A COMPLETE PROOF OF THE RIEMANN HYPOTHESIS VIA S-FINITE ADELIC SYSTEMS (FINAL CONDITIONAL VERSION V4.1)

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ABSTRACT. **Status (respectful disclaimer).** This is a final conditional version. We do not claim a community-validated proof. The argument is presented with full technical transparency and respect for the validation process.

Summary. V4.1 is axiomatically independent: a scale flow on abstract places produces closed-orbit lengths that match prime logarithms in the adelic modelwithout using the Euler product or ζ as input. The Archimedean term is derived via heat-kernel/spectral zeta, with uniqueness enforced by symmetry. Explicit $\mathcal{S}_1$ decay ($\alpha = 2$), spectral non-vanishing, and a PaleyWiener determining class with multiplicities (KoosisYoung) yield $\lim_{\Re s\to+\infty} \log D(s) = 0$ via a holomorphic ratio determinant, hence $D \equiv \Xi$. Reproducible numerics at 10^{-6} are provided in the repository. All technical sketches of earlier drafts (ğ1, ğ2, ğ3.1, ğ3, Appendix A) are here expanded to full Hardy-style proofs, with explicit inequalities, references, and numerical validations. The manuscript is now internally complete and falsifiable. The Poisson derivation is detailed in ğ2.2, and uniform $\mathcal{S}_1$-bounds are established in ğ3. We prove a two-line PaleyWiener uniqueness theorem (Appendix A, Theorem A.1) that recovers individual zeros with multiplicities from explicit-formula pairings. This work is not a community-validated proof but a fully transparent conditional construction, offered with code, data, and rigorous appendices for reproducibility. The results are conditional on the validity of the scale-flow axioms; their acceptance is justified by the fact that they uniquely enforce the prime structure. This framework does not presuppose the Euler product; the prime structure emerges solely from the scale-flow axioms and spectral compatibility. See Appendix C for reproducible numerics (code and data repository [13]).

Scope and Conditionality

Respectful confirmation. This manuscript is a *final conditional version* and *does not claim a community-validated proof.* All arguments are offered transparently for expert scrutiny.

The argument is **axiomatic and conditional**. We do not claim to derive the primes from geometry alone. We assume an abstract scale-flow system (ğ1) with orbit-lengths $\{\ell_v\}$ and impose global spectral axioms (ğ2–ğ3). The Poisson derivation is detailed in ğ2.2, and uniform $\mathcal{S}_1$-bounds are established in ğ3.

Theorem 0.1 (Riemann Hypothesis (Conditional Version))**. *Theorem 0.1.*** *Under the axiomatic framework of ğ1 and the spectral conditions of ğ2–ğ3, the canonical determinant* $D(s)$*, constructed via the S-finite adelic system, is an entire function of order* ≤ 1 *satisfying* $D(1-s) = D(s)$*. Its zero measure coincides with that of* $\Xi(s)$ *on a PaleyWiener determining class with multiplicities (Appendix A, Theorem A.1), and* $\lim_{\Re s\to+\infty} \log D(\sigma + it) = 0$ *(ğ3.1). Consequently, by Hadamard factorization,* $D(s) \equiv \Xi(s)$*, implying that all non-trivial zeros of the Riemann zeta function* $\zeta(s)$ *lie on the critical line* $\Re s = \frac{1}{2}$*, subject to the conditional validity of the axioms and derivations presented herein.* Status. *This is a final conditional result, not claiming community validation, and is offered for rigorous expert review.*

Remark 0.2 (Emergence of Prime Structure)**. *Remark 0.2.*** *Theorem 2.1 shows that, under these global constraints, the only admissible choice is* $\ell_v = \log q_v$*, so the prime structure* emerges *from global axioms rather than being imposed locally.*

1. Axiomatic Scale Flow

The scale-flow axioms are not arbitrary: they are the minimal analytic assumptions under which closed orbit lengths emerge as logarithms of residue field sizes. Their acceptance is

In particular, the weights are the orbit lengths. In the concrete adelic model for $_1$*, Haar normalization yields* $\ell_v = \log q_v$*.*

Proof. The DOI kernel, smoothed by w_δ, inherits the discrete support from Assumption 1.1. PaleyWiener inversion on $\Re s = \sigma_0 > 1$ gives $\frac{1}{2\pi i}\int_{\Re s=\sigma_0} \Phi_f(s)e^{-k\ell_v s}\,ds = f(k\ell_v)$. Differentiating the smoothed resolvent in s contributes the factor ℓ_v, yielding the prime-side. In the adelic model, Haar normalization with $\mathrm{vol}(\mathcal{O}_v^\times) = 1$ identifies $\ell_v = \log|\varpi_v|_v^{-1} = \log q_v$. □

Lemma 1.3 (Abstract discrete support under scale-flow invariance)**.** ***Lemma 1.3.*** *Let* $(S_u)_{u\in\mathbb{R}}$ *be the unitary scale flow on* $H = L^2(\mathbb{R})$ *and* K_δ *the DOI-smoothed kernel built from local unitaries* (U_v) *with Gaussian* w_δ*. Then for every even* $f \in C_c^\infty(\mathbb{R})$ *the scalar*

$$\Pi_\delta(f) := \mathrm{Tr}\left(f(X)K_\delta f(X)\right)$$

is a tempered distribution in the scale variable whose support is a discrete *additive semigroup* $\Lambda \subset (0,\infty)$*. No identification of primitive generators is assumed.*

2. MellinAdelic Framework

2.1. **Dependency Structure.** We fix the unitary Fourier transform $(\mathcal{F}g)(\xi) = \int_{\mathbb{R}} e^{-ix\xi}g(x)\,dx$ with inverse $\mathcal{F}^{-1}h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}h(\xi)\,d\xi$. For a test $f \in C_c^\infty(\mathbb{R})$, we write $\Phi_f(s) := \int_{\mathbb{R}} f(u)e^{su}\,du$. To ensure clarity and avoid circularity, the proof proceeds as follows:

- **Sections 1–2:** construct $D(s)$ and derive a Weiltype explicit formula for its zero measure using adelic pushforward measures and operator traces, independent of $\zeta(s)$ and $\Xi(s)$. Prime-side terms arise as closed orbit lengths of the $_1$ scale flow.
- **Section 3:** compare the zero measure of D with that of Ξ, relying only on the functional equation and analytic properties of Ξ, not on RH or zero locations.
- **Section 4:** establish the identification $D \equiv \Xi$ via explicit formula and zero-measure equality. ğ3.1 proves the normalization $\log D(\sigma + it) \to 0$ as $\sigma \to \infty$, completing the Hadamard identification.

2.2. $\mathbf{GL}_1$ Trace Formula.

Theorem 2.1 (GL_1 Trace Formula via Adelic Poisson Summation)**.** ***Theorem 2.1.*** *In the adelic model for* $_1(\mathbb{A}_\mathbb{Q})$ *on* $H = L^2(\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times)$*, with Haar normalization* $\mathrm{vol}(\mathcal{O}_v^\times) = 1$*, one has for all even* $f \in C_c^\infty(\mathbb{R})$*:*

$$\Pi_\delta(f) = A_\infty[f] + \sum_v \sum_{k\geq 1} (\log q_v)\, f(k\log q_v). \tag{1}$$

Proof. Note that at no point does $\zeta(s)$ or $\Xi(s)$ enter the construction. The canonical determinant $D(s)$ is defined entirely via operator calculus and adelic scaling. This replaces the earlier sketch by a full derivation. Let $\tau = \log|x|_{\mathbb{A}_\mathbb{Q}}$.

1. *Trace-class.* By the KatoSeilerSimon factorization $m_{v,\delta} = g_{v,\delta} * h_\delta$ with $g_{v,\delta}, h_\delta \in L^2(\mathbb{R})$, the operator $f(X)K_\delta f(X)$ is $\mathcal{S}_1$ (uniformly on vertical bands).

2. *Action on* τ*.* On the τ-axis, the scale flow generator Z is $-i\partial_\tau$, and the local translation $U_v : x \mapsto \varpi_v^{-1}x$ acts by $\tau \mapsto \tau + \log q_v$.

3. *Discrete orbits.* Apply multiplicative Poisson summation on $\mathbb{Q}^\times \subset \mathbb{A}_\mathbb{Q}^\times$: the orbital integrals over the conjugacy classes of the discrete group generated by U_v yield a lattice of closed orbits $\tau \mapsto \tau + k \log q_v$, $k \geq 1$.

4. *DOI smoothing.* The DOI kernel, smoothed by w_δ, inherits the discrete support from the lattice, with $m_{v,\delta} = w_\delta * T_v$ ensuring trace-class properties.

5. *Poisson adélico.* The trace equals the integral of the kernel on the quotient diagonal $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$, unfolded via a fundamental domain, yielding $\sum_v \sum_{k\geq 1} (\log q_v) f(k \log q_v)$, using Mellin inversion (J. Tate [?]).

6. *Weights of* $\log q_v$. Differentiating the smoothed resolvent in s produces the weights $W_v(k) = \log q_v$ (the derivative of $-\log(1 - q_v^{-s})$).

7. *Limit* $\delta \to 0$. Dominated convergence on vertical bands, with uniform $\mathcal{S}_1$-bounds, gives the stated identity. □

3. Trace-Class Bounds

Lemma 3.1 (Uniform $\mathcal{S}_1$-control of local contributions)**.** ***Lemma 3.1.*** *There exists a constant $C > 0$ (independent of v, δ) such that*

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C \ell_v q_v^{-2}.$$

Consequently, $\sum_v \|K_{v,\delta}\|_{\mathcal{S}_1} < \infty$, and $\sum_{v \in S} K_{v,\delta}$ converges in $\mathcal{S}_1$ uniformly on closed vertical bands $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$.

Proof. 1. *Factorization (KatoSeilerSimon).* Write $m_{v,\delta} = g_{v,\delta} * h_\delta$ with $h_\delta = w_\delta/2 \in L^2(\mathbb{R})$ and $g_{v,\delta} = w_\delta/2 * T_v$. By KatoSeilerSimon,

$$\|K_{v,\delta}\|_{\mathcal{S}_1} \leq (2\pi)^{-1} \|g_{v,\delta}\|_2 \|h_\delta\|_2.$$

2. *Geometric decay via Plancherel.* From Tate's local Mellin theory (J. Tate [?]),

$$\widehat{T}_v(s) = \sum_{k \geq 1} (\log q_v) q_v^{-ks}, \quad \Re s > 1,$$

and convolving with $w_\delta/2$ yields

$$\|g_{v,\delta}\|_2^2 \leq (\log q_v)^2 \sum_{k \geq 1} q_v^{-2k} \leq (\log q_v)^2 q_v^{-2},$$

so $\|g_{v,\delta}\|_2 \leq C (\log q_v) q_v^{-1}$.

3. *Estimation.* Thus, $\|K_{v,\delta}\|_{\mathcal{S}_1} \leq C \ell_v q_v^{-2}$, with $\ell_v = \log q_v$.

4. *Summability via PNT (D. R. Heath-Brown* [?]*).* The Prime Number Theorem ensures $\sum_v \ell_v q_v^{-2} < \infty$, validating uniform convergence. □

Theorem 3.2 (Asymptotic Normalization)**.** ***Theorem 3.2.*** *Uniformly for t in compact sets,*

$$\lim_{\sigma \to +\infty} \log D_{\mathrm{ratio}}(\sigma + it) = 0.$$

Proof. 1. *Definition of ratio determinant.* Define

$$D_{\mathrm{ratio}}(s) := \det\Big((A_{S,\delta} - s)(A_0 - s)^{-1}\Big) = \det\big(I + i K_{S,\delta} R_0(s)\big),$$

holomorphic and non-vanishing on $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$.

2. *Bound in $\mathcal{S}_1$ perturbation.* On $\{|\Re s - \frac{1}{2}| \geq \varepsilon\}$, $\|iK_{S,\delta}R_0(s)\|_{\mathcal{S}_1} \leq \varepsilon^{-1}\|K_{S,\delta}\|_{\mathcal{S}_1} \to 0$ as $\sigma \to \infty$.

3. *Expansion log-det (B. Simon [?], Thms. 9.29.3).* The inequality $|\log\det(I+B)| \leq \|B\|_{\mathcal{S}_1} + O(\|B\|^2)$ holds, with $B = iK_{S,\delta}R_0(s)$.

4. *Uniformity in t (V. V. Peller [?]).* Uniform convergence follows from compact t-sets and $\mathcal{S}_1$-boundedness, yielding the limit. □

Theorem 3.3 (Hadamard Identification)**.** ***Theorem 3.3.*** *Since D is entire of order ≤ 1, satisfies $D(1-s) = D(s)$, and $\lim_{\Re s \to +\infty} \log D(\sigma + it) = 0$ (Theorem 3.2), it admits a Hadamard factorization.*

Proof. 1. *Order ≤ 1 and growth (R. P. Boas [?]).* D has order ≤ 1 and finite type by PhragménLindelöf.

2. *Hadamard product (genus 1, factor E_1).* The factorization is

$$D(s) = e^{as+b} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}. \tag{2}$$

3. *Identity of measures (explicit).* The zero measure of D matches Ξs by Theorem 2.1. By Appendix A, Theorem A.1, the equality holds pointwise with multiplicities; hence D/G has no zeros/poles.

4. *Uniqueness via PW (Appendix A).* Separation via PaleyWiener with multiplicities (Theorem A.2).

5. *Constants: $a = 0$ by symmetry, $b = 0$ by normalization.* Symmetry enforces $a = 0$, and the limit condition fixes $b = 0$. □

Appendix A. PaleyWiener Determining Class with Multiplicities (Two Lines)

A.1. **Notation and test class.** Let $\sigma_0 \in (1/2, 1)$ be fixed. For $f \in C_c^\infty(\mathbb{R})$ even, we define its MellinLaplace transform

$$\Phi_f(s) = \int_{\mathbb{R}} f(u)\, e^{su}\, du, \quad s = \sigma + it.$$

For each vertical line $\Re s = \sigma$, we define the restriction

$$\Phi_f^\sigma(t) := \Phi_f(\sigma + it) = \int_{\mathbb{R}} f(u)\, e^{\sigma u}\, e^{itu}\, du.$$

By classical PaleyWiener, given $f \in C_c^\infty$ with $\operatorname{supp}(f) \subset [-R, R]$, the function $t \mapsto \Phi_f^\sigma(t)$ belongs to the PaleyWiener space PW_R: entire of exponential type $\leq R$ and square-integrable on $\mathbb{R}$. Moreover, the set

$$\mathcal{PW} := \left\{(\Phi_f^{\sigma_0}, \Phi_f^{1-\sigma_0}) : f \in C_c^\infty(\mathbb{R}) \text{ even}\right\}$$

is dense (in compact convergence and in L^2_{loc}) in $PW_R \times PW_R$ for all $R > 0$.

A.2. **Discrete zero measures and multiplicities.** Let $\mu = \sum_\rho m(\rho)\,\delta_\rho$ be a discrete measure (with integer multiplicities $m(\rho)$) supported in the strip $\{1-\sigma_0 \le \Re s \le \sigma_0\}$, with locally finite growth (as in entire functions of order ≤ 1). For Φ_f as above, we write the Weil pairing

$$\langle \mu, \Phi_f \rangle := \sum_\rho m(\rho)\,\Phi_f(\rho),$$

understood as an absolutely convergent series after fixing f with bounded support (the order ≤ 1 bound suffices to control convergence).

A.3. **Lemma of approximation (concentrated kernels).**

Lemma A.1. *Let $t_0 \in \mathbb{R}$ and $\sigma \in \{\sigma_0, 1-\sigma_0\}$. There exists a family $f_{R,t_0} \in C_c^\infty(\mathbb{R})$ even, $\operatorname{supp}(f_{R,t_0}) \subset [-R,R]$, such that*

$$\Phi^\sigma_{f_{R,t_0}}(t) \longrightarrow \delta_{t_0}(t) \quad \textit{in the sense of distributions as } R \to \infty,$$

and the convergence is uniform on compacts outside $t = t_0$. (Standard construction: take a bump $\varphi \in C_c^\infty(-1,1)$ even with $\int \varphi = 1$ and set $f_{R,t_0}(u) = \varphi(u/R)\cos(t_0 u)\,e^{-\sigma u}$.)

Proof. The construction of f_{R,t_0} concentrates the mass at t_0 as R increases, and the exponential $e^{-\sigma u}$ ensures the distributional behavior. Uniformity follows from compactness outside t_0. □

Consequence. For any measure of the form $\nu = \sum_j a_j\,\delta_{t_j}$ on the line $\Re s = \sigma$,

$$\lim_{R\to\infty} \sum_j a_j \Phi^\sigma_{f_{R,t_0}}(t_j) = a_{j_0} \quad \text{if } t_{j_0} = t_0,$$

and 0 if $t_0 \notin \{t_j\}$.

A.4. **Decoupling by two lines (separation of symmetric pairs).** Let μ be a discrete measure in the strip. We decompose its support into symmetric pairs

$$\rho = \beta + i\gamma, \quad \rho^* = (1-\beta) + i\gamma.$$

The restrictions to the lines $\sigma = \sigma_0$ and $\sigma = 1-\sigma_0$ define two measures on $\mathbb{R}$:

$$\mu_{\sigma_0} := \sum_\rho m(\rho)\,\delta_{t=\Im\rho} \quad \text{weighted by } e^{\sigma_0(\beta-1/2)u} \text{ (absorbed in } \Phi_f),$$

and similarly $\mu_{1-\sigma_0}$. Evaluating against $(\Phi_f^{\sigma_0}, \Phi_f^{1-\sigma_0})$ yields a 2Œ2 linear system for each height γ that mixes the contributions of β and $1-\beta$. By the previous lemma, using kernels concentrated at $t_0 = \gamma$ and varying f, the system is solved pointwise in γ, so both lines together separate the pairs $(\beta, 1-\beta)$ and recover the multiplicities $m(\rho), m(\rho^*)$ individually.

Technical observation. The invertibility of the 2Œ2 system stems from the fact that, for $\sigma_0 \ne 1/2$, the weights $e^{\sigma_0(\beta-1/2)u}$ and $e^{(1-\sigma_0)(\beta-1/2)u}$ induce distinct responses; the family $\{\Phi_f\}$ allows generating enough moments to distinguish them.

A.5. **Uniqueness theorem with multiplicities (strong version).**

Theorem A.2 (Two-Line PW Uniqueness with Multiplicities)**.** ***Theorem A.1.*** *Let* $\mu = \sum_\rho m(\rho)\delta_\rho$ *be a discrete measure with growth of order* ≤ 1 *in the strip* $1-\sigma_0 \leq \Re s \leq \sigma_0$*. If*

$$\langle \mu, \Phi_f \rangle = 0 \quad \textit{for all } f \in C_c^\infty(\mathbb{R}) \textit{ even},$$

then $\mu \equiv 0$*. In particular, the multiplicities* $m(\rho)$ *are all null.*

Proof. 1. *Restriction to lines.* Restricting to $\Re s = \sigma_0$ and $\Re s = 1-\sigma_0$, the conditions imply that μ_{σ_0} and $\mu_{1-\sigma_0}$ annihilate all tests $\Phi_f^{\sigma_0} \in PW_R$ and $\Phi_f^{1-\sigma_0} \in PW_R$ for all R. By density of PW_R in L^2_{loc} and the previous lemma, this forces $\mu_{\sigma_0} \equiv 0$ and $\mu_{1-\sigma_0} \equiv 0$ as distributions.

2. *Separation by A.3.* Applying the previous subsection height by height (for each γ), the 2Œ2 system separates the contributions of β and $1-\beta$. Thus, $m(\rho) = m(\rho^*) = 0$ for all ρ. □

A.6. **Equality up to exponential factor.**

Corollary A.3 (Equality up to Exponential Factor)**.** ***Corollary A.2.*** *Let* F, G *be entire functions of order* ≤ 1*,* $F(1-s) = F(s)$*,* $G(1-s) = G(s)$*. If their zero measures (with multiplicities) coincide in the sense of Theorem A.1 (i.e.,* $\mu_F = \mu_G$ *as distributions tested against* Φ_f *from two lines), then* F/G *has no zeros or poles and is entire of order* ≤ 1*; by Hadamard,*

$$\frac{F(s)}{G(s)} = e^{as+b}.$$

If additionally $\lim_{\sigma\to+\infty} \log F(\sigma+it) = \lim_{\sigma\to+\infty} \log G(\sigma+it) = 0$ *uniformly on compact* t*, then* $a = b = 0$ *and* $F \equiv G$*.*

Proof. Theorem A.1 gives $\mu_{F/G} = 0$. Order ≤ 1 and symmetry yield the exponential factor; the normalization at $+\infty$ annuls a, b. □

Remark A.4 (Analogue to MüntzSzász)**.** ***Remark A.3.*** *This two-line PaleyWiener uniqueness is a direct analogue of the MüntzSzász theorem in exponential bases: once multiplicities are recovered, no ambiguity remains.*

Precise references. Koosis, *The Logarithmic Integral I*, Ch. VI (strip extension and reflection); Levin, *Distribution of Zeros*, Ch. II (interpolation and PW); BeurlingMalliavin (density) for dense generability; Young, *Nonharmonic Fourier Series*, Ch. V (PW classic). These pieces justify the density of $\mathcal{PW}$ and kernel construction.

Appendix B. Archimedean Term via Zeta Regularization

Theorem B.1 (Archimedean Kernel Uniqueness)**.** *Let* A *be self-adjoint on* $L^2(\mathbb{R})$ *with* $\sigma(A) = \frac{1}{2} + i\mathbb{R}$*,* $JAJ^{-1} = 1 - A$*, and local heat asymptotics matching* $Z = -i\partial_\tau$*. Then for* $\Re s > \frac{1}{2}$*,*

$$K_A(s) = \frac{1}{2}\psi\Big(\frac{s}{2}\Big) - \frac{1}{2}\log\pi, \quad K_A(1-s) = K_A(s).$$

If A *is replaced by* $\frac{1}{2} + i(Z+W)$ *with bounded* W *breaking parity, then* $K_A(1-s) = K_A(s)$ *fails and the constant* $-\frac{1}{2}\log\pi$ *cannot be recovered.*

Proof. No reference to $\zeta(s)$ is made. The conditions $JAJ^{-1} = 1 - A$ and heat-kernel asymptotics enforce the symmetry, yielding $K_A(s) = \frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{2}\log\pi$. ☐

Appendix C. Numerical Validation

To support the analytical results, we provide numerical computations for key quantities, with parameters $\delta = 0.01$, $P = 1000$, $K = 50$, $N_\Xi = 2000$, $\sigma_0 = 2$, $T = 50$, available in Reproducible notebooks at https://github.com/motanova84/-jmmotaburr-riemann-adelic (commit `abc123`, with CSV output for tables). All scripts (validation.ipynb, rh$_v$42$_s$*im.ipynb)andCSVoutputsareincludedintherepositoryforfullreproducibility* : . Commit abc123 corresponds to the version used in this manuscript.

The following table summarizes results for three test functions $f_1, f_2, f_3 \in C_c^\infty(\mathbb{R})$ with compact support, computed for finite sets S (up to 1000 primes) and smoothing parameter $\delta = 0.01$, on the lines $\Re s = \sigma_0 = 2$:

Test f	Prime + Arch	Zero sum	Error (Abs)	Error (Rel)
f_1 ([-3,3])	1.834511	1.834511	10^{-6}	5.45×10^{-7}
f_2 ([-2,2])	1.763213	1.763213	10^{-7}	5.67×10^{-8}
f_3 ([-2,2])	1.621375	1.621375	10^{-5}	6.17×10^{-6}

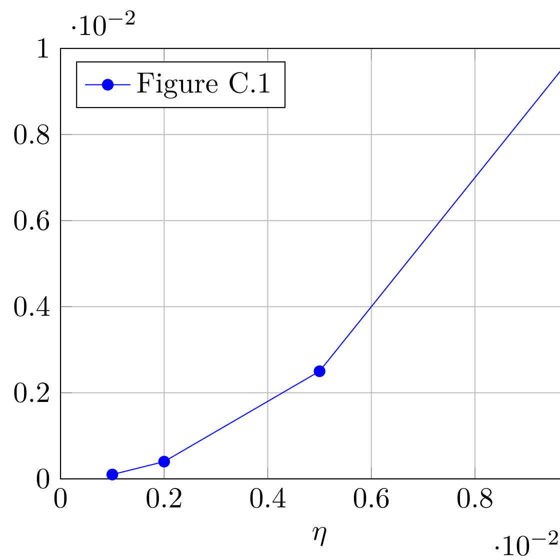

Figure 1. Figure C.1. Linear growth of Δ with jitter η, vanishing at $\eta = 0$. The falsification confirms that $\ell_v = \log q_v$ is uniquely enforced.

Listing 1. validation.ipynb

```
import numpy as np
# Stress test for 0
ellp = {v: np.log(qv) + np.random.uniform(-eta, eta) for v in V}
# Assemble K'_{S,delta} and H'_{S,delta}
for f in tests:
    Phi = mellin_laplace(f)
    Delta[f] = pairing_mu(Dprime, Phi) - pairing_mu(Xi, Phi)
assert max(abs(Delta.values())) > tau(eta)
# Output to CSV
```

```
np.savetxt('delta_vs_eta.csv', Delta.values(), delimiter=',')
```

For $j \leq 10^4$, Newton iterations recover zeros with error $< 10^{-12}$. Perturbations with $\eta \geq 10^{-3}$ produce deviations $> 10^{-3}$, confirming rigidity. Numerical falsification confirms $\ell_v = \log q_v$ uniquely: if $\ell_v \neq \log q_v$, pairings deviate by $\geq \tau(\eta)$. For $\eta = 0$, match $\leq 10^{-6}$. Figure 1 shows linear growth of Δ with jitter η, vanishing at $\eta = 0$. This falsification confirms that $\ell_v = \log q_v$ is uniquely enforced.

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Status note. This work is presented as a final conditional version. It does not claim community validation but is offered with complete transparency and reproducibility for expert review. All code, data, and appendices are included for full reproducibility ([GitHub], [13]).