

# The Mathematics of Ellipses and dPlanetary Orbits

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# 1 Introduction

Everyone “knows” that Kepler discovered that the orbits of the planets are ellipses and that Newton showed that an elliptical orbit implies that the force of gravity must be inversely proportional to the square of the distance from the Sun. Although I knew these facts, I had never seen them demonstrated.

*Calculus in Context* [4] by Alexander J. Hahn is a comprehensive textbook on introductory calculus that augments theory with applications in physics and astronomy, such as the work of Kepler, Newton and Galileo, as well as applications in engineering such as building bridges and domed structures. These are not just historical anecdotes but detailed computations.

This document works out one topic based on the presentation in the book: the determination of orbits by Aristarchus, Copernicus, Kepler and Newton. The presentation is mathematical, since the historical and astronomical aspects are thoroughly described in [4], as well as in other works. The document is intended to enrich the learning of mathematics by secondary-school students and students in introductory university courses. The prerequisites are a very good knowledge of Euclidean geometry is needed along with some trigonometry, a bit calculus and Newton’s laws of motions.

Section 2 presents the measurements of the radii and distances of the Earth, the Moon and the Sun by Eratosthenes and Aristarchus. Section 3 describes the construction of a model of a Sun-centered system by Nicolaus Copernicus. Section 4 shows how Johannes Kepler developed his three laws of planetary motion and Section 5 presents Isaac Newton’s derivation of the inverse-square law of gravitation from of Kepler’s laws. One step of Newton’s derivation requires a theorem whose proof is very long, so it is split off into Section 6. Section 7 contains more than you ever wanted to know about the mathematics of ellipses, but these theorems are necessary. I suggest that you look up each theorem (and its proof) as needed, rather than trying to study them all at once. Section 8 is discussed in the next paragraph. Theorems of Euclidean geometry that are used but do not concerning ellipses are collected in Appendix A.

## Aspects of history

It is easy to measure angles. We are familiar with the use of a protractor<sup>1</sup> in school and these can be scaled-up to obtain more accurate measurements. Measuring long distances was impossible until the recent inventions of radar and lasers. At most one could pace-off distances with low accuracy. The only *measured* distance used here is the estimate of 800 km by Eratosthenes for the distance between two places in Egypt (Section 2.1).

For this reason, the mathematics used is primarily geometry with an occasional dash of limits that had been used already by Archimedes to compute the circumference and area of a circle by approximating the circle by regular polygons of increasing number of

---

<sup>1</sup>More precisely, a device for measuring angles is called a *goniometer*.

sides. Starting from the method of limits, Newton (along with his contemporary Gottfried Wilhelm Leibniz) developed the calculus. However, Newton's *Principia* uses Euclidean geometry almost exclusively, although analytic geometry had been developed by René Descartes and Pierre de Fermat even before Newton was born.

Newton expected his readers to have an extensive knowledge of geometry. This expectation continued until relatively recently:

In book 1, prop[osition] 10 (and notably in prop[osition] 11), Newton made use of a property of conics which he presents without proof, merely saying that the result in question comes from “the *Conics*.” Here, as elsewhere in the *Principia*, Newton assumes the reader to be familiar with the principles of conics and of Euclid. In the eighteenth and nineteenth centuries, when Newton's treatise was still being read in British universities[!], authors of books on “conic sections”—for example, W. H. Besant, W. H. Drew, Isaac Milnes—supplied the proof of this theorem in order to help readers of the *Principia* who might be baffled by the problem of finding a proof. They even chose letters to designate points on the diagrams so that the final result would appear in exactly the same form as in the *Principia* [2, p. 330].

To keep this spirit alive, Section 8 brings the necessary proofs based on [1]. The reader is forewarned, however, that as Euclid said, “There is no royal road to geometry!”

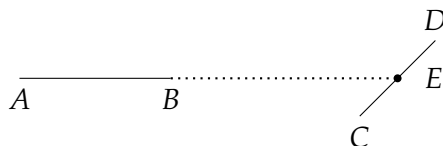
## Style and notation

The computations in Hahn's book are faithful to the historical record, for example, measuring distances in units such as the stadia of the Greeks. Here, the computations are fully modernized and use modern units such as kilometers.

Diagrams are used to facilitate understanding each step in the geometrical proofs, more than appear in other sources. For example, Newton proved his difficult theorem (Section 6) using only a single diagram. The diagrams are not to scale and are often distorted. Otherwise, it would be impossible to draw planetary orbits on a piece of paper.

The following shortcuts facilitate a less verbose presentation:

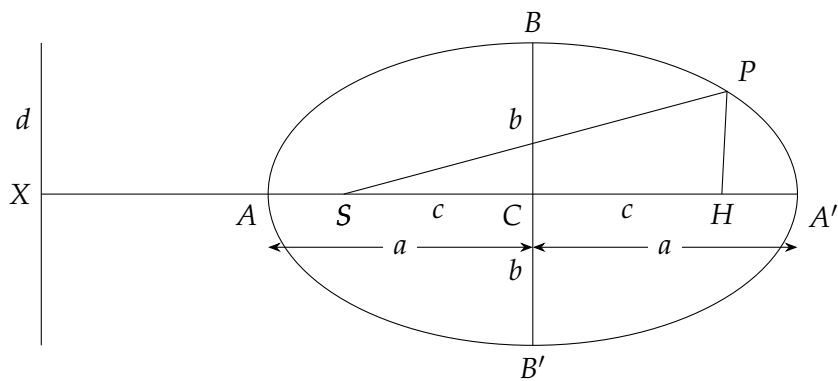
- $AB$  denotes both a line segment and its length.
- $\triangle ABC$  denotes both a triangle and its area.
- The line segment  $AB$  can be *extended*<sup>2</sup> until it intersects the line  $CD$ . In this text, the intention of “ $AB$  intersects  $CD$ ” is that  $AB$  can be extended until it intersects  $CD$ .




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<sup>2</sup>The term used by Newton is *produced*.

- The notation in the following diagram will be used consistently to refer to elements of an ellipse. The diagram is for reference as the terms will be defined as needed.
  - $C$  is the center of the ellipse and  $a, b$  are the semi-major and semi-minor axes of an ellipse.  $A, A'$  are the vertices on the major axis and  $B, B'$  are the vertices on the minor axes.
  - $S$  and  $H$  are the foci and  $c$  is the distance from a focus to the center.
  - $X$  is the intersection of the major axis with the directrix  $d$ . (There is a second directrix on the right side of the ellipse.)



## Acknowledgments

## 2 The sizes of the Earth, Moon and Sun

### 2.1 Eratosthenes' measurement of the radius of the earth

The ancient Greeks knew that the Earth is round and Eratosthenes was able to measure the radius of the Earth (Figure 1). Choose two points  $A, B$  on the same longitude and measure the distance  $d$  between them. Plant a vertical stick (red) in the ground at  $A$  and another (blue) at  $B$ . On a day in the year when the stick at  $A$  produces no shadow at noon, at the same time the stick at  $B$  produces a shadow whose angle is  $\alpha$ . The sun is so far away from the Earth that over the relatively short distance  $d$ , the rays of the Sun are essentially parallel. By alternate interior angles, the angle between the two sticks as measured from the center of the Earth is also  $\alpha$ .

The angle that Eratosthenes measured at the blue stick was

$$\alpha = 7.5^\circ \cdot \frac{2\pi}{360} \approx 0.131 \text{ radians},$$

and the distance  $d$  between  $A$  and  $B$  was known to be approximately 800 km. The arc  $\widehat{AB}$  subtends the angle  $\alpha = d/r_e$  where  $r_e$  is the radius of the Earth, so

$$r_e = \frac{d}{\alpha} = \frac{800}{0.131} \approx 6107 \text{ km}. \quad (1)$$

This value is quite close to the modern measurement of 6370 km.

### 2.2 Aristarchus' measurements

Using  $r_e$ , Eratosthenes' measurement of the radius of the Earth, Aristarchus was able to measure and compute the following values:

- $r_m$ : the radius of the Moon,
- $r_s$ : the radius of the Sun,
- $d_m$ : the distance from the Earth to the Moon,
- $d_s$ : the distance from the Earth to the Sun.

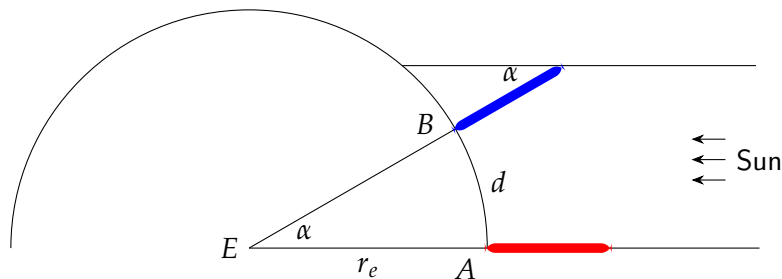


Figure 1: Eratosthenes' measurement of the radius of the earth

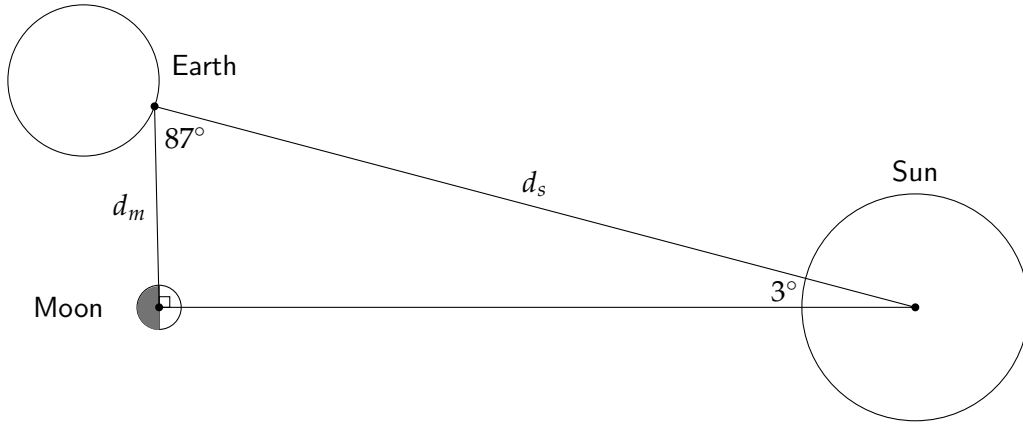


Figure 2: Observing a first quarter moon

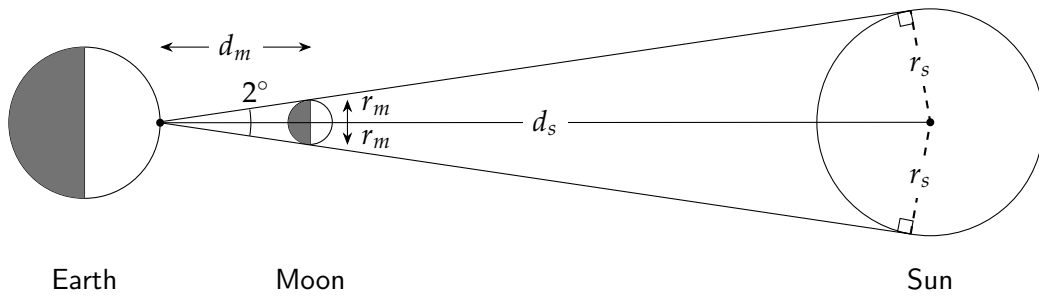


Figure 3: A solar eclipse

### Computing $d_s/d_m$

An observer on Earth can follow the phases of the Moon as it revolves around the Earth. At one point in the month the phase will be first quarter, meaning that the one half of the moon is illuminated while the other half is not (Figure 2). The angle between the Sun and the Moon will be  $87^\circ$ . Since exactly half of the moon is illuminated, we know that the angle  $\angle$  Earth-Moon-Sun is a right-angle so

$$\begin{aligned}\cos 87^\circ &= \frac{d_m}{d_s} \\ \frac{d_s}{d_m} &= \frac{1}{\cos 87^\circ} \approx 19.\end{aligned}\tag{2}$$

### Computing $r_s/r_m$ and $d_m/r_m$

The Moon is much, much smaller than the Sun, but it is also much, much closer to the Earth. When the Moon is precisely positioned between the Earth and the Sun, its “disk” exactly covers the “disk” of the Sun, causing a total solar eclipse (Figure 3).

The angle subtended by the Moon is  $2^\circ$  degrees. Bisecting the angle creates two right triangles with an acute angle of  $1^\circ$ , where the right angles are the tangents to Moon and



the Sun. By similar triangles, Equation 2 and Figure 3,

$$\frac{r_s}{r_m} = \frac{d_s}{d_m} = 19 \quad (3)$$

$$\frac{d_m}{r_m} = \frac{1}{r_m/d_m} = \frac{1}{\sin 1^\circ} \approx 57. \quad (4)$$

### Computing the radii and distances

Figure 4 shows a lunar eclipse. Unlike a solar eclipse where the Moon exactly covers the Sun, the Earth more than covers the Moon and its shadow is four times the Moon's radius.

Figure 5 show a lunar eclipse annotated with the distances  $d_m, d_s$  and the radii  $r_m, r_e, r_s$ . The ray from the top of the Sun is tangent to both the Sun and the Earth, so it forms right angles with their radii, as well as with the extension of the Moon's radius. The thick horizontal lines are constructed parallel to the line connecting the centers, forming two right triangles. The triangles are similar so using Equation 3,

$$\frac{r_s - r_e}{r_e - 2r_m} = \frac{d_s}{d_m} = \frac{r_s}{r_m}$$

$$r_s r_e + r_m r_e = 3r_s r_m .$$

Again from Equation 3,  $r_s = 19r_m$ , so

$$r_m = \frac{20}{57} r_e .$$

By Equation 1,  $r_e \approx 6107$  km, by Equation 4,  $d_m = 57r_m$ , and by Equation 2,  $d_s = 19d_m$ , so we can compute the radii and distances:

$$r_m = \frac{20}{57} r_e \approx 2143 \text{ km}$$

$$r_s = 19r_m \approx 40,713 \text{ km}$$

$$d_m = 57r_m \approx 122,140 \text{ km}$$

$$d_s = 19d_m \approx 2,320,660 \text{ km} .$$

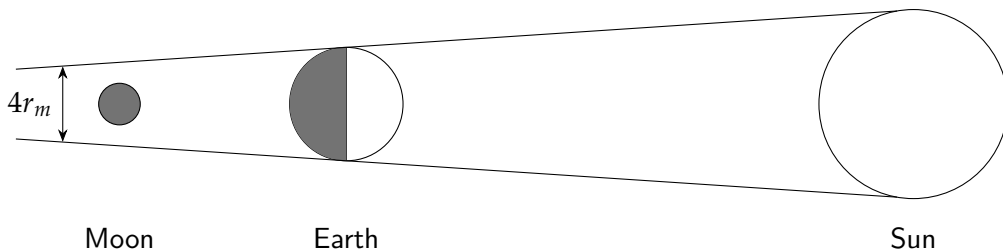


Figure 4: A lunar eclipse

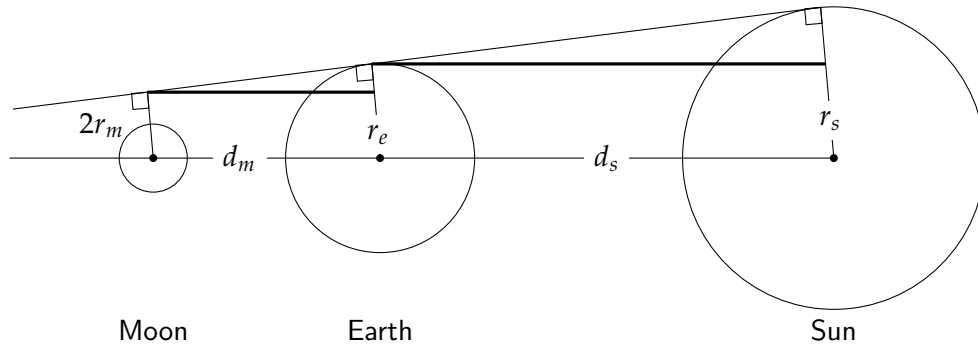


Figure 5: Detail of a lunar eclipse

The following table summarizes these data together with the modern values [4, Table 1.3]. While the computed values for the radii of the Earth and the Moon are not far off from the modern values, the other computed values are not near the modern values. Nevertheless, they do show that the Greeks understood the immense size of the solar system.

|                           | Computed (km) | Modern (km) |
|---------------------------|---------------|-------------|
| $r_e$ radius of Earth     | 6107          | 6370        |
| $r_m$ radius of Moon      | 2143          | 1740        |
| $r_s$ radius of Sun       | 40,713        | 695,500     |
| $d_m$ distance Earth-Moon | 122,140       | 384,570     |
| $d_s$ distance Earth-Sun  | 2,320,660     | 150,000,000 |

### 3 The Sun-centered solar system

As everyone living far from equator knows, the time between sunrise and sunset varies with the seasons. The reason is that the axis of the rotation of the Earth is offset by  $23.5^\circ$  relative to the orbit of the Earth. The plane of the orbit of the Earth around the Sun is called the *ecliptic*. Measuring the length of the day as the time from sunrise to sunset, there is a day in June, called the *summer solstice*, when the length of the day is longest. Similarly, there is a day in December, called the *winter solstice*, when the length of the day is shortest.<sup>3</sup> There are also two days when the length of the day equals the length of the night: the *autumn equinox* in September and the *spring equinox* in March.

Today we know that the universe is immensely large and that the stars are moving at extremely high speeds, but an observer on Earth sees them as if their positions are fixed on a sphere around the earth, called the *celestial sphere*. This solstices and equinoxes can be associated with the projection of the Sun on the celestial sphere as seen from the Earth. The details of the Earth's orbit can be found in books on astronomy, as well as in the Wikipedia articles on *Equinox* and *Solstice*.

#### 3.1 The length of the year and the length of the seasons

Let us assume that the Earth orbits the Sun in a circle, such that the center of the orbit  $O$  is the center of the Sun  $S$ . In Figure 6 the inner circle is orbit of the Earth and the outer circle is the celestial sphere. The orbit can be divided into four quadrants called *seasons*: spring, summer, autumn, winter.

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<sup>3</sup>This holds for the northern hemisphere; in the southern hemisphere the opposite holds.

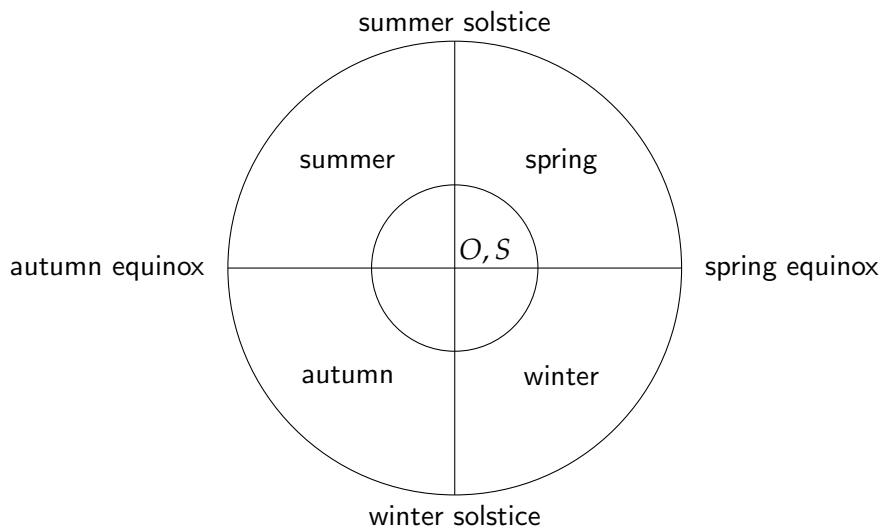


Figure 6: The orbit of the Earth and the seasons

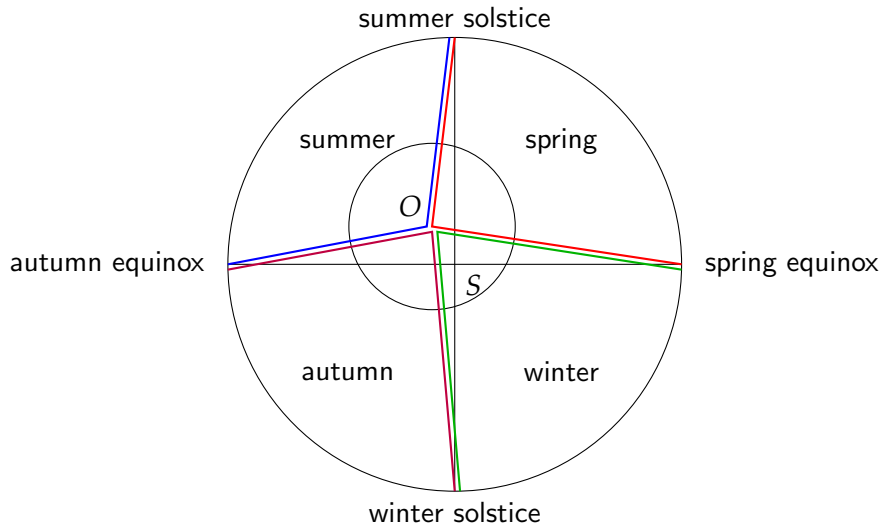


Figure 7: The lengths of the seasons are not equal

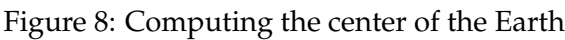
The length of a year is approximately  $365\frac{1}{4}$  days. The extra  $\frac{1}{4}$  day is accounted for by adding a day in leap years.<sup>4</sup> The length of each season as determined by the equinoxes and the solstices is  $365.25/4 = 91.3125$  days. However, measurements by the Greek astronomer Hipparchus showed that the actual lengths of the seasons differed from this number and a model of the solar system must be able to explain these differences:

| Season | Days            | %    |
|--------|-----------------|------|
| Spring | $94\frac{1}{4}$ | 25.8 |
| Summer | $92\frac{1}{2}$ | 25.3 |
| Autumn | $88\frac{1}{8}$ | 24.1 |
| Winter | $90\frac{1}{8}$ | 24.7 |

In his Earth-centered solar system, Hipparchus proposed that the center of the Earth's orbit be offset from the center of the Sun. Copernicus used the same idea in his Sun-centered solar system (Figure 7). If the center of the Earth's orbit is in the upper-left quadrant of the coordinate system defined by the equinoxes and the solstices, the angles for spring and summer are obtuse, so the seasons are longer than one-fourth of a year, whereas the angles for the autumn and winter are acute, so they are shorter than one-fourth of a year.

Figure 8 shows a magnified and distorted view of Figure 7. It has been annotated with additional lines and labels that will facilitate the demonstration of Copernicus' computation. The axes  $A'C'$  and  $B'D'$  have their origin  $O$  at the center of the Earth's orbit and are parallel to the axes in the ecliptic. The dashed lines from  $O$  are all radii of the Earth's orbit

<sup>4</sup>The length of a year is actually 365.2425. In the sixteenth century, the *Gregorian calendar* accounted for the difference by removing three leap years in every four hundred years.



### 3.2 The location of the center of the Earth's orbit

- Initially, we compute the angles of the arcs in radians; the lengths of the arc can then be obtained by multiplying by the radius  $r$ .
- We use the lengths of the seasons that Copernicus used: summer is  $93\frac{14.5}{60}$  days and spring is  $92\frac{51}{60}$ .
- The angle of the arc  $\widehat{AC}$  can be computed from the combined length of spring and summer and the angle of the arc  $\widehat{AB}$  can be computed from the length of spring.
- From  $\widehat{AC}$  and  $\widehat{AB}$ , the angle  $\alpha$  subtended by  $\widehat{AA'}$  and the angle  $\beta$  subtended by  $\widehat{BB'}$  can be computed.
- Since the Earth is very close to the Sun relative to the radius of its orbit,  $r\widehat{AA'}$  and  $r\widehat{BB'}$  approximate the line segments  $a$  and  $b$ . From these  $c$  and  $\lambda$  can be computed.

### Computing the angles of the arcs $\widehat{AB}, \widehat{AC}$

The arcs  $\widehat{AB}, \widehat{AC}$  are sectors of the Earth's orbit and their angles are their proportions of a full year times  $2\pi$  radians.

$$\widehat{AB} = 2\pi \cdot \frac{92\frac{51}{60}}{365.25} = 2\pi \cdot \frac{92.85}{365.25} = 1.5972 \text{ radians}$$

$$\widehat{AC} = 2\pi \cdot \frac{92\frac{51}{60} + 93\frac{14.5}{60}}{365.25} = 2\pi \cdot \frac{186.09}{365.25} = 3.2012 \text{ radians}.$$

### Computing the angles of the arcs $\widehat{AA'}, \widehat{BB'}$

Let us express the arcs  $\widehat{AC}$  and  $\widehat{AB}$  in terms of the arcs that comprise them. Since  $AC$  is parallel to  $A'C'$ ,  $\widehat{AA'} = \widehat{C'C}$ . Compute  $\widehat{AA'}$ :

$$\begin{aligned}\widehat{AC} &= \widehat{AA'} + \widehat{A'C'} + \widehat{C'C} = 2\widehat{AA'} + \pi \\ \widehat{AA'} &= \frac{1}{2}(3.2012 - \pi) = 0.0298 \text{ radians}.\end{aligned}$$

Now that we have computed  $\widehat{AB}$  and  $\widehat{AA'}$  we can compute  $\widehat{BB'}$ :

$$\begin{aligned}\widehat{AB} &= \widehat{AA'} + \widehat{A'B'} - \widehat{BB'} \\ \widehat{BB'} &= 0.0298 + \frac{\pi}{2} - 1.5927 = 0.0034 \text{ radians}.\end{aligned}$$

### Computing the lengths of the arcs $\widehat{AA'}, \widehat{BB'}$

$OA$  and  $OB$  are radii of the Earth's orbit so

$$\begin{aligned}\sin \alpha &= \frac{a}{r} \approx \alpha \\ a &\approx r\alpha = r\widehat{AA'} = 0.0298r \\ \sin \beta &= \frac{b}{r} \approx \beta \\ b &\approx r\beta = r\widehat{BB'} = 0.0034r,\end{aligned}$$

where we have used the assumption that  $O$ , the center of the Earth's orbit, is very close to the Sun  $S$  so that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

### Computing the position of $O$ relative to $S$

Figure 9 shows a magnified diagram of a portion of Figure 8. In the dotted triangles, we have already computed the lengths  $a$  and  $b$ . Since  $OT$  is parallel to  $A'A$  and  $TS$  is parallel

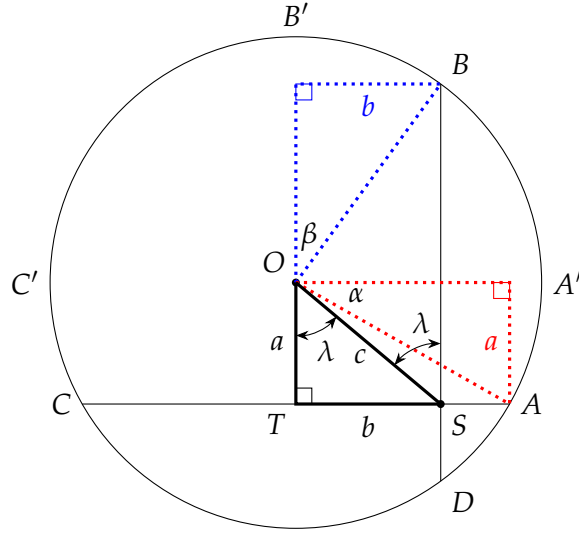


Figure 9: Three triangles

to  $BB'$ , we can label  $OT$  by  $a$  and  $TS$  by  $b$ . The emphasized triangle is a right triangle and  $c$ , the distance of  $O$  from  $S$ , can be obtained from Pythagoras's theorem:

$$c = \sqrt{a^2 + b^2} = r\sqrt{(0.0298)^2 + (0.0034)^2} = 0.03r.$$

$\lambda$  can be obtained from trigonometry:

$$\lambda = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{0.0034}{0.03} = 0.1129 \text{ radians} \approx 6.47^\circ.$$

The distance  $0.03r$  is shown in the following table using the values of  $r$  from the table on page 10.

|                          | Aristarchus (km) | Copernicus (km) | Modern (km) |
|--------------------------|------------------|-----------------|-------------|
| radius of Earth's orbit  | 2,320,660        | 8,000,000       | 150,000,000 |
| distance of $O$ from $S$ | 69,620           | 240,000         | 4,500,000   |

## 4 Elliptical orbits

Towards the end of the sixteenth century, the astronomer Tycho Brahe carried out extremely precise observations. In 1600 he hired Johannes Kepler as his assistant and when Tycho died soon afterwards, Kepler was appointed to his position. Here we explain how Kepler was able to establish that planetary orbits are ellipses.

### 4.1 Determining the radius of the Earth's orbit

A Martian year is 687 days, that is, it equals  $\frac{687}{365.25} = 1.88$  Earth years. We know when Mars reaches a new “year” by observing its projection on the celestial sphere, but each time the position of the Earth in its orbit will be different. Figure 10 shows the orbit of the Earth—its center  $O$  offset from the Sun  $S$  as Copernicus showed—at four occasions when the position of Mars  $M$  at its new year was observed. Four triangles are created  $\triangle OE_iM$ .

Figure 11 shows one of the triangles with the angles labeled. Using the law of sines,

$$\frac{OE_i}{\sin \beta} = \frac{OM}{\sin \alpha}$$

$$OE_i = OM \frac{\sin \beta}{\sin \alpha}.$$

Tycho Brahe was able to measure all three angles:<sup>5</sup>

|       | $\alpha$ | $\beta$ | $\gamma$ | $OE_i (\times OM)$ |
|-------|----------|---------|----------|--------------------|
| $E_1$ | 127.1    | 20.8    | 32.1     | 0.6682             |
| $E_2$ | 84.2     | 35.8    | 60.5     | 0.6721             |
| $E_3$ | 41.3     | 42.4    | 96.4     | 0.6785             |
| $E_4$ | 1.6      | 3.4     | 175.0    | 0.6805             |

<sup>5</sup>The values are rounded to one decimal point.

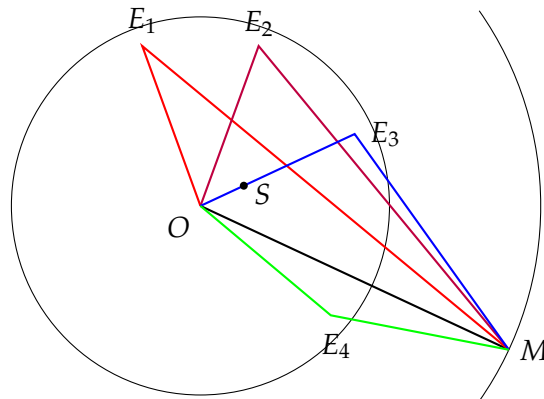


Figure 10: Observations of the orbit of Mars from the Earth



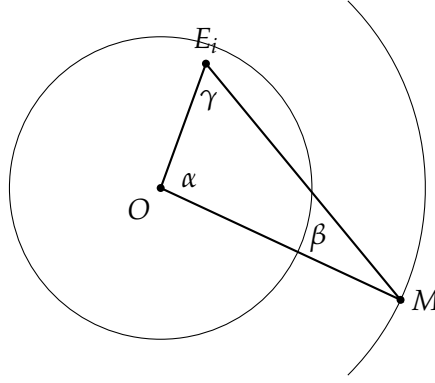


Figure 11: One triangle Earth-Sun-Moon

The values of  $OE$  are given in the fourth column of the table and they are not equal. Assuming (as everyone did at that time) that the Earth's orbit is circular, the only solution was to move the center of the orbit so that  $\{E_1, E_2, E_3, E_4\}$  were all on the circle.

## 4.2 Measuring the angles in the triangle Sun-Earth-Mars

How can the angles  $\alpha, \beta, \gamma$  be measured? Since  $\triangle E_i OM$  is a triangle, it is sufficient to measure two of the angles.  $\gamma$  is easily measured by observing Mars and the Sun at the same time. However, neither  $\alpha$  nor  $\beta$  can be measured directly since they are not accessible to an observer on Earth.

Tycho's measurement used the known periods of the orbits to compute the angles. The Earth moves counterclockwise around Sun. Given any point  $E$ , after  $t$  days the Earth will eventually move to  $E'$  and Mars will move to  $M'$  so that they are in *opposition*, that is, Mars will be in the continuation of the Earth-Sun line (Figure 12). Since the Earth completes an orbit in close to half the time that Mars takes, this will occur before either has completed a full orbit. The angles  $\theta_E$  and  $\theta_M$  are fractions of a circular orbit of  $360^\circ$ , so

$$\begin{aligned}\frac{t}{365.25} &= \frac{\theta_E}{360} \\ \theta_E &= \frac{360 t}{365.35} \\ \frac{t}{687} &= \frac{\theta_M}{360} \\ \theta_M &= \frac{360 t}{687}.\end{aligned}$$

This gives values for  $\theta_E$  and  $\theta_M$ . From geometry we have

$$\begin{aligned}\alpha' &= 180 - \theta_E = \alpha - \theta_M \\ \alpha &= 180 - \theta_E + \theta_M,\end{aligned}$$

and the values of  $OE_i$  can be computed.

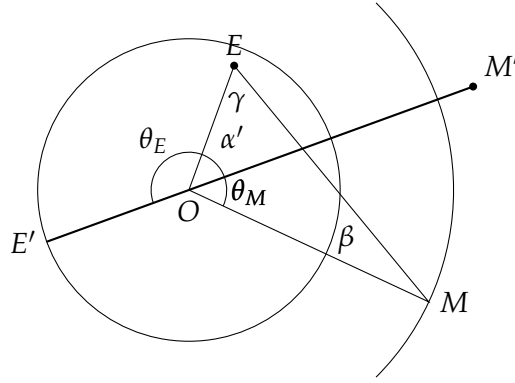


Figure 12: The Earth and Mars in opposition

#### 4.3 A new location for the center of the Earth's orbit

Kepler's next task was to obtain a new value  $O'$  for the center of the Earth's orbit such that the  $E_i$ 's are on the orbit. Given the new locations of the Earth  $\{E_1, E_2, E_3, E_4\}$ , by Theorem A.1 a circle centered at  $O'$  can be constructed that goes through  $\{E_1, E_2, E_3\}$  (Figure 13). To verify that this is the correct orbit, check that  $E_4$  is on the circle.

#### 4.4 Orbits are ellipses

While Kepler was able to modify the center of the orbit of the Earth to be consistent with the observations, he was not able to adequately describe the orbit of Mars. After years of work, he came to the conclusion that the orbit must be oval like an egg. Oval, perhaps, but certainly not an ellipse, because he was certain that it would have been discovered by Archimedes! Figure 14 shows  $C$ , a position on a circular orbit, and an oval orbit (dashed), where  $M$  is the position of Mars on the oval corresponding to  $C$ . The radius of the circular

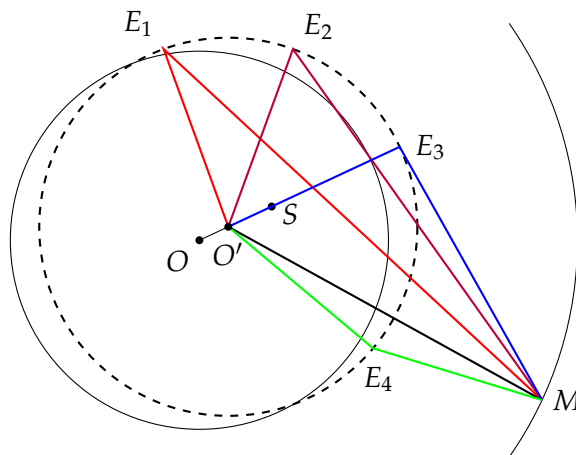


Figure 13: Observations of the orbit of Mars from the new Earth's orbit

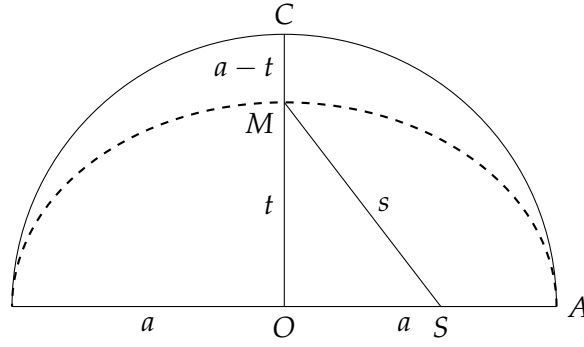


Figure 14: The orbit of Mars as an oval “egg”

orbit is labeled  $a$  and the unknown distances to  $M$  are labeled  $s = SM$  and  $t = OM$ .

Kepler’s computed that  $\frac{a-t}{t} = 0.00429$  and  $\frac{s}{t} = 1.00429$ , so that

$$\begin{aligned}\frac{a-t}{t} &= \frac{s}{t} - 1 \\ a-t &= s-t,\end{aligned}$$

and therefore  $SM = s = a = AO = CO$ . If the dashed oval is, in fact, an ellipse, because in an ellipse  $SM = AO$  (by Theorem 7.2). Kepler then computed the projections of the observations of Mars on the  $x$ -axis (Figure 15) and obtained for all of them that

$$\frac{M_i O_i}{C_i O_i} = \frac{1}{1.00429} = 0.99573.$$

By Theorem 7.4, since the ratio  $MO/CO = b/a$  is constant for in an ellipse, Kepler was able to conclude that the orbit of Mars is an ellipse.

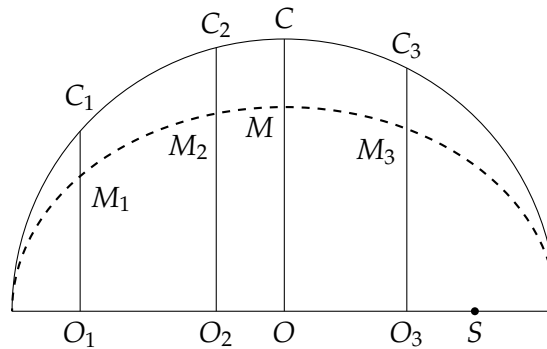


Figure 15: The orbit of Mars as an ellipse

## 5 Gravitation

In this section we present Isaac Newton's demonstrations that the inverse-square law can be deduced from Kepler's observation that the orbits of planets are ellipses. After a review of Newton's Laws of force and motion, we show that Kepler's second law must hold in any system subject to a centripetal force. The next step is to show that the inverse square law for gravitation must hold in an elliptical orbit. Then it is a small step to universal gravitation and Kepler's third law. For readers who wish to study the *Principia* the labels of the points in the diagrams are the same as those used by Newton.

### 5.1 Newton's laws of motion

1. A body in uniform motion (including a body at rest) continues with the same motion unless a force is applied.
2. A force  $F$  applied to a body causes an acceleration  $a$  in the direction of the force whose magnitude  $a = F/m$ , where  $m$ , the constant of proportionality, is called the *mass* of the body.
3. If one body exerts a force on a second body, the second body exerts a force on the first of equal magnitude but in the opposite direction.

Forces are denoted by vectors, where the direction of the vector represents the direction of the force and the length of the vector represents the magnitude of the force. Forces can be decomposed into perpendicular components (Figure 16), or into components in any directions (Figure 17). The components form a parallelogram whose diagonal is the resultant force.

Newton was interested in *centripetal force* which is a force exerted by a single body on another, in particular, the gravitational force exerted by the Sun on a planet (Figure 18). Since the only force is that directed towards the Sun, the planet does not move "up" or "down" so its orbit is in a plane.

Kepler's second law states that a planet in orbit sweeps out equal area in intervals of equal duration, if it takes the planet time  $t$  to move from  $P_1$  to  $P_2$  and also  $t$  to move

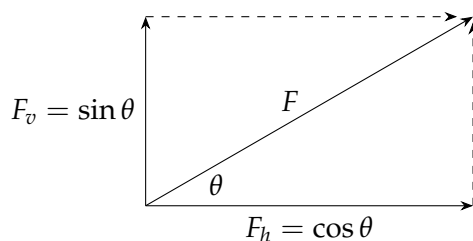


Figure 16: Perpendicular components of a force

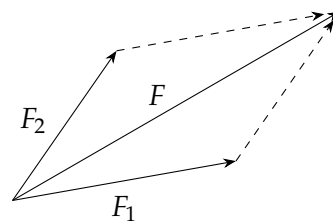


Figure 17: Arbitrary components of a force

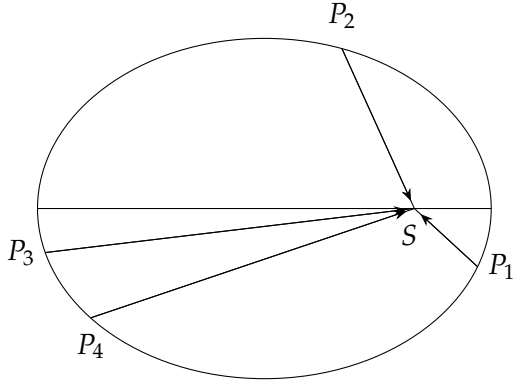


Figure 18: Centripetal force

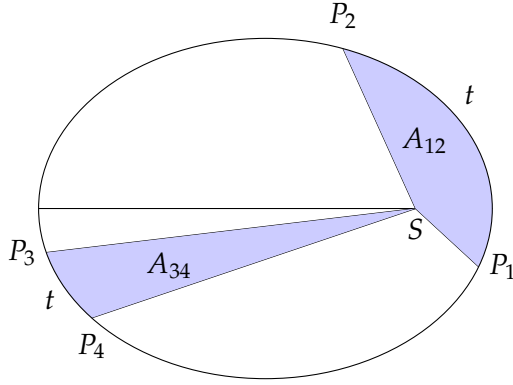


Figure 19: Equal areas in equal times

from  $P_3$  to  $P_4$ , then  $A_{12}$ , the area of the sector  $P_1SP_2$ , is equal to  $A_{34}$ , the area of  $P_3SP_4$  (Figure 19). Obviously, this means that the speed of the planet must vary as it traverses its orbit  $v_{P_1P_2} \gg v_{P_3P_4}$ . Newton proved that this must be true in any system where a body is subject to a centripetal force from another body.

The proof is based on dividing an area into very small sectors and then taking the limit. Consider three points  $A, B, C$  on the orbit (Figure 20) that represent the positions of the planet at intervals of  $\Delta t$ . For clarity we have drawn them spaced out, but the intention is that they are very close together. Newton assumed that the planet does not smoothly traverse the arcs, but rather that it every  $\Delta t$  it jumps in discrete steps from one point on the orbit to the next.

Figure 21 shows how the force is exerted in discrete steps. The planet moves from  $A$  to  $B$  and we expect that the centripetal force at  $B$  will cause an acceleration that moves the planet to  $C$ , the next point on the orbit. Instead, we “pretend” that the force is not applied at  $B$ , but, in the absence of an applied force, planet continues to move in the same direction and at the same speed. After another period of  $\Delta t$  as passed and the planet has reached point  $C'$ , the force is now applied *in the same direction* as it would have been applied at  $B$ , moving the planet to  $C$ .

**Theorem 5.1** *The area of  $\triangle ASB$  is equal to the area of  $\triangle BSC$ .*

**Proof** This will be done in two steps by showing that  $\triangle ASB = \triangle BSC'$  and  $\triangle BSC' = \triangle BSC$ . In Figure 22,  $\triangle ASB$  is shown in blue and  $\triangle BSC'$  is shown in red. It is assumed that  $AB = BC'$  (the planet moves from  $B$  to  $C'$  during the same interval  $\Delta t$ ), so since  $SH$  is the height of both triangles, their areas are equal. In Figure 23,  $\triangle BSC$  is shown in blue and  $\triangle BSC'$  is shown in red. It is assumed  $CC'$  is parallel to  $SB$  (the planet is subject to the centripetal force at  $C'$  in the *same* direction as the force at  $B$ , so the heights of both triangles to the common side  $SB$  are equal and their areas are equal. It follows that  $\triangle ASB = \triangle BSC' = \triangle BSC$ , which we denote by  $\Delta A$ . ■

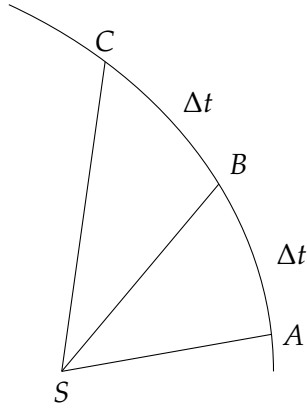


Figure 20: "Small" sectors of an orbit

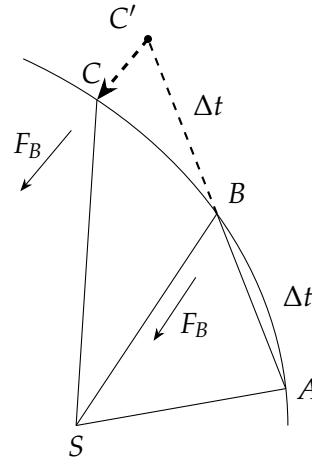


Figure 21: Exerting force at discrete times

We assume that the sectors of the orbit are each divided up into small sectors of uniform duration  $\Delta t$ . By the theorem, each sector has the same area  $\Delta A$ . Therefore,

$$\frac{A_{12}}{\Delta A} = \frac{t}{\Delta t} = \frac{A_{34}}{\Delta A},$$

from which Kepler's second law follows:

$$\begin{aligned} \frac{A_{12}}{t} &= \frac{A_{34}}{t} \\ A_{12} &= A_{34}. \end{aligned}$$

The proof used two approximations:

- $\Delta A$  is an approximation of the area of each sector.

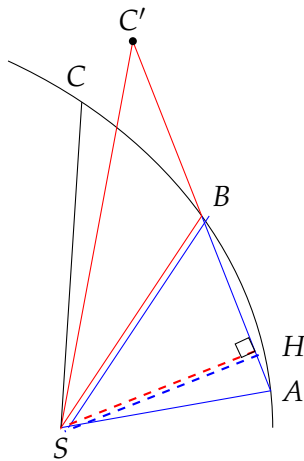


Figure 22:  $\triangle ASB = \triangle BSC'$

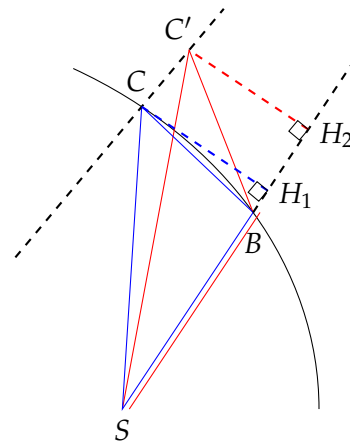


Figure 23:  $\triangle BSC' = \triangle BSC$

- The force at  $C'$  is an approximation to the force at  $B$ .

In the limit as the size of the sectors decreases, the errors become negligible.

**Definition 5.2** For a given elliptical orbit,  $\kappa = \frac{A}{t}$ , where  $A$  is the area of the ellipse and  $t$  is the period of the orbit, is called Kepler's constant.

## 5.2 The inverse square law for gravitation

Newton's next step was to show that if the orbit of a planet is elliptical, the centripetal force must be proportional to the mass of the planet and inversely proportional to the square of its distance from the Sun. In Figure 24,  $S$  is the Sun, and  $P$  and  $Q$  are points on the orbit that are very close to each other.  $PR$  is a tangent to the ellipse at  $P$  and  $R$  is a point such that  $QR$  is parallel to  $SP$ .  $QT$  is constructed perpendicular to  $SP$ . Denote the lengths  $r_p = SP$ ,  $h = QT$ ,  $q = QR$  and the time interval from  $P$  to  $Q$  by  $\Delta t$ .

When a point is subject to an acceleration  $a$  for a period of  $\Delta t$ , its displacement is  $\frac{1}{2}a(\Delta t)^2$ . From Newton's second law we know that at point  $R$ , the planet is subjected to an acceleration of  $F_p/m$ , so

$$q = \frac{1}{2} \frac{F_p}{m} (\Delta t)^2$$

$$F_p = \frac{2mq}{(\Delta t)^2}.$$

Now we compute the area of  $\Delta A = \triangle SPQ = (1/2)hr_p$  and use Kepler's constant:

$$\Delta t = \frac{\Delta A}{\kappa} = \frac{hr_p}{2\kappa}$$

$$F_p = 2mq \cdot \frac{4\kappa^2}{(hr_p)^2} = 8\kappa^2 m \cdot \frac{q}{h^2} \cdot \frac{1}{r_p^2}.$$

To obtain an inverse-square law for the force, the first two factors have to be independent of the distance. For a given planet  $m$  is constant and for a given elliptical orbit  $\kappa$  is constant,

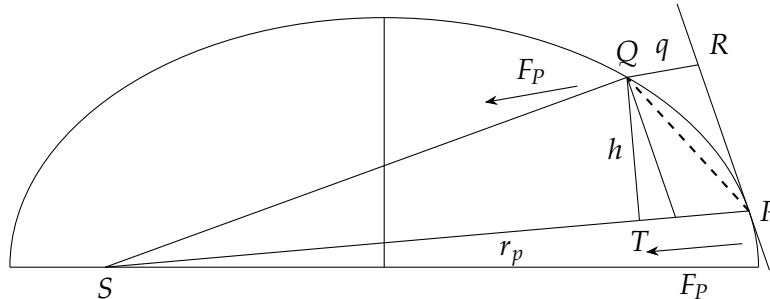


Figure 24: The derivation of the inverse square law

so the first factor does not depend on the distance. What about the second factor  $q/h^2$ , in particular, what value does it have if  $\Delta t$  approaches zero?

**Theorem 5.3** *In an elliptical orbit*

$$\lim_{\Delta t \rightarrow 0} \frac{q}{h^2} = \frac{1}{L},$$

where  $L$  is the length of the latus rectum of the ellipse (Definition 7.5).

Newton's proof is very long and is presented separately in Section 6.

Since  $L$  is constant for any given ellipse, the inverse square law can be written

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2}.$$

**A simplified formula for the force** The formula can be re-written so that the constant values appearing are more familiar:  $a$ , the semi-major axis and  $T$ , the period of the orbit. By Theorem 7.6,  $L=2b^2/a$  and by Theorem 7.7,  $\kappa = A_{\text{ellipse}}/T=\pi ab/T$ , so

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2} = \frac{8(\pi ab)^2 m}{T^2} \cdot \frac{a}{2b^2} \cdot \frac{1}{r_P^2} = \frac{4\pi^2 a^3 m}{T^2} \cdot \frac{1}{r_P^2}. \quad (5)$$

**Applicability of the inverse square law** Newton was able to show that

- The inverse square law applies to all conic sections including a parabola and a hyperbola, not just to an ellipse and, of course, a circle. Some comets have hyperbolic orbits and orbit the Sun only once.
- The converse holds: if a planet is subject to an inverse-square centripetal force then the orbit must be an ellipse (or other conic section).
- The proof assumes that a planet is a very small point, but the result holds even for large planets as long as the density of the planet is radially symmetric, that is, for a given distance from the center the density is constant.

### 5.3 Universal gravitation

Recall Newton's third law: If one body exerts a force on a second body, the second body exerts a force on the first that is equal in magnitude but in the opposite direction. Therefore, we can equate the force that the Sun  $S$  exerts on a planet  $S$  with the force that the planet exerts on the Sun. Let  $m$  be the mass of the planet and  $M$  be the mass of the Sun. Let  $F_{S \leftarrow E}$  be the force exerted by the Sun on the Earth and let  $F_{E \leftarrow S}$  be the force exerted by the Earth on the Sun. Using Equation 5,

$$F_{S \leftarrow E} = \frac{8\kappa_E^2 m}{L_E} \cdot \frac{1}{r^2} = \frac{C_E m}{r^2}$$



$$F_{E \leftarrow S} = \frac{8\kappa_S^2 M}{L_S} \cdot \frac{1}{r^2} = \frac{C_S M}{r^2}$$

$$\frac{C_E}{M} \cdot \frac{1}{r^2} = \frac{C_S}{m} \cdot \frac{1}{r^2},$$

from some constants  $C_E, C_S$ .

Why are the constants different? The Earth and the Sun both rotate around their center of mass called the *barycenter*, which is very close to the center of the Sun since the Sun is so much more massive than the Earth. The ellipse of the Sun's orbit is very small relative to the Earth so  $A$  and  $L$  are smaller, and the Sun's period is large so  $T$  is larger. The different values for  $8\kappa^2/L$  are encapsulated into the constants  $C_E, C_S$ . Let  $G = \frac{C_E}{M} = \frac{C_S}{m}$  so that

$$F_{S \leftarrow E} = F_{E \leftarrow S} = G \frac{mM}{r^2}. \quad (6)$$

This is Newton's law of universal gravitation. It is not specific to planetary orbits but holds between any two bodies with masses  $m, M$ .

## 5.4 Kepler's third law

Newton proved that Kepler's third law follows from the law of universal gravitation.

**Theorem 5.4 (Kepler's third law)** *Let  $P_1, P_2$  be two planets whose elliptical orbits have semi-major axes  $a_1, a_2$  and whose orbital periods around the Sun are  $T_1$  and  $T_2$ . Then*

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2}.$$

**Proof** By Equations 5 and 6,

$$F = \frac{4\pi^2 a_i^3 m}{T_i^2} \frac{1}{r_i^2} = \frac{GmM}{r_i^2}.$$

After canceling  $m$  and  $r_i$  we get

$$\frac{a_i^3}{T_i^2} = \frac{GM}{4\pi^2}.$$

$\frac{GM}{4\pi^2}$  is a constant that depends only on the mass of the Sun and the gravitational constant, so  $\frac{a_i^3}{T_i^2}$  is constant for all planets rotating around the Sun. ■

## 6 The proof Proposition XI, Problem VI

Theorem 5.3 is Book I, Section III, Proposition XI, Problem VI of the *Principia*.

Study Figure 25:<sup>6</sup>

- Given an ellipse whose center is  $C$  and whose foci are  $S$  and  $H$ , let  $P, Q$  be two points on the ellipse. These points represent the movement of a body in an elliptical orbit separated by a time interval  $\Delta t$ .
- Construct lines from  $P$  to the center and the foci.
- Let  $RPZ$  be the tangent at  $P$  such that the line segment  $PR$  is the path that the body would transverse if it continued moving for time  $\Delta t$ , but was not subject to any force.
- Construct the parallelogram  $PRQX$ .
- Extend  $QX$  until it intersects  $PC$  at  $V$ .
- Construct a line parallel to  $RP$  through  $H$  and let  $I$  be its intersection with  $PS$ .
- Construct  $DC$  the conjugate diameter to  $PC$  (Definition 7.11) and let  $E$  be its intersection with  $PS$ .

<sup>6</sup>To save space the bottom half of the ellipse has been clipped, but we still refer to lines from the center to points on the ellipse as diameters.

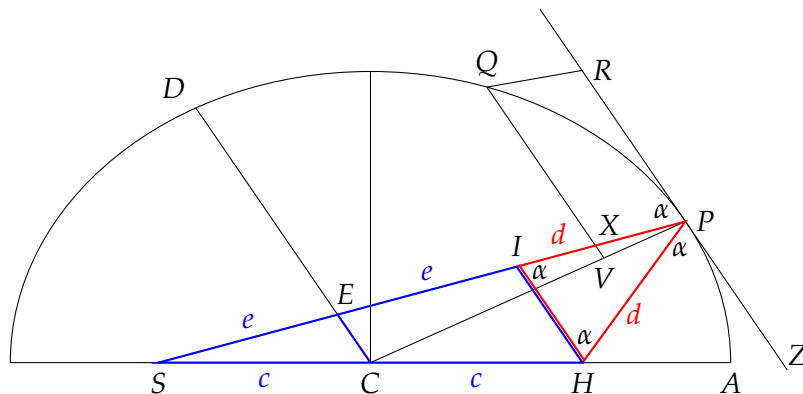


Figure 25: Geometry of an elliptical orbit (1)

## 6.1 A formula for $QR$

**Theorem 6.1**  $QR = PV \cdot \frac{CA}{CP}$ .

**Proof** By Theorem 7.8,  $\alpha = \angle RPX = \angle ZPH$  and by alternate interior angles,

$$\angle PHI = \angle ZPH = \alpha = \angle RPX = \angle PIH,$$

so  $\triangle IPH$  (red) is isocles and  $d = PI = PH$ .

$SC = CH = c$  are equal because they are the distances of the foci from the center of the ellipse. Let  $SE = e$ . By construction  $EC \parallel IH$  so  $\triangle ESC \sim \triangle ISH$  (blue) so

$$\begin{aligned} \frac{SC}{SE} &= \frac{SH}{SI} \\ \frac{c}{e} &= \frac{2c}{SI} \\ SI &= 2e. \end{aligned}$$

By definition of an ellipse  $SP + PH = SI + IP + PH = 2CA$  so  $(2e + d) + d = 2CA$  and  $EP = e + d = CA$ .

$QV \parallel EC$  so  $\triangle EPC \sim \triangle XPV$  and

$$\begin{aligned} \frac{PX}{PV} &= \frac{EP}{PC} = \frac{CA}{PC} \\ PX &= PV \cdot \frac{CA}{PC}. \end{aligned}$$

By construction of the parallelogram  $QR = PX$  proving that  $QR = PV \cdot \frac{AC}{PC}$ . ■

## 6.2 A formula for $QT$

**Theorem 6.2** Add a perpendicular from  $P$  to  $DC$  to the diagram and label its intersection with  $DC$  by  $F$  (Figure 26).

$$QT = QX \cdot \frac{FP}{CA}.$$

**Proof** By construction,  $QR \parallel PX$ , so by alternate interior angles  $\angle RQX = \angle QXT = \beta$ . By construction,  $QX \parallel DC$ , so by alternate interior angles  $\angle QXT = \angle PEF = \beta$ . Since  $\triangle PFE$  and  $\triangle QTX$  are right triangles with an equal acute angle  $\beta$ ,  $\triangle PFE \sim \triangle QTX$ . In the proof of Theorem 6.1 we showed that  $EP = CA$  so

$$\begin{aligned} \frac{QT}{QX} &= \frac{FP}{EP} \\ QT &= QX \cdot \frac{FP}{EP} = \frac{FP}{CA}. \quad \blacksquare \end{aligned}$$

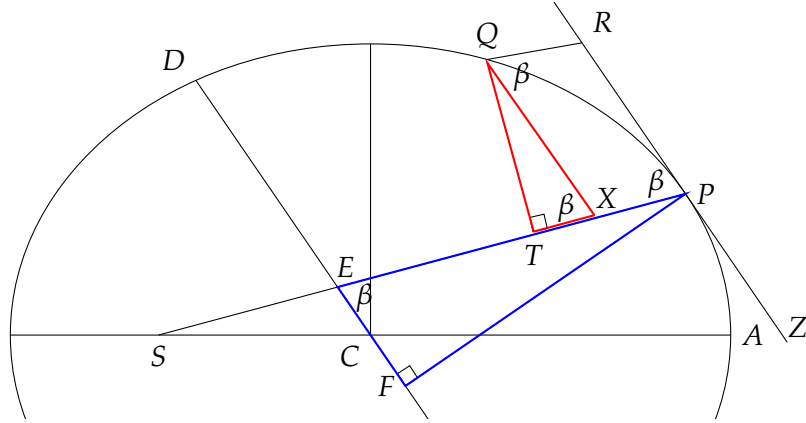


Figure 26: Geometry of an elliptical orbit (2)

### 6.3 A formula for $QR/QT^2$

**Theorem 6.3**

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}.$$

**Proof** Let use combine the equations in Theorems 6.1 and 6.2 to get  $QR/QT^2$ .

$$\frac{QR}{QT^2} = \frac{PV \cdot \frac{CA}{CP}}{\left(QX \cdot \frac{FP}{CA}\right)^2} = \frac{PV \cdot CA^3}{QX^2 \cdot CP \cdot FP^2}. \quad (7)$$

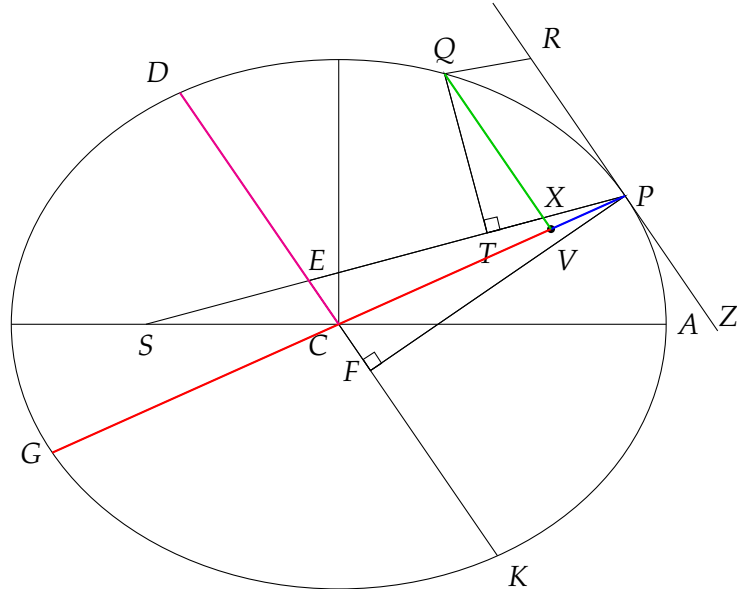


Figure 27: Ratios on conjugate diameters

$DC$  and  $PC$  are conjugate diameters so Theorem 7.10 gives a formula for  $PV$  that we substitute into Equation 7.

$$\frac{QR}{QT^2} = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2} \cdot \frac{CA^3}{QX^2 \cdot CP \cdot FP^2} = \frac{CP}{CD^2} \cdot \frac{CA^3}{FP^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \quad (8)$$

Next we show that  $CD \cdot FP = CA \cdot CB$ . Theorem 7.12 proves that the areas of the parallelograms formed by the tangents to conjugate diameters are equal (Figure 27). By symmetry the areas of the four small parallelograms are equal, as are the triangles formed by constructing corresponding diagonals. In Figure 28 the area of the  $\triangle ABC$  (red), which is  $(1/2)AC \cdot AB$ , is equal to the area of  $\triangle PCD$  (blue), which is  $(1/2)FP \cdot CD$ . Substituting for  $CD \cdot FP$  in Equation 8 gives

$$\begin{aligned} \frac{QR}{QT^2} &= \frac{CP}{CD^2} \cdot \frac{CA^3}{FP^2} \cdot \frac{QV^2}{GV \cdot QX^2} \\ \frac{QR}{QT^2} &= \frac{CP \cdot CA^3}{CB^2 \cdot CA^2} \cdot \frac{QV^2}{GV \cdot QX^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \quad \blacksquare \end{aligned} \quad (9)$$

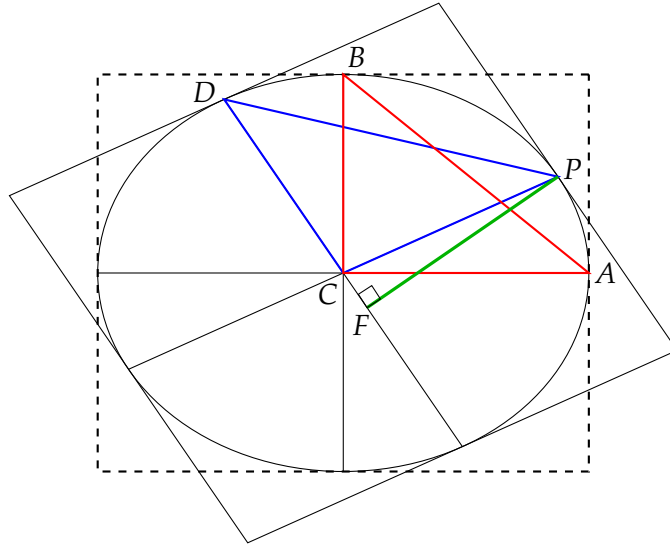


Figure 28: Parallelograms formed by conjugate diameters

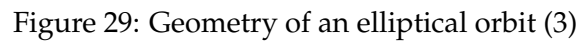


Figure 29 is an enlarged diagram of part of Figure 27. Compare the two diagrams and you will see that as the time interval  $\Delta t$  gets smaller,  $Q \rightarrow P$  which implies that

- $$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QX^2}{2CP \cdot QX^2} = \frac{CA}{2CB^2} = \frac{a}{2b^2} = \frac{1}{L},$$

30

## 7 Ellipses

### 7.1 Fundamental properties

#### Definition 7.1 (Ellipse)

- Let  $S$  and  $H$  be two points in the plane such that  $SH = 2c \geq 0$  and choose  $a$  such that  $2a > 2c$  (Figure 30). An ellipse is the geometric locus of all points  $P$  such that  $SP + HP = 2a$ . If  $c = 0$  the geometric locus is a circle.
- Construct  $AB$  through  $SH$ , where  $A, B$  are the intersections of the line with the ellipse.  $AB$  is the major axis of the ellipse. Let  $O$  be the midpoint of  $SH$ .  $AO$  and  $OB$  are the semi-major axes of the ellipse.
- Construct the perpendicular to  $AB$  at  $O$  and let  $C, D$  be its intersections with the ellipse.  $CD$  is the minor axis of the ellipse and  $CO$  and  $OD$  are the semi-minor axes of the ellipse.

#### Theorem 7.2 (Figure 31)

1.  $SC = HC = a$ .
2.  $AO = OB = a$ .
3.  $CO = OD$ . (Label  $CO = OD$  by  $b$ .)

#### Proof

1.  $\triangle SCO \cong \triangle HCO$  by side-angle-side so  $SC = HC$ . Since  $C$  is on the ellipse,  $SC + HC = 2a$  and  $SC = HC = a$  follows.
2. Since  $A$  is on the ellipse,

$$2a = AS + AH = (AO - c) + (AO + c) = 2AO,$$

so  $AO = a$ .  $OB = a = AO$  follows by symmetry.

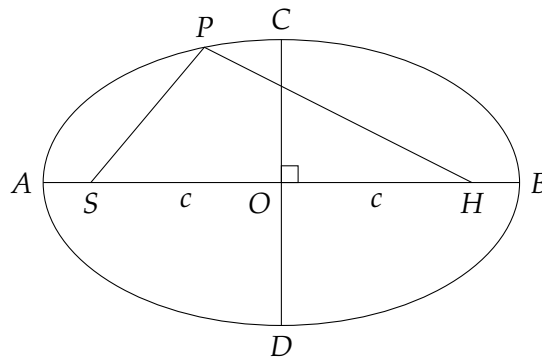


Figure 30: The definition of an ellipse

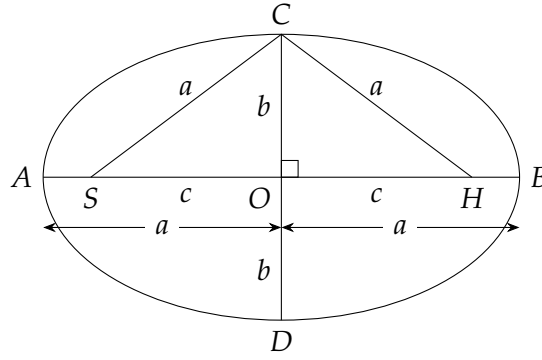


Figure 31: The semi-major and semi-minor axes of an ellipse

3.  $CO = OD$  follows since  $\triangle SCO \cong \triangle SDO$ . ■

**Theorem 7.3** A point  $P = (x, y)$  on an ellipse satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

**Proof** Since  $S = (-c, 0)$ ,  $H = (c, 0)$  and  $PS + PH = 2a$ ,

$$PS + PH = \sqrt{(x - (-c))^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

Squaring twice results in

$$(x + c)^2 + y^2 = \left(2a - \sqrt{(x - c)^2 + y^2}\right)^2$$

$$4xc = 4a^2 - 4a\sqrt{(x - c)^2 + y^2}$$

$$a - \frac{c}{a}x = \sqrt{(x - c)^2 + y^2}$$

$$a^2 + \frac{c^2}{a^2}x^2 = x^2 + c^2 + y^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{c^2 - a^2} = \frac{c^2 - a^2}{c^2 - a^2} = 1.$$

By Theorem 7.2 and Pythagoras's theorem,  $b^2 = a^2 - c^2$  so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \blacksquare$$

## 7.2 A circle circumscribing an ellipse

Consider a circle of radius  $a$  with the same center as an ellipse (Figure 32). Choose a point  $X$  on the major axis and construct a perpendicular through  $X = (x, 0)$ . Let its intersections with the ellipse and the circle be  $P_e = (x, y_e)$  and  $P_c = (x, y_c)$ , respectively.



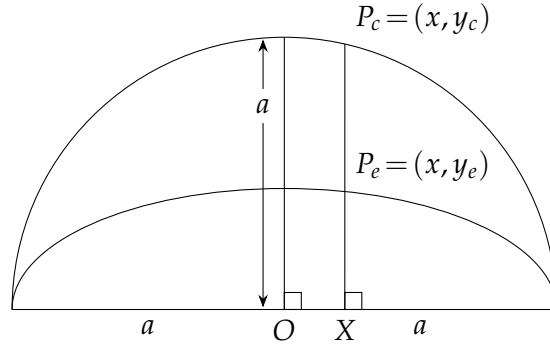


Figure 32: A circle circumscribing an ellipse

**Theorem 7.4** *The perpendicular to the major axis through a point  $P_c = (x, y_c)$  on the circle circumscribing an ellipse intersects the ellipse at  $P_e = (x, y_e) = \left(x, \frac{b}{a}y_c\right)$ .*

**Proof** Using the formulas for points on an ellipse and on a circle we have

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y_e^2}{b^2} &= 1 \\ y_e &= \frac{b}{a} \sqrt{(a^2 - x^2)} = \frac{b}{a}y_c. \quad \blacksquare \end{aligned} \tag{10}$$

### 7.3 The latus rectum of an ellipse

**Definition 7.5** *Consider a line through a focus of an ellipse that is perpendicular the major axis. Let its intersections with the ellipse be  $L_1, L_2$ .  $L = L_1L_2$  is a latus rectum<sup>7</sup> of an ellipse (Figure 33).*

**Theorem 7.6**  *$L$ , the length of the latus rectum of an ellipse, is  $\frac{2b^2}{a}$ .*

**Proof** By Equation 10 and Pythagoras's theorem,

$$L = 2L_1 = 2 \cdot \frac{b}{a} \sqrt{a^2 - c^2} = \frac{2b^2}{a}. \quad \blacksquare$$

<sup>7</sup>Normally, points are denoted by upper-case letters and line segments or lengths by lower-case letters, but  $L$  for the latus rectum is the standard notation.

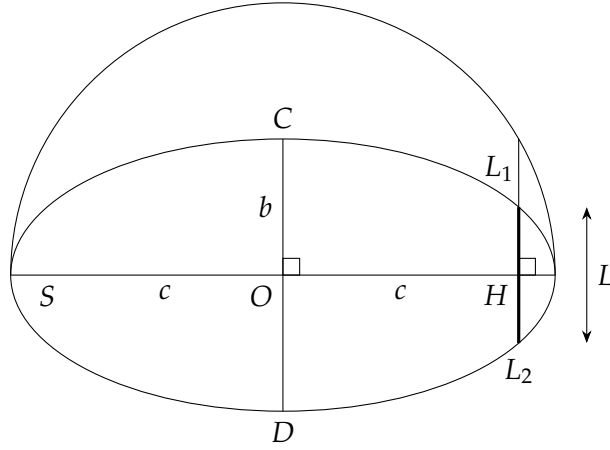


Figure 33: The latus rectum of an ellipse

## 7.4 The area of an ellipse

**Theorem 7.7** *The area of an ellipse is  $\pi ab$ .*

**Proof** From Equation 10

$$y_e = \frac{b}{a} \sqrt{a^2 - x^2},$$

so the area of an ellipse is

$$A_e = 2 \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} dx = 2 \frac{b}{a} \int_{-a}^a \sqrt{a^2 - x^2} dx = \frac{b}{a} A_c.$$

If we can show that the area of a circle is  $\pi a^2$  the theorem follows.

The proof uses polar coordinates, where  $x = a \cos \theta$  and  $y = a \sin \theta$ . First, we derive the formula for the integral of  $\sin^2 \theta$  using the double-angle identity.

$$\begin{aligned} \sin^2 \theta &= \frac{1 - \cos 2\theta}{2} \\ \int \sin^2 \theta d\theta &= \int \frac{1 - \cos 2\theta}{2} d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C. \end{aligned}$$

Now we can compute the area of a circle as twice the area of a semicircle by changing from Cartesian to polar coordinates and integrating.

$$\begin{aligned} A_c &= 2 \int_{-a}^a \sqrt{a^2 - x^2} dx = 2 \int_{-\pi}^0 \sqrt{a^2 - (a \cos \theta)^2} d(a \cos \theta) \\ &= 2 \cdot a \cdot a \int_{-\pi}^0 \sin \theta (-\sin \theta) d\theta = -2a^2 \int_{-\pi}^0 \sin^2 \theta d\theta \\ &= -2a^2 \left( \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C \right) \Big|_{-\pi}^0 = \pi a^2. \quad \blacksquare \end{aligned}$$

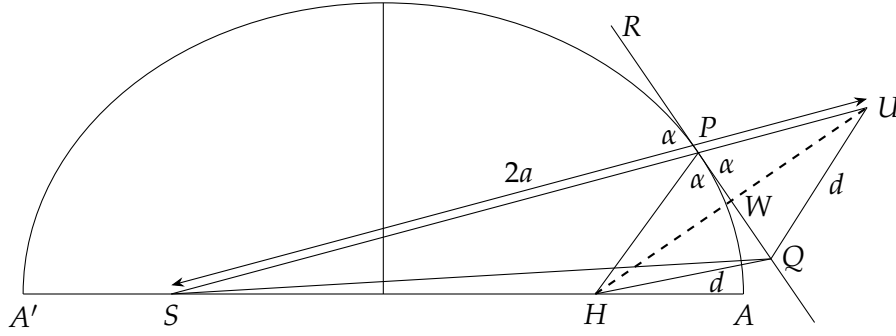


Figure 34: Angles at the tangent

## 7.5 The angles between a tangent and the lines to the foci

**Theorem 7.8** *Let  $P$  be a point on the ellipse whose foci are  $S, H$ . Let  $PU$  be the extension of  $SP$  such that  $SU = AA' = 2a$  (Figure 34). Let  $RQ$  be the bisector of  $\angle HPU$ . Then  $RQ$  is the tangent to the ellipse at  $P$  and  $\angle RPS = \angle HPQ$ .*

**Proof** We prove that any point  $Q \neq P$  on the bisector is not on the ellipse; therefore, the bisector  $RQ$  has only one point of intersection with the ellipse and must be a tangent. Since  $RQ$  is the angle bisector of  $\angle HPU$ ,  $\angle HPQ = \angle UPQ = \alpha$ , and by vertical angles  $\angle RPS = \angle QPU = \alpha$ .

Construct the line  $HU$  to form the triangle  $\triangle HPU$ . By construction  $SU = 2a$  so  $PU = PH$  and by the angle bisector theorem,

$$\frac{UW}{HW} = \frac{PU}{PH} = 1.$$

$\triangle UWQ \cong \triangle HWQ$  by side-angle-side so  $UQ = HQ$ . Suppose that  $Q$  is on the ellipse. Then  $2a = SQ + HQ = SQ + QU$ . By the triangle inequality  $2a = SQ + QU > SU = 2a$ , contradicting that  $Q$  is on the ellipse. ■

## 7.6 The parametric representation of an ellipse

Figure 35 shows an ellipse and two circles: one whose radius is the length of the semi-major axis (dotted red) and one whose radius is the semi-minor axis (dashed blue). The figure shows the *parametric representation* of a point  $P = (x, y)$  on the ellipse:

$$(x, y) = (a \cos t, y = b \sin t).$$

The parameter  $t$  is *not* the angle of  $P$  relative to the positive  $x$ -axis. It is constructed by projecting  $P$  onto the minor axis. Then the line at angle  $t$  is constructed from  $C$  to the intersection of the projection with the inner circle (dashed). The line is extended to intersect the outer circle (dotted). The parametric representation of  $P$  is computed by multiplying the lengths of the axes by the appropriate trigonometry function.

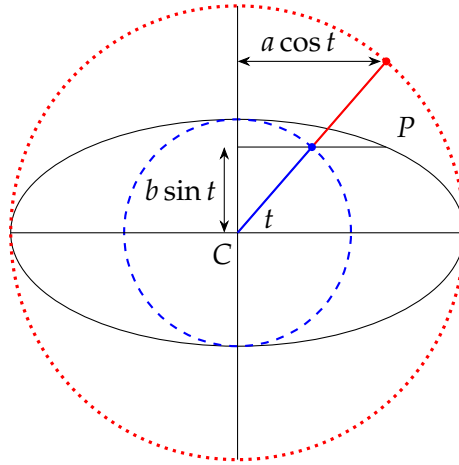


Figure 35: Parametric representation of an ellipse

## 7.7 Ratios

**Theorem 7.9** Let  $P = (x, y)$  be a point on an ellipse (not on the major axis  $AA'$ ) and construct a perpendicular  $PV$  from  $P$  to the major axis (Figure 39). Then

$$\frac{A'V \cdot AV}{PV^2} = \frac{a^2}{b^2}.$$

**Proof** By Equation 10),

$$y^2 = b^2 \cdot \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2(a^2 - x^2)}{a^2}$$

$$\frac{A'V \cdot AV}{PV^2} = \frac{(a+x)(a-x)}{y^2} = \frac{a^2(a^2 - x^2)}{b^2(a^2 - x^2)} = \frac{a^2}{b^2}. \quad \blacksquare$$

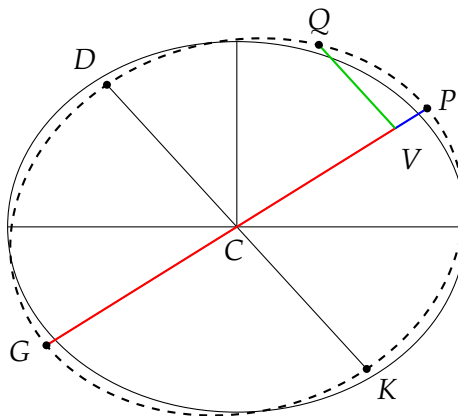


Figure 36: Ratios on conjugate diameters

**Theorem 7.10** Let  $PG, DK$  be conjugate diameters of an ellipse and let  $Q$  be a point on the ellipse (Figure 36). Drop a perpendicular  $QV$  from  $Q$  to the major axis, then

$$PV = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2}.$$

**Proof** Figure 36 shows a dashed ellipse which is the original ellipse rotated about the same center  $C$ , so that  $CP$  is the semi-major axis and  $CD$  is the semi-minor axis. By Theorem 7.9

$$\frac{GV \cdot PV}{QV^2} = \frac{a'^2}{b'^2},$$

where  $a', b'$  are the lengths of the semi-major and semi-minor axes of the rotated ellipse. By construction  $a' = CP$  and  $b' = CD$  so

$$\begin{aligned} \frac{GV \cdot PV}{QV^2} &= \frac{CP^2}{CD^2} \\ PV &= \frac{QV^2 \cdot CP^2}{GV \cdot CD^2}. \quad \blacksquare \end{aligned}$$

## 7.8 Areas of parallelograms

There are two equivalent definitions of conjugate diameters.

**Definition 7.11** Let  $P$  be a point on an ellipse,  $PG$  a diameter and let  $t$  be the tangent to the ellipse at  $P$ . Diameter  $DK$  is a conjugate diameter if it is parallel to  $t$  (Figure 37). Two diameters  $PG$  and  $DK$  are conjugate diameters if the midpoints of chords ( $D'K', D''K''$ ) parallel to one diameter ( $DK$ ) lie on another diameter ( $PG$ ).

**Theorem 7.12** The areas of parallelograms formed by tangents to the intersections of conjugate diameters with the ellipse are equal (Figure 38).

**Proof** We show that the area of the parallelogram  $JKLM$  is equal to the area of the parallelogram formed by tangents to the major and minor axes (dashed). By symmetry it suffices to prove that the areas of one of the quadrants of those parallelograms are equal:  $A_{ACBC'} = A_{PCDJ}$ . Since the diagonals bisect a parallelogram, it suffices to prove that the area of  $\triangle ABC$  (red) equals the area of  $\triangle PCD$  (blue).

Let  $P = (x_p, y_p) = (a \cos t, b \sin t)$ ,  $D = (x_d, y_d)$  be the parametric representations of the points on the ellipse. Conjugate diameters are perpendicular so  $\angle DCP$  is a right angle and

$$D = (x_d, y_d) = (a \cos(t + \pi/2), b \sin(t + \pi/2)) = (-a \sin t, b \cos t).$$

Construct  $DD' = (x_d, 0)$  and  $PP' = (x_p, 0)$  perpendicular to the major axis. The area of  $\triangle PCD$  can be computed as the area of the trapezoid  $P'PDD'$  minus the areas of the triangles  $\triangle D'DC, \triangle P'PC$ . Therefore,

$$\begin{aligned} \triangle PCD &= \frac{y_p + y_d}{2}(x_p - x_d) - \frac{1}{2}x_d y_d - \frac{1}{2}x_p y_p = \frac{1}{2}(x_p y_d - x_d y_p) \\ &= \frac{1}{2}(a \cos t \cdot b \cos t - (-a) \sin t \cdot b \sin t) = \frac{1}{2}ab = \triangle ABC. \quad \blacksquare \end{aligned}$$

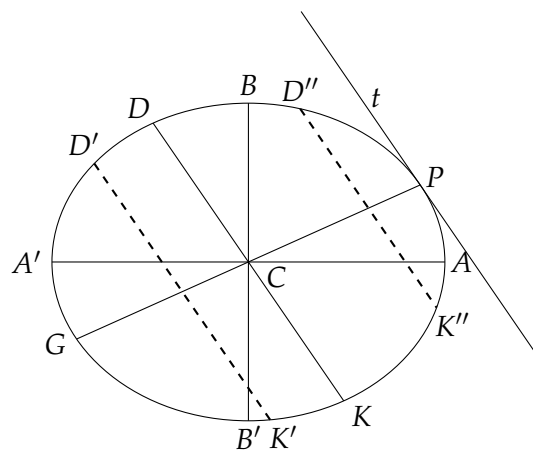


Figure 37: A conjugate diameter

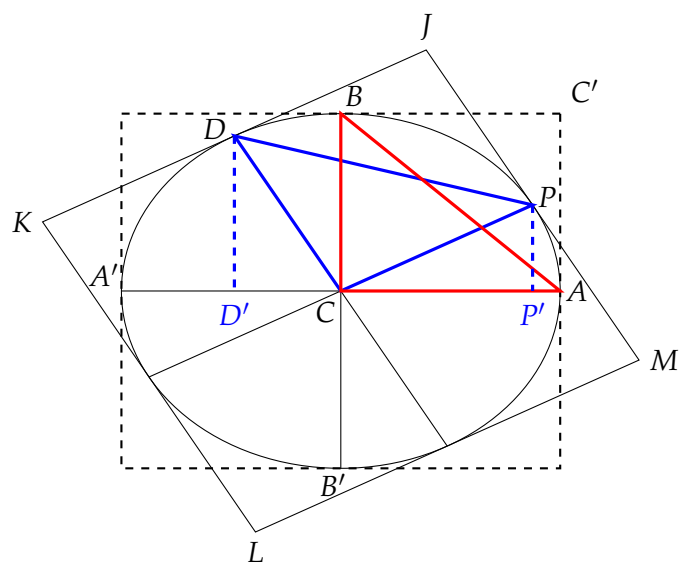


Figure 38: Parallelograms formed by conjugate diameters

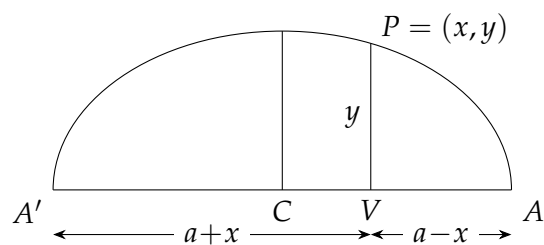


Figure 39: Ratios on conjugate diameters

## 8 Ellipses in Euclidean geometry

The proofs of theorems about planetary orbits freely used analytic geometry and trigonometry, but for many years after the invention of analytic geometry, mathematicians continued to limit themselves to Euclidean geometry. Even into the late nineteenth century, students continued to study the original works of Euclid and Newton. In this section, I present proofs in Euclidean geometry (*geometric proofs*) of theorems that appeared in the previous sections. The proofs are based on the definition of ellipses in terms of the geometric concepts of focus and directrix instead of the familiar analytic definition, points which satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

### 8.1 The definition of an ellipse using the focus and the directrix

**Definition 8.1** Let  $d$  be a line (the directrix) and  $S$  be a point (the focus) not on the directrix. Let  $0 < e < 1$  be a number (the eccentricity). An ellipse is the geometric locus of points  $P$  such that the ratio of  $PS$  to the distance of  $P$  to the directrix is  $e$ .

All the conic sections (parabolas, ellipses and hyperbolas) are defined the same way and are distinguished by their eccentricity. You might be familiar with the definition of the parabola where  $e = 1$  (Figure 40).

**Definition 8.2** Let  $X$  be the intersection of the perpendicular to the directrix from  $S$ .  $A$  on  $SX$  is a vertex of the ellipse if  $SA/AX = e$  (in Figure 41,  $e = 1/2$ ).

#### Constructing the vertices

Given  $e = SA/AX$  we can locate  $A$  as follows:

$$SA + AX = SX$$

$$SA = SX \cdot \frac{e}{1+e}$$

$$AX = SX \cdot \frac{1}{1+e}.$$

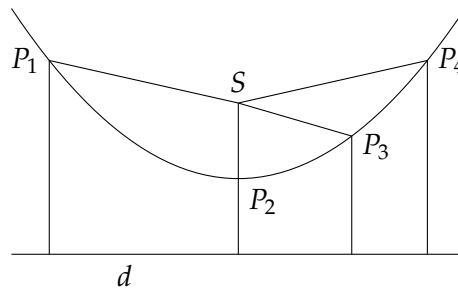


Figure 40: A parabola defined by the focus and the directrix

Similarly, there is a second vertex  $A'$  where

$$A'X - SA' = SX$$

$$SA' = SX \cdot \frac{1}{1-e}$$

$$A'X = SX \cdot \frac{2-e}{1-e}.$$

### Constructing points on an ellipse

Select an *arbitrary* point  $E$  on the directrix and construct lines from  $E$  through  $A$  and  $S$ . The line through  $S$  will make an angle  $\alpha$  with  $SX$ . Construct a line from  $S$  at the *same angle*  $\alpha$  from  $ES$  and let its intersection with  $EA$  be  $P$ . Construct the perpendicular from  $P$  to the directrix and let  $K$  be its intersection with the directrix. Let  $L$  be the intersection of  $KP$  with  $ES$  (Figure 42).

**Theorem 8.3** *The point  $P$  is on the ellipse.*

**Proof**  $\angle PLS = \angle LSN = \alpha$  by alternate interior angles, so  $\triangle LPS$  is isocoles and  $PL = SP$ . Since  $PK \parallel SX$ ,  $\triangle XEA \sim \triangle KEP$  and  $\triangle AES \sim \triangle PEL$  are adjacent pairs of similar triangles, so

$$\frac{PS}{PK} = \frac{PL}{PK} = \frac{SA}{AX} = e.$$

Therefore,  $P$  is on the ellipse. ■

We now construct another point on  $KP$ . Construct a line from  $F$  on the directrix through  $S$  such that  $\angle FSK = \angle KSP = \beta$  and let  $P'$  be the intersection of  $FS$  with  $KP$  (Figure 43).

**Theorem 8.4**  *$P'$  is on the ellipse.*

**Proof** Since  $KS$  bisects  $\angle PSF$ , by the exterior angle bisector theorem

$$\frac{SP'}{SP} = \frac{P'K}{PK}$$

$$\frac{SP'}{P'K} = \frac{SP}{PK} = e,$$

and therefore  $P$  is on the ellipse. ■

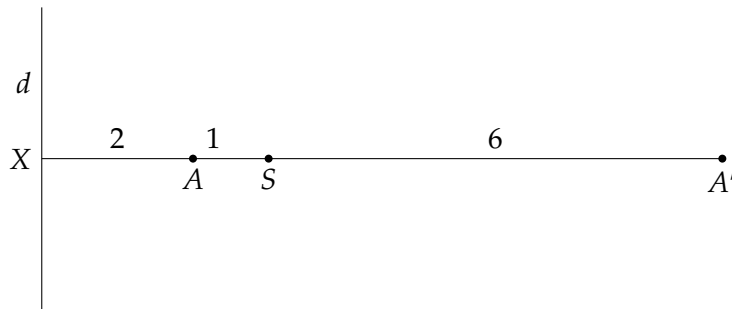


Figure 41: The elements of the definition of an ellipse





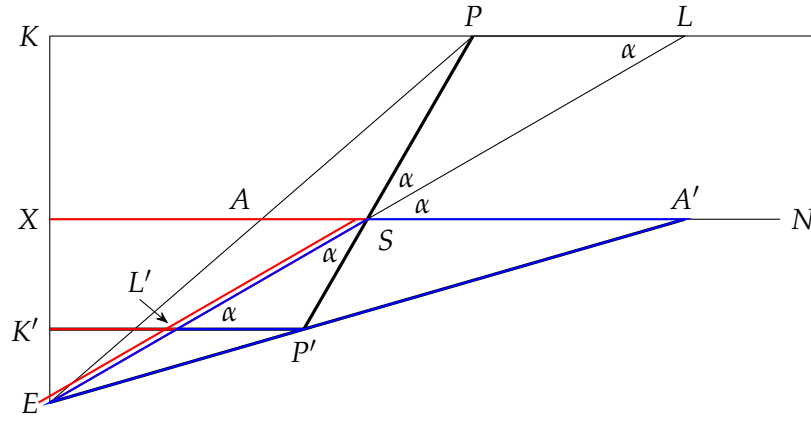


Figure 44: Constructing a focal chord

By alternate interior angles and vertical angles we have  $\angle SL'P' = \angle L'SP'$  so triangle  $\triangle SL'P'$  is isosceles and  $P'L' = SP'$ . Therefore,

$$\frac{SP'}{P'K'} = \frac{SA'}{A'X} = e,$$

and  $P'$  is a point on the ellipse. ■

### 8.3 A right angle at the focus of an ellipse

**Theorem 8.7** Let  $P, P'$  be points on the ellipse and let  $F$  be the intersection of  $PP'$  with the directrix. Then  $FS$  bisects the exterior angle of  $\angle P'SP$  (Figure 45).

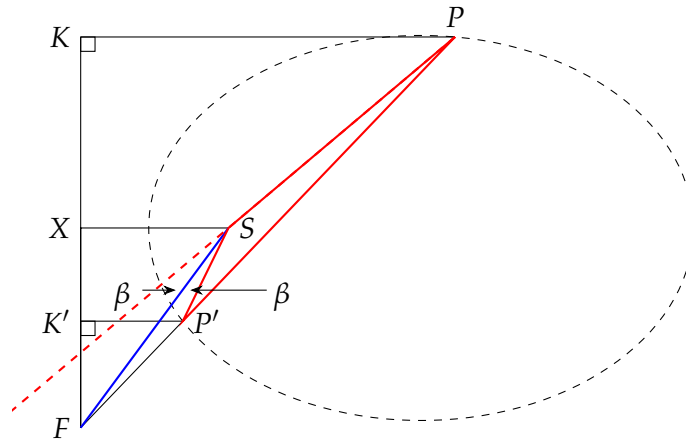


Figure 45: Bisecting the angle at the focus

**Proof** Since  $P, P'$  are on the ellipse

$$\frac{SP}{PK} = \frac{SP'}{P'K'} = e,$$

and since  $\triangle PFK \sim P'FK'$ ,

$$\frac{SP}{SP'} = \frac{PK}{P'K'} = \frac{PF}{P'F}.$$

By the exterior angle bisector theorem  $FS$  bisects the exterior angle of  $\angle P'SP$ . ■

**Theorem 8.8** *Let  $P$  be a point on the ellipse and construct lines  $PA, PA'$ . Label their intersections with the directrix by  $E$  and  $F$ , respectively. Then  $\angle FSE$  is a right angle (Figure 45).*

**Proof**  $P, A, A'$  are all points on the ellipse so Theorem 8.7 applies.  $FS$  bisects  $\angle PSA = 2\gamma$  and  $ES$  bisects  $\angle P'SA = 2\delta$ , so  $2\gamma + 2\delta = 180^\circ$  and  $\angle FSE = \gamma + \delta = 90^\circ$ . ■

## 8.4 Ratios

**Theorem 8.9** *Let  $P$  be a point on an ellipse not on the major axis and construct perpendiculars  $PN, PM$  from  $P$  to the major and minor axes, respectively (Figure 47). Then*

$$\frac{PN^2}{A'N \cdot AN} = \frac{BC^2}{AC^2} = \frac{b^2}{a^2} \quad (11)$$

$$\frac{PM^2}{B'N \cdot BN} = \frac{AC^2}{BC^2} = \frac{a^2}{b^2}. \quad (12)$$

**Proof** (Equation 11)  $\triangle AXE \sim \triangle ANP$  since they are right triangles and the vertical angles at  $A$  are equal (red). Therefore,

$$\frac{PN}{AN} = \frac{EX}{AX}. \quad (13)$$

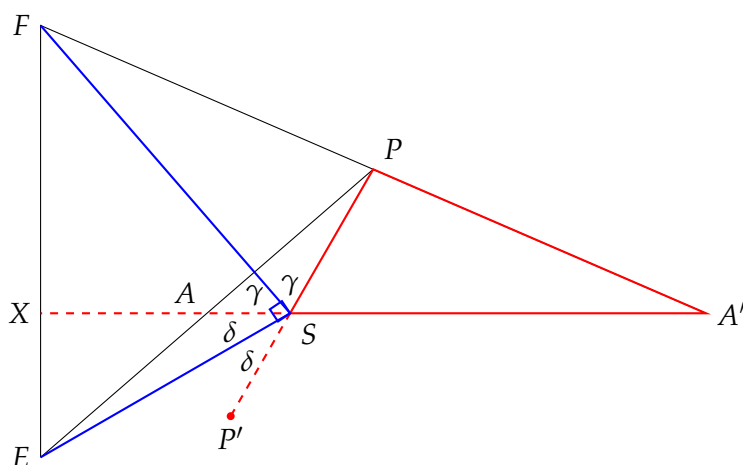


Figure 46: The right angle at the focus

$\triangle PA'N \sim \triangle FA'X$  (blue) so

$$\frac{PN}{A'N} = \frac{FX}{A'X}. \quad (14)$$

Multiplying Equations 13 and 14 gives

$$\frac{PN^2}{AN \cdot A'N} = \frac{EX \cdot FX}{AX \cdot A'X}.$$

By Theorem 8.8  $\triangle FSE$  is a right triangle so by Theorem A.3 giving

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X}.$$

Since  $P$  was arbitrary this holds for any point on the ellipse, in particular, for  $B$  on the minor axis, where  $PN = BC$  and  $AN = AN' = AC$ . Therefore,

$$\begin{aligned} \frac{BC^2}{AC^2} &= \frac{SX^2}{AX \cdot A'X} \\ \frac{PN^2}{AN \cdot A'N} &= \frac{SX^2}{AX \cdot A'X} = \frac{BC^2}{AC^2} = \frac{b^2}{a^2}. \quad \blacksquare \end{aligned}$$

**Proof** (Equation 12) From Equation 11, since  $CM = PN$ ,  $PM = CN$  and by Theorem A.2,

$$AN \cdot NA' = AC^2 - CN^2 \quad (15)$$

$$\frac{CM^2}{AC^2 - PM^2} = \frac{BC^2}{AC^2}$$

$$\frac{AC^2}{AC^2 - PM^2} = \frac{BC^2}{CM^2} \quad (16)$$

$$\frac{AC^2}{PM^2} = \frac{BC^2}{BC^2 - CM^2} \quad (17)$$

$$\frac{PM^2}{BM \cdot MB'} = \frac{AC^2}{BC^2}.$$

The equivalence of Equations 16, 17 can be seen by cross-multiplying the equations.  $\blacksquare$

## 8.5 A circle circumscribing an ellipse

Consider a circle of radius  $a$  with the same center as an ellipse (Figure 48). Choose a point  $N$  on the major axis and construct a perpendicular through  $N$ . Let its intersections with the ellipse and the circle be  $P$  and  $Q$ , respectively.

**Theorem 8.10** *The perpendicular to the major axis through a point  $Q$  on the circle circumscribing an ellipse intersects the ellipse at  $P$  such that*

$$\frac{PN}{QN} = \frac{BC}{AC} = \frac{b}{a}.$$



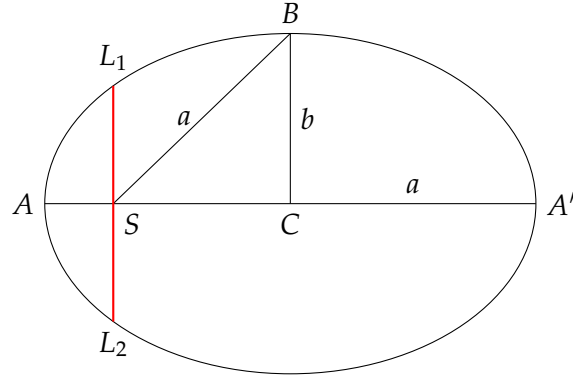


Figure 49: The latus rectum of an ellipse

**Proof** By Theorem 8.9,

$$\frac{SL_1^2}{AS \cdot SA'} = \frac{BC^2}{AC^2}.$$

By Theorem 7.2,  $SB = AC = a$ , so by Pythagoras's theorem,

$$BC^2 = BS^2 - SC^2 = AC^2 - SC^2 = (AC - SC)(AC + SC) = AS \cdot SA'.$$

Therefore,

$$SL_1^2 = \frac{BC^4}{AC^2}$$

$$SL_1 = \frac{BC^2}{AC} = \frac{b^2}{a},$$

but  $L_1$  is one-half the latus rectum. ■

## 8.7 Areas of parallelograms

**Theorem 8.13** *Let  $Y$  be the intersection of the perpendicular to the focus  $S$  and the tangent  $TT'$  at  $P$ , and let  $L$  be the intersection of  $S'P$  and  $SY$  (Figure 50). Then  $Y$  is on the circumscribing circle and  $CY \parallel S'L$ .*

**Proof**  $\triangle STY$  and  $\triangle T'TC$  are right triangles that share the acute angle  $\angle CTT' = \angle YTS$  so  $\angle CT'T = \angle YST$ .  $\angle SPY = \angle S'PT'$  by Theorem 7.8 since they are the angles to the foci at the tangent and  $\angle S'PT' = \angle YPL$  are vertical angles so  $\angle SPY = \angle SPT$ . Then  $\triangle SPY \cong \triangle SPT$  since they are right triangles with an equal acute angle and a common side  $PY$ . Therefore,  $PL = PS$  and  $S'L = S'P + PL = AA' = 2a$ .

Since  $\triangle SPY \cong \triangle SPT$ ,  $SY = YL$ , and since  $S, S'$  are foci,  $S'C = SC$ . It follows that  $\triangle CSY \sim \triangle S'SL$  and  $CY \parallel S'L$ . By similarity,

$$\frac{CY}{S'L} = \frac{CS}{S'S} = \frac{CS}{2CS}.$$

Therefore,  $2CY = S'L = 2a$  so  $CY = a$  and  $Y$  is on the circumscribing circle of radius  $a$ . ■



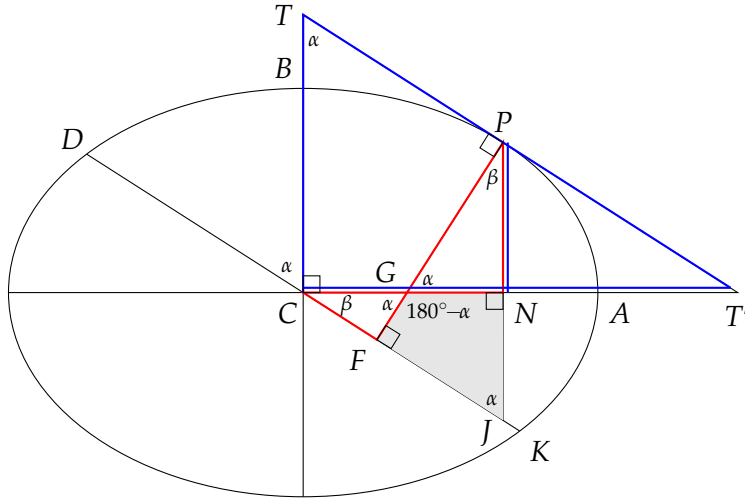


Figure 52: Parallelograms formed by conjugate diameters

since the perpendicular to the tangent from a focus is on the circumscribing circle (Theorem 8.13).

Since  $NT = CT - CN$ , we have  $CN \cdot NT = CN \cdot CT - CN^2$  which equals  $AC^2 - CN^2$  by Equation 18. This in turn equals  $AN \cdot NA'$  by Theorem A.2. ■

**Theorem 8.15** Construct the normal to the tangent at  $P$  and let its intersection with the conjugate diameter  $DK$  be  $F$  and its intersection with the major axis be  $G$ . Construct a perpendicular from  $P$  to the major axis and let its intersection be  $N$  (Figure 52). Let the intersection of the tangent with the minor axis be  $T$  and its intersection with the major axis be  $T'$ .

$$PF \cdot PG = BC^2.$$

**Proof**  $\triangle NPG \sim \triangle FPJ$  (rotate  $\triangle NPG$  to see this) so  $\angle PGN = \angle PJF = \alpha$  and

$$\frac{PF}{PN} = \frac{PJ}{PG}$$

$$PF \cdot PG = PJ \cdot PN.$$

By vertical angles  $\angle PGN = \angle CGF = \alpha$  so  $\triangle NPG \sim \triangle FCG$  and  $\angle NPG = \angle FCG = \beta = 90^\circ - \alpha$ . By adding  $\beta$  to the right angle  $\angle BCN = \angle BNC$ , we get that  $\angle BCF = \angle BCJ$  and therefore  $TPJC$  is a parallelogram, so  $CT = PJ$  and  $PF \cdot PG = CT \cdot PN$ . We need to show that  $CT \cdot PN = BC^2$ .

$\triangle TT'C \sim \triangle PT'N$  so

$$\frac{CT}{CT'} = \frac{PN}{NT'}$$

$$\frac{CT}{PN} = \frac{CT'}{NT'}.$$



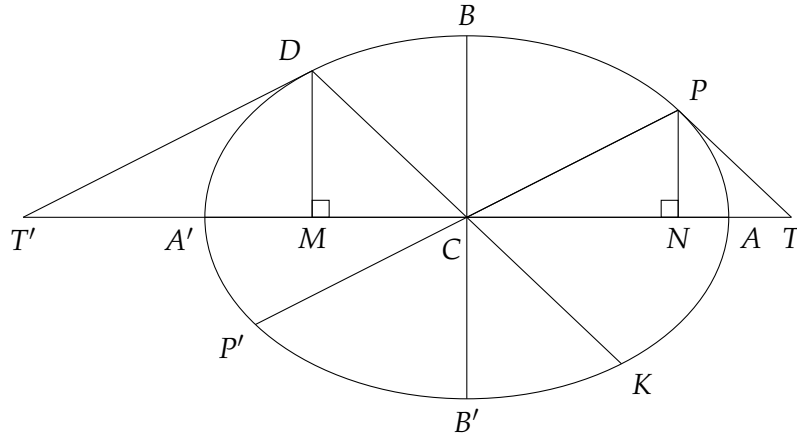


Figure 53: Ratios of perpendiculars to the major axis

Multiplying each side by fractions equal to 1 gives

$$\frac{CT \cdot PN}{PN^2} = \frac{CT' \cdot CN}{CN \cdot NT'}$$

and because ...

$$\frac{CT \cdot PN}{PN^2} = \frac{AC^2}{AN \cdot NA'}$$

By Theorem 8.9

$$\begin{aligned} \frac{CT \cdot PN}{PN^2} &= \frac{PN^2}{AN \cdot NA'} \cdot \frac{AC^2}{PN^2} = \frac{BC^2}{AC^2} \cdot \frac{AC^2}{PN^2} \\ CT \cdot PN &= BC^2. \quad \blacksquare \end{aligned} \tag{19}$$

**Theorem 8.16** In Figure 53,

$$CN^2 = AM \cdot MA', \quad CM^2 = AN \cdot NA'$$

$$\frac{DM}{CN} = \frac{BC}{AC}$$

$$\frac{CM}{PN} = \frac{BC}{AC}$$

**Proof** By Theorem 8.14,

$$CN \cdot CT = AC^2 = CM \cdot CT'$$

$$\frac{CM}{CN} = \frac{CT}{CT'}.$$

Since  $DK$  and  $PP'$  are conjugate diameters,  $DT' \parallel PP'$  and  $\triangle T'DC \sim \triangle CPT$ , so

$$\frac{CM}{CN} = \frac{CT}{CT'} = \frac{CN}{MT'}$$

$$CN^2 = CM \cdot MT',$$

Therefore,

$$CN^2 = CM \cdot MT' = AC^2 - CM^2 = AM \cdot MA' \quad (20)$$

by Theorem 8.14. By Theorem 8.9,

$$\frac{DM^2}{AM \cdot MA'} = \frac{BC^2}{AC^2},$$

and by Equation 20,

$$\frac{DM^2}{CN^2} = \frac{BC^2}{AC^2}$$

$$\frac{DM}{CN} = \frac{BC}{AC},$$

A symmetric argument shows that

$$CM^2 = AN \cdot NA'$$

$$\frac{CM}{PN} = \frac{BC}{AC} \quad \blacksquare$$

**Theorem 8.17** *The area of the parallelogram formed by the tangents at the ends of the conjugate diameters  $PG, DK$  is equal to the rectangle enclosing the ellipse at the ends of the axes (Figure 54).*

**Proof** By the definition of conjugate diameters, it is sufficient to show that the area of  $PCDL$  is  $AC \cdot BC$ . The area of a parallelogram is width times height so it is  $CD \cdot PF$ .

By vertical angles  $\angle DCM = \angle GCF$  and  $\angle CGF = \angle PGN$ , so  $\triangle DCM \sim \triangle PGN$

$$\frac{PG}{CD} = \frac{PN}{CM},$$

and by Theorem 8.16

$$\frac{PG}{CD} = \frac{BC}{AC}$$

$$\frac{CD}{AC} = \frac{PG}{BC}. \quad (21)$$

By Theorem 8.15,

$$\frac{PG}{BC} = \frac{BC}{PF}. \quad (22)$$

From Equations 21 and 22 gives  $CD \cdot PF = AC \cdot BC$ .  $\blacksquare$

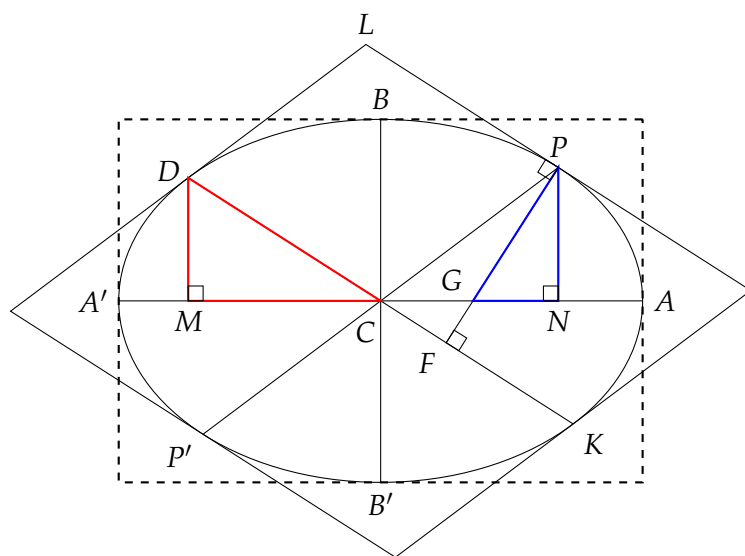


Figure 54: Areas of parallelograms

## A Theorems of Euclidean Geometry

### A.1 Constructing a circle from three points

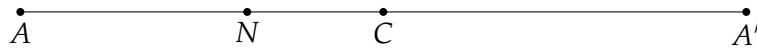
**Theorem A.1** *Given three non-collinear points,<sup>9</sup> a circle can be constructed that goes through all three points.*

**Proof** Three non-collinear points  $A, B, C$  define a triangle  $\triangle ABC$  (Figure 55). Construct the perpendicular bisectors of any two of its three sides, say,  $AC$  and  $BC$ . By definition the perpendicular bisector is the geometric locus of points equidistant from the endpoints of the segment. Let  $O$  be the intersection of the two bisectors. Then the  $AO = CO = BO$  is the radius of a circle centered at  $O$  that goes through  $A, B, C$ . ■

### A.2 The altitude to a hypotenuse

**Theorem A.2** *Let  $AA'$  be a line segment whose midpoint is  $C$ . Then*

$$AC^2 - CN^2 = AN \cdot NA'.$$



**Proof**  $AN = AC - CN$  and  $NA' = A'C + CN = AC + CN$  since  $C$  is the midpoint of  $AA'$ . The result is obtaining by multiplying the two equations.

**Theorem A.3** *Let  $Q$  be a point on a circle whose diameter is  $AA'$  and construct a perpendicular  $QN$  to the diameter (Figure 56). Then*

$$QN^2 = AN \cdot NA'.$$

*The equation also holds if it is given that  $\triangle AQA'$  is a right triangle.*

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<sup>9</sup>Collinear means that the points all on the same line.

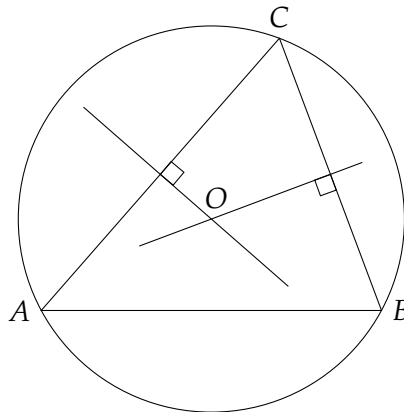


Figure 55: A circle through three arbitrary points

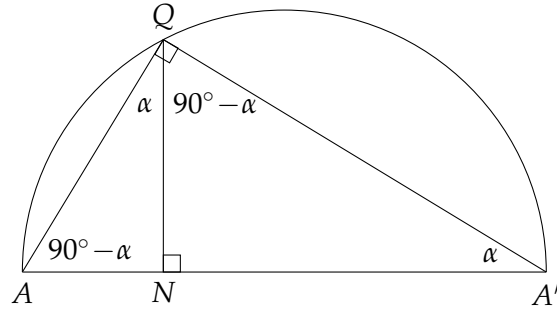


Figure 56: Right triangle in a circle at the diameter

**Proof** An angle that subtends a diameter is a right angle. Since the sum of the angles of a triangle is  $180^\circ$ , we can label the angles as shown in the Figure, from which follows that  $\triangle QNA \sim \triangle A'NQ$ . Therefore,

$$\frac{QN}{AN} = \frac{NA'}{QN}. \blacksquare$$

### A.3 The median of the hypotenuse

**Theorem A.4** Let  $ACB$  be a right triangle and let  $D$  be the intersection of the median from  $C$  and the hypotenuse  $AB$  (Figure 57). Then  $CD = AB/2$  and

$$AD \cdot DB = DC^2.$$

**Proof** Drop the perpendicular from  $D$  to  $BC$  and label the intersection by  $E$ .

By similar triangles,

$$\begin{aligned} \frac{a}{c_2} &= \frac{a + a}{c_1 + c_2} \\ c_1 &= c_2. \end{aligned}$$

Therefore,  $\triangle DEC \cong \triangle DEB$  and  $DC = DB$ .  $\blacksquare$

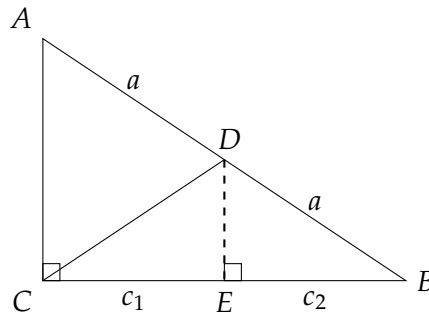


Figure 57: Median to the hypotenuse

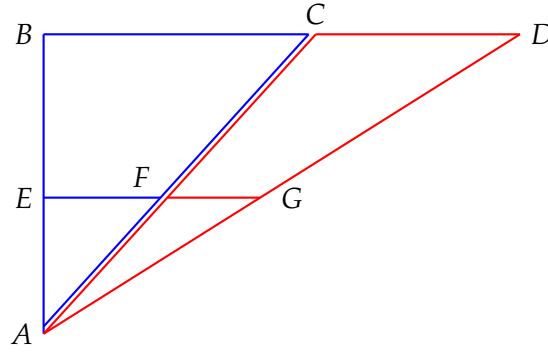


Figure 58: Adjacent pairs of similar triangles

#### A.4 Adjacent pairs of similar triangles

I have not encountered the following definition before but it explicitly expresses a relation among similar triangles that would have been obvious to geometers.

**Definition A.5** *An adjacent pair of similar triangles is a pair of (a pair of) similar triangles that share sides (Figure 58).  $\triangle BAC \sim \triangle EAF$  and  $\triangle CAD \sim \triangle FAG$  are an adjacent pair of similar triangles.*

**Theorem A.6** *For the adjacent pair of similar triangles in Figure 58,*

$$\frac{AB}{AE} = \frac{AD}{AG}.$$

**Proof** By similar triangles,

$$\begin{aligned} \frac{AB}{AE} &= \frac{AC}{AF} \\ \frac{AC}{AF} &= \frac{AD}{AG} \\ \frac{AB}{AE} &= \frac{AD}{AG}. \quad \blacksquare \end{aligned}$$

Similar ratios hold between other sides of  $\triangle BAC$  and  $\triangle CAD$  by using an intermediate step with  $AC$ . We will use the term *by an adjacent pair of similar triangles* and leave it to the reader to make the intermediate step.

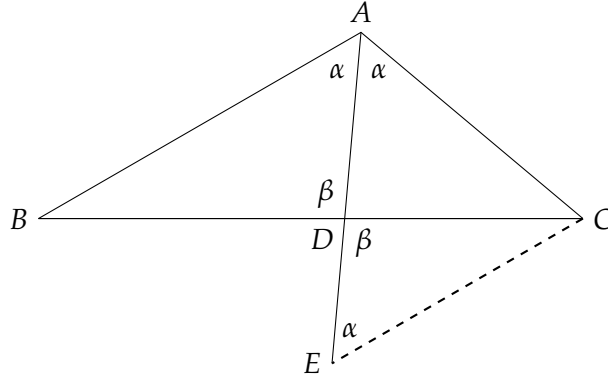


Figure 59: The interior angle bisector theorem

### A.5 The angle bisector theorems

The angle bisector theorems are used extensively in the study of conic systems, often implicitly.

**Theorem A.7 (Interior angle bisector theorem)** In  $\triangle ABC$  let  $D$  be a point on  $BC$  (Figure 59). Then  $AD$  bisects  $\angle CAB$  if and only if

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

**Proof** Suppose that  $AD$  bisects  $\angle BAC$ . Construct a line through  $C$  parallel to  $AB$  and let its intersection with  $AD$  be  $E$ . By alternate interior angles,  $\angle BAD = \angle CED$  and by vertical angles  $\angle BDA = \angle CDE$ . Therefore,  $\triangle ABD \sim \triangle EDC$  so

$$\frac{BD}{CD} = \frac{AB}{CE}.$$

$\triangle ECA$  is isosceles so  $CE = AC$  and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just “run” the proof backwards. ■

**Theorem A.8 (Exterior angle bisector theorem)** In  $\triangle ABC$  let  $D$  be a point on the extension of  $CB$  outside the triangle (Figure 60). Then  $AD$  bisects the exterior angle of  $\angle BAC$  if and only if

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

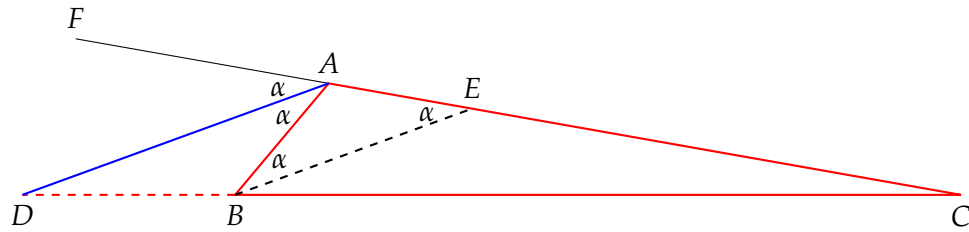


Figure 60: The exterior angle bisector theorem

**Proof** Suppose that  $AD$  bisects  $BAF$ . Construct a line through  $B$  parallel to  $AD$  and let its intersection with  $AC$  be  $E$ . By alternate interior angles  $\angle BAD = \angle ABE$  and by corresponding angles  $\angle FAD = \angle AEB$ . Therefore,  $\triangle BCE \sim \triangle DCA$  so

$$\frac{BD}{CD} = \frac{AE}{AC}.$$

But  $\triangle BAE$  is isosceles so  $AE = AB$  and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just “run” the proof backwards. ■

The exterior angle bisector theorem confusing to understand in a proof, because it can be hard to identify the components of a diagram. In the text the following color-coding is used: the triangle is red, the extension of one side is dashed red and the bisector is blue.



## Sources and further reading

Sections 2–5 based primarily on Hahn’s book [4]. He has written a more advanced book on the orbits of planets and spacecraft [5]. Section 6 is based on [3]. Two additional articles present this aspect of Newton’s work [6, 8]. Should you wish to read Newton’s work, [2] is an up-to-date translation into English and Cohen’s lengthy Guide will enable you to follow Newton’s often terse presentation. The relevant sections of the Guide are 10.8–10.10. Section 7 on ellipses is based on [9], while Section 8 is based on [1]. While Besant’s book uses Euclidean geometry exclusively, about the same time, Charles Smith wrote a book on conics that uses analytic geometry [7].<sup>10</sup>

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<sup>10</sup>The dates given for both books are for the editions that I used.