Surprising Geometric Constructions

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Contents

Chapter 1
Introduction

Geometric constructions are fundamental to Euclidean geometry and appear in secondary-school textbooks [6]. Most students of mathematics will also know that three constructions are impossible: trisecting angle, squaring a circle and doubling a cube. There are many interesting and surprising geometrical constructions that are probably unknown to students and teachers. This document presents these constructions in great detail using only secondary-school mathematics.

The LaTeX source can be found at `https://github.com/motib/constructions`.

Part I presents constructions with the familiar straightedge and compass, as well as constructions known to the Greeks that use extensions of the straightedge and compass. In recent years, the art of origami—paper folding—has been given a mathematical formalization as described in Part II. It may come as a surprise that constructions with origami are more powerful than constructions with straightedge and compass.

1.1 Constructions with straightedge and compass

Chapeter 2, the collapsing compass The modern compass is a *fixed compass* that maintains the distance between its legs when lifted off the paper. It can be used to construct a line segment of the same length as a given segment. The compass used in the ancient world was a *collapsing compass* that does not maintain the distance between its legs when lifted from the paper. Euclid showed that any construction that can be done with a fixed compass can be done with a collapsing compass. Numerous incorrect proofs have been given based on incorrect diagrams [17]. In order to emphasize that a proof must not depend on a diagram, I "prove" that *every* triangle is isoceles.

Chapter 3, trisecting an angle Trisecting an arbitrary angle is impossible, but the Greeks knew that any angle can be trisected using extensions of the straightedge and compass. This chapter presents constructions that trisect an angle using a neusis and a quadratrix. The quadratrix can also be used to square a circle.

Chapter 4, squaring a circle To square a circle requires the construction of a line segment of length π. This chapter presents three constructions of approximations to π, one by Adam Kochansky and two by Ramanujan.

Chapter 5, construction with only a compass Are both a straightedge and a compass necessary? Lorenzo Mascheroni and Georg Mohr showed that a compass only is sufficient.

Chapter 6 ,construction with only a straightedge Is a straightedge sufficient? The answer is no because a straightedge can "compute" only linear functions, whereas a compass can "compute" quadratic functions. Jacob Steiner proved that a straightedge is sufficient provided that somewhere in the plane a single circle exists.

1.2 Constructions with origami

Chapter 7, the axioms of origami The seven axioms of origami can be formalized in mathematics. This chapters derives formulas for the axioms, together with numerical examples.

Chapter 8, trisecting an angle Two methods for trisecting an angle with origami are given.

Chapter 9, doubling a cube Two methods for doubling a cube with origami are given.

Chapter 10, finding roots This chapter explains Eduard Lill's geometric method for finding real roots of any polynomial (actually, verifying that a value is a root). We demonstrate the method for cubic polynomials.

Chapter 11, constructing a cube root Margharita P. Beloch published an implementation of Lill's method that can find a root of a cubic polynomial with one fold.

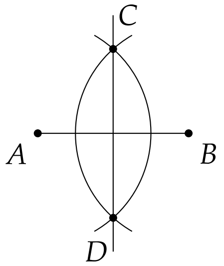
Part I

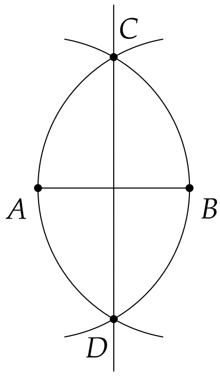
Straightedge and Compass

Chapter 2
Help, My Compass Collapsed!

2.1 Fixed compasses and collapsing compasses

A modern compass is a *fixed compass*: the distance between the two legs can be fixed so that it is possible to copy a line segment or a circle from one position to another. I have seen geometry textbooks that present the construction a perpendicular bisector to a line segment as follows: construct two circles centered at the ends of the line segment such that the radii are equal and *greater than half the length of the segment* (left diagram):

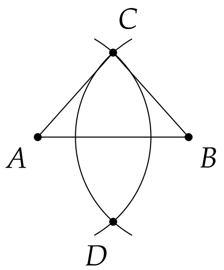


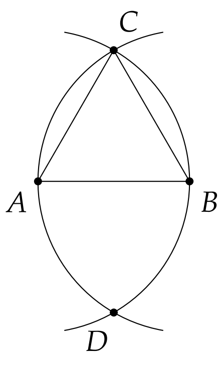


Euclid used a *collapsing compass* whose legs fold up when the compass is lifted off the paper. Teachers often use a collapsing compass consisting of a piece of chalk tied to a string. It is impossible to maintain a fixed radius when the chalk and the end of the string are removed from the blackboard. The right diagram above shows how to construct a perpendicular bisector with a collapsing compass: the length of the segment $\overline{AB}$ is, of course, equal to the length of the segment $\overline{BA}$, so the radii of the two circles are equal.

The proof that the line constructed is the perpendicular bisector is not at all elementary because relatively advanced concepts like congruent triangles have to be used. However, the proof that the same construction results in an equilateral triangle is very simple (right diagram below). The length of $\overline{AC}$ equals the length of $\overline{AB}$ since they are radii of the same circle, and for the same reason the length of $\overline{BC}$ is equal to the length of $\overline{BA}$. We have:

$$\overline{AC} = \overline{AB} = \overline{BA} = \overline{BC}\,.$$





The left diagram above shows that for the construction with the fixed compass the triangle will be isosceles, but not necessarily equilateral.

This construction of an equilateral triangle is the first proposition in Euclid's *Elements*. The second proposition shows how to copy a given line segment $\overline{AB}$ to a segment of the same length, one of whose end points is a given point C. Therefore, a fixed compass adds no additional capability. Toussaint [17] showed that many incorrect constructions for this proposition have been given. In fact, it was Euclid who gave a correct construction! The following section presents Euclid's construction and the proof of its correctness. Then I show an incorrect construction that can be found even in modern textbooks.

2.2 Euclid's construction for copying a line segment

Theorem: Given a line segment $\overline{AB}$ and a point C, a line segment can be constructed (using a collapsing compass) at C whose length is equal to the length of $\overline{AB}$:

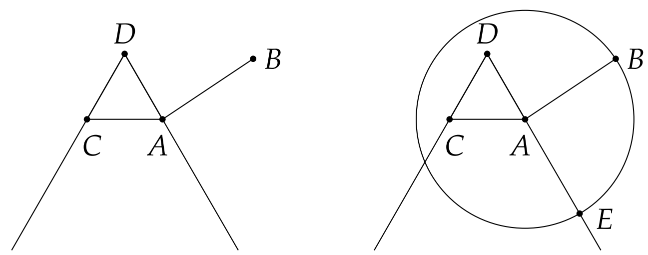
Construction:

Construct the line segment from A to C.

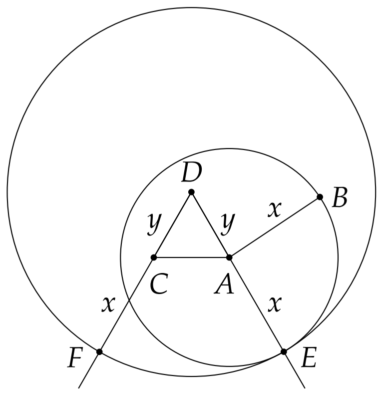
Construct an equilateral triangle whose base is $\overline{AC}$ (right diagram above). Label the third vertex D. By Euclid's first proposition, the triangle can be constructed using a collapsing compass.

Construct a ray that is a continuation of $\overline{DA}$ and a ray that is a continuation of $\overline{DC}$ (left diagram below).

Construct a circle centered at A with radius $\overline{AB}$. Label the intersection of the circle and the ray $\overline{DA}$ by E (right diagram below).



Construct a circle centered at D with radius $\overline{DE}$. Label the intersection of the circle and the ray $\overline{DC}$ by F:



Claim: The length of the line segment $\overline{CF}$ is equal to the length of $\overline{AB}$.

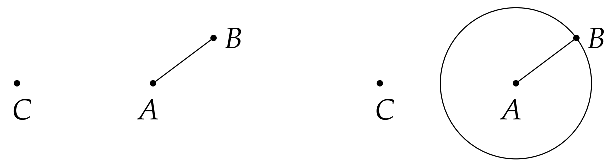
Proof: $\overline{DC} = \overline{DA}$ because $\triangle ACD$ is equilateral. $\overline{AE} = \overline{AB}$ because they are radii of the same circle centered at A. $\overline{DF} = \overline{DE}$ because they are radii of the same circle centered at D. Therefore, the length of the line segment $\overline{CF}$ is:

$$\overline{CF} = \overline{DF} - \overline{DC} = \overline{DE} - \overline{DC} = \overline{DE} - \overline{DA} = \overline{AE} = \overline{AB}\,.$$

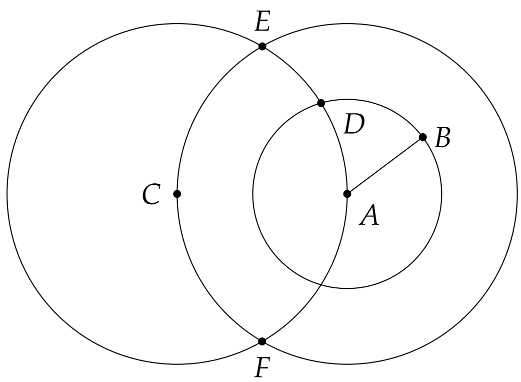
2.3 An incorrect construction for copying a line segment

Construction([13]):

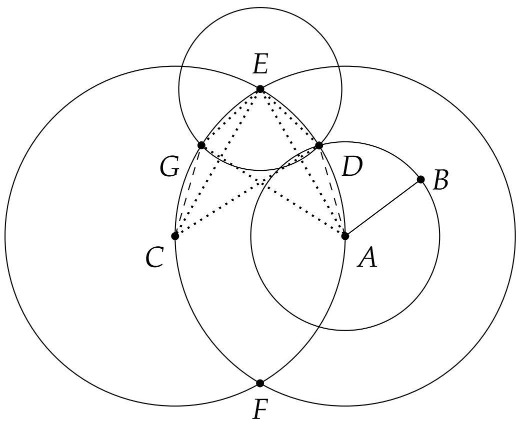
Construct a circle centered at A with radius $\overline{AB}$:



Construct a circle centered at A with radius $\overline{AC}$ and a circle centered at C with radius $\overline{AC} = \overline{CA}$. Label the intersections of the two circles E, F. Label the intersection of the circle centered at C and the circle centered at A with radius $\overline{AB}$ by D:



Construct a circle centered at E with radius ED. Label the intersection of this circle with the circle centered at A with radius AC by G:



Claim: The length of the line segment $\overline{GC}$ is equal to the length of $\overline{AB}$.

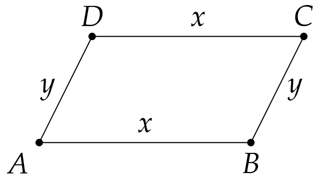
Proof: $\overline{CD} = \overline{CE}$ are radii of the circle centered at C. $\overline{AE} = \overline{AG}$ are radii of the (larger) circle centered at A. $\overline{CD} = \overline{CE} = \overline{AE} = \overline{AG}$ since the radii of the two circles are $\overline{AC} = \overline{CA}$. $\overline{EG} = \overline{ED}$ are radii of the circle centered at E. Therefore, $\triangle EAG \cong \triangle DCE$ by side-side-side so $\angle GEA = \angle DEC$.

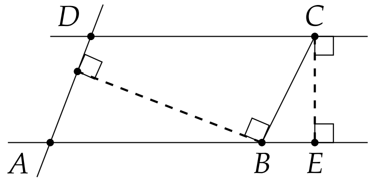
$\angle GEC = \angle GEA - \angle CEA = \angle DEC - \angle CEA = \angle DEA$. Therefore, $\triangle ADE \cong \triangle CGE$ by side-angle-side. $\overline{AB} = \overline{AD}$ are radii of the smaller circle centered at A, so $\overline{CG} = \overline{AD} = \overline{AB}$.

Is there an error in the proof? No! But there is a problem because the equality $\overline{AB} = \overline{GC}$ holds only when the length of $\overline{AB}$ is less that the length of $\overline{AC}$. In contrast, Euclid's construction and proof are true, independent of the relative lengths of $\overline{AB}$ and $\overline{AC}$, and independent of the position of the point C relative to the line segment $\overline{AB}$ [17].

2.4 A "simpler" construction for copying a line segment

Given a line segment $\overline{AB}$ and a point C, if we can build a parallelogram with these three points as its vertices, we obtain a line segment with C at one end whose length is equal to the length of $\overline{AB}$ (left diagram):





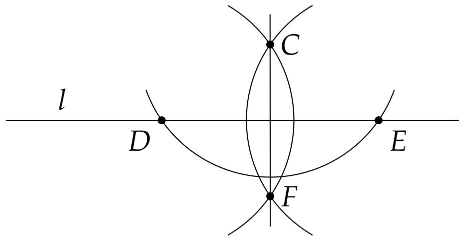
This construction can be found in [18, pp. 207–208].

Construction (right diagram):

Construct the line segment from B to C. Construct an altitude from C to the line containing the line segment $\overline{AB}$. Label the intersection by E. Construct an altitude to the line segment $\overline{CE}$ at C. This line is parallel to $\overline{AB}$. Use a similar method to construct a line parallel to $\overline{BC}$ through A. Label the intersection of the two lines by D.

$\overline{AD} \| \overline{BC}$, $\overline{AB} \| \overline{DC}$ and by definition $\overline{ABCD}$ is a parallelogram, so $\overline{AB} = \overline{CD}$ as required.

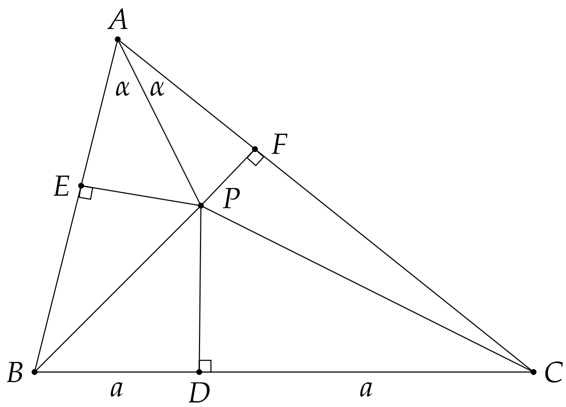
Construction with a collapsing compass: It is possible to construct an altitude to the line l through a given point C with a collapsing compass. Construct a circle centered at C with a radius that is greater than the distance of C from l. Label the intersections with l by D, E. Construct circles centered at D, E with radii $\overline{DC} = \overline{EC}$. The line connecting the intersections C, F of the circles centered at D, E is an altitude through C.



The proof the correctness of this construction is much more difficult than Euclid's proof of his construction.

2.5 Don't trust a diagram

We can prove that *all* triangles are isosceles!

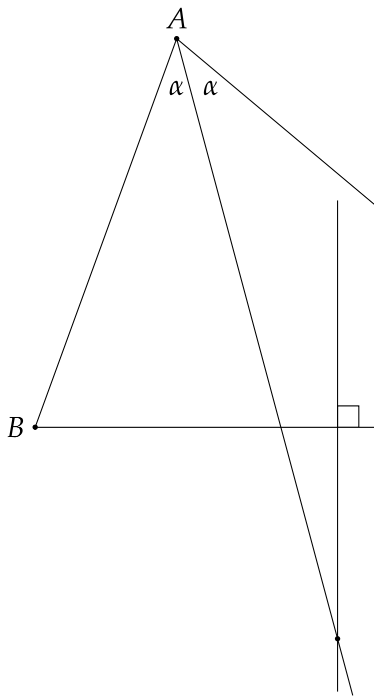


Given an arbitrary triangle $\triangle ABC$, let P be the intersection of the angle bisector of $\angle BAC$ and the perpendicular bisector of $\overline{BC}$. Label by D, E, F the intersections of the altitudes from P to the sides $\overline{BC}, \overline{AB}, \overline{AC}$. $\triangle APF \cong \triangle APE$ because they are right triangles with equal angles α and a common side $\overline{AP}$.

$\triangle DPC \cong \triangle DPB$ by side-angle-side because $\overline{PD}$ is a common side, $\angle PDB = \angle PDC$ are right angles and $\overline{BD} = \overline{DC} = a$ because $\overline{PD}$ is the perpendicular bisector of $\overline{BC}$. $\triangle EPB \cong \triangle FPC$ by side-side-angle in a right triangle, because $\overline{EP} = \overline{PF}$ by the first congruence and $\overline{PB} = \overline{PC}$ by the second congruence. By combining the equations we get that $\triangle ABC$ is isoceles:

$$\overline{AB} = \overline{AE} + \overline{EB} = \overline{AF} + \overline{FC} = \overline{AC}\,.$$

The problem with the proof is that the diagram is incorrect because point P is *outside* the triangle, as can be seen from the following diagram:



Chapter 3

How to Trisect an Angle (With a Little Help from a Friend)

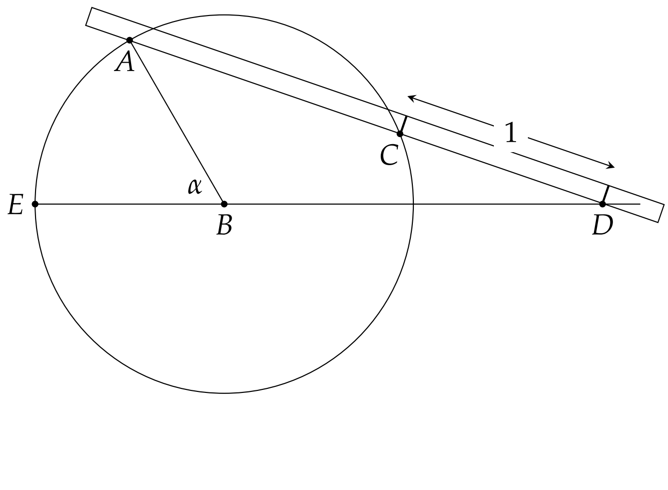
The reason is that it is impossible to trisect an arbitrary angle is that it requires the construction of cube roots, but the compass and straightedge can only construct lengths that are expressions built from the four arithmetic operators and square roots.

Greek mathematicians discovered that if other instruments are allowed, angles can be trisected [20]. Section 3.1 presents a construction of Archimedes using a simple instrument called a *neusis* [19]. Section 3.2 shows a more complex construction of Hippias using the *quadratrix* [22]. As a bonus, Section 3.3 shows that the quadratrix can square a circle.

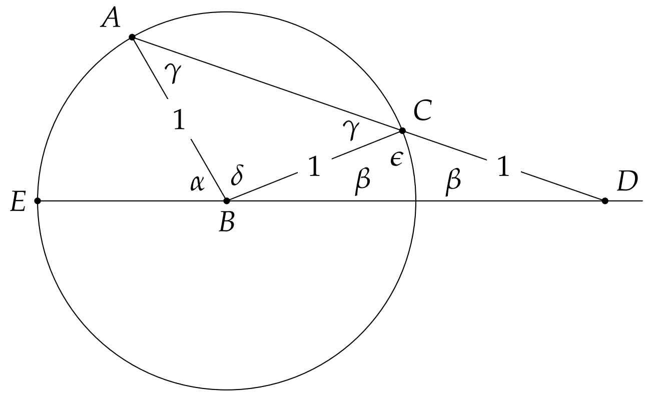
3.1 Trisection using the neusis

The term straightedge is used instead of "ruler" because a straightedge has no marks on it. The only operation it can perform is to construct a straight line between two points, while a ruler can measure distances. To trisect an angle all we need is a straightedge with two marks that are a fixed distance apart, called a *neusis*. We define the distance between the marks as 1:

Let α be an arbitrary angle $\angle ABE$ within a circle with center B and radius 1. The circle can be constructed by setting the compass to the distance between the marks on the neusis. Extend the radius $\overline{EB}$ beyond the circle. Place an edge of the neusis on A and move it until it intersects the extension of $\overline{EB}$ at D and the circle at C, using the marks so that the length of the line segment $\overline{CD}$ is 1. Draw the line $\overline{AD}$:



Draw line $\overline{BC}$ and label the angles and line segments as shown:

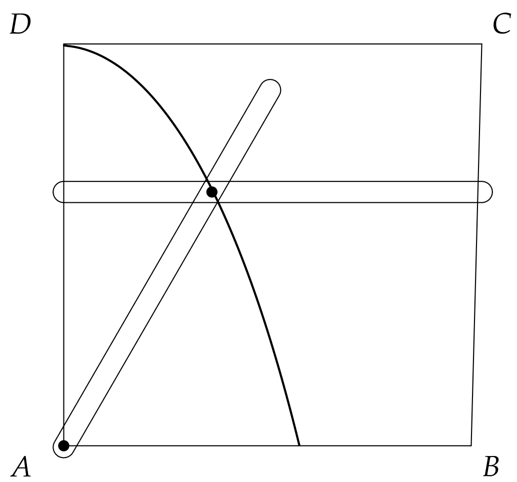


Both $\triangle ABC$ and $\triangle BCD$ are isoceles: $\overline{AB} = \overline{BC}$ since both are radii and $\overline{BC} = \overline{CD}$ by construction using the neusis. A computation (using the facts that the angles of a triangle and supplemenary angles add up to π radians) shows that β trisects α:

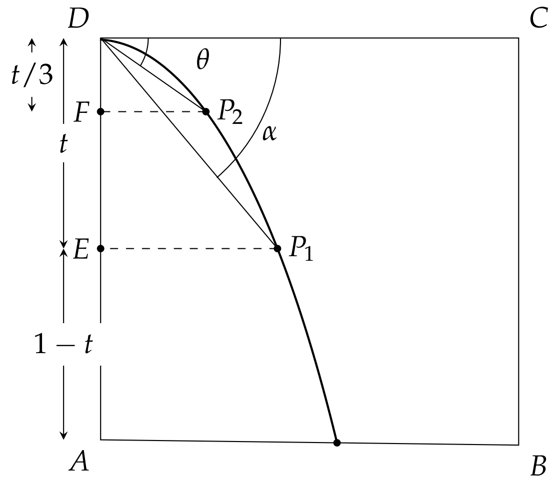
$$\begin{aligned}
\epsilon &= \pi - 2\beta \\
\gamma &= \pi - \epsilon = 2\beta \\
\delta &= \pi - 2\gamma = \pi - 4\beta \\
\alpha &= \pi - \delta - \beta \\
&= 4\beta - \beta \\
&= 3\beta\,.
\end{aligned}$$

3.2 Trisection using the quadratrix

The following diagram shows a *quadratrix compass*: two (unmarked) straightedges connected by a joint that constrains them to move together. One straightedge moves parallel to the x-axis from $\overline{DC}$ to $\overline{AB}$, while the second straightedge is allowed to rotate around the origin at A until it lies horizontally along $\overline{AB}$. The curve traced by the joint of the two straightedges is called the *quadratrix curve* or simply the *quadratrix*.



As the horizontal straightedge is moved down at a constant velocity, the other straightedge is constrained to move at a constant angular velocity. In fact, that is the definition of the quadratrix curve. As the y-coordinate of the horizontal straightedge decreases from 1 to 0, the angle of the other straightedge relative to the x-axis decreases from 90° to 0°. The following diagram shows how this can be used to trisect an arbitrary angle α:

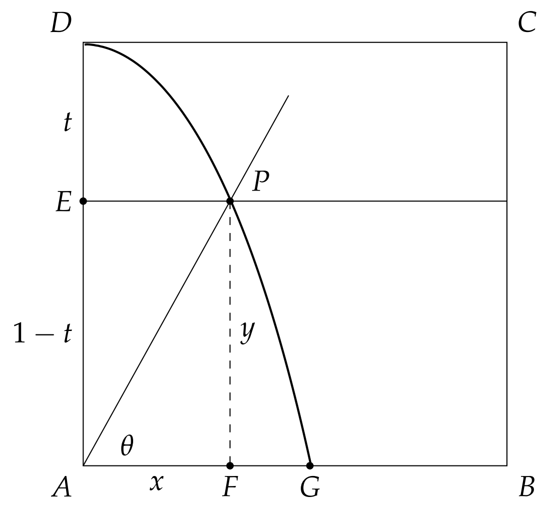


P_1 is the intersection of the line defining the angle α relative to $\overline{DC}$ and the quadratrix. This point has y-coordinate $1-t$, where t is the distance that the horizontal straightedge has moved from its initial position $\overline{DC}$. Now trisect the *line segment* $\overline{DE}$ to obtain point F. (It is easy to trisect a line segment using Thales theorem.) Let P_2 be the intersection of a line from F parallel to $\overline{DC}$ and the quadratrix. By the equality of the velocities, we have:

$$\frac{\theta}{\alpha} = \frac{t/3}{t}$$

$$\theta = \alpha/3\,.$$

3.3 Squaring the circle using the quadratrix



Suppose that the horizontal straightedge has moved t down the y-axis to point E and the rotating straightedge forms an angle of θ with the x-axis. P is the intersection of the quadratrix and the horizontal straightedge, and F is the projection of P on the x-axis. What are the coordinates of P? Clearly, $y = \overline{PF} = \overline{EA} = 1 - t$. On the quadratrix, θ decreases at the same rate that t increases:

$$\begin{aligned}\frac{1-t}{1} &= \frac{\theta}{\pi/2} \\ \theta &= \frac{\pi}{2}(1-t)\,.\end{aligned}$$

Check if this makes sense: when $t = 0$, $\theta = \pi/2$ and when $t = 1$, $\theta = 0$.

The x-coordinate of P follows from trigonometry:

$$\tan\theta = \frac{y}{x}\,.$$

which gives:

$$x = \frac{y}{\tan\theta} = y\cot\theta = y\cot\frac{\pi}{2}(1-t) = y\cot\frac{\pi}{2}y\,.$$

We usually express a function as $y = f(x)$ but it can also be expressed as $x = f(y)$.

As the horizontal straightedge moves down, it will merge with $\overline{AB}$ and P with lie on a point on the x-axis, which we denote G. Let us compute the x-coordinate of the point G. We can't simply plug in $y = 0$ because $\cot 0$ is not defined, but we might get lucky by computing the limit of x as y goes to 0. First, multiply and divide by $\pi/2$:

$$x = y\cot\frac{\pi}{2}y = \frac{2}{\pi}\cdot\frac{\pi}{2}y\cot\frac{\pi}{2}y\,.$$

For convenience, perform a change of variable $z = \dfrac{\pi}{2}y$ and compute the limit:

$$\lim_{z\to 0} z\cot z = \lim_{z\to 0}\frac{z\cos z}{\sin z} = \lim_{z\to 0}\frac{\cos z}{\dfrac{\sin z}{z}} = \frac{\cos 0}{1} = 1\,,$$

using the well-known fact that $\displaystyle\lim_{z\to 0}\frac{\sin z}{z} = 1$. Therefore, as $y \to 0$:

$$x \to \frac{2}{\pi}\cdot\lim_{y\to 0}\frac{\pi}{2}y\cot\frac{\pi}{2}y = \frac{2}{\pi}\cdot 1 = \frac{2}{\pi}\,.$$

Using the quadratrix we have constructed a line segment $\overline{AG}$ whose length is $x = \dfrac{2}{\pi}$. With an ordinary straightedge and compass it is easy to construct a line segment of length $\sqrt{\dfrac{2}{x}} = \sqrt{\dfrac{2}{2/\pi}} = \sqrt{\pi}$ and then construct a square whose area is π.

Chapter 4
How to (Almost) Square a Circle

4.1 Approximations to π

To square a circle the length $\sqrt{\pi}$ must be constructed, however, π is *transcendental*, meaning that it is not the solution of any algebraic equation.

This chapter brings three constructions of approximations to π. The following table shows the formulas of the lengths that are constructed, their approximate values, the difference between these values and the value of π, and the error in meters that results if the approximation is used to compute the circumference of the earth given that its radius is 6378 km.

Construction	Formula	Value	Difference	Error (m)
π		3.14159265359	–	–
Kochansky	$\sqrt{\dfrac{40}{3}-2\sqrt{3}}$	3.14153338705	5.932×10^{-5}	756
Ramanujan 1	$\dfrac{355}{113}$	3.14159292035	2.667×10^{-7}	3.4
Ramanujan 2	$\left(9^2+\dfrac{19^2}{22}\right)^{1/4}$	3.14159265258	1.007×10^{-9}	0.013

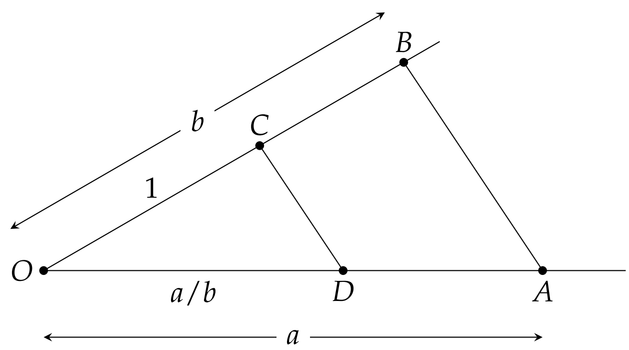
Kochansky's construction from 1685 can be found in [2].

Ramanujan's constructions from 1913 can be found in [14, 15].

In the constructions in this chapter we need to divide a line segment into three parts; here we show how the division of two lengths can be constructed. Given a line segment of length 1 and line segments of lengths a, b, by similar triangles:

$$\frac{1}{b}=\frac{\overline{OD}}{a},$$

so $\overline{OD}=\dfrac{a}{b}$.



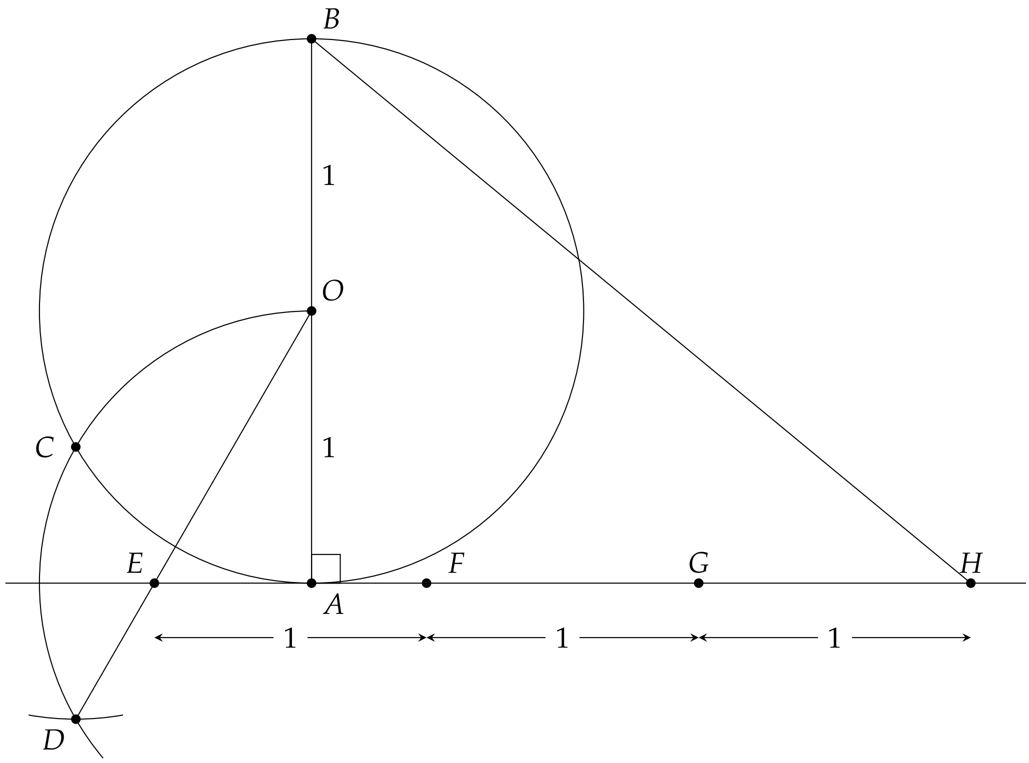
4.2 Kochansky's construction

4.2.1 The construction

Construct three circles:

1. Construct a unit circle centered at O, let $\overline{AB}$ be a diameter and construct a tangent to the circle at A.
2. Construct a unit circle centered at A. Its intersection with the first circle is C.[1]
3. Construct a unit circle centered at C. Its intersection with the second circle is D.

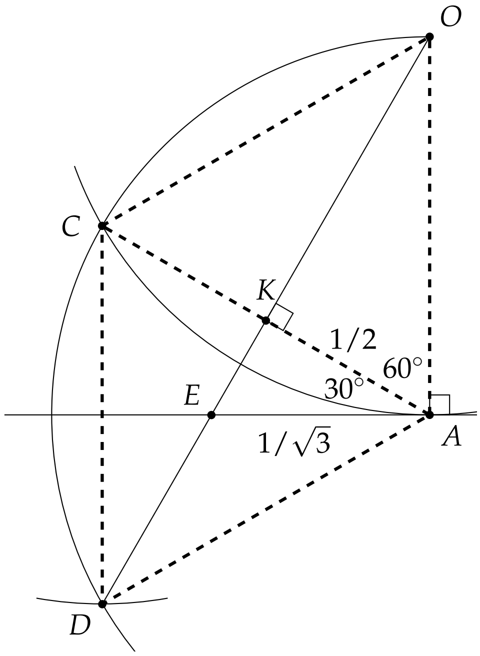
Construct $\overline{OD}$ and denote its intersection with the tangent by E.
From E construct F, G, H, each at distance 1 from the previous point; then $\overline{AH} = 3 - \overline{EA}$.
Construct $\overline{BH}$.
Claim: $\overline{BH} = \sqrt{\frac{40}{3} - 2\sqrt{3}} \approx \pi$.



[1]For the second and third circles, the diagram only shows the arc that intersects the previous circle.

4.2.2 The proof

Extract the following diagram from the first one. Dashed line segments have been added. Since all the circles are unit circles, it is easy to see that the length of each dashed line segment is 1. It follows that $\overline{AOCD}$ is a rhombus so its diagonals are perpendicular to and bisect each other at K and $\overline{AK} = \frac{1}{2}$.



The diagonal $\overline{AC}$ forms two equilateral triangles $\triangle OAC, \triangle DAC$ so $\angle OAC = 60°$. Since tangent forms a right angle with the radius $\overline{OA}$, $\angle KAE = 30°$. Now:

$$
\begin{aligned}
\frac{1/2}{\overline{EA}} &= \cos 30° = \frac{\sqrt{3}}{2} \\
\overline{EA} &= \frac{1}{\sqrt{3}} \\
\overline{AH} &= 3 - \overline{EA} = \left(3 - \frac{1}{\sqrt{3}}\right) = \frac{3\sqrt{3}-1}{\sqrt{3}}
\end{aligned}
$$

Returning to the first diagram, we see that $\triangle ABH$ is a right triangle:

$$
\begin{aligned}
\overline{BH}^2 &= \overline{OB}^2 + \overline{AH}^2 \\
&= 4 + \frac{9 \cdot 3 - 6\sqrt{3} + 1}{3} = \frac{40}{3} - 2\sqrt{3} \\
\overline{BH} &= \sqrt{\frac{40}{3} - 2\sqrt{3}} \approx 3.141533387 \approx \pi\,.
\end{aligned}
$$

4.3 Ramanujan's first construction

4.3.1 The construction

Construct a unit circle centered at O and let $\overline{PR}$ be a diameter.

H bisects $\overline{PO}$ and T trisects $\overline{RO}$.

Construct a perpendicular at T that intersects the circle at Q.

Construct a chord $\overline{RS} = \overline{QT}$.

Construct $\overline{PS}$.

Construct a line parallel to $\overline{RS}$ from T that intersects $\overline{PS}$ at N.

Construct a line parallel to $\overline{RS}$ from O that intersects $\overline{PS}$ at M.

Construct the chord $\overline{PK} = \overline{PM}$.

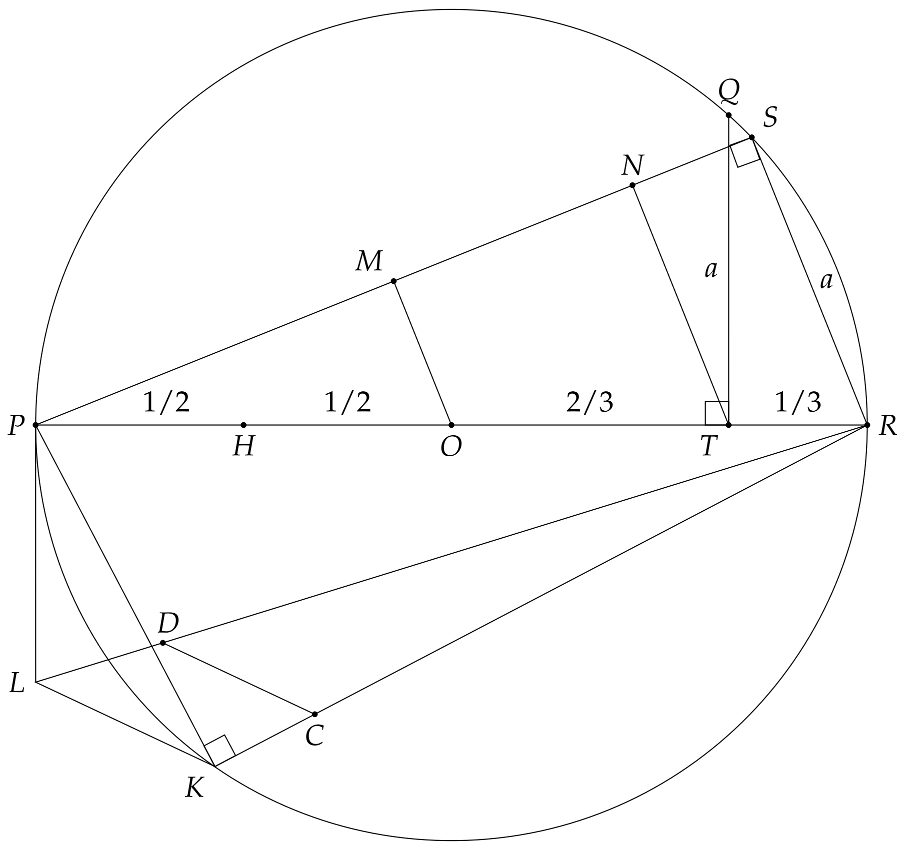
Construct the tangent at P of length $\overline{PL} = \overline{MN}$.

Connect the points K, L, R.

Find point C such that $\overline{RC}$ is equal to $\overline{RH}$.

Construct $\overline{CD}$ parallel to $\overline{KL}$ that intersects $\overline{LR}$ at D.

Claim: $\overline{RD}^2 = \dfrac{355}{113} \approx \pi$.



4.3.2 The proof

By Pythagoras' theorem on $\triangle QOT$:

$$\overline{QT} = \sqrt{1^2 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}.$$

$\triangle PSR$ is a right triangle because it subtends a diameter. By Pythagoras theorem:

$$\overline{PS} = \sqrt{2^2 - \left(\frac{\sqrt{5}}{3}\right)^2} = \sqrt{4 - \frac{5}{9}} = \frac{\sqrt{31}}{3}.$$

$\triangle MPO \sim \triangle SPR$ so:

$$\begin{aligned}
\frac{\overline{PM}}{\overline{PO}} &= \frac{\overline{PS}}{\overline{PR}} \\
\frac{\overline{PM}}{1} &= \frac{\sqrt{31}/3}{2} \\
\overline{PM} &= \frac{\sqrt{31}}{6} \\
\overline{PK} &= \overline{PM} = \frac{\sqrt{31}}{6}.
\end{aligned}$$

$\triangle NPT \sim \triangle SPR$ so:

$$\begin{aligned}
\frac{\overline{PN}}{\overline{PT}} &= \frac{\overline{PS}}{\overline{PR}} \\
\frac{\overline{PN}}{5/3} &= \frac{\sqrt{31}/3}{2} \\
\overline{PN} &= \frac{5\sqrt{31}}{18} \\
\overline{MN} &= \overline{PN} - \overline{PM} = \sqrt{31}\left(\frac{5}{18} - \frac{1}{6}\right) = \frac{\sqrt{31}}{9} \\
\overline{PL} &= \overline{MN} = \frac{\sqrt{31}}{9}.
\end{aligned}$$

$\triangle PKR$ is a right triangle because it subtends a diameter. By Pythagoras's theorem:

$$\overline{RK} = \sqrt{2^2 - \left(\frac{\sqrt{31}}{6}\right)^2} = \frac{\sqrt{113}}{6}.$$

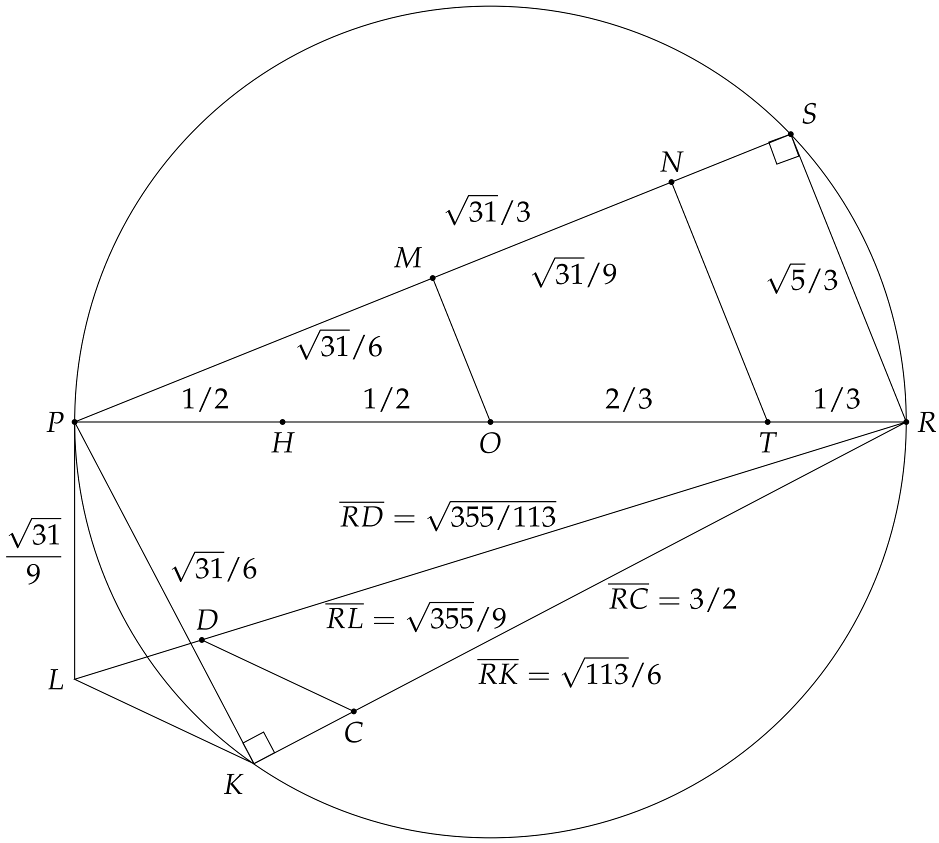
$\triangle PLR$ is a right triangle because it $\overline{PL}$ is a tangent. By Pythagoras's theorem:

$$\overline{RL} = \sqrt{2^2 + \left(\frac{\sqrt{31}}{9}\right)^2} = \frac{\sqrt{355}}{9}.$$

$\overline{RC} = \overline{RH} = \dfrac{1}{3} + \dfrac{2}{3} + \dfrac{1}{2} = \dfrac{3}{2}$. Since $\overline{CD}$ is parallel to $\overline{LK}$, by similar triangles:

$$\begin{aligned}
\frac{\overline{RD}}{\overline{RC}} &= \frac{\overline{RL}}{\overline{RK}} \\
\frac{\overline{RD}}{3/2} &= \frac{\sqrt{355}/9}{\sqrt{113}/6} \\
\overline{RD} &= \sqrt{\frac{355}{113}}\,.
\end{aligned}$$

Here is the construction with line segments labeled with their lengths:



The value $\dfrac{355}{113}$ could be constructed by constructing two line segments of length 355 and 113 and then using the division construction shown in Section 4.1, but that is rather tedious!

4.4 Ramanujan's second approximation

4.4.1 The construction

Construct a unit circle centered at O with diameter $\overline{AB}$, and let C be the intersection of the perpendicular at O with the circle.

Trisect $\overline{AO}$ so that $\overline{AT} = 1/3$ and $\overline{TO} = 2/3$.

Construct $\overline{BC}$ and find points M, N such that $\overline{CM} = \overline{MN} = \overline{AT} = 1/3$.

Construct $\overline{AM}$ and $\overline{AN}$ and let P be the point on $\overline{AN}$ such that $\overline{AP} = \overline{AM}$.

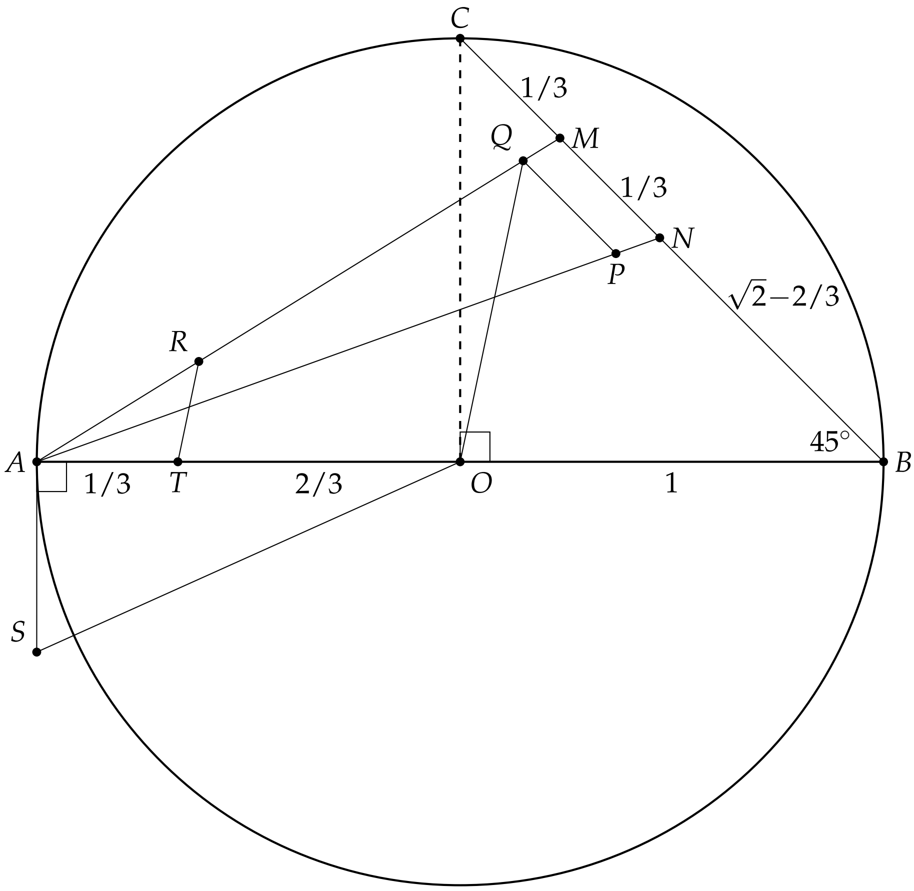
From P construct a line parallel to $\overline{MN}$ that intersects $\overline{AM}$ at Q.

Construct $\overline{OQ}$ and then from T construct a line parallel to $\overline{OQ}$ that intersects $\overline{AM}$ at R.

Construct a line segment $\overline{AS}$ tangent to A of length $\overline{AR}$.

Construct $\overline{SO}$.

Claim: $3\sqrt{\overline{SO}} = \left(9^2 + \frac{19^2}{22}\right)^{\frac{1}{4}} \approx \pi$.



4.4.2 The proof

$\triangle COB$ is a right triangle, $\overline{OB} = \overline{OC} = 1$, so by Pythagoras' theorem $\overline{CB} = \sqrt{2}$ and $\overline{NB} = \sqrt{2} - 2/3$. The triangle is isoceles so $\angle NBA = \angle MBA = 45°$.

We use the law of cosines on $\triangle NBA$ to compute $\overline{AN}$:

$$\begin{aligned}
\overline{AN}^2 &= \overline{BA}^2 + \overline{BN}^2 - 2 \cdot \overline{BA} \cdot \overline{BN} \cdot \cos \angle NBA \\
&= 2^2 + \left(\sqrt{2} - \frac{2}{3}\right)^2 - 2 \cdot 2 \cdot \left(\sqrt{2} - \frac{2}{3}\right) \cdot \frac{\sqrt{2}}{2} \\
&= \left(4 + 2 + \frac{4}{9} - 4\right) + \sqrt{2} \cdot \left(-\frac{4}{3} + \frac{4}{3}\right) = \frac{22}{9} \\
\overline{AN} &= \sqrt{\frac{22}{9}}.
\end{aligned}$$

Similarly, we use the law of cosines on $\triangle MBA$ to compute $\overline{AM}$:

$$\begin{aligned}
\overline{AM}^2 &= \overline{BA}^2 + \overline{BM}^2 - 2 \cdot \overline{BA} \cdot \overline{BM} \cdot \cos \angle MBA \\
&= 2^2 + \left(\sqrt{2} - \frac{1}{3}\right)^2 - 2 \cdot 2 \cdot \left(\sqrt{2} - \frac{1}{3}\right) \cdot \frac{\sqrt{2}}{2} \\
&= \left(4 + 2 + \frac{1}{9} - 4\right) + \sqrt{2} \cdot \left(-\frac{2}{3} + \frac{2}{3}\right) = \frac{19}{9} \\
\overline{AM} &= \sqrt{\frac{19}{9}}.
\end{aligned}$$

By construction $\overline{QP} \| \overline{MN}$ so $\triangle MAN \sim \triangle QAP$, and by construction $\overline{AP} = \overline{AM}$ giving:

$$\begin{aligned}
\frac{\overline{AQ}}{\overline{AM}} &= \frac{\overline{AP}}{\overline{AN}} = \frac{\overline{AM}}{\overline{AN}} \\
\overline{AQ} &= \frac{\overline{AM}^2}{\overline{AN}} = \frac{19/9}{\sqrt{22/9}} = \frac{19}{3\sqrt{22}}.
\end{aligned}$$

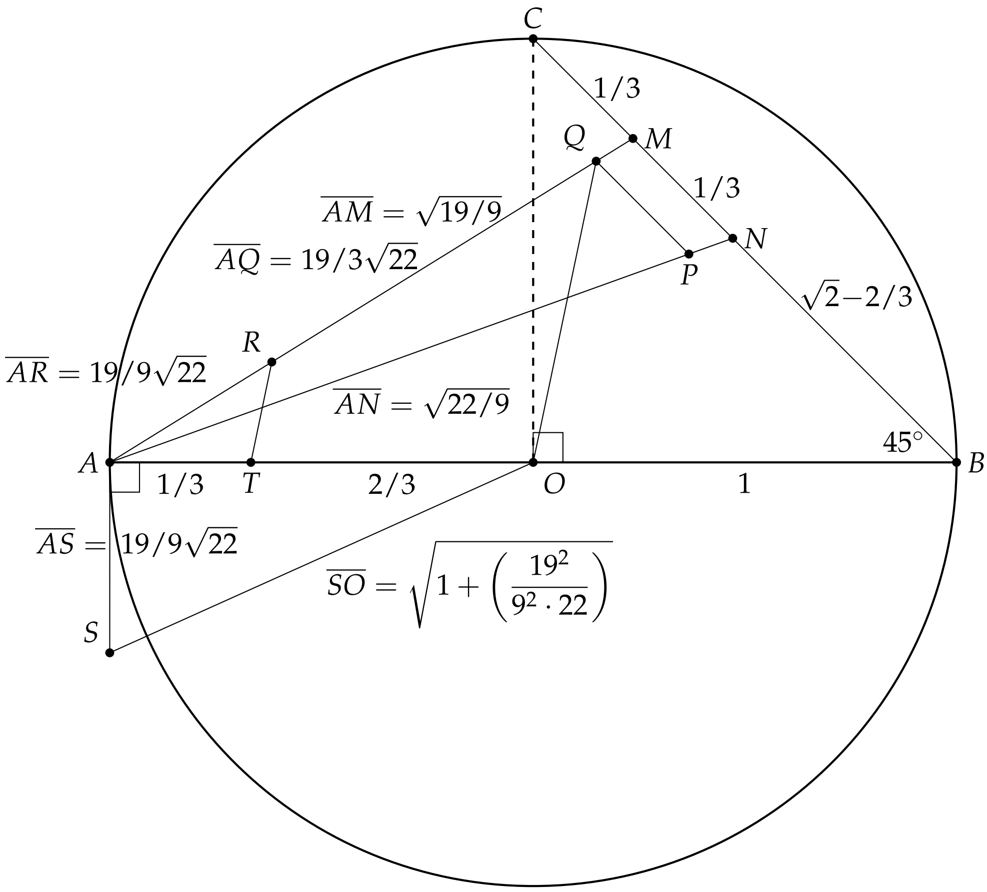
By construction $\overline{TR} \| \overline{OQ}$ so $\triangle RAT \sim \triangle QAO$ giving:

$$\begin{aligned}
\frac{\overline{AR}}{\overline{AQ}} &= \frac{\overline{AT}}{\overline{AO}} \\
\overline{AR} &= \overline{AQ} \cdot \frac{\overline{AT}}{\overline{AO}} = \frac{19}{3\sqrt{22}} \cdot \frac{1/3}{1} = \frac{19}{9\sqrt{22}}.
\end{aligned}$$

By construction $\overline{AS} = \overline{AR}$ and $\triangle OAS$ is a right triangle. By Pythagoras' theorem:

$$
\begin{aligned}
\overline{SO} &= \sqrt{1^2 + \left(\frac{19}{9\sqrt{22}}\right)^2} \\
3\sqrt{\overline{SO}} &= 3\left(1 + \frac{19^2}{9^2 \cdot 22}\right)^{\frac{1}{4}} \\
&= \left(3^4 + \frac{3^4 \cdot 19^2}{9^2 \cdot 22}\right)^{\frac{1}{4}} \\
&= \left(9^2 + \frac{19^2}{22}\right)^{\frac{1}{4}} \approx 3.14159265262 \approx \pi\,.
\end{aligned}
$$

Here is the construction with line segments labeled with their lengths:

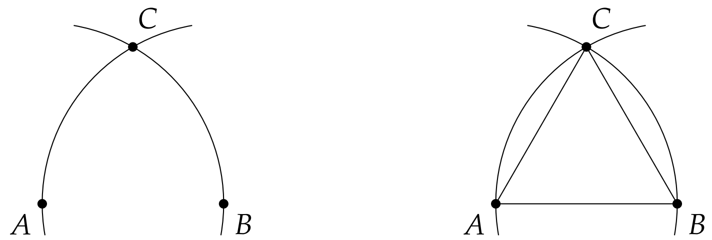


Chapter 5
A Compass is Sufficient

In 1797 the Italian mathematician Lorenzo Mascheroni proved that any construction carried out with a compass and straightedge can be carried out with the compass alone! Later it came to light that the construction had already been discovered by the Danish mathematician Georg Mohr 1672. The theorem is now called the Mohr-Mascheroni Theorem.

In this chapter I present a proof of the theorem based on the proof in problem 33 of [4] and reworked by Michael Woltermann [5].[1] A different proof can be found in [8].

What does it mean to perform a construction with only a compass? The right diagram below shows the construction of an equilateral triangle using a straightedge and compass. How can we construct a triangle without the line segments $\overline{AB}, \overline{AC}, \overline{BC}$? In fact, there is no need to *see* the lines. A line is defined by two points, so it is sufficient to construct the points in order to obtain a construction equivalent to the one with a straightedge (left diagram):



In the diagrams we will draw lines, but they are used only to understand the construction and the proof of its correctness. It is important that you convince yourself that the construction itself uses only a compass.

A construction by straightedge and compass is a sequence taken from these three operations:

- Find the point of intersection of two straight lines.
- Find the point of intersection of a straight line and a circle.
- Find the point(s) of intersection of two circles.

It is clear that the third operation can be done with only a compass. We need to show that the first two operations can be done with a compass alone.

Notation:

- $c(O, A)$: the circle with center O through point A.
- $c(O, r)$: the circle with center O and radius r.
- $c(O, AB)$: the circle with center O and radius the length of line segment $\overline{AB}$.

First we will solve four preliminary problems (Sections 5.1–5.4). Next, we show the construction for finding the intersection of two lines (Section 5.5), and finally the construction of the intersection of a line and a circle (Section 5.6).

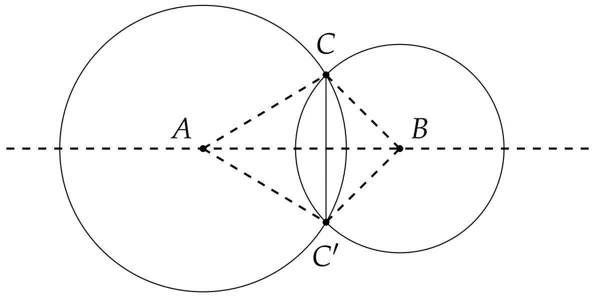
[1] I would like to thank Woltermann for permission to use his work.

5.1 Reflection of a point

Given a line $\overline{AB}$ and a point C not on $\overline{AB}$, build a point C' which is a reflection of C about $\overline{AB}$.

C' is a *reflection* about a line segment $\overline{AB}$ if $\overline{AB}$ (or the line containing $\overline{AB}$) is the perpendicular bisector of the line CC'.

Construct a circle centered on A passing through C and circle centered on B passing through C. The intersection of the two circles is the point C' which is the reflection of C.

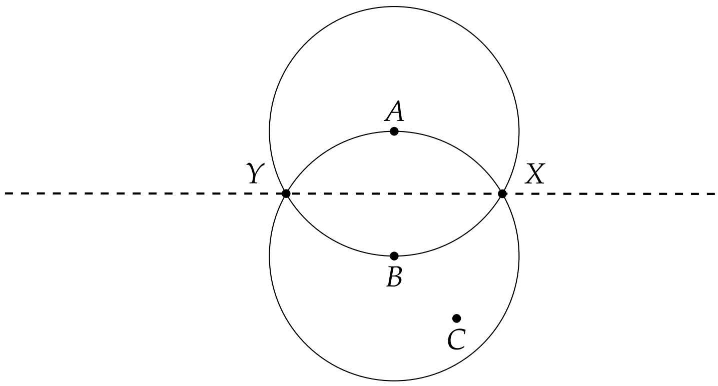


Proof: $\triangle ABC \cong \triangle ABC'$ by side-side-side since $\overline{AC}, \overline{AC'}$ are radii of the same circle as are $\overline{BC}, \overline{BC'}$, and $\overline{AB}$ is a common side. Therefore, $\angle CAB = \angle C'AB$ so $\overline{AB}$ is the angle bisector of $\angle CAC'$. But $\triangle CAC'$ is an isosceles triangle and the angle bisector $\overline{AB}$ is also the perpendicular bisector of the base of $\triangle CAC'$. By definition C' is the reflection of C around $\overline{AB}$.

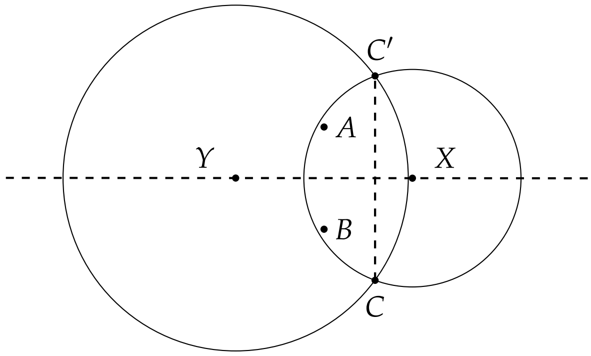
5.2 Construct a circle with a given radius

Given points A, B, C, construct $c(A, BC)$.

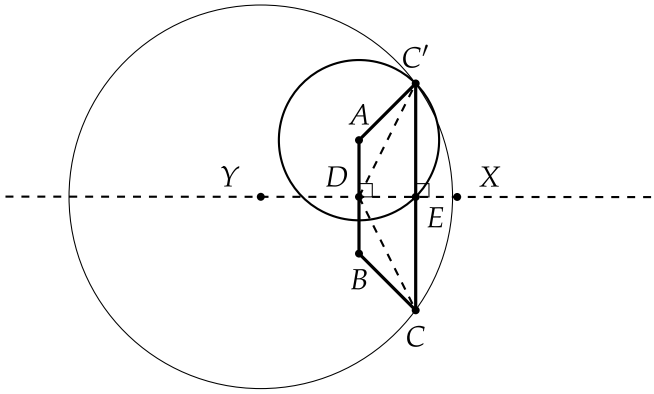
Construct $c(A, B)$ and $c(B, A)$ and let X and Y be their points of intersection.



Construct C', the reflection of C about line $\overline{XY}$ as described in Section 5.1.



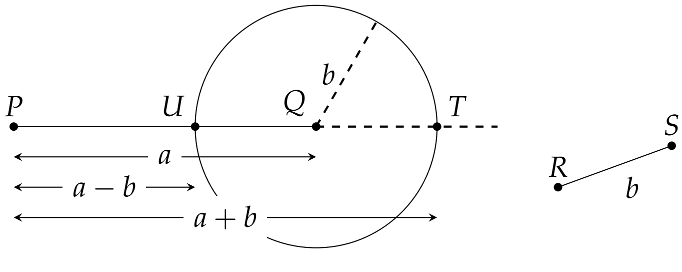
$c(A, C')$ is the desired circle.



Proof: A is the reflection of B around $\overline{XY}$ (since $\triangle YAX \cong \triangle YBX$) and C' is the reflection of C around the $\overline{XY}$. By definition, $\overline{XY}$ is the perpendicular bisector of $\overline{CC'}$ and $\overline{AB}$, so $\overline{C'E} = \overline{EC}$, $\overline{AD} = \overline{DB}$ and $\angle DEC = \angle DEC'$ are right angles. $\triangle DEC \cong \triangle DEC'$ by side-angle-side, so $\overline{DC} = \overline{DC'}$ and $\angle ADC' = \angle BDC$ (they are complementary to $\angle EDC'$, $\angle EDC$). Therefore, $\triangle ADC' \cong \triangle BDC$ by side-angle-side, so $\overline{AC'} = \overline{BC}$.

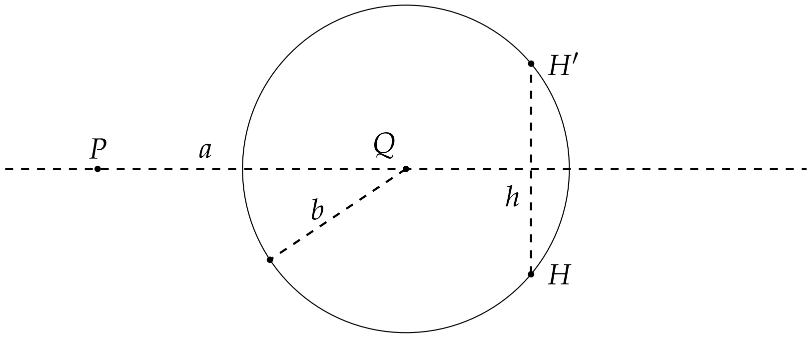
5.3 Addition and subtraction of line segments

Given line segment $\overline{PQ}$ of length a and line segment $\overline{RS}$ of length b, construct line segments $\overline{QT}, \overline{QU}$ such that $\overline{PUQT}$ is a line segment, where the length of $\overline{PU}$ is $a - b$ and the length of $\overline{PT}$ is $a + b$.

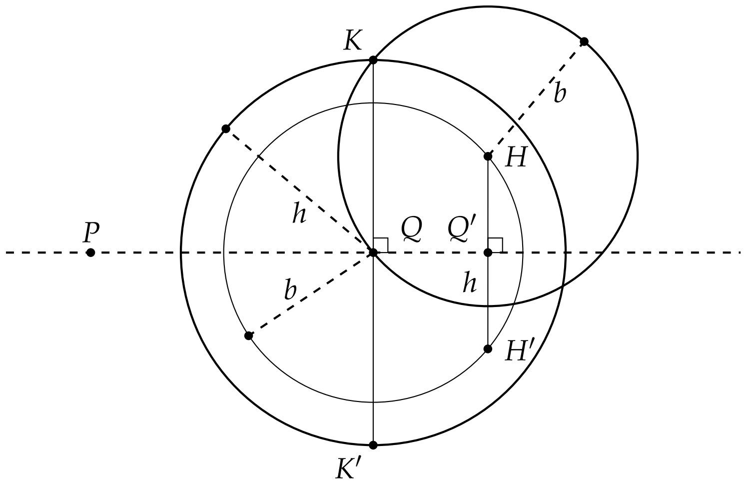


Constructing an isosceles trapezoid

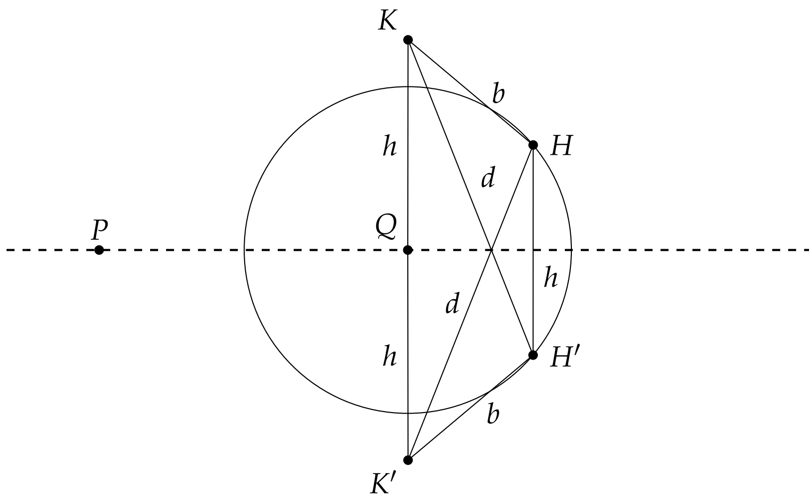
Let H be any point on $c(Q, b)$. Construct H', its reflection about $\overline{PQ}$. h is the length of $\overline{HH'}$:



Construct the circles $c(H, b)$, $c(Q, h)$. Let K be the intersection of the circles and construct K', the reflection of K about line $\overline{PQ}$:



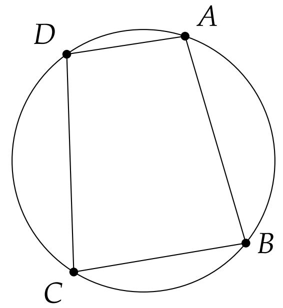
$\overline{PQ}$ is the perpendicular bisector of both $\overline{HH'}$ and $\overline{KK'}$, so these line segments are parallel. $\overline{KH} = b$ since it is the radius of the circle centered on H. K', H' are reflections of K, H and it is not hard to show that $\overline{K'H'} = \overline{KH}$ ($\triangle QQ'H \cong \triangle QQ'H'$ and then $\triangle KQH \cong \triangle K'QH'$). Therefore, $\overline{KHH'K'}$ is an isosceles trapezoid whose bases are $\overline{HH'} = h$, $\overline{KK'} = 2h$. Let d be the length of the diagonals $\overline{K'H} = \overline{KH'}$:



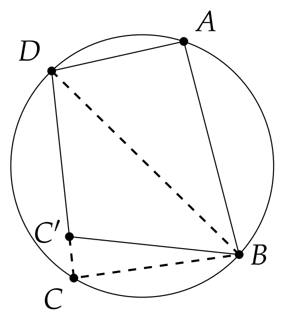
Circumscribing the trapezoid by a circle

We want to proof that it is possible to circumscribe $\overline{KHH'K'}$ by a circle. We will prove that: if the opposite angles of a quadrilateral are supplementary, then the trapezoid can be circumscribed by a circle, and that in an isosceles trapezoid the opposite angles are supplementary. Geometry textbooks give the simple proof that the opposite angles of a quadrilateral circumscribed by a circle are supplementary, but it is hard to find a proof of the converse, so I present both proofs here.

If a quadrilateral can be circumscribed by a circle then the opposite angles are supplementary: An inscribed angle equals half the subtended arc, so $\angle DAB$ is half of the arc $\widehat{DCB}$ and $\angle DCB$ is half of the arc $\widehat{DAB}$. The two arcs subtend the entire circumference of the circle, so their sum is 360°. Therefore, $\angle DAB + \angle DCB = \frac{1}{2} \cdot 360^\circ = 180^\circ$, and similarly $\angle ADC + \angle ABC = 180^\circ$



A quadrilateral whose opposite angles are supplementary can be circumscribed by a circle: Any triangle can be circumscribed by a circle. Circumscribe $\triangle DAB$ by a circle and suppose that C' is a point such that $\angle DAB + \angle DC'B = 180^\circ$, but C' is *not* on the circumference of the circle. Without loss of generality, let C' be within the circle:

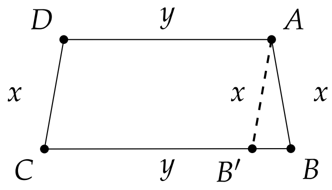


Construct a ray that extends $\overline{DC'}$ and let C be its intersection with the circle. $\overline{ABCD}$ is circumscribed by a circle so:

$$\begin{aligned}
\angle DAB + \angle DCB &= 180^\circ \\
\angle DAB + \angle DCB &= \angle DAB + \angle DC'B \\
\angle DCB &= \angle DC'B\,,
\end{aligned}$$

which is impossible if C is on the circle and C' is inside the circle.

Finally, we show that the opposite angles of an isosceles trapezoid are supplementary.



Construct the line $\overline{AB'}$ parallel to $\overline{CD}$. $\overline{AB'CD}$ is a parallelogram and $\triangle ABB'$ is an isosceles triangle, so $\angle C = \angle ABB' = \angle AB'B = \angle B$. Similarly, $\angle A = \angle D$. Since the sum of the internal angles of any quadrilateral is equal to $360°$:

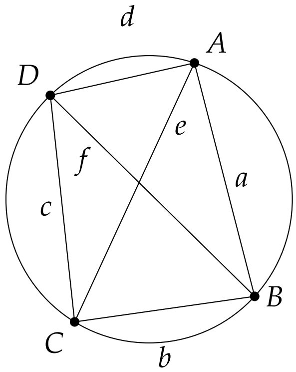
$$\begin{aligned} \angle A + \angle B + \angle C + \angle D &= 360° \\ 2\angle A + 2\angle C &= 360° \\ \angle A + \angle C &= 180°\,. \end{aligned}$$

and similarly $\angle B + \angle D = 180°$.

Ptolemy's theorem

Ptolemy's theorem relates the lengths of the diagonals and the lengths of the sides of a quadrilateral that is circumscribed by a circle:

$$ef = ac + bd\,.$$



There is a geometric proof of the theorem (see Wikipedia), but I will present a simple trigonometric proof. The law of cosines for the four triangles $\triangle ABC$, $\triangle ADC$, $\triangle DAB$, $\triangle DCB$ gives the following equations:

$$\begin{aligned} e^2 &= a^2 + b^2 - 2ab\cos\angle B \\ e^2 &= c^2 + d^2 - 2cd\cos\angle D \\ f^2 &= a^2 + d^2 - 2ad\cos\angle A \\ f^2 &= b^2 + c^2 - 2bc\cos\angle C\,. \end{aligned}$$

$\angle C = 180^\circ - \angle A$ and $\angle D = 180^\circ - \angle B$ because they are opposite angles of a quadrilateral circumscribed by a circle, so:

$$\begin{aligned}\cos \angle D &= -\cos \angle B \\ \cos \angle C &= -\cos \angle A .\end{aligned}$$

We can eliminate the cosine term from the first two equations and from the last two equations. After some messy arithmetic, we get:

$$\begin{aligned}e^2 &= \frac{(ac+bd)(ad+bc)}{(ab+cd)} \\ f^2 &= \frac{(ab+cd)(ac+bd)}{(ad+bc)} .\end{aligned}$$

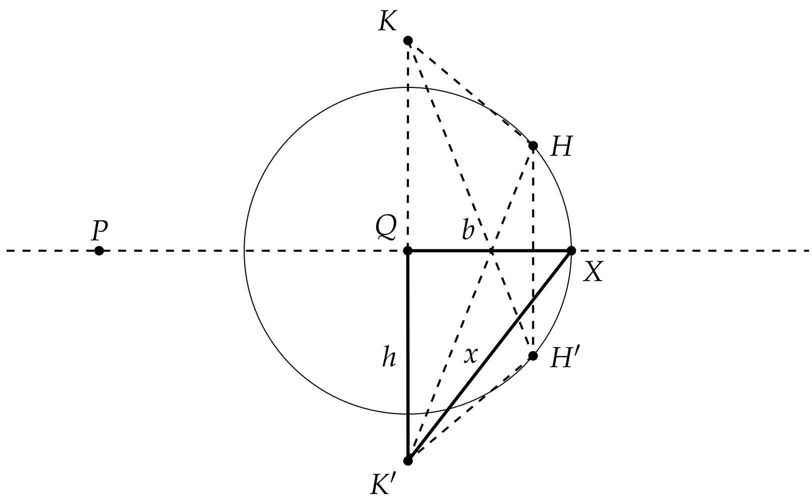
Multiply the two equations and simplify to get Ptolemy's theorem:

$$\begin{aligned}e^2 \cdot f^2 &= (ac+bd)^2 \\ ef &= (ac+bd) .\end{aligned}$$

Using Ptolemy's theorem

For the construction on page 32, the diagonals are of length d, the legs are of length b, and the bases are of lengths h and $2h$, so Ptolemy's theorem gives $d \cdot d = b \cdot b + h \cdot 2h$ or $d^2 = b^2 + 2h^2$.

Let X be the point on line $\overline{PQ}$ that extends $\overline{PQ}$ by b. (We will eventually construct X; now we're just imagining it.) Define $x = \overline{K'X}$. Since $\triangle QK'X$ is a right triangle, $x^2 = b^2 + h^2$:

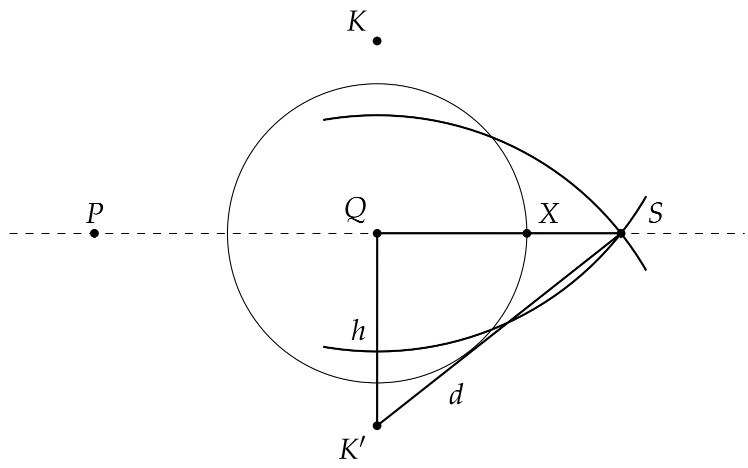


From the computation of Ptolemy's theorem theorem above:

$$\begin{aligned}d^2 &= b^2 + 2h^2 \\ &= (x^2 - h^2) + 2h^2 \\ &= x^2 + h^2 .\end{aligned}$$

Don't look for a right triangle in the diagram. We are claiming that *it is possible to construct* a triangle with sides x, h, d.

Let us construct the point S as the intersection of the circles $c(K, d), c(K', d)$:

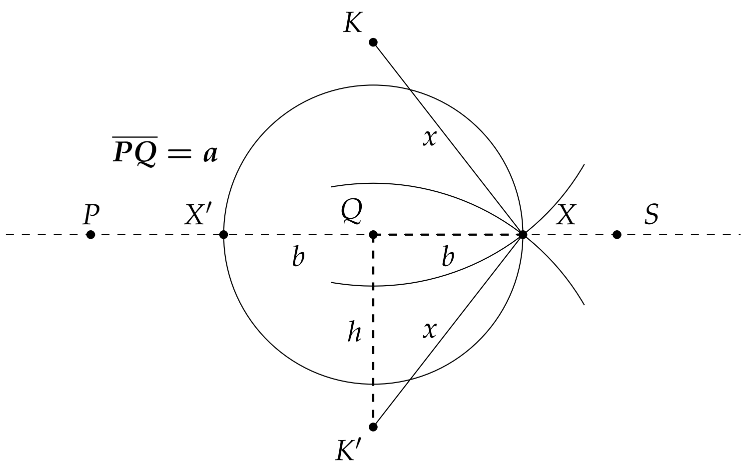


We obtain a right triangle $\triangle QSK'$. By Pythagoras' theorem $\overline{QS}^2 + h^2 = d^2$, so:

$$\overline{QS}^2 = d^2 - h^2 = x^2\,,$$

and $\overline{QS} = x$.

It is possible to construct the point X as the intersection of the circles $c(K, x), c(K', x)$:



Recall that we want to extend $\overline{PQ}$ of length a by a length b, or decrease its length by b. Since the length of $\overline{QX}$ is $\sqrt{x^2 - h^2} = b$, the length of $\overline{PX}$ is $a + b$ and the length of $\overline{PX'}$ is $a - b$.

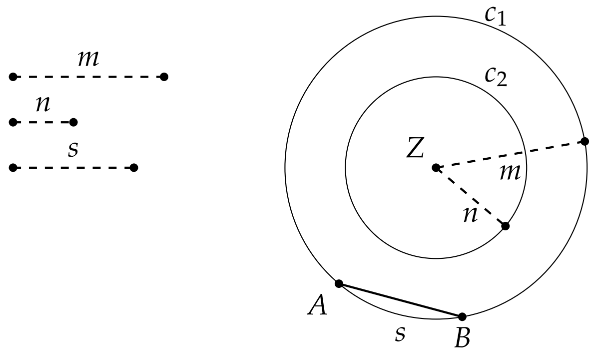
5.4 Construct a line segment relative to three other line segments

Given line segments of length n, m, s, construct a line segment of length:

$$x = \frac{n}{m} s\,.$$

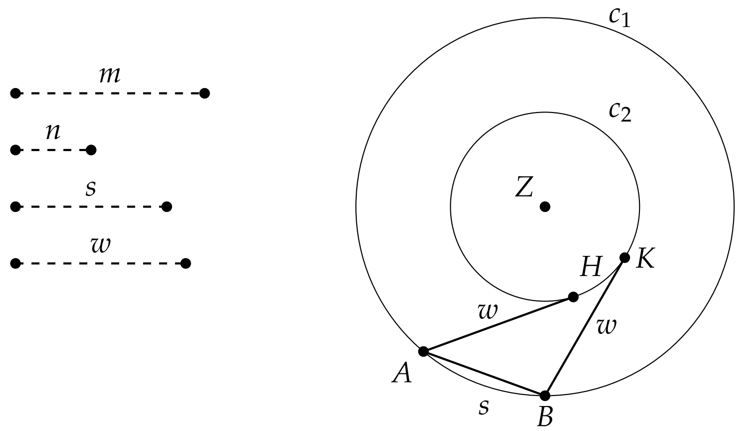
Construct two concentric circles $c_1 = c(Z, m)$ and $c_2 = c(Z, n)$, and chord $\overline{AB} = s$ on c_1. (A chord can be constructed using only a compass as shown in Section 5.2.)

We assume that $m > n$. If not, exchange the notation.

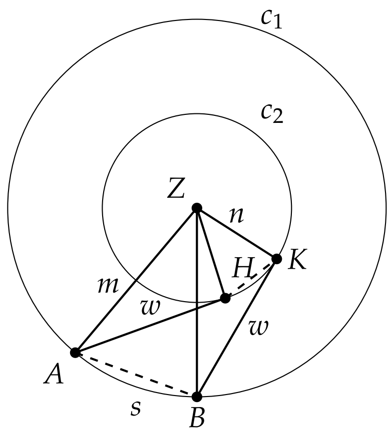


We also assume that s does not intersect c_2. If not, use the construction in Section 5.3 to multiply m, n by a number k so that the chord does not intersect the circle. Note that this does not change the value that we are trying to construct since $x = \dfrac{kn}{km}s = \dfrac{n}{m}s$.

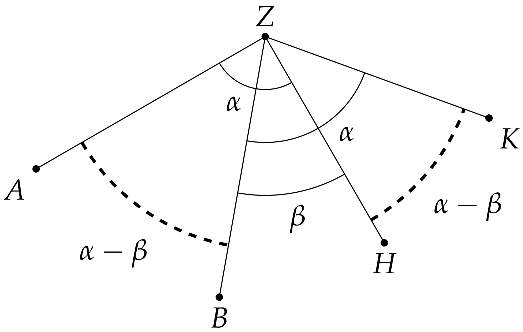
Choose any point H on circle c_2. Label the length of $\overline{AH}$ by w. Construct point K on c_2 such that the length of $\overline{BK}$ is w.



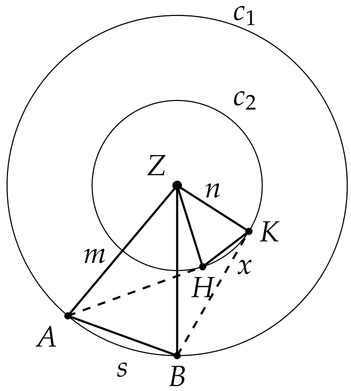
$\triangle AHZ \cong \triangle BZK$ by side-side-side: $\overline{ZA} = \overline{ZB} = m$, the radius of c_1, $\overline{ZH} = \overline{ZK} = n$, the radius of c_2, and $\overline{AH} = \overline{BK} = w$ by construction.



From $\triangle AHZ \cong \triangle BZK$, we get $\angle AZB = \angle HZK$. It is difficult to see this equality from the diagram, but the following diagram clarifies the relation among the angles. Define $\alpha = \angle AZH = \angle BZK$ and $\beta = \angle BZH$. It is easy to see that $\angle AZB = \angle HZK = \alpha - \beta$.



$\triangle ZAB \sim \triangle ZHK$ by side-angle-side, since both are isosceles triangles and we have shown that they have the same vertex angle.



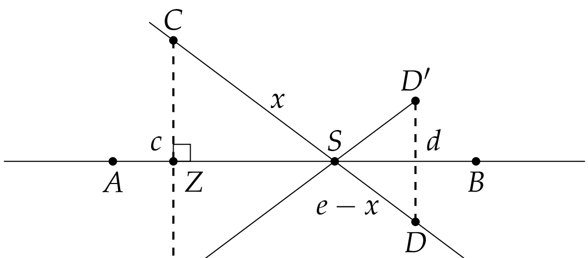
Label $\overline{HK}$ by x. Then:

$$\begin{aligned}\frac{m}{s} &= \frac{n}{x}\\ x &= \frac{n}{m}s\,.\end{aligned}$$

5.5 Find the intersection of two lines

Given two lines containing the line segments $\overline{AB}, \overline{CD}$, it is possible to construct their intersection using only a compass.

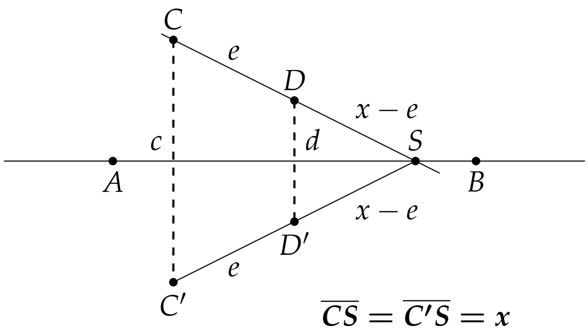
Let C', D' be the reflections of C, D around $\overline{AB}$. S, the point of intersection of $\overline{CD}$ and $\overline{C'D'}$, lies on $\overline{AB}$, because $\triangle CZS \cong \triangle C'ZS$ by side-angle-side: $\overline{CZ} = \overline{C'Z}$, $\angle CZS = \angle C'ZS$ are right triangles and $\overline{ZS}$ is a common side. Therefore, $\overline{C'S} = \overline{CS}$ and similarly $\overline{D'S} = \overline{DS}$.



$\boldsymbol{\overline{CD} = \overline{C'D'} = e}$

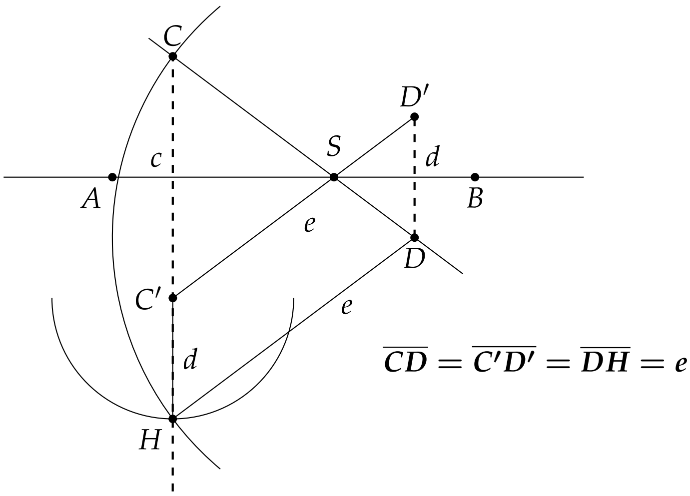
Label $x = \overline{CS}, c = \overline{CC'}, d = \overline{DD'}, e = \overline{CD}$. $\triangle CSC' \sim \triangle DSD'$ are similar so $\dfrac{x}{e-x} = \dfrac{c}{d}$. Solving the equation for x gives $x = \dfrac{c}{c+d}e$.

If C, D are on the same side of $\overline{AB}$:



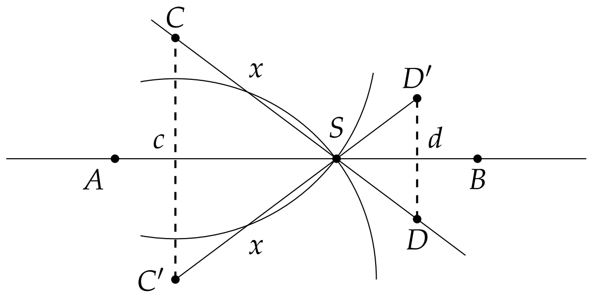
$\triangle CSC' \sim \triangle DSD'$ gives $\dfrac{x}{x-e} = \dfrac{c}{d}$. Solving for x gives $x = \dfrac{c}{c-d}e$.

Construct the circles $c(C', d), c(D, e)$ and label their intersection by H. The sum of the line segments $\overline{CC'}, \overline{C'H}$ is $c+d$. We have to show that H is on the extension of $\overline{CC'}$ so that $\overline{CH}$ is a line segment of length $c+d$. ($\overline{CH} = c-d$ in case D is on the same side of $\overline{AB}$ as C.)



From the definition of H as the intersection of $c(C', d), c(D, e)$, we get $\overline{DH} = e$, $\overline{C'H} = d$, but $\overline{C'D'} = e$, $\overline{D'D} = d$, so the quadrilateral $\overline{C'D'DH}$ is a parallelogram, since the lengths of both pairs of opposite sides are equal. By construction, the line segment $\overline{DD'}$ is parallel to $\overline{CC'}$, so $\overline{C'H}$ is parallel to $\overline{DD'}$ is also parallel to $\overline{CC'}$. Since one of its end points is C', it must be on the line containing $\overline{CC'}$.

The lengths c, d, e are given and we proved in Section 5.3 that a line segment of length $c+d$ can be construction and in Section 5.4 that a line segment of length $x = \dfrac{c}{c+d}e$ can be constructed. S, the intersection of $c(C', x)$ and $c(C, x)$, is the intersection of $\overline{AB}, \overline{CD}$ has been constructed.

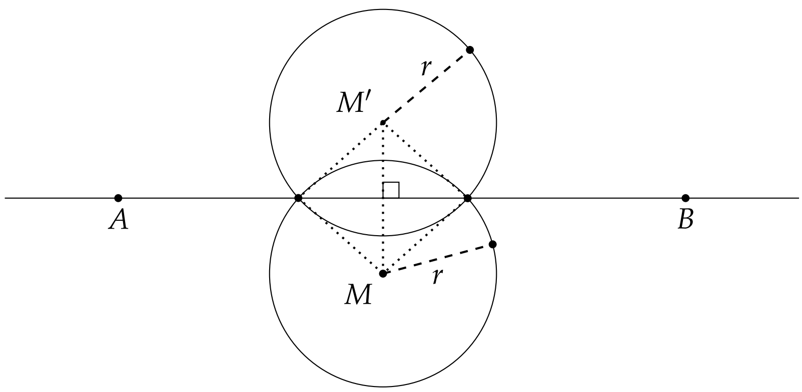


$\overline{CD} = \overline{C'D'} = \overline{DH} = e$

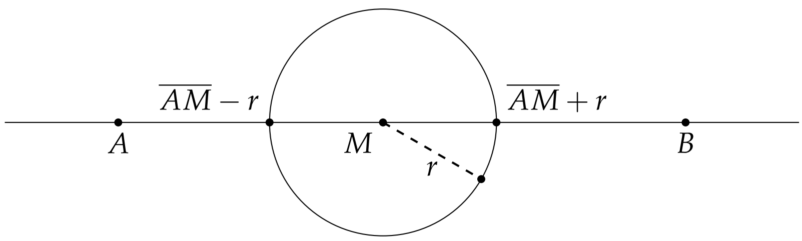
5.6 Finding the intersection of a line and a circle

Given a circle $k = C(M, r)$ and a line $\overline{AB}$, construct their intersections using only a compass.

Construct M', be the reflection of M about $\overline{AB}$ and the circle $k' = c(M', r)$. The points of intersection of k, k' are the points of intersection of the line $\overline{AB}$ and the circle k. This can be shown by congruent triangles, as indicated by the dotted lines in the diagram.



This construction cannot be done if M is on the line $\overline{AB}$. In that case, extend and shorten $\overline{AM}$ by length r as described in Section 5.3. The end points of the extended and shortened segments are the intersections of k with $\overline{AB}$.



Chapter 6
A Straightedge (with Something Extra) is Sufficient

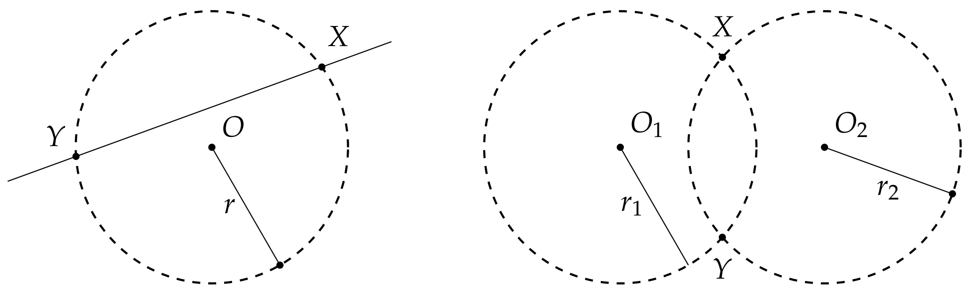
Can every construction with straightedge and compass be done with only a straightedge? The answer is no because lines represent linear equations and cannot represent quadratic equations like circles. In 1822 the French mathematician Jean-Victor Poncelet conjectured that a straightedge only is sufficient, provided that one circle exists in the plane. This was proved in 1833 by the Swiss mathematician Jakob Steiner. In this chapter I present a proof of the theorem based on the proof in problem 34 in [4] as reworked by Michael Woltermann [5].[1]

Every step of a construction with straightedge and compass is one of these three operations:

- Find the point of intersection of two straight lines.
- Find the point(s) of intersection of a straight line and a circle.
- Find the point(s) of intersection of two circles.

It is clear that the first operation can be performed with a straightedge only.

What does it mean to perform a construction with straight-edge alone? A circle is defined by a point O, its center, and a line segment whose length is the radius r, one of whose endpoints is the center. If we can construct the points labeled X and Y in the diagram below, we can claim to have successfully constructed the intersection of a given circle with a given line and of two given circles. The circles drawn with dashed lines in the diagram do not actually appear in the construction. In this chapter, the single existing circle is drawn with a regular line, and the dashed circles are only used to help understand a construction and its proof.



First we present five auxiliary constructions (Sections 6.1–6.5), and then show how to find the intersection of a line with a circle (Section 6.6) and the intersection of two circles (Section 6.7).

[1] I would like to thank Woltermann for permission to use his work.

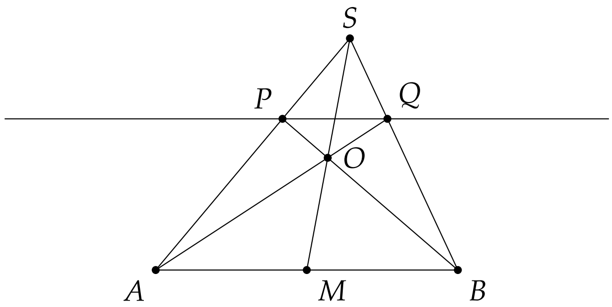
6.1 Construct a line parallel to a given line

Given a line l defined by two points A, B and a point P not on the line, construct a line through P that is parallel to $\overline{AB}$.

There are two cases:

- A "directed line": the midpoint M of $\overline{AB}$ is given.
- Any other line.

Case 1, directed line: Construct a ray that extends AP and choose any point S on the ray beyond P. Construct the lines $\overline{BP}, \overline{SM}, \overline{SB}$. Label by O the intersection of $\overline{BP}$ with $\overline{SM}$. Construct a ray that extends $\overline{AO}$ and label by Q the intersection of the ray $\overline{AO}$ with $\overline{SB}$.



Claim: $\overline{PQ}$ is parallel to $\overline{AB}$.

Proof: We will use Ceva's theorem that we proof later.

Theorem (Ceva): Given three line segments from the vertices of a triangle to the opposite edges that intersect in a point (O in the diagram, but M is not necessarily the midpoint of the side), the lengths of the segments satisfy:

$$\frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BQ}}{\overline{QS}} \cdot \frac{\overline{SP}}{\overline{PA}} = 1\,.$$

In the construction above, M is the midpoint of $\overline{AB}$, so $\dfrac{\overline{AM}}{\overline{MB}} = 1$, so the equation becomes:

$$\frac{\overline{BQ}}{\overline{QS}} = \frac{\overline{PA}}{\overline{SP}} = \frac{\overline{AP}}{\overline{PS}}\,. \tag{6.1}$$

We will prove that $\triangle ABS \sim \triangle PQS$, so that $\overline{PQ}$ is parallel to $\overline{AB}$ because $\angle ABS = \angle PQS$.

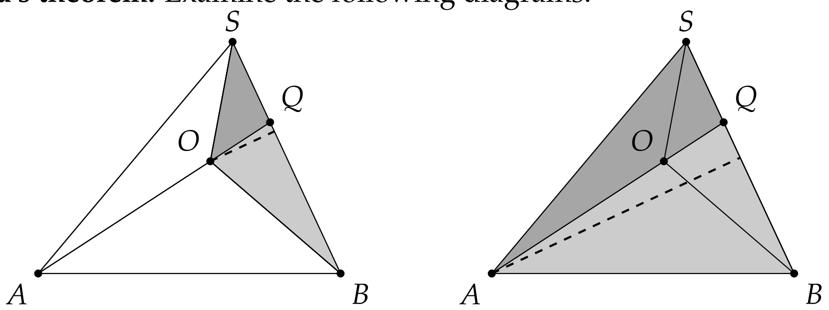
$$\begin{aligned}
\overline{BS} &= \overline{BQ} + \overline{QS} \\
\frac{\overline{BS}}{\overline{QS}} &= \frac{\overline{BQ}}{\overline{QS}} + \frac{\overline{QS}}{\overline{QS}} = \frac{\overline{BQ}}{\overline{QS}} + 1 \\
\overline{AS} &= \overline{AP} + \overline{PS} \\
\frac{\overline{AS}}{\overline{PS}} &= \frac{\overline{AP}}{\overline{PS}} + \frac{\overline{PS}}{\overline{PS}} = \frac{\overline{AP}}{\overline{PS}} + 1\,.
\end{aligned}$$

Using Equation 6.1:

$$\frac{\overline{BS}}{\overline{QS}} = \frac{\overline{BQ}}{\overline{QS}} + 1 = \frac{\overline{AP}}{\overline{PS}} + 1 = \frac{\overline{AS}}{\overline{PS}},$$

so that $\triangle ABS \sim \triangle PQS$.

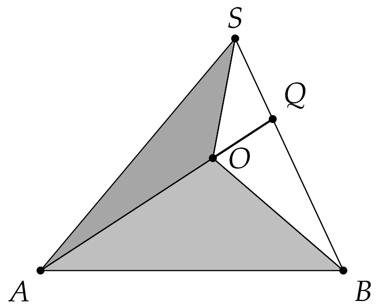
Proof of Ceva's theorem: Examine the following diagrams:



If the altitudes of two triangles are equal, their areas are proportional to the bases. In both diagrams, the altitudes of the gray triangles are equal, so:[2]

$$\frac{\triangle BQO}{\triangle SQO} = \frac{\overline{BQ}}{\overline{QS}}, \qquad \frac{\triangle BQA}{\triangle SQA} = \frac{\overline{BQ}}{\overline{QS}}.$$

By subtracting the areas of the indicated triangles, we get the proportion between the gray triangles:



$$\frac{\triangle BOA}{\triangle SOA} = \frac{\triangle BQA - \triangle BQO}{\triangle SQA - \triangle SQO} = \frac{\overline{BQ}}{\overline{QS}}.$$

This might look strange at first. We explain it using a simpler notation:

$$\begin{aligned} \frac{c}{d} &= \frac{a}{b} \\ \frac{e}{f} &= \frac{a}{b} \\ c - e &= \frac{ad}{b} - \frac{af}{b} = \frac{a}{b}(d-f) \\ \frac{c-e}{d-f} &= \frac{a}{b}. \end{aligned}$$

[2]We use the name of a triangle as a shortcut for its area

Similarly, we can prove:

$$\frac{\overline{AM}}{\overline{MB}} = \frac{\triangle AOS}{\triangle BOS}\,, \qquad \frac{\overline{SP}}{\overline{PA}} = \frac{\triangle SOB}{\triangle AOB}\,,$$

so:

$$\frac{\overline{AM}}{\overline{MB}}\frac{\overline{BQ}}{\overline{QS}}\frac{\overline{SP}}{\overline{PA}} = \frac{\triangle AOS}{\triangle BOS}\frac{\triangle BOA}{\triangle SOA}\frac{\triangle SOB}{\triangle AOB} = 1\,,$$

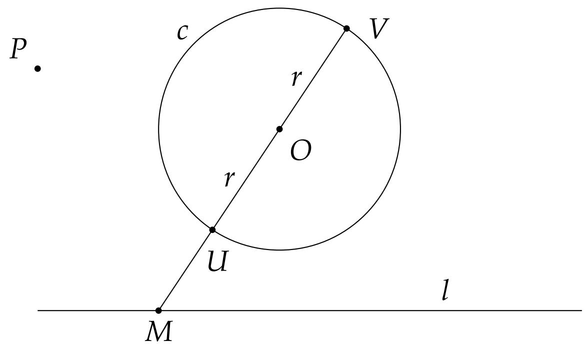
because the order of the vertices in a triangle makes no difference:

$$\triangle AOS = \triangle SOA,\ \triangle BOA = \triangle AOB,\ \triangle SOB = \triangle BOS\,.$$

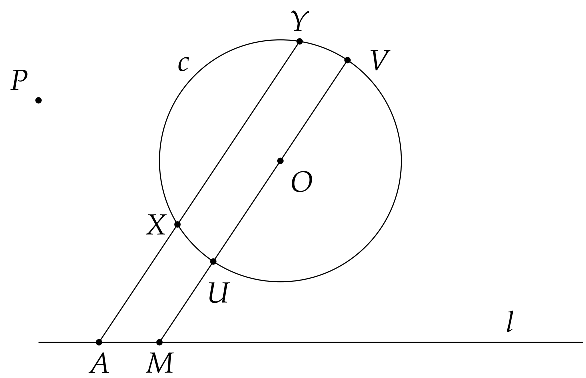
End of the proof of Ceva's theorem

Case 2, any other line: Label the line by l and the existing circle, which we will call the **fixed circle**, by c, where the center of c is O and its radius is r. P is the point not on the line through which it is required to construct a line parallel to l. Convince yourself that the construction does not depend on the location of the center of the circle or its radius.

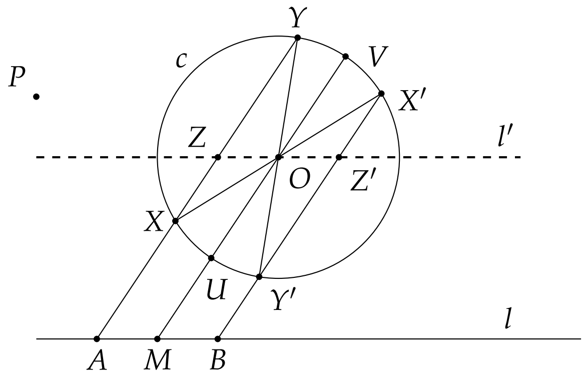
Choose M, any point on l, and construct a ray extending $\overline{MO}$ that intersects the circle at U, V.



This line is a **directed line** because O, the center of the circle, bisects the diameter $\overline{UV}$. Choose a point A on l and use the construction for a directed line to construct a line parallel to $\overline{UV}$, which intersects the circle at X, Y.



Construct the diameters $\overline{XX'}$ and $\overline{YY'}$. Construct the ray from $\overline{X'Y'}$ and label by B its intersection with l.

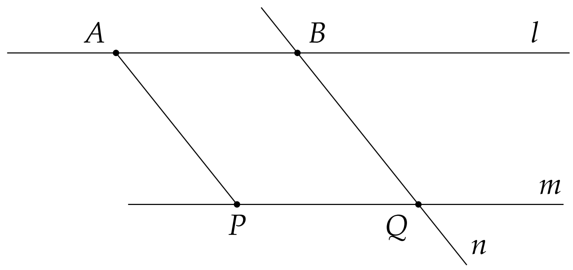


l is a directed line because M is the bisector of $\overline{AB}$, so a line can be constructed through P parallel to l.

Proof: $\overline{OX}, \overline{OX'}, \overline{OY}, \overline{OY'}$ are all radii of the circle and $\angle XOY = \angle X'OY'$ since they are vertical angles. $\triangle XOY \cong \triangle X'OY'$ by side-angle-side. Define (not construct!) l' to be a line through O parallel to l that intersects $\overline{XY}$ at Z and $\overline{X'Y'}$ at Z'. $\angle XOZ = \angle X'OZ'$ because they are vertical angles, so $\triangle XOZ \cong \triangle X'OZ'$ by angle-side-angle and $\overline{ZO} = \overline{OZ'}$. $\overline{AMOZ}$ and $\overline{BMOZ'}$ are parallelograms (quadrilaterals with opposite sides parallel), so $\overline{AM} = \overline{ZO} = \overline{OZ'} = \overline{MB}$.

Corollary: Given a line segment $\overline{AB}$ and a point P not on the line. It is possible to construct a line through P that is parallel to $\overline{AB}$ and whose length is equal to the length of $\overline{AB}$. In other words, it is possible to copy $\overline{AB}$ parallel to itself with P as one of its endpoints.

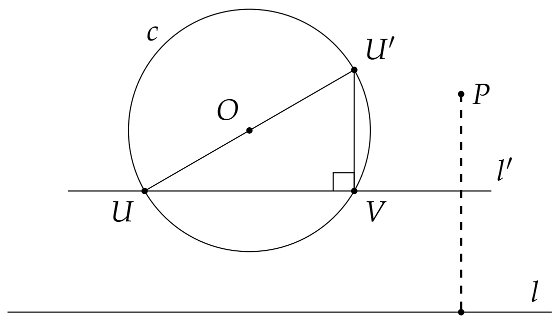
Proof: We have proved that it is possible to construct a line m through P parallel to $\overline{AB}$ and a line n through B to parallel to $\overline{AP}$. The quadrilateral $\overline{ABQP}$ is a parallelogram so opposite sides are equal $\overline{AB} = \overline{PQ}$.



6.2 Construct a perpendicular to a given line

Construct a perpendicular line segment through a point P to a given line l (P is not on l).

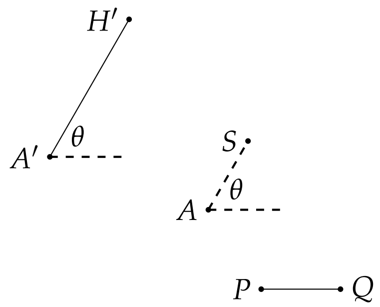
Construct (Section 6.1) a line l' parallel to l that intersects the **fixed circle** at U, V. Construct the diameter $\overline{UOU'}$ and chord $\overline{VU'}$.



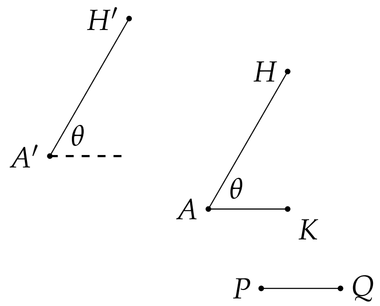
$\angle UVU'$ is an angle that subtends a semicircle so it is a right angle. Therefore, $\overline{VU'}$ is perpendicular to $\overline{UV}$ and l. Construct (Section 6.1) the parallel to $\overline{VU'}$ through P.

6.3 Copy a line segment in a given direction

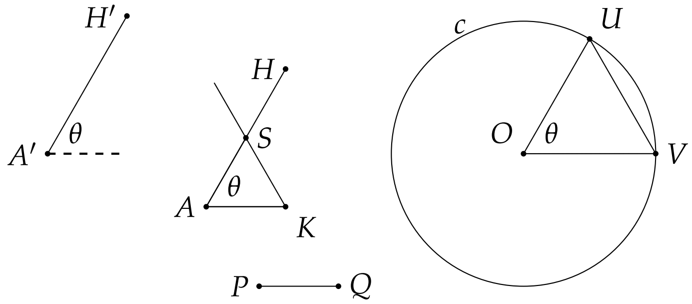
The corollary at the end of Section 6.1 shows that it is possible to copy a line segment parallel to itself. Here we show that it is possible to copy a line segment in the direction of another line. The meaning of "direction" is that the line defined by two points A', H' defines an angle θ relative to some axis. The task is to copy $\overline{PQ}$ to $\overline{AS}$ so that $\overline{AS}$ will have the same angle θ relative to the same axis. In the diagram $\overline{PQ}$ is on the x-axis but that is of no importance.



By Section 6.1 it is possible to construct a line segment $\overline{AH}$ so that $\overline{AH} \| \overline{A'H'}$ and to construct a line segment $\overline{AK}$ so that $\overline{AK} \| \overline{PQ}$.



$\angle HAK = \theta$ so it remains to find a point S on $\overline{AH}$ so that $\overline{AS} = \overline{PQ}$.



Construct two radii $\overline{OU}, \overline{OV}$ of the fixed circle which are parallel to $\overline{AH}, \overline{AK}$, respectively, and construct a ray through K parallel to $\overline{UV}$. Label its intersection with $\overline{AH}$ by S.

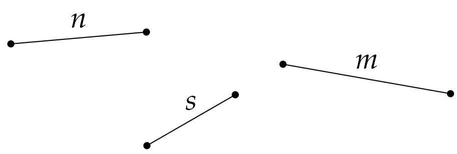
Claim: $\overline{AS} = \overline{PQ}$

Proof: By construction, $\overline{AH} \| \overline{OU}$ and $\overline{AK} \| \overline{OV}$, so $\angle SAK = \theta = \angle UOV$. $\overline{SK} \| \overline{UV}$ and $\triangle SAK \sim \triangle UOV$ by angle-angle-angle, $\triangle UOV$ is isosceles, because $\overline{OU}, \overline{OV}$ are radii of the same circle. Therefore, $\triangle SAK$ is isosceles and $\overline{AS} = \overline{AK} = \overline{PQ}$.

6.4 Construct a line segment relative to three other line segments

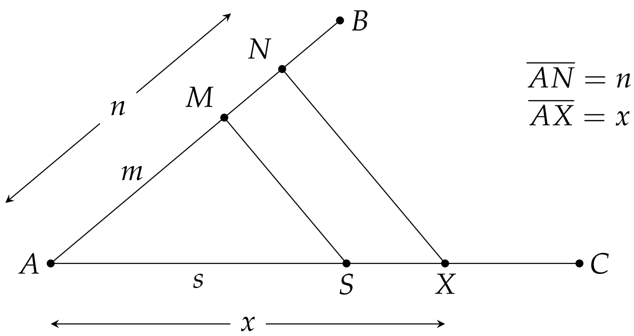
Given line segments of lengths n, m, s, construct a segment of length $x = \dfrac{n}{m}s$.

The three line segments are located at arbitrary positions and directions in the plane.



Choose a point A and construct two rays $\overline{AB}, \overline{AC}$. By the construction in Section 6.3 it is possible to construct points M, N, S such that $\overline{AM} = m$, $\overline{AN} = n$, $\overline{AS} = s$. Construct a line through N parallel to $\overline{MS}$ which intersects $\overline{AC}$ at X and label its length by x. $\triangle MAS \sim \triangle NAX$ by angle-angle-angle so:

$$\frac{m}{n} = \frac{s}{x} \qquad x = \frac{n}{m}s\,.$$



6.5 Construct a square root

Given line segments of lengths a, b, construct a segment of length $\sqrt{ab}$.

We want to express $x = \sqrt{ab}$ in the form $\dfrac{n}{m}s$ in order to use the result of Section 6.4.

- For n we use d, the diameter of the fixed circle.
- For m we use $t = a + b$ which can be constructed from a, b as shown in Section 6.3.
- We define $s = \sqrt{hk}$ where h, k are defined as expressions on the lengths a, b, t, d, and we will show how it is possible to construct a line segment of length $\sqrt{ab}$.

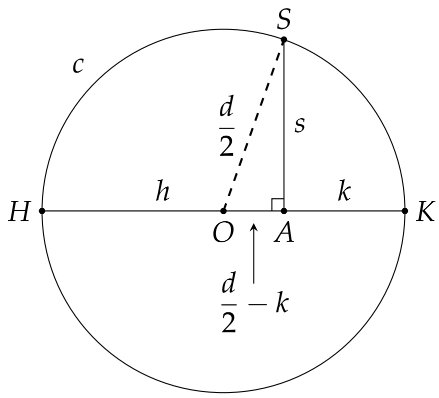
Define $h = \dfrac{d}{t}a$, $k = \dfrac{d}{t}b$ and and compute:

$$x = \sqrt{ab} = \sqrt{\frac{th}{d}\frac{tk}{d}} = \sqrt{\left(\frac{t}{d}\right)^2 hk} = \frac{t}{d}hk = \frac{t}{d}s\,.$$

We also compute:

$$h + k = \frac{d}{t}a + \frac{d}{t}b = \frac{d(a+b)}{t} = \frac{dt}{t} = d\,.$$

By Section 6.3 construct $\overline{HA} = h$ on a diameter $\overline{HK}$ of the fixed circle. From $h + k = d$ we have $\overline{AK} = k$:



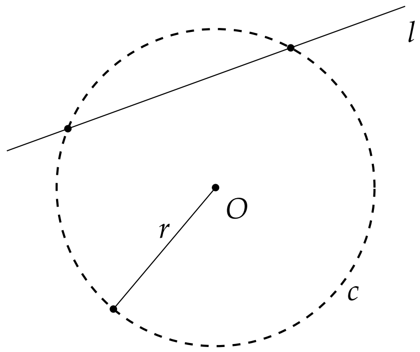
By Section 6.2 construct a perpendicular to $\overline{HK}$ at A and label the intersection of this line with the circle by S. $\overline{OS} = \overline{OK} = \dfrac{d}{2}$ because they are radii of the circle and $\overline{OA} = \dfrac{d}{2} - k$. By Pythagoras's theorem:

$$\begin{aligned} s^2 = \overline{SA}^2 &= \left(\frac{d}{2}\right)^2 - \left(\frac{d}{2} - k\right)^2 \\ &= \left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2 + 2\frac{dk}{2} - k^2 \\ &= k(d-k) = kh \\ s &= \sqrt{hk}\,. \end{aligned}$$

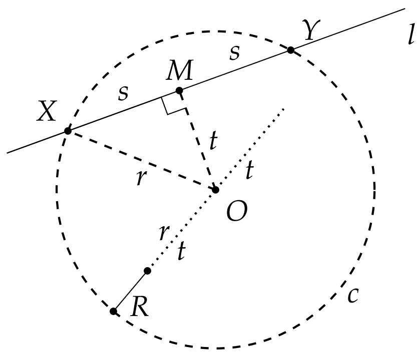
Now $x = \dfrac{t}{d}s$ can be constructed by Section 6.4.

6.6 Construct the points of intersection of a line and a circle

Given a line l and a circle $c(O, r)$, construct their points of intersection.



By Section 6.2 it is possible to construct a perpendicular from the center of the circle O to the line l. Label the intersection of l with the perpendicular by M. M bisects the chord $\overline{XY}$, where X, Y are the intersections of the line with the circle. $2s$ is the length of the chord $\overline{XY}$. Note that s, X, Y in the diagram are just definitions: we haven't constructed them yet.



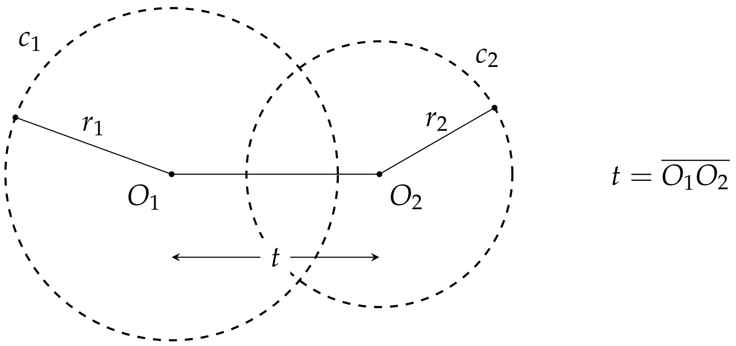
$\triangle OMX$ is a right triangle, so $s^2 = r^2 - t^2 = (r+t)(r-t)$. r is the given radius of the circle and t is defined s the length of $\overline{OM}$, the line segment between O and M. By Section 6.3 it is possible to construct a line segment of length t from O in the two directions $\overline{OR}$ and $\overline{RO}$. The result is two line segments of length $r+t, r-t$.

Section 6.5 shows how to construct a line segment of length $s = \sqrt{(r+t)(r-t)}$. By Section 6.3 it is possible to construct line segments of length s from M along the given line l in both directions. Their other endpoints are the points of intersection of l and c.

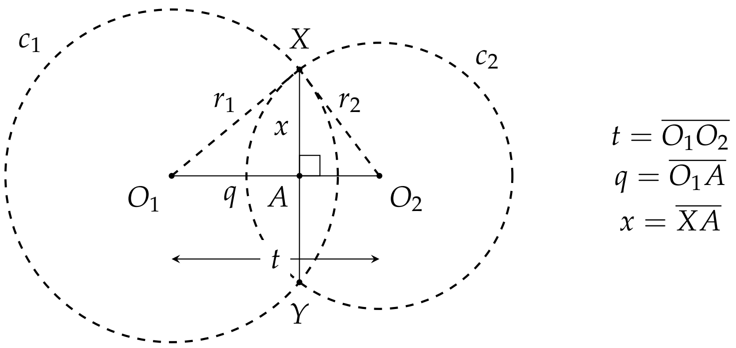
6.7 Construct the points of intersection of two circles

Given two circles $c(O_1, r_1), c(O_2, r_2)$, construct their points of intersection.

With a straightedge it is possible to construct a line segment $\overline{O_1O_2}$ that connects the two centers. Label its length t.



Label by A be the point of intersection of $\overline{O_1O_2}$ and $\overline{XY}$, and label the lengths $q = \overline{O_1A}$, $x = \overline{XA}$. Note that we have not constructed A, but if we succeed in constructing the lengths q, x, by Section 6.3 we can construct A at length q from O_1 in the direction $\overline{O_1O_2}$.



By Section 6.2 a perpendicular to $\overline{O_1O_2}$ at A can be constructed and again by Section 6.3 it is possible to construct line segments of length x from A is both directions along the perpendicular. Their other endpoints are the points of intersection of the circles.

Constructing the length q: Define $d = \sqrt{r_1^2 + t^2}$, the hypotenuse of a right triangle. It can be constructed from r_1, t, the known lengths of the other two sides. On any line construct a line segment $\overline{RS}$ of length r_1, then a perpendicular to $\overline{RS}$ through R, and finally, a line segment $\overline{RT}$ of length t through R on the perpendicular. The length of the hypotenuse $\overline{ST}$ is d. This right triangle can be constructed anywhere in the plane, not necessarily near the circles.

By the law of cosines for $\triangle O_1O_2X$:

$$\begin{aligned} r_2^2 &= r_1^2 + t^2 - 2r_1 t \cos \angle XO_1O_2 \\ &= r_1^2 + t^2 - 2tq \\ q &= \frac{(d+r_2)(d-r_2)}{2t}. \end{aligned}$$

By Section 6.3 these lengths can be constructed and by Section 6.4 q can be constructed from $d + r_2, d - r_2, 2t$.

Constructing the length x: $\triangle AO_1X$ is a right triangle, so $x = \sqrt{r_1^2 - q^2} = \sqrt{(r_1+q)(r_1-q)}$. By Section 6.3 $h = r_1 + q, k = r_1 - q$ can be constructed, as can $x = \sqrt{hk}$ by Section 6.5.

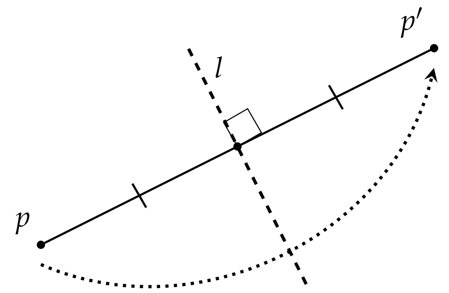
Part II

Origami

Chapter 7
Axioms

Each axiom states that a *fold* exists that will place given points and lines onto points and lines, such that certain properties hold. The term fold comes from the origami operation of folding a piece of paper, but here it is used to refer to the geometric line that would be created by folding the paper.

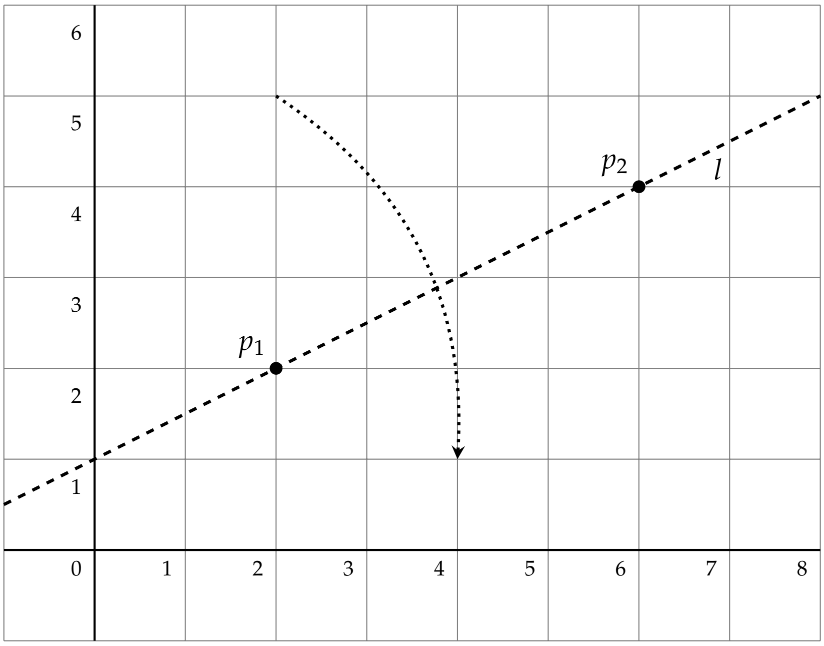
The axioms are called the *Huzita-Hatori axioms* [21], although their final form resulted from the work of several mathematicians. Lee [10, Chapter 4] is a good overview of the mathematics of origami, while Martin [11, Chapter 10] is a formal development. The reader should be aware that, *by definition*, folds result in *reflections*. Given a point p, its reflection around a fold l results in a point p', such that l is the perpendicular bisector of the line segment $\overline{pp'}$:



In the diagrams, given lines are solid, folds are dashed, auxiliary lines are dotted, and dotted arrows indicate the direction of folding the paper.

7.1 Axiom 1

Axiom Given two distinct points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, there is a unique fold l that passes through both of them.



Derivation of the equation of the fold

The equation of fold l is derived from the coordinates of p_1 and p_2: the slope is the quotient of the differences of the coordinates and the y-intercept is derived from p_1:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\,. \tag{7.1}$$

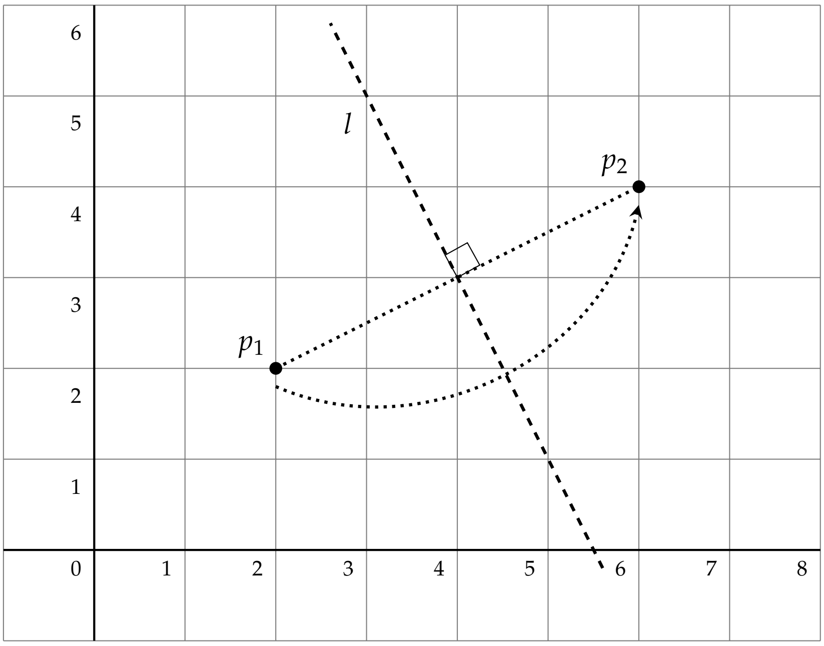
Example

Let $p_1 = (2, 2), p_2 = (6, 4)$. The equation of l is:

$$\begin{aligned} y - 2 &= \frac{4-2}{6-2}(x-2) \\ y &= \frac{1}{2}x + 1\,. \end{aligned}$$

7.2 Axiom 2

Axiom Given two distinct points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, there is a unique fold l that places p_1 onto p_2.



Derivation of the equation of the fold

The fold l is the perpendicular bisector of $\overline{p_1 p_2}$. Its slope is the negative reciprocal of the slope of the line connecting p_1 and p_2. l passes through the midpoint between the points:

$$y - \frac{y_1 + y_2}{2} = -\frac{x_2 - x_1}{y_2 - y_1}\left(x - \frac{x_1 + x_2}{2}\right). \tag{7.2}$$

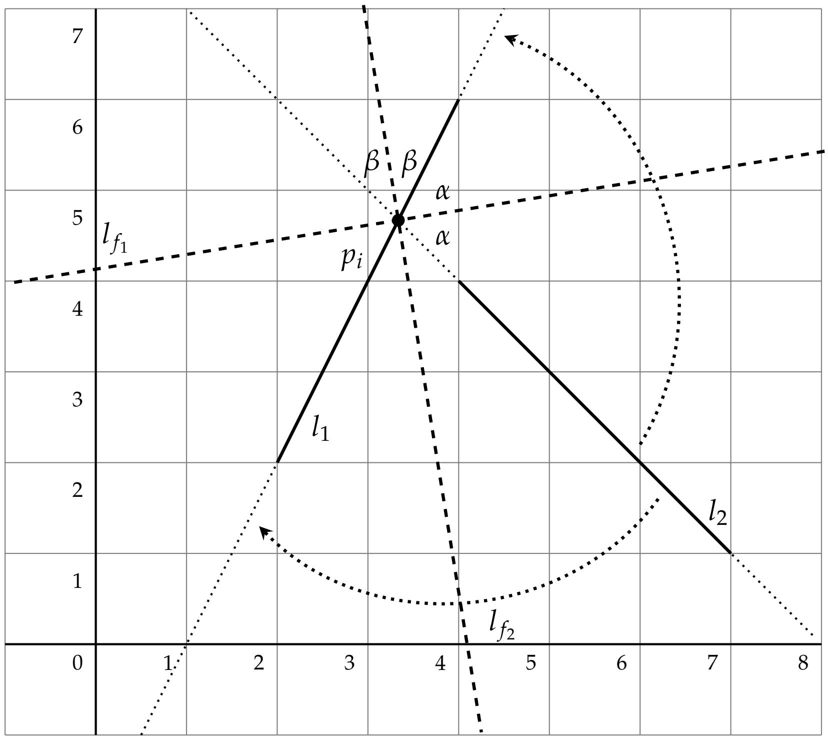
Example

Let $p_1 = (2, 2)$, $p_2 = (6, 4)$. The equation of l is:

$$\begin{aligned} y - \left(\frac{2+4}{2}\right) &= -\frac{6-2}{4-2}\left(x - \left(\frac{2+6}{2}\right)\right) \\ y &= -2x + 11\,. \end{aligned}$$

7.3 Axiom 3

Axiom Given two lines l_1 and l_2, there is a fold l that places l_1 onto l_2.



If the lines are parallel, let l_1 be $y = mx + b_1$ and let l_2 be $y = mx + b_2$. The fold is the line parallel to l_1, l_2 and halfway between them $y = mx + \dfrac{b_1 + b_2}{2}$.
If the lines intersect, let l_1 be $y = m_1 x + b_1$ and let l_2 be $y = m_2 x + b_2$.
Derivation of the point of intersection
$p_i = (x_i, y_i)$, the point of intersection of the two lines, is:

$$
\begin{aligned}
m_1 x_i + b_2 &= m_2 x_i + b_2 \\
x_i &= \frac{b_2 - b_1}{m_1 - m_2} \\
y_i &= m_1 x_i + b_1\,.
\end{aligned}
$$

Example
Let l_1 be $y = 2x - 2$ and let l_2 be $y = -x + 8$. The point of intersection is:

$$
\begin{aligned}
x_i &= \frac{8 - (-2)}{2 - (-1)} = \frac{10}{3} \\
y_i &= 2 \cdot \frac{10}{3} - 2 = \frac{14}{3}\,.
\end{aligned}
$$

Derivation of the equation of the slope of the angle bisector

The two lines form an angle at their point of intersection, actually, two pairs of vertical angles. The folds are the bisectors of these angles.

If the angle of line l_1 relative to the x-axis is θ_1 and the angle of line l_2 relative to the x-axis is θ_2, then the fold is the line which makes an angle of $\theta_b = \dfrac{\theta_1 + \theta_2}{2}$ with the x-axis.

$\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$ are given so m_b, the slope of the angle bisector, is:

$$m_b = \tan\theta_b = \tan\frac{\theta_1 + \theta_2}{2}\,.$$

The derivation requires the use of two trigonometric identities that we derive here:

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)}\\
&= \frac{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}\\
&= \frac{\sin\theta_1 + \cos\theta_1\tan\theta_2}{\cos\theta_1 - \sin\theta_1\tan\theta_2}\\
&= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2}\,.
\end{aligned}$$

We use this formula to obtain a quadratic equation in $\tan(\theta/2)$:

$$\tan\theta = \frac{\tan(\theta/2) + \tan(\theta/2)}{1 - \tan^2(\theta/2)}$$

$$\tan\theta\,(\tan(\theta/2))^2 \;+\; 2\,(\tan(\theta/2)) \;-\; \tan\theta = 0\,,$$

whose solutions are:

$$\tan(\theta/2) = \frac{-1 \pm \sqrt{1 + \tan^2\theta}}{\tan\theta}\,.$$

First derive m_s, the slope of $\theta_1 + \theta_2$:

$$m_s = \tan(\theta_1 + \theta_2) = \frac{m_1 + m_2}{1 - m_1 m_2}\,.$$

Then derive m_b, the slope of the angle bisector:

$$\begin{aligned}
m_b &= \tan\frac{\theta_1 + \theta_2}{2}\\
&= \frac{-1 \pm \sqrt{1 + \tan^2(\theta_1 + \theta_2)}}{\tan(\theta_1 + \theta_2)}\\
&= \frac{-1 \pm \sqrt{1 + m_s^2}}{m_s}\,.
\end{aligned}$$

Example

For the lines $y = 2x - 2$ and $y = -x + 8$, the slope of the angle bisector is:

$$m_s = \frac{2 + (-1)}{1 - (2 \cdot -1)} = \frac{1}{3}$$

$$m_b = \frac{-1 \pm \sqrt{1 + (1/3)^2}}{1/3} = -3 \pm \sqrt{10}\,.$$

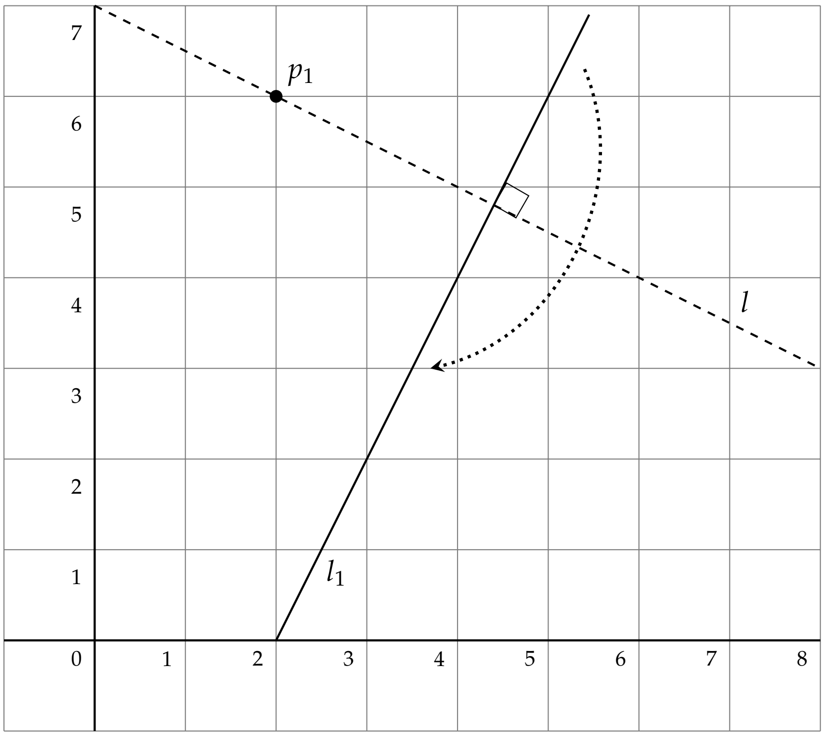
Derivation of the equation of the fold

Let us derive equation of the fold l_{f_1} with the positive slope; we know the coordinates of the intersection of the two lines $m_i = \left(\frac{10}{3}, \frac{14}{3}\right)$:

$$\begin{aligned}
\frac{14}{3} &= (-3 + \sqrt{10}) \cdot \frac{10}{3} + b \\
b &= \frac{44 - 10\sqrt{10}}{3} \\
y &= (-3 + \sqrt{10})x + \frac{44 - 10\sqrt{10}}{3} \approx 0.162x + 4.13\,.
\end{aligned}$$

7.4 Axiom 4

Axiom Given a point p_1 and a line l_1, there is a unique fold l perpendicular to l_1 that passes through point p_1.



Derivation of the equation of the fold

Let l_1 be $y = m_1 x + b_1$ and let $p_1 = (x_1, y_1)$. l is perpendicular to l_1 so its slope is $-\frac{1}{m_1}$. Since it passes through p_1, we can compute the intercept b and write down its equation:

$$y_1 = -\frac{1}{m}x_1 + b$$

$$b = \frac{(my_1 + x_1)}{m}$$

$$y = -\frac{1}{m}x + \frac{(my_1 + x_1)}{m}.$$

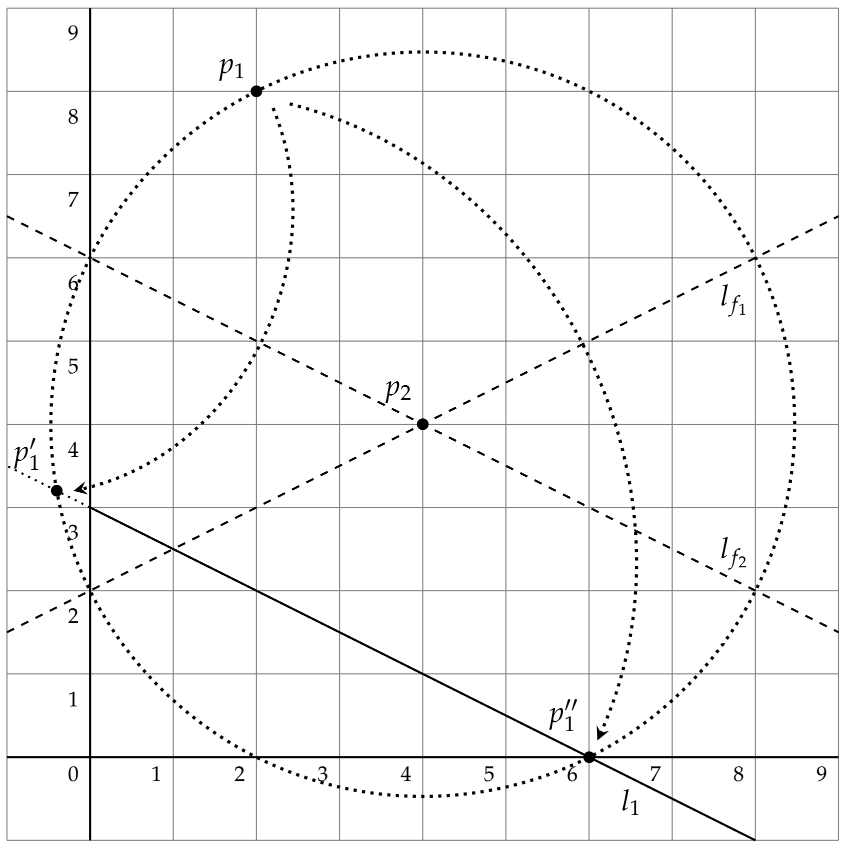
Example

Let $p_1 = (2, 6)$ and let l_1 be $y = 2x - 4$. The equation of the fold l is:

$$y = -\frac{1}{2}x + \frac{2 \cdot 6 + 2}{2} = -\frac{1}{2}x + 7.$$

7.5 Axiom 5

Axiom Given two points p_1, p_2 and a line l_1, there is a fold l that places p_1 onto l_1 and passes through p_2.



For a given pair of points and a line, there may be zero, one or two folds.

Derivation of the equations of the reflections

Let l be a fold through p_2 and p_1' be the reflection of p_1 around l. The length of $\overline{p_1p_2}$ equals the length of $\overline{p_1'p_2}$. The locus of points at distance $\overline{p_1p_2}$ *from* p_2 is the circle centered at p_2 whose radius is the length of $\overline{p_1p_2}$. The intersections of this circle with the line l_1 give the possible points p_1'.

Let l_1 be $y = m_1x + b_1$ and let $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$. The equation of the circle centered at p_2 with radius the length of $\overline{p_1p_2}$ is:

$$(x - x_2)^2 + (y - y_2)^2 = r^2, \quad \text{where}$$

$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Substituting the equation of the line into the equation for the circle:

$$(x - x_2)^2 + ((m_1x + b_1) - y_2)^2 = (x - x_2)^2 + (m_1x + (b_1 - y_2))^2 = r^2,$$

we obtain a quadratic equation for the x-coordinates of the possible intersections:

$$x^2(1+m_1^2) \;+\; 2(-x_2+m_1b-m_1y_2)x \;+\; (x_2^2+(b_1^2-2b_1y_2+y_2^2)-r^2)=0\,. \tag{7.3}$$

The quadratic equation has at most two solutions x_1', x_1'' and we can compute $y_1', y_1'')$ from $y = m_1x + b_1$. The reflected points are $p_1' = (x_1', y_1')$, $p_1'' = (x_1'', y_1'')$.

Example

Let $p_1 = (2,8)$, $p_2 = (4,4)$ and let l_1 be $y = -\dfrac{1}{2}x + 3$. The equation of the circle is:

$$(x-4)^2 + (y-4)^2 = r^2 = (4-2)^2 + (4-8)^2 = 20\,.$$

Substitute the equation of the line into the equation of the circle and simplify to obtain a quadratic equation for the x-coordinates of the intersections (or use Equation 7.3):

$$\begin{aligned}
(x-4)^2 + \left(\left(-\frac{1}{2}x+3\right)-4\right)^2 &= 20\\
5x^2-28x-12 &= 0\\
(5x+2)(x-6) &= 0\,.
\end{aligned}$$

The two points of intersection are:

$$p_1' = \left(-\frac{2}{5}, \frac{16}{5}\right) = (-0.4, 3.2)\,, \qquad p_1'' = (6,0)\,.$$

Derivation of the equations of the folds

The folds will be the perpendicular bisectors of $\overline{p_1p_1'}$ and $\overline{p_1p_1''}$. The equation of a perpendicular bisector is given by Equation 7.2, repeated here with for p_1':

$$y - \frac{y_1+y_1'}{2} = -\frac{x_1'-x_1}{y_1'-y_1}\left(x - \frac{x_1+x_1'}{2}\right)\,. \tag{7.4}$$

Example

For $p_1 = (2,8)$ and $p_1' = \left(-\dfrac{2}{5}, \dfrac{16}{5}\right)$, the equation of the fold l_{f_1} is:

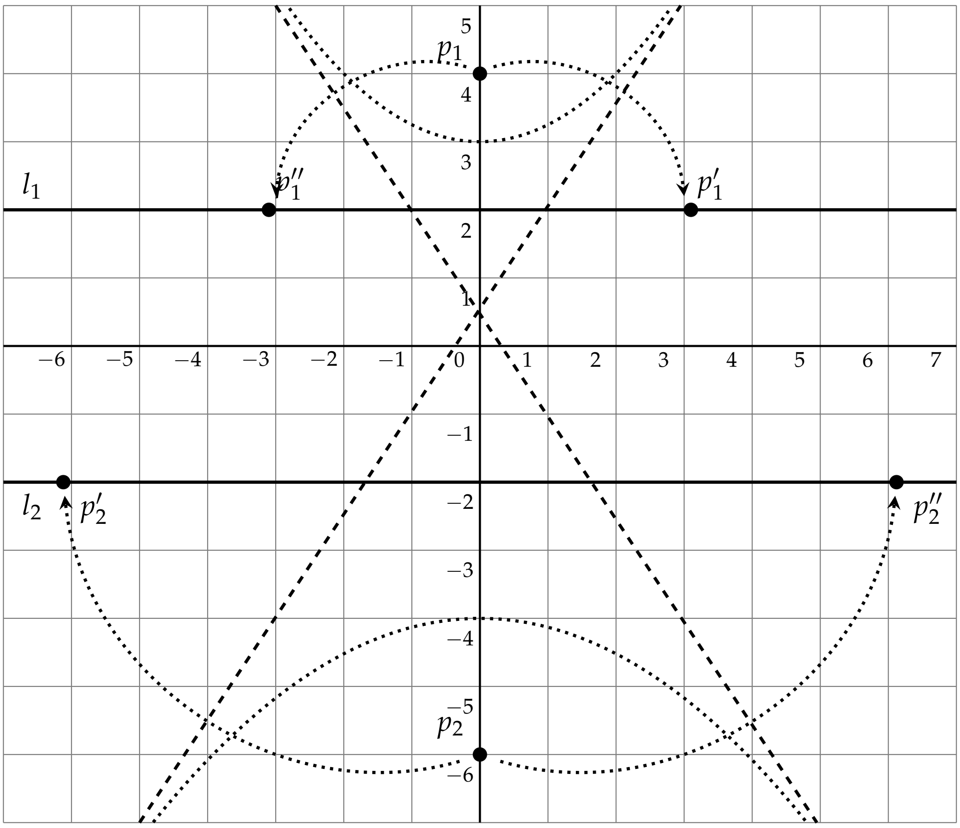
$$\begin{aligned}
y - \frac{8+(16/5)}{2} &= -\frac{(-2/5)-2}{(16/5)-8}\left(x - \frac{2+(-2/5)}{2}\right)\\
y &= -\frac{1}{2}x+6\,.
\end{aligned}$$

For $p_1 = (2,8)$ and $p_1'' = (6,0)$, the equation of the fold l_{f_2} is:

$$\begin{aligned}
y - \frac{8+0}{2} &= -\frac{6-2}{0-8}\left(x - \frac{2+6}{2}\right)\\
y &= \frac{1}{2}x+2\,.
\end{aligned}$$

7.6 Axiom 6

Axiom Given two points p_1 and p_2 and two lines l_1 and l_2, there is a fold l that places p_1 onto l_1 and p_2 onto l_2.



For a given pair of points and pair of lines, there may be zero, one, two or three folds.

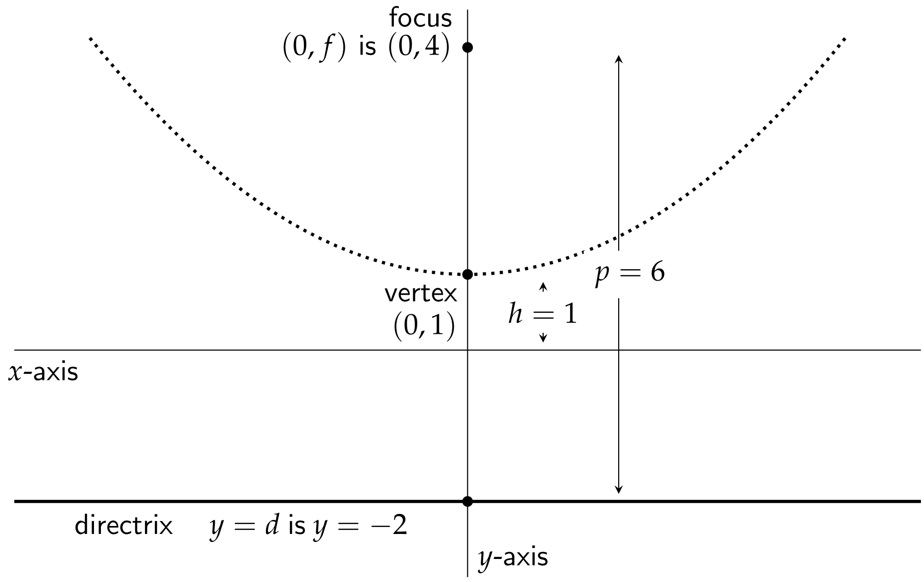
If a fold places p_i onto l_i, the distance from p_i to the fold is equal to the distance from l_i to the fold. The locus of points that are equidistant from a point p_i and a line l_i is a parabola with focus p_i and directrix l_i. A fold is any line tangent to that parabola. A detailed justification of this claim is given below.

For a fold to simultaneously place p_1 onto l_1 and p_2 onto l_2, it must be a tangent common to the two parabolas.

The formula for an arbitrary parabola is quite complex, so we limit the presentation to parabolas with the the y-axis as the axis of symmetry. This is not a significant limitation because for any parabola there is a rigid motion that moves the parabola so that its axis of symmetry is the y-axis. An example will also be given where one of the parabolas has the x-axis as its axis of symmetry.

Derivation of the equation a fold

Let $(0, f)$ be the focus of a parabola with directrix $y = d$. Define $p = f - d$, the signed length of the line segment between the focus and the directrix.[1] If the vertex of the parabola is on the x-axis, the equation of the parabola is $y = \dfrac{x^2}{2p}$. To move the parabola up or down the y-axis so that its vertex is at $(0, h)$, add h to the equation of the parabola: $y = \dfrac{x^2}{2p} + h$.



Define $a = 2ph$ so that the equation of the parabola is:

$$y = \frac{x^2}{2p} + \frac{a}{2p}$$

$$x^2 - 2py + a = 0\,.$$

The equation of the parabola in the diagram above is:

$$x^2 - 2 \cdot 6\, y + 2 \cdot 6 \cdot 1 = 0$$

$$x^2 - 12y + 12 = 0\,.$$

Substitute the equation of an *arbitrary* line $y = mx + b$ into the equation for the parabola to obtain an equation for the points of intersection of the line and the parabola:

$$x^2 - 2p(mx + b) + a = 0$$

$$x^2 + (-2mp)x + (-2pb + a) = 0\,.$$

[1] We have been using the notation p_i for points; the use of p here might be confusing but it is the standard notation. The formal name for p is one-half the *latus rectum*.

The line will be a *tangent* to the parabola *if and only if* this quadratic equation has *exactly one* solution *if and only if* its discriminant is zero:

$$(-2mp)^2 - 4 \cdot 1 \cdot (-2pb + a) = 0\,,$$

which simplifies to:

$$m^2p^2 + 2pb - a = 0\,. \tag{7.5}$$

This is the quadratic equation with variable m for the slopes of tangents to the parabola. There are an infinite number of tangents because for each m, there is some b that makes the line a tangent by moving it up or down.[2]

To obtain the common tangents to both parabolas, the equations for the two parabolas have two unknowns and can be solved for m and b.

Example

Parabola 1: focus $(0,4)$, directrix $y = 2$, vertex $(0,3)$, $p = 2$, $a = 2 \cdot 2 \cdot 3 = 12$. The equation of the parabola is:

$$x^2 - 2 \cdot 2y + 12 = 0\,.$$

Substituting into Equation 7.5 and simplifying:

$$m^2 + b - 3 = 0\,.$$

Parabola 2: focus $(0,-4)$, directrix $y = -2$, vertex $(0,-3)$, $p = -2$, $a = 2 \cdot -2 \cdot -3 = 12$. The equation of the parabola is:

$$x^2 - 2 \cdot (-2)y + 12 = 0\,.$$

Substituting into Equation 7.5 and simplifying:

$$m^2 - b - 3 = 0\,. \tag{7.6}$$

The solutions of the two equations:

$$m^2 + b - 3 = 0$$

$$m^2 - b - 3 = 0\,,$$

are $m = \pm\sqrt{3} \approx \pm 1.73$ and $b = 0$. There are two common tangents that are the folds:

$$y = \sqrt{3}x\,, \quad y = -\sqrt{3}x\,.$$

Example

Parabola 1 is unchanged.

Parabola 2: focus $(0,-6)$, directrix $y = -2$, vertex $(0,-4)$, $p = -4$, $a = 2 \cdot -4 \cdot -4 = 32$. The equation of the parabola is:

$$x^2 - 2 \cdot (-4)y + 32 = 0\,.$$

[2]Except of course for a line parallel to the axis of symmetry.

Substituting into Equation 7.5 and simplifying:

$$2m^2 - b - 4 = 0\,.$$

The solutions of the two equations (using Equation 7.6 for the first parabola):

$$m^2 + b - 3 = 0$$

$$2m^2 - b - 4 = 0\,,$$

are $m = \pm\sqrt{\frac{7}{3}} \approx \pm 1.53$ and $b = \frac{2}{3}$. There are two common tangents that are folds:

$$y = \sqrt{\frac{7}{3}}x + \frac{2}{3}\,, \quad y = -\sqrt{\frac{7}{3}}x + \frac{2}{3}\,.$$

Example

Let us now define a parabola whose axis of symmetry is the x-axis.

Parabola 1 is unchanged.

Parabola 2: focus $(4, 0)$, directrix $x = 2$, vertex $(3, 0)$, $p = 2$, $a = 2 \cdot 2 \cdot 3 = 12$. The equation of the parabola is:

$$y^2 - 4x + 12 = 0\,.$$

This is an equation with y^2 and x instead of x^2 and y, so we can't use Equation 7.5 and we must perform the derivation again.

Substitute the equation for a line:

$$(mx + b)^2 - 4x + 12 = 0$$

$$m^2x^2 + (2mb - 4)x + (b^2 + 12) = 0\,,$$

set the discriminant equal to zero and simplify:

$$(2mb - 4)^2 \;-\; 4m^2(b^2 + 12) = 0$$

$$-3m^2 - mb + 1 = 0\,.$$

If we try to solve the two equations (using Equation 7.6 for the first parabola):

$$m^2 + b - 3 = 0$$

$$-3m^2 - mb + 1 = 0\,,$$

we obtain a cubic equation with variable m:

$$m^3 - 3m^2 - 3m + 1 = 0\,. \tag{7.7}$$

Since a cubic equation has at least one and at most three (real) solutions, there can be one, two or three common tangents. There can also be no common tangents if the two equations have no solution, for example, if one parabola is "contained" with another: $y = x^2$, $y = x^2 + 1$.

The formula for solving cubic equations is quite complicated, so I used a calculator on the internet and obtained three solutions:

$$m = 3.73\,,\ m = -1\,,\ m = 0.27\,.$$

Choosing $m = 0.27$, $b = 3 - m^2 = 2.93$, and the equation of the fold is:

$$y = 0.27x + 2.93\,.$$

From the form of Equation 7.7, we might guess that 1 or -1 is a solution:

$$1^3 - 3 \cdot 1^2 - 3 \cdot 1 + 1 = -4$$

$$(-1)^3 - 3 \cdot (-1)^2 - 3 \cdot (-1) + 1 = 0\,.$$

Divide Equation 7.7 by $m - (-1) = m + 1$ to obtain the quadratic equation $m^2 - 4m + 1$ whose roots are $2 \pm \sqrt{3} \approx 3.73, 0.27$.

Derivation of the equations of the reflections

We derive the position of the reflection $p'_1 = (x'_1, y'_1)$ of $p_1 = (x_1, y_1)$ around some tangent line l_t whose equation is $y = m_t x + b_t$. The derivation is identical for any tangent and for p_2.

To reflect p_1 around l_t, we find the line l_p with equation $y = m_p x + b_p$ that is perpendicular to l_t and passes through p_1:

$$y = -\frac{1}{m_t} x + b_p$$

$$y_1 = -\frac{1}{m_t} x_1 + b_p$$

$$y = \frac{-x}{m_t} + \left(y_1 + \frac{x_1}{m_t} \right)\,.$$

Next we find the intersection $p_t = (x_t, y_t)$ of l_t and l_p:

$$m_t x_t + b_t = \frac{-x_t}{m_t} + \left(y_1 + \frac{x_1}{m_t} \right)$$

$$x_t = \frac{\left(y_1 + \dfrac{x_1}{m_t} - b_t \right)}{\left(m_t + \dfrac{1}{m_t} \right)}$$

$$y_t = m_t x_t + b_t\,.$$

The reflection p'_1 is easy to derive because the intersection p_t is the midpoint between p_1 and its reflection p'_1:

$$x_t = \frac{x_1 + x'_1}{2}\,, \quad y_t = \frac{y_1 + y'_1}{2}$$

$$x'_1 = 2x_t - x_1\,, \quad y'_1 = 2y_t - y_1\,.$$

Example

$p_1 = (0,4)$, l_1 is $y = \sqrt{3}x$:

$$x_t = \frac{\left(4 + \frac{0}{\sqrt{3}} - 0\right)}{\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)} = \sqrt{3}$$

$$y_t = \sqrt{3}\sqrt{3} + 0 = 3$$

$$x_1' = 2x_t - x_1 = 2\sqrt{3} - 0 = 2\sqrt{3} \approx 3.46$$

$$y_1' = 2y_t - y_1 = 2 \cdot 3 - 4 = 2\,.$$

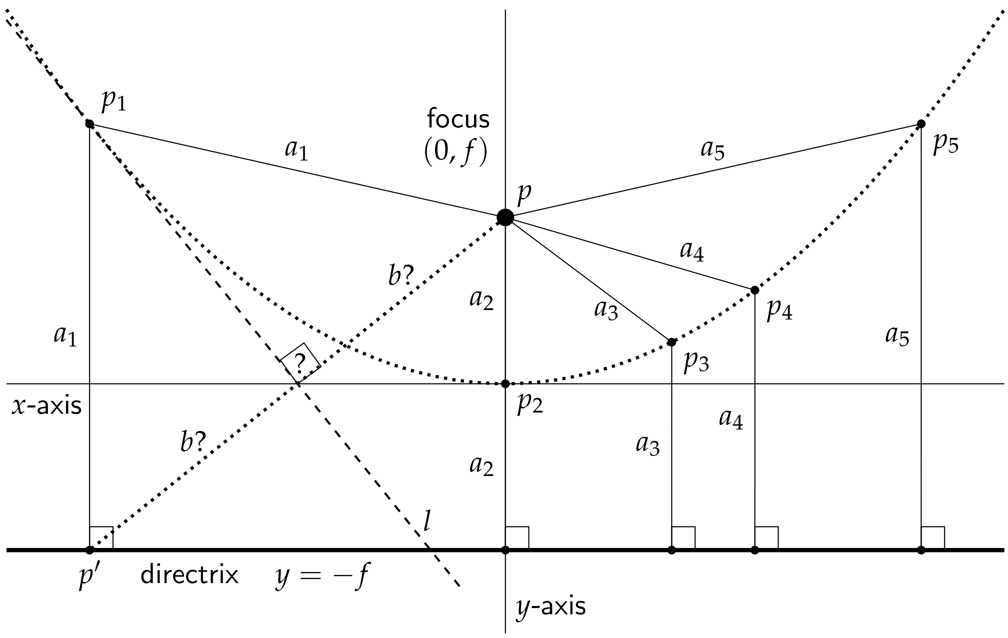
A fold is any tangent to a parabola

Students are usually introduced to parabolas as the graphs of second degree equations:

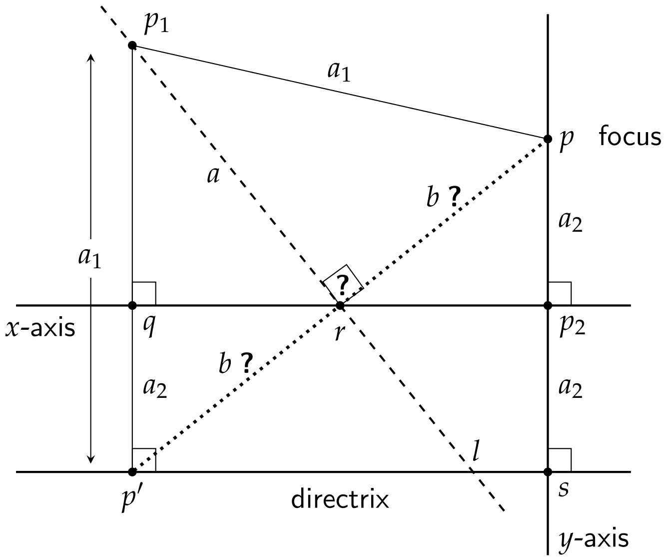
$$y = ax^2 + bx + c\,.$$

However, parabolas can be defined geometrically: given a point, the *focus*, and a line, the *directrix*, the locus of points equidistant from the focus and the directrix defines a parabola.

The following diagram shows the focus—the large dot at $p = (0, f)$, and the directrix—the thick line whose equation is $y = -f$. The resulting parabola is shown as a dotted curve. Its vertex p_2 is at the origin of the axes.



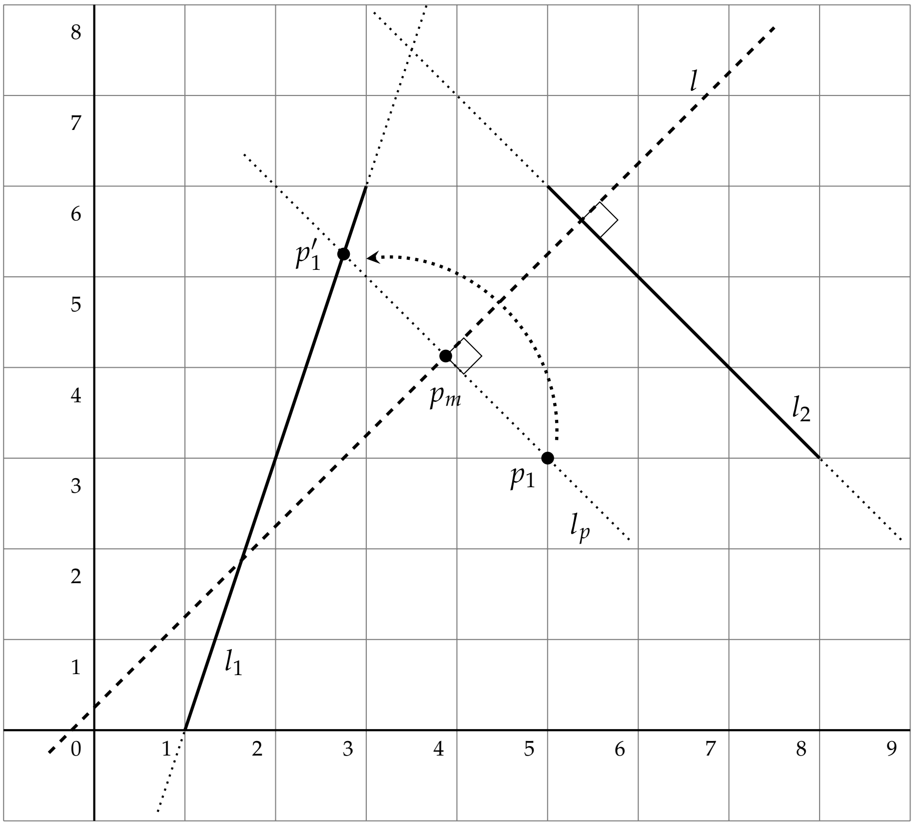
We have selected five points p_i, $i = 1, \ldots, 5$ on the parabola. Each point p_i is at a distance of a_i from both the focus and from the directrix. Drop a perpendicular from p_1 to the directrix and let p' be its intersection with the directrix. Since p_1 is on the parabola, $\overline{p'p_1} = \overline{p_1p} = a_1$. We claim that the tangent l to the parabola at p_1 (dashed line) is a fold that reflects p onto p'. We have to prove the l is the perpendicular bisector of $\overline{pp'}$. Let us extract a simplified diagram:



- The directrix is parallel to the x-axis, the focus p is on the y-axis and $\overline{p_1p'}$ is perpendicular to the directrix. Therefore, $\angle p'qr$ and $\angle pp_2r$ are right angles.
- $\overline{qp'}$ and $\overline{p_2s}$ are opposite sides of a rectangle, so $\overline{qp'} = \overline{p_2s}$, which in turn is equal to $\overline{pp_2}$ since p_2 is on the parabola and thus equidistant from p and s.
- $\angle qrp'$ and $\angle p_2rp$ are equal vertical angles.
- The right triangles $\triangle qrp'$ and $\triangle p_2rp$ have one acute angle equal and one side equal so they are congruent. Therefore, $\overline{p'r} = \overline{rp}$ and $\overline{p_1r}$ is the median of $\triangle pp_1p'$.
- p_1 is on the parabola so $\overline{pp_1} = \overline{p_1p'}$. Therefore, $\triangle pp_1p'$ is an isoceles triangle.
- In the isoceles triangle $\triangle pp_1p'$, the median $\overline{p_1r}$ is also the perpendicular bisector of $\overline{pp'}$.
- Therefore, l is a fold because it contains the line segment $\overline{p_1r}$.

7.7 Axiom 7

Axiom Given one point p_1 and two lines l_1 and l_2, there is a fold l that places p_1 onto l_1 and is perpendicular to l_2.



Derivation of the equation of the fold

Let $p_1 = (x_1, y_1)$, let l_1 be $y = m_1 x + b_1$ and let l_2 be $y = m_2 x + b_2$.

Since the fold l is perpendicular to l_2, and the line l_p containing $\overline{p_1 p_1'}$ is perpendicular to l, it follows that l_p parallel to l_2:

$$y = m_2 x + b_p \, .$$

l_p passes through p_1 so $y_1 = m_2 x_1 + b_p$ and its equation is:

$$y = m_2 x + (y_1 - m_2 x_1) \, .$$

$p_1' = (x_1', y_1')$, the reflection of p_1 around the fold l, is the intersection of l_1 and l_p:

$$m_1 x_1' + b_1 = m_2 x_1' + (y_1 - m_2 x_1)$$

$$x_1' = \frac{y_1 - m_2 x_1 - b_1}{m_1 - m_2}$$

$$y_1' = m_1 x_i' + b_1 \, .$$

The midpoint $p_m = (x_m, y_m)$ of l_p between p_1 and p_1' is on the fold l:

$$(x_m, y_m) = \left(\frac{x_1 + x_1'}{2}, \frac{y_1 + y_1'}{2} \right) .$$

The equation of the fold l is the perpendicular bisector of $\overline{p_1 p_1'}$. First compute the intercept of l which passes through p_m:

$$y_m = -\frac{1}{m_2} x_m + b_m$$

$$b_m = y_m + \frac{x_m}{m_2} .$$

The equation of the fold l is:

$$y = -\frac{1}{m_2} x + \left(y_m + \frac{x_m}{m_2} \right) .$$

Example

Let $p_1 = (5, 3)$, let l_1 be $y = 3x - 3$ and let l_2 be $y = -x + 11$.

$$x_1' = \frac{3 - (-1) \cdot 5 - (-3)}{3 - (-1)} = \frac{11}{4}$$

$$y_1' = 3 \cdot \frac{11}{4} + (-3) = \frac{21}{4}$$

$$p_m = \left(\frac{5 + \frac{11}{4}}{2}, \frac{3 + \frac{21}{4}}{2} \right) = \left(\frac{31}{8}, \frac{33}{8} \right) .$$

The equation of the fold l is:

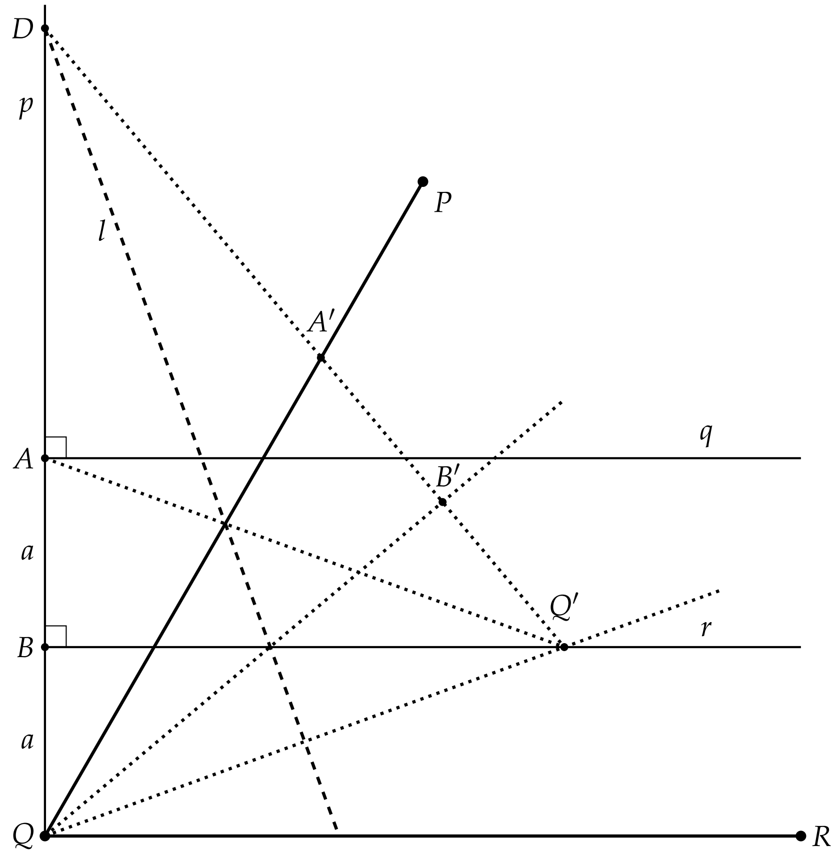
$$y = -\frac{1}{-1} \cdot x + \left(\frac{33}{8} + \frac{\frac{31}{8}}{-1} \right) = x + \frac{1}{4} .$$

Chapter 8
Trisecting an Angle

8.1 Abe's trisection of an angle

This construction is based upon the presentation in [12]. The second proof is based upon [1].

8.1.1 The construction

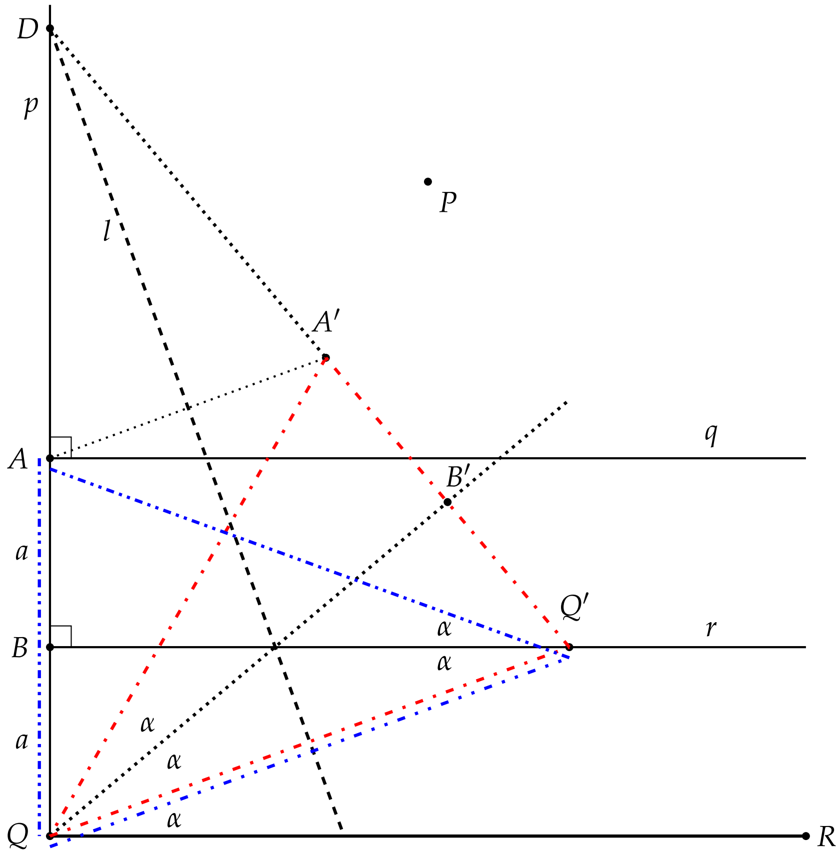


Given an acute angle $\angle PQR$, let p be the perpendicular to $\overline{QR}$ at Q. Let q be a perpendicular to p that intersects $\overline{PQ}$ at point A, and let r be the perpendicular to p at B that is halfway between Q and A.

Using Axiom 6, construct a fold l that places A at A' on $\overline{PQ}$ and Q at Q' on r. Let B' be the reflection of B around l.

Draw the lines $\overline{QB'}$ and QQ'. We claim that $\angle PQB'$, $\angle B'QQ'$ and $\angle Q'QR$ are a trisection of $\angle PQR$.

8.1.2 First proof



Since A', B', Q' are all reflections around the same line l of the points A, B, Q on one line DQ, they are all on one line $\overline{DQ'}$. By construction, $\overline{AB} = \overline{BQ}$, $\overline{BQ'}$ is perpendicular to AQ; $\overline{BQ'}$ is a common side, so $\triangle ABQ' \cong \triangle QBQ'$ by side-angle-side. Therefore, $\angle AQ'B = \angle QQ'B = \alpha$, since $\overline{Q'B}$ is the perpendicular bisector of the isoceles triangle $\triangle AQ'Q$.

By alternating interior angles, $\angle Q'QR = \angle QQ'B = \alpha$.

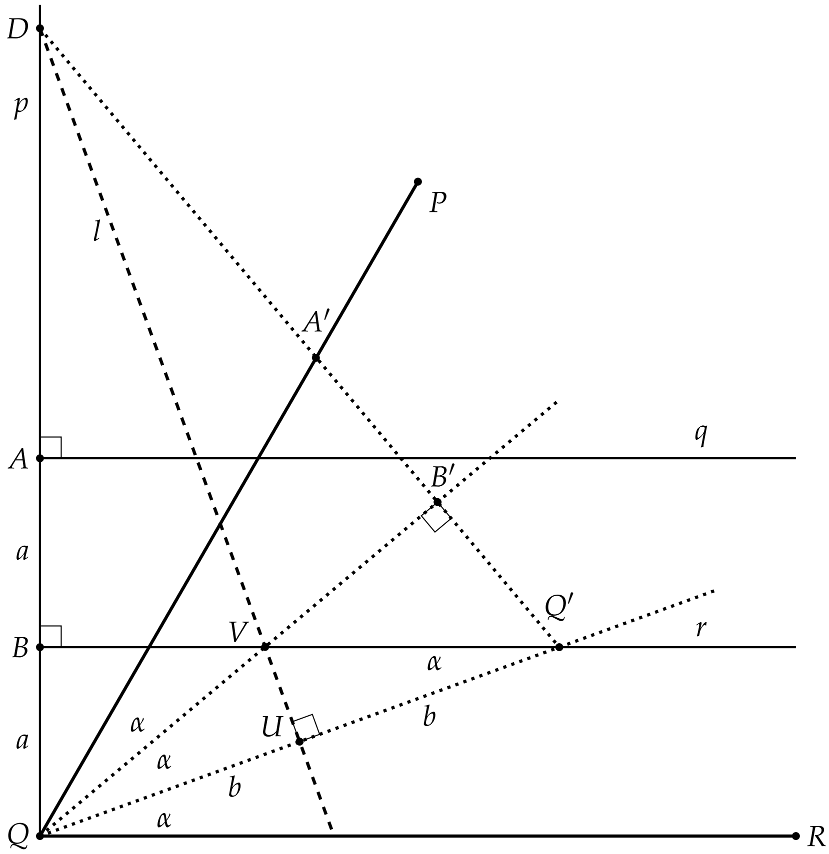
By reflection, $\triangle AQ'Q \cong \triangle A'QQ'$.[1]

> The fold l is the perpendicular bisector of both $\overline{AA'}$ and $\overline{QQ'}$; drop perpendiculars from A and A' to $\overline{QQ'}$; then $\overline{AQ} = \overline{A'Q'}$ follows by congruent right triangles. $\overline{AA'Q'Q}$ is an isoceles trapezoid so its diagonals are equal $\overline{AQ'} = \overline{A'Q}$.

Therefore, $\overline{QB'}$, the reflection of $\overline{Q'B}$, is the perpendicular bisector of an isoceles triangle and $\angle A'QB' = \angle B'QQ' = \angle QQ'B = \angle Q'QR = \alpha$.

[1]The two triangles have been emphasized using different patterns of dashes and dots, as well as using color.

8.1.3 Second proof

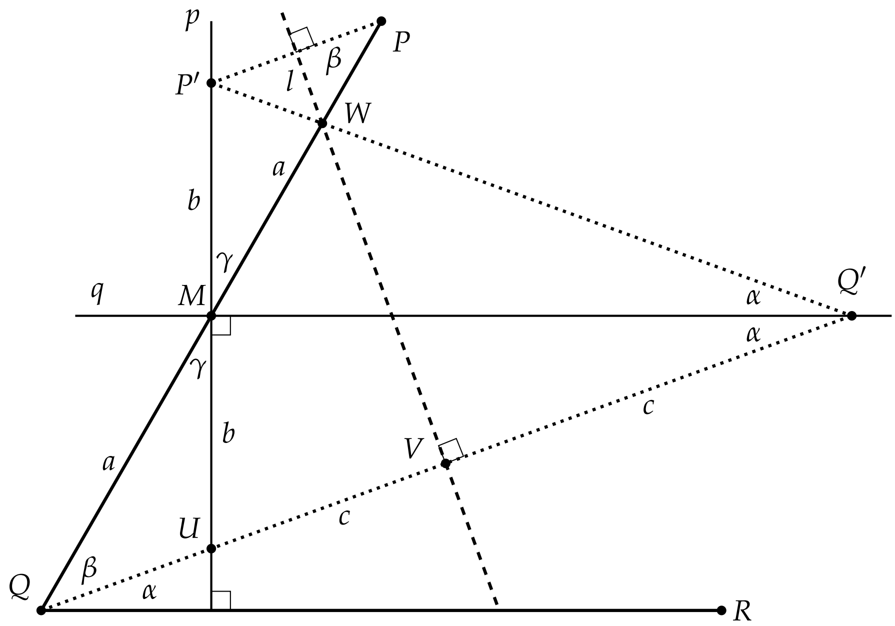


Since l is a fold, it is the perpendicular bisector of $\overline{QQ'}$. Denote the intersection of l with $\overline{QQ'}$ by U, and its intersection with $\overline{QB'}$ by V. $\triangle VUQ \cong \triangle VUQ'$ by side-angle-side since $\overline{VU}$ is a common side, the angles at U are right angles and $\overline{QU} = \overline{Q'U} = b$. Therefore, $\angle VQU = \angle VQ'U = \alpha$ and then $\angle Q'QR = \angle VQ'U = \alpha$ by alternating interior angles.

As in Proof 1, A', B', Q' are all reflections around l, so they are all on one line $\overline{DQ'}$, and $\overline{A'B'} = \overline{AB} = \overline{BQ} = \overline{B'Q'} = a$. Then $\triangle A'B'Q \cong \triangle Q'B'Q$ and $\angle A'QB' = \angle Q'QB' = \alpha$.

8.2 Martin's trisection of an angle

8.2.1 The construction



Given the acute angle $\angle PQR$, let M be the midpoint of $\overline{PQ}$. Construct p the perpendicular to $\overline{QR}$ through M and construct q perpendicular to p through M. q is parallel to $\overline{QR}$.

Using Axiom 6, construct a fold l that places P at P' on p and Q at Q' on q. More than one fold may be possible; choose the one that intersects $\overline{PM}$.

Draw the lines $\overline{PP'}$ and $\overline{QQ'}$. Denote the intersection of $\overline{QQ'}$ with p by U and its intersection with l by V. Denote the intersection of $\overline{PQ}$ and $P'Q'$ with l by W.[2]

8.2.2 Proof

$\triangle QMU \cong \triangle PMP'$ by angle-side-angle: $\angle P'PM = \angle UQM = \beta$ by alternate interior angles; $\overline{QM} = \overline{MP} = a$ since M is the midpoint of $\overline{PQ}$; $\angle QMU = \angle PMP'$ are vertical angles. Therefore, $\overline{P'M} = \overline{MU} = b$.

$\triangle P'MQ' \cong \triangle UMQ'$ by side-angle-side: we have shown that $\overline{P'M} = \overline{MU} = b$; the angles at M are right angles; $\overline{MQ'}$ is a common side. Since the altitude of the isoceles triangle $\triangle P'Q'U$ is the bisector of $\angle P'Q'U$, so $\angle P'Q'M = \angle UQ'M = \alpha$.

$\triangle QWV \cong \triangle Q'WV$ by side-angle-side: $\overline{QV} = \overline{VQ'} = c$; the angles at V are right angles since the fold is the perpendicular bisector of $\overline{QQ'}$; $\overline{VW}$ is a common side. Therefore, $\angle WQV = \beta = \angle WQ'V = 2\alpha$. By alternate interior angles $\angle Q'QR = \angle MQ'Q = \alpha$. We have $\angle PQR = \beta + \alpha = 2\alpha + \alpha = 3\alpha$ so $\angle Q'QR$ is one-third of $\angle PQR$.

[2] It is not immediate that both $\overline{PQ}$ and $P'Q'$ intersect l at the same point. $\triangle PP'W \sim \triangle QQ'W$ so the altitudes divide the vertical angles $\angle PWP'$, $\angle QWQ'$ similarly and thus must be on the same line.

Chapter 9
Doubling a Cube

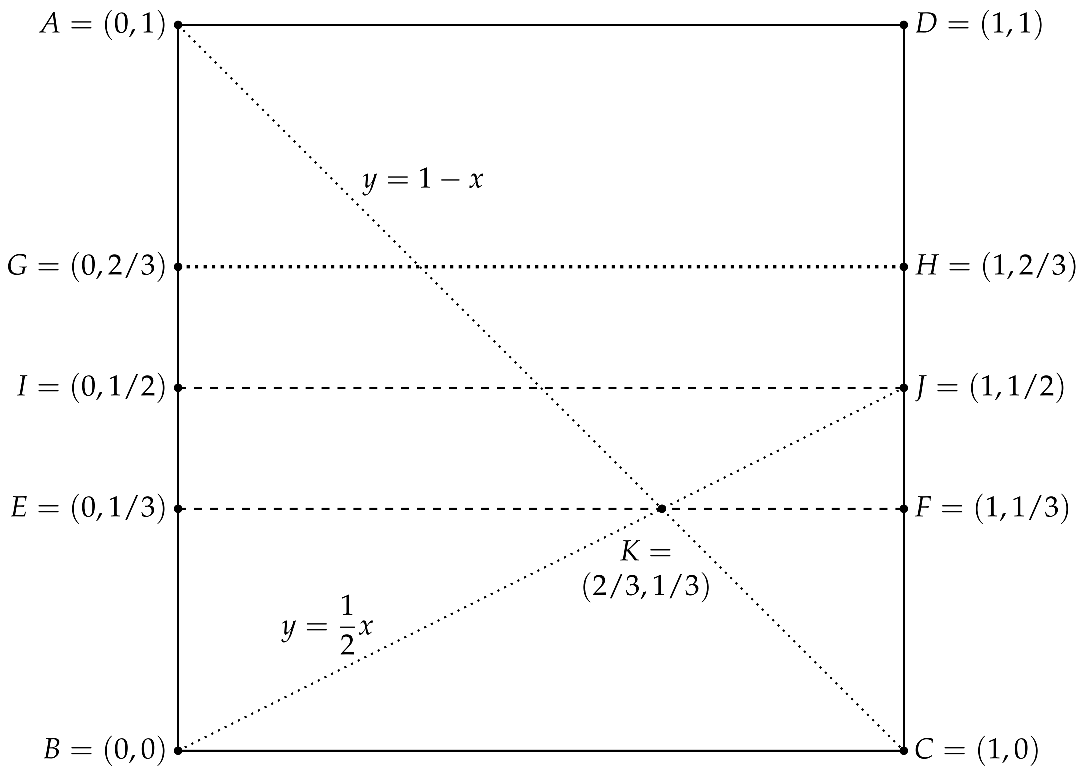
9.1 Messer's doubling of a cube

To double a cube we need to construct $\sqrt[3]{2}$. This construction is based on [12, 10].

9.1.1 Dividing a length into thirds

Lang [9] shows efficient constructs for rational fractions of the length of the side of a square (piece of paper). Here, we need to divide the side of the square into thirds.

First, fold the square in half to locate the point $J = (1, 1/2)$. Next, draw the lines $\overline{AC}$ and $\overline{BJ}$.



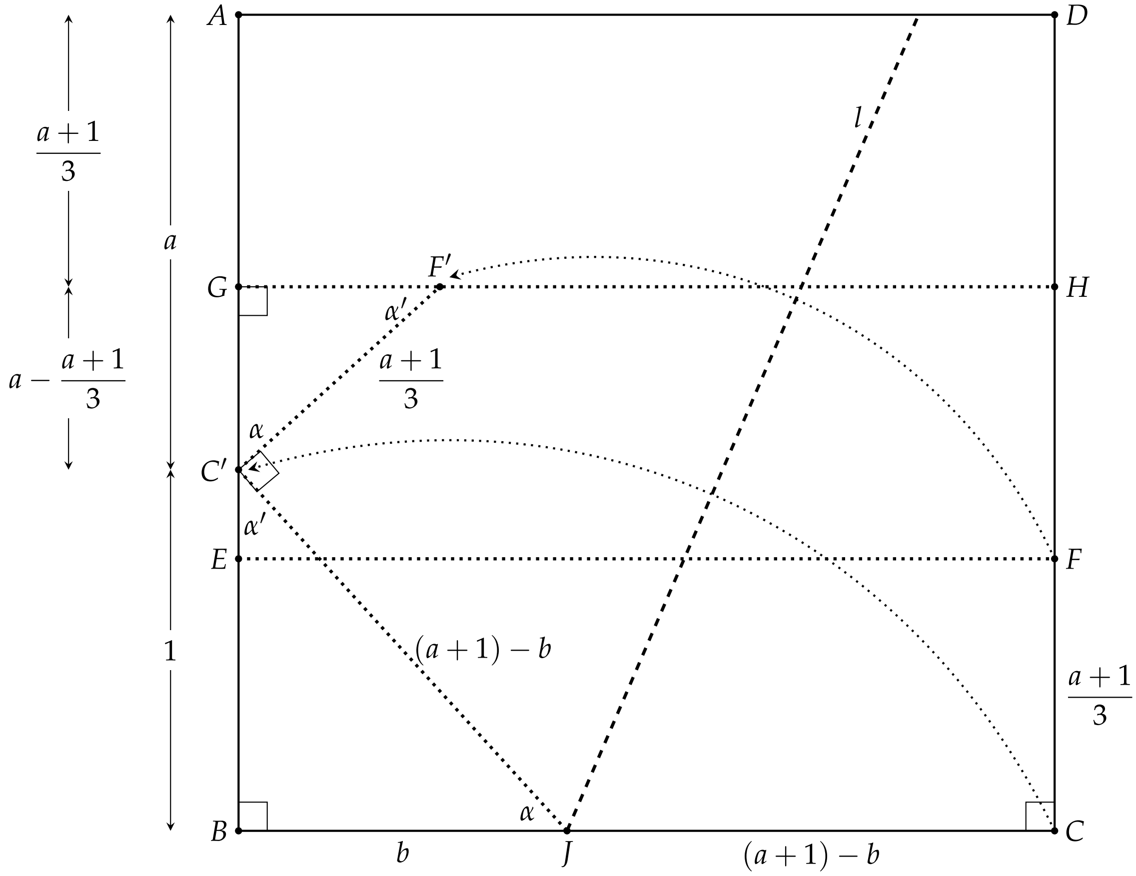
The coordinates of their point of intersection K are obtained by solving the two equations:

$$\begin{aligned} y &= 1 - x \\ y &= \frac{1}{2}x\,. \end{aligned}$$

The result is $x = 2/3, y = 1/3$.

Construct the line $\overline{EF}$ perpendicular to $\overline{AB}$ through K, and construct the reflection $\overline{GH}$ of $\overline{BC}$ around $\overline{EF}$. The side of the square has been divided into thirds.

9.1.2 Building $\sqrt[3]{2}$



Label the side of the square by $a+1$. The construction will show that $a = \sqrt[3]{2}$.

Using Axiom 6 place C at C' on $\overline{AB}$ and F at F' on $\overline{GH}$. Denote by J the point intersection of the fold with $\overline{BC}$ and denote by b the length of $\overline{BJ}$. The length of $\overline{JC}$ is $(a+1)-b$.

When the fold is performed, the line segment $\overline{JC}$ is reflected onto the line segment $\overline{JC'}$ of the same length, and $\overline{CF}$ is folded onto the line segment $\overline{C'F'}$ of the same length. A simple computation shows that the length of $\overline{GC'}$ is:

$$a - \frac{a+1}{3} = \frac{2a-1}{3}\,. \tag{9.1}$$

Finally, since $\angle FCJ$ is a right angle, so is $\angle F'C'J$.

$\triangle C'BJ$ is a right triangle so by Pythagoras' theorem:

$$\begin{aligned}
1^2 + b^2 &= ((a+1)-b)^2 \\
a^2 + 2a - 2(a+1)b &= 0 \\
b &= \frac{a^2+2a}{2(a+1)}\,.
\end{aligned}$$

$\angle GC'F' + \angle F'C'J + \angle JC'B = 180°$ since they form the straight line $\overline{GB}$. Denote $\angle GC'F'$ by α.

$$\angle JC'B = 180° - \angle F'C'J - \angle GC'F' = 180° - 90° - \alpha = 90° - \alpha\,,$$

which we denote by α'. The triangles $\triangle C'BJ$, $\triangle F'GC'$ are right triangles, so $\angle C'JB = \alpha$ and $\angle C'F'G = \alpha'$. Therefore, the triangles are similar and using Equation 9.1 we have:

$$\frac{b}{(a+1)-b} = \frac{\dfrac{2a-1}{3}}{\dfrac{a+1}{3}}\,.$$

Substituting for b:

$$\begin{aligned}
\frac{\dfrac{a^2+2a}{2(a+1)}}{(a+1)-\dfrac{a^2+2a}{2(a+1)}} &= \frac{2a-1}{a+1} \\[2em]
\frac{a^2+2a}{a^2+2a+2} &= \frac{2a-1}{a+1}\,.
\end{aligned}$$

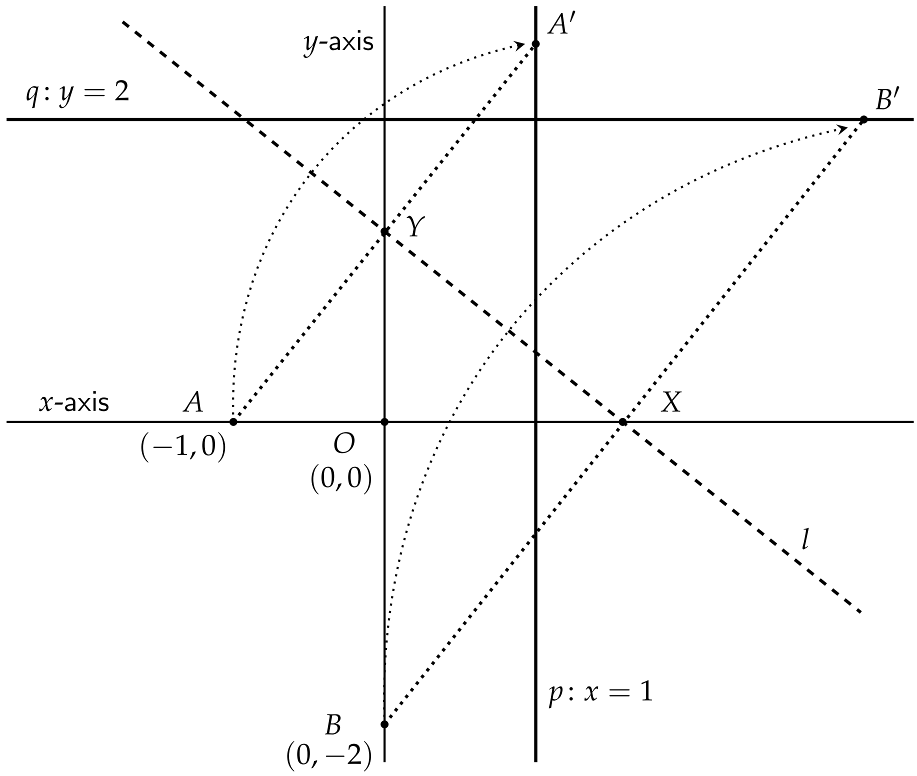
Simplifying results in $a^3 = 2$ and $a = \sqrt[3]{2}$.

9.2 Beloch's doubling of a cube

Margharita P. Beloch formalized Axiom 6 (Section 7.6) and showed that it could be used to solve cubic equations. Here we give her construction for doubling the cube. The solution of cubic equations is discussed in Chapters 10, 11.

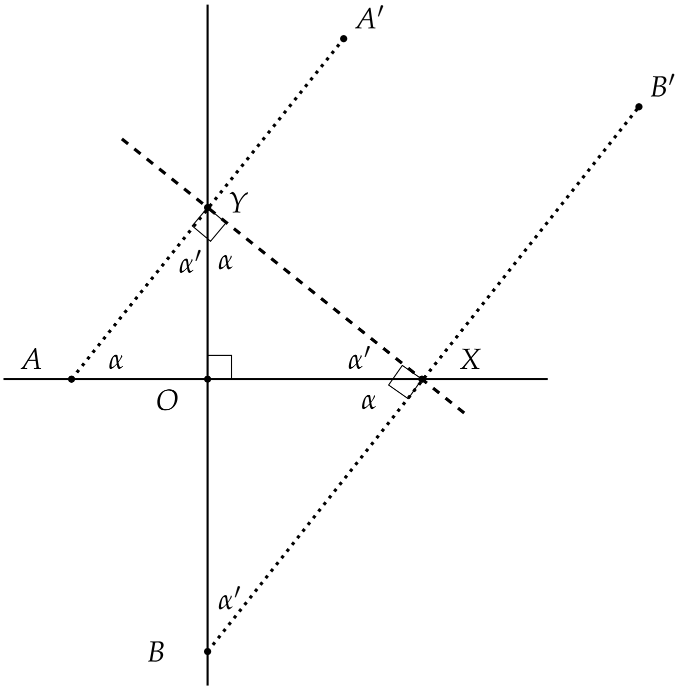
9.2.1 The construction

Place point A at $(-1, 0)$ and point B at $(0, -2)$. Let p be the line with equation $x = 1$ and let q be the line with equation $y = 2$. Using Axiom 6 construct a fold l that places A at A' on p and B at B' on q. Denote the intersection of the fold and the y-axis by Y and the intersection of the fold and x-axis by X.



9.2.2 Proof

Let us extract a simplified diagram:



The fold is the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$. Therefore, $\angle AYX$ and $\angle YXB$ are right angles and $\overline{AA'}$ is parallel to $\overline{BB'}$. By alternate interior angles $\angle YAO = \angle BXO = \alpha$. If an acute angle in a right triangle is α, the other acute angle must be $90° - \alpha$, which we denote α'. The labeling of the angles in all the triangles in the diagram follows immediately.

We have three similar triangles $\triangle AOY \sim \triangle YOX \sim \triangle XOB$. $\overline{OA} = 1$, $\overline{OB} = 2$ are given, so:

$$\frac{\overline{OY}}{\overline{OA}} = \frac{\overline{OX}}{\overline{OY}} = \frac{\overline{OB}}{\overline{OX}}$$

$$\frac{\overline{OY}}{1} = \frac{\overline{OX}}{\overline{OY}}$$

$$\overline{OY}^2 = \overline{OX}$$

$$\frac{\overline{OY}}{1} = \frac{2}{\overline{OX}}$$

$$\overline{OY}^2 = \overline{OX} = \frac{2}{\overline{OY}},$$

resulting in $\overline{OY}^3 = 2$ and $\overline{OY} = \sqrt[3]{2}$.

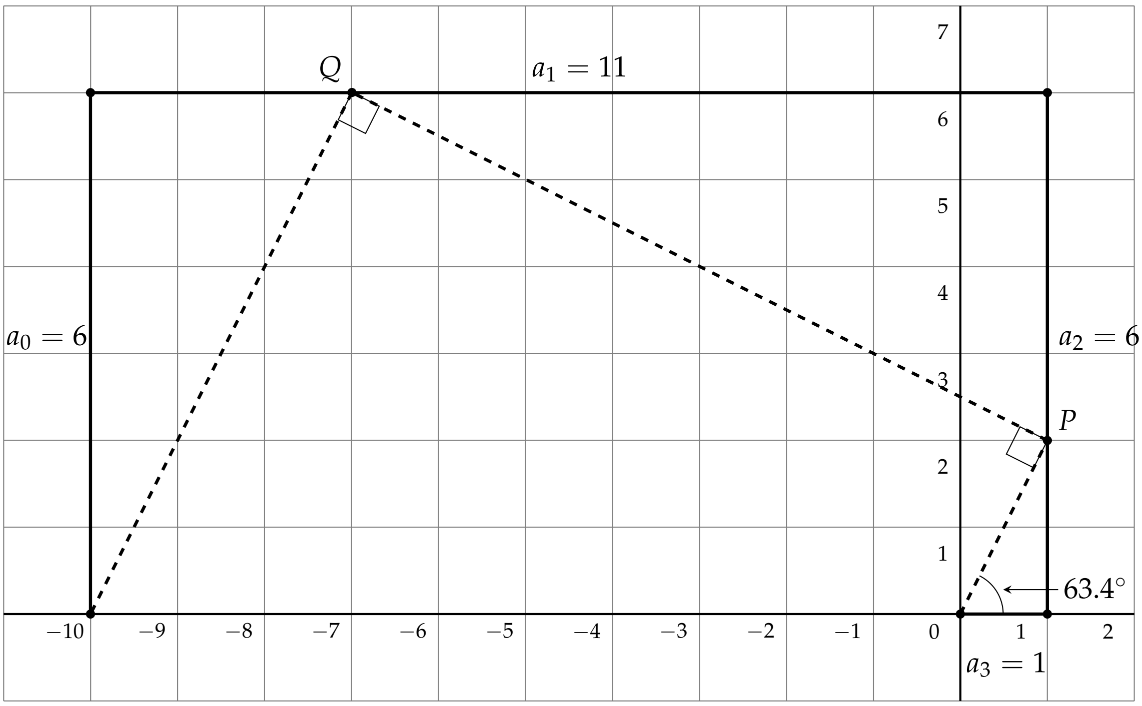
Chapter 10
Lill's Method for Finding Roots

10.1 Magic

Construct a path consisting of four line segments $\{a_3, a_2, a_1, a_0\}$ of lengths:

$$\{a_3 = 1, a_2 = 6, a_1 = 11, a_0 = 6\},$$

starting from the origin, first along the positive direciton of the x-axis and turning 90° counterclockwise between segments. Construct a second path as follows: draw a line from the origin at an angle of 63.4° and mark its intersection with a_2 by P. Turn left 90°, draw a line and and mark its intersection with a_1 by Q. Turn left 90° once again, draw a line and note that it intersects the end of the first path at $(-10, 0)$.



Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = x^3 + 6x^2 + 11x + 6$. Compute $\tan 63.4° = 2$, the tangent of the angle at the start of the second path. Then:

$$p(-\tan 63.4°) = (-2)^3 + 6(-2)^2 + 11(-2) + 6 = 0\,.$$

Congratulations! You have found a root of the cubic polynomial $x^3 + 6x^2 + 11x + 6$.

10.2 Introduction

This example demonstrates a method discovered by Eduard Lill in 1867 for graphically finding (or more accurately, verifying) the real roots of any polynomial [3, 7, 16]. We limit the presentation to cubic polynomials. Lill's method has seen renewed interest because it can be implemented using origami, as we shall see in Chapter 11.

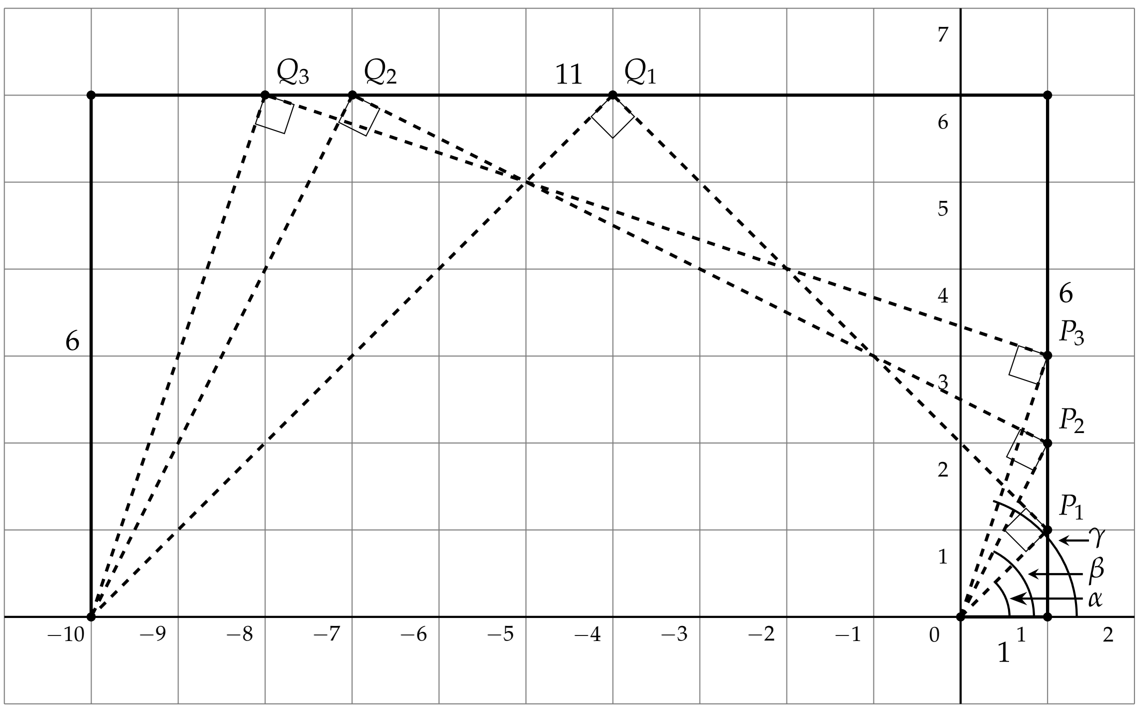
In Sections 10.3–10.4 we continue the initial example to find additional roots and to show that if an angle α is such that $(-\tan\alpha)$ is *not* a root, then the construction doesn't work. Section 10.5 presents the full specification of Lill's method. Special cases of the method are demonstrated by the examples in Sections 10.6–10.8. Since Lill's method can find a real root of any cubic polynomial, it can be used to trisect an angle. By computing $\sqrt[3]{2}$ as a root of $x^3 - 2$, it can double a cube (Section 10.9). Section 10.10 gives a proof that Lill's method can find the real roots of any cubic polynomial. The proof for arbitrary polynomials is similar.

10.3 Multiple roots

Let us continue the example above. The polynomial $p(x) = a_3x^3 + 6x^2 + 11x + 6$ has three roots $-1, -2, -3$. Computing the arc tangent of the negation of the roots gives:

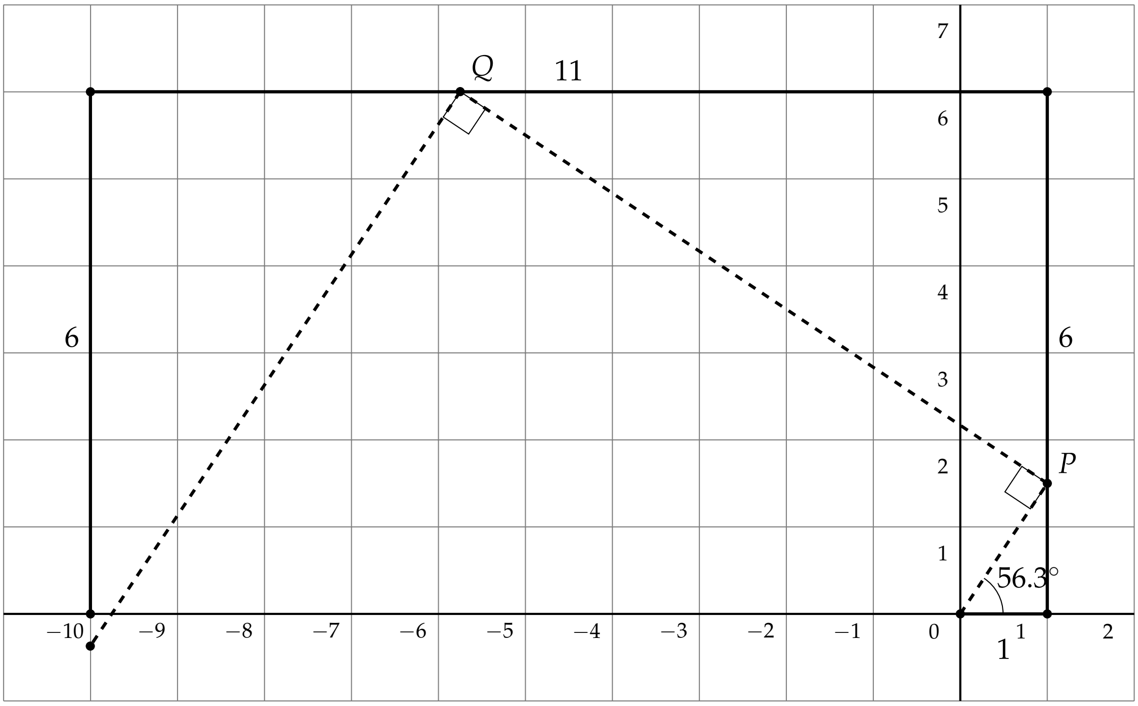
$$\alpha = -\tan^{-1}(-1) = 45°, \quad \beta = -\tan^{-1}(-2) = 63.4°, \quad \gamma = -\tan^{-1}(-3) = 71.6°\,.$$

In the diagram below we see that for each of the three angles, the second path intersects the end of the first path.



10.4 Paths that do not lead to roots

Perhaps the second path intersects the end of the first path for *any* initial angle, for example, $56.3°$. In the following diagram, the second path intersects the extension of the line segment for the coefficient a_0, but not at $(-10, 0)$, the end of the first path. We conclude that $-\tan 56.3° = -1.5$ is *not* a root of the equation.



10.5 Specification of Lill's method

Examining the examples below will help understand the details.

- Start with an arbitrary cubic polynomial: $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.
- Construct the first path as follows:
  - For each coefficient a_3, a_2, a_1, a_0 (in that order) draw a line segment starting at the origin $O = (0, 0)$ in the positive direction of the x-axis. Turn 90° counterclockwise between each segment.
- Construct the second path as follows:
  - We use the symbol for a coefficient a_i to also denote the corresponding side of the first path.
  - Construct a line from O at an angle of θ with the positive x-axis that intersects a_2 at point P.
  - Turn $\pm 90°$ and construct a line from P that intersects a_1 at Q.
  - Turn $\pm 90°$ and construct a line from Q that intersects a_0 at R.
  - If R is the end point of the first path, then $-\tan\theta$ is a root of $p(x)$.
- Special cases:
  - When drawing the line segments of the first path, if a coefficient is negative, draw the line segment *backwards*.
  - When drawing the line segments of the first path, if a coefficient is zero, do not draw a line segment but continue with the next $\pm 90°$ turn.
- Notes:
  - "Intersects a_i" means "intersects the line that contains the line segment a_i".
  - When building the second path, choose to turn left or right by 90° so that there is an intersection with the next segment of the first path.

10.6 Negative coefficients

Section 7.6 gave an example of the use of Axiom 6 that resulted in the polynomial $p(x) = x^3 - 3x^2 - 3x + 1$ with negative coefficients.

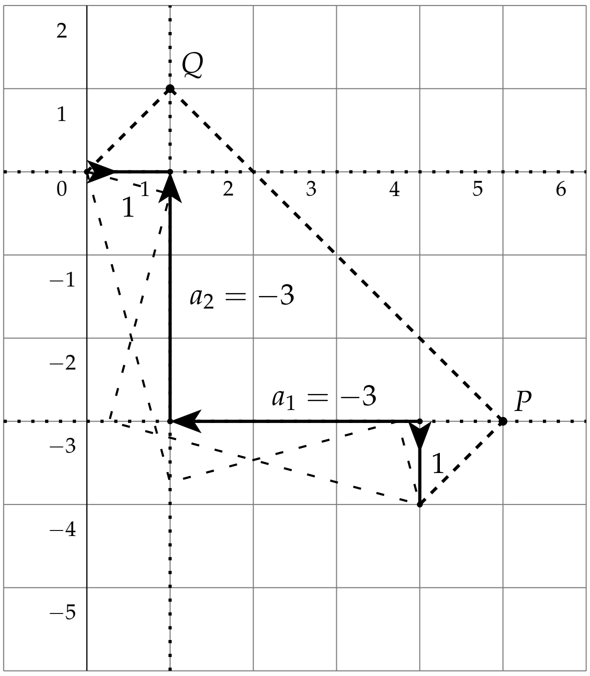
We start by drawing a segment of length 1 to the right. Next we turn 90° to face up, but the coefficient is negative, so we draw a segment of length 3 *down*. After turning 90° to the left, the coefficient is again negative, so we draw a segment of length 3 to the right. Finally, we turn down and draw a segment of length 1.

We start the second path with a line angled 45° with the x-axis. It intersects the *extension* of the line segment for a_2 at $(1, 1)$. Turning $-90°$ (to the right), the line intersects the *extension* of the line segment for a_1 at $(5, -3)$. Turning $-90°$ again, the line intersects the end of the first path at $(4, -4)$.

Since $-\tan 45° = -1$, a real root of the polynomial is -1:

$$p(-1) = (-1)^3 - 3(-1)^2 - 3(-1) + 6 = 0\,.$$

The loosely dashed lines in the diagram will be discussed in Section 10.8.



10.7 Zero coefficients

a_2, the coefficient of the x^2 term in the polynomial $x^3 - 7x - 6 = 0$, is zero. For a zero coefficient, we "draw" a line segment of length 0, that is, we do not draw a line, but we still make the $\pm 90°$ turn before "drawing" it, as indicated by the arrow pointed up at point $(1,0)$ in the diagram. Next make an additional turn and draw a line of length -7, that is, of length 7 backwards, to point $(8,0)$. Finally, turn again and draw a line of length -6 to point $(8,6)$.

There are three second paths that intersect the end of the first path. They start with angles of:

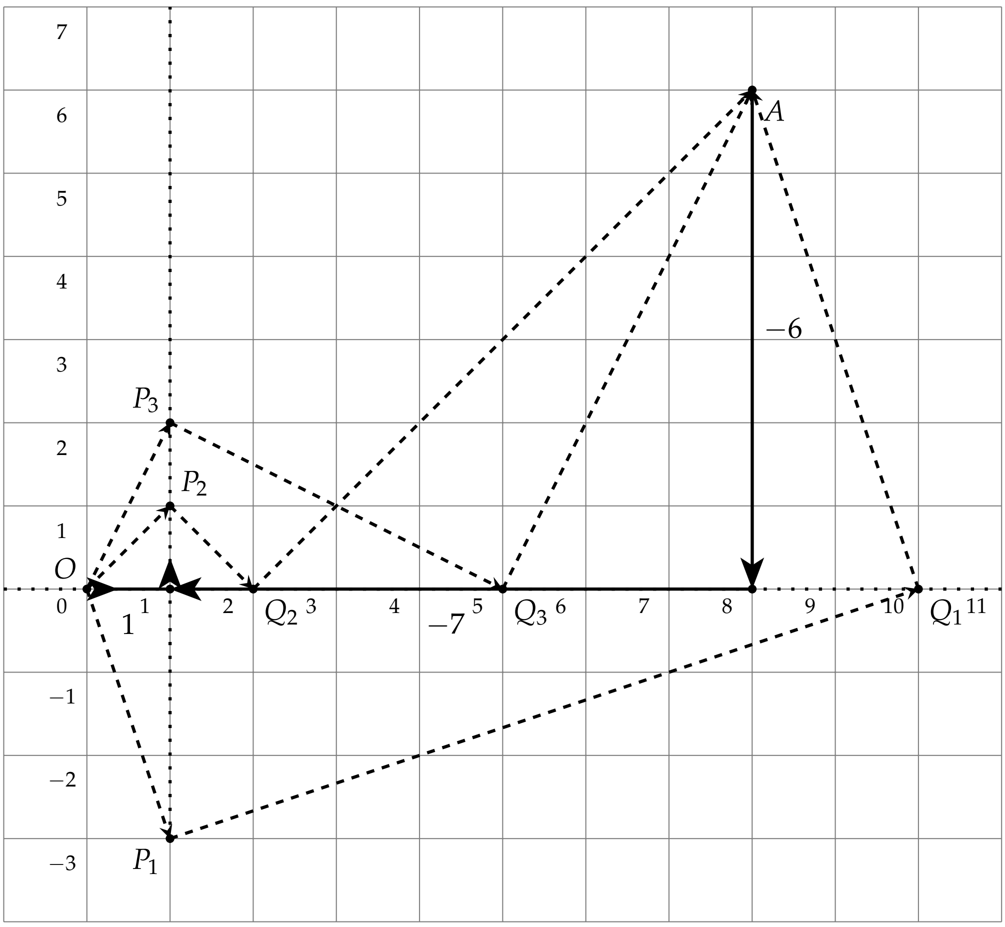
$$\alpha = 45°, \quad \beta = 63.4°, \quad \gamma = -71.6°\,.$$

We conclude that there are three real roots:

$$-\tan 45° = -1, \quad -\tan 63.4° = -2, \quad -\tan(-71.6°) = 3\,.$$

Check:

$$(x+1)(x+2)(x-3) = x^3 - 7x - 6\,.$$

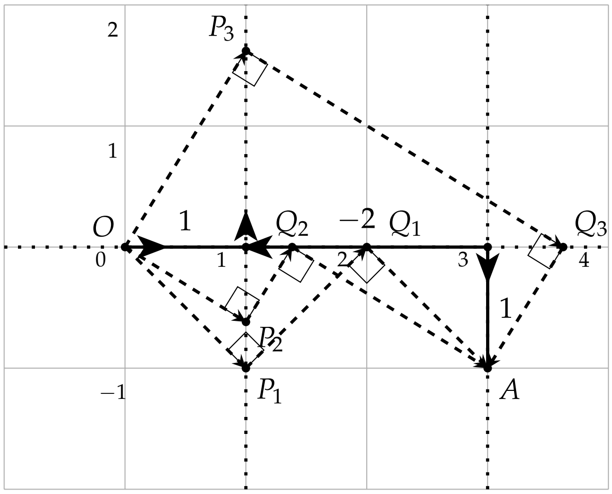


10.8 Non-integer roots

Consider the polynomial $p(x) = x^3 - 2x + 1$. The first segment is from $(0,0)$ to $(1,0)$ and turns up. The coefficient of x^2 is zero so no segment is drawn and turns left. The next coefficient is negative so the segment it goes backwards from $(1,0)$ to $(3,0)$ and turns right. Finally, a segment is drawn from $(3,0)$ to $(3,-1)$. Clearly, 1 is a root of $p(x)$ and since $-\tan^{-1}(-45°) = 1$, there is a path $\overline{OP_1Q_1A}$.

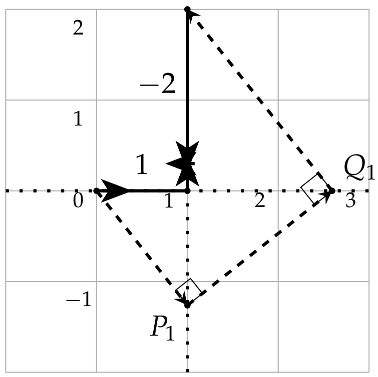
If we divide $p(x)$ by $x-1$, we obtain the quadratic polynomial $x^2 + x - 1$ whose roots are $\frac{-1\pm\sqrt{5}}{2} \approx 0.62,\ -1.62$. There are two additional second paths: one starting at $-31.8°$ since $-\tan^{-1} 0.62 = -31.8°$, and one starting at $58.3°$ since $-\tan^{-1} 1.62 = 58.3°$.

Similarly, the polynomial in Section 10.6 has roots $2 \pm \sqrt{3} \approx 3.73, 0.27$. The corresponding angles are $-75°$ and $-15°$, because $-\tan(-75°) \approx 3.73$ and $-\tan(-15°) \approx 0.27$.



10.9 The cube root of two

$\sqrt[3]{2}$ is a root of the cubic polynomial $x^3 - 2$. In the first path, we turn left twice without drawing any line segments, because a_2 and a_1 are both zero. Then we turn left again (to face down) and draw backwards because $a_0 = -2$ is negative. The first segment of the second path is drawn at an angle of $-51.6°$ and $-\tan(-51.6°) \approx 1.26 \approx \sqrt[3]{2}$.



10.10 Proof of Lill's method

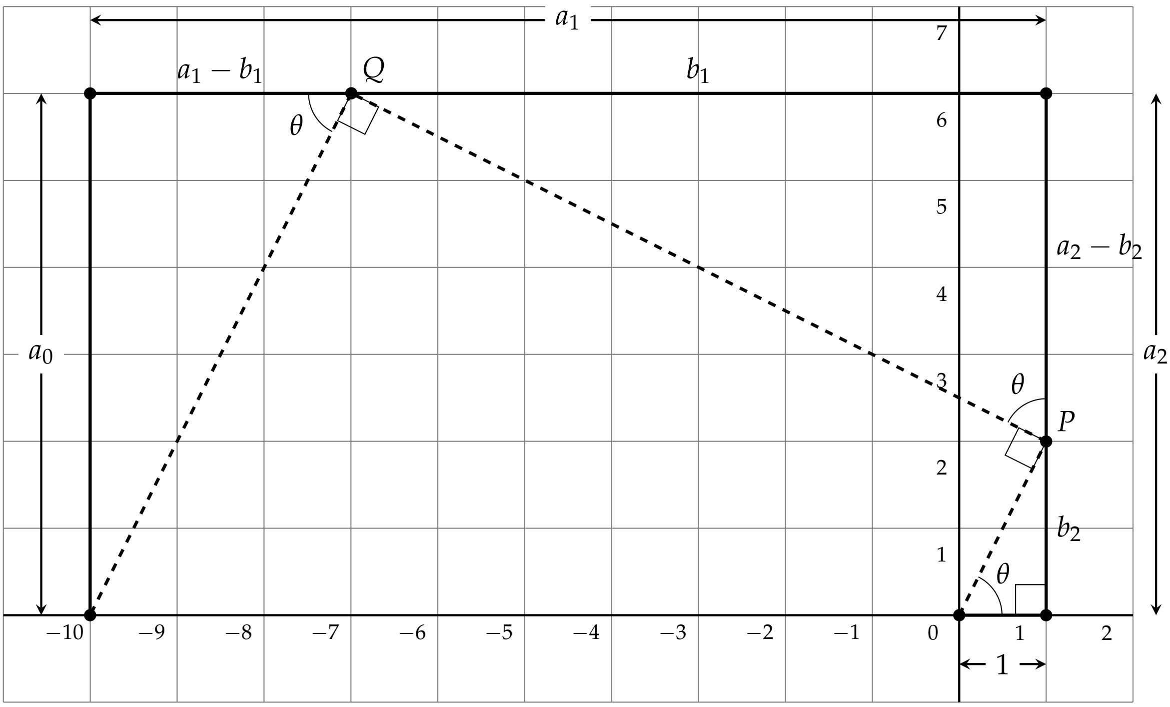
We limit ourselves to monic cubic polynomials $p(x) = x^3 + a_2x^2 + a_1x + a_0$.[1] In the diagram below, segments of the first path are labeled with coefficients and with $b_2, b_1, a_2 - b_2, a_1 - b_1$. Since the sum of the angles of a triangle is $180°$, in a right triangle if one acute angle is θ, the other is $90° - \theta$. Therefore, the angle above P and the angle to the left of Q are equal to θ. We now derive a sequence of formulas for $\tan\theta$:

$$\begin{aligned}
\tan\theta &= \frac{b_2}{1} = b_2 \\
\tan\theta &= \frac{b_1}{a_2 - b_2} = \frac{b_1}{a_2 - \tan\theta} \\
b_1 &= \tan\theta(a_2 - \tan\theta) \\
\tan\theta &= \frac{a_0}{a_1 - b_1} = \frac{a_0}{a_1 - \tan\theta(a_2 - \tan\theta)}\,.
\end{aligned}$$

Simplifying the last equation gives:

$$\begin{aligned}
(\tan\theta)^3 - a_2(\tan\theta)^2 + a_1(\tan\theta) - a_0 &= 0 \\
-(\tan\theta)^3 + a_2(\tan\theta)^2 - a_1(\tan\theta) + a_0 &= 0 \\
(-\tan\theta)^3 + a_2(-\tan\theta)^2 + a_1(-\tan\theta) + a_0 &= 0\,.
\end{aligned}$$

We conclude that $-\tan\theta$ is a real root of $p(x) = x^3 + a_2x^2 + a_1x + a_0$.



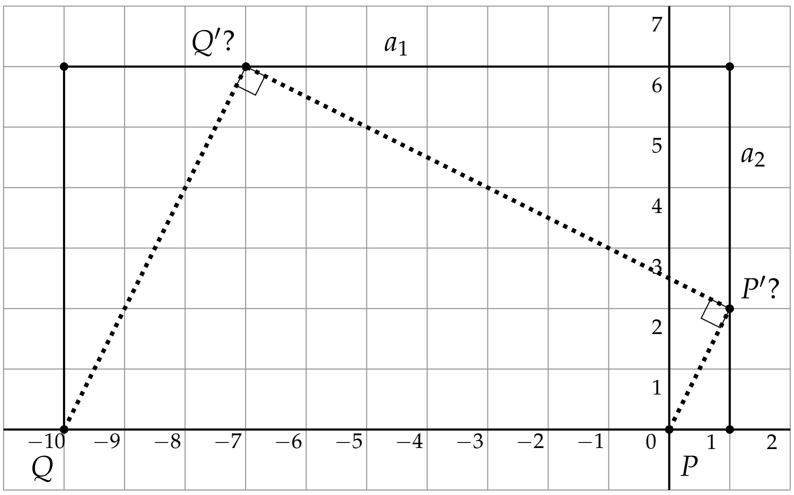
[1] If the polynomial is not monic, divide it by a_3 and the resulting monic polynomial has the same roots.

Chapter 11
Beloch's Fold and Beloch's Square

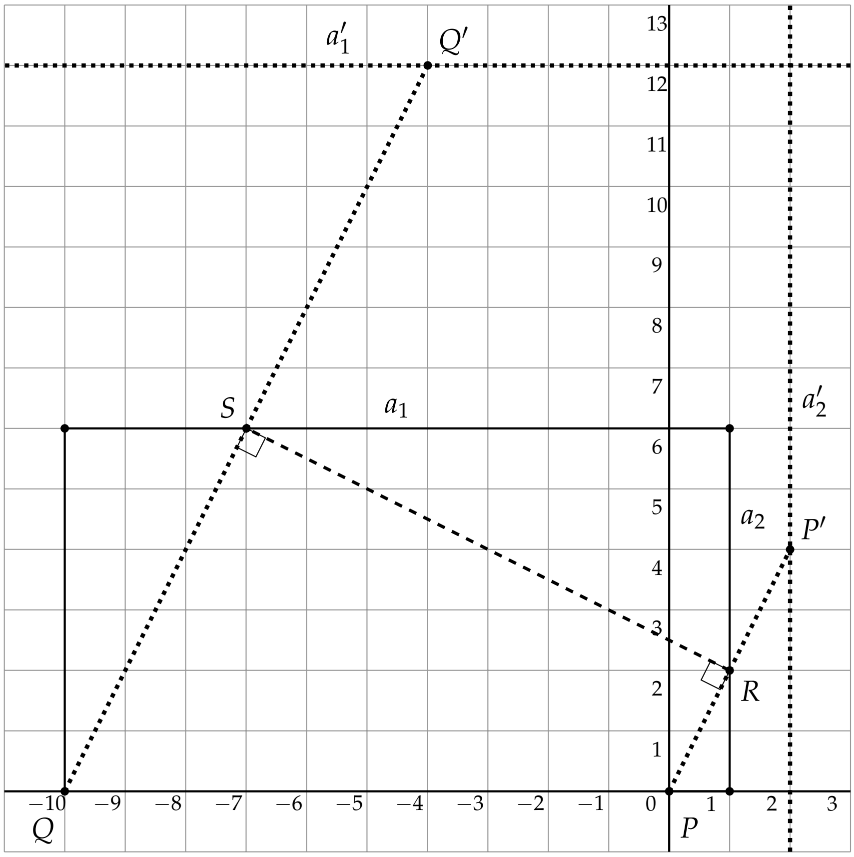
11.1 The Beloch fold

Margharita P. Beloch discovered a remarkable connection between origami and Lill's method for finding roots of polynomials [7]. She found that one application of the operation of origami Axiom 6 (Section 7.6) applied to the first path of Lill's method can obtain a real root of any cubic polynomial. The operation is often called the *Beloch fold*.

Consider the polynomial $p(x) = x^3 + 6x^2 + 11x + 6$ (Section 10.1). In the following diagram we have emphasized the second path and renamed some vertices. To solve the equation we perform a Beloch fold to simultaneously place the points P, Q at P', Q' on the line segments of lengths a_2, a_1, respectively. Unfortunately, if you perform the fold, the path does not solve the equation: Q' is way off to the right, so the angles at P' and Q' are not right angles.

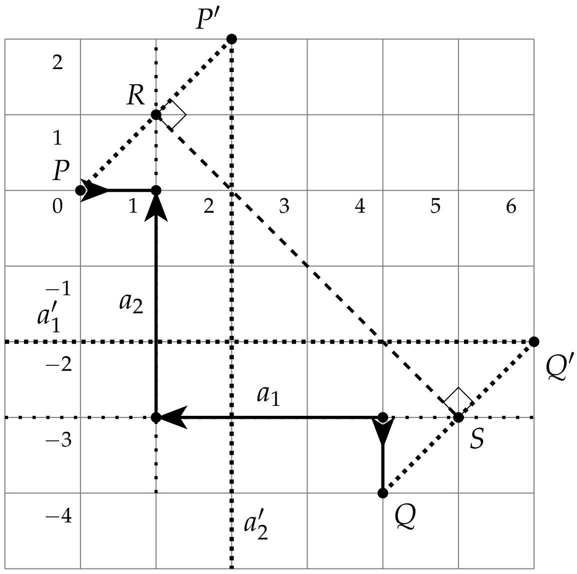


Recall that a fold is the perpendicular bisector of the line segment between any point and its reflection around the fold. We want $\overline{P'?Q'?}$ to be a fold so that it will be perpendicular to both $\overline{QQ'}$ and $\overline{PP'}$. The fold is the perpendicular bisector of $\overline{QQ'}$ and $\overline{PP'}$, so P', Q', the reflections of P, Q, must be the same distance away from the fold as P and Q, respectively. With some change of notation we have the following diagram.



A line a_2' is drawn so that it is parallel to a_2 and the same distance from a_2 as a_2 is from P. Similarly, line a_1' is drawn so that it is parallel to a_1 and the same distance from a_1 as a_1 is from Q. Apply Axiom 6 to simultaneously place P at P' on a_2' and to place Q at Q' on a_1'. The fold $\overline{RS}$ is the perpendicular bisector of the lines $\overline{PP'}$ and $\overline{QQ'}$; therefore, the angles at R and S are right angles as required.

Let us try the Beloch fold on the polynomial $x^3 - 3x^2 - 3x + 1$ from Section 10.6. a_2 is the vertical line segment of length 3 whose equation is $x = 1$, and its parallel line is a'_2 whose equation is $x = 2$, because P is at a distance of 1 from a_2. a_1 is the horizontal line segment of length 3 whose equation is $y = -3$, and its parallel line is a'_1 whose equation is $y = -2$ because Q is at a distance of 1 from a_1. The fold RS is the perpendicular bisector of both $\overline{PP'}$ and $\overline{QQ'}$. The line $\overline{PRSQ}$ is the same as the second path in Section 10.6.

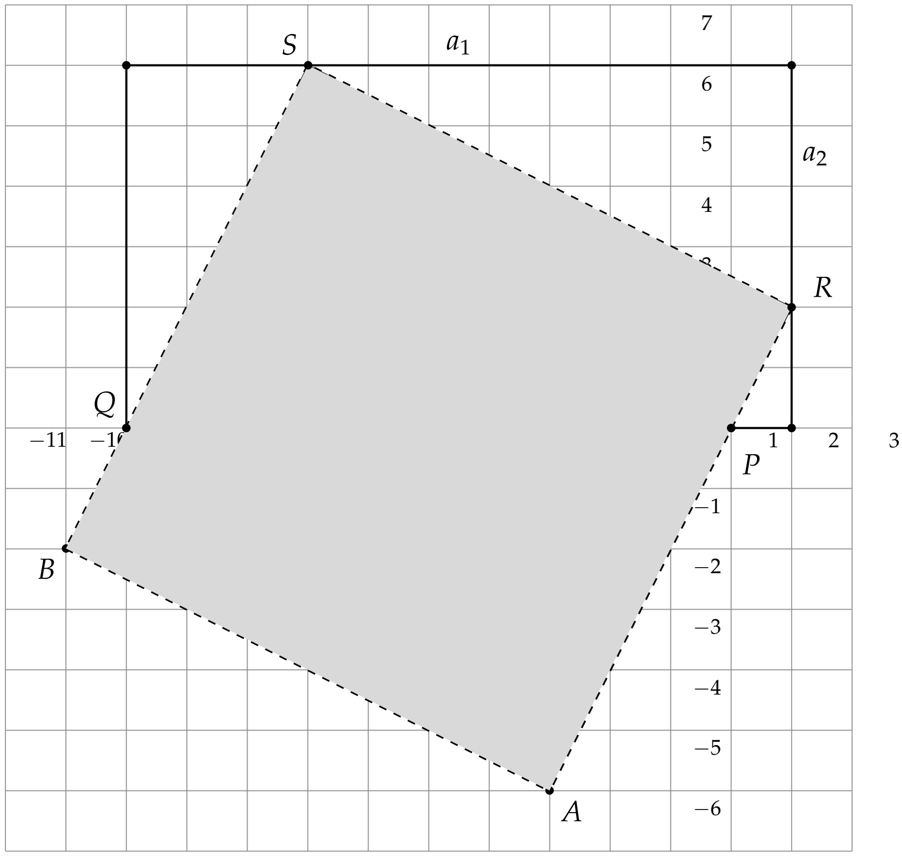


11.2 The Beloch square

This construction in the previous section can be expressed in terms of a *Beloch square*: Given two points P, Q and two lines a_2, a_1, construct a square $\overline{ARSB}$, such that:

- One side is $\overline{RS}$ where R lies on a_2 and S lies on a_1;
- P lies on $\overline{RA}$ and Q lies on $\overline{SB}$.

The following diagram extends the construction for $x^3 + 6x^2 + 11x + 6$ to show the Beloch square. The length of RS is $\sqrt{80} = 4\sqrt{5} \approx 8.94$. We can construct the square by adding three sides of this length.



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