

M4401 Differential Equations

Tutorial 1: Picard's Iteration and IVP Solutions

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M4401 – Ordinary Differential Equations

PROBLEM 1: USE PICARD'S ITERATION TO SOLVE IVPS

We apply Picard's iteration method:

$$y_0(x) = y_0, \quad y_{n+1}(x) = y_0 + \int_{x_0}^x f(t, y_n(t)) dt.$$

Solution 1. (a) $\frac{dy}{dx} = x + y - 1, \quad y(0) = 1$

Let $y_0(x) = 1$.

$$\begin{aligned} y_1(x) &= 1 + \int_0^x (t + 1 - 1) dt = 1 + \int_0^x t dt = 1 + \frac{x^2}{2} \\ y_2(x) &= 1 + \int_0^x \left(t + \left(1 + \frac{t^2}{2} \right) - 1 \right) dt = 1 + \int_0^x \left(t + \frac{t^2}{2} \right) dt = 1 + \frac{x^2}{2} + \frac{x^3}{6} \\ y_3(x) &= 1 + \int_0^x \left(t + \left(1 + \frac{t^2}{2} + \frac{t^3}{6} \right) - 1 \right) dt = 1 + \int_0^x \left(t + \frac{t^2}{2} + \frac{t^3}{6} \right) dt \\ &= 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned}$$

Pattern: $y_n(x) = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}$. As $n \rightarrow \infty$, this tends to:

$$y(x) = 1 + (e^x - 1 - x) = e^x - x$$

So the exact solution is $\boxed{y(x) = e^x - x}$.

(b) $\frac{dy}{dx} = y - x, \quad y(0) = 2$
 $y_0(x) = 2$

$$\begin{aligned}
y_1(x) &= 2 + \int_0^x (2 - t) dt = 2 + \left[2t - \frac{t^2}{2} \right]_0^x = 2 + 2x - \frac{x^2}{2} \\
y_2(x) &= 2 + \int_0^x \left(\left(2 + 2t - \frac{t^2}{2} \right) - t \right) dt = 2 + \int_0^x \left(2 + t - \frac{t^2}{2} \right) dt \\
&= 2 + \left[2t + \frac{t^2}{2} - \frac{t^3}{6} \right]_0^x = 2 + 2x + \frac{x^2}{2} - \frac{x^3}{6} \\
y_3(x) &= 2 + \int_0^x (y_2(t) - t) dt = 2 + \int_0^x \left(2 + 2t + \frac{t^2}{2} - \frac{t^3}{6} - t \right) dt \\
&= 2 + \int_0^x \left(2 + t + \frac{t^2}{2} - \frac{t^3}{6} \right) dt = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}
\end{aligned}$$

Pattern suggests $y(x) = x + 1 + e^x$. Check: $y(0) = 1 + 1 = 2$. Try $y = x + 1 + e^x$. Then: $y' = 1 + e^x$, and $y - x = 1 + e^x + x - x = 1 + e^x$. Yes! So $\boxed{y(x) = x + 1 + e^x}$.

(c) $\frac{dy}{dx} = 2(1 + y), \quad y(0) = 0$
 $y_0(x) = 0$

$$\begin{aligned}
y_1(x) &= 0 + \int_0^x 2(1 + 0) dt = 2x \\
y_2(x) &= 0 + \int_0^x 2(1 + 2t) dt = 2 \int_0^x (1 + 2t) dt = 2 \left[t + t^2 \right]_0^x = 2x + 2x^2 \\
y_3(x) &= 0 + \int_0^x 2(1 + 2t + 2t^2) dt = 2 \int_0^x (1 + 2t + 2t^2) dt = 2 \left[t + t^2 + \frac{2t^3}{3} \right]_0^x = 2x + 2x^2 + \frac{4x^3}{3}
\end{aligned}$$

Pattern: coefficients resemble $e^{2x} - 1$. Recall $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots$

So $y_n(x) \rightarrow e^{2x} - 1$. Check: $y(0) = 0$, $y' = 2e^{2x}$, $2(1 + y) = 2(1 + e^{2x} - 1) = 2e^{2x}$. Good.

So $\boxed{y(x) = e^{2x} - 1}$.

(d) $\frac{dy}{dx} = 2x - y, \quad y(0) = 1$
 $y_0(x) = 1$

$$y_1(x) = 1 + \int_0^x (2t - 1) dt = 1 + [t^2 - t]_0^x = 1 + x^2 - x$$

$$\begin{aligned} y_2(x) &= 1 + \int_0^x (2t - (1 + t^2 - t)) dt = 1 + \int_0^x (-t^2 + 3t - 1) dt \\ &= 1 + \left[-\frac{t^3}{3} + \frac{3t^2}{2} - t \right]_0^x = 1 - \frac{x^3}{3} + \frac{3x^2}{2} - x \end{aligned}$$

$$\begin{aligned} y_3(x) &= 1 + \int_0^x (2t - y_2(t)) dt = 1 + \int_0^x \left(3t - 1 + \frac{t^3}{3} - \frac{3t^2}{2} \right) dt \\ &= 1 + \left[\frac{3t^2}{2} - t + \frac{t^4}{12} - \frac{3t^3}{2} \right]_0^x = 1 + \frac{3x^2}{2} - x + \frac{x^4}{12} - \frac{3x^3}{2} \end{aligned}$$

Solving via integrating factor: $y' + y = 2x$, IF: e^x , so

$$(e^x y)' = 2xe^x \Rightarrow e^x y = 2(xe^x - e^x) + C \Rightarrow y = 2x - 2 + Ce^{-x}$$

$$y(0) = 1 \Rightarrow -2 + C = 1 \Rightarrow C = 3. \text{ So } \boxed{y(x) = 2x - 2 + 3e^{-x}}.$$

(e) $\frac{dy}{dx} = 2x(1 - y), \quad y(0) = 2$
 $y_0(x) = 2$

$$y_1(x) = 2 + \int_0^x 2t(1 - 2) dt = 2 - \int_0^x 2t dt = 2 - x^2$$

$$y_2(x) = 2 + \int_0^x 2t(1 - (2 - t^2)) dt = 2 + \int_0^x 2t(-1 + t^2) dt = 2 - x^2 + \frac{x^4}{2}$$

$$y_3(x) = 2 + \int_0^x 2t \left(1 - \left(2 - t^2 + \frac{t^4}{2} \right) \right) dt = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$$

Try $y = 1 + e^{-x^2}$. Then $y' = -2xe^{-x^2}$, $2x(1 - y) = 2x(-e^{-x^2}) = -2xe^{-x^2}$. Yes!
 $y(0) = 1 + 1 = 2$. So $\boxed{y(x) = 1 + e^{-x^2}}$.

(f) $\frac{dy}{dx} = xy + 1, \quad y(0) = 0$
 $y_0(x) = 0$

$$y_1(x) = \int_0^x 1 dt = x$$

$$y_2(x) = \int_0^x (t^2 + 1) dt = x + \frac{x^3}{3}$$

$$y_3(x) = \int_0^x \left(t^2 + \frac{t^4}{3} + 1 \right) dt = x + \frac{x^3}{3} + \frac{x^5}{15}$$

Linear ODE: $y' - xy = 1$. IF: $e^{-x^2/2}$. Then:

$$\frac{d}{dx}(ye^{-x^2/2}) = e^{-x^2/2} \Rightarrow y = e^{x^2/2} \int_0^x e^{-t^2/2} dt$$

$$\text{So } \boxed{y(x) = e^{x^2/2} \int_0^x e^{-t^2/2} dt}.$$

PROBLEM 2: VERIFY SOLUTIONS USING ELEMENTARY METHODS

Solution 2. We verify each solution from Problem 1 using standard ODE techniques.

(a) $y' - y = x - 1$. IF: e^{-x} . Then:

$$(ye^{-x})' = (x - 1)e^{-x} \Rightarrow y = e^x \int (x - 1)e^{-x} dx = e^x(-xe^{-x} + C) = -x + Ce^x$$

$$y(0) = 1 \Rightarrow C = 1. \quad y = e^x - x. \quad \text{Verified.}$$

(b) $y' - y = -x$. IF: e^{-x} . Then:

$$(ye^{-x})' = -xe^{-x} \Rightarrow y = e^x(xe^{-x} + e^{-x} + C) = x + 1 + Ce^x$$

$$y(0) = 2 \Rightarrow C = 1. \quad y = x + 1 + e^x. \quad \text{Verified.}$$

(c) Separable: $\frac{dy}{1+y} = 2dx \Rightarrow \ln|1+y| = 2x + C \Rightarrow y = e^{2x} - 1$. Verified.

(d) $y' + y = 2x$. IF: e^x . Then:

$$(ye^x)' = 2xe^x \Rightarrow y = 2x - 2 + 3e^{-x}. \quad \text{Verified.}$$

(e) $\frac{dy}{1-y} = 2x dx \Rightarrow -\ln|1-y| = x^2 + C \Rightarrow y = 1 + e^{-x^2}$. Verified.

(f) $y' - xy = 1$. IF: $e^{-x^2/2}$. Then:

$$ye^{-x^2/2} = \int_0^x e^{-t^2/2} dt \Rightarrow y = e^{x^2/2} \int_0^x e^{-t^2/2} dt. \quad \text{Verified.}$$

PROBLEM 3: THREE PICARD'S ITERATIONS

Solution 3. (a) $\frac{dy}{dx} = x + y^2, \quad y(0) = 0$

$$y_0(x) = 0$$

$$y_1(x) = \frac{x^2}{2}$$

$$y_2(x) = \frac{x^2}{2} + \frac{x^5}{20}$$

$$y_3(x) = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

$$(b) \quad \frac{dy}{dx} = 2 - \frac{y}{x}, \quad y(1) = 2$$
$$y_0(x) = 2$$

$$y_1(x) = 2x - 2 \ln x$$
$$y_2(x) = 2 + (\ln x)^2$$
$$y_3(x) = 2x - 2 \ln x - \frac{(\ln x)^3}{3}$$

$$(c) \quad \frac{dy}{dx} = 1 - xy, \quad y(0) = 0$$
$$y_0(x) = 0$$

$$y_1(x) = x$$
$$y_2(x) = x - \frac{x^3}{3}$$
$$y_3(x) = x - \frac{x^3}{3} + \frac{x^5}{15}$$

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