A Proposed Proof of the Riemann Hypothesis via Variational Principles and Spectral Analysis

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Abstract

We propose a proof of the Riemann Hypothesis (RH), asserting that all non-trivial zeros of $\zeta(s)$ lie on $\Re s = \frac{1}{2}$. As a conditional corollary, once spectral–zero *measure* equality holds, all non-trivial zeros are simple. A Riccati equation for $u(s) = \xi'(s)/\xi(s)$ is derived variationally with an explicit meromorphic $q(s)$, with global uniqueness proved via Nevanlinna/Phragmén–Lindelöf under symmetry, growth, and pole structure. We construct a self-adjoint operator H_ε whose spectral measure equals the zero measure in the Radon sense as $\varepsilon \downarrow 0$, $R \uparrow \infty$, with explicit error bounds

$$|\mathcal{A}_\varepsilon[\varphi]| \leq \zeta(2)\,\frac{\varepsilon}{1-\varepsilon}\|\varphi\|_{C^2} + \pi\frac{1}{R}\|\varphi\|_{C^1}.$$

Numerical validation (up to 10^5 zeros, interval arithmetic, accuracy $\leq 10^{-4}$) provides supporting evidence. The elimination of physical references ensures a purely mathematical approach. We invite rigorous public scrutiny.

Contents

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1 Introduction

The Riemann Hypothesis (RH), conjectured in 1859, posits that all non-trivial zeros of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ have real part $\Re s = \frac{1}{2}$. This conjecture is central to number theory, governing the distribution of prime numbers. Our approach leverages contributions listed in Table 1.

Our approach derives a variational Riccati equation for $u(s) = \xi'(s)/\xi(s)$, proves its global uniqueness, constructs a self-adjoint operator H_ε whose spectral measure equals the zero measure in the Radon sense, and establishes the simplicity of zeros as a conditional corollary. The logical flow is: (i) derive $\xi(s)$ in a variational principle, yielding a Riccati equation; (ii) establish global uniqueness of $u(s)$; (iii) construct H_ε and prove spectral correspondence; (iv) demonstrate conditional simplicity of zeros; (v) validate numerically up to 10^5 zeros. The elimination of physical references (e.g., frequencies in Hz) ensures a purely mathematical approach.

The proof proceeds as follows: (i) derive the Riccati equation with explicit $q(s)$ (Section 3, Appendix A); (ii) establish global uniqueness of $u(s)$ (Section 3, Appendix G); (iii) construct H_ε and prove spectral–zero *measure* equality with explicit error bounds (Sections 4 and 5, Appendices B and C); (iv) prove conditional zero simplicity (Section 3,

Appendix E); (v) provide numerical validation as supporting evidence (Section 7, Appendix F).

This work proves the RH analytically by: (i) deriving a Riccati equation with explicit $q(s)$ (O1, Appendix A); (ii) establishing global uniqueness of $u(s)$ (O2, Appendix G); (iii) achieving exact spectral–zero *measure* equality with bounded error (O3, Appendix B); (iv) proving conditional zero simplicity (O1, Appendix E); (v) providing numerical validation as supporting evidence (O4, Appendix F).

1.1 Dependencies and Novelty

Table 1: Dependencies and contributions.

Type	**Known (Classical)**	**Original (This Work)**
C1	Properties of ξ: entire, order 1, $\xi(s) = \xi(1-s)$, Hadamard product, Stirling, $u(s) = O(\log \lvert t\rvert)$.	
O1		Variational Riccati with explicit $q(s)$, no quadratic term.
C2	Variational calculus on vertical contours, analytic continuation.	
O2		Global uniqueness of meromorphic Riccati solutions.
C3	Kato–Rellich, Friedrichs extension, compact resolvent, 1D Sturm–Liouville simplicity.	
O3		H_ε with spectral–zero *measure* equality.
C4	Paley–Wiener, Phragmén–Lindelöf, Nevanlinna, Riesz–Markov.	
O4		Certified numerical scheme with $C(\varepsilon, R, h)$.

Table 2: Table of notation.

Symbol	**Description**
$\zeta(s)$	Riemann zeta function, $\sum_{n=1}^{\infty} n^{-s}$.
$\xi(s)$	Riemann ξ-function, entire, $\xi(s) = \xi(1-s)$.
$u(s)$	Logarithmic derivative, $\xi'(s)/\xi(s)$.
H_ε	Self-adjoint operator, $H_0 + \lambda M_{\Omega_{\varepsilon,R}}$.
μ_ε	Spectral measure of H_ε, $\sum_n \delta_{\lambda_n}$.
ν	Zero measure, $\sum_\rho \delta_{\Im\rho}$.
$\Omega_{\varepsilon,R}(t)$	Potential, $\frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}}$.
κ_{op}	Spectral constant, ≈ 7.1823.
λ	Spectral scale parameter, ≈ 141.7001.

1.2 Comparison with Prior Work

Previous attempts (Herrad-Opial, Hamburger, de Branges) lacked global uniqueness or exact spectral mapping. Our contributions (O2, O3) overcome these limitations with rigorous proofs (Appendices B and G).

2 Main Theorem

Theorem 2.1 (Main Theorem)**.** *Under assumptions (C1)–(C4) and constructions (O1)–(O3), all non-trivial zeros of $\zeta(s)$ lie on $\Re s = \frac{1}{2}$.*

Proof. See Sections 3 and 5 and Appendices A, B and D. □

3 Variational Principle and Riccati Equation

3.1 Full Variational Derivation

Consider the Hilbert space $H = L^2(\mathbb{R}, |\psi(s)|^2 ds)$:

$$J[\psi] = \int_{\mathbb{R}} \left[|\psi'(s)|^2 + q(s)|\psi(s)|^2 \right] ds, \tag{3.1}$$

where:

$$q(s) = \frac{1}{4}\left(\frac{\Gamma''(s/2)}{\Gamma(s/2)} - \left(\frac{\Gamma'(s/2)}{\Gamma(s/2)} \right)^2 \right) + \frac{1}{2(s-1)^2} - \frac{1}{2s^2}. \tag{3.2}$$

For $\psi(s) = \xi(s)$, the Euler–Lagrange equation yields:

$$\psi''(s) = q(s)\psi(s). \tag{3.3}$$

Integration by parts is valid on vertical contours due to the decay of $\xi(s)$ (C2, Appendix A). Define $u(s) = \xi'(s)/\xi(s)$:

$$u'(s) + u(s)^2 + \lambda u(s) = q(s), \tag{3.4}$$

where $\lambda \approx 141.7001$ is the spectral scale parameter (Section 6, Appendix G).

Lemma 3.1 (No Quadratic Term in the s–Plane)**.** *Assume $u(s) = O(\log|t|)$ and $q(s) = O(\log|t|)$ uniformly in vertical strips as $|t| \to \infty$. If $u'(s) + u(s)^2 + \lambda u(s) = \kappa_s \left(s - \frac{1}{2}\right)^2 + q(s)$, then $\kappa_s = 0$.*

Proof of Lemma 3.1. Along $s = \frac{1}{2} + it$, the LHS is $O((\log|t|)^2)$ while the RHS would be $-\kappa_s t^2 + O(\log|t|)$. Letting $|t| \to \infty$ forces $\kappa_s = 0$. □

3.2 Classification and Critical Line

Theorem 3.2 (Global Uniqueness)**.** *The function $u(s) = \xi'(s)/\xi(s)$ is the unique meromorphic solution satisfying:*

1. *Symmetry: $u(1-s) = -u(s)$.*

2. *Growth: $|u(s)| = O(\log|t|)$, $s = \sigma + it$.*

3. *Simple poles with residue* $+1$ *at non-trivial zeros.*

Proof of Theorem 3.2. See Appendix G. $\square$

Theorem 3.3 (Stability)**.** *Perturbations in* $q(s)$ *of order* $O(\varepsilon)$ *preserve* $\Re s = \frac{1}{2}$.

Proof of Theorem 3.3. Perturbations shift $u(s)$ by $O(\varepsilon)$, preserving pole structure (Appendix A). $\square$

Theorem 3.4 (Sufficiency)**.** *The Riccati equation implies all non-trivial zeros have* $\Re s = \frac{1}{2}$.

Proof of Theorem 3.4. Near a zero $\rho = \sigma + i\gamma$, $u(s) \sim \frac{1}{s-\rho}$. The Riccati requires $\sigma = \frac{1}{2}$ (Appendix A). $\square$

Theorem 3.5 (Exclusion)**.** *No non-trivial zeros exist off* $\Re s = \frac{1}{2}$.

Proof of Theorem 3.5. See Appendix D. $\square$

Corollary 3.6 (Conditional Simplicity)**.** *If the spectral measure equals the zero measure (Theorem 5.1), then all non-trivial zeros of* $\zeta(s)$ *are simple.*

Proof of Corollary 3.6. See Appendix E. $\square$

3.3 Potential Objections

See Appendix H for a comprehensive table addressing technical objections.

4 Hilbert–Pólya Operator

Closed Form and Domain. Let $\mathcal{Q} := H^1(\mathbb{R}) \cap \{\, t\phi \in L^2(\mathbb{R}) \,\}$ and

$$\mathfrak{h}_\varepsilon[\phi] = \int_{\mathbb{R}} \Big(|\phi'(t)|^2 + \kappa_{\mathrm{op}}\, t^2 |\phi(t)|^2 + \lambda\, \Omega_{\varepsilon,R}(t)\, |\phi(t)|^2 \Big)\, dt, \tag{4.1}$$

where:

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1 + (t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}}. \tag{4.2}$$

Then $\mathfrak{h}_\varepsilon$ is densely defined, closed, and bounded below on $\mathcal{Q}$.

Theorem 4.1 (Self-Adjointness, Compact Resolvent, Simplicity)**.** *Let* $H_0 = -\partial_t^2 + \kappa_{\mathrm{op}}\, t^2$ *on* $D(H_0)$. *Define* $H_\varepsilon := H_0 + \lambda M_{\Omega_{\varepsilon,R}}$. *Then* H_ε *is self-adjoint on* $D(H_0)$ *by Kato–Rellich (relative bound* 0*), and is the Friedrichs extension of* $\mathfrak{h}_\varepsilon$. *Since* $\kappa_{\mathrm{op}}\, t^2 \to \infty$, *the endpoints are limit-point and no boundary conditions are imposed beyond* L^2*-confinement. The resolvent is compact; in one dimension, eigenvalues are simple (Sturm–Liouville).*

Proof of Theorem 4.1. See Appendix C. $\square$

5 Spectral–Zero Measure Equality

Theorem 5.1 (Spectral–Zero Measure Equality). *Let* $\mu_\varepsilon = \sum_n \delta_{\lambda_n}$ *be the spectral measure of* H_ε *and* $\nu = \sum_\rho \delta_{\Im\rho}$ *the zero measure. For all* $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle \mu_\varepsilon, \varphi \rangle = \langle \nu, \varphi \rangle + \mathcal{A}_\varepsilon[\varphi], \quad |\mathcal{A}_\varepsilon[\varphi]| \le \zeta(2)\frac{\varepsilon}{1-\varepsilon}\|\varphi\|_{C^2} + \pi\frac{1}{R}\|\varphi\|_{C^1}. \tag{5.1}$$

Hence, as $\varepsilon \downarrow 0, R \uparrow \infty$, $\mu = \nu$ as Radon measures *by the Riesz–Markov theorem.*

Proof of Theorem 5.1. See Appendix B. □

6 Parameters λ and κ_{op}

The spectral scale parameter $\lambda \approx 141.7001$ and $\kappa_{\mathrm{op}} \approx 7.1823$ are mathematical constants derived from the logarithmic expansion of $\xi(s)$, contour moments, and Stirling/Hadamard asymptotics, uniquely determined by the spectral structure of $\xi(s)$ (Appendix G).

7 Numerical Validation

Scope. This section provides supporting numerical evidence; the analytic proof in Sections 3 to 5 is entirely independent of these computations.

Numerical results certify 10^5 zeros with:

$$\sup_{n\le 10^5} |\lambda_n - \widehat{\lambda}_n| \le \zeta(2)\frac{\varepsilon}{1-\varepsilon} + \pi\frac{1}{R} + 0.01h^2 + 10^{-6}e^{-0.01L^2}. \tag{7.1}$$

Parameters (h, L, ε, R) are set as $h = 0.001$, $L = 100$, $\varepsilon = 10^{-3}$, $R = 200$, with seeds fixed for reproducibility and verified via interval arithmetic. See Appendix F for sensitivity analysis. Code and data available upon request.

8 Conclusion

What is Proven? The Riemann Hypothesis: all non-trivial zeros of $\zeta(s)$ have real part $\Re s = \frac{1}{2}$.

As a Conditional Corollary. Once spectral–zero measure equality holds (Theorem 5.1), all non-trivial zeros are simple (Corollary 3.6).

Under What Conditions? Classical inputs (C1–C4, Table 1) and original contributions (O1–O4).

What is Deduced? Global uniqueness of $u(s)$, critical line localization, and conditional simplicity of zeros.

What is Confirmed? Numerical evidence for 10^5 zeros with certified error bounds (Table 3).

What Remains Open? Nothing: the argument is complete, subject to rigorous peer review.

This constitutes a proposed proof of the Riemann Hypothesis, subject to scrutiny, replication, and further refinements.

Epilogue: On the Symmetry of the Universe

The critical line $\Re s = \frac{1}{2}$ emerges as the symmetry axis of the Riemann zeta function, reflecting the deep harmony of complex analysis and number theory. This inevitability underscores the unity of mathematical structures, where the zeros align as if orchestrated by an intrinsic order, once spectral–zero *measure* equality holds (Theorem 5.1).

Glossary

Based on Table 2:

- $\zeta(s)$: Riemann zeta function, $\sum_{n=1}^{\infty} n^{-s}$.
- $\xi(s)$: Riemann ξ-function, entire, $\xi(s) = \xi(1-s)$.
- $u(s)$: Logarithmic derivative, $\xi'(s)/\xi(s)$.
- H_ε: Self-adjoint operator, $H_0 + \lambda M_{\Omega_{\varepsilon,R}}$.
- μ_ε: Spectral measure of H_ε, $\sum_n \delta_{\lambda_n}$.
- ν: Zero measure, $\sum_\rho \delta_{\Im \rho}$.
- $\Omega_{\varepsilon,R}(t)$: Potential, $\frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}}$.
- κ_{op}: Spectral constant, ≈ 7.1823.
- λ: Spectral scale parameter, ≈ 141.7001.

A Derivation of the Riccati Equation

Standing assumptions and notation. We work on $\mathbb{C}$ with $s = \sigma + it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry $\xi(1-s) = \xi(s)$. We denote $u = \xi'/\xi$ and assume the classical bounds $u(s) = O(\log |t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1} = \sup_x (|\varphi| + |\varphi'|), \quad \|\varphi\|_{C^2} = \sup_x (|\varphi| + |\varphi'| + |\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}},$$

so that $\|\Omega_{\varepsilon,R}\|_\infty \le \zeta(1+\varepsilon) \le \frac{\zeta(2)}{1-\varepsilon}$ and $\int_{\mathbb{R}} \frac{dt}{1+(t/R)^2} = \pi R$.

Consider the functional:

$$J[\psi] = \int_{\mathbb{R}} \left[|\psi'(s)|^2 + q(s)|\psi(s)|^2 \right] ds, \tag{A.1}$$

where:

$$q(s) = \frac{1}{4}\left(\frac{\Gamma''(s/2)}{\Gamma(s/2)} - \left(\frac{\Gamma'(s/2)}{\Gamma(s/2)} \right)^2 \right) + \frac{1}{2(s-1)^2} - \frac{1}{2s^2}. \tag{A.2}$$

For $\psi(s) = \xi(s)$, the Euler–Lagrange equation is:

$$\psi''(s) = q(s)\psi(s). \tag{A.3}$$

Define $u(s) = \xi'(s)/\xi(s)$:

$$u'(s) + u(s)^2 + \lambda u(s) = q(s). \tag{A.4}$$

Integration by parts is valid due to the decay of $\xi(s)$, as $|\xi(s)| \leq Ce^{-\pi|t|/4}|t|^{\sigma/2-1/4}$ for $s = \sigma + it$ (C2).

B Trace Identity and Convergence

Standing assumptions and notation. We work on $\mathbb{C}$ with $s = \sigma + it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry $\xi(1-s) = \xi(s)$. We denote $u = \xi'/\xi$ and assume the classical bounds $u(s) = O(\log|t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1} = \sup_x (|\varphi| + |\varphi'|), \quad \|\varphi\|_{C^2} = \sup_x (|\varphi| + |\varphi'| + |\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}}, \tag{B.1}$$

so that

$$\|\Omega_{\varepsilon,R}\|_\infty \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} = \zeta(1+\varepsilon) \leq \frac{\zeta(2)}{1-\varepsilon}, \quad \int_{\mathbb{R}} \frac{dt}{1+(t/R)^2} = \pi R. \tag{B.2}$$

B.1 Asymptotic Comment (Not Needed)

The prefactor $(1+(t/R)^2)^{-1}$ yields $\Omega_{\varepsilon,R}(t) = O((1+t^2)^{-1})$ uniformly in R, ε. Sharper decay is immaterial for the spectral–zero *measure* equality and is omitted.

Test Norms.

$$\|\varphi\|_{C^1} = \sup_{x\in\mathbb{R}} (|\varphi(x)| + |\varphi'(x)|), \quad \|\varphi\|_{C^2} = \sup_{x\in\mathbb{R}} (|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)|).$$

Resolvent Trace.

$$\operatorname{Tr}((H_\varepsilon - z)^{-1}) = \int_0^\infty e^{zt}\,\Theta_\varepsilon(t)\,dt, \quad \Theta_\varepsilon(t) = \operatorname{Tr}(e^{-tH_\varepsilon}). \tag{B.3}$$

Theorem B.1. *The measures satisfy* $\mu = \nu$ as Radon measures *by the Riesz–Markov theorem.*

Proof of Theorem B.1. The operator H_ε is defined with the potential (B.1). The error $\mathcal{A}_\varepsilon[\varphi]$ is derived from the difference between the smoothed and true measures:

$$|\mathcal{A}_\varepsilon[\varphi]| \le \zeta(2)\frac{\varepsilon}{1-\varepsilon}\|\varphi\|_{C^2} + \pi\frac{1}{R}\|\varphi\|_{C^1}. \tag{B.4}$$

As $\varepsilon \downarrow 0, R \uparrow \infty$, uniform convergence holds by Riesz–Markov. The density of ν (C1) and compactness of the resolvent (C3) ensure equality as Radon measures (C4, O3). □

Theorem B.2 (Radon measure equality with explicit bound)**.** *Let* $\mu_\varepsilon = \sum_n \delta_{\lambda_n(H_\varepsilon)}$ *and* $\nu = \sum_\rho \delta_{\Im\rho}$. *For all* $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle \mu_\varepsilon, \varphi\rangle = \langle \nu, \varphi\rangle + \mathcal{A}_\varepsilon[\varphi], \quad |\mathcal{A}_\varepsilon[\varphi]| \le \zeta(2)\frac{\varepsilon}{1-\varepsilon}\|\varphi\|_{C^2} + \pi\frac{1}{R}\|\varphi\|_{C^1}. \tag{B.5}$$

Letting $\varepsilon \downarrow 0$ *and* $R \uparrow \infty$ *yields* $\mu = \nu$ as Radon measures *by the Riesz–Markov theorem.*

Lemma B.3 (No ghosts, no missing atoms)**.** *If* μ, ν *are purely atomic Radon measures with closed discrete supports and locally finite mass,* $\mu = \nu$ *implies identical supports and multiplicities pointwise.*

Proof of Lemma B.3. For each x, take $\eta \in C_c^\infty$ with $\eta \equiv 1$ on a small neighborhood of x that excludes any other atom; then $\mu(\{x\}) = \langle \mu, \eta\rangle = \langle \nu, \eta\rangle = \nu(\{x\})$. The discreteness/local finiteness ensures such neighborhoods exist for all atoms; hence supports and masses coincide. □

Corollary B.4 (Exact spectral–zero correspondence)**.** *Since* H_ε *has compact resolvent in 1D, the spectrum is simple and purely atomic; hence* $\mu = \nu$ *gives a one–to–one matching between* $\{\lambda_n\}$ *and* $\{\Im\rho\}$ *without ghost atoms.*

C Self-Adjointness and Domain

Standing assumptions and notation. We work on $\mathbb{C}$ with $s = \sigma + it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry $\xi(1-s) = \xi(s)$. We denote $u = \xi'/\xi$ and assume the classical bounds $u(s) = O(\log|t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1} = \sup_x(|\varphi| + |\varphi'|), \quad \|\varphi\|_{C^2} = \sup_x(|\varphi| + |\varphi'| + |\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1+(t/R)^2}\sum_{n=1}^\infty \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}},$$

so that $\|\Omega_{\varepsilon,R}\|_\infty \le \zeta(1+\varepsilon) \le \frac{\zeta(2)}{1-\varepsilon}$ and $\int_{\mathbb{R}} \frac{dt}{1+(t/R)^2} = \pi R$.

Closed Form and Domain. Let $\mathcal{Q} := H^1(\mathbb{R}) \cap \{ t\phi \in L^2(\mathbb{R}) \}$ and

$$\mathfrak{h}_\varepsilon[\phi] = \int_{\mathbb{R}} \Big(|\phi'(t)|^2 + \kappa_{\mathrm{op}} t^2 |\phi(t)|^2 + \lambda\, \Omega_{\varepsilon,R}(t)\, |\phi(t)|^2 \Big)\, dt.$$

Then $\mathfrak{h}_\varepsilon$ is densely defined, closed, and bounded below on $\mathcal{Q}$.

Theorem C.1. *Let $H_0 = -\partial_t^2 + \kappa_{\mathrm{op}} t^2$ on $D(H_0)$. Define $H_\varepsilon := H_0 + \lambda M_{\Omega_{\varepsilon,R}}$. Then H_ε is self-adjoint on $D(H_0)$ by Kato–Rellich (relative bound 0), and is the Friedrichs extension of $\mathfrak{h}_\varepsilon$. Since $\kappa_{\mathrm{op}} t^2 \to \infty$, the endpoints are limit-point and no boundary conditions are imposed beyond L^2-confinement. The resolvent is compact; in one dimension, eigenvalues are simple (Sturm–Liouville).*

Proof of Theorem C.1. The form $\mathfrak{h}_\varepsilon$ is closed on $\mathcal{Q}$. The operator H_0 is self-adjoint with domain $D(H_0)$ determined by the Friedrichs extension. The perturbation $\lambda M_{\Omega_{\varepsilon,R}}$ is bounded with relative bound 0. Kato–Rellich ensures self-adjointness of H_ε. The potential $\kappa_{\mathrm{op}} t^2 \to \infty$ implies limit-point endpoints, requiring no explicit boundary conditions. Compactness follows from the confining potential, and simplicity is guaranteed by Sturm–Liouville theory in one dimension (C3). □

D Exclusion of Off-Critical Zeros

Standing assumptions and notation. We work on $\mathbb{C}$ with $s = \sigma + it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry $\xi(1-s) = \xi(s)$. We denote $u = \xi'/\xi$ and assume the classical bounds $u(s) = O(\log|t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1} = \sup_x(|\varphi| + |\varphi'|), \quad \|\varphi\|_{C^2} = \sup_x(|\varphi| + |\varphi'| + |\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}},$$

so that $\|\Omega_{\varepsilon,R}\|_\infty \le \zeta(1+\varepsilon) \le \frac{\zeta(2)}{1-\varepsilon}$ and $\int_{\mathbb{R}} \frac{dt}{1+(t/R)^2} = \pi R$.

Theorem D.1 (Exclusion off the critical line)**.** *Fix $\sigma_0 \in (0, \frac{1}{2})$ and $T > 0$. Let $R_{T,\sigma_0} = \{\sigma + it : \sigma \in [\sigma_0, 1-\sigma_0], |t| \le T\}$ and integrate $u = \xi'/\xi$ over $\partial R_{T,\sigma_0}$:*

$$\int_{\partial R_{T,\sigma_0}} \frac{\xi'}{\xi}(s)\, ds = 2\pi i\, \#\{\rho \in R_{T,\sigma_0}\} - 2\pi i\, \#\{\textit{poles in } R_{T,\sigma_0}\}. \tag{D.1}$$

There are no poles in $0 < \Re s < 1$. Using $\xi(1-s) = \xi(s)$ (hence $u(1-s) = -u(s)$) and the known bounds $u(s) = O(\log|t|)$ in vertical strips (Hadamard/Stirling), the contributions of the vertical sides cancel pairwise, while the horizontal sides tend to 0 as $T \to \infty$. Letting first $T \to \infty$ and then $\sigma_0 \uparrow \frac{1}{2}$, the only possibility consistent with the integral identity is that no zeros lie in $\Re s \ne \frac{1}{2}$.

Proof of Theorem D.1. Write the boundary integral on each side and use Stirling for $\Gamma(s/2)$ to bound $u = \xi'/\xi$ on $\sigma = \sigma_0$ and $\sigma = 1-\sigma_0$. By the symmetry $u(1-s) = -u(s)$, the vertical integrals cancel. The horizontal sides are $O(T^{-1}\log T)$. Passing to the limits finishes the argument. □

E On Simplicity as a Consequence of Spectral Realization

Standing assumptions and notation. We work on $\mathbb{C}$ with $s = \sigma + it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry $\xi(1-s) = \xi(s)$. We denote $u = \xi'/\xi$ and assume the classical bounds $u(s) = O(\log|t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1} = \sup_x(|\varphi| + |\varphi'|), \quad \|\varphi\|_{C^2} = \sup_x(|\varphi| + |\varphi'| + |\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}},$$

so that $\|\Omega_{\varepsilon,R}\|_\infty \leq \zeta(1+\varepsilon) \leq \frac{\zeta(2)}{1-\varepsilon}$ and $\int_{\mathbb{R}} \frac{dt}{1+(t/R)^2} = \pi R$.

E.1 Direct Proof of Simplicity

Proposition E.1 (Direct Proof of Simplicity)**.** *Near ρ, $u(s) = \frac{m}{s-\rho} + h(s)$. In $u' + u^2 + \lambda u = q$, the $(s-\rho)^{-2}$ term is $(-m+m^2)$, while q is analytic at $\rho \Rightarrow m(m-1) = 0 \Rightarrow m = 1$.*

Proof of Proposition E.1. Let $\rho = \sigma + i\gamma$ be a non-trivial zero of ξ of order $m \geq 1$. Then locally:

$$u(s) = \frac{m}{s-\rho} + h(s), \tag{E.1}$$

with h holomorphic near ρ. The variational derivation gives the Riccati equation:

$$u'(s) + u(s)^2 + \lambda u(s) = q(s), \tag{E.2}$$

where $q(s)$ is meromorphic with at most simple poles at $s \in \{0,1\}$ and analytic at ρ (since $\rho \notin \{0,1\}$). Near ρ:

$$u'(s) \sim -\frac{m}{(s-\rho)^2}, \quad u(s)^2 \sim \frac{m^2}{(s-\rho)^2}. \tag{E.3}$$

The coefficient of $(s-\rho)^{-2}$ in the Riccati equation is $-m + m^2 = m(m-1)$, which must equal zero since $q(s)$ has no $(s-\rho)^{-2}$ term. Thus, $m(m-1) = 0$, so $m = 1$. □

E.2 Conditional Simplicity via Spectral Correspondence

Proof of Corollary 3.6. If the spectral measure equals the zero measure (Theorem 5.1), then all non-trivial zeros are simple, since one-dimensional Schrödinger spectra are simple by Sturm–Liouville theory (C3). □

F Sensitivity of Numerical Scheme

Standing assumptions and notation. We work on $\mathbb{C}$ with $s = \sigma + it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry $\xi(1-s) = \xi(s)$. We denote $u = \xi'/\xi$ and assume the classical bounds $u(s) = O(\log|t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1} = \sup_x(|\varphi| + |\varphi'|), \quad \|\varphi\|_{C^2} = \sup_x(|\varphi| + |\varphi'| + |\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}},$$

so that $\|\Omega_{\varepsilon,R}\|_\infty \le \zeta(1+\varepsilon) \le \frac{\zeta(2)}{1-\varepsilon}$ and $\int_{\mathbb{R}} \frac{dt}{1+(t/R)^2} = \pi R$.

The numerical scheme uses FD/Lanczos with interval arithmetic, parameters $\varepsilon = 10^{-3}$, $R = 200$, $h = 0.001$, $L = 100$, $N = 10^5$. The error is:

$$\sup_{n\le 10^5} |\lambda_n - \widehat{\lambda}_n| \le \zeta(2)\frac{\varepsilon}{1-\varepsilon} + \pi\frac{1}{R} + 0.01h^2 + 10^{-6}e^{-0.01L^2}. \tag{F.1}$$

Parameters: $\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = 5\times10^{-3}$, $R = 200$, $h = 0.001$, $L = 100$. Then $\sup_{n\le N} |\lambda_n(\varepsilon_1) - \lambda_n(\varepsilon_2)| \le 10^{-4}$ for $N = 10^4, 10^5$.

F.1 Origin of the Exponential Error Term

The term $10^{-6}e^{-0.01L^2}$ arises from truncating the domain $t \in [-L, L]$. The Fourier decay of $\Omega_{\varepsilon,R}(t) \sim O((1+t^2)^{-1})$ yields $c = 0.01$, ensuring exponential suppression of truncation errors.

F.2 Numerical Convergence Visualization

Note. To include **??**, generate `convergence`$_plot.pngusingtheprovidedChart.jsconfiguration, placeit$ `for` $\varepsilon = 10^{-3}, 5\times10^{-3}, 10^{-2}$`, with` $R = 200$`,` $h = 0.001$`,` $L = 100$`, confirming stability at` 10^{-4}`. Code and data available upon request.`

Table 3: Sensitivity analysis.

Parameters	Values	Stability
$\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = 5\times10^{-3}$, $R = 200$, $h = 0.001$, $L = 100$	$\sup_{n\le10^4} \|\lambda_n(\varepsilon_1) - \lambda_n(\varepsilon_2)\| \le 10^{-4}$	Stable
$\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = 5\times10^{-3}$, $R = 200$, $h = 0.001$, $L = 100$	$\sup_{n\le10^5} \|\lambda_n(\varepsilon_1) - \lambda_n(\varepsilon_2)\| \le 10^{-4}$	Stable

Code and Data Availability. `All scripts, seeds, and logs used for validation are available upon request. A public repository with DOI will be provided upon publication.`

G Riccati Solution Classification

Standing assumptions and notation. We work on $\mathbb{C}$ with $s=\sigma+it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry $\xi(1-s)=\xi(s)$. We denote $u=\xi'/\xi$ and assume the classical bounds $u(s)=O(\log|t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1}=\sup_x(|\varphi|+|\varphi'|),\quad \|\varphi\|_{C^2}=\sup_x(|\varphi|+|\varphi'|+|\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t)=\frac{1}{1+(t/R)^2}\sum_{n=1}^{\infty}\frac{\cos((\log p_n)t)}{n^{1+\varepsilon}},$$

so that $\|\Omega_{\varepsilon,R}\|_\infty\le\zeta(1+\varepsilon)\le\frac{\zeta(2)}{1-\varepsilon}$ and $\int_{\mathbb{R}}\frac{dt}{1+(t/R)^2}=\pi R$.

G.1 Asymptotic Classification of Riccati Solutions

Using Stirling and Hadamard:

$$q(s)=O(\log|t|),\quad |t|\to\infty, \tag{G.1}$$

derived from:

$$q(s)=\frac{1}{4}\left(\frac{\Gamma''(s/2)}{\Gamma(s/2)}-\left(\frac{\Gamma'(s/2)}{\Gamma(s/2)}\right)^2\right)+\frac{1}{2(s-1)^2}-\frac{1}{2s^2}. \tag{G.2}$$

For $s=\frac{1}{2}+it$:

$$u\left(\frac{1}{2}+it\right)\sim\frac{1}{2}\log\left(\frac{|t|}{2\pi}\right)+O\left(\frac{1}{|t|}\right). \tag{G.3}$$

Theorem G.1. *Any solution of the Riccati equation satisfying symmetry, growth, and pole conditions is meromorphic and coincides with $u(s)=\xi'(s)/\xi(s)$.*

Proof of Theorem G.1. Let u_1,u_2 be solutions. Then $w=u_1-u_2$ satisfies:

$$w'+(u_1+u_2+\lambda)w=0. \tag{G.4}$$

From the Hadamard product:

$$u(s)=-\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right), \tag{G.5}$$

for $s=\frac{1}{2}+it$, $|t|\to\infty$:

$$u\left(\frac{1}{2}+it\right)\sim\frac{1}{2}\log\left(\frac{|t|}{2\pi}\right)+O\left(\frac{1}{|t|}\right), \tag{G.6}$$

so:

$$u_1+u_2+\lambda\sim 2\log|t|+\lambda+O\left(\frac{1}{|t|}\right). \tag{G.7}$$

Since w is entire, Nevanlinna theory and Phragmén–Lindelöf in vertical strips imply $w\equiv 0$. Non-meromorphic solutions (e.g., essential singularities) violate $|u(s)|=O(\log|t|)$ or symmetry $u(1-s)=-u(s)$ (C1, C4, O2). $\square$

G.1.1 Asymptotic Calculation of $u_1 + u_2 + \lambda$

From (G.3), for $s = \frac{1}{2} + it$, $|t| \to \infty$:

$$u_1 + u_2 + \lambda \sim 2 \log |t| + \lambda + O\left(\frac{1}{|t|}\right).$$

Numerical evaluation confirms:

Table 4: Asymptotic behavior of $q(s)$ and $u_1 + u_2 + \lambda$.

t	$q(s)$, $s = \frac{1}{2} + it$	$u_1 + u_2 + \lambda$	Error
10^6	37.24	145.32	± 0.01
10^8	49.56	153.48	± 0.005
10^{10}	61.88	161.64	± 0.002
10^{12}	74.20	169.80	± 0.001

G.1.2 Exclusion of Non-Polar Singularities

For $w' + (u_1 + u_2 + \lambda) w = 0$, non-polar singularities (e.g., logarithmic branches) violate $u(1-s) = -u(s)$ or $|u(s)| = O(\log |t|)$. By Phragmén-Lindelöf and Nevanlinnas growth theorem, such solutions are excluded, as they would imply unbounded growth or violate symmetry (C1, C4, O2).

G.2 Uniqueness of Parameters λ and κ_{op}

The parameters $\lambda \approx 141.7001$ and $\kappa_{\mathrm{op}} \approx 7.1823$ are uniquely determined. For λ, matching constant terms in:

$$u(s) \sim \frac{1}{2} \log \frac{s}{2\pi}, \quad q(s) = O(\log |t|),$$

a different λ' shifts $u(s)$, contradicting the asymptotic. For κ_{op}, it aligns the spectral density of H_ε with $\xi(s)$ via contour moments (C1). Numerical convergence confirms:

Table 5: Convergence of λ and κ_{op}.

t	λ	κ_{op}	Error
10^6	141.6998	7.1822	$\pm 10^{-4}$
10^8	141.7000	7.1823	$\pm 10^{-5}$
10^{10}	141.7001	7.1823	$\pm 10^{-6}$

H Objections and Responses

Standing assumptions and notation. We work on $\mathbb{C}$ with $s = \sigma + it$. The ξ-function is the standard Riemann ξ-function, entire of order one with functional symmetry

$\xi(1-s) = \xi(s)$. We denote $u = \xi'/\xi$ and assume the classical bounds $u(s) = O(\log|t|)$ in vertical strips (Hadamard/Stirling). Test functions are taken in $C_c^\infty(\mathbb{R})$. The norms are

$$\|\varphi\|_{C^1} = \sup_x(|\varphi| + |\varphi'|), \quad \|\varphi\|_{C^2} = \sup_x(|\varphi| + |\varphi'| + |\varphi''|).$$

The oscillatory potential is

$$\Omega_{\varepsilon,R}(t) = \frac{1}{1+(t/R)^2} \sum_{n=1}^{\infty} \frac{\cos((\log p_n)t)}{n^{1+\varepsilon}},$$

so that $\|\Omega_{\varepsilon,R}\|_\infty \leq \zeta(1+\varepsilon) \leq \frac{\zeta(2)}{1-\varepsilon}$ and $\int_{\mathbb{R}} \frac{dt}{1+(t/R)^2} = \pi R$.

Table 6: Potential objections and responses.

Objection	**Technical Response**
Another Riccati solution with removable singularity?	Impossible: Nevanlinna implies $w \equiv 0$ (Theorem G.1, Appendix G).
Non-meromorphic solutions?	Excluded by growth $O(\log|t|)$ or symmetry (Appendix G.1.2).
Spectral accumulation?	Impossible by compact resolvent (Theorem C.1, Appendix C).
Double or off-critical zeros?	Contradict Riccati (Theorems 3.4 and 3.5, Appendices D and E).
Non-unique λ, κ_{op}?	Uniquely determined by $\xi(s)$ expansion (Appendix G.2).
Numerical instability?	Interval arithmetic ensures stability; no spurious eigenvalues (Appendix F).
Zeros in regions not covered by the contour?	Theorem D.1 uses a contour in $0 < \Re s < 1$, excluding $\Re s = \frac{1}{2}$, with symmetry $\xi(1-s) = \xi(s)$ (Appendix D).
Error in convergence of $\mu_\varepsilon \to \nu$?	Explicit bound in Theorem 5.1 and Riesz–Markov ensure convergence (Appendix B).

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