The Geometry of Ellipses and Planetary Orbits

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Contents

Chapter 1

Introduction

Everyone "knows" that Kepler discovered that the orbits of the planets are ellipses and that Newton showed that an elliptical orbit implies that the force of gravity must be inversely proportional to the square of the distance from the Sun. Although I knew these facts, I had never seen them demonstrated.

Calculus in Context [5] by Alexander J. Hahn is a comprehensive textbook on introductory calculus that augments theory with applications in physics and astronomy, such as the work of Kepler, Newton and Galileo, as well as applications in engineering such as building bridges and domed structures. These are not just historical anecdotes but detailed computations.

This document contains a detailed explanation of one topic from Hahn's book: the determination of orbits by Aristarchus, Copernicus, Kepler and Newton. The presentation is mathematical, since the historical and astronomical aspects are thoroughly described in [5], as well as in other works. The document is intended to enrich the learning of mathematics by secondary-school students and students in introductory university courses. The prerequisites are a very good knowledge of Euclidean geometry along with some trigonometry, a bit calculus and Newton's laws of motions.

Chapter 2 presents the measurements of the radii and distances of the Earth, Moon and Sun by Eratosthenes and Aristarchus. Chapter 3 describes the construction of a model of a Sun-centered system by Nicolaus Copernicus. Chapter 4 shows how Johannes Kepler developed his three laws of planetary motion and Chapter 5 presents Isaac Newton's derivation of the inverse-square law of gravitation from of Kepler's laws. One step of Newton's derivation requires a theorem whose proof is very long, so it is split off into Chapter 6.

Chapter 7 contains more than you ever wanted to know about the mathematics of ellipses, but these theorems are necessary. I suggest that you look up each theorem (and its proof) as needed, rather than trying to study them all at once. Chapter 8 gives a different definition of ellipses and different proofs of the theorems, as explained in the next paragraph. The relationship between the two definitions of ellipses is shown in Appendix A. Theorems of Euclidean geometry that are used but do not concern ellipses are collected in Appendix B.

Euclidean geometry

It is easy to measure angles. We are familiar with the use of a protractor in school and these can be scaled-up to obtain more accurate measurements. Measuring long distances was impossible until the recent inventions of radar and lasers. At most one could pace-off distances with low accuracy. The only *measured* distance used here is the estimate of 800 km by Eratosthenes for the distance between two places in Egypt (Chapter 2.1). For this reason, the mathematics used is primarily Euclidean geometry, in particular, similar triangles and ratios of their sides.

The final steps in Newton's derivation requires the use of limits, which had been used already by Archimedes to compute the circumference and area of a circle by approximating the circle. Newton (along with his contemporary Gottfried Wilhelm Leibniz) developed the calculus from the concept of limits. However, Newton's *Principia* uses Euclidean geometry almost exclusively, although analytic geometry had been developed by René Descartes and Pierre de Fermat even before Newton was born.

Newton expected his readers to have an extensive knowledge of geometry. This expectation continued until relatively recently:

> In book 1, prop[osition] 10 (and notably in prop[osition] 11), Newton made use of a property of conics which he presents without proof, merely saying that the result in question comes from "the *Conics*." Here, as elsewhere in the *Principia*, Newton assumes the reader to be familiar with the principles of conics and of Euclid. In the eighteenth and nineteenth centuries, when Newton's treatise was still being read in British universities, authors of books on "conic sections"—for example, W. H. Besant, W. H. Drew, Isaac Milnes—supplied the proof of this theorem in order to help readers of the *Principia* who might be baffled by the problem of finding a proof. They even chose letters to designate points on the diagrams so that the final result would appear in exactly the same form as in the *Principia* [2, p. 330].

To keep this spirit alive, Chapter 8 brings the necessary proofs of ellipses using Euclidean geometry. The reader is forewarned, however, that as Euclid said, "There is no royal road to geometry!"

Style and notation

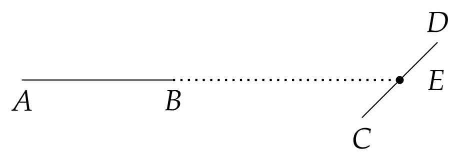
The computations in Hahn's book are faithful to the historical record, for example, measuring distances in units such as the stadia of the Greeks. Here, the computations are fully modernized and use modern units such as kilometers.

Diagrams are used to facilitate understanding each step in the geometrical proofs, more than appear in other sources. For example, Newton proved his difficult theorem (Chapter 6) using only a single diagram. The diagrams are not to scale and are often distorted. Otherwise, it would be impossible to draw planetary orbits on a piece of paper.

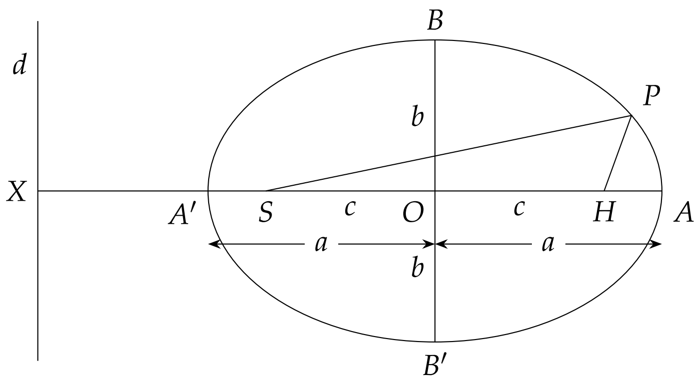
The following shortcuts facilitate a less verbose presentation:

- AB denotes both a line segment and its length.
- $\triangle ABC$ denotes both a triangle and its area.
- The phrase "AB intersects CD" is used even if AB needs to be *extended*[1] until it intersects CD:

[1] The term used by Newton is *produced*.



- The notation in the following diagram will be used consistently to refer to elements of an ellipse. The diagram is for reference since the terms will be defined needed.
  - O is the center of the ellipse and a, b are the semi-major and semi-minor axes of an ellipse. A, A' are the vertices on the major axis and B, B' are the vertices on the minor axes.
  - S and H are the foci and c is the distance from a focus to the center.
  - X is the intersection of the major axis with the directrix d. (There is a second directrix on the right side of the ellipse.)



Chapter 2

The sizes of the Earth, Moon and Sun

2.1 Eratosthenes's measurement of the radius of the earth

The ancient Greeks knew that the Earth is round and Eratosthenes was able to measure the radius of the Earth (Figure 2.1). Choose two points A, B on the same longitude and measure the distance d between them. Plant a vertical stick (red) in the ground at A and another (blue) at B. On a day in the year when the stick at A produces no shadow at noon, at the same time the stick at B produces a shadow whose angle is α. The sun is so far away from the Earth that over the relatively short distance d, the rays of the Sun are essentially parallel. By alternate interior angles, the angle between the two sticks as measured from the center of the Earth is also α.

The angle that Eratosthenes measured at the blue stick was

$$\alpha = 7.5^\circ \cdot \frac{2\pi}{360} \approx 0.131 \text{ radians},$$

and the distance d between A and B was known to be approximately 800 km. The arc $\widehat{AB}$ subtends the angle $\alpha = d/r_e$ where r_e is the radius of the Earth, so

$$r_e = \frac{d}{\alpha} = \frac{800}{0.131} \approx 6107 \text{ km}. \tag{2.1}$$

This value is quite close to the modern measurement of 6370 km.

2.2 Aristarchus's measurements

Using r_e, Eratosthenes's measurement of the radius of the Earth, Aristarchus was able to measure and compute the following values:

- r_m: the radius of the Moon,

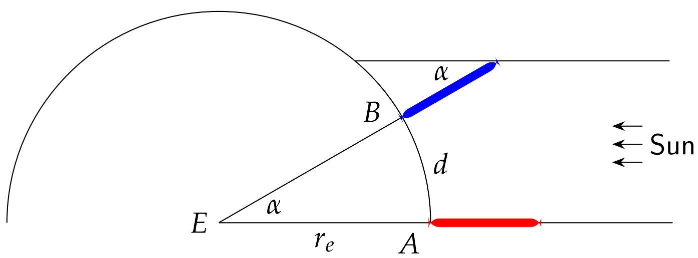


Figure 2.1: Eratosthenes's measurement of the radius of the earth

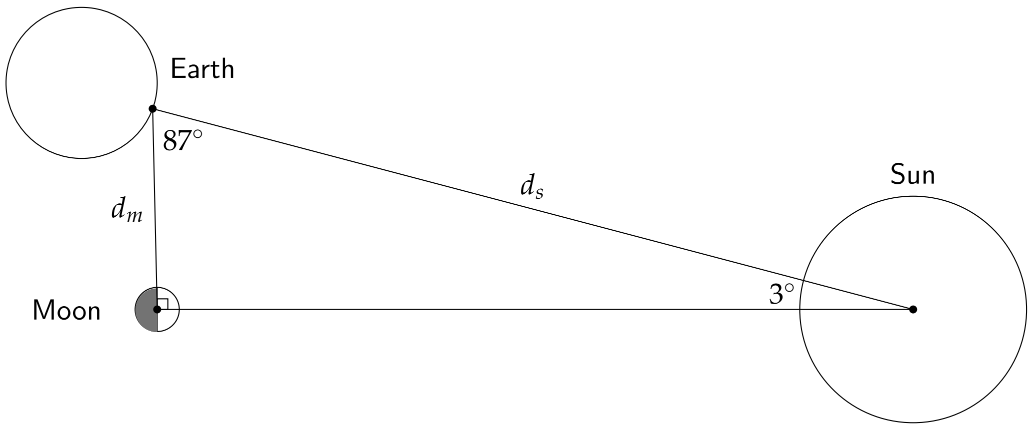


Figure 2.2: Observing a first quarter moon

- r_s: the radius of the Sun,
- d_m: the distance from the Earth to the Moon,
- d_s: the distance from the Earth to the Sun.

Computing d_s/d_m

An observer on Earth can follow the phases of the Moon as it revolves around the Earth. At one point in the month the phase will be first quarter, meaning that the one half of the moon is illuminated while the other half is not (Figure 2.2). The angle between the Sun and the Moon will be 87°. Since exactly half of the moon is illuminated, we know that the angle $\angle$ Earth-Moon-Sun is a right-angle so

$$\begin{aligned} \cos 87^\circ &= \frac{d_m}{d_s} \\ \frac{d_s}{d_m} &= \frac{1}{\cos 87^\circ} \approx 19\,. \end{aligned} \tag{2.2}$$

Computing r_s/r_m and d_m/r_m

The Moon is much, much smaller than the Sun, but it is also much, much closer to the Earth. When the Moon is precisely positioned between the Earth and the Sun, its "disk" exactly covers the "disk" of the Sun, causing a total solar eclipse (Figure 2.3).

The angle subtended by the Moon is 2° degrees. Bisecting the angle creates two right triangles with an acute angle of 1°, where the right angles are the tangents to Moon and the Sun. By similar triangles, Equation 2.2 and Figure 2.3,

$$\frac{r_s}{r_m} = \frac{d_s}{d_m} = 19 \tag{2.3}$$

$$\frac{d_m}{r_m} = \frac{1}{r_m/d_m} = \frac{1}{\sin 1^\circ} \approx 57\,. \tag{2.4}$$

Computing the radii and distances

Figure 2.4 shows a lunar eclipse. Unlike a solar eclipse where the Moon exactly covers the Sun, the Earth more than covers the Moon and its shadow is four times the Moon's radius.

Figure 2.5 show a lunar eclipse annotated with the distances d_m, d_s and the radii r_m, r_e, r_s. The ray from the top of the Sun is tangent to both the Sun and the Earth, so it forms right angles with their radii, as well as with the extension of the Moon's radius. The thick horizontal lines are constructed parallel to the line connecting the centers, forming two similar right triangles, so using Equation 2.3,

$$\begin{aligned} \frac{r_s - r_e}{r_e - 2r_m} &= \frac{d_s}{d_m} = \frac{r_s}{r_m} \\ r_s r_e + r_m r_e &= 3 r_s r_m\,. \end{aligned}$$

Again from Equation 2.3, $r_s = 19 r_m$, so

$$r_m = \frac{20}{57} r_e\,.$$

By Equation 2.1, $r_e \approx 6107$ km, by Equation 2.4, $d_m = 57 r_m$, and by Equation 2.2, $d_s = 19 d_m$, so we can compute the radii and distances:

$$r_m = \frac{20}{57} r_e \approx 2143 \text{ km}$$

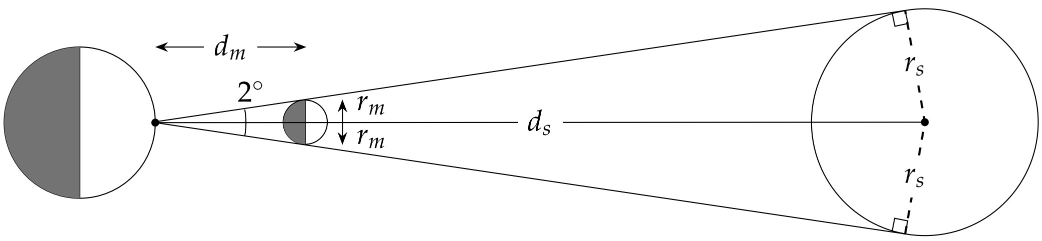


Figure 2.3: A solar eclipse

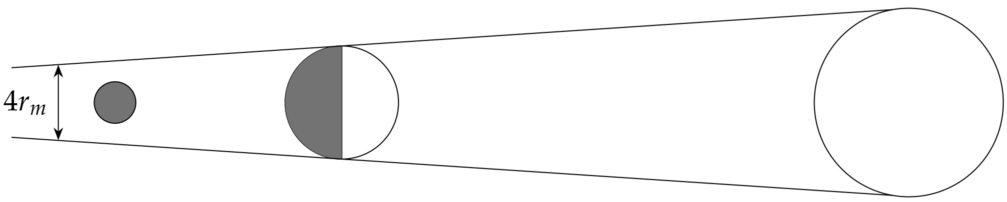


Figure 2.4: A lunar eclipse

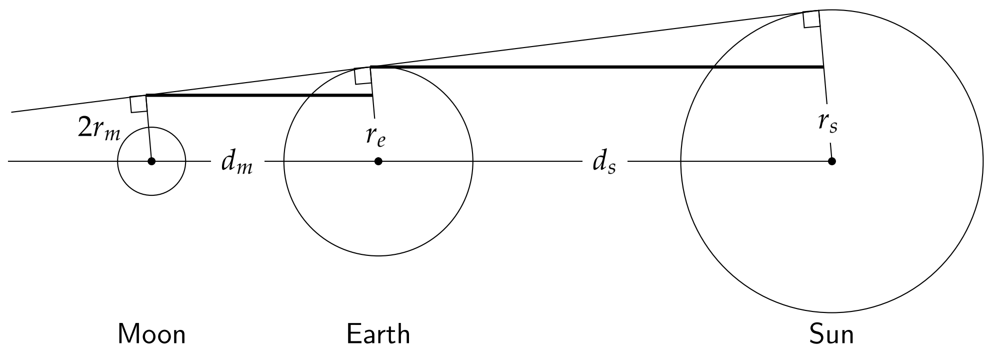


Figure 2.5: Detail of a lunar eclipse

$$
\begin{aligned}
r_s &= 19r_m \approx 40,713 \text{ km} \\
d_m &= 57r_m \approx 122,140 \text{ km} \\
d_s &= 19d_m \approx 2,320,660 \text{ km}\,.
\end{aligned}
$$

The following table summarizes these data together with the modern values [5, Table 1.3]. While the computed values for the radii of the Earth and the Moon are not far off from the modern values, the other computed values are not anywhere near the modern values. Nevertheless, they do show that the Greeks understood the immense size of the solar system.

		Computed (km)	Modern (km)
r_e	radius of Earth	6107	6370
r_m	radius of Moon	2143	1740
r_s	radius of Sun	40,713	695,500
d_m	distance Earth-Moon	122,140	384,570
d_s	distance Earth-Sun	2,320,660	150,000,000

Chapter 3

The Sun-centered solar system

As everyone living far from equator knows, the time between sunrise and sunset varies with the seasons. The reason is that the axis of the rotation of the Earth is offset by 23.5° relative to the orbit of the Earth. The plane of the orbit of the Earth around the Sun is called the *ecliptic*. Measuring the length of the day as the time from sunrise to sunset, there is a day in June, called the *summer solstice*, when the length of the day is longest. Similarly, there is a day in December, called the *winter solstice*, when the length of the day is shortest.[1] There are also two days when the length of the day equals the length of the night: the *autumn equinox* in September and the *spring equinox* in March.

Today we know that the universe is immensely large and that the stars are moving at extremely high speeds, but an observer on Earth sees them as if their positions are fixed on a sphere around the earth, called the *celestial sphere*. This solstices and equinoxes can be associated with the projection of the Sun on the celestial sphere as seen from the Earth. The details of the Earth's orbit can be found in books on astronomy, as well as in the Wikipedia articles on *Equinox* and *Solstice*.

3.1 The length of the year and the length of the seasons

Let us assume that the Earth orbits the Sun in a circle, such that the center of the orbit O is the center of the Sun S. In Figure 3.1 the inner circle is orbit of the Earth and the outer circle is the celestial sphere. The orbit can be divided into four quadrants called *seasons*: spring, summer, autumn, winter.

The length of a year is approximately $365\frac{1}{4}$ days. The extra $\frac{1}{4}$ day is accounted for by adding a day in leap years.[2] The length of each season as determined by the equinoxes and the solstices is $365.25/4 = 91.3125$ days. However, measurements by the Greek astronomer Hipparchus showed that the actual lengths of the seasons differed from this number and a model of the solar system must be able to explain these differences:

Season	Days	%
Spring	$94\frac{1}{4}$	25.8
Summer	$92\frac{1}{2}$	25.3
Autumn	$88\frac{1}{8}$	24.1
Winter	$90\frac{1}{8}$	24.7

[1] This holds for the northern hemisphere; in the southern hemisphere the opposite holds.

[2] The length of a year is actually 365.2425. In the sixteenth century, the *Gregorian calendar* accounted for the difference by removing three leap years in every four hundred years.

In his Earth-centered solar system, Hipparchus proposed that the center of the Earth's orbit is offset from the center of the Sun. Copernicus used the same idea in his Sun-centered solar system (Figure 3.2). If the center of the Earth's orbit is in the upper-left quadrant of the coordinate system defined by the equinoxes and the solstices, the angles for spring and summer are obtuse, so the seasons are longer than one-fourth of a year, whereas the angles for the autumn and winter are acute, so they are shorter than one-fourth of a year.

Figure 3.3 shows a magnified and distorted view of Figure 3.2. It has been annotated with additional lines and labels that will facilitate the demonstration of Copernicus's computation. The axes $A'C'$ and $B'D'$ have their origin O at the center of the Earth's orbit and are parallel to the axes in the ecliptic. The dashed lines from O are all radii of the Earth's orbit that will be denoted r. The dotted right triangles will be used in the computation.

3.2 The location of the center of the Earth's orbit

Copernicus's task was to locate the position of the center of the Earth's orbit O relative to the center of the Sun S. This will be given in polar coordinates $OS = c$ and $\angle FSB = \lambda$ (the label is on the large circle of the ecliptic). The strategy of the computation is as follows:

- Initially, we compute the angles of the arcs in radians; the lengths of the arc can then be obtained by multiplying by the radius r.
- We use the lengths of the seasons that Copernicus used: summer is $93\frac{14.5}{60}$ days and spring is $92\frac{51}{60}$.
- The angle of the arc $\widehat{AC}$ can be computed from the combined length of spring and summer and the angle of the arc $\widehat{AB}$ can be computed from the length of spring.

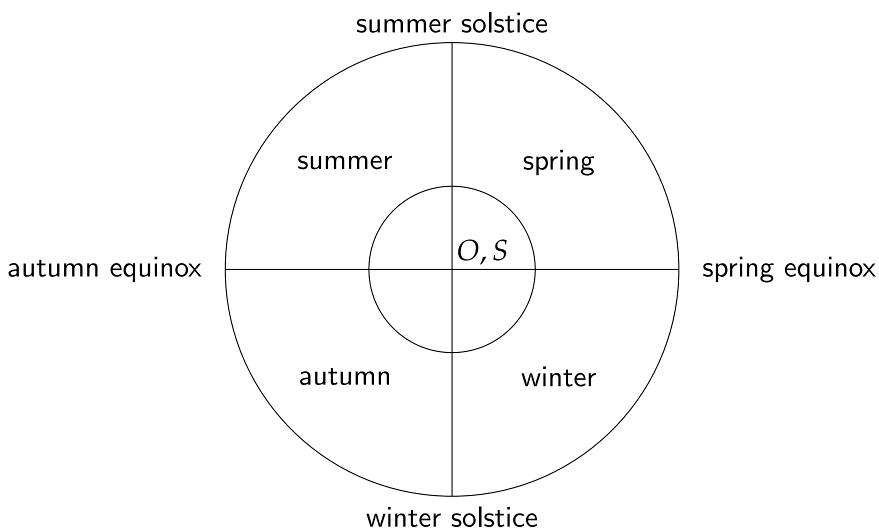


Figure 3.1: The orbit of the Earth and the seasons

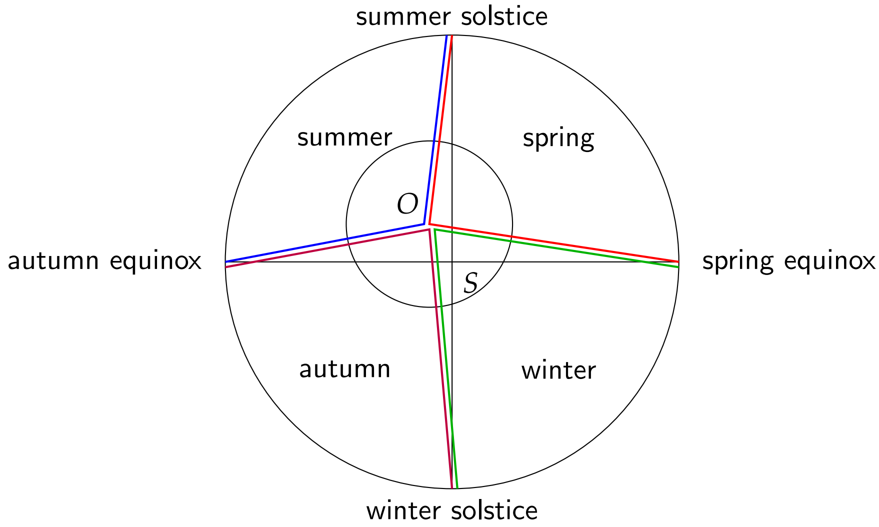


Figure 3.2: The lengths of the seasons are not equal

- From $\widehat{AC}$ and $\widehat{AB}$, the angle α subtended by $\widehat{AA'}$ and the angle β subtended by $\widehat{BB'}$ can be computed.
- Since the Earth is very close to the Sun relative to the radius of its orbit, $r\,\widehat{AA'}$ and $r\,\widehat{BB'}$ approximate the line segments a and b. From these c and λ can be computed.

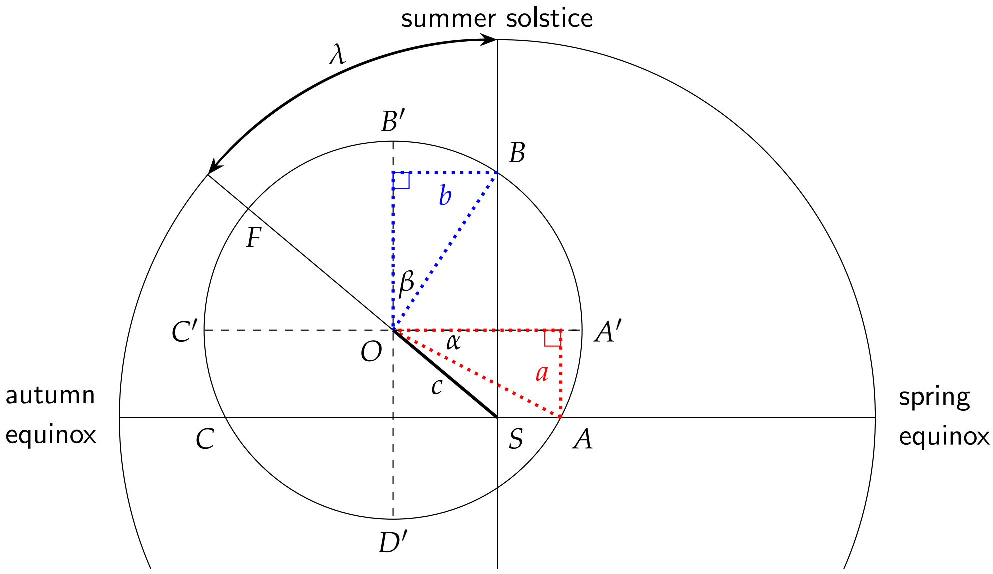


Figure 3.3: Computing the center of the Earth

Computing the angles of the arcs $\widehat{AB}, \widehat{AC}$

The arcs $\widehat{AB}, \widehat{AC}$ are sectors of the Earth's orbit and their angles are their proportions of a full year times 2π radians.

$$\widehat{AB} = 2\pi \cdot \frac{92\frac{51}{60}}{365.25} = 2\pi \cdot \frac{92.85}{365.25} = 1.5972 \text{ radians}$$

$$\widehat{AC} = 2\pi \cdot \frac{92\frac{51}{60} + 93\frac{14.5}{60}}{365.25} = 2\pi \cdot \frac{186.09}{365.25} = 3.2012 \text{ radians}\,.$$

Computing the angles of the arcs $\widehat{AA'}, \widehat{BB'}$

Let us express the arcs $\widehat{AC}$ and $\widehat{AB}$ in terms of the arcs that comprise them. Since AC is parallel to $A'C'$, $\widehat{AA'} = \widehat{C'C}$. Compute $\widehat{AA'}$:

$$\begin{aligned}\widehat{AC} &= \widehat{AA'} + \widehat{A'C'} + \widehat{C'C} = 2\widehat{AA'} + \pi \\ \widehat{AA'} &= \frac{1}{2}(3.2012 - \pi) = 0.0298 \text{ radians}\,.\end{aligned}$$

Now that we have computed $\widehat{AB}$ and $\widehat{AA'}$ we can compute $\widehat{BB'}$:

$$\begin{aligned}\widehat{AB} &= \widehat{AA'} + \widehat{A'B'} - \widehat{BB'} \\ \widehat{BB'} &= 0.0298 + \frac{\pi}{2} - 1.5927 = 0.0034 \text{ radians}\,.\end{aligned}$$

Computing the lengths of the arcs $\widehat{AA'}, \widehat{BB'}$

OA and OB are radii of the Earth's orbit so

$$\begin{aligned}\sin\alpha &= \frac{a}{r} \approx \alpha \\ a &\approx r\alpha = r\widehat{AA'} = 0.0298r \\ \sin\beta &= \frac{b}{r} \approx \beta \\ b &\approx r\beta = r\widehat{BB'} = 0.0034r\,,\end{aligned}$$

where we have used the assumption that O, the center of the Earth's orbit, is very close to the Sun S so that $\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = \lim_{\beta\to 0} \frac{\sin\beta}{\beta} = 1$.

Computing the position of O relative to S

Figure 3.4 shows a magnified diagram of a portion of Figure 3.3. In the dotted triangles, we have already computed the lengths a and b. Since OT is parallel to $A'A$ and TS is

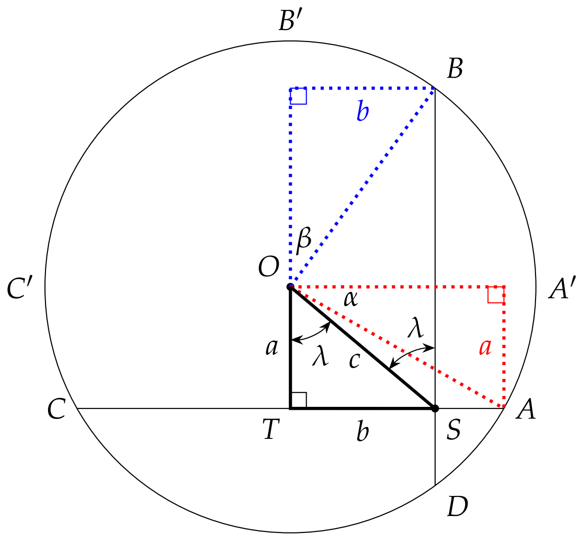


Figure 3.4: Three triangles

parallel to BB', we can label OT by a and TS by b. The emphasized triangle is a right triangle and c, the distance of O from S, can be obtained from Pythagoras's theorem:

$$c = \sqrt{a^2 + b^2} = r\sqrt{(0.0298)^2 + (0.0034)^2} = 0.03r\,.$$

λ can be obtained from trigonometry:

$$\lambda = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0.0034}{0.03} = 0.1129 \text{ radians} \approx 6.47^\circ\,.$$

The distance $0.03r$ is shown in the following table using the values of r from the table on page 7.

	Aristarchus (km)	Copernicus (km)	Modern (km)
radius of Earth's orbit	2,320,660	8,000,000	150,000,000
distance of O from S	69,620	240,000	4,500,000

Chapter 4

Elliptical orbits

Towards the end of the sixteenth century, the astronomer Tycho Brahe carried out extremely precise observations. In 1600 he hired Johannes Kepler as his assistant and when Tycho died soon afterwards, Kepler was appointed to his position. Here we explain how Kepler was able to establish that planetary orbits are ellipses.

4.1 Determining the radius of the Earth's orbit

A Martian year is 687 days, that is, it equals $\frac{687}{365.25} = 1.88$ Earth years. We know when Mars reaches a "new year" by observing its projection on the celestial sphere, but each time the position of the Earth in its orbit will be different. Figure 4.1 shows the orbit of the Earth—its center O offset from the Sun S as Copernicus showed—at four occasions when the position of Mars M at its new year was observed. Four triangles are created $\triangle OE_iM$.

Figure 4.2 shows one of the triangles with the angles labeled. Using the law of sines,

$$\frac{OE_i}{\sin\beta} = \frac{OM}{\sin\alpha}$$

$$OE_i = OM\,\frac{\sin\beta}{\sin\alpha}\,.$$

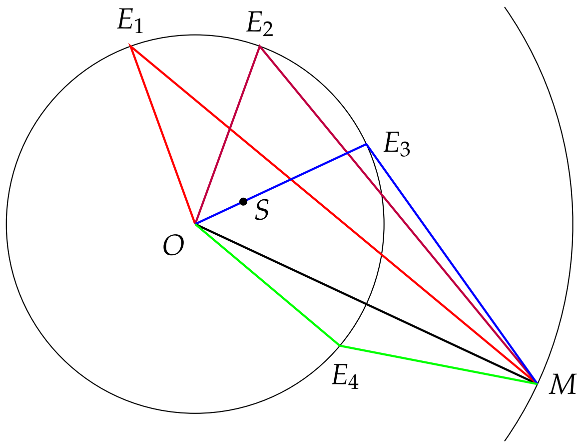


Figure 4.1: Observations of the orbit of Mars from the Earth

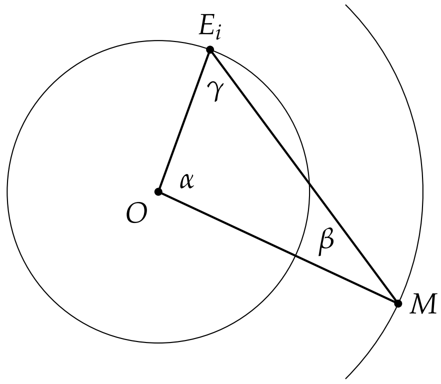


Figure 4.2: One triangle Earth-Sun-Moon

Tycho Brahe was able to measure all three angles:[1]

	α	β	γ	OE_i
E_1	127.1	20.8	32.1	$0.6682 \cdot OM$
E_2	84.2	35.8	60.5	$0.6721 \cdot OM$
E_3	41.3	42.4	96.4	$0.6785 \cdot OM$
E_4	1.6	3.4	175.0	$0.6805 \cdot OM$

The values of OE_i given in the fourth column of the table are not equal. Assuming (as everyone did at that time) that the Earth' orbit is circular, the only solution was to move the center of the orbit so that $\{E_1, E_2, E_3, E_4\}$ were all on the circle.

4.2 Measuring the angles in the triangle Sun-Earth-Mars

How can the angles α, β, γ be measured? Since $\triangle E_iOM$ is a triangle, it is sufficient to measure two of the angles. γ is easily measured by observing Mars and the Sun at the same time. However, neither α nor β can be measured directly since they are not accessible to an observer on Earth.

Tycho's measurement used the known periods of the orbits to compute the angles. The Earth moves counterclockwise around Sun. Given any point E, for some t, t days later the Earth will have moved to E' and Mars will have moved to M', so that they are in *opposition*, that is, Mars will be on the continuation of the Earth-Sun line (Figure 4.3). Since the Earth completes an orbit in about half the time that Mars takes to complete an orbit, t will be such that neither the Earth nor Mars has completed a full orbit. The angles θ_E and θ_M are fractions of a circular orbit of 360°, so

$$\frac{t}{365.25} = \frac{\theta_E}{360}$$
$$\theta_E = \frac{360}{365.35}\,t$$

[1]The values are rounded to one decimal point.

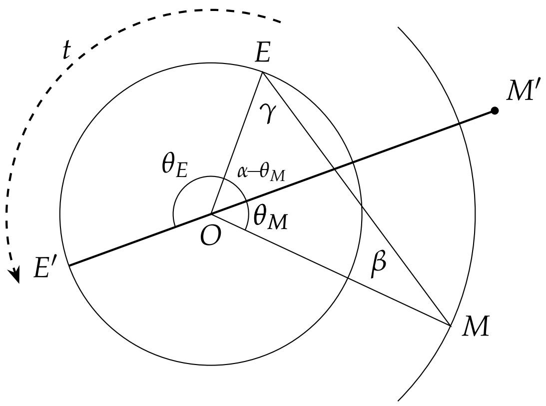


Figure 4.3: The Earth and Mars in opposition

$$\frac{t}{687} = \frac{\theta_M}{360}$$
$$\theta_M = \frac{360}{687}\,t\,.$$

This gives values for θ_E and θ_M. Since $E'M'$ is a straight line, we have that $\alpha - \theta_M = 180^\circ - \theta_E$, so that $\alpha = 180 - \theta_E + \theta_M$, and the values of OE_i can be computed.

4.3 A new location for the center of the Earth's orbit

Kepler's next task was to obtain a new value O' for the center of the Earth's orbit such that the E_i's are on the orbit. Given the new locations of the Earth $\{E_1, E_2, E_3, E_4\}$, by Theorem B.1.1 a circle centered at O' can be constructed that goes through $\{E_1, E_2, E_3\}$ (Figure 4.4). To verify that this is the correct orbit, check that E_4 is on the circle.

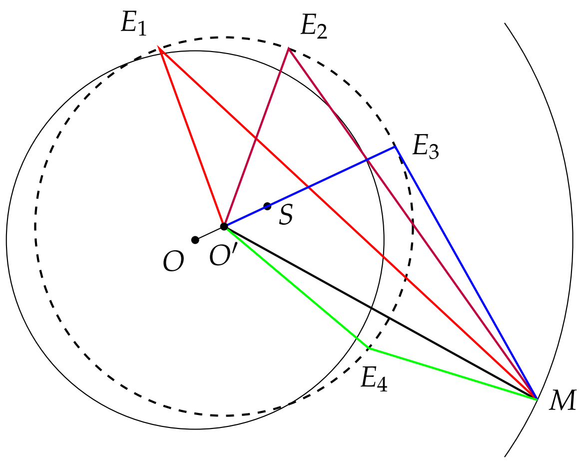


Figure 4.4: Observations of the orbit of Mars from the new Earth's orbit

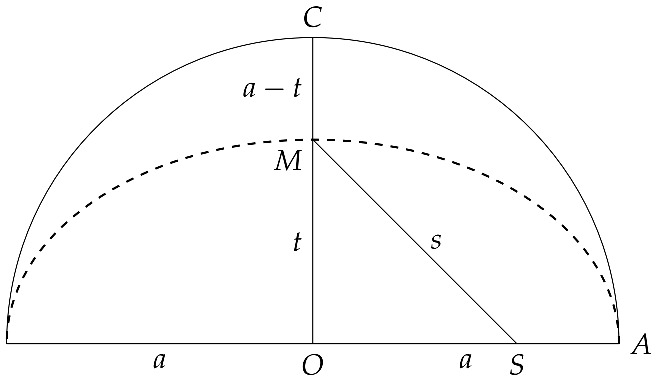


Figure 4.5: The orbit of Mars as an oval "egg"

4.4 Orbits are ellipses

While Kepler was able to modify the center of the orbit of the Earth to be consistent with the observations, he was not able to adequately describe the orbit of Mars. After years of work, he came to the conclusion that the orbit must be oval like an egg. Oval, perhaps, but certainly not an ellipse, because he was certain that it would have been discovered by Archimedes! Figure 4.5 shows C, a position on a circular orbit, and an oval orbit (dashed), where M is the position of Mars on the oval corresponding to C. The radius of the circular orbit is labeled a and the unknown distances to M are labeled $s = SM$ and $t = OM$. Kepler's computed that $\dfrac{a-t}{t} = 0.00429$ and $\dfrac{s}{t} = 1.00429$, so that

$$\begin{aligned}\frac{a-t}{t} &= \frac{s}{t} - 1 \\ a - t &= s - t\,,\end{aligned}$$

and therefore $SM = s = a = AO = CO$. The dashed oval is likely an ellipse, because in an ellipse $SM = AO$ (by Theorem 7.1.2). Kepler then computed the projections of the observations of Mars on the x-axis (Figure 4.6) and obtained for all of them that

$$\frac{M_iO_i}{C_iO_i} = \frac{1}{1.00429} = 0.99573\,.$$

By Theorem 7.2.1, since the ratio $MO/CO = b/a$ is constant for in an ellipse, Kepler was able to conclude that the orbit of Mars is an ellipse.

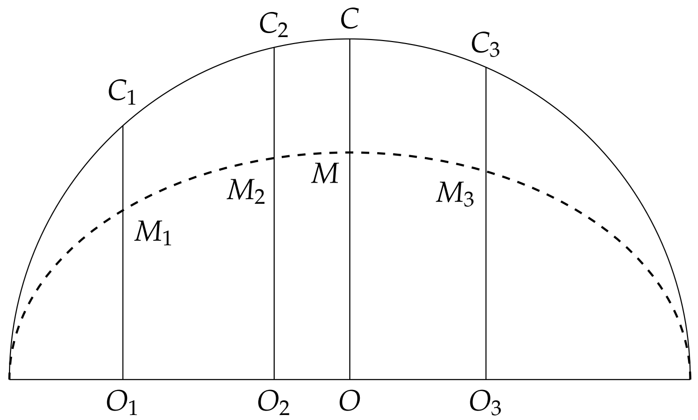


Figure 4.6: The orbit of Mars as an ellipse

Chapter 5

Gravitation

In the *Principia* Isaac Newton proved the following theorem led to the theory of universal gravitation.

> **Theorem 5.0.1** *If a planet subject to a centripetal force follows an elliptical orbit around the Sun, then the force decreases as the inverse square of the distance from the Sun.*

After a review of Newton's Laws of force and motion, we show that Kepler's second law must hold in *any* system subject to a centripetal force. The next step is to show the inverse square law and then it is a small step to universal gravitation and Kepler's third law.

5.1 Newton's laws of motion

1. A body in uniform motion (including a body at rest) continues with the same motion unless a force is applied.

2. A force F applied to a body causes an acceleration a in the direction of the force whose magnitude $a = F/m$, where m, the constant of proportionality, is called the *mass* of the body.

3. If one body exerts a force on a second body, the second body exerts a force on the first of equal magnitude but in the opposite direction.

Forces are denoted by vectors, where the direction of the vector represents the direction of the force and the length of the vector represents the magnitude of the force. Forces can be decomposed into perpendicular components (Figure 5.1), or into components in any directions (Figure 5.2). The components form a parallelogram whose diagonal is the resultant force.

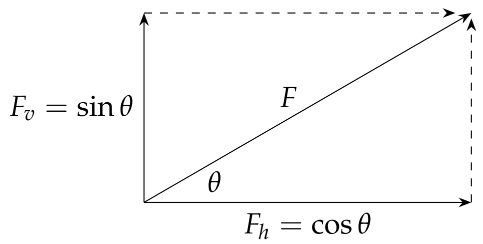


Figure 5.1: Perpendicular components of a force

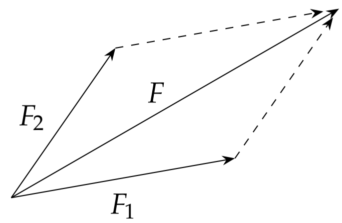


Figure 5.2: Arbitrary components of a force

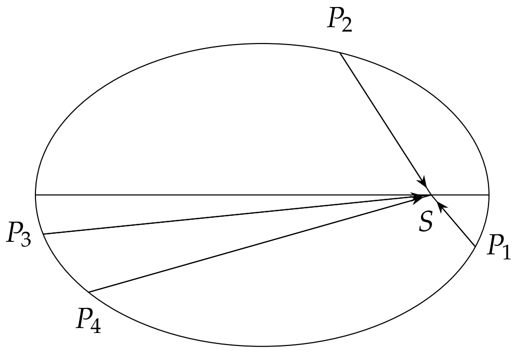


Figure 5.3: Centripetal force

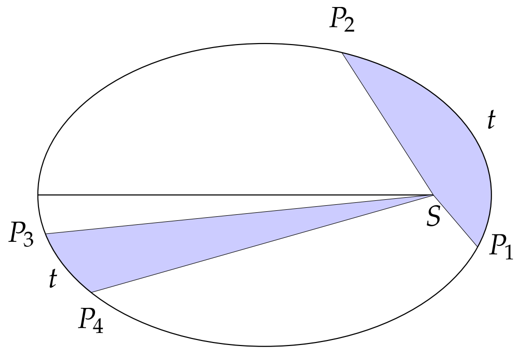


Figure 5.4: Equal areas in equal times

Newton was interested in *centripetal force* which is a force exerted by a single body on another, in particular, the gravitational force exerted by the Sun on a planet (Figure 5.3). Since the only force is that directed towards the Sun, the planet does not move "up" or "down" so its orbit is in a plane.

Kepler's second law states that a planet in orbit sweeps out equal area in intervals of equal duration, that is, if it takes the planet time t to move from P_1 to P_2 and also t to move from P_3 to P_4, then the area of the sector P_1SP_2 is equal to the area of P_3SP_4 (Figure 5.4). (Obviously, this means that the speed of the planet must vary as it traverses its orbit $v_{P_1P_2} \gg v_{P_3P_4}$.) Newton proved that this must be true in any system where a body is subject to a centripetal force from another body.

The proof is based on dividing an area into very small sectors and then taking the limit. Consider three points A, B, C on the orbit (Figure 5.5) that represent the positions of the planet at intervals of Δt. For clarity we have drawn them spaced out, but the intention is that they are very close together. Newton assumed that the planet does not smoothly traverse the arcs, but rather that it every Δt it jumps in discrete steps from one point on the orbit to the next.

Figure 5.6 shows how the force is exerted in discrete steps. The planet moves from A to B and we expect that the centripetal force at B will cause an acceleration that moves the planet to C, the next point on the orbit. Instead, we "pretend" that the force is not applied at B, but, in the absence of an applied force, planet continues to move in the same direction and at the same speed. After another period of Δt as passed and the planet has reached point C', the force is now applied *in the same direction* as it would have been applied at B, moving the planet to C.

Theorem 5.1.1 *The area of $\triangle ASB$ is equal to the area of $\triangle BSC$.*

Proof The proof will be done in two steps by showing that $\triangle ASB = \triangle BSC'$ and then that $\triangle BSC' = \triangle BSC$.

- In Figure 5.7, $\triangle ASB$ is shown in blue and $\triangle BSC'$ is shown in red. It is assumed that $AB = BC'$ (the planet moves from B to C' during the same interval Δt), so since SH is the height of both triangles, their areas are equal.

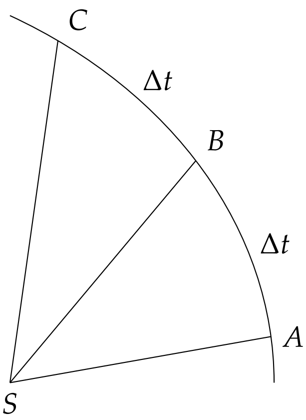


Figure 5.5: “Small” sectors of an orbit

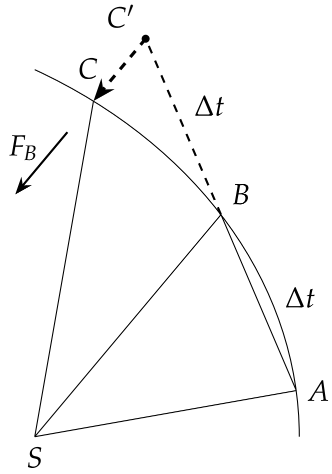


Figure 5.6: Exerting force at discrete times

- In Figure 5.8, $\triangle BSC$ is shown in blue and $\triangle BSC'$ is shown in red. It is assumed CC' is parallel to SB (the planet is subject to the centripetal force at C' in the *same* direction as the force at B), so the heights of both triangles to the common side SB are equal and their areas are equal. It follows that $\triangle ASB = \triangle BSC' = \triangle BSC$, which we denote by ΔA. ∎

We assume that the sectors of the orbit are each divided up into small sectors of uniform duration Δt. By the theorem, each sector has the same area ΔA. Therefore (see Figure 5.4),

$$\frac{A_{P_1SP_2}}{\Delta A} = \frac{t}{\Delta t} = \frac{A_{P_3SP_4}}{\Delta A},$$

from which Kepler's second law follows: $A_{P_1SP_2} = A_{P_3SP_4}$.

The proof used two approximations:

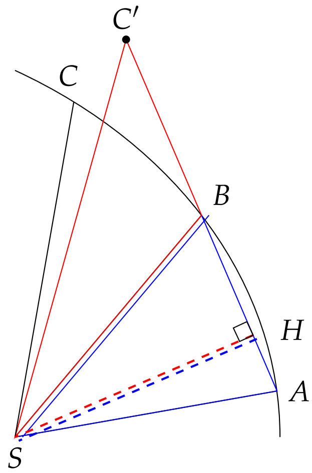


Figure 5.7: $\triangle ASB = \triangle BSC'$

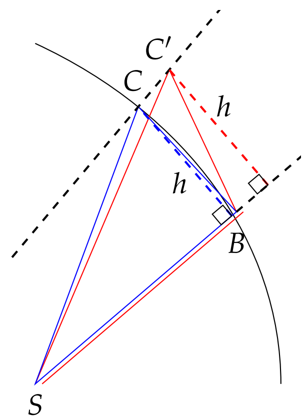


Figure 5.8: $\triangle BSC' = \triangle BSC$

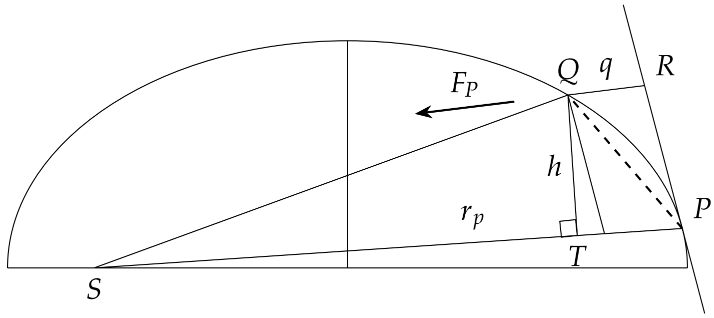


Figure 5.9: The derivation of the inverse square law

- ΔA is an approximation of the area of each sector.
- The force at C' is an approximation to the force at B.

In the limit as the size of the sectors decreases, the errors become negligible.

Definition 5.1.2 *For a given elliptical orbit,* $\kappa = \dfrac{A}{t}$*, where* A *is the area of the ellipse and* t *is the period of the orbit, is called* Kepler's constant.

5.2 The inverse square law for gravitation

Newton's next step was to show that if the orbit of a planet is elliptical, the centripetal force must be proportional to the mass of the planet and inversely proportional to the square of its distance from the Sun. In Figure 5.9, S is the Sun, and P and Q are points on the orbit that are very close to each other. PR is the tangent to the ellipse at P, and R is chosen to that QR is parallel to SP. QT is constructed perpendicular to SP. Denote the lengths $r_P = SP$, $h = QT$, $q = QR$ and the time interval from P to Q by Δt.

When a point is subject to an acceleration a for a period of Δt, its displacement is $\frac{1}{2}a(\Delta t)^2$. From Newton's second law we know that at point R, the planet is subjected to an acceleration of F_P/m, so

$$q = \frac{1}{2}\frac{F_P}{m}(\Delta t)^2$$

$$F_P = \frac{2mq}{(\Delta t)^2}\,.$$

Now we compute the area of $\Delta A_{PSQ} \approx \triangle SPQ = (1/2)hr_p$ and use Kepler's constant:

$$\Delta t = \frac{\Delta A_{PSQ}}{\kappa} = \frac{hr_P}{2\kappa}$$

$$F_P = 2mq \cdot \frac{4\kappa^2}{(hr_P)^2} = 8\kappa^2 m \cdot \frac{q}{h^2} \cdot \frac{1}{r_P^2}\,.$$

To obtain an inverse-square law for the force, the first two factors have to be independent of the distance. For a given planet m is constant and for a given elliptical orbit κ is constant, so the first factor does not depend on the distance. What about the second factor q/h^2, in particular, what value does it have as Δt approaches zero?

Theorem 5.2.1 *In an elliptical orbit*

$$\lim_{\Delta t \to 0} \frac{q}{h^2} = \frac{1}{L},$$

where L is the length of the latus rectum of the ellipse (Definition 7.3.1).

Newton's proof is very complex and is presented separately in Chapter 6.

Since L is constant for any given ellipse, the inverse square law can be written

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2}. \tag{5.1}$$

The formula can be re-written so that the constant values appearing are more familiar: a, the semi-major axis and T, the period of the orbit. By Theorem 7.3.2, $L{=}2b^2/a$ and by Theorem 7.4.1, $\kappa = A_{ellipse}/T{=}\pi ab/T$, so

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2} = \frac{8(\pi ab)^2 m}{T^2} \cdot \frac{a}{2b^2} \cdot \frac{1}{r_P^2} = \frac{4\pi^2 a^3 m}{T^2} \cdot \frac{1}{r_P^2}. \tag{5.2}$$

Newton was able to show that:

- The inverse square law applies to all conic sections including a parabola and a hyperbola, not just to an ellipse and, of course, to a circle. Some comets have hyperbolic orbits and orbit the Sun only once.
- The converse holds: if a planet is subject to an inverse-square centripetal force then the orbit must be an ellipse (or another conic section).
- The proof assumes that a planet is a very small point, but the result holds even for large planets as long as the density of the planet is radially symmetric, that is, for a given distance from the center the density is constant.

5.3 Universal gravitation

By Newton's third law, we can equate the force $F_{S\leftarrow E}$ that the Sun S exerts on the Earth E with the force $F_{E\leftarrow S}$ that the Earth exerts on the Sun. Let m be the mass of the Earth and M be the mass of the Sun, then by Equation 5.1,

$$F_{S\leftarrow E} = \frac{8\kappa_E^2 m}{L_E} \cdot \frac{1}{r^2} = \frac{C_E m}{r^2}$$

$$F_{E\leftarrow S} = \frac{8\kappa_S^2 M}{L_S} \cdot \frac{1}{r^2} = \frac{C_S M}{r^2}$$

$$\frac{C_E}{M} \cdot \frac{1}{r^2} = \frac{C_S}{m} \cdot \frac{1}{r^2},$$

from some constants C_E, C_S.

Why are the constants different? The Earth and the Sun both rotate around their center of mass called the *barycenter*, which is very close to the center of the Sun since the Sun is so much more massive than the Earth. The ellipse of the Sun's orbit is very small relative to the Earth so A and L are smaller, and the Sun's period is large so T is larger. The different values for $8\kappa^2/L$ are encapsulated into the constants C_E, C_S. Let $G = \dfrac{C_E}{M} = \dfrac{C_S}{m}$ so that

$$F_{S \leftarrow E} = F_{E \leftarrow S} = G\frac{mM}{r^2}\,. \tag{5.3}$$

This is Newton's law of universal gravitation. It is not specific to planetary orbits but holds between any two bodies with masses m, M.

5.4 Kepler's third law

Theorem 5.4.1 (Kepler's third law) *Let P_1, P_2 be two planets whose elliptical orbits have semi-major axes a_1, a_2 and whose orbital periods around the Sun are T_1 and T_2. Then*

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2}\,.$$

Proof By Equations 5.2 and 5.3,

$$F = \frac{4\pi^2 a_i^3 m}{T_i^2}\frac{1}{r_i^2} = \frac{GmM}{r_i^2}\,. \tag{5.4}$$

After canceling m and r_i we get

$$\frac{a_i^3}{T_i^2} = \frac{GM}{4\pi^2}\,.$$

$GM/4\pi^2$ is a constant that depends only on the mass of the Sun and the gravitational constant, so a_i^3/T_i^2 is constant for all planets rotating around the Sun. ∎

Chapter 6

A proof Proposition XI, Problem VI

Theorem 5.2.1 is Book I, Section III, Proposition XI, Problem VI of the *Principia*.

Study Figure 6.1:[1]

- Let P, Q be two points on the ellipse that represent the movement of a body in an elliptical orbit separated by a time interval Δt. Construct lines from P to the center C and the foci S, H.
- Construct the tangent at P and choose R on the tangent such that the body would move from P to R if it continued for time Δt not subject to any force. Construct the parallelogram $PRQX$ and extend QX until it intersects PC at V.
- Construct a line parallel to RP through H and let I be its intersection with PS.
- Construct DC the conjugate diameter to PC (Definition 7.6.1), and let E be its intersection with PS.

6.1 A formula for QR

Theorem 6.1.1 $QR = PV \cdot \dfrac{CA}{CP}$.

Proof By Theorem 7.5.1, $\angle RPX = \angle ZPH = \alpha$ and by alternate interior angles,

$$\angle PHI = \angle ZPH = \alpha = \angle RPX = \angle PIH\,,$$

so $\triangle IPH$ (red) is isosceles and $PI = PH = d$.

[1]The bottom half of the ellipse is not shown, but we still refer to lines DC, PC as diameters.

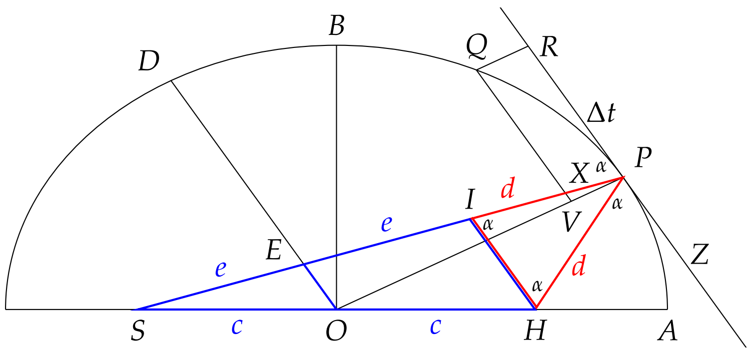


Figure 6.1: Geometry of an elliptical orbit (1)

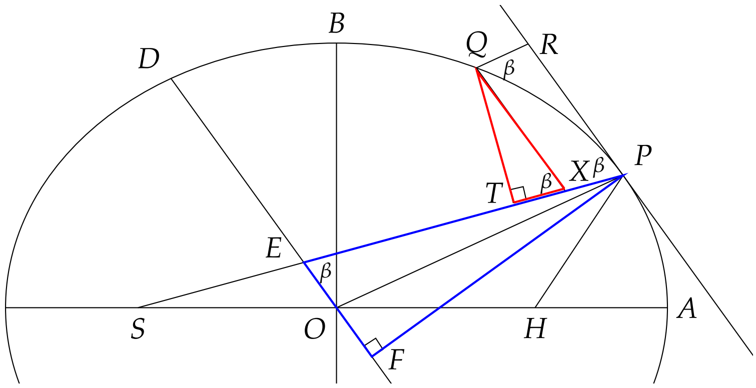


Figure 6.2: Geometry of an elliptical orbit (2)

$SC = CH = c$ are equal because they are the distances of the foci from the center of the ellipse. Let $SE = e$. By construction $EC \parallel IH$ so $\triangle ESC \sim \triangle ISH$ (blue) and

$$\frac{SC}{SE} = \frac{SH}{SI}$$
$$SI = \frac{SH \cdot SE}{SC} = \frac{2c \cdot e}{c} = 2e\,.$$

By definition of an ellipse $SP + PH = SI + IP + PH = 2CA$ so $2e + d + d = 2CA$ and $EP = e + d = CA$.

$QV \parallel EC$ so $\triangle EPC \sim \triangle XPV$ and

$$\frac{PX}{PV} = \frac{EP}{PC} = \frac{CA}{PC}$$
$$PX = PV \cdot \frac{CA}{PC}\,.$$

Since $PRQX$ is a parallelogram $QR = PX$, $QR = PV \cdot \dfrac{AC}{PC}$. ∎

6.2 A formula for QT

Construct a perpendicular from P to DC and label its intersection with DC by F. Construct a perpendicular from Q to SP and label its intersection with SP by T (Figure 6.2).

Theorem 6.2.1

$$QT = QX \cdot \frac{FP}{CA}\,.$$

Proof By construction, $QR \parallel PX$, so by alternate interior angles $\angle RQX = \angle QXT = \beta$. By construction, $QX \parallel DC$, so by alternate interior angles $\angle QXT = \angle PEF = \beta$. Since $\triangle PFE$ and $\triangle QTX$ are right triangles with an equal acute angle β, $\triangle PFE \sim \triangle QTX$. In the proof of Theorem 6.1.1 we showed that $EP = CA$ so

$$\frac{QT}{QX} = \frac{FP}{EP}$$
$$QT = QX \cdot \frac{FP}{EP} = QX \cdot \frac{FP}{CA}. \quad \blacksquare$$

6.3 A formula for QR/QT^2

Theorem 6.3.1

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \tag{6.1}$$

Proof Let use combine the equations in Theorems 6.1.1 and 6.2.1 to get QR/QT^2.

$$\frac{QR}{QT^2} = \frac{PV \cdot \dfrac{CA}{CP}}{\left(QX \cdot \dfrac{FP}{CA}\right)^2} = \frac{PV \cdot CA^3}{QX^2 \cdot CP \cdot FP^2}. \tag{6.2}$$

DC and PC are conjugate diameters so Theorem 7.6.3 gives a formula for PV that we substitute into Equation 6.3.

$$\frac{QR}{QT^2} = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2} \cdot \frac{CA^3}{QX^2 \cdot CP \cdot FP^2} = \frac{CP \cdot CA^3}{CD^2 \cdot FP^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \tag{6.3}$$

Next we show that $CD \cdot FP = CA \cdot CB$. By Theorem 7.8.1 the areas of the parallelograms formed by the tangents to conjugate diameters are equal. By symmetry the areas of the four small parallelograms are equal, as are the triangles formed by constructing diagonals. In Figure 6.3 the area of the $\triangle ABC$ (red), which is $(1/2)CA \cdot CB$, is equal to the area of $\triangle PCD$ (blue), which is $(1/2)CD \cdot FP$. Substituting for $CD \cdot FP$ in Equation 6.3 gives

$$\frac{QR}{QT^2} = \frac{CP \cdot CA^3}{CB^2 \cdot CA^2} \cdot \frac{QV^2}{GV \cdot QX^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \quad \blacksquare$$

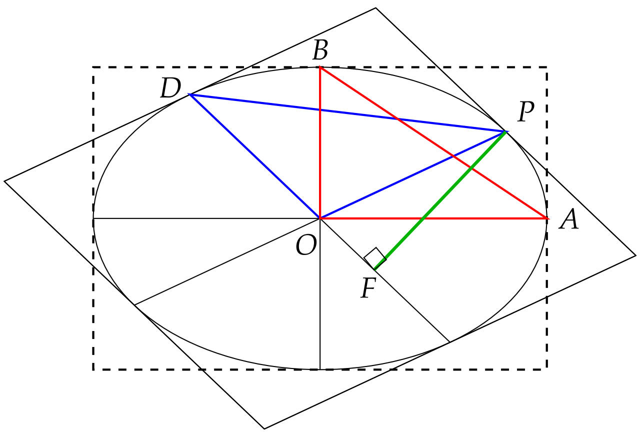


Figure 6.3: Parallelograms formed by conjugate diameters

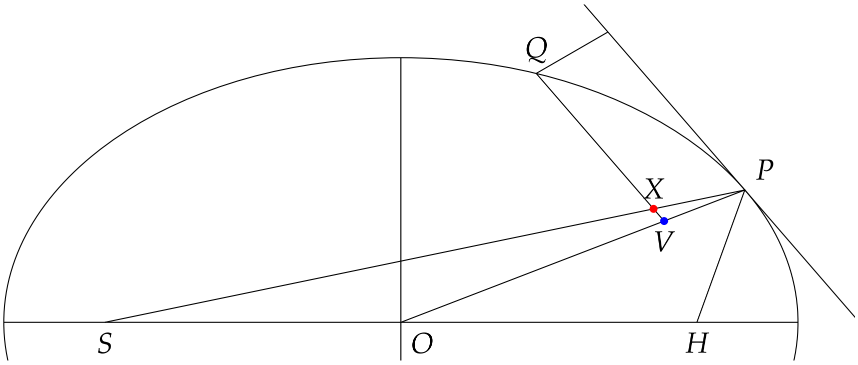


Figure 6.4: Geometry of an elliptical orbit (3)

6.4 Approaching the limit

Figure 6.4 is an enlarged diagram of part of Figure 6.2. As the time interval Δt gets smaller, $Q \to P$ which implies that

- $X \to V$ so that $QX \to QV$.
- $V \to P$ so that $CV \to CP$ and hence $GV \to 2CP$.
- In the limit $QX = QV$ and $GV = 2CP$. Substituting into Equation 6.1 gives

$$\lim_{Q \to P} \frac{QR}{QT^2} = \lim_{Q \to P} \frac{CP \cdot CA}{CB^2} \cdot \frac{QX^2}{2CP \cdot QX^2} = \frac{CA}{2CB^2} = \frac{a}{2b^2} = \frac{1}{L},$$

using the result of Theorem 7.3.2 for the length of the latus rectum.

Chapter 7

Ellipses

7.1 Fundamental properties

Definition 7.1.1 (Ellipse)

- *Let S and H be two points in the plane such that $SH = 2c \geq 0$ and choose a such that $2a > 2c$ (Figure 7.1). An* ellipse *is the geometric locus of all points P such that $SP + PH = 2a$. If $c = 0$ the geometric locus is a* circle*.*
- *Construct AA' through SH, where A, B are the intersections of the line with the ellipse. AA' is the* major axis *of the ellipse. Let O be the midpoint of SH. AO, OA' are the* semi-major axes *of the ellipse.*
- *Construct the perpendicular to AA' at O and let B, B' be its intersections with the ellipse. BB' is the* minor axis *of the ellipse and BO, OB' are the* semi-minor axes *of the ellipse.*

Theorem 7.1.2 *(Figure 7.2)*

1. *$SB = HB = a$.*
2. *$AO = OA' = a$.*
3. *$BO = OB'$. (Label $BO = OB'$ by b.)*

Proof

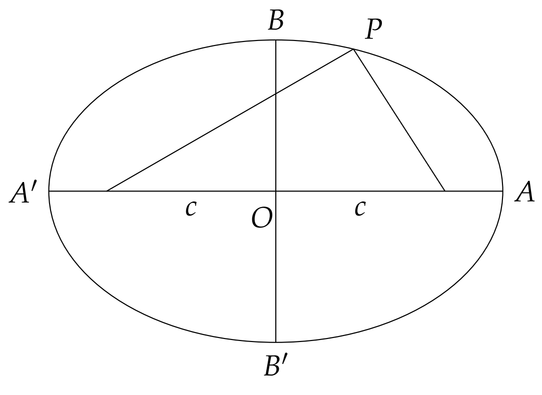


Figure 7.1: The definition of an ellipse

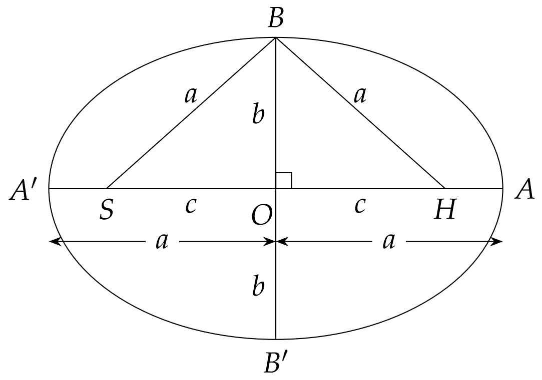


Figure 7.2: The semi-major and semi-minor axes of an ellipse

1. $\triangle SBO \cong \triangle HBO$ by side-angle-side so $SB = HB$. Since B is on the ellipse, $SB + HB = 2a$ and $SB = HB = a$ follows.

2. Since A is on the ellipse,
$$2a = AS + HA = (AO - c) + (AO + c) = 2AO\,,$$
so $AO = a$. $OA' = a = AO$ follows by symmetry.

3. $BO = OB'$ follows from $\triangle SBO \cong \triangle SB'O$. ∎

Theorem 7.1.3 *A point $P = (x, y)$ on an ellipse satisfies the equation*
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{7.1}$$

Proof Since $S = (-c, 0)$, $H = (c, 0)$ and $SP + PH = 2a$,
$$PS + PH = \sqrt{(x-(-c))^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a\,.$$
Squaring twice results in
$$\begin{aligned}
(x+c)^2 + y^2 &= \left(2a - \sqrt{(x-c)^2 + y^2}\right)^2 \\
4xc &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} \\
a - \frac{c}{a}x &= \sqrt{(x-c)^2 + y^2} \\
a^2 + \frac{c^2}{a^2}x^2 &= x^2 + c^2 + y^2 \\
\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} &= \frac{a^2 - c^2}{a^2 - c^2} = 1\,.
\end{aligned}$$
By Theorem 7.1.2 and Pythagoras's theorem, $b^2 = a^2 - c^2$ so
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \quad ∎$$

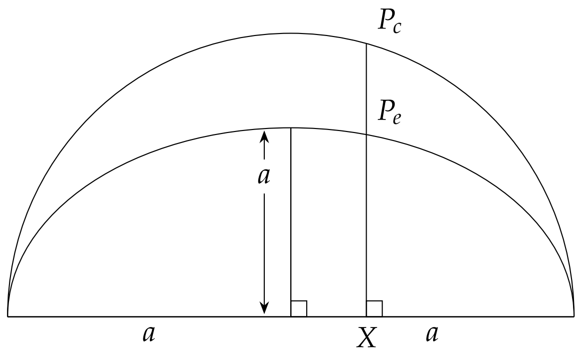


Figure 7.3: A circle circumscribing an ellipse

7.2 A circle circumscribing an ellipse

Consider a circle of radius a with the same center as an ellipse (Figure 7.3). Choose a point X on the major axis and construct a perpendicular through $X = (x, 0)$. Let its intersections with the ellipse and the circle be $P_e = (x, y_e)$ and $P_c = (x, y_c)$, respectively.

Theorem 7.2.1 *The perpendicular to the major axis through a point $P_c = (x, y_c)$ on the circle circumscribing an ellipse intersects the ellipse at $P_e = (x, y_e) = \left(x, \frac{b}{a} y_c\right)$.*

Proof From Equation 7.1 and the formula $x^2 + y^2 = a^2$ for the circle,

$$y_e = \frac{b}{a}\sqrt{(a^2 - x^2)} = \frac{b}{a} y_c \,. \quad \blacksquare \tag{7.2}$$

7.3 The latus rectum of an ellipse

Definition 7.3.1 *Consider a line through a focus of an ellipse that is perpendicular the major axis. Let its intersections with the ellipse be L_1, L_2. Then $L = L_1 L_2$ is a* latus rectum *of an ellipse (Figure 7.4).*[1]

Theorem 7.3.2 *L, the length of the latus rectum of an ellipse, is $\dfrac{2b^2}{a}$.*

Proof By Equation 7.2 and Pythagoras's theorem,

$$L = 2L_1 = 2 \cdot \frac{b}{a}\sqrt{a^2 - c^2} = \frac{2b^2}{a} \,. \quad \blacksquare$$

[1]Usually, lines are denoted by lower-case letters, but L for the latus rectum is the standard notation.

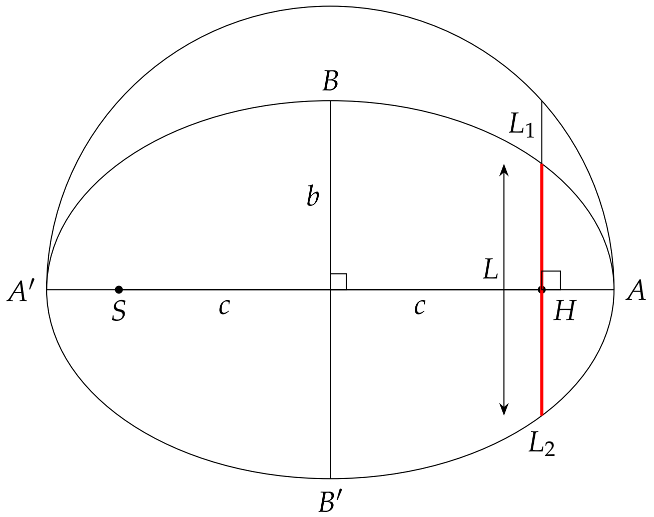


Figure 7.4: The latus rectum of an ellipse

7.4 The area of an ellipse

Theorem 7.4.1 *The area of an ellipse is πab.*

Proof From Equation 7.2

$$y_e = \frac{b}{a}\sqrt{a^2 - x^2}\,,$$

so the area of an ellipse is

$$A_e = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{b}{a}\cdot 2\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \frac{b}{a}A_c\,.$$

If we can show that the area of a circle is πa^2 the theorem follows.

The proof uses polar coordinates, where $x = a\cos\theta$ and $y = a\sin\theta$. First, we derive the formula for the integral of $\sin^2\theta$ using the double-angle identity.

$$\begin{aligned}
\sin^2\theta &= \frac{1-\cos 2\theta}{2} \\
\int \sin^2\theta\,d\theta &= \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C\,.
\end{aligned}$$

Now we can compute the area of a circle as twice the area of a semicircle by changing from Cartesian to polar coordinates and integrating.

$$\begin{aligned}
A_c &= 2\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = 2\int_{-\pi}^{0} \sqrt{a^2 - (a\cos\theta)^2}\;d(a\cos\theta) \\
&= 2\cdot a\cdot a\int_{-\pi}^{0} \sin\theta(-\sin\theta)\,d\theta = -2a^2\int_{-\pi}^{0} \sin^2\theta\,d\theta \\
&= -2a^2\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} + C\right)\Bigg|_{-\pi}^{0} = \pi a^2\,. \quad \blacksquare
\end{aligned}$$

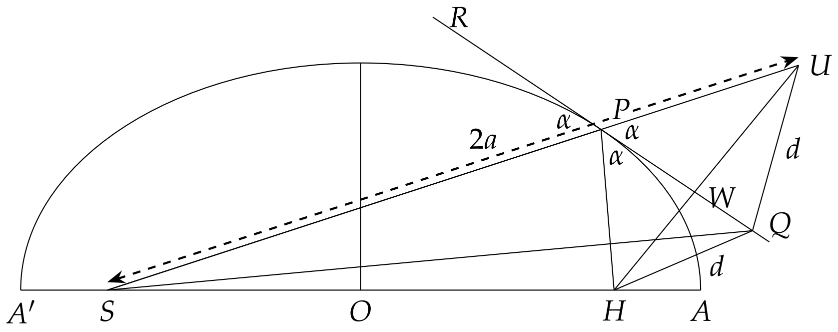


Figure 7.5: Angles at the tangent

7.5 The angles between a tangent and the lines to the foci

Theorem 7.5.1 *Let P be a point on the ellipse whose foci are S, H. Let PU be the extension of SP such that $SU = AA' = 2a$. Let RQ be the bisector of $\angle HPU$. Then $\angle RPS = \angle QPH$ and RQ is the tangent to the ellipse at P (Figure 7.5).*

Proof We prove that any point $Q \neq P$ on the bisector is not on the ellipse, so the bisector RQ has only one point of intersection with the ellipse and it must be the tangent at P. Since RQ is the angle bisector of $\angle HPU$ (the exterior angle of $\angle SPH$), $\angle QPH = \angle QPU = \alpha$, and by vertical angles $\angle QPU = \angle RPS = \alpha$.

Construct the line HU to form the triangle $\triangle HPU$ which intersects PQ at W. By construction $PH = PU$ so $\triangle HPW \cong \triangle UPW$ by side-angle-side and $UW = HW$. But HU is a straight line, therefore, if $\angle HWQ = \angle UWQ$, then they are both right angles and $\triangle HWQ = \triangle UWQ$ by side-angle-side, so $UQ = HQ = d$. Suppose that Q is on the ellipse so that $2a = SQ + QH = SQ + QU$. By the triangle inequality $2a = SQ + QU > SU = 2a$, contradicting that Q is on the ellipse. ∎

7.6 Conjugate diameters

Definition 7.6.1 *There are two equivalent definitions of conjugate diameters.*

- *Let P be a point on an ellipse, PG a diameter and let t be the tangent to the ellipse at P. Diameter DK is a* conjugate diameter *if it is parallel to t (Figure 7.6).*
- *Two diameters PG and DK are* conjugate diameters *if the midpoints of chords $(D'K', D''K'')$ parallel to one diameter (DK) lie on another diameter (PG).*

Theorem 7.6.2 *Let $P = (x, y)$ be a point on an ellipse (not on the major axis AA') and construct a perpendicular PV from P to the major axis (Figure 7.7). Then*

$$\frac{A'V \cdot AV}{PV^2} = \frac{a^2}{b^2}.$$

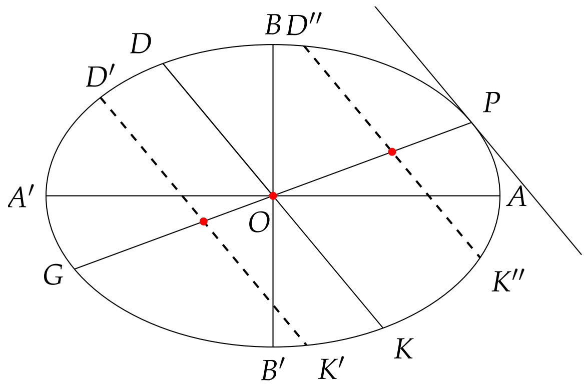


Figure 7.6: Conjugate diameters

Proof By Equation 7.2,

$$y^2 = b^2 \cdot \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2(a^2 - x^2)}{a^2}$$

$$\frac{A'V \cdot AV}{PV^2} = \frac{(a+x)(a-x)}{y^2} = \frac{a^2(a^2-x^2)}{b^2(a^2-x^2)} = \frac{a^2}{b^2}\,. \quad \blacksquare$$

Theorem 7.6.3 *Let PG, DK be conjugate diameters of an ellipse and let Q be a point on the ellipse (Figure 7.8). Constrct the perpendicular QV from Q to the major axis, then*

$$PV = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2}\,.$$

Proof Figure 7.8 shows a dashed ellipse which is the original ellipse rotated about the same center O, so that OP is the semi-major axis and OD is the semi-minor axis. By Theorem 7.6.2,

$$\frac{GV \cdot PV}{QV^2} = \frac{a'^2}{b'^2}\,,$$

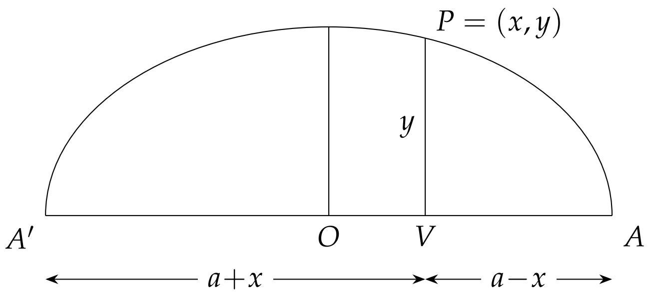


Figure 7.7: Ratios on conjugate diameters

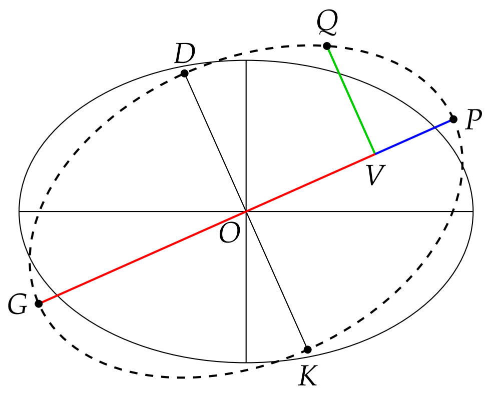


Figure 7.8: Ratios on conjugate diameters

where a', b' are the lengths of the semi-major and semi-minor axes of the rotated ellipse. By construction $a' = OP$ and $b' = OD$ so

$$\frac{GV \cdot PV}{QV^2} = \frac{CP^2}{OD^2}$$

$$PV = \frac{QV^2 \cdot CP^2}{GV \cdot OD^2}. \quad \blacksquare$$

7.7 The parametric representation of an ellipse

Figure 7.9 shows an ellipse and two circles: one whose radius is the length of the semi-major axis (dotted red) and one whose radius is the semi-minor axis (dashed blue). The figure shows the *parametric representation* of a point $P = (x, y)$ on the ellipse:

$$(x, y) = (a \cos t,\ y = b \sin t)\,.$$

The parameter t is *not* the angle of P relative to the positive x-axis. Construct the perpendicular through P to the minor axis and let P_I be its intersection with the inner circle so that CP_I defines an angle t. Extend C_I until it intersects the outer circle at P_O. The parametric representation of P is computed from the lengths of the axes and trigonometry functions of t.

7.8 Areas of parallelograms

Theorem 7.8.1 *The areas of the parallelograms formed by tangents to the intersections of and pair of conjugate diameters with the ellipse are equal (Figure 7.10).*

Proof We show that the area of the parallelogram $JKLM$ is equal to the area of the parallelogram formed by tangents to the major and minor axes (dashed). By symmetry it suffices to prove that the areas of one of the quadrants of those parallelograms are equal: $A_{ACBC'} = A_{PCDJ}$. Since the diagonals bisect a parallelogram, it suffices to prove that that the area of $\triangle ABC$ (red) equals the area of $\triangle PCD$ (blue).

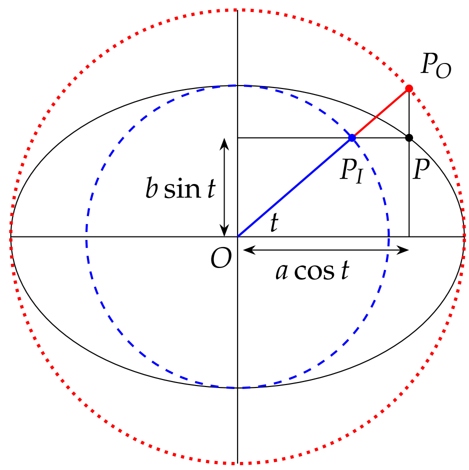


Figure 7.9: Parametric representation of an ellipse

Let $P = (x_p, y_p) = (a \cos t, b \sin t), D = (x_d, y_d)$ be the parametric representations of the points on the ellipse. Conjugate diameters are perpendicular so $\angle DCP$ is a right angle and

$$D = (x_d, y_d) = (a \cos(t + \pi/2), b \sin(t + \pi/2)) = (-a \sin t, b \cos t) .$$

Construct $DD' = (x_d, 0)$ and $PP' = (x_p, 0)$ perpendicular to the major axis. The area of $\triangle PCD$ can be computed as the area of the trapezoid $P'PDD'$ minus the areas of the triangles $\triangle D'DC, \triangle P'PC$. Therefore,

$$\begin{aligned} \triangle PCD &= \frac{y_p + y_d}{2}(x_p - x_d) - \frac{1}{2}x_d y_d - \frac{1}{2}x_p y_p = \frac{1}{2}\left(x_p y_d - x_d y_p\right) \\ &= \frac{1}{2}\left(a \cos t \cdot b \cos t - (-a) \sin t \cdot b \sin t\right) = \frac{1}{2}ab = \triangle ABC . \quad \blacksquare \end{aligned}$$

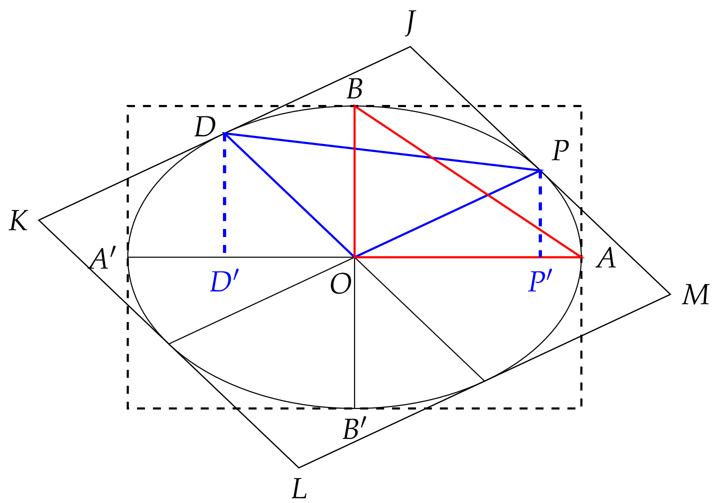


Figure 7.10: Parallelograms formed by conjugate diameters

Chapter 8

Ellipses in Euclidean geometry

The proofs of theorems about planetary freely used analytic geometry and trigonometry, but for many years after the invention of analytic geometry, mathematicians continued to limit themselves to Euclidean geometry. In this section, I present proofs in Euclidean geometry of theorems that appeared in Chapter 7.[1] The proofs are based on the definition of ellipses in terms of the geometric concepts of focus and directrix instead of the familiar analytic definition (Equation 7.1).

8.1 The definition of an ellipse using the focus and the directrix

Definition 8.1.1 *Let d be a line (the* directrix*) and S be a point (the* focus*) not on the directrix. Let $0 < e < 1$ be a number (the* eccentricity*). An* ellipse *is the locus of points P such that the ratio of PS to the distance of P to the directrix is e.*

All the conic sections (parabolas, ellipses and hyperbolas) are defined the same way and are distinguished by their eccentricity.

Definition 8.1.2 *Let X be the intersection of the perpendicular to the directrix from S. A on SX is a* vertex *of the ellipse if $SA/AX = e$ (in Figure 8.1, $e = 1/2$).*

The computation of AX and $A'X$ from the given SX is presented in Appendix A, along with an explanation how to obtain a, b, c of Definition 7.1.1 from e and SX.

Definition 8.1.1 is non-constructive. It states that the ellipse is the locus of points satisfying a certain property, but aside from the vertices we have not constructed any such points. Here we show how to construct any of the points on the ellipse.

[1] Theorems 7.5.1, 7.6.3 were proved using Euclidean geometry and Theorem 7.4.1 requires taking limits.

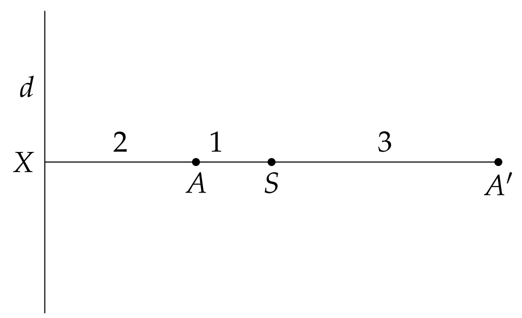


Figure 8.1: The elements of the definition of an ellipse

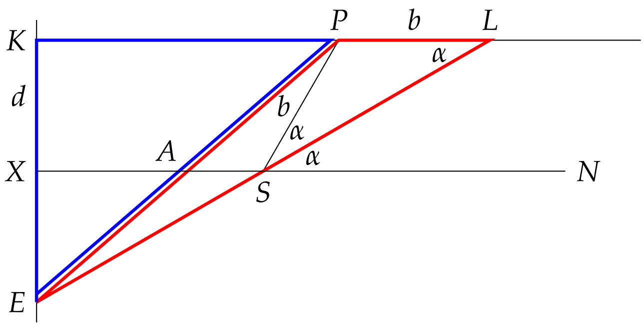


Figure 8.2: Constructing points on the ellipse

Select an *arbitrary* point E on the directrix and construct lines from E through A and S. The line through S will make an angle α with SX. Construct a line from S at the *same angle* α from ES and let its intersection with EA be P. Construct the perpendicular from P to the directrix and let K be its intersection with the directrix. Let L be the intersection of KP with ES (Figure 8.2).

Theorem 8.1.3 *The point P is on the ellipse.*

Proof $\angle PLS = \angle LSN = \alpha$ by alternate interior angles, so $\triangle LPS$ is isosceles and $PL = SP$. Since $PK \parallel SX$, $\triangle XEA \sim \triangle KEP$ and $\triangle AES \sim \triangle PEL$ are adjacent pairs of similar triangles, so

$$\frac{PS}{PK} = \frac{PL}{PK} = \frac{SA}{AX} = e\,.$$

Therefore, P is on the ellipse. ∎

By choosing different points E on the directrix, any point on the ellipse can be constructed.

8.2 A right angle at the focus of an ellipse

Theorem 8.2.1 *Let P, P' be points on the ellipse and let F be the intersection of PP' with the directrix. Then FS bisects the exterior angle of $\angle P'SP$(Figure 8.3).*

Proof Since P, P' are on the ellipse

$$\frac{SP}{PK} = \frac{SP'}{P'K'} = e\,,$$

and since $\triangle PFK \sim P'FK'$,

$$\frac{SP}{SP'} = \frac{PK}{P'K'} = \frac{PF}{P'F}\,.$$

By the exterior angle bisector theorem (Theorem B.4.2), FS bisects the exterior angle of $\angle P'SP$. ∎

Theorem 8.2.2 *Let P be a point on the ellipse and construct lines PA, PA'. Label their intersections with the directrix by E and F, respectively. Then $\angle FSE$ is a right angle (Figure 8.4).*

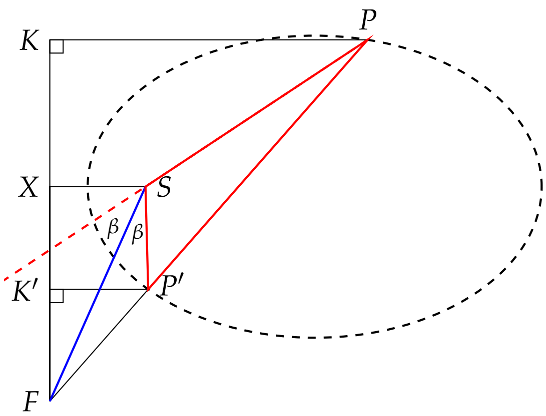


Figure 8.3: Bisecting the angle at the focus

Proof P, A, A' are all points on the ellipse so Theorem 8.2.1 applies. FS bisects $\angle PSX = 2\gamma$ and ES bisects $\angle P'SX = 2\delta$, so $2\gamma + 2\delta = 180°$ and $\angle FSE = \gamma + \delta = 90°$. ∎

8.3 Ratios of perpendiculars to the axes

The following theorem proves Theorem 7.6.2 using Euclidean geometry.

Theorem 8.3.1 *Let P be a point on an ellipse not on the major axis and construct perpendiculars PN, PM from P to the major and minor axes, respectively (Figure 8.5). Then*

$$\frac{PN^2}{A'N \cdot NA} = \frac{BC^2}{AC^2} = \frac{b^2}{a^2} \tag{8.1}$$

$$\frac{PM^2}{B'N \cdot NA} = \frac{AC^2}{BC^2} = \frac{a^2}{b^2}\,. \tag{8.2}$$

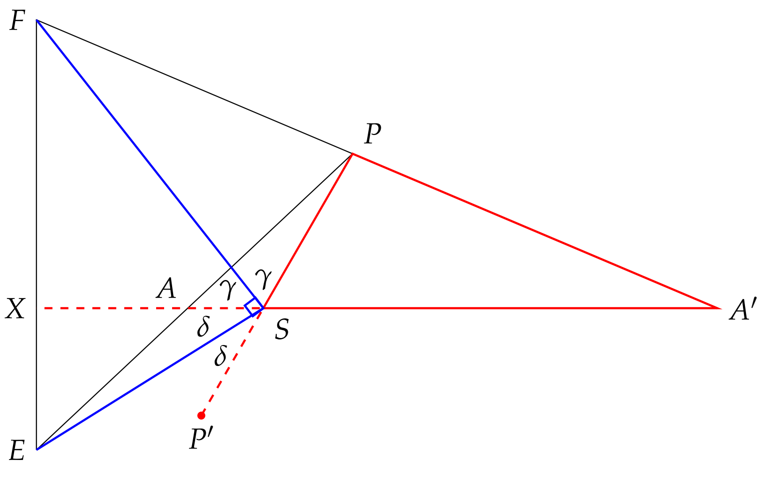


Figure 8.4: The right angle at the focus

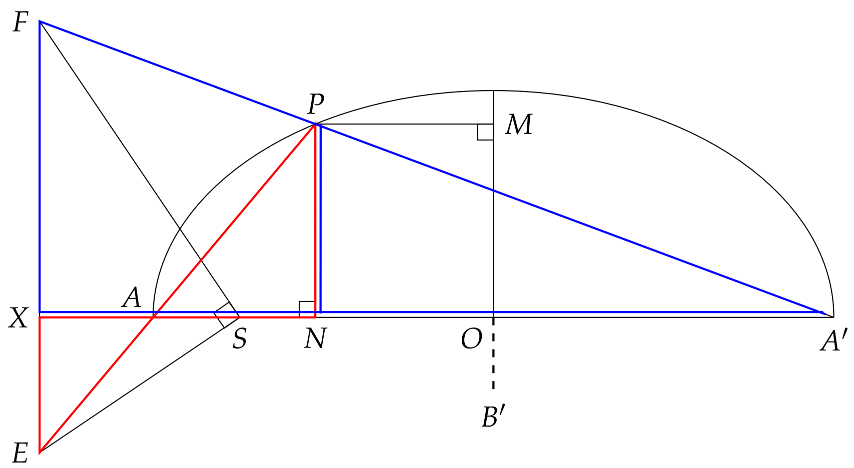


Figure 8.5: Ratio of an ordinate

Proof (Equation 8.1) $\triangle AXE \sim \triangle ANP$ since they are right triangles and the vertical angles at A are equal (red). Therefore,

$$\frac{PN}{AN} = \frac{EX}{AX}\,. \tag{8.3}$$

$\triangle PA'N \sim \triangle FA'X$ (blue) so

$$\frac{PN}{A'N} = \frac{FX}{A'X}\,. \tag{8.4}$$

Multiplying Equations 8.3 and 8.4 gives

$$\frac{PN^2}{AN \cdot A'N} = \frac{EX \cdot FX}{AX \cdot A'X}\,.$$

By Theorem 8.2.2 $\triangle FSE$ is a right triangle so by Theorem B.2.2,

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X}\,.$$

Since P was arbitrary this holds for any point on the ellipse, in particular, for B on the minor axis, where $PN = BC$ and $AN = AN' = AC$. Therefore,

$$\frac{BC^2}{AC^2} = \frac{SX^2}{AX \cdot A'X}$$

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X} = \frac{BC^2}{AC^2} = \frac{b^2}{a^2}\,. \quad \blacksquare$$

Proof (Equation 8.2) Since $CM = PN$, $PM = CN$, by Theorem B.2.1, Equation 8.1 becomes

$$\frac{CM^2}{AC^2 - PM^2} = \frac{BC^2}{AC^2}$$

$$\frac{AC^2}{AC^2 - PM^2} = \frac{BC^2}{CM^2}\,. \tag{8.5}$$

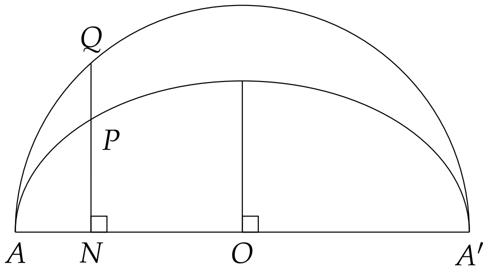


Figure 8.6: A circle circumscribing an ellipse

It follows that

$$\frac{AC^2}{PM^2} = \frac{BC^2}{BC^2 - CM^2}, \tag{8.6}$$

By cross-multiplying the Equations 8.5 and 8.6. By Theorem B.2.1, Equation 8.6 implies

$$\frac{PM^2}{BM \cdot MB'} = \frac{AC^2}{BC^2}. \quad \blacksquare$$

8.4 A circle circumscribing an ellipse

The following theorem proves Theorem 7.2.1 in Euclidean geometry.

Theorem 8.4.1 *Consider a circle of radius a with the same center as an ellipse (Figure 8.6). Choose a point N on the major axis and construct a perpendicular through N. Let its intersections with the ellipse and the circle be P and Q, respectively. Then*

$$\frac{PN}{QN} = \frac{BC}{AC} = \frac{b}{a}.$$

Proof From Theorem 8.3.1,

$$\frac{PN^2}{AN \cdot NA'} = \frac{BC^2}{AC^2},$$

and by Theorem B.2.2, $AN \cdot NA = QN^2$. $\blacksquare$

8.5 The latus rectum of an ellipse

The following theorem proves Theorem 7.3.2 in Euclidean geometry.

Theorem 8.5.1 *L, the length of the latus rectum of an ellipse, is* $\frac{2b^2}{a}$ *(Figure 8.7).*

Proof By Theorem 8.3.1,

$$\frac{SL_1^2}{AS \cdot SA'} = \frac{BC^2}{AC^2}.$$

By Theorem 7.2, $SB = AC = a$, so by Pythagoras's theorem,

$$BC^2 = BS^2 - SC^2 = AC^2 - SC^2 = (AC - SC)(AC + SC) = AS \cdot SA'.$$

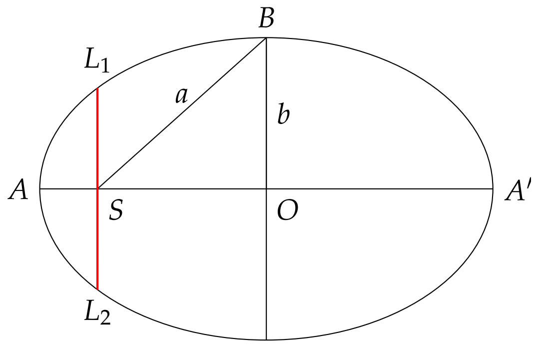


Figure 8.7: The latus rectum of an ellipse

Therefore, the length of one-half the latus rectum is

$$SL_1^2 = \frac{BC^4}{AC^2}$$

$$SL_1 = \frac{BC^2}{AC} = \frac{b^2}{a} . \quad \blacksquare$$

8.6 Areas of parallelograms

Theorem 8.6.1 *Let Y be the intersection the perpendicular through the focus S to the tangent TT' at P, and let L be the intersection of $S'P$ and SY (Figure 8.8). Then Y is on the circumscribing circle and $CY \parallel S'L$.*

Proof

$\triangle STY \sim \triangle T'TC$ since they are right triangles that share the acute angle $\angle CTT' = \angle YTS$, so $\angle CT'T = \angle YST = \beta$.

By Theorem 7.5.1, $\angle SPY = \angle S'PT' = \alpha$ since they are the angles to the foci at the tangent. $\angle S'PT' = \angle YPL = \alpha$ are vertical angles, so $\angle SPY = \angle LPY = \alpha$.

Then $\triangle SPY \cong \triangle LPY$ since they are right triangles with an equal acute angle and a common side PY. Therefore, $PL = PS$ and $S'L = S'P + PL = AA' = 2a$.

Since $\triangle SPY \cong \triangle LPY$, $SY = YL$, and since S, S' are foci, $S'C = SC$. It follows that $\triangle CSY \sim \triangle S'SL$ and $CY \parallel S'L$. By similarity,

$$\frac{CY}{S'L} = \frac{CS}{S'S} = \frac{CS}{2CS} .$$

Therefore, $2CY = S'L = 2a$ so $CY = a$ and Y is on the circumscribing circle of radius a. $\blacksquare$

Theorem 8.6.2 *Let N be the intersection of the perpendicular through P to the major axis (Figure 8.9). Then $CN \cdot NT = AC^2 = AN \cdot NA'$.*

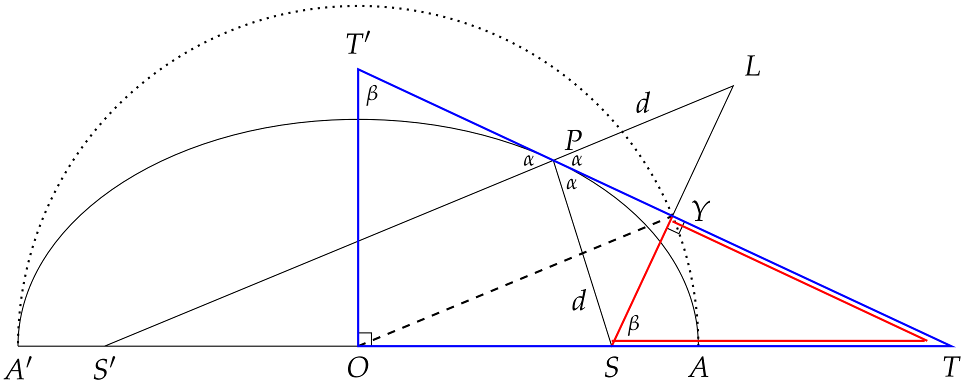


Figure 8.8: The perpendicular from a focus to a tangent

Proof Continuing with the construction from Figure 8.8, we focus on the segments AN, NA' (Figure 8.9). We can deduce the angles that are shown.

- $CY \parallel S'P$ (Theorem 8.6.1) so $\angle CYP = \angle S'PT'$ by corresponding angles.
- $\angle S'PT' = \angle SPY$ by Theorem 7.5.1 since they are the angles to the foci at the tangent.
- $\angle SYP$ and $\angle SNP$ are right angles and therefore $SYPN$ is quadrilateral that can be circumscribed by a circle whose diameter is PS.[2] Therefore, $\angle SPY = \angle SNY$ since they are subtended by the same chord YS.

Since $\angle CYT \sim \angle CNY$ and $\angle YCT$ is a common angle, $\triangle CYT \sim \triangle CNY$ and

$$\frac{CN}{CY} = \frac{CY}{CT}$$

[2]It can be proven that a quadrilateral whose opposite angle are supplementary can be circumscribed by a circle. If two opposite angles are right angles that sum to 180°, then the other two angles must also sum to 180°.

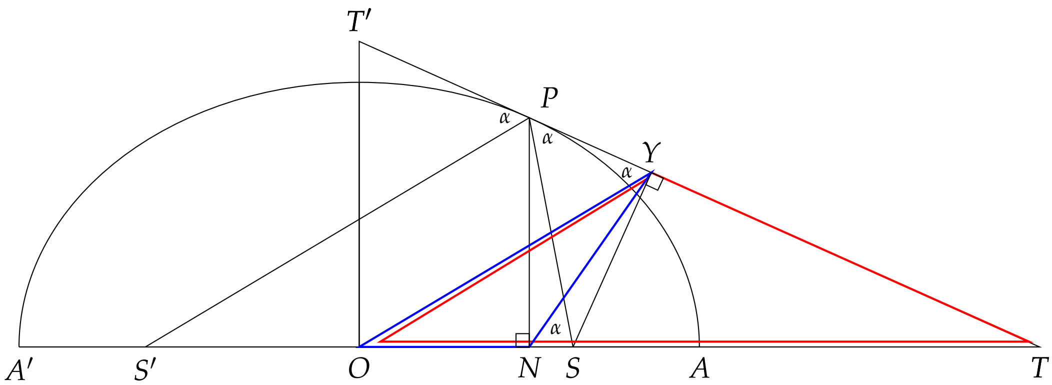


Figure 8.9: Ratios of segments of the major axis

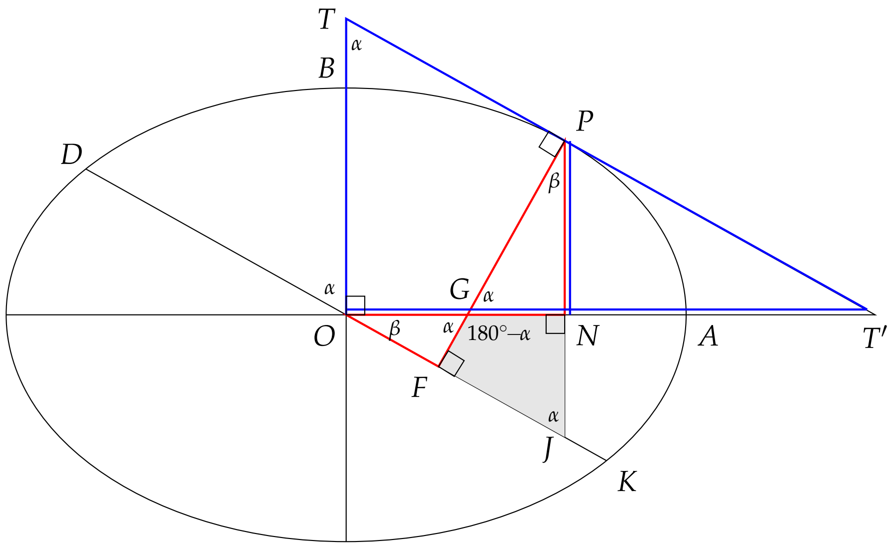


Figure 8.10: Parallelograms formed by conjugate diameters

$$CN \cdot CT = CY^2 = AC^2 , \tag{8.7}$$

since the perpendicular to the tangent from a focus is on the circumscribing circle (Theorem 8.6.1). Since $NT = CT - CN$, we have $CN \cdot NT = CN \cdot CT - CN^2$ which equals $AC^2 - CN^2$ by Equation 8.7. This in turn equals $AN \cdot NA'$ by Theorem B.2.1. ▮

Construct the normal to the tangent at P and let its intersection with the conjugate diameter DK be F and its intersection with the major axis be G. Construct a perpendicular from P to the major axis and let its intersection be N. Let the intersection of the tangent with the minor axis be T and its intersection with the major axis be T' (Figure 8.10).

Theorem 8.6.3

$$PF \cdot PG = BC^2 .$$

Proof $\triangle NPG \sim \triangle FPJ$ (rotate $\triangle NPG$ to see this) so $\angle PGN = \angle PJF = \alpha$ and

$$\frac{PF}{PN} = \frac{PJ}{PG}$$

$$PF \cdot PG = PJ \cdot PN . \tag{8.8}$$

By vertical angles $\angle PGN = \angle CGF = \alpha$ so $\triangle NPG \sim \triangle FCG$ and $\angle NPG = \angle FCG = \beta = 90^\circ - \alpha$. By adding β to the right angles $\angle BCN$ and $\angle TPF$, we get that $\angle TCJ = \angle TPJ$ and therefore $TPJC$ is a parallelogram, so $CT = PJ$ and $PF \cdot PG = CT \cdot PN$. By Equation 8.8, the theorem will be proven if we can show that $CT \cdot PN = BC^2$.

$\triangle TT'C \sim \triangle PT'N$ so

$$\frac{CT}{CT'} = \frac{PN}{NT'}$$

$$\frac{CT}{PN} = \frac{CT'}{NT'} .$$

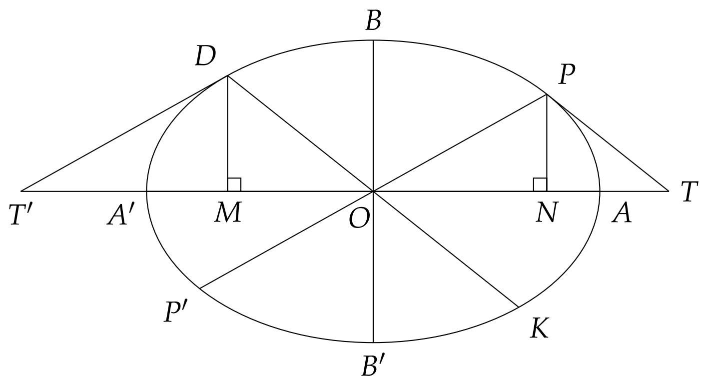


Figure 8.11: Ratios of perpendiculars to the major axis

Multiplying each side by fractions equal to 1 gives

$$\frac{CT \cdot PN}{PN^2} = \frac{CT' \cdot CN}{CN \cdot NT'} .$$

By Equation 8.7, $CN \cdot CT' = AC^2$, and by Theorem 8.6.2, $CN \cdot NT' = AN \cdot NA'$, so

$$\frac{CT \cdot PN}{PN^2} = \frac{AC^2}{AN \cdot NA'} .$$

Multiplying by PN^2/PN^2 and using Theorem 8.3.1 gives

$$\begin{aligned} \frac{CT \cdot PN}{PN^2} &= \frac{PN^2}{AN \cdot NA'} \cdot \frac{AC^2}{PN^2} = \frac{BC^2}{AC^2} \cdot \frac{AC^2}{PN^2} \\ CT \cdot PN &= BC^2 . \quad \blacksquare \end{aligned} \tag{8.9}$$

Theorem 8.6.4 *In Figure 8.11,*

$$\begin{aligned} CN^2 &= AM \cdot MA', \quad CM^2 = AN \cdot NA' \\ \frac{DM}{CN} &= \frac{BC}{AC} \\ \frac{CM}{PN} &= \frac{BC}{AC} \end{aligned}$$

Proof By Theorem 8.6.2,

$$\begin{aligned} CN \cdot CT &= AC^2 = CM \cdot CT' \\ \frac{CM}{CN} &= \frac{CT}{CT'} . \end{aligned}$$

Since DK and PP' are conjugate diameters, $DT' \parallel PP'$ and $\triangle T'DC \sim \triangle CPT$, so

$$\begin{aligned} \frac{CM}{CN} &= \frac{CT}{CT'} = \frac{CN}{MT'} \\ CN^2 &= CM \cdot MT' , \end{aligned}$$

Therefore,

$$CN^2 = CM \cdot MT' = AC^2 = AM \cdot MA' \tag{8.10}$$

by Theorem 8.6.2. By Theorem 8.3.1,

$$\frac{DM^2}{AM \cdot MA'} = \frac{BC^2}{AC^2},$$

and by Equation 8.10,

$$\begin{aligned}\frac{DM^2}{CN^2} &= \frac{BC^2}{AC^2}\\ \frac{DM}{CN} &= \frac{BC}{AC},\end{aligned}$$

A symmetric argument shows that

$$\begin{aligned}CM^2 &= AN \cdot NA'\\ \frac{CM}{PN} &= \frac{BC}{AC} \quad \blacksquare.\end{aligned}$$

This theorem proves Theorem 7.8.1 in Euclidean geometry.

Theorem 8.6.5 *The area of the parallelogram formed by the tangents at the ends of the conjugate diameters PP', DK is equal to the area of the rectangle enclosing the ellipse at the ends of the axes (Figure 8.12).*

Proof By the definition of conjugate diameters, it is sufficient to show that the area of $PCDL$ is $AC \cdot BC$. The area of a parallelogram is width times height so it is $CD \cdot PF$.

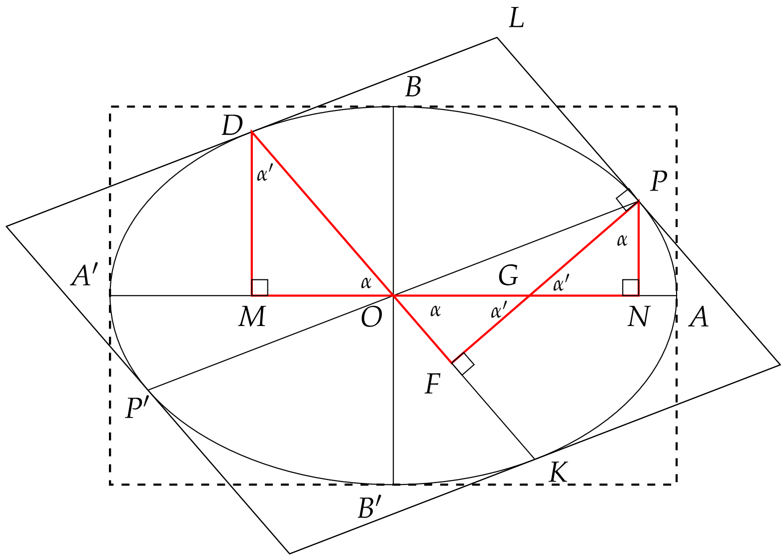


Figure 8.12: Areas of parallelograms ($\alpha' = 90° - \alpha$)

By vertical angles $\angle DCM = \angle GCF$ and $\angle CGF = \angle PGN$, so $\triangle DCM \sim \triangle PGN$

$$\frac{PG}{CD} = \frac{PN}{CM},$$

and by Theorem 8.6.4

$$\frac{PG}{CD} = \frac{AC}{BC}$$

$$\frac{CD}{AC} = \frac{PG}{BC}. \tag{8.11}$$

By Theorem 8.6.3,

$$\frac{PG}{BC} = \frac{BC}{PF}. \tag{8.12}$$

From Equations 8.11 and 8.12 gives $CD \cdot PF = AC \cdot BC$. ∎

Chapter 9

Lagrange points

Consider a spacecraft orbiting the Sun and subject to the gravitational force of both the Earth and the Sun. Joseph-Louis Lagrange and Leonhard Euler discovered that there are five points where the spacecraft rotates with the same orbital period as the Earth and thus appears to maintain a fixed position as viewed from the Earth. These points are called the *Lagrange points* $L1, L2, L3, L4, L5$ and their positions are shown in Figure 9.1.[1]

In this section we present an approximate derivation of the locations of $L1, L2, L3$. The most significant approximation is that we assume that the spacecraft orbits around the center of the Sun, whereas it actually orbits around the barycenter of the Sun and the Earth. The derivation of the locations of $L4, L5$ is beyond the scope of this document. The final subsection describes the objects that exist at the Lagrange points.

9.1 Lagrange point $L1$

We assume that the Earth is in a circular orbit of radius r_E around the Sun and that the masses of the two satisfy $m_E \ll m_S$. Let us suppose that we wish to place a space telescope T of mass $m_T \ll m_E$ in a circular orbit at distance $r_1 \ll r_E$ from the Earth (Figure 9.2), such that T has the same orbital period as the Earth (one year). Is this possible?

[1] The orbits are clearly shown in the gif in the Wikipedia entry *Lagrange point*.

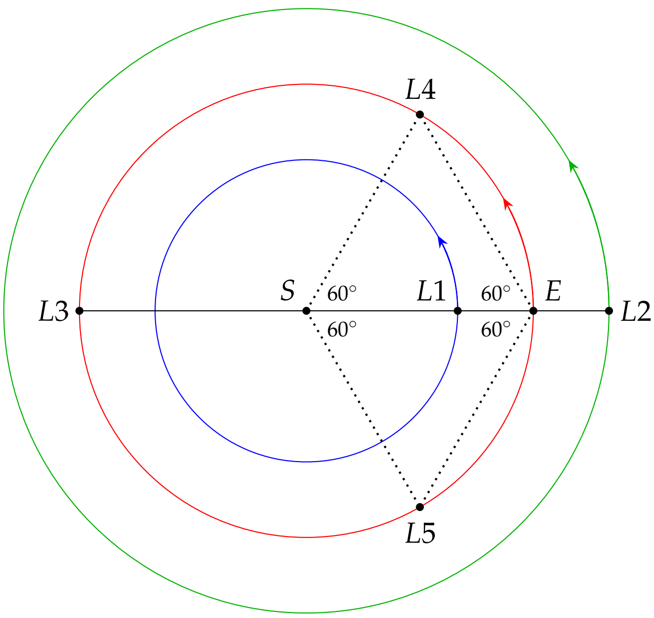


Figure 9.1: The five Lagrange points

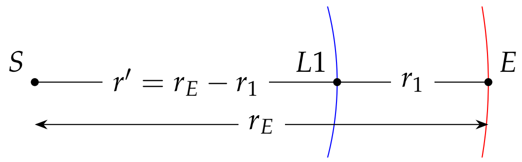


Figure 9.2: Lagrange point $L1$

If we ignore the gravitational force exerted by E on T at $L1$, obviously not, because by Kepler's third law

$$\frac{A_E^3}{T_E^2} = \frac{A_1^3}{T_E^2},$$

so $A_1 = A_E$ and T must be located at the center of the Earth. Can the gravitational force exerted by E on T cause $T_1 = T_E$ while r_1 is greater than the radius of the Earth?

You might think that T should be placed so that the gravitational force exerted by the Sun is exactly balanced by the gravitational force exerted by the Earth, but, of course, if there is no net force on T, by Newton's first law T would simply move in a straight line off into space. Instead, we want the net centripetal force on T to be

$$F = \frac{Gm_S m_T}{r'^2} - \frac{Gm_E m_T}{r_1^2}, \tag{9.1}$$

so that it moves in an orbit with period T_E. To simplify notation we let $r' = r_E - r_1$.

Since the length of the orbit of T at $L1$ is $2\pi r'$, T's velocity is

$$v_1 = \frac{2\pi r'}{T_1}. \tag{9.2}$$

As Newton did in his investigation of elliptical orbits, the motion from A to D is separated into a tangential motion AC where there is no force on T, followed by motion towards the center with a centripetal force from C to S (Figure 9.3). The distances indicated are those of the non-accelerated motion $v_t\Delta t$ and the accelerated motion $\frac{1}{2}a(\Delta t)^2$ during a very small time period Δt. We need to find the acceleration a that will cause T to reach D. By Pythagoras's theorem ,

$$\begin{aligned} r'^2 + (v_1\Delta t)^2 &= (r' + x)^2 = r'^2 + 2r'x + x^2 \\ (v_1\Delta t)^2 &= x(2r' + x). \end{aligned}$$

Since Δt is assumed to be very small and since r_T is close to r_E, $2r' + x \approx 2r'$, so by Newton's second law,

$$\begin{aligned} \frac{1}{2}a(\Delta t)^2 &= \frac{1}{2}\frac{v_1^2}{r'}(\Delta t)^2 \\ F &= m_T a = \frac{m_T v_1^2}{r'}. \end{aligned}$$

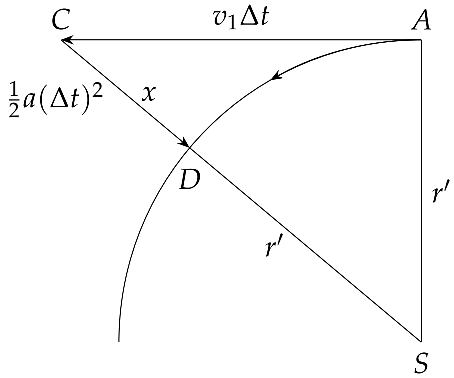


Figure 9.3: Non-accelerated and accelerated motion

By Equation 9.1 the force needed to keep T in the desired orbit is

$$F = \frac{m_T v_T^2}{r'} = \frac{Gm_S m_T}{r'^2} - \frac{Gm_E m_T}{r_1^2},$$

and the velocity of T at $L1$ must satisfy

$$v_1^2 = \frac{Gm_S}{r'} - \frac{Gm_E r'}{r_1^2}.$$

The period of the desired orbit is $T_1 = T_E$ so by Equation 9.2,

$$\begin{aligned} \frac{4\pi^2 r'^2}{T_E^2} &= \frac{Gm_S}{r'} - \frac{Gm_E r'}{r_t^2} \\ \frac{4\pi^2}{T_E^2} &= \frac{Gm_S}{r'^3} - \frac{Gm_E}{r' r_t^2}. \end{aligned}$$

But by Kepler's third law (Equation 5.4), where the elliptical semi-major axis a_i is the circular radius r_E,

$$\begin{aligned} \frac{4\pi^2 r_E^3 m}{T_E^2}\frac{1}{r_E^2} &= \frac{GmM}{r_E^2} \\ \frac{Gm_S}{r_E^3} &= \frac{Gm_S}{r'^3} - \frac{Gm_E}{r' r_1^2} \\ \frac{1}{r_E^3} &= \frac{1}{r'^3} - \frac{m_E/m_S}{r' r_1^2}. \end{aligned} \tag{9.3}$$

Let $y = m_E/m_S$ and $z = r_1/r_E$ so $r' = r_E - r_1 = r_E(1-z)$. Multiply by r_E^3 and make the substitutions.

$$\begin{aligned} \frac{r_E^3}{r'^3} - \frac{m_E/m_S r_E^3}{r' r_1^2} &= 1 \\ \frac{1}{(1-z)^3} - \frac{y r_E^3}{r_E(1-z)z^2 r_E^2} &= 1 \\ \frac{1}{(1-z)^3} - \frac{y}{z^2(1-z)} &= 1. \end{aligned}$$

Since $z = r_1/r_E$ is very small, we get the following approximations from the Taylor series [5, Chapter 11.8]:

$$\begin{aligned}
\frac{1}{(1-z)} &= 1 + z + z^2 + \cdots \approx 1 + z \\
\frac{1}{(1-z)^3} &= 1 + 3z + 6z^2 + \cdots \approx 1 + 3z \\
1 + 3z - \frac{y}{z^2}(1+z) &\approx 1 \\
3z^3 &\approx y(1+z) \approx y\,.
\end{aligned}$$

Let us plug in the numbers $m_S \approx 2 \times 10^{30}$ kg, $m_E \approx 6 \times 10^{24}$ kg, $r_E \approx 1.5 \times 10^8$ km.

$$\begin{aligned}
\left(\frac{r_1}{1.5 \times 10^8}\right)^3 &\approx \frac{6 \times 10^{24}}{3 \times 2 \times 10^{30}} = 10^{-6} \\
r_1 &\approx 1.5 \times 10^8 \cdot \sqrt[3]{10^{-6}} \approx 1.5 \times 10^6\,.
\end{aligned}$$

If an object is placed 1.5 million km from the Earth, the period of its orbit around the Sun will be approximately one year. This is quite far—the Moon is less than 400, 000 km from the Earth—but still relatively far from the Sun which is 150 million km away.

9.2 Lagrange point $L2$

The computation for $L2$ is similar using $r' = r_E + r_2$ (Figure 9.4). With the appropriate modifications to Equation 9.1 we get

$$\begin{aligned}
F &= \frac{Gm_S m_T}{r'^2} + \frac{Gm_E m_T}{r_2^2} \\
\frac{1}{r_E^3} &= \frac{1}{r'^3} + \frac{m_E/m_S}{r' r_2^2} \\
1 &= \frac{1}{(1+z)^3} + \frac{y}{z^2(1+z)}\,.
\end{aligned}$$

The approximations based on the Taylor series are $(1+z)^{-3} \approx 1 - 3z$ and $(1+z)^{-1} \approx 1 - z$, leading to the same equation $3z^3 \approx y$. Therefore, $L2$ is the same distance from the Earth as $L1$ but on the opposite side of the Earth.

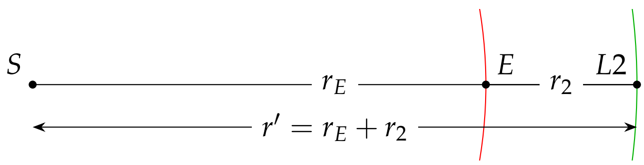


Figure 9.4: Lagrange point $L2$

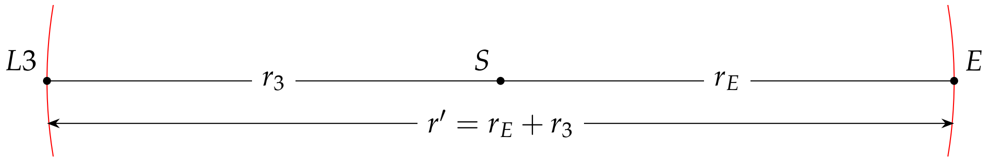


Figure 9.5: Lagrange point $L3$

9.3 Lagrange point $L3$

The Lagrange point $L3$ is on the other side of the Sun (Figure 9.5). The modifications to Equation 9.3 give

$$\begin{aligned}
\frac{1}{r_E^3} &= \frac{1}{r_3^3} + \frac{m_E/m_S}{r'^3} \\
1 &= \frac{1}{z^3} + \frac{y}{z^3(r_E + r_3)^3} \\
1 &= \frac{1}{z^3} + \frac{y}{(1+z)^3} \\
z^3 &= \frac{1}{1-y},
\end{aligned}$$

since $z \ll 1$. But $y \approx 10^{-6}$ so $z^3 \approx 1$, $r_3 \approx r_E$ and T is approximately the same distance from the Sun as it is from the Earth.

9.4 Objects at the Lagrange points

Although the orbits objects at $L1$, $L2$ and $L3$ are not stable, they are relatively stable so a spacecraft can be placed into a small orbit around one of these points. Even when it drifts, the force required to return it to the Lagrange point is small, which means that the propellant in the spacecraft can maintain it on station for a long time.

The *Deep Space Climate Observatory (DSCOVR)* was placed at Lagrange point $L1$. It continually observes the Sun and the sunlit side of the Earth. The *James Webb Space Telescope* with its 6.5 meter diameter infrared telescope was placed at $L2$ in 2022. $L2$ is ideal for telescopes: if a sun shield is placed facing the Earth and the Sun, the spacecraft itself can remain at the very low temperature that its sensors require. Lagrange point $L3$ is not useful for spacecraft because the line-of-sight to the Earth is blocked by the Sun.

The orbits of objects at $L4$ and $L5$ are stable. Asteroids that are stable at a Lagrange point are called *trojans* and most are located at the $L4$ and $L5$ points of Jupiter. There are two extremely small trojans at the Earth's $L4$ point.

Appendix A

The Two Definitions of an Ellipse

Ellipses were defined in two ways (refer to Figure A.1).

Definition 7.1.1: Given S and H, two points (the *foci*) such that $SH = 2c > 0$, and $2a > 2c > 0$, an *ellipse* is the locus of points P such that $SP + PH = 2a$ (red). The *eccentricity* is c/a.

Definition 8.1.1: Given a line (the *directrix*), a point S (a *focus*) at distance $SX = d > 0$ from the directrix and $0 < e < 1$ (the *eccentricity*), an *ellipse* is the locus of points P such that $SP/PK = e$ (blue). A on SX is a *vertex* of the ellipse if $SA/AX = e$.

In this appendix we show how to compute the parameters of Definition 7.1.1 from those of Definition 8.1.1 and conversely. First, we compute a and c from d and e.

AX and $A'X$ can be computed from d and e:

$$\begin{aligned} SA + AX &= SX = d \\ SA &= d - \frac{SA}{e} = d \cdot \frac{e}{1+e} \\ AX &= d \cdot \frac{1}{1+e} \\ A'X - SA' &= d \\ SA' &= \frac{SA'}{e} - d = d \cdot \frac{e}{1-e} \\ A'X &= d \cdot \frac{1}{1-e}\,. \end{aligned} \tag{A.1}$$

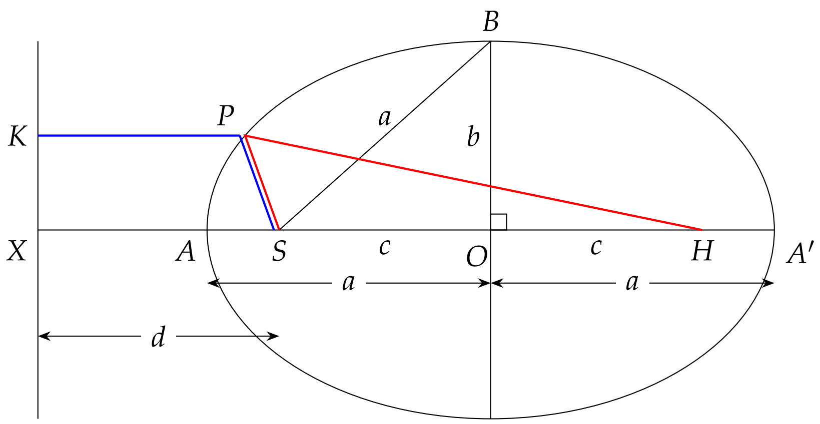


Figure A.1: Two definitions of an ellipse

$a = AA'/2$ can now be computed from $A'X - AX$:

$$
\begin{aligned}
a = \frac{AA'}{2} &= \frac{1}{2}(A'X - AX) = \frac{d}{2}\left(\frac{1}{1-e} - \frac{1}{1+e}\right) \\
&= \frac{d}{2} \cdot \frac{2e}{1-e^2} = d \cdot \frac{e}{1-e^2}\,.
\end{aligned} \tag{A.2}
$$

$c = OS$ is $a - SA$ so by Equations A.1, A.2,

$$
c = OS = a - SA = d \cdot \frac{e}{1-e^2} - d \cdot \frac{e}{1+e} = d \cdot \frac{e^2}{1-e^2}\,.
$$

Finally, b can be computed from $b = \sqrt{a^2 - c^2}$:

$$
b = d \cdot \frac{e}{\sqrt{1-e^2}}\,.
$$

Conversely, by Equation A.2, d can be computed from a and e:

$$
d = a \cdot \frac{1-e^2}{e}\,.
$$

As an example, we compute the e-factors for $e = \sqrt{5}/3$ and then multiply it by various values of d:

$$
\begin{aligned}
a/d &= \frac{e}{1-e^2} = \frac{\sqrt{5}/3}{4/9} = \frac{3\sqrt{5}}{4} \\
c/d &= \frac{e^2}{1-e^2} = \frac{5/9}{4/9} = \frac{5}{4} \\
b/d &= \frac{e}{\sqrt{1-e^2}} = \frac{\sqrt{5}/3}{2/3} = \frac{\sqrt{5}}{2}\,.
\end{aligned}
$$

For $d = 4/\sqrt{5}$ we have:

$$
a = 3, \qquad b = 2, \qquad c = \sqrt{5}\,.
$$

Conversely,

$$
d = a \cdot \frac{1-e^2}{e} = 3 \cdot \frac{4/9}{\sqrt{5}/3} = \frac{4}{\sqrt{5}}\,.
$$

Figure A.1 was drawn with $d = \sqrt{5}$ giving:

$$
a = 15/4 = 3.75, \qquad b = 5/2 = 2.5, \qquad c = 5\sqrt{5}/4 \approx 2.8\,.
$$

Appendix B

Theorems of Euclidean Geometry

B.1 Constructing a circle from three points

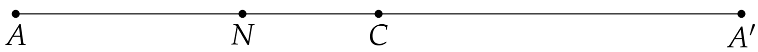
Theorem B.1.1 *Given three non-collinear points,*[1] *a circle can be constructed that goes through all three points.*

Proof Three non-collinear points A, B, C define a triangle $\triangle ABC$ (Figure B.1). Construct the perpendicular bisectors of any two of its three sides, say, AC and BC. By definition the perpendicular bisector is the geometric locus of points equidistant from the endpoints of the segment. Let O be the intersection of the two bisectors. Then the $AO = CO = BO$ is the radius of a circle centered at O that goes through A, B, C. ∎

B.2 The product of two subsegments

Theorem B.2.1 *Let AA' be a line segment whose midpoint is C. Then*

$$AC^2 - CN^2 = AN \cdot NA' .$$



Proof $AN = AC - CN$ and $NA' = A'C + CN = AC + CN$ since C is the midpoint of AA'. The result is obtaining by multiplying the two equations. ∎

[1]*Collinear* means that the points all on the same line.

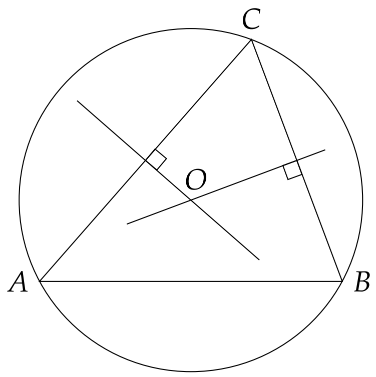


Figure B.1: A circle through three arbitrary points

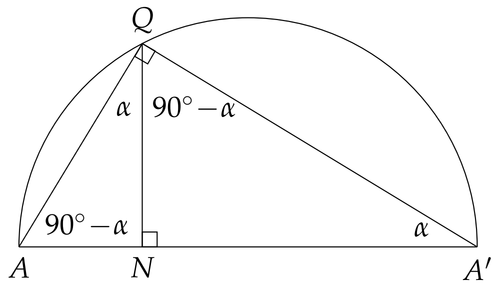


Figure B.2: Right triangle in a circle at the diameter

Theorem B.2.2 *Let Q be a point on a circle whose diameter is AA' and construct a perpendicular QN to the diameter (Figure B.2). Then*

$$QN^2 = AN \cdot NA' .$$

The equation also holds if it is given *that $\triangle AQA'$ is a right triangle.*

Proof An angle that subtends a diameter is a right angle. Since the sum of the angles of a triangle is 180°, we can label the angles as shown in the Figure, from which follows that $\triangle QNA \sim \triangle A'NQ$. Therefore,

$$\frac{QN}{AN} = \frac{NA'}{QN} . \quad \blacksquare$$

B.3 Adjacent pairs of similar triangles

I have not encountered the following definition before but it explicitly expresses a relation among similar triangles that would have been obvious to geometers.

Definition B.3.1 *An* adjacent pair of similar triangles *is a pair of (a pair of) similar triangles that share sides. In Figure B.3, $\triangle BAC \sim \triangle EAF$ and $\triangle CAD \sim \triangle FAG$ are an adjacent pair of similar triangles.*

Theorem B.3.2 *For the adjacent pair of similar triangles in Figure B.3,*

$$\frac{AB}{AE} = \frac{AD}{AG} .$$

Proof By similar triangles,

$$\frac{AB}{AE} = \frac{AC}{AF}$$

$$\frac{AC}{AF} = \frac{AD}{AG}$$

$$\frac{AB}{AE} = \frac{AD}{AG} . \quad \blacksquare$$

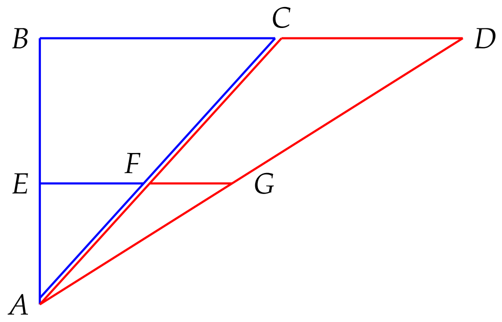


Figure B.3: Adjacent pairs of similar triangles

Similar ratios hold between other sides of $\triangle BAC$ and $\triangle CAD$ by using an intermediate step with AC. We will use the term *by an adjacent pair of similar triangles* and leave it to the reader to make the intermediate step.

B.4 The angle bisector theorems

Theorem B.4.1 (Interior angle bisector theorem) *In $\triangle ABC$ let D be a point on BC (Figure B.4). Then AD bisects $\angle CAB$* if and only if

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Proof Suppose that AD bisects $\angle BAC$. Construct a line through C parallel to AB and let its intersection with AD be E. By alternate interior angles, $\angle BAD = \angle CED =$ and by vertical angles $\angle BDA = \angle CDE$. Therefore, $\triangle ABD \sim \triangle EDC$ so

$$\frac{BD}{CD} = \frac{AB}{CE}.$$

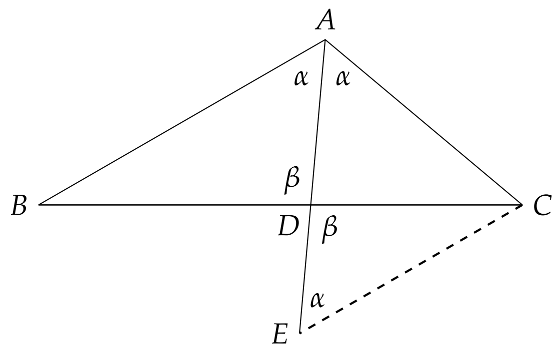


Figure B.4: The interior angle bisector theorem

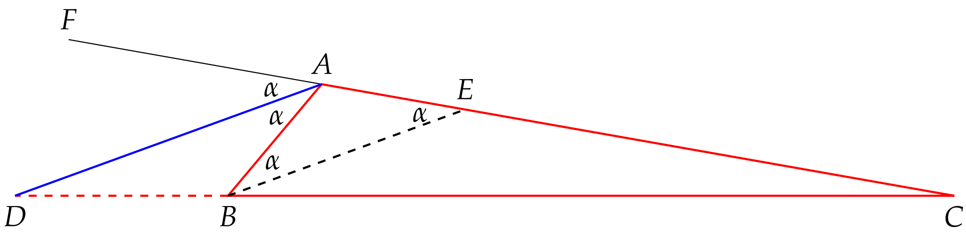


Figure B.5: The exterior angle bisector theorem

$\triangle ECA$ is isosceles so $CE = AC$ and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just "run" the proof backwards. ∎

Theorem B.4.2 (Exterior angle bisector theorem) *In $\triangle ABC$ let D be a point on the extension of CB outside the triangle (Figure B.5). Then AD bisects the exterior angle of $\angle BAC$ if and only if*

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Proof Suppose that AD bisects $\angle BAF$. Construct a line through B parallel to AD and let its intersection with AC be E. By alternate interior angles $\angle BAD = \angle ABE$ and by corresponding angles $\angle FAD = \angle AEB$. Therefore, $\triangle BCE \sim \triangle DCA$ so

$$\frac{BD}{CD} = \frac{AE}{AC}.$$

But $\triangle BAE$ is isosceles so $AE = AB$ and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just "run" the proof backwards. ∎

> The exterior angle bisector theoren confusing to understand in a proof, because it can be hard to identify the components of a diagram. In the text the following color-coding is used: the triangle is red, the extension of one side is dashed red and the bisector is blue.

Sources and further reading

Sections 2–5 are primarily on Hahn's book [5]. He has written a more advanced book on the orbits of planets and spacecraft [6]. The proof in Chapter 6 is from [4]. Two additional articles present this aspect of Newton's work [7, 9]. A modern translation of Newton's *Principia* is [2] and Cohen's lengthy *Guide* will facilitate understanding Newton's often terse presentation. The relevant sections of the Guide are 10.8–10.10. Chapter 8 is based on Besant's textbook [1]. Drew wrote a similar shorter textbook [3]. Compare these textbooks with [8] that uses analytic geometry.[2] The computations of Chapter 9 are from [10].

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[2] The dates given for these books are for the editions that I used.