0.1 The perfect compass

The *perfect compass* is an instrument that is capable of constructing any conic section. It consists of an *axis* whose position is fixed but which can rotate. At the end of the axis is an *arm* which contains a pencil that can slide within the arm so that it is always in contact with the base. As the arm rotates around the axis it draws a conic section (Figure 1). Figure 2 shows the parameters that can be set to configure a perfect compass.[1]

Figure 1: A perfect compass

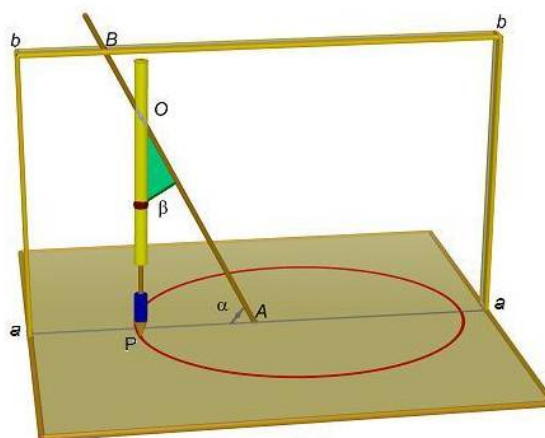

Figure 2: The configuration of a perfect compass

The axis AB is fixed to the base at A at an angle α. The arm is attached to the axis at O at an angle β and the pencil touches the bases at P. The parameters α, β, AB determine which conic section will be drawn. Here we limit ourselves to ellipses by specifying that $0 < \beta < \alpha < 90$.

The notation in the figures in the sequel conform to the notation in [**?**]. B denotes the point of attachment of the to the axis while O is reserved for the center of the conic section. The axis is drawn in red and the arm in blue.

Constructing the major axis

Consider an arbitrary point A on an arbitrary line l that we have drawn to be horizontal (Figure 3). Place the base of the axis AB at A and rotate the axis until the arm intersects the line l. There are two intersections labeled E and D. Let the midpoint of ED be O. Construct the (outer) circle centered at O with radius $OD = OE = r_o$.

The diagrams will be drawn in two dimensions; you have to imagine that B is actually above the plane and the axis AB and the arms BD, BE have been "laid flat."

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[1]The images in the Figures are courtesy of the Associazione Macchine Matematiche.
https://www.macchinematematiche.org/en/conics/compasso-perfetto.html.
Animations of the perfect compass can be found at https://www.geogebra.org/m/tMxr4Mah and
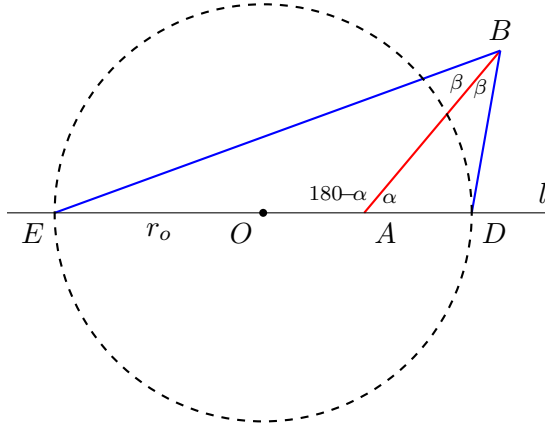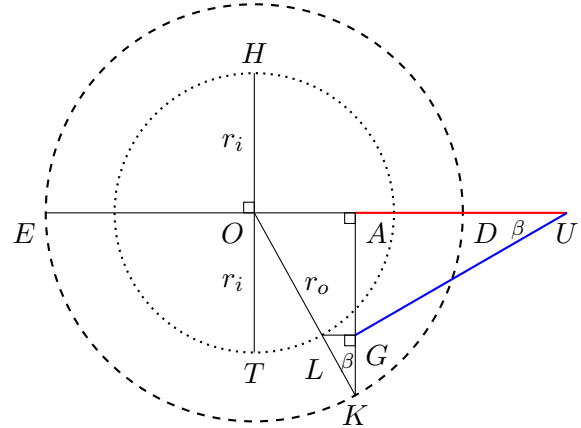https://www.geogebra.org/m/uj2xauej.

Figure 3: Constructing the major axis

Figure 4: Constructing the minor axis

Constructing the minor axis

Consider now the situation where axis is rotated by $-90°$ and denote by G the intersection of the arm with the plane. If you lay the compass flat, the axis AB becomes U (Figure 4). Extend AG until it intersects the circle at K and construct $OK = r_o$. Construct the perpendicular to AK at G and let its intersection with OK be L. Draw the (inner) circle centered at O with radius $OL = r_i$. Construct the perpendicular to ED at O and let its intersections with the inner circle be H, T.

Theorem 0.1 (Prop. VIII) $\dfrac{AG}{AK} = \dfrac{HO}{OD}$, *where $HO = r_i$ and $OD = r_o$ are the semi-minor and semi-major axes of an ellipse.*

Proof By Theorem **??**,
$$\frac{AG^2}{DA \cdot AE} = \frac{HO^2}{OD^2},$$
and by Theorem **??**, $AK^2 = DA \cdot AE$. ∎

G is a point the ellipse with semi-major axis r_o and semi-minor axis r_i

Since $\triangle AKO \sim \triangle AGO$,
$$\frac{AG}{AK} = \frac{OL}{OK} = \frac{r_i}{r_o} = \frac{HO}{OD},$$
and G is a point on an ellipse with semi-minor axis r_i and semi-major axis r_o.

An alternate method of proving that G is on an ellipse is to show that the coordinates of G satisfy the parametric representation (Definition **??**). Let t be the angle between OD and OK. $OA = r_o \cos t$ and $AG = r_i \sin t$ (Figure 5) so G is a point on the ellipse.

Figure 6 shows the ellipse (green) and you can check that G is on the ellipse. It also shows that a symmetric construction gives the point G' on the ellipse. Another symmetric construction shows that there are an additional pair of points to the left of the minor-axis. An easy way to see this is that moving A, the base of the axis to the symmetrical position on OE will cause the
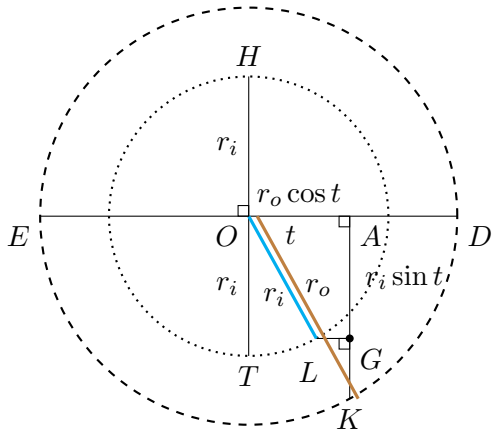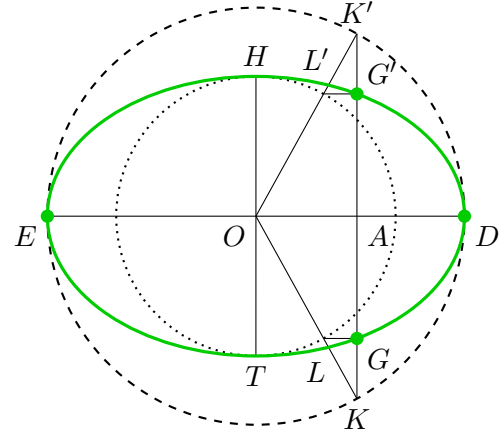
Figure 5: The parametric representation

Figure 6: Checking that G is on the ellipse

same curve to be constructed. We have now located six points on an ellipse. Form the following surprising theorem (proof omitted) we conclude that the perfect compass constructs an ellipse.[2]

Theorem 0.2 *Given five points, no three of which are collinear, there is a unique conic section containing the points.*

Add to sources and bibliography:

Section 0.1 is based on [**?**]; the authors develop formulas for computing the parameters of the ellipse drawn by a compass with given AB, α, β, and conversely.

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@unpublished{henk,
  author={Henk Hietbrink and Hazal Yürük},
  title={Why do we need a conic-section compass?},
  howpublished={submitted},
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```

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[2]Approaches to proving the theorem are discussed in the Wikipedia article *Five points determine a conic*.