M4501 Differential Equations
Tutorial 1: Picard's Iteration and IVP Solutions

Prepared by: Motaung Mandla>Motaung Mandla

National University of Lesotho

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Department of Mathematics and Computer Science
M4501 – Differential Equations

PROBLEM 1: USE PICARD'S ITERATION TO SOLVE IVPS

We apply Picard's iteration method:

$$y_0(x) = y_0, \quad y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt.$$

Solution 1. *(a)* $\dfrac{dy}{dx} = x + y - 1, \quad y(0) = 1$

Let $y_0(x) = 1$.

$$\begin{aligned}
y_1(x) &= 1 + \int_0^x (t + 1 - 1)\, dt = 1 + \int_0^x t\, dt = 1 + \frac{x^2}{2} \\
y_2(x) &= 1 + \int_0^x \left(t + \left(1 + \frac{t^2}{2}\right) - 1\right) dt = 1 + \int_0^x \left(t + \frac{t^2}{2}\right) dt = 1 + \frac{x^2}{2} + \frac{x^3}{6} \\
y_3(x) &= 1 + \int_0^x \left(t + \left(1 + \frac{t^2}{2} + \frac{t^3}{6}\right) - 1\right) dt = 1 + \int_0^x \left(t + \frac{t^2}{2} + \frac{t^3}{6}\right) dt \\
&= 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}
\end{aligned}$$

Pattern: $y_n(x) = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}$. *As* $n \to \infty$, *this tends to:*

$$y(x) = 1 + (e^x - 1 - x) = e^x - x$$

So the exact solution is $\boxed{y(x) = e^x - x}$.

(b) $\dfrac{dy}{dx} = y - x, \quad y(0) = 2$

$y_0(x) = 2$

$$y_1(x) = 2 + \int_0^x (2-t)\,dt = 2 + \left[2t - \frac{t^2}{2}\right]_0^x = 2 + 2x - \frac{x^2}{2}$$

$$y_2(x) = 2 + \int_0^x \left(\left(2 + 2t - \frac{t^2}{2}\right) - t\right) dt = 2 + \int_0^x \left(2 + t - \frac{t^2}{2}\right) dt$$

$$= 2 + \left[2t + \frac{t^2}{2} - \frac{t^3}{6}\right]_0^x = 2 + 2x + \frac{x^2}{2} - \frac{x^3}{6}$$

$$y_3(x) = 2 + \int_0^x (y_2(t) - t)\,dt = 2 + \int_0^x \left(2 + 2t + \frac{t^2}{2} - \frac{t^3}{6} - t\right) dt$$

$$= 2 + \int_0^x \left(2 + t + \frac{t^2}{2} - \frac{t^3}{6}\right) dt = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$$

Pattern suggests $y(x) = x + 1 + e^x$. *Check:* $y(0) = 1 + 1 = 2$. *Try* $y = x + 1 + e^x$. *Then:* $y' = 1 + e^x$, *and* $y - x = 1 + e^x + x - x = 1 + e^x$. *Yes! So* $\boxed{y(x) = x + 1 + e^x}$.

(c) $\dfrac{dy}{dx} = 2(1+y), \quad y(0) = 0$

$y_0(x) = 0$

$$y_1(x) = 0 + \int_0^x 2(1+0)\,dt = 2x$$

$$y_2(x) = 0 + \int_0^x 2(1+2t)\,dt = 2\int_0^x (1+2t)\,dt = 2\left[t + t^2\right]_0^x = 2x + 2x^2$$

$$y_3(x) = 0 + \int_0^x 2(1+2t+2t^2)\,dt = 2\int_0^x (1+2t+2t^2)\,dt = 2\left[t + t^2 + \frac{2t^3}{3}\right]_0^x = 2x + 2x^2 + \frac{4x^3}{3}$$

Pattern: coefficients resemble $e^{2x} - 1$. *Recall* $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots$

So $y_n(x) \to e^{2x} - 1$. *Check:* $y(0) = 0$, $y' = 2e^{2x}$, $2(1+y) = 2(1 + e^{2x} - 1) = 2e^{2x}$. *Good.*

So $\boxed{y(x) = e^{2x} - 1}$.

(d) $\dfrac{dy}{dx} = 2x - y, \quad y(0) = 1$

$y_0(x) = 1$

$$\begin{aligned}
y_1(x) &= 1 + \int_0^x (2t-1)\,dt = 1 + \left[t^2 - t\right]_0^x = 1 + x^2 - x \\
y_2(x) &= 1 + \int_0^x \big(2t - (1 + t^2 - t)\big)\,dt = 1 + \int_0^x (-t^2 + 3t - 1)\,dt \\
&= 1 + \left[-\frac{t^3}{3} + \frac{3t^2}{2} - t\right]_0^x = 1 - \frac{x^3}{3} + \frac{3x^2}{2} - x \\
y_3(x) &= 1 + \int_0^x (2t - y_2(t))\,dt = 1 + \int_0^x \left(3t - 1 + \frac{t^3}{3} - \frac{3t^2}{2}\right)dt \\
&= 1 + \left[\frac{3t^2}{2} - t + \frac{t^4}{12} - \frac{t^3}{2}\right]_0^x = 1 + \frac{3x^2}{2} - x + \frac{x^4}{12} - \frac{x^3}{2}
\end{aligned}$$

Solving via integrating factor: $y' + y = 2x$, *IF:* e^x, *so*

$$(e^x y)' = 2xe^x \Rightarrow e^x y = 2(xe^x - e^x) + C \Rightarrow y = 2x - 2 + Ce^{-x}$$

$y(0) = 1 \Rightarrow -2 + C = 1 \Rightarrow C = 3$. *So* $\boxed{y(x) = 2x - 2 + 3e^{-x}}$.

(e) $\dfrac{dy}{dx} = 2x(1-y), \quad y(0) = 2$

$y_0(x) = 2$

$$\begin{aligned}
y_1(x) &= 2 + \int_0^x 2t(1-2)\,dt = 2 - \int_0^x 2t\,dt = 2 - x^2 \\
y_2(x) &= 2 + \int_0^x 2t(1-(2-t^2))\,dt = 2 + \int_0^x 2t(-1+t^2)\,dt = 2 - x^2 + \frac{x^4}{2} \\
y_3(x) &= 2 + \int_0^x 2t\left(1 - \left(2 - t^2 + \frac{t^4}{2}\right)\right)dt = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}
\end{aligned}$$

Try $y = 1 + e^{-x^2}$. *Then* $y' = -2xe^{-x^2}$, $2x(1-y) = 2x(-e^{-x^2}) = -2xe^{-x^2}$. *Yes!* $y(0) = 1 + 1 = 2$. *So* $\boxed{y(x) = 1 + e^{-x^2}}$.

(f) $\dfrac{dy}{dx} = xy + 1, \quad y(0) = 0$

$y_0(x) = 0$

$$\begin{aligned}
y_1(x) &= \int_0^x 1\,dt = x \\
y_2(x) &= \int_0^x (t^2 + 1)\,dt = x + \frac{x^3}{3} \\
y_3(x) &= \int_0^x \left(t^2 + \frac{t^4}{3} + 1\right)dt = x + \frac{x^3}{3} + \frac{x^5}{15}
\end{aligned}$$

Linear ODE: $y' - xy = 1$. *IF:* $e^{-x^2/2}$. *Then:*

$$\frac{d}{dx}(ye^{-x^2/2}) = e^{-x^2/2} \Rightarrow y = e^{x^2/2} \int_0^x e^{-t^2/2}\,dt$$

So $\boxed{y(x) = e^{x^2/2} \int_0^x e^{-t^2/2}\,dt}$.

PROBLEM 2: VERIFY SOLUTIONS USING ELEMENTARY METHODS

Solution 2. *We verify each solution from Problem 1 using standard ODE techniques.*

(a) $y' - y = x - 1$. *IF:* e^{-x}. *Then:*

$$(ye^{-x})' = (x-1)e^{-x} \Rightarrow y = e^x \int (x-1)e^{-x}dx = e^x(-xe^{-x} + C) = -x + Ce^x$$

$y(0) = 1 \Rightarrow C = 1$. $y = e^x - x$. *Verified.*

(b) $y' - y = -x$. *IF:* e^{-x}. *Then:*

$$(ye^{-x})' = -xe^{-x} \Rightarrow y = e^x(xe^{-x} + e^{-x} + C) = x + 1 + Ce^x$$

$y(0) = 2 \Rightarrow C = 1$. $y = x + 1 + e^x$. *Verified.*

(c) Separable: $\frac{dy}{1+y} = 2dx \Rightarrow \ln|1+y| = 2x + C \Rightarrow y = e^{2x} - 1$. *Verified.*

(d) $y' + y = 2x$. *IF:* e^x. *Then:*

$$(ye^x)' = 2xe^x \Rightarrow y = 2x - 2 + 3e^{-x}. \text{ Verified.}$$

(e) $\frac{dy}{1-y} = 2xdx \Rightarrow -\ln|1-y| = x^2 + C \Rightarrow y = 1 + e^{-x^2}$. *Verified.*

(f) $y' - xy = 1$. *IF:* $e^{-x^2/2}$. *Then:*

$$ye^{-x^2/2} = \int_0^x e^{-t^2/2}dt \Rightarrow y = e^{x^2/2} \int_0^x e^{-t^2/2}dt. \text{ Verified.}$$

PROBLEM 3: THREE PICARD'S ITERATIONS

Solution 3. *(a)* $\frac{dy}{dx} = x + y^2, \quad y(0) = 0$

$y_0(x) = 0$

$$\begin{aligned}
y_1(x) &= \frac{x^2}{2} \\
y_2(x) &= \frac{x^2}{2} + \frac{x^5}{20} \\
y_3(x) &= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}
\end{aligned}$$

(b) $\dfrac{dy}{dx} = 2 - \frac{y}{x}, \quad y(1) = 2$

$y_0(x) = 2$

$$
\begin{aligned}
y_1(x) &= 2x - 2\ln x \\
y_2(x) &= 2 + (\ln x)^2 \\
y_3(x) &= 2x - 2\ln x - \frac{(\ln x)^3}{3}
\end{aligned}
$$

(c) $\dfrac{dy}{dx} = 1 - xy, \quad y(0) = 0$

$y_0(x) = 0$

$$
\begin{aligned}
y_1(x) &= x \\
y_2(x) &= x - \frac{x^3}{3} \\
y_3(x) &= x - \frac{x^3}{3} + \frac{x^5}{15}
\end{aligned}
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Prepared for the Department of Mathematics and Computer Science, National University of Lesotho.
Contact: your.email@nul.ls