Theory of Computation

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Outline

- Sets, Strings, Languages
- Languages and Problems
- Concepts about Finite Automata (FAs)
- Deterministic Finite Automata (DFAs)
- Notion of regular languages
- Operations with FAs

Concepts

- **Alphabet** (Σ) is a non-empty finite set of symbols:
  - $\Sigma = \{0, 1\}$, binary alphabet
  - $\Sigma = \{a, b, \ldots, z\}$, set of lower case letters
  - Set of ASCII chars
- **String** is a finite sequence of symbols selected from an alphabet
  - 01101 is a string over $\Sigma = \{0, 1\}$
  - Empty string (ε) has zero occurrences of symbols
  - Length of a string is the number of occurrences of symbols: $|01101| = 5$, $|\varepsilon| = 0$
  - Power of an alphabet Σ^k is the set of strings with length k, consisting of symbols of Σ (Cartesian product)
    - $\Sigma^0 = \{\varepsilon\}$
    - If $\Sigma = \{0, 1\}$ then $\Sigma^1 = \{0, 1\}$, $\Sigma^2 = \{00, 01, 10, 11\}$, $\Sigma^3 = \{000, 001, \ldots, 111\}$
    - Distinction between $\Sigma = \{0, 1\}$, set of symbols, and $\Sigma^1 = \{0, 1\}$, set of strings

Language

- The set of all strings over an alphabet Σ is denoted as Σ^*
  - $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$
  - $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \ldots$
  - $\Sigma^* = \Sigma^0 \cup \Sigma^+$
- **Language** L over an alphabet Σ is the subset of Σ^* ($L \subseteq \Sigma^*$)
- Examples of Languages:
  - Language of the strings with n 0s followed by n 1s:
    - $\{\varepsilon, 01, 0011, 000111, \ldots\}$
  - Set of binary prime numbers s
    - $\{10, 11, 101, 111, 1011, \ldots\}$
  - Empty language:
    - $\varnothing$
  - Language with only the empty string:
    - $\{\varepsilon\}$

Problem

- Decide if a given string belongs to a language
  - Given $w \in \Sigma^*$ and $L \subseteq \Sigma^*$, $w \in L$?
- It is common to describe a language using a set constructor notation:
  - {w | w consists of an equal number of 0s and 1s}
    - Set of strings referred as w such that w….
  - {w | w is a program in C syntactically correct}
- Example: primality testing
  - $w \in L_p$?   Where w is a string with the binary representation of a number and L_p is the language that contains all the strings representing the prime numbers in binary

Language or a Problem?

- Problem in a common sense:
  - Request to calculate or transform an input (e.g., compiler)
  - Not a yes/no decision
- In the context of complexity study, defining a problem in terms of a language is adequate
  - It is of similar difficulty to solve the decision as the problem
  - If it is as difficult as to decide if a string belongs to language L_X (set of valid strings in language X) than to translate programs in X to object code

If it was not, we could execute the translator, and then decide if the string belongs to L_X according to the success of the translator to produce object code. The problem of the decision would be easier which contradicts the supposition (proof by contradiction).

Language or a Problem?

- Languages and problems are essentially the same thing!
- Any Problem can be converted to a Language, and vice-versa:
  - Problem: Determine if a number is prime
  - Language: $L = \{p : p \text{ is prime}\}$

Finite Automata (FAs)

Example 1

(a) Let’s think of an automaton that recognizes only strings over {1}

with an even number of 1s

Set constructor notation: $\{w \mid w \in \{1\}^* \text{ and } |w| \text{ is even}\}$ or $\{w \in \{1\}^* \mid |w| \text{ is even}\}$

(b) Let’s think of an automaton that recognizes only strings over {0, 1}

with an even number of 1s

Set constructor notation: $\{w \in \{0,1\}^* \mid n_1(w) \text{ is even}\}$

Deterministic Finite Automata (DFAs)

- Deterministic
  - In a state, for each input there is only a single possible transition
- A DFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where:
  - Q is the set of finite states
  - Σ is the set of finite input symbols (the alphabet)
  - δ is the transition function, of states and inputs to states (e.g., $p = \delta(q, a)$)
    - $\delta : Q \times \Sigma \rightarrow Q$
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of final (accept) states
- DFA: $A = (Q, \Sigma, \delta, q_0, F)$

Tuple: a finite ordered list of elements

String Processing

- The language of a DFA $A = (Q, \Sigma, \delta, q_0, F)$ is the set of all the strings accepted/recognized by the DFA A
  - Input string: $a_1a_2\ldots a_n$
  - Initial state: q_0
  - Step: $\delta(q_{i-1},a_i) = q_i$
  - If $q_n \in F$ then the string is accepted

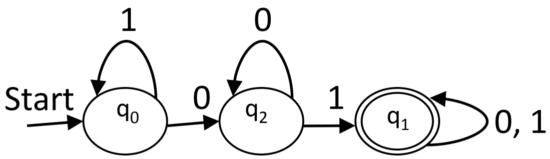
Defining an DFA

- Example: recognizer of the binary strings that contain the substring 01
  - {x01y | x and y are strings of 0s and 1s}
  - Σ = {0,1}
  - Q={q_0, q_1, q_2} has to memorize if it has already seen 01 (q_1), if the last one was 0 (q_2), or if didn't see nothing relevant (q_0)
  - Start state: q_0
  - Transition function
    - $\delta(q_0,1) = q_0$ $\delta(q_0,0) = q_2$ $\delta(q_2,0) = q_2$ $\delta(q_2,1) = q_1$ $\delta(q_1,0) = q_1$ $\delta(q_1,1) = q_1$
  - Final states: {q_1}
  - $A = (Q, \Sigma, \delta, q_0, F) = (\{q_0, q_1, q_2\}, \{0,1\}, \delta, q_0, \{q_1\})$

Transition (state) Diagrams

Example: recognizer of the binary strings that contain the substring 01

- The transition diagram of a DFA $A = (Q, \Sigma, \delta, q_0, F)$ is a graph
  - State in Q $\Rightarrow$ node/vertex
  - $\delta(q, a) = p$ where $q, p \in Q$ and $a \in \Sigma$ $\Rightarrow$ edge from q to p with label a
    - Edges from q to p can be merged and represented using a list of labels
  - Initial state $\Rightarrow$ arrow with Start (we will sometimes omit the Start label)
  - States in F $\Rightarrow$ double circle in the node



Transition Tables

Example: recognizer of the binary strings that contain the substring 01

- A transition table is the tabular representation of the δ function
  - States $\Rightarrow$ rows
  - Inputs $\Rightarrow$ columns
  - Initial State $\Rightarrow$ arrow
  - Final States $\Rightarrow$ *

	0	1
$\rightarrow q_0$	q_2	q_0
$*q_1$	q_1	q_1
q_2	q_2	q_1

Exercise 2

- Give a DFA that accepts the following language
  - L = { w | w has an even number of 0s and an even number of 1s}

Extended Transition Funtion $\hat{\delta}$

- Language of a DFA: set of strings formed by the sequence of labels for all the paths from the start node to one of the accept nodes
- Extended transition function, $\hat{\delta}$ (q,w) = p
  - q, state
  - w, input string
  - p, reached state when we start in q and process w
- Inductive definition in|w|
  - Basis: $\hat{\delta}$ (q,ε) = q
  - Induction: assuming w=xa then $\hat{\delta}$(q,w) = δ($\hat{\delta}$(q,x), a)
    - If $\hat{\delta}$ (q,x) = p and δ(p,a) = r, to go from q to r, we go from q to p and then with a step to r
    - $\hat{\delta}$(q,w) = δ(p, a)

Language of a DFA

- Processing in the DFA that recognizes strings with even number of 0s and 1s for the input w = 110101
  - $\hat{\delta}(q_0,\varepsilon) = q_0$
  - $\hat{\delta}(q_0,1) = \delta(\hat{\delta}(q_0, \varepsilon),1) = \delta(q_0,1) = q_1$
  - $\hat{\delta}(q_0,11) = \delta(\hat{\delta}(q_0,1),1) = \delta(q_1,1) = q_0$
  - …
  - $\hat{\delta}(q_0,110101) = \delta(\hat{\delta}(q_0,11010),1) = \delta(q_1,1) = q_0$
- Language of a DFA $A = (Q, \Sigma , \delta, q_0, F)$ is
  - $L(A) = \{w \mid \hat{\delta}(q_0,w) \in F\}$
- If a language L is L(A) for a DFA A then it is a **regular language**

Exercise 3 (homework)

- Give a DFA to recognize strings over the alphabet {0,1} with a ‘1’ in the third from last position.

Operations over Finite Automata

- Example of a Cartesian (cross) product
- Discuss the possible applications of the Cartesian product between finite automata

- Example:
  - DFA1 that recognizes: $\{w \in \{0,1\}^* \mid n_1(w) \text{ is even}\}$
  - DFA2 that recognizes: {x01y | x and y are strings of 0's and 1's}
  - What can give the Cartesian product of the two DFAs?

Summary

- Deterministic Finite Automata (DFAs)
- Use of DFAs to recognize strings
- Use of DFAs to represent regular languages
- Product of DFAs