Figure 1: Equal angles between tangents and lines to the foci

Theorem 11.7 - "intersection of the perpendicular"

statement of theorem 11.8

The following theorem is used in Section 0.1.

Theorem 0.1 (Prop. XIII) *If two tangents are constructed to an ellipse from an external point, the angle between them and lines to the foci are equal (Figure 5).*

Proof Let QP, QP' be the tangents, $SY, S'Y', SZ, S'Z'$ the perpendiculars from the foci on the tangents, and construct $SQ, S'Q$. We need to show that the blue and red angles are equal.

By Theorem **??**, $SY \cdot S'Y' = BC^2 = SZ \cdot S'Z'$, so

$$\frac{SY}{SZ} = \frac{S'Z'}{S'Y'}.\tag{1}$$

The quadrilateral $SYQZ$ can be circumscribed by a circle since two opposite vertices are right angles, and similarly for $S'Y'QZ'$. Then $\angle YSZ$ is supplementary to $\angle YQZ$ and $\angle Z'S'Y'$ is supplementary to $\angle Z'QY' = \angle YQZ$. Therefore, $\angle YSZ = \angle Z'S'Y'$. Together with Equation 1 we conclude that $\triangle YSZ \sim Z'S'Y'$. Now $\angle SQP = \angle SZY$ since they are both inscribed angles subtended by the chord SY and similarly $\angle S'Y'Z' = \angle S'QP'$. Therefore, $\angle SQP = \angle SZY = \angle S'Y'Z' = \angle S'QP'$. ∎

0.1 Marden's Theorem

According to Dan Kalman, Marden's Theorem is "the Most Marvelous Theorem in Mathematics" [**?**]. This section explains his proof of this theorem; it assumes that the reader is familiar with complex numbers.

The statement of the theorem

Theorem 0.2 (Marden) *Let $p(z)$ be a cubic polynomial with complex coefficients whose roots z_1, z_2, z_3 are non-collinear in the complex plane. Let T be the triangle whose vertices are z_1, z_2, z_3. Then there is a unique ellipse E (called the* Steiner inellipse*) inscribed within T such that the sides of T are tangent to the ellipse at their midpoints. f_1, f_2, the foci of the ellipse, are the roots of $p'(z)$ (Figures 2, 3).*

We omit the relatively simple proof that without loss of generality we can assume that two of the roots are 1 and -1 [**?** , p. 332], and that the third root w is above the x-axis: $w = u + vi, v > 0$. This follows since the theorem remains true if the triangle is translated, rotated or scaled.

There are three steps to the proof. First, we show that there is *an* ellipse which is tangent to one side of the triangle at its midpoint. Then we show that this ellipse is tangent to *all* three sides of the triangle, and finally, that the points of contact are the midpoints of the side.

There is an ellipse that is tangent to one side of the triangle

Theorem 0.3 *Let $p(z), p'(z), T, E, f_1, f_2$ be as defined in the statement of Marden's theorem. Then there is* an *ellipse E which is tangent to one side of T at its midpoint.*

First example Suppose that the third root is on the y-axis, say at i (Figure 2). Then

$$p(z) = (z + 1)(z - 1)(z - i) = z^3 - iz^2 - z + i)$$
$$p'(z) = 3z^2 - 2iz - 1 = 3\left(z^2 - \frac{2}{3}iz - \frac{1}{3}\right).$$

Using the quadratic formula we can compute the roots of $p'(z)$ $f_{1,2} = (1/3)(i \pm \sqrt{2})$, and it is easy to verify that the following relationship holds between the roots f_1, f_2 and the coefficients of $p'(z)$.

$$f_1 + f_2 = \frac{2}{3}i$$
$$f_1 f_2 = -\frac{1}{3},$$

c, one-half the distance between the foci, is $\sqrt{2}/3$, and b, the semi-minor axis, is $1/3$, so a, the semi-major axis, is

$$a = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{\sqrt{3}}.$$

The general case The equations that were the developed at essentially the same for any position of the third root w above the x-axis.

$$p(z) = (z + 1)(z - 1)(z - w) = z^3 - wz^2 - z + i)$$
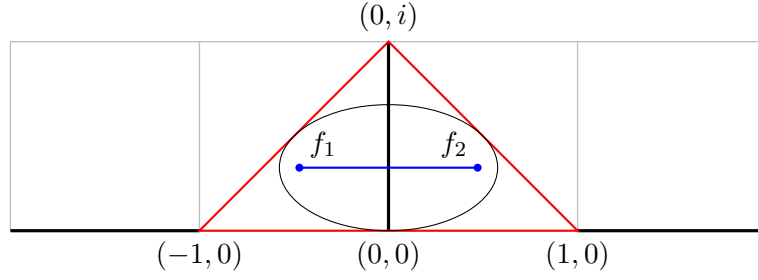$$p'(z) = 3z^2 - 2wz - 1 = 3\left(z^2 - \frac{2}{3}wz - \frac{1}{3}\right)$$

Figure 2: The third root is on the y-axis

$$f_{1,2} = \frac{1}{3}(w \pm \sqrt{w^2 + 3})$$
$$f_1 + f_2 = \frac{2}{3}w$$
$$f_1 f_2 = -\frac{1}{3}.$$

Second example Let $w = 1 + 2i$ and substitute in the equations for the foci.

$$f_{1,2} = \frac{1}{3}((1 + 2i) \pm 2\sqrt{i}) = \frac{1}{3}((1 \pm \sqrt{2}) + (2 \pm \sqrt{2})i)$$
$$f_1 + f_2 = \frac{2}{3}(1 + 2i)$$
$$f_1 f_2 = -\frac{1}{3}.$$

where we used the fact that $\sqrt{i} = (1 + i)/\sqrt{2}$ (check!).

One-half the distance between the foci is $2/3$ and the semi-minor axis is $\sqrt{2}/3$, so the semi-major axis is $\sqrt{2/3}$. The ellipse is rotated by $45°$ so that its major axis connects the foci (Figure 3).

Proof (Theorem 0.3) Assume for now that w is not on the y-axis.

By assumption $w = u + vi, v > 0$, so $f_1 + f_2 = \frac{2}{3}w > 0$ and at least one of f_1, f_2 is greater than zero and above the x-axis. Since $f_1 f_2 = -\frac{1}{3}$, the imaginary part of the product is zero. Let $f_1 = c_1 + d_1 i, f_2 = c_2 + d_2 i$; then the imaginary part of the product is $c_1 d_2 + c_2 d_1 = 0$. Construct the lines from the foci to the origin (Figure 4). Then,

$$\frac{d_1}{c_1} + \frac{d_2}{c_2} = 0$$
$$\tan\theta_1 + \tan\theta_2 = \frac{\sin(\theta_1 + \theta_2)}{\cos\theta_1 \cos\theta_2} = 0,$$

using the trigonometric identity for the sum of two tangents.

The foci are not on y-axis so the cosines are non-zero, and, therefore, $\sin(\theta_1 + \theta_2) = 0$. $\theta_1 + \theta_2$ cannot be $0°$ since $w > 0$ and we conclude that $\theta_1 + \theta_2 = 180°$.

Referring again to the Figure, since θ_1, θ_2 are supplementary, the angle between the line from f_2 and the origin forms an angle θ_1 with the negative x-axis. Consider the ellipse with foci

f_1, f_2 that passes through the origin, which is also the midpoint of one side of the triangle. By Theorem **??**, the ellipse is tangent to this side.

If the third root is on the y-axis, the two angles are right angles and the theorem follows immediately. ∎

The ellipse is tangent to all three sides of the triangle

We showed that the ellipse whose foci are the roots of $p'(z)$ has a tangent that is the side of the triangle on the x-axis and that the point of contact is the midpoint of that side. To prove Marden's theorem, we must show that the other sides of the triangle are tangent at their midpoints to the *same* ellipse.

Theorem 0.4 *Let $p(z), p'(z), T, f_1, f_2$ be as defined in the statement of Marden's theorem and*
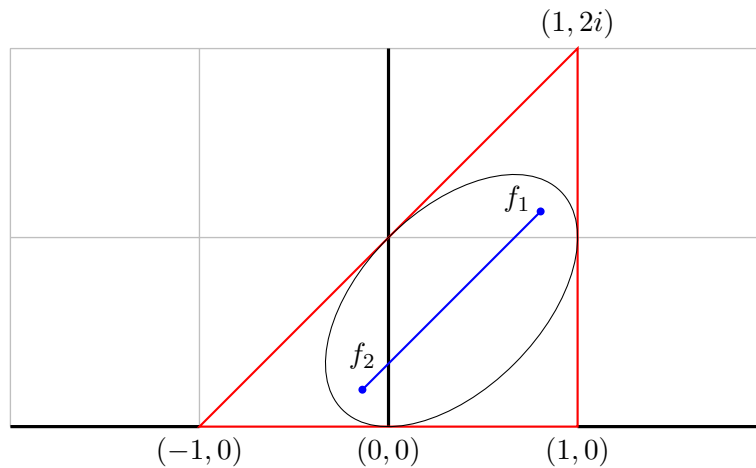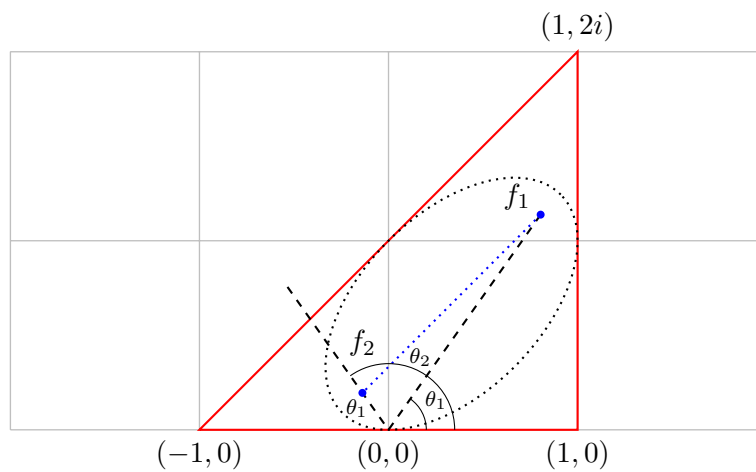

Figure 3: The third root is not on the y-axis

Figure 4: The ellipse is tangent to the line OW

the ellipse E constructed in Theorem 0.3. Then the other two sides of T are tangents to E and their points of contact are the their midpoints.

Proof By symmetry it is sufficient to prove for one of the other sides OW (Figure 5). In the Figure, the vertices of the triangle on the x-axis are now $(0,0), (2,0)$, and the third vertex is at an arbitrary position $w = u + iv, v > 0$.

We now compute the foci as before.

$$p(z) = p(p-1)(p-w) = z^3 - (1+w)z^2 + wz$$
$$p'(z) = 3z^2 - 2(1+w)z + w$$
$$f_1 + f_2 = \frac{2}{3}(1+w)$$
$$f_1 f_2 = \frac{w}{3}.$$

From the first equation and $w > 0$ we know that at least one focus is above the x-axis and since the ellipse is tangent to the x-axis the other focus is too.

Let us express $f_1 f_2 = w/3$ in polar coordinates (notation as in Figure 5).

$$f_1 f_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} = \frac{r}{3}e^{i\theta}$$
$$\theta_1 + \theta_2 = \theta$$
$$\theta - \theta_2 = \theta_1.$$

Therefore, $\angle f_2 OV = \theta = \angle f_1 OW$.

Now $O = (0,0)$ is *external* to the ellipse because OV is tangent to the ellipse at $(1,0)$. By Theorem 0.1, OW is tangent to the ellipse. ∎

The proof of Marden's Theorem

Proof We must show that the tangent OW contacts the ellipse at its midpoint M. By construction, ellipse E is tangent to the *midpoint* of OV and by Theorem 0.4 it is also tangent to OW, VW. Using the construction of Theorem 0.3 again, construct an ellipse E' that is tangent to the OW at M, which by Theorem 0.4 is also tangent to the other sides of T. But E, E' have the same foci and the same tangents, so they must be the same ellipse. ∎

%%%%%%%%%%%%%%%%%%%%%%%%%%%

Section 0.1 is based on [**?**].

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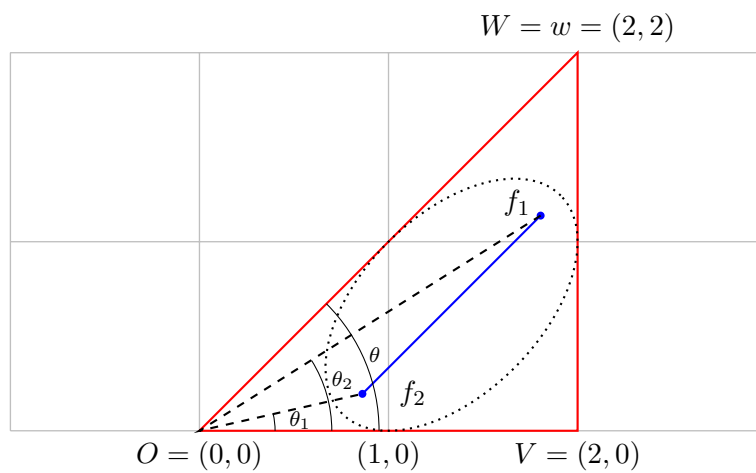
Figure 5: The ellipse is also tangent to OW

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