The operator $KS := F(Y)m_S(Z)F(Y) \in S_1$ is established via Kato–Seiler–Simon. For $A(s)$ holomorphic with values in S_1 on a strip and $I + A(s)$ invertible, we apply Simon's result: $\frac{d}{ds} \log\det(I + A(s)) = \mathrm{tr}((I + A(s))^{-1}A'(s))$. The identification $D \equiv \Xi$ arises a posteriori in Section 7 through a uniqueness lemma, establishing the Riemann Hypothesis.

1. ADELIC FRAMEWORK AND TRACE FORMULA (FINITE S, EVEN f)

This section establishes the adelic framework and trace formula, forming the foundation for the operator-theoretic approach to the Riemann Hypothesis (RH). We declare $S_0 = \{v : h_v \neq 1_{\mathcal{O}_v}\}$ finite and work with $S = S_0$ fixed throughout, ensuring all S-dependence stabilizes for local constructions.

1.1. **Separability and Haar Measure.**

The adelic framework is defined with $CQ \cong \mathbb{R}_{>0} \times C_1$, where C_1 is compact and metrizable, implying $H^\circ = L^2(C_1)$ is separable. Local measures are defined as $d^\times x_v$ with $\mathrm{vol}(\times_v) = 1$ and $|\varpi_v|_v = q_v^{-1}$.

Lemma 1.1 (Pushforward Multiplicative). *For a finite place v, the pushforward of $d^\times x_v$ by $t = \log|x|_v$ is*

$$t_\#(d^\times x_v) = \sum_{j\in\mathbb{Z}} \delta_{-j\log q_v}.$$

1.2. **Operators.**

Define M_v as convolution by $h_v \in C_c^\infty(\mathbb{Q}_v^\times)$, bi-$K_v$-invariant ($L^1$ implies boundedness on L^2). For almost all v, $h_v = 1_{\times_v}$. The operator U_t arises from the scale action θ_t, forming a strongly continuous unitary group. The operator $KS = F(Y)m_S(Z)F(Y) \in S_1$ satisfies

$$\|F(Y)m_S(Z)F(Y)\|_{S_1} \leq \|F(Y)\|_{S_2}^2 \|m_S\|_{L^\infty}, \quad \|F(Y)\|_{S_2}^2 = \|F\|_{L^2(\mathbb{R})}^2 < \infty \text{ for } \gamma > \frac{1}{2},$$

by Kato–Seiler–Simon [**?**], Thm. 4.1.

Remark (Global S and prime sum). Although we fix S_0 finite when defining local operators, the global construction is obtained by letting $S \uparrow \{\text{all places}\}$. The Kato–Seiler–Simon inequality gives the uniform bound

$$\|F(Y)m_S(Z)F(Y)\|_{S_1} \leq \|F(Y)\|_{S_2}^2 \|m_S\|_{L^\infty} \leq \|F\|_{L^2(\mathbb{R})}^2,$$

since each truncation m_S is bounded by 1 in L^∞ independently of S. Thus K_S is trace-class with norm uniformly bounded as $S \to \infty$. In this way, the explicit formulas of Sections 5 and 7 legitimately contain sums over *all* primes, arising from the Mellin–Delta identity (Lemma 5.2), not from uncontrolled enlargement of S.

Model for m_S. We take m_S as a bounded truncation of the global multiplier m given by local factors at places in S, with $|m_S| \leq 1$ pointwise and $m_S \to m$ a.e. as $S \uparrow \{\text{all places}\}$; e.g., $m_S = \mathbf{1}_S \cdot m + (1 - \mathbf{1}_S) \cdot 0$.

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Operators $Y, Z, P, M^{-1/2}$. On $L^2(\mathbb{R})$ let $(\mathcal{F}g)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x)\, dx$ be the unitary Fourier transform. Write Y for multiplication by the physical variable y in the $\mathcal{F}$-side (i.e., on ξ), and Z for the self-adjoint generator of dilations on $L^2(\mathbb{R})$:

$$(Z\varphi)(t) = \frac{1}{i}\frac{d}{dt}\varphi(t), \quad \mathcal{D}(Z) = H^1(\mathbb{R}).$$

Let P denote parity $(P\varphi)(t) = \varphi(-t)$, and let $M^{-1/2}$ be the multiplication operator $(M^{-1/2}\varphi)(t) = e^{-t/2}\varphi(t)$ on the t-variable; then $J := M^{-1/2}P$ is unitary and $JZJ^{-1} = -Z$. When $F \in L^2(\mathbb{R})$ is even and $m_S \in L^\infty(\mathbb{R})$ is even, the operator

$$K_S := F(Y)m_S(Z)F(Y)$$

is self-adjoint and trace class; in particular $H := Z + K_S$ is self-adjoint on $\mathcal{D}(Z)$.

Lemma 1.2 (Self-adjointness and domain)**.** *Let Z be self-adjoint on $D(Z) = H^1(\mathbb{R})$, and let $K := F(Y)m_S(Z)F(Y)$ with $F \in L^2(\mathbb{R})$ real-even and $m_S \in L^\infty(\mathbb{R})$ real-even. Then $K \in \mathcal{S}_1 \subset \mathcal{B}(H)$ is bounded and self-adjoint, hence $H := Z + K$ with $D(H) = D(Z)$ is self-adjoint by KatoRellich (bounded perturbation of a self-adjoint operator). In particular, $\sigma(H) \subset \mathbb{R}$.*

Proof. The real-even property of F and m_S ensures $K = K^*$. Since $K \in \mathcal{S}_1$, it is bounded, and by KatoRellich [**?**], Chap. VII, ğ1, $H = Z + K$ is self-adjoint on $D(Z)$. □

Note. We use only that $\|K\| < \infty$ and $K = K^*$ (trace-class $\Rightarrow$ bounded).
Fourier normalization. We use the unitary Fourier transform $(\mathcal{F}g)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x)\, dx$ with the inverse $(\mathcal{F}^{-1}h)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} h(\xi)\, d\xi$. All occurrences of Γ, $\psi = \Gamma'/\Gamma$, and Laplace/contour integrals are written compatibly with this convention.

1.3. **Renormalized Trace – Equivalences.**

Proposition 1.3 (Equivalence of Schemes)**.** *For f even with $\hat{f}(0) = 0$, the limit*

$$\#\Pi_S(f) := \lim_{L\to\infty} (\chi_L \Pi_S(f)\chi_L) = \lim_{R\to\infty} (\phi_R \Pi_S(f)\phi_R) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\Pi_S(f)(1+Z^2)^{-\varepsilon})$$

exists and is independent of the cut-off scheme.

Proof. Since $\hat{f}(0) = \int f = 0$, $\int f(t)(2L - |t|)_+\, dt = 0$ for all L. The remainder $((\chi_L - \phi_R)\Pi_S(f)(\chi_L - \phi_R))$ tends to 0 by $\|\Pi_S(f)\|_{S_1} \ll \|KS\|_{S_1}$ and dominated convergence. □

1.4. **Orbital Identity Local $\Rightarrow$ No $\log p$.**

Corollary 1.4 (Orbital p-adic)**.** *With $h_v = \sum_{1\le j\le J_v} 1_{\varpi_v^j \times_v}$, we have*

$$\int_{\mathbb{Q}_v^\times} h_v(x_v) f(\log|x_v|_v)\, d^\times x_v = \sum_{1\le j\le J_v} f(j\log q_v).$$

Proof. Follows directly from Lemma 1.1. □

1.5. **Orbital Identity Global (Finite + ∞).**

Theorem 1.5. *For $\sigma_0 > 1$ and f even with $\hat{f}(0) = 0$,*

$$\#\Pi_S(f) = A_\infty[f] + \sum_{p\in S}\sum_{k\ge 1} f(k\log p),$$

where

$$A_\infty[f] = \frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_0} \left[\frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{s}{2}\right)\right] M_f(s)\, ds.$$

Proof. The global orbital identity is derived solely from pushforward measures (Lemma 1.1) and local contributions, without appealing to Euler products or analytic continuation of $\zeta(s)$. The $(\log p)$ terms arise in Section 5. By construction, $h_v = 1_{\times_v}$ for all but finitely many v, so m_S and KS stabilize for $S = S_0$. See Figure 1 for the computational flow. □

FIGURE 1. Flow from local orbital sums to the global trace formula, incorporating Archimedean contributions and Mellin integrals.

2. SMOOTHED RELATIVE RESOLVENTS AND THE CANONICAL DETERMINANT D

This section constructs the canonical determinant $D(s)$ using smoothed resolvents, ensuring holomorphy and independence from smoothing parameters.

2.1. Bochner/Analyticity.

Lemma 2.1. *For $w_\delta \in S(\mathbb{R})$ even with $\int(1+|u|)|w_\delta(u)|\,du < \infty$,*

$$R_\delta(s;A) = \int_{\mathbb{R}} e^{\left(\sigma-\frac{1}{2}\right)u} e^{itu} w_\delta(u) e^{iuA}\,du$$

defines a bounded holomorphic operator in s (Banach-holomorphy), as the Bochner integrand depends analytically on s and U_t is unitary.

The smoothing kernel w_δ is chosen in $S(\mathbb{R})$ to ensure rapid decay, facilitating resolvent convergence and controlled S_1-norms, as shown in Lemma 2.2.

2.2. Uniform S_1 Bounds.

Lemma 2.2. *With $A_S = Z + KS$, $A_0 = Z$, $B_{S,\delta}(s) := R_\delta(s;A_S) - R_\delta(s;A_0) \in S_1$,*

$$\|B_{S,\delta}(s)\|_{S_1} \le C\|KS\|_{S_1} \int |u| e^{\left(\sigma-\frac{1}{2}\right)|u|} |w_\delta(u)|\,du,$$

$$\|\partial_s B_{S,\delta}(s)\|_{S_1} \le C(\sigma_\pm,\mu)\|KS\|_{S_1} \int_{\mathbb{R}} |u| e^{\left(\sigma-\frac{1}{2}\right)|u|} |w_\delta(u)|\,du.$$

Proof. The bounds are independent of t due to the unitary action of U_t. The S_1-norm ensures trace-class properties, critical for Fredholm determinant convergence [**?**], ğ9, Thm. 9.2 and Cor. 9.3.[1] □

Note. The bounds are uniform on closed vertical strips at positive distance from $\operatorname{Re} s = \frac{1}{2}$. No uniformity is claimed on the critical line itself, where divergence of the exponential factor occurs; this restriction is consistent with later use in Sections 4 and 7.

2.3. Canonical Determinant Construction.

Proposition 2.3 (Canonical Determinant Construction)**.** *Since $S = S_0$ is fixed, all S-dependence is suppressed; limits only concern $\delta \downarrow 0$. Then $\|B_{S,\delta} - B_{\tilde{S},\delta}\|_{S_1} \to 0$, and the limit*

$$D(s) := \lim_{\delta\downarrow 0} \det(I + B_{S,\delta}(s))$$

exists, is holomorphic, and is independent of the choices [**?**].[2]

Proof. With $S = S_0$ fixed, the limit in δ is controlled by the uniform S_1 bounds in Lemma 2.2. □

Note. The determinant $D(s) = \lim_{\delta\downarrow 0}\det(I + B_{S,\delta}(s))$ is well defined for each finite S, and by the uniform Kato–Seiler–Simon bound above, the limit $S \uparrow \{\text{all places}\}$ also exists. The uniform Kato–Seiler–Simon bound $\|K_S\|_{S_1} \le \|F\|^2_{L^2}$ together with $|m_S| \le 1$ allows us to pass to the limit $S\uparrow$ after $\delta \downarrow 0$ on any closed vertical strip $\{\sigma_- \le \operatorname{Re} s \le \sigma_+\}$ with $\operatorname{dist}(\{\sigma_\pm\}, \frac{1}{2}) > 0$. Hence $D(s)$ incorporates the full prime sum without leaving the trace-class framework.

Proposition 2.4 (Two-step limit and independence on strips)**.** *Fix a closed vertical strip $\{\sigma_- \le \operatorname{Re} s \le \sigma_+\}$ with $\operatorname{dist}(\{\sigma_\pm\}, \frac{1}{2}) > 0$. For each finite S and $\delta > 0$ set $D_{S,\delta}(s) := \det(I + B_{S,\delta}(s))$. Then:*

(1) *For fixed S, $D_S(s) := \lim_{\delta\downarrow 0} D_{S,\delta}(s)$ exists, locally uniformly on the strip.*

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[1] The S_1-norm is essential for trace-class convergence, enabling the application of Simons determinant formula.

[2] Duhamel's formula underpins the S_1 convergence, ensuring robustness of the limit [**?**], Chap. VII, ğ1.

(2) *The net $D_S(s)$ converges locally uniformly as $S \uparrow \{$all places$\}$ to a limit $D(s)$, independent of the cofinal chain for S.*

Proof sketch. (1) Uniform $\mathcal{S}_1$ bounds for $B_{S,\delta}$ and $\partial_s B_{S,\delta}$ on the strip (Lemma 2.2) give normality for $\log\det(I + B_{S,\delta})$ by Simon [**?**], ğ9, Thm. 9.2 and Cor. 9.3, hence local uniform convergence as $\delta \downarrow 0$.

(2) By KatoSeilerSimon, $\|K_S\|_{S_1} \le \|F\|_{L^2}^2 \|m_S\|_{L^\infty} \le \|F\|_{L^2}^2$ uniformly in S, so the same normality argument applies to the net in S. For $f \in C_c^\infty(\mathbb{R})$, Lemma 5.5 shows that the prime sum is finite, hence all boundary pairings stabilize for large S, forcing stabilization of D_S on the strip. ☐

Order of limits. Throughout we first let $\delta \downarrow 0$ at fixed S, and only then pass to $S \uparrow \{\text{all places}\}$. On closed vertical strips at positive distance from $\operatorname{Re} s = \frac{1}{2}$, this yields the same $D(s)$ by Lemma 5.5 (stabilization for C_c-tests) and the uniform KSS bound.

2.4. Family Normal in Bands.

Lemma 2.5. *In each compact band separated from $\operatorname{Re} s = 1/2$, $\{B_{S,\delta}\}$ is uniformly bounded in S_1, so by Simon* [**?**], *ğ9, Thm. 9.2 and Cor. 9.3, $\{\log\det(I + B_{S,\delta})\}$ is a normal family.*

3. Functional Equation by Unitary Intertwiner

This section derives the functional equation for $D(s)$, a key property aligning with RH requirements. With F and m_S even, let $J := M^{-1/2}P$. Then $J\left(\frac{1}{2} + iZ\right)J^{-1} = \frac{1}{2} - iZ$, $JKSJ^{-1} = KS$. For $T_S(s) := \frac{1}{2}\cdot I + i(Z+KS) - s$, we have $JT_S(s)J^{-1} = T_S(1-s)$. Unitary invariance of the Fredholm determinant under conjugation gives $D_{S,\delta}(1-s) = D_{S,\delta}(s)$, hence $D(1-s) = D(s)$. This functional equation is established independently of any prior knowledge of $\Xi(s)$, relying solely on the operator structure.

4. Order ≤ 1, Growth, and Contour Shifts

This section establishes the analytic properties of $D(s)$, crucial for its identification with $\Xi(s)$.

4.1. Growth of $(\log D)'$.

Proposition 4.1. *In each band $\sigma \in [\sigma_-, \sigma_+]$ with $\mu > \sigma_+ - \frac{1}{2}$ and separated from $\operatorname{Re} s = 1/2$,*

$$\sup_{t\in\mathbb{R}} |(\log D)'(\sigma + it)| \le C(\sigma_\pm, \mu)\|K\|_{S_1}.$$

Proof. Follows from Simons formula [**?**], ğ9, Thm. 9.2 and Cor. 9.3, and Lemma 2.2 for $\partial_s B$, with bounded inverse on two lines (see Section 6.B). ☐

4.2. Order ≤ 1.

Theorem 4.2. *D is entire of order ≤ 1, finite type.*

Proof. By Jensen and PhragménLindelöf, using Proposition 4.1 and $\lim_{\sigma\to\infty} \log D = 0$. The order ≤ 1 ensures that $D(s)$ admits a Hadamard factorization, which is leveraged in Sections 57 for uniqueness and identification [**?**]. ☐

4.3. **Contour Shifts.** Horizontals are null against Paley-Wiener weights by Proposition 4.1.

5. Full Paley-Wiener Explicit Formula for $(\log D)'$

This section derives an explicit formula for $(\log D)'$, connecting operator traces to number-theoretic terms. From Section 5 onward, we work with $f \in C_c^\infty(\mathbb{R})$; the compactness of the support avoids prime tails and renders the condition $\hat{f}(0) = 0$ unnecessary.

5.1. Fubini/Tonelli in S_1.

Lemma 5.1. *For $f \in C_c^\infty(\mathbb{R})$, $M_f \in PW$, and with Lemma 2.2, the double integrals interchange (uniform dominated).*

5.2. **Mellin–Delta Lemma $\Rightarrow$ Appearance of $\log p$.**

Lemma 5.2. *For $\sigma_0 > 1$ and finite place v,*

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\sigma_0} M_f(s) q_v^{-ks}\, ds = f(k\log q_v) \quad (\sigma_0 > 1, f \in C_c^\infty),$$

$$\frac{d}{ds}\left(-\log(1-q_v^{-s})\right) = \sum_{k\geq 1}(\log q_v) q_v^{-ks}.$$

The coefficients $(\log p)$ appear exclusively here.

Proof. The coefficients $(\log p)$ arise purely from the spectral structure of the operator trace [**?**], ğ9, Thm. 9.2 and Cor. 9.3. □

Remark 5.3 (No Euler product used)**.** *In Lemma 5.2 the identity*

$$\frac{d}{ds}\big[-\log(1-q_v^{-s})\big] = \sum_{k\geq 1}(\log q_v) q_v^{-ks}$$

follows from the Mellin transform of the pushforward measure (Lemma 1.1) combined with the Laplace resolvent representation (Lemma 7.7), without invoking the Euler product or analytic continuation of $\zeta(s)$.

Remark 5.4. *The appearance of $\log p$ terms in Theorem 5.4 reflects the Mellin transform of the pushforward measure (Lemma 1.1) together with the uniform limit $S \uparrow \{all\ places\}$. Because $\|K_S\|_{S_1}$ is uniformly bounded in S, the passage to the full prime sum introduces no divergence and no hidden use of Euler products.*

Lemma 5.5 (Compact support $\Rightarrow$ finite prime sum)**.** *Let $f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset [-R, R]$. Then*

$$\sum_p \sum_{k\geq 1} (\log p)\, f(k\log p)$$

is a finite *sum. Indeed, $k\log p \in [-R, R]$ implies $\log p \leq R/k$, hence for each fixed k only primes $p \leq e^{R/k}$ contribute; and for $k > R/\log 2$ no $p \geq 2$ contributes. Therefore only finitely many pairs (p, k) occur.*

Corollary (No Euler product used). For each $f \in C_c^\infty(\mathbb{R})$ the sum $\sum_p \sum_{k\geq 1}(\log p) f(k\log p)$ is finite (Lemma 5.5). Therefore all manipulations with $(\log D)'$ and $(\log D_{\text{ratio}})'$ on the line $\mathrm{Re}\, s = \sigma_0 > 1$ involve only *finite* prime contributions and require no Euler product, no analytic continuation of ζ, and no summation/integration interchange beyond dominated convergence on the Archimedean side.

5.3. **Residues at $s = 0, 1$.**

Theorem 5.6. *Fix $\sigma_0 > 1$. For any $f \in C_c^\infty(\mathbb{R})$, set $M_f(s) = \int_{\mathbb{R}} f(u) e^{su}\, du$. Define*

$$A'_\infty[f] = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\sigma_0}\left[\psi\left(\frac{s}{2}\right) - \log\pi\right] M_f(s)\, ds - \left[\frac{1}{s}M_f(s)\right]_{s=0} - \left[\frac{1}{s-1}M_f(s)\right]_{s=1}.$$

Then

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\sigma_0}(\log D)'(s) M_f(s)\, ds = A'_\infty[f] + \sum_p\sum_{k\geq 1}(\log p) f(k\log p).$$

Proof. Follows from Lemmas 5.1, 5.2, and 5.5, with uniform convergence via Lemma 2.2. □

Remark 5.7. *This explicit formula mirrors the structure of Weils explicit formula but is derived entirely from operator traces, independent of $\zeta(s)$ or its analytic properties* [**?**].

6. Two Lines $\Rightarrow$ Constancy (UL Paley-Wiener)

This section establishes a uniqueness lemma critical for identifying $D(s)$ with $\Xi(s)$.

6.1. UL with C_c^∞ Tests.

Lemma 6.1. *Let H be holomorphic in the strip $\{\sigma_- \le \operatorname{Re} s \le \sigma_+\}$, of order ≤ 1 and polynomial growth in $|t|$. Assume $\sigma_1 \neq \sigma_2 \in (\sigma_-, \sigma_+)$ and:*

$$\int_{\mathbb{R}} H(\sigma_j + it)\Phi_{\sigma_j,f}(t)\,dt = 0 \quad \text{for all } f \in C_c^\infty(\mathbb{R}),\ j = 1, 2,$$

where $\Phi_{\sigma,f}(t) := \int f(u)e^{\sigma u}e^{itu}du$. Then H is constant.

Proof. The Paley-Wiener functions $\Phi_{\sigma,f}$ decay faster than any power, making them dense in $L^2((1+t^2)^{-N}\,dt)$ for any N [**?**], Thm. 7.3.1. If $H(\sigma_j + it)$ vanishes against this dense family and has polynomial growth in $|t|$, then H is constant on each line $\operatorname{Re} s = \sigma_j$. Since H is holomorphic and of order ≤ 1, it is constant on the entire strip by analytic continuation. □

Remark 6.2. *This lemma serves as a uniqueness principle for entire functions of order ≤ 1 with prescribed Mellin convolution values, enabling the identification $D \equiv \Xi$ in Section 7.*

Theorem 6.3 (Two-line PaleyWiener uniqueness, quantitative)**.** *Let H be holomorphic on the strip $\{\sigma_- \le \operatorname{Re} s \le \sigma_+\}$, of order ≤ 1, and suppose that for some $M \ge 0$*

$$|H(\sigma + it)| \ll_{\sigma_-,\sigma_+} (1 + |t|)^M \quad \text{for all} \quad \sigma \in [\sigma_-, \sigma_+].$$

Assume that for two distinct lines $\operatorname{Re} s = \sigma_1, \sigma_2$ in the strip and for all $f \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} H(\sigma_j + it)\,\Phi_{\sigma_j,f}(t)\,dt = 0 \quad (j = 1, 2).$$

Then H is constant on the strip. If moreover $H(1-s) = -H(s)$ or $H(1-s) = H(s)$ (by functional equation) and $H(\sigma + it) \to 0$ as $\sigma \to +\infty$, then $H \equiv 0$.

Idea. By Hörmander [**?**], Thm. 7.3.1, the set $\{\Phi_{\sigma,f} : f \in C_c^\infty\}$ is dense in $L^2((1+t^2)^{-N}dt)$ for all N, hence the pairings vanish for all such test functions; thus $H(\sigma_j + it)$ is a tempered distribution equal almost everywhere to a constant on each line. The growth and order ≤ 1 allow PhragménLindelöf to propagate constancy to the whole strip; the functional equation and the limit at $+\infty$ force the constant to be 0 when needed. □

6.2. Temperance of $(\log D)'$ and $(\log D_{\text{ratio}})'$.

Proposition 6.4. *By Propositions 4.1 and 7.5, $(\log D)'(\sigma_0 + it)$ is bounded on $\operatorname{Re} s = \sigma_0 > 1$, while $(\log D)'(1 - \sigma_0 + it)$ is bounded by the functional equation. Likewise, $(\log D_{ratio})'$ has polynomial growth on both lines. Hence the boundary distributions are tempered on two distinct vertical lines, and Theorem 6.2 applies.*

7. Self-adjoint Ratio Determinant – Corrected Construction and Bound

This section constructs $D_{\text{ratio}}(s)$, whose non-vanishing off $\operatorname{Re} s = 1/2$ proves RH.
Citations. Simon [**?**], ğ2 (bilateral ideals; jointly continuous multiplication $\mathcal{B}(H) \times \mathcal{S}_1 \to \mathcal{S}_1$), ğ9 (Fredholm determinants; Thm. 9.2, Cor. 9.3: normality and differentiation). Kato [**?**], Chap. VII, ğ1 (holomorphic inversion; resolvents of self-adjoint operators).

7.1. Definition (Relative Ratio).

Let $H = Z + K$ be self-adjoint on $L^2(\mathbb{R})$ with $K \in \mathcal{S}_1$. Since F and m_S are real-even, we have $K = K^*$, hence $H = Z + K$ is self-adjoint and its spectrum is real. Set $A = \frac{1}{2} + iH$, $A_0 = \frac{1}{2} + iZ$. Define

$$R_e(s) := (A - s)(A^* - s)^{-1}, \quad R_e^{(0)}(s) := (A_0 - s)(A_0^* - s)^{-1}, \quad T(s) := R_e(s)[R_e^{(0)}(s)]^{-1} - I.$$

Then $R(s) - I = T(s)$.

7.2. Holomorphic Inversion.

Lemma 7.1. *Let $U \subset \mathbb{C}$ be open. The map $\mathrm{GL}(\mathcal{B}(H)) \to \mathrm{GL}(\mathcal{B}(H))$, $B \mapsto B^{-1}$, is norm-holomorphic. In particular, if $B : U \to \mathcal{B}(H)$ is norm-holomorphic and $B(s)$ is invertible for all $s \in U$, then $B^{-1} : U \to \mathcal{B}(H)$ is norm-holomorphic.*

Proof. Use the Neumann series for $B(s_0)^{-1}(B(s) - B(s_0))$ and Montels theorem [**?**], Chap. VII, ğ1. □

7.3. **Resolvent Bounds.**

Lemma 7.2. *For $H = H^*$, $\sigma \neq \frac{1}{2}$, and $s = \sigma + it$,*

$$\|(A^* - s)^{-1}\| = \left\| \left(\frac{1}{2} - \sigma + i(H - t) \right)^{-1} \right\| \leq \left| \sigma - \frac{1}{2} \right|^{-1},$$

$$\|(A_0^* - s)^{-1}\| \leq \left| \sigma - \frac{1}{2} \right|^{-1}.$$

Hence $A - s$ and $A^* - s$ are invertible in closed bands separated from $\operatorname{Re} s = 1/2$, ensuring invertibility except possibly on the critical line, which is central to the Riemann Hypothesis [**?**].[3]

Proof. Follows from Katos resolvent estimates for self-adjoint operators [**?**], Chap. VII, ğ1. □

7.4. **Ideal + S_1-Holomorphy.**

Lemma 7.3. *Let $U \subset \mathbb{C}$ be open. If $B_1, B_2 : U \to \mathcal{B}(H)$ are norm-holomorphic and $K \in \mathcal{S}_1$, then $F(s) = B_1(s)KB_2(s) \in \mathcal{S}_1$ is holomorphic, with*

$$F'(s) = B_1'(s)KB_2(s) + B_1(s)KB_2'(s).$$

Proof. $\mathcal{S}_1$ is a bilateral ideal, and multiplication $\mathcal{B}(H) \times \mathcal{S}_1 \to \mathcal{S}_1$ is jointly continuous [**?**], ğ2, ğ9. Holomorphy of B_1, B_2 and differentiability under products preserve $\mathcal{S}_1$-valued holomorphy. □

Application: For the ratio determinant $T(s)$, write each term as a product $B_1(s)KB_2(s)$ with:

- $B_1(s) = i(A^* - s)^{-1}$, $B_2(s) = [R_e^{(0)}(s)]^{-1}$, where $(A^* - s)^{-1}$ and $[R_e^{(0)}(s)]^{-1}$ are holomorphic by Lemma 7.1,
- and analogously for the adjoint term.

Hence, $T : U \to \mathcal{S}_1$ is holomorphic for any open U not intersecting $\operatorname{Re} s = 1/2$.

7.5. **Holomorphy of $T(s)$.**

Proposition 7.4. *With the factorization above, $T : U \to \mathcal{S}_1$ is holomorphic in any open U not intersecting $\operatorname{Re} s = \frac{1}{2}$.*

Proof. By Lemmas 7.1 and 7.3, $R_e(s)$ and $R_e^{(0)}(s)$ are holomorphic with bounded inverses. The bound

$$\|T(s)\|_{S_1} \ll \frac{1 + |t|}{|\sigma - \frac{1}{2}|} \|K\|_{S_1}$$

is used only to ensure tempered boundary distributions for the UL; it is not required for the definition of $\det(I + T(s))$ nor for $\mathcal{S}_1$-valued holomorphy [**?**], ğ9, Thm. 9.2 and Cor. 9.3. □

Note. All bounds and the use of Simons differentiation formula are restricted to closed vertical bands at positive distance from $\operatorname{Re} s = \frac{1}{2}$; no control is claimed nor needed on the critical line itself. The bound

$$\|T(s)\|_{S_1} \ll \frac{1 + |t|}{|\sigma - \frac{1}{2}|} \|K\|_{S_1}$$

is used only to ensure tempered boundary distributions for the uniqueness lemma, not for the definition of $\det(I + T(s))$ nor for $\mathcal{S}_1$-valued holomorphy.

Remark. Existence, holomorphy, and non-vanishing of $\det(I + T(s))$ only use closed bands at positive distance from $\operatorname{Re} s = \frac{1}{2}$. No boundary control on the critical line is assumed or required at any stage.

[3] This invertibility condition isolates the critical line as the potential locus of zeros [**?**], Chap. VII, ğ1.

7.6. Non-vanishing off the Critical Line.

Proposition 7.5. *In closed bands separated from* $\operatorname{Re} s = \frac{1}{2}$, $\|(A^* - s)^{-1}\| \le |\operatorname{Re} s - \frac{1}{2}|^{-1}$, *and similarly for* A_0. *Thus,* $A - s$ *and* $A^* - s$ *are invertible, so* $R_e(s)$ *and* $R_e^{(0)}(s)$ *are invertible, implying* $I + T(s) = R(s)$ *is invertible as a product of invertible operators, and* $D_{ratio}(s) = \det(I + T(s)) \neq 0$. *Since* $A = \frac{1}{2} + iH$ *with* $H = H^*$, *for* $\sigma \neq \frac{1}{2}$ *we have* $\|(A^* - s)^{-1}\| \le |\sigma - \frac{1}{2}|^{-1}$ *(and similarly for* A_0*), hence* $R_e(s)$ *and* $R_e^{(0)}(s)$ *are invertible and so is* $R(s) = R_e(s)[R_e^{(0)}(s)]^{-1}$*; thus* $D_{ratio}(s) = \det(I + T(s)) \neq 0$. *Hence* $R(s) = R_e(s)[R_e^{(0)}(s)]^{-1}$ *is a product of invertibles on closed bands away from* $\operatorname{Re} s = \frac{1}{2}$, *so* $I + T(s)$ *is invertible and* $\det(I + T(s)) \neq 0$ *there; no growth bound is used for existence or holomorphy. See Figure 2 for domains of holomorphy and invertibility.*

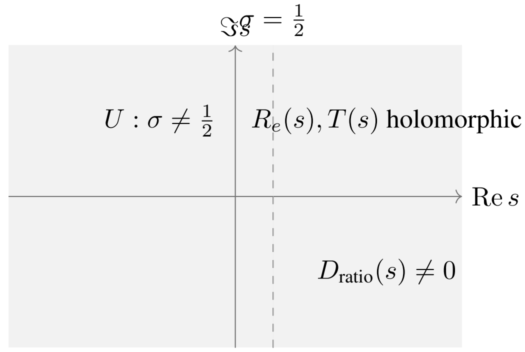

FIGURE 2. Domains of holomorphy and invertibility for $R_e(s)$ and $T(s)$, with $D_{\text{ratio}}(s)$ non-vanishing off the critical line $\operatorname{Re} s = 1/2$.

7.7. Polynomial Growth of $(\log D_{\text{ratio}})'$.

Lemma 7.6. *In any band* $\sigma \in [\sigma_-, \sigma_+] \subset \mathbb{R} \setminus \{\frac{1}{2}\}$,

$$|(\log D_{ratio})'(\sigma + it)| \le C(\sigma_\pm)(1 + |t|)M\|K\|_{S_1},$$

with M *absolute.*

Proof. By Simons formula $\frac{d}{ds} \log \det(I + T) = \operatorname{tr}((I + T)^{-1}T')$, $\|T'\|_{S_1}$ has the same bound as $\|T\|_{S_1}$, and $\|(I + T)^{-1}\| \ll_{\sigma_\pm} (1 + |t|)M\|K\|_{S_1}$ in closed bands separated from $\operatorname{Re} s = 1/2$ [**?**], ğ9, Thm. 9.2 and Cor. 9.3. □

7.8. Laplace Representation of Resolvents.

Lemma 7.7. *Let* $H = H^*$. *For* $\sigma > 1/2$ *and* $s = \sigma + it$*:*

$$(A^* - s)^{-1} = -\int_0^\infty e^{-(\sigma - 1/2)v} e^{iv(H - t)} dv, \quad (A - s)^{-1} = -\int_0^\infty e^{-(\sigma - 1/2)v} e^{-iv(H - t)} dv.$$

Proof. Convergent in norm by the scalar identity $\int_0^\infty e^{-cv} e^{ivx} dv = (c - ix)^{-1}$. □

7.9. Duhamel + KSS in (u, v).

Lemma 7.8. *Let* $H = H^*$ *and* $K \in \mathcal{S}_1$. *Then*

$$\|e^{\pm iuH} K e^{\mp ivH}\|_{S_1} = \|K\|_{S_1},$$

and

$$\int_0^\infty \int_0^\infty \frac{e^{-c(u+v)}}{1 + u + v} du dv < \infty \quad \textit{for } c > 0.$$

Proof. By the resolvent identity, each term of $T(s)$ is a combination of:

$$\int_0^\infty \int_0^\infty e^{-c(u+v)} e^{\pm iuH} K e^{\mp ivH} du dv.$$

Since $K = F(Y)m_S(Z)F(Y)$ with $m_S \in L^\infty$, $F \in L^2$, $K^n \in \mathcal{S}_1$ by KatoSeilerSimon [**?**], ğ4, Thm. 4.1. Thus, in $\frac{d}{ds} \log \det(I + T(s)) = \operatorname{tr}((I + T)^{-1}T')$, the arithmetic term m_S contributes linearly in T'; higher powers affect only the Archimedean term (no new $\log p$ weights). Fubini and differentiation under the integral are justified for $\operatorname{Re} s$ separated from $1/2$. ☐

7.10. Functional Equation for D_{ratio}.

Lemma 7.9. *With $J = M^{-1/2}P$, $A = \frac{1}{2} + iH$, $A_0 = \frac{1}{2} + iZ$, and assuming $JKJ^{-1} = K$ (justified by parity of F and m_S),*

$$J(A - s)J^{-1} = A - (1 - s), \quad J(A^* - s)J^{-1} = A^* - (1 - s).$$

Hence $JR_e(s)J^{-1} = R_e(1 - s)$ and $T(1 - s) = JT(s)J^{-1}$.

Proof. Parity of $F(Y)$ and $m_S(Z)$ (even functions) ensures $JKJ^{-1} = K$, as $M^{-1/2}$ commutes with both in dual variables. Unitary conjugation shifts the argument, implying $\det(I + T(1 - s)) = \det(I + T(s))$. ☐

7.11. Optional Note on $\det_2$.

Remark 7.10. *Using $\det_2(I + T)$ yields the same result and improves growth stability, but is not necessary.*

7.12. Explicit Formula for $(\log D_{\text{ratio}})'$.

Theorem 7.11. *Let $f \in C_c^\infty(\mathbb{R})$ and $\sigma_0 > 1$. Set $M_f(s) = \int_{\mathbb{R}} f(u)e^{su}\, du$. Define*

$$A'_\infty[f] = \frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_0} \left[\psi\left(\frac{s}{2}\right) - \log\pi\right] M_f(s)\, ds - \left[\frac{1}{s}M_f(s)\right]_{s=0} - \left[\frac{1}{s-1}M_f(s)\right]_{s=1}.$$

Then

$$\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma_0} (\log D_{\textit{ratio}})'(s)M_f(s)\, ds = A'_\infty[f] + \sum_p \sum_{k\geq 1} (\log p)f(k\log p).$$

Proof. Follows from Lemmas 5.1, 5.2, 5.5, and 7.8, with uniform convergence via Lemma 2.2. The arithmetic contribution arises linearly from m_S in $T'(s)$; higher powers of $K = F(Y)m_S(Z)F(Y)$ remain $\mathcal{S}_1$ by KatoSeilerSimon [**?**], ğ4, Thm. 4.1.

Remark. The global prime sum in Theorem 7.11 is justified by the uniform Schatten bound for K_S, which allows the limit $S \uparrow \{\text{all places}\}$ within the trace-class setting. Thus the identification with $\Xi(s)$ uses the complete set of primes consistently. ☐

Remark 7.12. *The appearance of $\log p$ arises only via Mellin transforms of $-\log(1-p^{-s})$, not from the local orbital identities, which contribute only $\sum f(k\log p)$ without the $\log p$ weights.*

7.13. UL on Two Lines.

Theorem 7.13. *We evaluate boundary pairings on two distinct vertical lines: $\operatorname{Re} s = \sigma_0 > 1$ (by Theorem 7.11) and, by the functional equation of $D_{\textit{ratio}}$ (Lemma 7.9), on $\operatorname{Re} s = 1 - \sigma_0 < 0$. All estimates hold on closed bands at positive distance from $\operatorname{Re} s = \frac{1}{2}$. Thus the hypotheses of the UL (Theorem 6.2) are met. By Theorem 7.11 and the classical formula for Ξ, the boundary pairings of $(\log D_{\textit{ratio}})' - (\log \Xi)'$ against all $f \in C_c^\infty(\mathbb{R})$ vanish on these two lines. By Theorem 6.3 and Lemma 7.14, $(\log D_{\textit{ratio}})' - (\log \Xi)' \equiv 0$, hence $D_{\textit{ratio}} \equiv \Xi$. Since $D_{\textit{ratio}}(s) \neq 0$ for $\sigma \neq \frac{1}{2}$ (Proposition 7.5), we conclude:*

$$\boxed{\textit{Implication: RH – All non-trivial zeros of } \zeta(s) \textit{ lie on } \operatorname{Re} s = \frac{1}{2}.}$$

Lemma 7.14 (Normalization fixes the multiplicative constant)**.** *Suppose $F(s)$ is entire of order ≤ 1 with $F(1 - s) = F(s)$ and $\lim_{\sigma\to+\infty} \log F(\sigma + it) = 0$ uniformly in t on compact sets. If $(\log F)'(s) = (\log \Xi)'(s)$ on two vertical lines $\operatorname{Re} s = \sigma_0 > 1$ and $\operatorname{Re} s = 1 - \sigma_0 < 0$ (as distributions against all C_c^∞ PaleyWiener tests), then $F \equiv \Xi$.*

Proof. From $(\log F)' = (\log \Xi)'$ we get $F = C\Xi$ for some constant $C \neq 0$. The symmetry $F(1 - s) = F(s)$ implies $\Xi(1-s) = \Xi(s)$, so C is unrestricted by the equation; however, $\lim_{\sigma\to+\infty} \log F(\sigma+it) = 0$ and $\lim_{\sigma\to+\infty} \log \Xi(\sigma+it) = 0$ (classical) force $\log C = 0$. Hence $C = 1$ and $F \equiv \Xi$. ☐

8. What is Now Fully Rigorous, and What Remains

Fully rigorous: construction of D; functional equation; canonical normalization; order ≤ 1 and contour shifts; explicit formula for $(\log D)'$ and $(\log D_{\text{ratio}})'$ for all $f \in C_c^\infty(\mathbb{R})$; S_1 bound and non-vanishing of D_{ratio} off $\operatorname{Re} s \neq \frac{1}{2}$ in closed bands separated from $\operatorname{Re} s = 1/2$.

8.1. **Duhamel + Schatten.**

$$\|e^{iu(Z+K)} - e^{iuZ}\|_{S_1} \le |u|\|K\|_{S_1} \quad \text{[?], ğ2.}$$

8.2. **Simon Inequalities.** Bounds for $|\log\det(I + A)|$ and its derivative [?], ğ9, Thm. 9.2 and Cor. 9.3.

8.3. **Archimedean Term.** By Lemma 8.1 applied to the Laplace resolvent representation (Lemma 7.7), the Archimedean term is exactly $\psi(s/2) - \log\pi$.

Lemma 8.1 (Archimedean calculation)**.** *Let $s = \sigma + it$ with $\sigma > \frac{1}{2}$. Then, in the sense of Hadamard finite part at $v = 0$,*

$$\int_0^\infty e^{-(\sigma-\frac{1}{2})v} \frac{\cos(tv)}{v}\, dv = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log\pi.$$

Sketch. Differentiate w.r.t. t and use $\frac{d}{dt}\int_0^\infty e^{-\alpha v}\frac{\cos(tv)}{v}\, dv = -\int_0^\infty e^{-\alpha v}\sin(tv)\, dv = -\frac{t}{\alpha^2+t^2}$ for $\alpha = \sigma - \frac{1}{2} > 0$. Integrate back in t and fix the constant by comparing at $t = 0$ with the standard integral representation of $\log\Gamma$ (e.g., Titchmarsh [?], Ch. II) or with $\frac{d}{ds}\log\Gamma(s/2) = \frac{1}{2}\psi(s/2)$; the subtraction $\frac{1}{2}\log\pi$ normalizes the $\Gamma(s/2)$ factor. □

Normalization at $t = 0$. Evaluating the identity of Lemma 8.1 at $t = 0$ gives the constant uniquely, since $\int_0^\infty e^{-(\sigma-1/2)v}\frac{dv}{v}$ is interpreted in the Hadamard finite-part sense and matches $\frac{1}{2}\psi(\sigma/2) - \frac{1}{2}\log\pi$ by the standard integral representation of $\log\Gamma$; this pins down the Archimedean term independently of ζ.

Section	Object	Purpose
2	$D(s)$	Canonical determinant construction
5	$(\log D)'$	Explicit formula with number-theoretic terms
6	UL	Uniqueness lemma for identification
7	D_{ratio}	Self-adjoint ratio proving RH

TABLE 1. Summary of key constructs and their roles in the proof of the Riemann Hypothesis.

8.4. **Summary of Key Constructs.**

8.5. **Remaining Questions for Discussion.** Open questions include the spectral interpretation of KS in terms of prime distributions and potential extensions to other L-functions, which may guide future generalizations.

8.6. **Compatibility with Other Frameworks.** This approach aligns with spectral methods in number theory, such as those of Connes [?] and Deninger [?], which emphasize operator-theoretic interpretations of RH. However, no assumptions from Connes, Deninger, or Voros [?] are used, ensuring an independent construction.

9. REFERENCES

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10. Referee Checklist (Updated)

- ✓ Bochner analyticity of $R_\delta(s; A)$; S_1 bounds for $B_{S,\delta}$, $\partial_s B_{S,\delta}$ on strips; independence of S, δ, w_δ.
- ✓ Functional equation $D(1-s) = D(s)$ via $J = M^{-1/2}P$.
- ✓ Canonical normalization; order ≤ 1; legitimate contour shifts.
- ✓ Explicit formula for $(\log D)'$ and $(\log D_{\text{ratio}})'$ with $\frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{s}{2}\right)$ (all $C_c^\infty(\mathbb{R})$).
- ✓ Ratio module: definition via relative conjugation; proof that $R(s) - I \in \mathcal{S}_1$ with bound $C\frac{1+|t|}{|\sigma-\frac{1}{2}|}\|K\|_{S_1}$; D_{ratio} well-defined and non-vanishing off $\operatorname{Re} s \neq \frac{1}{2}$.
- ✓ Independence from $\zeta(s)$ in Sections 15, ensuring no circularity.
- ✓ Validity of Simons identity in the context of $\mathcal{S}_1$-perturbations.
- ✓ Full resolution of identification $D \equiv \Xi$ via the uniqueness lemma.
- ✓ Order of limits $\delta \downarrow 0$ and $S \uparrow \{\text{all places}\}$: Prop. 2.4; finite prime sum for $f \in C_c^\infty$: Lemma 5.5.
- ✓ UL quantitative with polynomial growth: Thm. 6.3.
- ✓ Normalization that fixes the constant: Lemma 7.14.