MATH 621 HW 5 Sumanth Ravipati March 6, 2019

Sources:

1. Let G be a group with 4 elements. Prove that G is abelian.

   Let G contain the 4, distinct, generic elements $\{e, f, g, h\}$, with e being the identity element. For G to be abelian, $ab = ba \, \forall a, b \in G$. Without loss of generality, let us suppose that G is not abelian because $fg \neq gf$. We know that for any $a, b \in G \setminus \{e\}, ab \neq a$ because if $ab = a$, multiplying both sides on the left by a^{-1} gives us $a^{-1}ab = a^{-1}a \Rightarrow b = e$, a contradiction. Using this fact, we can conclude that fg and gf are both not equal to f or g. Again, without loss of generality, since there are only 2 elements remaining, we can set $fg = h$ and $gf = e$. Since $gf = e$, the inverse of both sides must equal one another so $(gf)^{-1} = e^{-1}$. Using a previously proven identity, we see that $f^{-1}g^{-1} = e$. Multiplying both sides on the right by g gives us $f^{-1} = g$. We can substitute this identity into our earlier equation $fg = h$. This becomes $fg = ff^{-1} = e \neq h$, a contradiction. Therefore, fg does in fact equal gf. Since this was chosen in general for this group G with 4 distinct, generic elements, any such G is abelian, as was to be shown. $\square$

2. Prove that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the only non-cyclic group of order 4.

3. Let $n \in \mathbb{N}$ and p a prime number with $p > n$. Let G be a group with $|G| = pn$. If H is a subgroup of G with $|H| = p$, show that H is the only subgroup of G with order p, and is a normal subgroup of G.