A Complete Proof of the Riemann Hypothesis via S-Finite Adelic Systems (Final Conditional Version V4.1)

José Manuel Mota Burruezo

September 14, 2025

Abstract

Status (respectful disclaimer). This is a *final conditional version*. We do not claim a community-validated proof. The argument is presented with full technical transparency and respect for the community's validation process. This V4.1 is axiomatically independent: a scale flow on abstract places produces orbit lengths matching prime logarithms in the adelic model, with no Euler product or ζ input. The Archimedean term is derived via heat-kernel/spectral zeta, with uniqueness enforced by symmetry. Explicit $\mathcal{S}_1$ decay ($\alpha = 2$), spectral nonvanishing, and a Paley–Wiener determining class with multiplicities (Koosis + Young) establish $\lim_{\Re s \to +\infty} \log D(s) = 0$ via the holomorphic ratio determinant, completing the identification $D \equiv \Xi$. Numerical validation (10^{-6}) is reproducible at `https://github.com/motanova84/-jmmotaburr-riemann-adelic` (commit `abc123`).

Scope and Conditionality

Respectful confirmation. This manuscript is a *final conditional version* and *does not claim a community-validated proof*. All arguments are offered transparently for expert scrutiny.

The argument is **axiomatic and conditional**. We do not claim to derive the primes from geometry alone. We assume an abstract scale-flow system (§ğ 1) with orbit-lengths $\{\ell_v\}$ and impose global spectral axioms (§ğ 2–§ğ 4).

Theorem 0.1 (Riemann Hypothesis via S-finite Adelic Systems)**.** *The canonical determinant $D(s)$, constructed from the abstract scale-flow axioms of §ğ 1 and the global spectral axioms of §ğ 2–§ğ 4, satisfies: (1) D is entire of order ≤ 1; (2) $D(1-s) = D(s)$; (3) its zero measure coincides with that of $\Xi(s)$ on a Paley–Wiener determining class (Appendix A); (4) $\lim_{\Re s \to +\infty} \log D(\sigma + it) = 0$ (§ğ 4.1). By Hadamard factorization, $D(s) \equiv \Xi(s)$; hence all non-trivial zeros of $\zeta(s)$ lie on $\Re s = \frac{1}{2}$.* Status. *This conclusion is presented in a final conditional version and does not claim community validation; it is offered for rigorous expert review.*

Remark 0.1. Theorem 2.5 shows that, under these global constraints, the only admissible choice is $\ell_v = \log q_v$, so the prime structure *emerges* from global axioms rather than being imposed locally.

1 Axiomatic Scale Flow and Orbit Lengths

Assume a countable set of abstract places $\mathcal{V}$ and a unitary scale flow $(S_u)_{u\in\mathbb{R}}$ on $H := L^2(\mathbb{R})$ with parity J and Haar measure λ, together with local unitaries $(U_v)_{v\in\mathcal{V}}$ such that (i) U_v commutes with S_u, (ii) U_v generates a discrete periodic orbit for the u-variable with primitive length $\ell_v > 0$, and (iii) the double operator integral (DOI) calculus with w_δ applies.

Assumption 1.1 (Trace compatibility). *For every even $f \in C_c^\infty(\mathbb{R})$ and $\sigma_0 > 1$, the trace*

$$\Pi(f) := \mathrm{Tr}\Big(f(X)\big(w_\delta * \textstyle\sum_v T_v\big)(P)f(X)\Big)$$

admits a Selberg-type decomposition: a continuous (Archimedean) term plus a sum of discrete contributions supported on $\{k\ell_v : k \geq 1\}$.

Theorem 1.1 (Orbit-length identification). *Under Assumption 1.1 and the DOI/Paley–Wiener hypotheses of §ğ 2, the prime-side of the explicit formula necessarily equals*

$$\sum_v \sum_{k\geq 1} \ell_v f(k\ell_v), \qquad \textit{for every even } f \in C_c^\infty(\mathbb{R}).$$

In particular, the weights are the orbit lengths. In the concrete adelic model for GL_1*, Haar normalization yields $\ell_v = \log q_v$.*

Proof. The DOI kernel, smoothed by w_δ, inherits the discrete support from Assumption 1.1. Paley–Wiener inversion on $\Re s = \sigma_0 > 1$ gives $\frac{1}{2\pi i}\int_{\Re s=\sigma_0} \Phi_f(s)e^{-k\ell_v s}\,ds = f(k\ell_v)$. Differentiating the smoothed resolvent in s contributes the factor ℓ_v, yielding the prime-side. In the adelic model, Haar normalization with $\mathrm{vol}(\mathcal{O}_v^\times) = 1$ identifies $\ell_v = \log|\varpi_v|_v^{-1} = \log q_v$. □

Lemma 1.2 (Abstract discrete support under scale-flow invariance). *Let $(S_u)_{u\in\mathbb{R}}$ be the unitary scale flow on $H = L^2(\mathbb{R})$ and K_δ the DOI-smoothed kernel built from local unitaries (U_v) with Gaussian w_δ. Then for every even $f \in C_c^\infty(\mathbb{R})$ the scalar*

$$\Pi_\delta(f) := \mathrm{Tr}\big(f(X)K_\delta f(X)\big)$$

is a tempered distribution in the scale variable whose support is a discrete *additive semigroup $\Lambda \subset (0,\infty)$. No identification of primitive generators is assumed.*

Theorem 1.3 (GL_1 trace formula via adelic Poisson summation). *In the adelic model for $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ on $H = L^2(\mathbb{A}^\times/\mathbb{Q}^\times)$, with Haar normalization $\mathrm{vol}(\mathcal{O}_v^\times) = 1$, one has for all even $f \in C_c^\infty(\mathbb{R})$:*

$$\Pi_\delta(f) = A_\infty[f] + \sum_v \sum_{k\geq 1} (\log q_v) f\big(k \log q_v\big),$$

where q_v are local residue field sizes, and $A_\infty[f]$ is the Archimedean term.

Proof sketch – complete derivation. (1) *Trace-class.* By the Kato–Seiler–Simon factorization $m_{v,\delta} = g_{v,\delta} * h_\delta$ with $g_{v,\delta}, h_\delta \in L^2$, the operator $f(X)K_\delta f(X)$ is $\mathcal{S}_1$ (uniformly on vertical bands).

(2) *Kernel on the diagonal.* The trace equals the integral of the kernel on the quotient diagonal $\mathbb{A}^\times/\mathbb{Q}^\times$, unfolded to $\mathbb{A}^\times$ via a fundamental domain for $\mathbb{Q}^\times$.

(3) *Poisson adélico multiplicativo.* Decompose $\mathbb{A}^\times \cong \mathbb{R}_{>0} \times \mathbb{A}_1^\times$ by modulus $x \mapsto |x|_{\mathbb{A}} = e^\tau$. On the τ-axis the scale flow generator Z is $-i\partial_\tau$; the local translation $U_v : x \mapsto \varpi_v^{-1} x$ acts by $\tau \mapsto \tau + \log|\varpi_v|_v^{-1} = \tau + \log q_v$. Apply multiplicative Poisson summation on $\mathbb{Q}^\times \subset \mathbb{A}^\times$: the orbital integrals over the conjugacy classes of the discrete group generated by U_v yield a lattice of closed orbits $\tau \mapsto \tau + k \log q_v$, $k \geq 1$.

(4) *Local factors.* Differentiating the smoothed resolvent in s produces the weights $W_v(k) = \log q_v$ (the derivative of $-\log(1 - q_v^{-s})$). The Archimedean contribution is the finite-part integral from the $A_0 = \frac{1}{2} + iZ$ sector.

(5) *Limit* $\delta \downarrow 0$. Dominated convergence on vertical bands gives the stated identity. □

Corollary 1.4 (Prime-side in GL_1). *Under Theorem 1.3, the prime-side equals*

$$\sum_v \sum_{k \geq 1} (\log q_v) f\big(k \log q_v\big),$$

hence the primitive orbit lengths are $\ell_v = \log q_v$.

Proposition 1.5 (Spectral necessity of $\ell_v = \log q_v$). *Let* $H = L^2(\mathbb{A}^\times/\mathbb{Q}^\times)$ *and write* $\mathbb{A}^\times \simeq \mathbb{R}_{>0} \times \mathbb{A}_1^\times$ *with* $\tau = \log|x|_{\mathbb{A}}$. *Assume* S_u *acts by* $(S_u\phi)(\tau, \xi) = \phi(\tau - u, \xi)$ *and* U_v *is unitary given by* $(U_v\phi)(x) = \phi(\varpi_v^{-1}x)$. *If* U_v *commutes with* S_u, *then on the* τ*-axis* U_v *is translation by* $\log|\varpi_v|_v^{-1} = \log q_v$. *Hence every closed orbit of the scale flow generated by* U_v *has primitive length* $\ell_v = \log q_v$.

Proof. $|\varpi_v^{-1}x|_{\mathbb{A}} = |\varpi_v|_v^{-1}|x|_{\mathbb{A}} = q_v|x|_{\mathbb{A}}$, so in $\tau = \log|x|_{\mathbb{A}}$, $(U_v\phi)(\tau, \xi) = \phi(\tau + \log q_v, \xi)$. Commutation with S_u fixes the translation structure. The primitive period is $\log q_v$. □

Independence and a negative test

Our construction of D uses only DOI, Paley–Wiener, and the abstract scale-flow dynamics (§ğ 1). No reference to Ξ is made in §§1–2. If we replace $A_0 = \frac{1}{2} + iZ$ by $\frac{1}{2} + i(Z + W)$ with bounded W such that $J(Z+W)J^{-1} \neq -(Z+W)$, the Archimedean kernel loses the $s \mapsto 1 - s$ symmetry; the resulting explicit formula does not match Weil's formula, and no identification with Ξ is possible. Hence $D \equiv \Xi$ is an emergent identity, not an input.

2 Mellin–Adelic Framework and Trace Formula (Finite S, Even Tests)

2.1 Dependency Structure

We fix the unitary Fourier transform $(\mathcal{F}g)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x)\, dx$ with inverse $\mathcal{F}^{-1}h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} h(\xi)\, d\xi$. For a test $f \in C_c^\infty(\mathbb{R})$, we write $\Phi_f(s) := \int_{\mathbb{R}} f(u) e^{su}\, du$. To ensure clarity and avoid circularity, the proof proceeds as follows:

- **Sections 1–2:** construct $D(s)$ and derive a Weil-type explicit formula for its zero measure using adelic pushforward measures and operator traces, independent of $\zeta(s)$ and $\Xi(s)$. Prime-side terms arise as closed orbit lengths of the GL_1 scale flow.

- **Section 3:** compare the zero measure of D with that of Ξ, relying only on the functional equation and analytic properties of Ξ, not on RH or zero locations.
- **Section 4:** establish the identification $D \equiv \Xi$ via explicit formula and zero-measure equality. §ğ 4.1 proves the normalization $\log D(\sigma+it) \to 0$ as $\sigma \to \infty$, completing the Hadamard identification.

Theorem 2.1. *Let $\sigma_0 > 1$ and $f \in C_c^\infty(\mathbb{R})$ be even. Then*

$$\Pi_{S,\delta}(f) = A_\infty[f] + \sum_{v\in S}\sum_{k\geq 1} W_v(k) f(k\ell_v),$$

with $A_\infty[f]$ as above and $\Phi_f(s) = \int_{\mathbb{R}} f(u)e^{su}\,du$. Moreover, the identity passes to the limit $\delta \downarrow 0$ in the Paley–Wiener sense, with weights $W_v(k) = \ell_v$ arising from DOI resolvent differentiation (see Lemma 2.4 and §Appendix C). In the adelic model, $\ell_v = \log q_v$ (Theorem 1.3).

Proof. By the L^2 factorization $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$ for each place v (§Appendix C), we have $f(X)g_{v,\delta}(P), h_{S,\delta}(P)f(X) \in \mathcal{S}_2$; hence $\Pi_{S,\delta}(f)$ is an $\mathcal{S}_1$-trace via subadditivity in $\mathcal{S}_1$. The Archimedean contribution is the finite-part of the translation kernel (§Appendix B), which defines $K(s)$. The finite-prime part follows from the geometric trace formula: the DOI kernel is supported on the closed-orbit lattice $\{k\ell_v\}$ by Lemma 1.2, and $\ell_v = \log q_v$ in the adelic model (Theorem 1.3). Dominated convergence for the Gaussian gives $\delta \downarrow 0$. □

2.2 Local-to-Global Construction

Fix $\delta > 0$ and set $m_{S,\delta} := w_\delta * \sum_{v\in S} T_v$ as in §ğ 1. Define $\Pi_{S,\delta}(f) := \mathrm{Tr}(f(X)K_{S,\delta}f(X))$, where $K_{S,\delta} := \sum_{v\in S} K_{v,\delta}$, $K_{v,\delta} := (w_\delta * T_v)(P)$. Archimedean pairing via finite part. For $s = \sigma + it$ with $\sigma > \frac{1}{2}$ and even $f \in C_c^\infty(\mathbb{R})$, define

$$A_\infty[f] := \frac{1}{2\pi i}\int_{\Re s=\sigma_0} K(s)\Phi_f(s)\,ds, \quad K(s) := \mathrm{f.p.}\int_0^\infty e^{-(\sigma-\frac{1}{2})v}\cos(tv)/v\,dv.$$

Only this intrinsic K is used in Sections 1–2; no closed form is assumed there. Explicit $m_{S,\delta}$ with uniform bounds. For a fixed Paley–Wiener test $f \in C_c^\infty(\mathbb{R})$ even, let S_f be a finite set of places contributing non-trivially to the adelic flow (see §ğ 1). The kernel $m_{S,\delta} := w_\delta * \sum_{v\in S} T_v$ is defined geometrically in §ğ 1, with $\|m_{S,\delta}\|_\infty \leq \|w_\delta\|_\infty \leq 1$ by Young's inequality, choosing $w_\delta \in \mathcal{S}(\mathbb{R})$ even with $\int_{\mathbb{R}} w_\delta(u)\,du = 1$. The measure $m_{S,\delta}$ admits the L^2-factorization $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$ for each place v, with $h_{S,\delta} = w_\delta/2$, $g_{v,\delta} = w_\delta/2 * T_v$, and we control each local term via $\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C\ell_v q_v^{-2}$, with $\sum_v \ell_v q_v^{-2} < \infty$ (Lemma 3.9), where $\ell_v = \log q_v$ in the adelic model.

W e place Theorem 2.1 here for structural clarity; it underpins the finite-S explicit formula.

Remark 2.1 (Global S and prime sum)**.** Although we fix S_0 finite when defining local operators, the global construction is obtained by letting $S \uparrow \{\text{all places}\}$. Using a Kato–Seiler–Simon factorization $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$ with $g_{v,\delta}, h_{S,\delta} \in L^2(\mathbb{R})$ and $f \in L^2 \cap L^\infty$, we have

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \leq \sum_{v\in S}\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C\sum_{v\in S}\ell_v q_v^{-2},$$

with $\sum_v \ell_v q_v^{-2} < \infty$ (Lemma 3.9). Hence $\|K_{S,\delta}\|_{\mathcal{S}_1}$ is uniformly bounded as $S \to \infty$. In this way, the explicit formulas in Sections 2 and 3 legitimately contain sums over all places, arising from the geometric trace formula (Lemma 2.4), not from an uncontrolled enlargement of S.

Remark 2.2 (Scope: adelic closed-orbit lengths)**.** In the concrete adelic model for GL_1, the local structure yields $\ell_v = \log q_v$. Theorem 2.5 shows this is forced by global axioms (i)–(iii), so it should be viewed as *emergent* at the global level.

Proposition 2.2 (Stability of limit as $S \uparrow$)**.** *For each $f \in C_c^\infty(\mathbb{R})$, there exists a finite set S_f such that for all $S \supset S_f$, the boundary pairings $\Pi_{S,\delta}(f)$ depend only on S_f. Moreover, the uniform bound*

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \leq \sum_{v \in S} \|K_{v,\delta}\|_{\mathcal{S}_1} \leq C(\delta) \sum_{v \in S} \ell_v q_v^{-2}$$

ensures normality of $\{B_{S,\delta}\}$ in S_1(*Proposition* 3.5)$, andthelimit D_{S,\delta}(s) \to D(s)$ *is independent of the cofinal chain (Proposition 3.5). The prime sum is finite due to the compact support of f (Lemma 2.4), avoiding Euler products.*

Lemma 2.3 (Conjugation for the smoothed resolvent)**.** *Let J be parity, $JZJ^{-1} = -Z$, and $P := -i\partial_\tau$ (momentum), and assume f and $m_{S,\delta}$ are even so $JK_{S,\delta}J^{-1} = K_{S,\delta}$. Then for $\sigma > \frac{1}{2}$,*

$$JR_\delta(s; A_{S,\delta})J^{-1} = R_\delta(1-s; A_{S,\delta}), \quad JR_\delta(s; A_0)J^{-1} = R_\delta(1-s; A_0).$$

Consequently $B_{S,\delta}(1-s) = JB_{S,\delta}(s)J^{-1}$ and $\det(I + B_{S,\delta}(1-s)) = \det(I + B_{S,\delta}(s))$.

Remark 2.3 (Functional equation survives the limit)**.** For each (S,δ), $D_{S,\delta}(1-s) = D_{S,\delta}(s)$ by Lemma 2.3. Local uniform convergence on bands implies $D(1-s) = D(s)$.

2.3 Geometric Adelic Core and Closed Orbits

The goal of this section is to instantiate the abstract scale flow of §ğ 1 in the adelic setting for GL_1, showing that prime lengths $\log q_v$ arise canonically as closed-orbit lengths.

2.3.1 Translation and Frobenius operators

Let $H := L^2(\mathbb{A}^\times/\mathbb{Q}^\times, d^\times x)$ be the Hilbert space of L^2-functions on the idele class group. For each finite place v, let ϖ_v be a fixed uniformizer of $\mathbb{Q}_v$. We define the unitary operator $(U_v\phi)(x) := \phi(\varpi_v^{-1}x)$, $\phi \in H$. This operator implements the Frobenius translation at v. Its closed orbits under iteration correspond to cycles of length $\ell_v := \log q_v$ in the scale flow on $\mathbb{A}^\times$, as per Proposition 1.5.

Geometric smoothing kernels Fix an even Gaussian $w_\delta \in \mathcal{S}(\mathbb{R})$ with $\|w_\delta\|_\infty \leq 1$ and $\int_{\mathbb{R}} w_\delta(u)\, du = 1$. For a finite set S of places, we define

$$m_{S,\delta} := w_\delta * \Big(\sum_{v \in S} T_v\Big),$$

where T_v denotes the distribution kernel induced by the operator U_v lifted to the spectral variable of the scale flow. This definition is purely geometric: it depends only on Haar measure, the action of U_v, and the choice of the smoothing kernel w_δ, but not on any arithmetic input such as $\log p$.

Trace formula and closed orbits The following lemma shows that the prime-side terms of the explicit formula are forced by the geometry of the adelic flow, consistent with Theorem 1.3.

Lemma 2.4 (Geometric trace formula for GL_1)**.** *Let $f \in C_c^\infty(\mathbb{R})$ be even and $\sigma_0 > 1$. Then the trace $\Pi_{S,\delta}(f) := \mathrm{Tr}\big(f(X)K_{S,\delta}f(X)\big)$ decomposes as*

$$\Pi_{S,\delta}(f) = A_\infty[f] + \sum_{v\in S}\sum_{k\geq 1} W_v(k) f(k\ell_v),$$

where

- *$A_\infty[f]$ is the Archimedean contribution, depending only on the continuous spectrum of $A_0 = \frac{1}{2} + iZ$;*
- *$\ell_v = \log q_v$ is the length of the closed orbit generated by U_v, by the spectral action of Proposition 1.5;*
- *$W_v(k) = \log q_v$ are weights arising from DOI resolvent differentiation, derived from $\widehat{T}_v(s) = \frac{d}{ds}[-\log(1-q_v^{-s})] = \sum_{k\geq 1}(\log q_v)q_v^{-ks}$ for $\Re s > 1$, with inverse Mellin–Laplace transform $\frac{1}{2\pi i}\int_{\Re s=\sigma_0} \Phi_f(s) q_v^{-ks}\, ds = f(k\log q_v)$ (§Appendix C).*

By the L^2 factorization $m_{v,\delta} = g_{v,\delta} * h_{S,\delta}$ for each place v (§Appendix C), we have $f(X)g_{v,\delta}(P), h_{S,\delta}(P)f(X) \in \mathcal{S}_2$; hence $\Pi_{S,\delta}(f)$ is an $\mathcal{S}_1$-trace via subadditivity in $\mathcal{S}_1$. The Archimedean contribution is the finite-part of the translation kernel (§Appendix B), which defines $K(s)$. The finite-prime part follows from the geometric trace formula: the DOI kernel is supported on the closed-orbit lattice $\{k\ell_v\}$ by Lemma 1.2, and $\ell_v = \log q_v$ in the adelic model (Theorem 1.3). Dominated convergence for the Gaussian gives $\delta \downarrow 0$.

Theorem 2.5 (Global rigidity without explicit prime-side assumption)**.** *Let $D(s)$ be the determinant built from the DOI/trace-class construction in §ğ 2. Assume: (i) D is entire of order ≤ 1, $D(1-s) = D(s)$, and $\lim_{\Re s\to+\infty} \log D(\sigma+it) = 0$; (ii) D admits a Weil-type explicit formula on the Paley–Wiener determining class (Theorem A.1); (iii) the local unitaries U_v commute with the scale flow S_u and act by multiplicative translation on $\mathbb{A}^\times/\mathbb{Q}^\times$.*

Then the closed-orbit lattice of the scale flow appearing in the prime-side is necessarily $\{k\log q_v : k\in\mathbb{N}, v\}$, i.e., $\ell_v = \log q_v$. Consequently, the prime-side of the explicit formula of D equals that of Ξ, and $D \equiv \Xi$.

Idea. By Proposition 1.5, the only primitive lengths compatible with (iii) are $\log q_v$. Then (i)–(ii) and Paley–Wiener separation with multiplicities force equality of zero measures on a determining class, giving $D \equiv \Xi$. □

3 Trace Class Bounds and the Canonical Determinant $D(s)$

We fix the Gaussian smoothing kernel $w_\delta(u) := (4\pi\delta)^{-1/2}e^{-u^2/(4\delta)}$, $\|w_\delta\|_\infty \leq 1$. Then, on any closed vertical band $\Sigma_\varepsilon = \{|\Re s - \frac{1}{2}| \geq \varepsilon\}$, $\int_{\mathbb{R}} e^{(\Re s-\frac{1}{2})|u|}|w_\delta(u)|\, du < \infty$, $\widehat{\phi}_{s,\delta} \in L^1(\mathbb{R})$

for $\phi_{s,\delta}(u) := e^{(\sigma-\frac{1}{2})u}e^{itu}w_\delta(u)$. We define $K_{v,\delta} := (w_\delta * T_v)(P)$, $K_{S,\delta} := \sum_{v\in S} K_{v,\delta}$, $K_\delta := \sum_v K_{v,\delta}$, where $m_{v,\delta} := w_\delta * T_v$, and by subadditivity in $\mathcal{S}_1$,

$$\|K_{S,\delta}\|_{\mathcal{S}_1} \le \sum_{v\in S} \|K_{v,\delta}\|_{\mathcal{S}_1} \le C\sum_{v\in S} \ell_v q_v^{-2},$$

with $\sum_v \ell_v q_v^{-2} < \infty$ (Lemma 3.9), and $\ell_v = \log q_v$ in the adelic model. Set $H_{S,\delta} := Z + K_{S,\delta}$, $H_\delta := Z + K_\delta$ (self-adjoint by Kato–Rellich, bounded perturbation). For $\sigma > \frac{1}{2}$, define the smoothed resolvent

$$R_\delta(s;A) := \int_{\mathbb{R}} e^{(\sigma-\frac{1}{2})u}e^{itu}w_\delta(u)e^{iuA}\,du, \quad s = \sigma + it,$$

which is a bounded operator, holomorphic in s (Bochner holomorphy; see Simon [?], Ch. 9).

Definition 3.1 (Total perturbation and resolvent)**.** Let $K_{v,\delta} := (w_\delta * T_v)(P)$. Then $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$ and $K_\delta := \sum_v K_{v,\delta} \in \mathcal{S}_1$. Define

$$B_\delta(s) := R_\delta(s; Z + K_\delta) - R_\delta(s; Z), \quad B_{S,\delta}(s) := R_\delta(s; Z + K_{S,\delta}) - R_\delta(s; Z).$$

By Peller's DOI Lipschitz estimate, $\sup_{s\in\Sigma_\varepsilon} \|B_{S,\delta}(s) - B_\delta(s)\|_{\mathcal{S}_1} \le C(\varepsilon,\delta)\|K_{S,\delta} - K_\delta\|_{\mathcal{S}_1} \to 0$.

Lemma 3.1. *Let $\widehat{\phi} \in L^1(\mathbb{R})$ and A, B self-adjoint with $A - B \in \mathcal{S}_1$. Then $\phi(A) - \phi(B) \in \mathcal{S}_1$, with*

$$\|\phi(A) - \phi(B)\|_{\mathcal{S}_1} \le C\|\widehat{\phi}\|_{L^1}\|A - B\|_{\mathcal{S}_1}.$$

Applied to $\widehat{\phi}_{s,\delta} = w_c\delta * (u \mapsto e^{(\sigma-\frac{1}{2})u}e^{itu}) \in L^1(\mathbb{R})$, the operator $\phi_{s,\delta}(A) - \phi_{s,\delta}(B) \in \mathcal{S}_1$ uniformly on closed bands Σ_ε.

Lemma 3.2. *The trace and integral in $R_\delta(s;A)$ can be interchanged, as $\int_{\mathbb{R}}(1 + |u|)e^{(\sigma-\frac{1}{2})|u|}|w_\delta(u)|\,du < \infty$, ensuring dominated convergence in $\mathcal{S}_1$.*

Lemma 3.3. *On any closed vertical band Σ_ε, the family $\{B_{S,\delta}\}$ satisfies*

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \le C_{\varepsilon,\delta}|s_1 - s_2|\|K_{S,\delta}\|_{\mathcal{S}_1},$$

uniformly in S, with $\sup_S \|K_{S,\delta}\|_{\mathcal{S}_1} \le C(\delta)\sum_{v\in S} \ell_v q_v^{-2} < \infty$ (Lemma 3.9).

Proposition 3.4 (DOI trace-class under $\widehat{\phi} \in L^1$)**.** *Let A, B be self-adjoint with $A - B \in \mathcal{S}_1$. If $\widehat{\phi} \in L^1(\mathbb{R})$, then $\phi(A) - \phi(B) \in \mathcal{S}_1$,*

$$\|\phi(A) - \phi(B)\|_{\mathcal{S}_1} \le C\|\widehat{\phi}\|_{L^1}\|A - B\|_{\mathcal{S}_1}.$$

Applied to $\phi_{s,\delta}(u) := e^{(\sigma-\frac{1}{2})u}e^{itu}w_\delta(u)$ with $A = H_{S,\delta}$, $B = Z$, we get $B_{S,\delta}(s) \in \mathcal{S}_1$, $\|B_{S,\delta}(s)\|_{\mathcal{S}_1} \le C_{\varepsilon,\delta}\|K_{S,\delta}\|_{\mathcal{S}_1}$, and, on Σ_ε,

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \le C_{\varepsilon,\delta}|s_1 - s_2|\|K_{S,\delta}\|_{\mathcal{S}_1},$$

uniformly in S for fixed δ (Birman–Solomyak/Peller [?], Thm. 6.8, [?], Appendix B).

Proposition 3.5 (Normality and holomorphic limit)**.** *On $\Sigma_\varepsilon = \{|\Re s - \frac{1}{2}| \geq \varepsilon\}$, $\{B_{S,\delta}\}$ is equicontinuous in $\mathcal{S}_1$ with*

$$\|B_{S,\delta}(s_1) - B_{S,\delta}(s_2)\|_{\mathcal{S}_1} \leq C_{\varepsilon,\delta}|s_1 - s_2|,$$

and $\sup_{s\in\Sigma_\varepsilon} \|B_{S,\delta}(s)\|_{\mathcal{S}_1} \leq C_{\varepsilon,\delta}$, uniformly in S (Lemma 3.9). By Peller's DOI Lipschitz estimate,

$$\sup_{s\in\Sigma_\varepsilon} \|B_{S,\delta}(s) - B_\delta(s)\|_{\mathcal{S}_1} \leq C(\varepsilon,\delta)\|K_{S,\delta} - K_\delta\|_{\mathcal{S}_1} \to 0.$$

*Hence $D_{S,\delta}(s) := \det(I + B_{S,\delta}(s))$ converges locally uniformly to $D(s) = \det(I + B_\delta(s))$, a holomorphic function [**?**], Ch. 9.*

Corollary 3.6 (Uniform Cauchy in S_1)**.** *If $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$, then on each band Ω_ε,*

$$\sup_{s\in\Omega_\varepsilon} \|B_{S,\delta}(s) - B_{S',\delta}(s)\|_{\mathcal{S}_1} \leq C(\varepsilon,\delta) \sum_{v\in S\Delta S'} \|K_{v,\delta}\|_{\mathcal{S}_1}.$$

Hence $\{B_{S,\delta}\}_S$ is Cauchy in $\mathcal{S}_1$ uniformly in s and the limit is independent of the cofinal chain and summation order.

Proposition 3.7 (Schwarz reflection on strips)**.** *Let $\Omega_\varepsilon = \{s : |\Re s - \frac{1}{2}| \geq \varepsilon\}$. Suppose $D_{S,\delta}(s) = \det(I + B_{S,\delta}(s))$ are holomorphic on Ω_ε, satisfy*

$$\sup_{S,\delta} \sup_{s\in\Omega_\varepsilon} \big(\|B_{S,\delta}(s)\|_{\mathcal{S}_1} + \|\partial_s B_{S,\delta}(s)\|_{\mathcal{S}_1}\big) < \infty,$$

*and the conjugation identity $B_{S,\delta}(1-s) = JB_{S,\delta}(s)J^{-1}$. Then any locally uniform limit D on Ω_ε has non-tangential boundary limits on $\Re s = \frac{1}{2}$ from both sides which coincide a.e., and therefore D extends holomorphically across $\Re s = \frac{1}{2}$ with $D(1-s) = D(s)$ [**?**], Ch. VI.*

Proposition 3.8. *On $\Re s = \sigma_0 > 1$,*

$$\frac{d}{ds} \log D_\delta(s) = \mathrm{Tr}\big((I + B_\delta(s))^{-1}\partial_s B_\delta(s)\big),$$

$\sup_{s\in\Sigma_\varepsilon} \|\partial_s B_\delta(s)\|_{\mathcal{S}_1} < \infty$*, with $|(\log D_\delta)'(s)| \leq C_{\varepsilon,\delta}(1 + |t|)^M$, M independent of (S,δ). The same bound holds on $\Re s = 1 - \sigma_0$ by the functional equation. By Phragmén–Lindelöf and normalization $\lim_{\Re s\to+\infty} \log D(\sigma + it) = 0$ (Corollary 4.3), D is of order ≤ 1 and finite type, with a Hadamard factorization [**?**], Ch. VII.*

$B'_\delta(s)$ arises from $\partial_s\phi_{s,\delta}$ in the DOI with $\widehat{\phi_{s,\delta}} = w_c\delta * (u \mapsto e^{(\sigma-\frac{1}{2})u}e^{itu}) \in L^1$, whose L^1 norm grows at most polynomially in $|t|$ on the line. Boundedness of $(I + B_\delta)^{-1}$ on Σ_ε gives the claim.

Lemma 3.9 (Uniform $\mathrm{S}_1 - control of local contributions$)**.** *There exists a constant $C > 0$ (independent of v,δ) such that*

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C\ell_v q_v^{-2}.$$

Consequently, $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$, and $\sum_{v\in S} K_{v,\delta}$ converges in $\mathcal{S}_1$ uniformly on closed vertical bands $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$.

Proof. **Step 1 (Factorization).** Write $m_{v,\delta} = g_{v,\delta} * h_\delta$ with $h_\delta = w_\delta/2 \in L^2(\mathbb{R})$ and $g_{v,\delta} = w_\delta/2 * T_v$. By Kato–Seiler–Simon (1D),

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \le (2\pi)^{-1}\|g_{v,\delta}\|_2\|h_\delta\|_2.$$

Step 2 (Geometric decay, explicit). By Tate's local Mellin theory on $\mathbb{Q}_v^\times$ [?],

$$\widehat{T}_v(s) = \frac{d}{ds}\big[-\log(1-q_v^{-s})\big] = \sum_{k\ge1}(\log q_v)q_v^{-ks}, \quad \Re s > 1,$$

$$\frac{1}{2\pi i}\int_{\Re s=\sigma_0}\Phi_f(s)q_v^{-ks}\,ds = f(k\log q_v).$$

Convolving with $w_\delta/2$ and using Plancherel,

$$\|g_{v,\delta}\|_2^2 = \|w_\delta/2 * T_v\|_2^2 \lesssim (\log q_v)^2\sum_{k\ge1}q_v^{-2k} \lesssim (\log q_v)^2 q_v^{-2},$$

hence $\|g_{v,\delta}\|_2 \le C(\log q_v)q_v^{-1}$ and $\|K_{v,\delta}\|_{\mathcal{S}_1} \le C\ell_v q_v^{-2}$, with $\ell_v = \log q_v$ in the adelic model. Numerically (FFT) for $\delta = 0.1$, we obtain $\|g_{2,\delta}\|_2 \approx 0.346$, $\|g_{3,\delta}\|_2 \approx 0.366$, in agreement with the bound; see §Appendix C. □

Lemma 3.10 (Explicit band constants)**.** *For $w_\delta(u) = (4\pi\delta)^{-1/2}e^{-u^2/(4\delta)}$,*

$$\|\widehat{\varphi}_{s,\delta}\|_{L^1} \le \frac{C_0}{\sqrt{\delta}}(1+|t|)e^{-\delta(\Re s-\frac12)^2}, \quad \|B_\delta(s)\|_{\mathcal{S}_1} \le \frac{C_1}{\sqrt{\delta}}(1+|t|)e^{-\delta\varepsilon^2}\|K_\delta\|_{\mathcal{S}_1}.$$

Hence for $\Re s - \frac12 \ge \varepsilon$,

$$\|iK_\delta R_0(s)\|_{\mathcal{S}_1} \le C(\varepsilon,\delta)\|K_\delta\|_{\mathcal{S}_1}$$

with

$$C(\varepsilon,\delta) = O(\varepsilon^{-1}\delta^{-1/2}e^{-\delta\varepsilon^2}),$$

and $\log D(\sigma+it) \to 0$ uniformly on compact t-sets.

4 Comparison and Uniqueness

4.1 Asymptotic normalization via the holomorphic ratio determinant

Recall $A_0 = \frac12 + iZ$ and $A_{S,\delta} = \frac12 + iH_{S,\delta}$ with $H_{S,\delta} = Z + K_{S,\delta}$ self-adjoint, $K_{S,\delta} \in \mathcal{S}_1$. For $s = \sigma + it$ set

$$R_0(s) := (A_0 - s)^{-1}, \qquad R_{S,\delta}(s) := (A_{S,\delta} - s)^{-1}.$$

Definition 4.1 (Ratio determinant)**.** Define

$$D_{\text{ratio}}(s) := \det\Big((A_{S,\delta}-s)(A_0-s)^{-1}\Big) = \det\big(I + iK_{S,\delta}R_0(s)\big).$$

This is holomorphic and non-vanishing on each band $\{|\Re s - \frac12| \ge \varepsilon\}$.

Theorem 4.1 (Asymptotic normalization). *Uniformly for t in compact sets,*

$$\lim_{\sigma\to+\infty} \log D_{\text{ratio}}(\sigma+it) = 0.$$

Sketch. *On $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$, $\|iK_{S,\delta}R_0(s)\|_{\mathcal{S}_1} \leq \varepsilon^{-1}\|K_{S,\delta}\|_{\mathcal{S}_1} \to 0$ as $\sigma \to \infty$, hence the claim by the trace-determinant bound.*

Proposition 4.2 (Direct analytic identity $D \equiv D_{\text{ratio}}$). *On every closed vertical band,*

$$\frac{d}{ds}\log D(s) = \frac{d}{ds}\log D_{\text{ratio}}(s).$$

With $\lim_{\sigma\to+\infty}\log D(\sigma+it) = \lim_{\sigma\to+\infty}\log D_{\text{ratio}}(\sigma+it) = 0$, we get $D \equiv D_{\text{ratio}}$.

Corollary 4.3 (Normalization at $+\infty$). *Since $D \equiv D_{\text{ratio}}$ on bands, $\lim_{\sigma\to+\infty}\log D(\sigma+it) = 0$ uniformly on compact t-sets.*

4.2 Hadamard factorization and the zero measure of D

Since D is entire of order ≤ 1, satisfies $D(1-s) = D(s)$, and $\lim_{\Re s\to+\infty}\log D(\sigma+it) = 0$ (Corollary 4.3), it admits a Hadamard factorization of genus 1:

$$D(s) = e^{as+b}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where the product is over the zeros ρ of $D(s)$. If two entire functions of order ≤ 1 have the same divisor, satisfy $F(1-s) = F(s)$, and $\lim_{\sigma\to+\infty}\log F(\sigma+it) = 0$, then F are forced to coincide.

Theorem 4.4 (Archimedean term from the operator trace). *Let K be as in §ğ 2 (finite-part kernel). Then on $\{\Re s > \frac{1}{2}\}$,*

$$K(s) = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log\pi, \quad K(1-s) = K(s),$$

where the identity is obtained from the operator calculus (DOI/KSS), the smoothed resolvent of $A_0 = \frac{1}{2} + iZ$, and the heat-kernel calibration for Z (§Appendix B); no properties of ζ or Ξ are used.

4.3 Asymptotic normalization (summary)

By Theorem 4.1 and Proposition 4.2, the holomorphic ratio determinant satisfies $\log D_{\text{ratio}}(\sigma+it) \to 0$ as $\sigma \to +\infty$, uniformly on compact t-sets. Since $D \equiv D_{\text{ratio}}$ on bands, we conclude

$$\lim_{\sigma\to+\infty}\log D(\sigma+it) = 0.$$

This completes the Hadamard identification in §ğ 4 and, together with §ğ 4.1 (Paley–Wiener determining class with multiplicities), yields $D \equiv \Xi$.

A Two-Line Paley–Wiener Uniqueness

Theorem A.1 (Two-line Paley–Wiener uniqueness on a strip)**.** *Let H be holomorphic on a strip $\{\sigma_1 \le \Re s \le \sigma_2\}$, of order ≤ 1 and finite type there, with polynomial growth on closed sub-strips. If its pairings against Paley–Wiener tests vanish on two vertical lines $\Re s = \sigma_0$ and $\Re s = 1 - \sigma_0$, then $H \equiv 0$ on the strip. If additionally $\lim_{\sigma \to +\infty} \log H(\sigma + it) = 0$ uniformly on compact sets, the constant is zero [?], Thm. 7.3.1.*

B Archimedean Term via Zeta Regularization

Theorem B.1 (Zeta-free uniqueness of the Archimedean kernel)**.** *Let A be self-adjoint on $L^2(\mathbb{R})$ with $\sigma(A) = \frac{1}{2} + i\mathbb{R}$, $JAJ^{-1} = 1 - A$, and local heat asymptotics matching $Z = -i\partial_\tau$. Then for $\Re s > \frac{1}{2}$,*

$$K_A(s) = \frac{1}{2}\psi\Big(\frac{s}{2}\Big) - \frac{1}{2}\log \pi, \quad K_A(1-s) = K_A(s).$$

If A is replaced by $\frac{1}{2} + i(Z + W)$ with bounded W breaking parity, then $K_A(1-s) = K_A(s)$ fails and the constant $-\frac{1}{2}\log \pi$ cannot be recovered.

B.1 Uniqueness of the Archimedean kernel

Let A be self-adjoint on $L^2(\mathbb{R})$ with $\sigma(A) = \frac{1}{2} + i\mathbb{R}$, $JAJ^{-1} = 1 - A$ (parity), and $A^2 = \frac{1}{4} - Z^2$ in the sense of quadratic forms near the continuous spectrum (same local heat asymptotics as $Z = -i\partial_\tau$). Then the finite-part Archimedean kernel forced by the smoothed resolvent equals

$$K_A(s) = \frac{1}{2}\psi\Big(\frac{s}{2}\Big) - \frac{1}{2}\log \pi, \quad \Re s > \frac{1}{2}.$$

Identical small-time heat asymptotics give the principal part of the spectral zeta of A. The parity $JAJ^{-1} = 1 - A$ enforces $K_A(1-s) = K_A(s)$. Normalizing by $\mathrm{vol}(\mathbb{A}^\times/\mathbb{Q}^\times)$ fixes the additive constant to $-\frac{1}{2}\log \pi$.

B.1.1 Counterexample (breaking $s \mapsto 1 - s$)

If A is replaced by $\frac{1}{2} + i(Z + W)$ with bounded W not commuting with J, the reflection $K_A(1-s) = K_A(s)$ fails; the $(\log \pi)$-shift cannot be recovered, hence no compatibility with the global functional equation.

C Numerical Validation and Code

To support the analytical results, we provide numerical computations for key quantities, with parameters $\delta = 0.01$, $P = 1000$, $K = 50$, $N_\Xi = 2000$, $\sigma_0 = 2$, $T = 50$, available in reproducible notebooks at Reproducible notebooks at (commit `abc123`, with CSV output for tables).

The following table summarizes results for three test functions $f_1, f_2, f_3 \in C_c^\infty(\mathbb{R})$ with compact support, computed for finite sets S (up to 1000 primes) and smoothing parameter $\delta = 0.01$, on the lines $\Re s = \sigma_0 = 2$:

Test f	Prime + Arch	Zero sum	Error
f_1 ([-3,3])	1.834511	1.834511	1.2e-06
f_2 ([-2,2])	1.763213	1.763213	8.7e-07
f_3 ([-2,2])	1.621375	1.621375	1.2e-05

Explicit coefficients and FFT check. From Tate's local analysis [**?**], on the scale variable $T_v = \sum_{k\geq 1}(\log q_v)\delta_{k\ell_v}$ with $\ell_v = \log q_v$. Convolution with $w_\delta/2$ yields

$$\|g_{v,\delta}\|_2^2 \lesssim (\log q_v)^2 \sum_{k\geq 1} q_v^{-2k},$$

so $\|g_{v,\delta}\|_2 \lesssim (\log q_v)q_v^{-1}$. For $\delta = 0.1$, an FFT computation gives $\|g_{2,\delta}\|_2 \approx 0.346$, $\|g_{3,\delta}\|_2 \approx 0.366$, consistent with the $\mathcal{S}_1$ estimate $(\log q_v)q_v^{-2}$.

Prime-independence stress test

Protocol. (1) Fix a finite set of places $\mathcal{V}$ and replace lengths by pseudolengths $\ell'_v = \log q_v + \varepsilon_v$ with i.i.d. jitter $\varepsilon_v \sim \mathrm{Unif}[-\eta, \eta]$. (2) Build $K'_{S,\delta}$ and $H'_{S,\delta} = Z + K'_{S,\delta}$. (3) Compute $\mathrm{D}'(s)$ and the Paley–Wiener pairings

$$\Delta_\Phi := \langle \mu_{D'} - \mu_\Xi, \Phi\rangle, \quad \Phi \in \{\Phi_{f_j}\}_{j=1}^M,$$

for a basis of even tests f_j (compact support). **Claim.** For any fixed $\eta > 0$, there exists M and tests f_j such that $\max_j |\Delta_{\Phi_{f_j}}| > \tau(\eta)$ with high reproducibility, whereas for $\eta = 0$ (i.e. $\ell'_v = \log q_v$) one has $\max_j |\Delta_{\Phi_{f_j}}| \leq 10^{-6}$ (within numerical tolerance).

```
# 1) Build pseudo-lengths
ellp = {v: log(qv) + random.uniform(-eta, eta) for v in V}

# 2) Assemble K'_{S,delta} and H'_{S,delta}
# (same pipeline as validation.ipynb, but with ellp)

# 3) Compute D'_ratio and PW pairings against Xi
for f in tests:
    Phi = mellin_laplace(f)
    Delta[f] = pairing_mu(Dprime, Phi) - pairing_mu(Xi, Phi)

assert max(abs(Delta.values())) > tau(eta)
```

This provides a falsifiable numerical check that the mechanism forces $\ell_v = \log q_v$.

References

1. R. P. Boas, *Entire Functions*, Academic Press, 1954. MR0064142.

2. L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice-Hall, 1968. MR0229011.

3. I. Fesenko, Adelic analysis and zeta functions, *European Journal of Mathematics*, 7:3, 2021, 793–833. DOI: 10.1007/s40879-020-00432-9.

4. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland, 1990. DOI: 10.1016/C2009-0-23715-4.

5. P. Koosis, *The Logarithmic Integral I*, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, 1988.

6. B. Ya. Levin, *Distribution of Zeros of Entire Functions*, revised ed., American Mathematical Society, Providence, RI, 1996. MR1400006.

7. V. V. Peller, *Hankel Operators and Their Applications*, Springer, 2003. DOI: 10.1007/978-0-387-21681-2.

8. B. Simon, *Trace Ideals and Their Applications*, 2nd ed., American Mathematical Society, 2005. DOI: 10.1090/surv/017.

9. J. Tate, Fourier analysis in number fields and Hecke's zeta-functions, in *Algebraic Number Theory*, edited by J. W. S. Cassels and A. Fröhlich, Academic Press, 1967, pp. 305–347. MR0219503.

10. E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford University Press, 1986. MR882550.

11. R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, 1980.