Universidade do Porto
Faculdade de Engenharia
FEUP

Mestrado Integrado em Engenharia Informática e Computação

Métodos Formais em Engenharia de Software 2020/21

TP6. First Order Logic and Set Theory

1. First Order Logic.

a. Which of the following assertions are true:

(a) $3 < 5 \wedge 21 < 5^2$;
(b) $3 > 5 \vee 21 < 5^2$;
(c) $3 > 5 \Rightarrow 7 < 19$;
(d) $3 > 5 \Rightarrow 19 < 7$;
(e) $\forall n \in \mathbb{N}. n^2 \geq 0$;
(f) $\exists m \in \mathbb{N}. \forall n \in \mathbb{N}. n \geq m$;
(g) $\forall n \in \mathbb{N}. \exists m \in \mathbb{N}. n \geq m$;
(h) $\forall m \in \mathbb{N}. \exists n \in \mathbb{N}. m^2 < m \times n$;
(i) $\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. m^2 < m \times n$;
(j) $(\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. m^2 < m \times n) \Rightarrow (7 + 2 = 12)$.

(a)T
(b)T
(c)T
(d)T
(e)T
(f)T
(g)T
(h)F
(i)F
(j)T

b. Which of the following assertions are true:

(a) $\exists m \in \mathbb{N}. \forall n \in \mathbb{N}. n > m$;
(b) $\exists m \in \mathbb{N}. \forall n \in \mathbb{N}_1. n > m$;
(c) $\exists m \in \mathbb{N}_1. \forall n \in \mathbb{N}. n \geq m$;
(d) $\forall m \in \mathbb{N}_1. \exists n \in \mathbb{N}. m^2 < m \times n$;

(a)F
(b)T
(c)F
(d)T

c. Apply the following substitutions:

(a) $(x < y)[y + 1/y]$;
(b) $(x < y)[x + 1, y + 1/x, y]$;
(c) $(x < y)[y + 1, x - 2/x, y]$;
(d) $(x < y \Rightarrow x < z)[z + 1/y]$;
(e) $(\forall\, n \in \mathbb{N}.(m < n \vee n < w))[m + 1/n]$;
(f) $(\forall\, n \in \mathbb{N}.(m < n \vee n < w))[m + 1/w]$;
(g) $(\forall\, n \in \mathbb{N}.(m < w \vee n < w))[m + 1/m]$;
(h) $(\forall\, n \in \mathbb{N}.(m < w \vee n < w))[n - 3/w]$;
(i) $(n = 5 \wedge \forall\, n \in \mathbb{N}.n^2 \leq m)[m + 1/n]$.

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(a)x < y + 1
(b)x + 1 < y + 1 |=| x < y
(c)y + 1 < x - 2 |=| y < x - 3
(d)x < z + 1
(e)no changes
(f)∀n in N. (m < n or n < m + 1)
(g)∀n in N. (m + 1 < w or n < w)
(h)∀n in N. (m < n - 3 or n < n - 3) |=| (m < n - 3)
(i)m + 1 = 5 and ∀n in N. n^2 <= m
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2. Set Theory

Consider the following sets:

- DOGS = {Charlie, Max, Oscar, Rocky}
- CATS = {Bella, Garfield}
- BREED = {Bassethound, Sheepdog, Yorkshire, Siamese}

Calculate the result of the following operations and state their intuitive meaning:

(a) $DOGS \cup CATS$;
(b) $DOGS \cap CATS$;
(c) $DOGS - CATS$;
(d) $\mathbb{P}(DOGS)$;
(e) $\mathbb{P}(DOGS) \cup \mathbb{P}(CATS)$;
(f) $\mathbb{P}(DOGS \cup CATS)$;
(g) $\mathbb{P}(DOGS) \cap \mathbb{P}(CATS)$;
(h) $DOGS \times CATS$;
(i) $\mathbb{P}(DOGS \times BREED)$.

(a) `{Charlie, Max, Oscar, Rocky, Bella, Garfield}`
(b) `{}`
(c) `DOGS`
(d) `{ {}, {Charlie}, {Max}, {Oscar}, {Rocky}, {Charlie, Max}, … {Charlie, Max, Oscar, Rocky, Bella, Garfield}}`
(e) `{ {}, P(DOGS), P(CATS) }`
(f) `{ {}, {Charlie}, …., DOGS union CATS }`
(g) `{{}}`
(h) `{(Charlie, Bella), (Max, Garfield), ... , (Rocky, Garfield)}`
(i) `{ {}, {(Charlie, Bassethound)}, ... , {(Charlie, Siamese), (Max, Bassethound), ... , (Rocky, Siamese)} }`

Which of the following claims are true:

(a) $Charlie \in DOGS$;
(b) $Bella \subseteq CATS$;
(c) $Max \in \mathbb{P}(DOGS)$;
(d) $\{Oscar, Garfield\} \in \mathbb{P}(DOGS)$;
(e) $\{Rocky, Garfield\} \in \mathbb{P}(DOGS \times CATS)$;
(f) $\{(Charlie, Bella)\} \in \mathbb{P}(DOGS \times CATS)$;
(g) $\{Charlie, Garfield\} \in \mathbb{P}(DOGS \cup CATS)$;
(h) $\{Max, Garfield\} \subseteq \mathbb{P}(DOGS \cup CATS)$;
(i) $\{Rocky, Bella\} \subseteq \mathbb{P}(DOGS) \cup \mathbb{P}(CATS)$;
(j) $\{\{Garfield\}, \{Oscar\}\} \subseteq \mathbb{P}(DOGS) \cup \mathbb{P}(CATS)$;
(k) $\{\{Bella\}, \{Rocky\}\} \subseteq \mathbb{P}(DOGS \cup CATS)$;
(l) $\{\{Bella\}, \{Rocky\}\} \in \mathbb{P}(DOGS \cup CATS)$;
(m) $\{\{Charlie, Garfield\}\} \subseteq \mathbb{P}(DOGS \cup CATS)$;
(n) $\{\{Charlie, Garfield\}\} \in \mathbb{P}(DOGS \cup CATS)$;
(o) $\{\} \in \mathbb{P}(DOGS \times CATS)$;
(p) $\{\} \subseteq \mathbb{P}(DOGS \times CATS)$;
(q) $\{\} \in \mathbb{P}(DOGS \cap CATS)$;
(r) $\{\} \subseteq \mathbb{P}(DOGS \cap CATS)$.

(a) T
(b) F
(c) F
(d) F
(e) F
(f) T
(g) T
(h) F
(i) F
(j) T
(k) T
(l) F
(m) T
(n) F
(o) T
(p) T
(q) T
(r) T

Which of the following claims are true for an arbitrary instantiation of free variables:

(a) $(member \subseteq list \wedge new \in list) \Rightarrow (member \cup \{new\} \subseteq list)$;

(b) $(member \subseteq list \wedge new \in list) \Rightarrow (new \in member)$;

(c) $new \in list \Rightarrow \{new\} \subseteq list$;

(d) $\forall n \in member. \exists s \in \mathbb{P}(member). n \in s$;

(e) $\forall n \in member. \exists s \in \mathbb{P}(member). s \neq \{\} \wedge n \notin s$;

(f) $(member \cap list = \{\} \wedge new \notin member) \Rightarrow (member \cap list = \{\})[list \cup \{new\}/list]$;

(g) $x < y \Rightarrow (x \leq y)[x + 1/x]$.

(a) T
(b) F
(c) T
(d) T
(e) F
(f) T
(g) T

3. Relations

Consider the following sets:

- DOGS = {Charlie, Max, Oscar, Rocky, Zoey}
- CATS = {Bella, Garfield, Simba, Tigger}
- BREED = {Bassethound, Sheepdog, Yorkshire, Siamese}

Let *ANIMALS* = *DOGS* ∪ *CATS* and consider the relation $R_1 \subseteq ANIMALS \times BREED$ defined as

$$R_1 = \left\{ \begin{array}{l} (Max, Yorkshire), (Oscar, Sheepdog), (Zoey, Sheepdog), \\ (Garfield, Siamese), (Simba, Siamese) \end{array} \right\}$$

1) Calculate:

(a) $dom(R_1)$;

(b) $ran(R_1)$;

(c) $DOGS \lhd R_1$;

(d) $R_1 \rhd \{Bassethound, Yorkshire, Siamese\}$;

(e) $R_1[\{Oscar, Simba, Rocky\}]$;

(f) $R_1[dom(DOGS \lhd R_1)]$;

(g) $R_1^{-1}[ran(R_1 \rhd \{Sheepdog, Bassethound, Siamese\})]$;

(h) $R_1^{-1}[ran(DOGS \ntriangleleft R_1)]$;

(i) $R_1 \ntriangleright \{Bassethound\}$;

(j) $R_1 \ntriangleright \{Yorkshire\}$;

(k) $(DOGS \lhd R_1) \ntriangleright \{Yorkshire\}$;

(l) $R_1^{-1} \ntriangleright CATS$;

(m) $R_1[\{Max, Oscar\}] \ntriangleleft (R_1^{-1} \rhd DOGS)$.

a. `{Max,Oscar,Zoey,Garfield,Simba}`
b. `{Yorkshire,Sheepdog,Siamese}`
c. `{(Max,Yorkshire),(Oscar,Sheepdog),(Zoey,Sheepdog)}`
d. `{(Max,Yorkshire),(Garfield,Siamese),(Simba,Siamese)}`
e. `{Sheepdog,Siamese}`
f. `{Yorkshire,Sheepdog}` (porque `dom(DOGS◁R1) = {Max,Oscar,Zoey}`)

g. {Oscar,Zoey,Garfield,Simba} (porque ran(R1▷{Sheepdog,Bassethound,Siamese}) = {Sheepdog,Siamese})
h. {Garfield,Simba} (porque ran(DOGS⩤R1) = {Siamese})
i. {(Max,Yorkshire),(Oscar,Sheepdog),(Zoey,Sheepdog),(Garfield,Siamese),(Simba,Siamese)}
j. {(Oscar,Sheepdog),(Zoey,Sheepdog),(Garfield,Siamese),(Simba,Siamese)}
k. {(Oscar,Sheepdog),(Zoey,Sheepdog)}
l. {Garfield,Simba}
m. {} (porque $R1^{-1}$▷DOGS = {(Yorkshire,Max),(Sheepdog,Oscar),(Sheepdog,Zoey)})

2) Consider now relation $R_2 \subseteq$ *BREED* X *{dog, cat}* defined as

$$R_2 = \{(Bassethound, dog), (Sheepdog, dog), (Yorkshire, dog), (Siamese, cat)\}.$$

Calculate R1; R2.

{(Max,dog),(Oscar,dog),(Zoey,dog),(Garfield,cat),(Simba,cat)}

3) Let *R: ANIMALS* ⇸ *BREED* be a function that returns the breeds of our animals. Identify the formulae that better capture the following intuitive meanings:

(a) There exist animals that do not have a breed;
(b) All our cats share the same breed;
(c) We do not have any Bassethound;
(d) We have more dogs than cats;
(e) Tigger has no breed;
(f) We have a dog of breed Sheepdog;
(g) Which are the breeds of our dogs?;
(h) Which are the names of all our animals that are not dogs?

a) ∃a ∈ ANIMALS . R[{a}] = {} (ou dom(R) ≠ ANIMALS)
b) ∃b ∈ BREED . {b} = R[CATS]
c) Bassethound ∉ ran(R) (ou R^{-1}[{Bassethound}] = {})
d) |R[CATS]| < |R[DOGS]|
e) R[{Tiger}] = {}
f) R^{-1}[{Sheepdog}] ≠ {} (ou ∃a ∈ ANIMALS . R[{a}] = {Sheepdog})
g) DOGS ◁ R
h) DOGS ⩤ R1

4) Consider the following relation

$$eats = \left\{ \begin{array}{l} (jan, eggs), (jan, cheese), (jan, pizza), (jim, eggs), (jim, salad), \\ (ken, pizza), (lisa, cheese), (lisa, salad), (lisa, pizza) \end{array} \right\}$$

(a) Which is the relation $\{jan\} \lhd eats$?
(b) Which is the relation $\{jim\} \lhd\!\!\!- \, eats$?
(c) Which is the relation $eats \rhd \{cheese, pizza\}$?
(d) What is $dom(eats \rhd \{eggs\})$?
(e) Identify the relation that better captures "the set of persons that either eat *eggs* or *pizza*".
(f) Identify the relation that better captures "the set of persons that both eats *eggs* and *pizza*".

a) {(jan,eggs),(jan,cheese),(jan,pizza)}
b) {(jan,eggs),(jan,cheese),(jan,pizza),(ken,pizza),(lisa,cheese),(lisa,salad),(lisa,pizza)}
c) {(jan,cheese),(jan,pizza),(ken,pizza),(lisa,cheese),(lisa,pizza)}
d) {jan,jim}
e) R^{-1}[{eggs,pizza}] (ou R^{-1}[{eggs}] ∪ R^{-1}[{pizza}])
f) R^{-1}[{eggs}] ∩ R^{-1}[{pizza}]

5) Consider now the relation

$$cost = \left\{ \begin{array}{l} (eggs, cheap), (cheese, cheap), (pizza, expensive), \\ (salad, cheap), (expensive, expensive), (chips, cheap) \end{array} \right\}$$

Calculate the following:

(a) $eats[\{jan, lisa\}]$;

(b) $eats^{-1}$;

(c) $eats^{-1}[\{cheese, eggs\}]$;

(d) $eats;\ cost$;

(e) $eats;\ (cost \rhd \{expensive\})$;

(f) $eats^{-1}[cost^{-1}[\{expensive\}]]$;

(g) $eats \oplus \{(lisa, expensive)\}$.

a) `{eggs,cheese,pizza,eggs,salad}`
b) `{(eggs,jan),(cheese,jan),(pizza,jan),(eggs,jim),(salad,jim),(pizza,ken),(cheese,lisa),(salad,lisa),(pizza,lisa)}`
c) `{jan,jim,lisa}`
d) `{(jan,cheap),(jan,expensive),(jim,cheap),(ken,expensive),(lisa,cheap),(lisa,expensive)}`
e) `{(jan,cheap),(jim,cheap),(lisa,cheap)}`
f) `{jan,ken,pizza}`
g) `{(jan,eggs),(jan,cheese),(jan,pizza),(jim,eggs),(jim,salad),(ken,pizza),(lisa,expensive)}`

4. Specification of Properties

Suppose that a Zoo has several habitats, with animals of different species. Suppose defined sets Animals, Species, Habitats that contain respectively all animals, species and habitats. Suppose also that there is a set Veterinarians containing all veterinarians of the Zoo and relations

$$\begin{array}{l} \text{animal-species} : Animals \leftrightarrow Species \\ \text{animal-habitat} : Animals \leftrightarrow Habitats \\ \text{co-exist} : Animals \leftrightarrow Animals \\ \text{knows-cure} : Veterinaries \leftrightarrow Species \\ \text{coordinator-habitat} : Habitats \leftrightarrow Veterinaries \\ \text{veterinaries-habitat} : Habitats \leftrightarrow Veterinaries \end{array}$$

with the intuitive meaning. Write down formulas that reflect the following requisites (you may need more than one formula to represent a single requisite):

1. Every animal is of a single species.
2. An animal of the Zoo may be, at most, in a single habitat.
3. Each habitat has at least one animal.
4. Each habitat has at most 100 animals.
5. Each habitat may contain only animals that may coexist.
6. Each veterinarian knows how to cure at least one species of animals.
7. Each habitat of the Zoo has a single coordinator.
8. Each veterinarian can be at most coordinator of a single habitat.
9. All species in the zoo have a veterinarian that knows how to heal them.
10. The coordinator of a habitat is a veterinarian of that habitat.
11. The veterinarians associated with each habitat know how to heal all the species of the habitat.
12. Each veterinarian is associated with at most 2 habitats.

1. $(\forall a \in Animals,\ s1,s2 \in Species\ .\ (a,s1) \in \text{animal-species} \wedge (a,s2) \in \text{animal-species} \Rightarrow s1 = s2)$ (função parcial)
$\wedge$

$\text{dom}(\text{animal-species}) = \text{Animals}$ (função total)

2. $(\forall a \in \text{Animals}, h1,h2 \in \text{Habitat} \,.\, (a,h1) \in \text{animal-habitat} \land (a,h2) \in \text{animal-habitat} \Rightarrow h1 = h2)$ (função parcial)
3. $\text{ran}(\text{animal-habitat}) = \text{Habitats}$ (relação sobrejectiva)
4. $\forall h \in \text{Habitats} \,.\, |\text{animal-habitat}^{-1}[\{h\}]| < 100$
5. $\forall h \in \text{Habitats}, a1,a2 \in \text{Animals} \,.\, ((a1,h) \in \text{animal-habitat} \land (a2,h) \in \text{animal-habitat}) \Rightarrow (a1, a2) \in \text{co-exist}$
6. $\text{dom}(\text{knows-cure}) = \text{Veterinarians}$ (função total)
7. $\forall h \in \text{Habitats} \; \exists v \in \text{Veterinarians} \,.\, \text{coordinator-habitat}[\{h\}] = \{v\}$
8. $\forall v \in \text{Veterinarians} \,.\, |\text{coordinator-habitat};\{v\}| \leq 1$
9. $\text{ran}(\text{knows-cure}) = \text{Species}$
10. $\text{coordinator-habitat} \subseteq \text{veterinaries-habitat}$
11. $\forall h \in \text{Habitats} \,.\, \text{animal-species}[\text{animal-habitat};\{h\}] \subseteq \text{veterinaries-habitat}[\{h\}];\text{knows-cure}$
12. $\forall v \in \text{Veterinarians} \,.\, |\text{veterinaries-habitat};\{v\}| \leq 2$