The Geometry of Planetary Orbits and Ellipses

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On a cloth untrue
With a twisted cue
And elliptical billiard balls.

William S. Gilbert
The Mikado

He seemed to live in some high abstract region of surds and conic sections, with little to connect him with ordinary life.

Arthur Conan Doyle
The Adventure of the Lion's Mane

Contents

Chapter 1

Introduction

Everyone "knows" that Kepler discovered that the orbits of the planets are ellipses and everyone "knows" that Newton showed that a planet in an elliptical orbit is subject to the force of gravity that is inversely proportional to the square of the distance from the Sun. Although I knew these facts, I had never seen them demonstrated until I read *Calculus in Context* [8] by Alexander J. Hahn. This is a comprehensive textbook on introductory calculus that augments theory with applications in physics and astronomy, such as the work of Kepler, Newton and Galileo, as well as applications in engineering such as building bridges and domed structures. These are not just historical anecdotes but detailed computations.

This document is a tutorial on the planetary orbits with an emphasis on proofs using Euclidean geometry. Although Newton invented the calculus and used it to study motion, from the time of the Greeks, "proof" meant proof by geometry. Newton's proof requires a depth of knowledge of Euclidean geometry and conic sections that is no longer studied today.

The tutorial is intended to enrich the learning of mathematics by secondary-school students and students in introductory university courses. The prerequisites are a good knowledge of Euclidean geometry along with some trigonometry, a bit calculus and Newton's laws of motions.

Overview

Part I of this document contains an explanation of the determination of orbits by Aristarchus, Copernicus, Kepler and Newton. The presentation is mathematical, since the historical and astronomical aspects are thoroughly described in [8], as well as in other works. Chapter 2 presents the measurements of the radii of the Earth, Moon and Sun, and of the distances between them, as determined by Eratosthenes and Aristarchus. Chapter 3 describes the construction of a model of a Sun-centered system by Nicolaus Copernicus. Chapter 4 shows how Johannes Kepler developed his three laws of planetary motion. Chapter 5 presents Isaac Newton's derivation of the inverse-square law of gravitation from of Kepler's laws. One step of Newton's derivation requires a theorem whose proof is very long, so it is split off into Chapter 6. Even Nobel laureate Richard P. Feynman found Newton's proof daunting, so he invented his own proof which we present in Chapter 7, along with a earlier proof by James Clerk Maxwell that uses the same technique. Chapter 8 describes Lagrange points, points in the solar system where spacecraft can be placed so that the periods of their orbits are the same as the Earth's.

Part II brings the definitions and theorems (and their proofs) required to understand Newton's proof. Chapter 9 contains *four* definitions of ellipses and shows that the definitions are equivalent. Do not read this chapter straight through! Instead, refer to it as needed. The theorems you need to prove Newton's theorem appear in Chapter 10, but the proofs are in most cases modernized using analytic geometry. Chapter 11 contains proofs of these theorems in Euclidean geometry.

Chapter 12 is a bonus chapter on generating ellipses by roulettes and glissettes.

Theorems of Euclidean geometry that may be unfamiliar but do not concern ellipses are collected in Appendix A.

Euclidean Geometry and William H. Besant

The fundamental importance of Euclidean geometry in mathematics continued until relatively recently, as shown by this amazing quote.

> In book 1, prop[osition] 10 (and notably in prop[osition] 11), Newton made use of a property of conics which he presents without proof, merely saying that the result in question comes from "the *Conics*." Here, as elsewhere in the *Principia*, Newton assumes the reader to be familiar with the principles of conics and of Euclid. In the eighteenth and nineteenth centuries, when Newton's treatise was still being read in British universities, authors of books on "conic sections"—for example, W. H. Besant, W. H. Drew, Isaac Milnes—supplied the proof of this theorem in order to help readers of the *Principia* who might be baffled by the problem of finding a proof. They even chose letters to designate points on the diagrams so that the final result would appear in exactly the same form as in the *Principia* [4, p. 330].

William H. Besant, FRS (1828–1917) was a British mathematician who studied at Cambridge University, where he was Senior Wrangler, the student with the highest grade on the Cambridge Mathematical Tripos examination. In addition to his mathematical achievements, he was well-known as a coach for students taking the Tripos. This led to the publication of *Conic Sections Treated Geometrically* in nine editions from 1869–1895.

The book has fifteen chapters, starting with a general chapter on conic sections followed by chapters on the parabola, the ellipse and the hyperbola. The chapter on ellipses has 31 propositions (theorems), 21 corollaries and 110 examples (exercises), more than you ever wanted to know about ellipses! I studied the chapter in detail and was struck by his deep knowledge of Euclidean geometry of the sort one learns in secondary school.

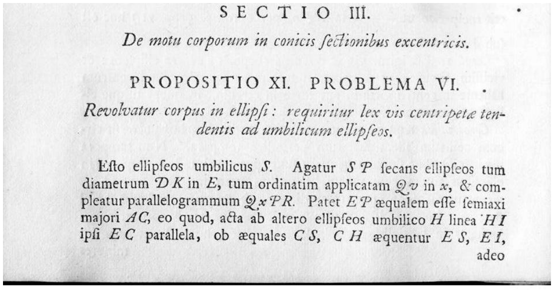
Project Gutenberg has published a PDF which is a transcription of the ninth edition [3]. Words like "trigonometry," "coordinate" and "equation" simply do not appear.

While detailed proofs are given, Besant's style is terse, indicating that he expected his students to have an intimate familiarity with Euclidean geometry. The book has numerous diagrams (reproduced as-is in the transcription), but even the most complicated ones do not have angles and line segments labeled. This tutorial expands Besant's proofs with additional details and more elaborate diagrams.

Newton's *Principia*

The final steps in Newton's derivation require the use of limits, which had been used already by Archimedes to compute the circumference and area of a circle by approximating the circle. Newton (along with his contemporary Gottfried Wilhelm Leibniz) developed the calculus from the concept of limits. However, the *Principia* uses Euclidean geometry almost exclusively, although analytic geometry had already been developed by René Descartes and Pierre de Fermat.

The following reproduction of Newton's presentation of the infamous Book I, Section III, Proposition XI, Problem VI of the *Principia* will give the reader a taste of his terse presentation.



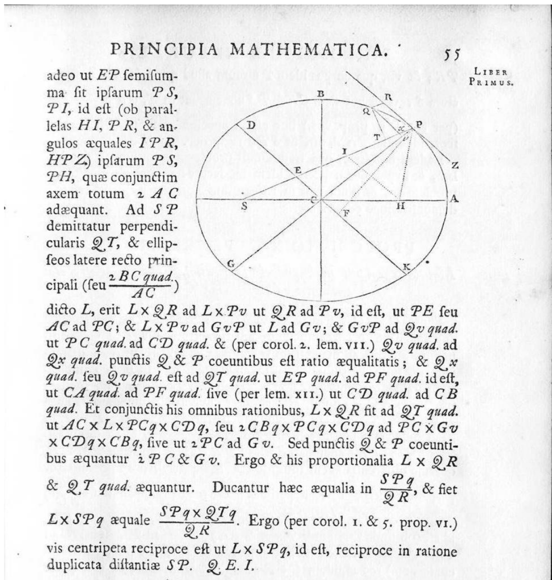
SECTIO III.

De motu corporum in conicis ſectionibus excentricis.

PROPOSITIO XI. PROBLEMA VI.

Revolvatur corpus in ellipſi: requiritur lex vis centripetæ tendentis ad umbilicum ellipſeos.

Eſto ellipſeos umbilicus S. Agatur SP ſecans ellipſeos tum diametrum DK in E, tum ordinatim applicatam Qv in x, & compleatur parallelogrammum $QxPR$. Patet EP æqualem eſſe ſemiaxi majori AC, eo quod, acta ab altero ellipſeos umbilico H linea HI ipſi EC parallela, ob æquales CS, CH æquentur ES, EI, adeo



PRINCIPIA MATHEMATICA. 55

LIBER PRIMUS.

adeo ut EP ſemiſumma ſit ipſarum PS, PI, id eſt (ob parallelas HI, PR, & angulos æquales IPR, HPZ) ipſarum PS, PH, quæ conjunctim axem totum $2AC$ adæquant. Ad SP demittatur perpendicularis QT, & ellipſeos latere recto principali (ſeu $\frac{2BC\,quad.}{AC}$) dicto L, erit $L \times QR$ ad $L \times Pv$ ut QR ad Pv, id eſt, ut PE ſeu AC ad PC; & $L \times Pv$ ad GvP ut L ad Gv; & GvP ad Qv *quad.* ut PC *quad.* ad CD *quad.* & (per corol. 2. lem. VII.) Qv *quad.* ad Qx *quad.* punctis Q & P coeuntibus eſt ratio æqualitatis; & Qx *quad.* ſeu Qv *quad.* eſt ad QT *quad.* ut EP *quad.* ad PF *quad.* id eſt, ut CA *quad.* ad PF *quad.* ſive (per lem. XII.) ut CD *quad.* ad CB *quad.* Et conjunctis his omnibus rationibus, $L \times QR$ fit ad QT *quad.* ut $AC \times L \times PCq \times CDq$, ſeu $2CBq \times PCq \times CDq$ ad $PC \times Gv \times CDq \times CBq$, ſive ut $2PC$ ad Gv. Sed punctis Q & P coeuntibus æquantur $2PC$ & Gv. Ergo & his proportionalia $L \times QR$ & QT *quad.* æquantur. Ducantur hæc æqualia in $\frac{SPq}{QR}$, & fiet $L \times SPq$ æquale $\frac{SPq \times QTq}{QR}$. Ergo (per corol. 1. & 5. prop. VI.) vis centripeta reciproce eſt ut $L \times SPq$, id eſt, reciproce in ratione duplicata diſtantiæ SP. $Q.E.I.$

Isaac Newton published *Philosophiæ Naturalis Principia Mathematica* in Latin in 1687. Subsequent editions appeared in 1723 and 1726. The third edition was translated into English as *The Mathematical Principles of Natural Philosophy* by Andrew Motte in 1729. This translation has been modernized several times, but truly new translations have only appeared recently. The translation by I. Bernard Cohen is very useful because of his extensive *Guide* that precedes the translation [4]. Should you wish to attempt to understand it, a detailed explanation is given in [4, pp. 324–329]. A comprehensive list of links to editions of the *Principia* can be found in the Wikipedia entry for *Philosophiæ Naturalis Principia Mathematica*.

Part I

Planetary orbits

Chapter 2

The sizes of the Earth, Moon and Sun

2.1 Eratosthenes's measurement of the radius of the earth

The ancient Greeks knew that the Earth is round and Eratosthenes was able to measure the radius of the Earth (Figure 2.1). Choose two points A, B on the *same longitude* and measure the distance d between them. Plant a vertical stick in the ground at A (red) and another at B (blue). On a day in the year when the stick at A produces no shadow at noon, at the same time the stick at B produces a shadow whose angle is α. The sun is so far away from the Earth that over the relatively short distance d, the rays of the Sun are essentially parallel. By alternate interior angles, the angle between the two sticks as measured from the center of the Earth is also α.

The angle that Eratosthenes measured was

$$\alpha = 7.5^\circ \cdot \frac{2\pi}{360} \approx 0.131 \text{ radians}\,,$$

and the distance d between A and B was known to be approximately 800 km. The arc $\widehat{AB}$ subtends the angle $\alpha = d/r_e$ where r_e is the radius of the Earth, so

$$r_e = \frac{d}{\alpha} = \frac{800}{0.131} \approx 6107 \text{ km}\,. \tag{2.1}$$

This value is close to the modern measurement of 6370 km.

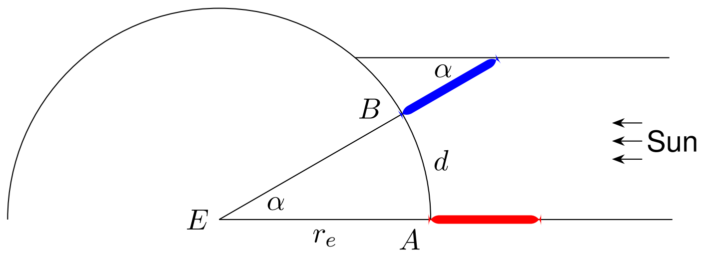


Figure 2.1: Eratosthenes's measurement of the radius of the earth

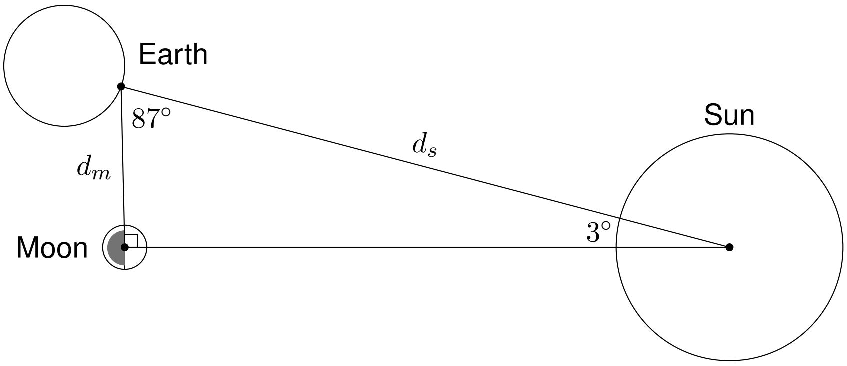


Figure 2.2: Observing a first quarter moon

2.2 Aristarchus's measurements of radii and distances

Using r_e Aristarchus was able to measure and compute the following values.

- r_m: the radius of the Moon,
- r_s: the radius of the Sun,
- d_m: the distance from the Earth to the Moon,
- d_s: the distance from the Earth to the Sun.

Computing d_s/d_m

An observer on Earth can follow the phases of the Moon as it revolves around the Earth. Consider a moon in the first quarter: one half of the moon is illuminated while the other half is not (Figure 2.2). The angle between the Sun and the Moon will be $87°$. Since exactly half of the moon is illuminated, we know that the angle Earth-Moon-Sun is a right-angle so

$$\begin{aligned} \cos 87° &= \frac{d_m}{d_s} \\ \frac{d_s}{d_m} &= \frac{1}{\cos 87°} \approx 19\,. \end{aligned} \tag{2.2}$$

Computing r_s/r_m **and** d_m/r_m

The Moon is much, much smaller than the Sun, but it is also much, much closer to the Earth. When the Moon is precisely positioned between the Earth and the Sun, its "disk" exactly covers the "disk" of the Sun, causing a total solar eclipse (Figure 2.3).

The angle subtended by the Moon is $2°$ degrees. Bisecting the angle creates two right triangles with an angle of $1°$ and the right angle at the tangents. By similar triangles,

$$\frac{r_s}{r_m} = \frac{d_s}{d_m} = 19 \tag{2.3}$$

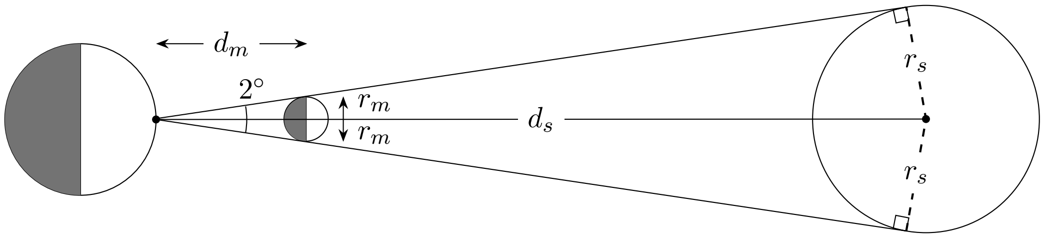


Figure 2.3: A solar eclipse

$$\frac{d_m}{r_m} = \frac{1}{r_m/d_m} = \frac{1}{\sin 1°} \approx 57\,. \tag{2.4}$$

Computing the radii and distances

Figure 2.4 shows a lunar eclipse. Unlike a solar eclipse where the Moon exactly covers the Sun, the Earth more than covers the Moon and its shadow is four times the Moon's radius.

Figure 2.5 show a lunar eclipse annotated with the distances d_m, d_s and the radii r_m, r_e, r_s. The ray from the top of the Sun is tangent to both the Sun and the Earth, so it forms right angles with their radii, as well as with the extension of the Moon's radius. The thick horizontal lines are constructed parallel to the line connecting the centers, forming two similar right triangles, so using Equation 2.3,

$$\begin{aligned}
\frac{r_s - r_e}{r_e - 2r_m} &= \frac{d_s}{d_m} = \frac{r_s}{r_m} \\
r_s r_e + r_m r_e &= 3 r_s r_m \\
19 r_m r_e + r_m r_e &= 3 \cdot 19 r_m^2 \\
r_m &= \frac{20}{57} r_e\,.
\end{aligned}$$

By Equation 2.1, $r_e \approx 6107$ km, by Equation 2.4, $d_m = 57r_m$, and by Equation 2.2, $d_s = 19d_m$, so we can compute the radii and distances.

While the computed values for the radii of the Earth and the Moon are not far off from the modern values [8, Table 1.3], the other computed values are not near the modern values. Nevertheless, they do show that the Greeks understood the immense size of the solar system.

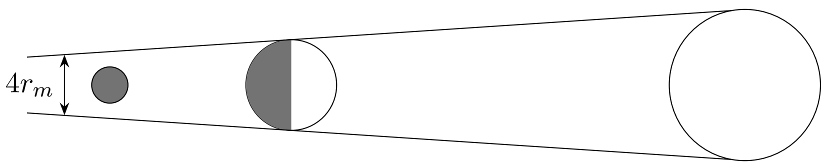


Figure 2.4: A lunar eclipse

		Formula	Computed (km)	Modern (km)
r_e	radius of Earth		6107	6370
r_m	radius of Moon	$(20/57)r_e$	2143	1740
r_s	radius of Sun	$19r_m$	$40,713$	$695,500$
d_m	distance Earth-Moon	$57r_m$	$122,140$	$384,570$
d_s	distance Earth-Sun	$19d_m$	$2,320,660$	$150,000,000$

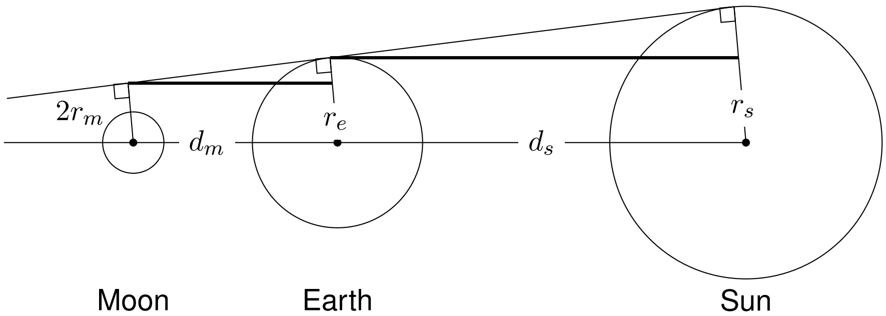


Figure 2.5: Detail of a lunar eclipse

Chapter 3

The Sun-centered solar system

3.1 The length of the seasons

The time between sunrise and sunset varies with the seasons because the axis of the rotation of the Earth is offset by $23.5°$ relative to the orbit of the Earth. The plane of the orbit of the Earth around the Sun is called the *ecliptic*. Measuring the length of the day as the time from sunrise to sunset, there is a day in June, called the *summer solstice*, when the length of the day is longest. Similarly, there is a day in December, called the *winter solstice*, when the length of the day is shortest (in the northern hemisphere). There are also two days when the length of the day equals the length of the night: the *autumn equinox* in September and the *spring equinox* in March.

Today we know that the universe is immensely large and that the stars are moving at extremely high speeds, but an observer on Earth sees them as if their positions are fixed on a sphere around the earth, called the *celestial sphere*. This solstices and equinoxes can be associated with the projection of the Sun on the celestial sphere as seen from the Earth.

Let us assume that the Earth orbits the Sun in a circle such that the center of the orbit O is the center of the Sun S. In Figure 3.1 the inner circle is orbit of the Earth and the outer circle is the

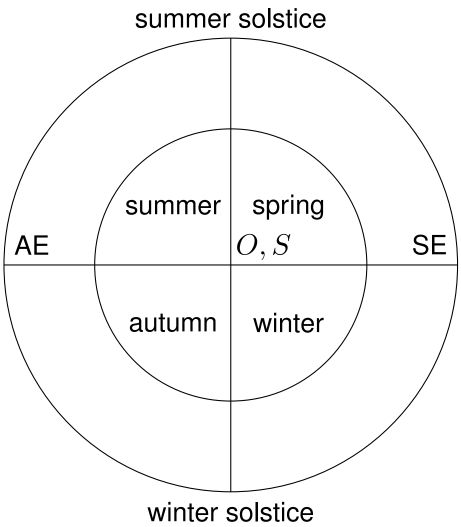


Figure 3.1: The orbit of the Earth and the seasons
AE=autumn equinox, SE=spring equinox

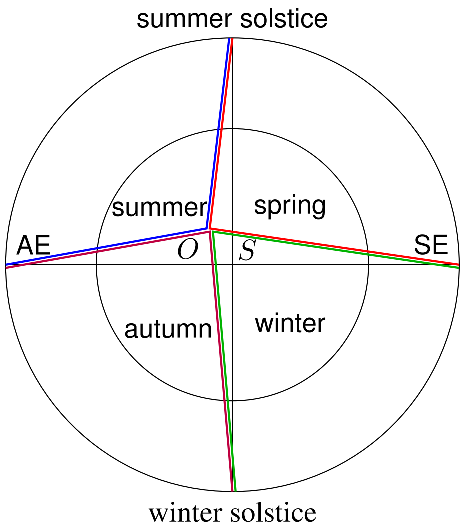


Figure 3.2: The lengths of the seasons are not equal
AE=autumn equinox, SE=spring equinox

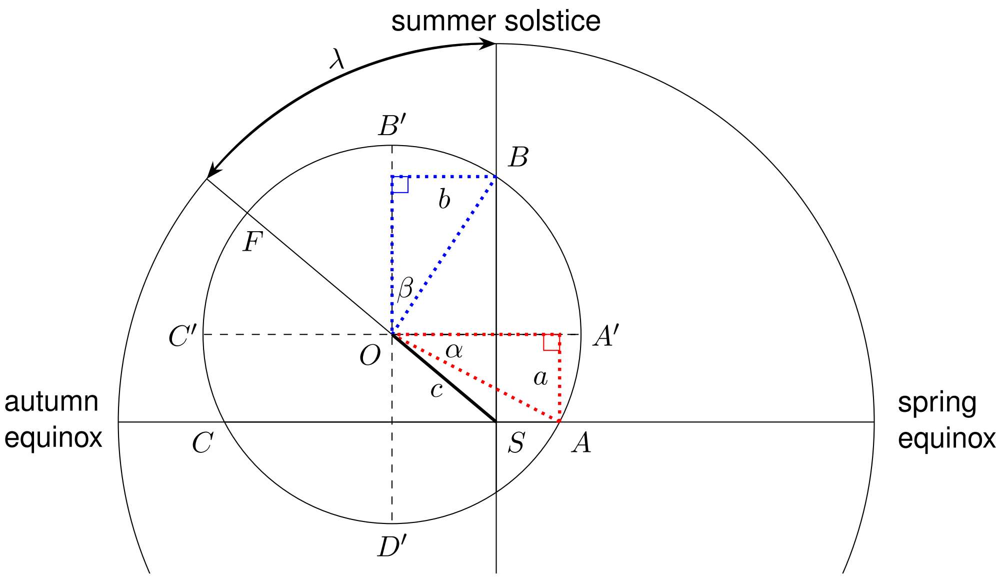


Figure 3.3: Computing the center of the Earth

celestial sphere. The orbit can be divided into four quadrants called *seasons*: spring, summer, autumn, winter.

The length of a year is approximately $365\frac{1}{4}$ days. The extra $1/4$ day is accounted for by adding a day in leap years.[1] The length of each season as determined by the equinoxes and the solstices is $365.25/4 = 91.3125$ days. However, measurements by the Greek astronomer Hipparchus showed that the actual lengths of the seasons differed from this number and a model of the solar system must be able to explain these differences.

Season	Days	%
Spring	$94\frac{1}{4}$	25.8
Summer	$92\frac{1}{2}$	25.3
Autumn	$88\frac{1}{8}$	24.1
Winter	$90\frac{1}{8}$	24.7

3.2 The location of the center of the Earth's orbit

Figure 3.3 shows a magnified view of Figure 3.2 that has been annotated with additional lines and labels that will facilitate the demonstration of Copernicus's computation. The axes $A'C'$ and $B'D'$ have their origin O at the center of the Earth's orbit and are parallel to the axes in the ecliptic. The dashed lines from O are all radii of the Earth's orbit that will be denoted r. The dotted right triangles will be used in the computation.

Copernicus's task was to locate the position of the center of the Earth's orbit O relative to the center of the Sun S. This will be given in polar coordinates $OS = c$ and $\angle FSB = \lambda$ (the label is on the large circle of the ecliptic). The strategy of the computation is as follows:

- Initially, we compute the angles of the arcs in radians; the lengths of the arc can then be obtained by multiplying by the radius r.

[1]The length of a year is actually 365.2425. In the sixteenth century, the *Gregorian calendar* accounted for the difference by removing three leap years in every four hundred years.

- We use the lengths of the seasons that Copernicus used: summer is $93\frac{14.5}{60}$ days and spring is $92\frac{51}{60}$.
- The angle of the arc $\widehat{AC}$ can be computed from the combined length of spring and summer, and the angle of the arc $\widehat{AB}$ can be computed from the length of spring.[2]
- From $\widehat{AC}$ and $\widehat{AB}$, the angle α subtended by $\widehat{AA'}$ and the angle β subtended by $\widehat{BB'}$ can be computed.
- Since the Earth is very close to the Sun relative to the radius of its orbit, $r\,\widehat{AA'}$ and $r\,\widehat{BB'}$ approximate the line segments a and b. From these c and λ can be computed.

Computing the angles of the arcs $\widehat{AB}, \widehat{AC}$

The arcs $\widehat{AB}, \widehat{AC}$ are sectors of the Earth's orbit and their angles are their proportions of a full year times 2π radians.

$$\begin{aligned}\widehat{AB} &= 2\pi \cdot \frac{92\frac{51}{60}}{365.25} = 2\pi \cdot \frac{92.85}{365.25} = 1.5972 \text{ radians}\\ \widehat{AC} &= 2\pi \cdot \frac{92\frac{51}{60} + 93\frac{14.5}{60}}{365.25} = 2\pi \cdot \frac{186.09}{365.25} = 3.2012 \text{ radians}\,.\end{aligned}$$

Computing the angles of the arcs $\widehat{AA'}, \widehat{BB'}$

Let us express the arcs $\widehat{AC}$ and $\widehat{AB}$ in terms of the arcs that comprise them. Since $AC \parallel A'C'$, $\widehat{AA'} = \widehat{C'C}$ and we can now compute $\widehat{AA'}$.

$$\begin{aligned}\widehat{AC} &= \widehat{AA'} + \widehat{A'C'} + \widehat{C'C} = 2\widehat{AA'} + \pi\\ \widehat{AA'} &= \frac{1}{2}(3.2012 - \pi) = 0.0298 \text{ radians}\,.\end{aligned}$$

From $\widehat{AB}$ and $\widehat{AA'}$ we can compute $\widehat{BB'}$.

$$\begin{aligned}\widehat{AB} &= \widehat{AA'} + \widehat{A'B'} - \widehat{BB'}\\ \widehat{BB'} &= 0.0298 + \frac{\pi}{2} - 1.5927 = 0.0034 \text{ radians}\,.\end{aligned}$$

Computing the lengths of the arcs $\widehat{AA'}, \widehat{BB'}$

Using the assumption that O, the center of the Earth's orbit, is very close to the Sun S, $\sin\alpha \approx \alpha$ and $\sin\beta \approx \beta$. OA and OB are radii of the Earth's orbit so

$$\begin{aligned}a &= r\sin\alpha \approx r\alpha = r\widehat{AA'} = 0.0298r\\ b &= r\sin\beta \approx r\beta = r\widehat{BB'} = 0.0034r\,.\end{aligned}$$

[2] All arcs are measured counterclockwise.

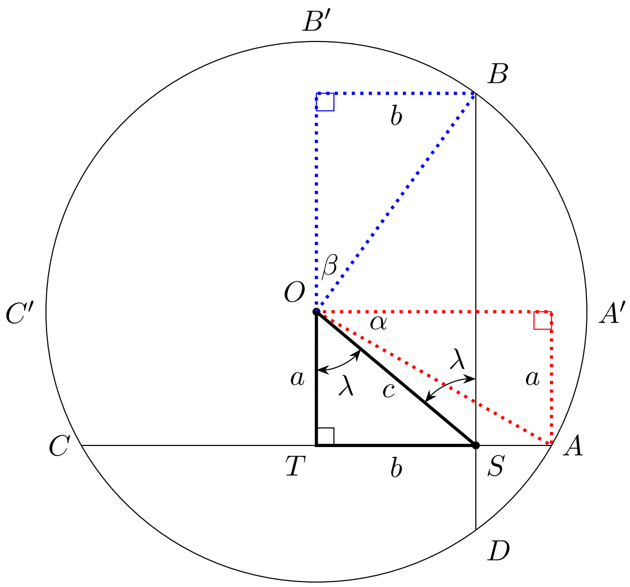


Figure 3.4: Three triangles

Computing the position of O relative to S

Figure 3.4 shows a portion of Figure 3.3. In the dotted triangles, we have already computed the lengths a and b. Since $OT \parallel A'A$ and $TS \parallel BB'$, we can label OT by a and TS by b. $\triangle OTS$ is a right triangle and $OS = c$ can be obtained from Pythagoras's theorem.

$$c = \sqrt{a^2 + b^2} = r\sqrt{(0.0298)^2 + (0.0034)^2} = 0.03r\,.$$

λ can be obtained from trigonometry.

$$\lambda = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0.0034}{0.03} = 0.1129 \text{ radians} \approx 6.47^\circ\,.$$

The distance $0.03r$ is shown in the following table using the values of r from the table on page 7.

	Aristarchus (km)	Copernicus (km)	Modern (km)
radius of Earth's orbit	$2,320,660$	$8,000,000$	$150,000,000$
distance of O from S	$69,620$	$240,000$	$4,500,000$

Chapter 4

Elliptical orbits

Towards the end of the sixteenth century, the astronomer Tycho Brahe carried out extremely precise observations. In 1600 he hired Johannes Kepler as his assistant and when Tycho died soon afterwards, Kepler was appointed to his position. Here we explain how Kepler was able to establish that planetary orbits are ellipses.

4.1 Determining the radius of the Earth's orbit

A Martian year is 687 days, that is, it equals $\frac{687}{365.25} = 1.88$ Earth years. We know when Mars reaches a "new year" by observing its projection on the celestial sphere, but each time the position of the Earth in its orbit will be different. Figure 4.1 shows the orbit of the Earth—its center O offset from the Sun S as Copernicus showed—at four occasions when the position of Mars M at its new year was observed. Four triangles are created $\triangle OE_iM$.

Figure 4.2 shows one of the triangles with the angles labeled. Using the law of sines,

$$\begin{aligned}\frac{OE_i}{\sin\beta} &= \frac{OM}{\sin\gamma}\\ OE_i &= OM\,\frac{\sin\beta}{\sin\gamma}\,.\end{aligned}$$

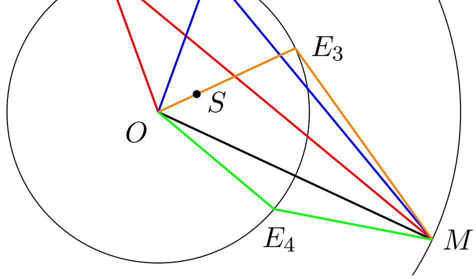


Figure 4.1: Observations of the orbit of Mars from the Earth

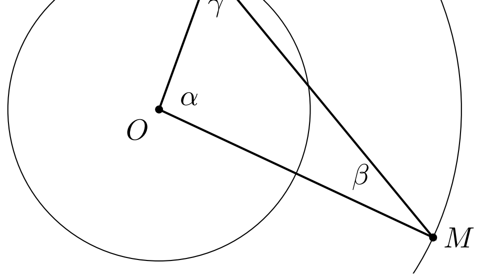


Figure 4.2: Angles in the Earth-Sun-Moon triangle

	α	β	γ	OE_i
E_1	127.1	20.8	32.1	$0.6682 \cdot OM$
E_2	84.2	35.8	60.5	$0.6721 \cdot OM$
E_3	41.3	42.4	96.4	$0.6785 \cdot OM$
E_4	1.6	3.4	175.0	$0.6805 \cdot OM$

Figure 4.3: Angles between the Earth and Mars, and distance OE_i

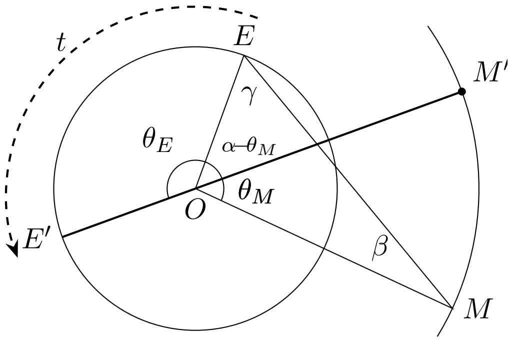


Figure 4.4: The Earth and Mars in opposition

Tycho Brahe measured all three angles and the values of OE_i computed from the angles were not the same (Figure 4.3). If the Earth' orbit is circular, he had to move the center of the orbit so that $\{E_1, E_2, E_3, E_4\}$ were all on the circle.

4.2 Measuring the angles in the triangle Sun-Earth-Mars

How can the angles α, β, γ be measured? $\triangle E_iOM$ is a triangle so it is sufficient to measure two of the angles. γ is easily measured by observing Mars and the Sun at the same time, but neither α nor β can be measured directly since they are not accessible to an observer on Earth.

	α	β	γ	OE_i
E_1	127.1	20.8	32.1	$0.6682 \cdot OM$
E_2	84.2	35.8	60.5	$0.6721 \cdot OM$
E_3	41.3	42.4	96.4	$0.6785 \cdot OM$
E_4	1.6	3.4	175.0	$0.6805 \cdot OM$

Tycho's measurement used the known periods of the orbits to compute the angles. The Earth moves counterclockwise around Sun. Given any point E, for some t, t days later the Earth will have moved to E' and Mars will have moved to M', so that they are in *opposition*, that is, Mars will be on the continuation of the Earth-Sun line (Figure 4.4). Since the Earth completes an orbit in about half the time that Mars takes to complete an orbit, t will be such that neither the Earth nor Mars has completed a full orbit. The angles θ_E and θ_M are fractions of a circular orbit of 360°, so

$$\begin{aligned}\theta_E &= \frac{360}{365.25}t\\ \theta_M &= \frac{360}{687}t\,.\end{aligned}$$

This gives values for θ_E and θ_M. Since $E'M'$ is a straight line, we have that $\alpha - \theta_M = 180° - \theta_E$, so that $\alpha = 180 - \theta_E + \theta_M$, and the values of OE_i can be computed.

Kepler's next task was to obtain a new value O' for the center of the Earth's orbit such that the E_i's are on the orbit. Given the new locations of the Earth $\{E_1, E_2, E_3, E_4\}$, by Theorem A.1 a circle centered at O' can be constructed that goes through $\{E_1, E_2, E_3\}$ (Figure 4.5). To verify that this is the correct orbit, check that E_4 is on the circle.

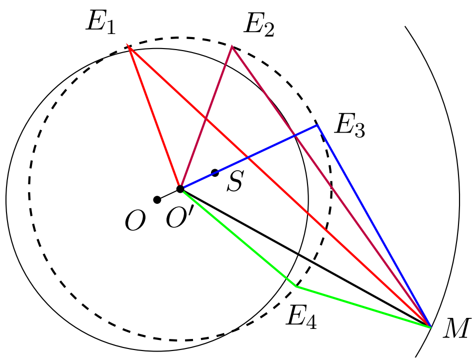


Figure 4.5: Observations of the orbit of Mars from the new Earth's orbit

4.3 Orbits are ellipses

While Kepler was able to modify the center of the orbit of the Earth to be consistent with the observations, he was not able to adequately describe the orbit of Mars. After years of work, he came to the conclusion that the orbit must be oval like an egg. Oval, perhaps, but certainly not an ellipse, because "the job would have been done by Archimedes" [8, p. 94]. Figure 4.6 shows C, a position on a circular orbit, and an oval orbit (dashed), where M is the position of Mars on the oval corresponding to C. The radius of the circular orbit is labeled $a = OA = OC$ and the unknown distances to M are labeled $s = SM$ and $t = OM$.

Kepler's computed that $\frac{a-t}{t} = 0.00429$ and $\frac{s}{t} = 1.00429$, so that $(a-t)+t = s$ and therefore $SM = s = a = AO$. The dashed oval is likely an ellipse, because in an ellipse $SM = AO$ (Theorem 9.4). Kepler then computed the projections of the observations of Mars on the x-axis (Figure 4.7) and obtained for all of them that

$$\frac{M_iO_i}{C_iO_i} = \frac{1}{1.00429} = 0.99573\,.$$

By Theorem 10.1, $MO/CO = b/a$ is constant for an ellipse, and Kepler concluded that the orbit of Mars is an ellipse.

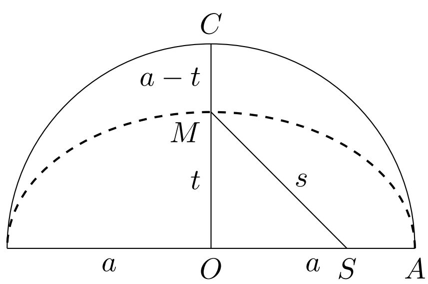


Figure 4.6: The orbit of Mars as an oval "egg"

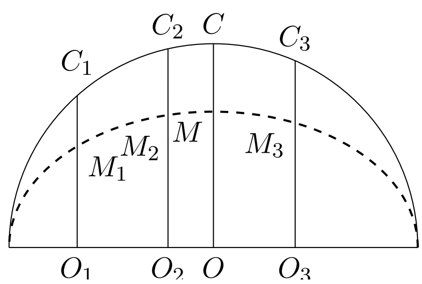


Figure 4.7: The orbit of Mars as an ellipse

Chapter 5

Gravitation

In the *Principia* Isaac Newton proved the following theorem.

> **Theorem 5.1** *If a planet subject to a centripetal force follows an elliptical orbit around the Sun, then the force decreases as the inverse square of the distance from the Sun.*

After a review of Newton's Laws of force and motion, we show that Kepler's second law must hold in *any* system subject to a centripetal force. The next step is to show the inverse-square law and then it is a small step to universal gravitation and Kepler's third law.

5.1 Newton's laws of motion and Kepler's second law

1. A body in uniform motion (including a body at rest) continues with the same motion unless a force is applied.
2. A force F applied to a body causes an acceleration a in the direction of the force whose magnitude is $a = F/m$, where m, the constant of proportionality, is the *mass* of the body.
3. If one body exerts a force on a second body, the second body exerts a force on the first of equal magnitude but in the opposite direction.

Forces are denoted by vectors, where the direction of the vector represents the direction of the force and the length of the vector represents the magnitude of the force. Forces can be decomposed into perpendicular components (Figure 5.1), or into components in any directions (Figure 5.2). The components form a parallelogram whose diagonal is the resultant force.

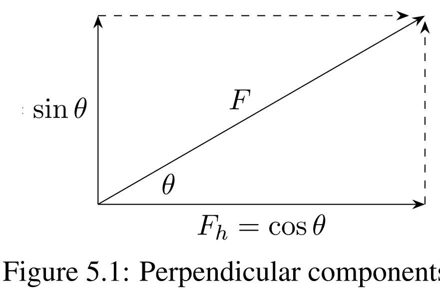


Figure 5.1: Perpendicular components of a force

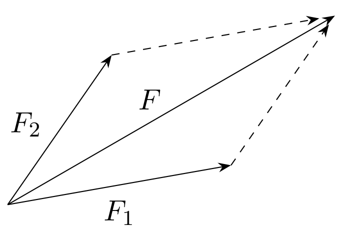


Figure 5.2: The parallelogram formed by two forces

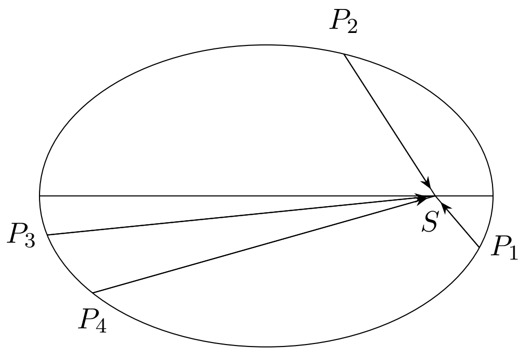


Figure 5.3: Centripetal force

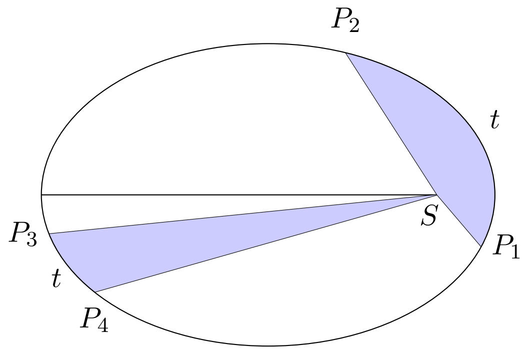


Figure 5.4: Equal areas in equal times

A *centripetal force* is an attractive force exerted by a single body on another, in particular, the gravitational force exerted by the Sun on a planet (Figure 5.3). Since this is the only force exerted on the planet, it does not move "up" or "down" and its orbit is in a plane.

Kepler's second law states that a planet in orbit sweeps out equal area in intervals of equal duration, that is, if it takes the planet time t to move from P_1 to P_2 and also t to move from P_3 to P_4, then the area of the sector P_1SP_2 is equal to the area of P_3SP_4 (Figure 5.4). It follows that the speed of the planet must vary as it traverses its orbit ($v_{P_1P_2} \gg v_{P_3P_4}$). Kepler's second law holds for any two-body system subject to a centripetal force; the force need not be inverse-square.

The proof is based on dividing an area into very small sectors and then taking the limit. Consider three points A, B, C on the orbit (Figure 5.5) that represent the positions of the planet at intervals of Δt. For clarity we have drawn them spaced out, but the intention is that they are very close together. Newton assumed that the planet does not smoothly traverse the arcs, but rather that it every Δt it jumps in discrete steps from one point on the orbit to the next.

Figure 5.6 shows how the force is exerted in discrete steps. The planet moves from A to B and we expect that the centripetal force at B will cause an acceleration that moves the planet to C, the next point on the orbit. Instead, we "pretend" that the force is not applied at B, but, in the

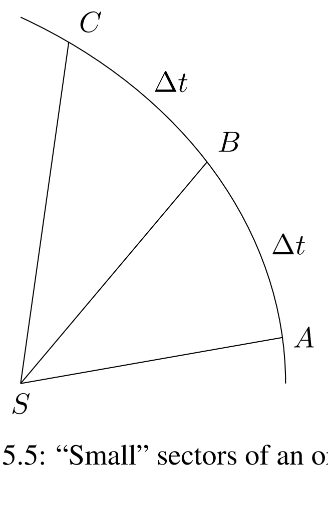


Figure 5.5: "Small" sectors of an orbit

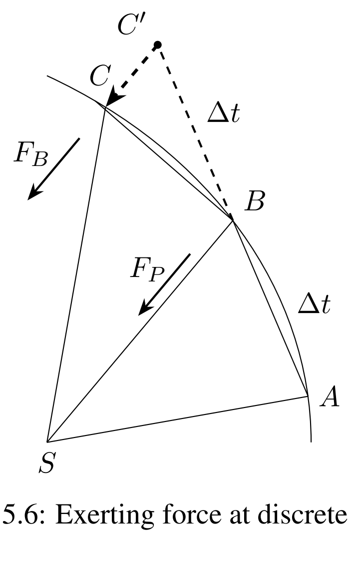


Figure 5.6: Exerting force at discrete times

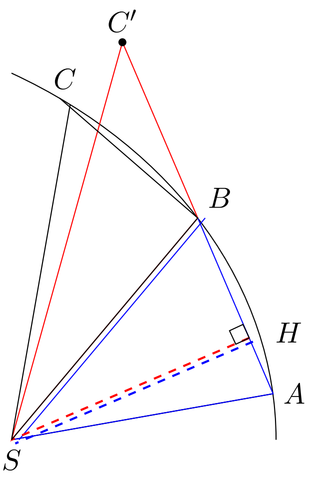


Figure 5.7: $\triangle ASB = \triangle BSC'$

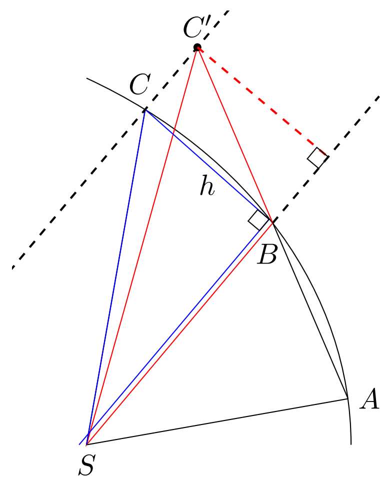


Figure 5.8: $\triangle BSC' = \triangle BSC$

absence of an applied force, planet continues with the same velocity. After another period of Δt as passed and the planet has reached point C', the force is now applied *in the same direction* as it would have been applied at B, moving the planet to C.

Theorem 5.2 *The area of $\triangle ASB$ is equal to the area of $\triangle BSC$.*

Proof The proof will be done in two steps by showing that $\triangle ASB = \triangle BSC'$ and then that $\triangle BSC' = \triangle BSC$.

- In Figure 5.7, $\triangle ASB$ is shown in blue and $\triangle BSC'$ is shown in red. It is assumed that $AB = BC'$ (the planet moves from B to C' during the same interval Δt), so since SH is the height of both triangles, their areas are equal.
- In Figure 5.8, $\triangle BSC$ is shown in blue and $\triangle BSC'$ is shown in red. It is assumed CC' is parallel to SB (the planet is subject to the centripetal force at C' in the *same* direction as the force at B), so the heights of both triangles to the common side SB are equal and their areas are equal. It follows that $\triangle BSC' = \triangle BSC = \triangle ASB$. ∎

We assume that the sectors of the orbit are each divided up into small sectors of uniform duration Δt. By the theorem, each sector has the same area ΔA. Therefore (see Figure 5.4),

$$\frac{A_{P_1SP_2}}{\Delta A} = \frac{t}{\Delta t} = \frac{A_{P_3SP_4}}{\Delta A},$$

from which Kepler's second law follows: $A_{P_1SP_2} = A_{P_3SP_4}$.

The proof used two approximations:

- ΔA is an approximation of the area of each sector.
- The force at C' is an approximation to the force at B.

In the limit as the size of the sectors decreases, the errors become negligible.

Definition 5.3 *For a given elliptical orbit, $\kappa = A/t$, where A is the area of the ellipse and t is the period of the orbit, is called* Kepler's constant.

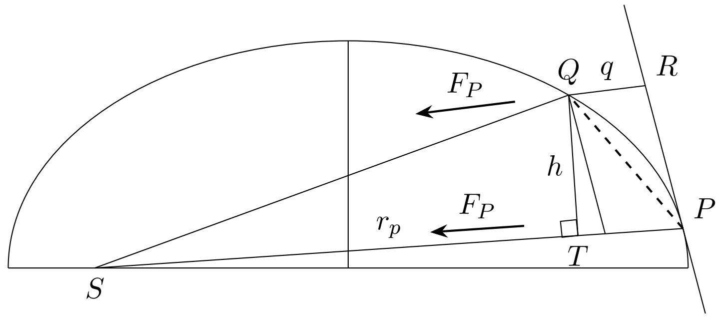


Figure 5.9: The derivation of the inverse square law

5.2 The inverse square law for gravitation

Newton's next step was to show that if the orbit of a planet is elliptical, the centripetal force must be proportional to the mass of the planet and inversely proportional to the square of its distance from the Sun. In Figure 5.9, S is the Sun, and P and Q are points on the orbit that are very close to each other. PR is the tangent to the ellipse at P, and R is chosen to that QR is parallel to SP. QT is constructed perpendicular to SP. Denote the lengths $r_P = SP$, $h = QT$, $q = QR$ and the time interval from P to Q by Δt.

When a point is subject to an acceleration a for a period of Δt, its displacement is $\frac{1}{2}a(\Delta t)^2$. From Newton's second law we know that at point R, the planet is subjected to an acceleration of F_P/m, so

$$\begin{aligned} q &= \frac{1}{2}\frac{F_P}{m}(\Delta t)^2 \\ F_P &= \frac{2mq}{(\Delta t)^2}\,. \end{aligned}$$

Now we compute the area of the sector PSQ which is approximately the area of $\triangle SPQ = (1/2)hr_p$ and use Kepler's constant:

$$\begin{aligned} \Delta t &= \frac{\Delta A_{PSQ}}{\kappa} = \frac{hr_P}{2\kappa} \\ F_P &= 2mq\cdot\frac{4\kappa^2}{(hr_P)^2} = 8\kappa^2 m\cdot\frac{q}{h^2}\cdot\frac{1}{r_P^2}\,. \end{aligned}$$

To obtain an inverse-square law for the force, the first two factors have to be independent of the distance. For a given planet m is constant and for a given elliptical orbit κ is constant, so the first factor does not depend on the distance. What about the second factor q/h^2, in particular, what value does it have as Δt approaches zero?

Theorem 5.4 *In an elliptical orbit*

$$\lim_{\Delta t\to 0}\frac{q}{h^2} = \frac{1}{L}\,,$$

where L is the length of the latus rectum of the ellipse (Definition 9.8).

Newton's proof is very complex and is presented separately in Chapter 6.

Since L is constant for any given ellipse, the inverse square law can be written

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2} \,. \tag{5.1}$$

The formula can be re-written so that the constant values appearing are more familiar: a, the semi-major axis and T, the period of the orbit. By Theorem 10.2, $L{=}2b^2/a$ and by Theorem 10.3, A_e, the area of the ellipse is πab. Therefore, $\kappa = A_e/T{=}\pi ab/T$ and

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2} = \frac{8(\pi ab)^2 m}{T^2} \cdot \frac{a}{2b^2} \cdot \frac{1}{r_P^2} = \frac{4\pi^2 a^3 m}{T^2} \cdot \frac{1}{r_P^2} \,. \tag{5.2}$$

Newton was able to show that:

- The inverse square law applies to all conic sections including parabolas and hyperbolas and, of course, a circle which is a special case of an ellipse.
- The converse holds: if a planet is subject to an inverse-square centripetal force then the orbit must be an ellipse (or another conic section).
- The proof assumes that a planet is a very small point, but the result holds even for large planets as long as the density of the planet is radially symmetric, that is, for a given distance from the center the density is constant.

5.3 Universal gravitation

By Newton's third law, we can equate the force $F_{S\leftarrow E}$ that the Sun S exerts on the Earth E with the force $F_{E\leftarrow S}$ that the Earth exerts on the Sun. Let m be the mass of the Earth and M be the mass of the Sun, then by Equation 5.1,

$$\begin{aligned}
F_{S\leftarrow E} &= \frac{8\kappa_E^2 m}{L_E} \cdot \frac{1}{r^2} = \frac{C_E m}{r^2} \\
F_{E\leftarrow S} &= \frac{8\kappa_S^2 M}{L_S} \cdot \frac{1}{r^2} = \frac{C_S M}{r^2} \\
\frac{C_E}{M} \cdot \frac{1}{r^2} &= \frac{C_S}{m} \cdot \frac{1}{r^2} \,,
\end{aligned}$$

from some constants C_E, C_S.

Why are the constants different? The Earth and the Sun both rotate around their center of mass called the *barycenter*, which is very close to the center of the Sun since the Sun is so much more massive than the Earth. The ellipse of the Sun's orbit is very small relative to the Earth so A and L are smaller, and the Sun's period is large so T is larger. The different values for $8\kappa^2/L$ are encapsulated into the constants C_E, C_S. Let $G = \dfrac{C_E}{M} = \dfrac{C_S}{m}$ so that

$$F_{S\leftarrow E} = F_{E\leftarrow S} = G\frac{mM}{r^2} \,. \tag{5.3}$$

This is Newton's law of universal gravitation. It is not specific to planetary orbits but holds between any two bodies with masses m, M.

5.4 Kepler's third law

Theorem 5.5 (Kepler's third law) *Let P_1, P_2 be two planets whose elliptical orbits have semi-major axes a_1, a_2 and whose orbital periods around the Sun are T_1 and T_2. Then*

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2}\,.$$

Proof By Equations 5.2 and 5.3,

$$F = \frac{4\pi^2 a_i^3 m}{T_i^2}\frac{1}{r_i^2} = \frac{GmM}{r_i^2}\,. \tag{5.4}$$

After canceling m and r_i we get

$$\frac{a_i^3}{T_i^2} = \frac{GM}{4\pi^2}\,.$$

$GM/4\pi^2$ is a constant that depends only on the mass of the Sun and the gravitational constant, so a_i^3/T_i^2 is constant for all planets rotating around the Sun. ∎

Chapter 6

A proof Proposition XI, Problem VI

Theorem 5.4 (repeated here) is Book I, Section III, Proposition XI, Problem VI of the *Principia*.

Theorem 6.1 *In an elliptical orbit*

$$\lim_{\Delta t \to 0} \frac{q}{h^2} = \frac{1}{L},$$

where L is the length of the latus rectum of the ellipse.

The construction of the diagram in Figure 6.1 is as follows.[1]

- Let P, Q be two points on the elliptical orbit separated by a time interval Δt. Construct lines from P to the center C and the foci S, H.
- Construct the tangent at P and choose R on the tangent such that the body would move from P to R if it continued for time Δt not subject to any force. Construct the parallelogram $PRQX$ and extend QX until it intersects PC at V.
- Construct a line parallel to RP through H and let I be its intersection with PS.
- Construct DC, the conjugate diameter to PC (Definition 9.9), and let E be its intersection with PS.

6.1 A formula for QR

Theorem 6.2 $QR = PV \cdot \dfrac{CA}{CP}$.

Proof By Theorem 10.4, $\angle RPX = \angle ZPH = \alpha$ and $IH \parallel RZ$, so by alternate interior angles

$$\angle PHI = \angle ZPH = \alpha = \angle RPX = \angle PIH\,.$$

Therefore, $\triangle IPH$ (red) is isosceles and $PI = PH = d$.

[1]The bottom half of the ellipse is not shown, but we still refer to lines DC, PC as diameters.

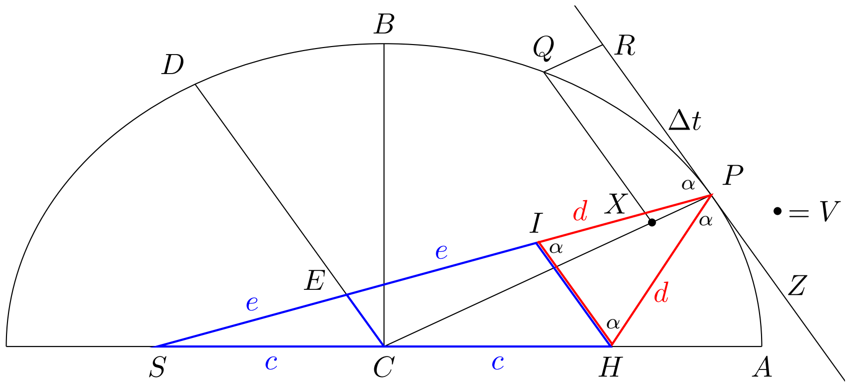


Figure 6.1: Geometry of an elliptical orbit (1)

$SC = CH = c$ are equal because they are the distances of the foci from the center of the ellipse. Let $SE = e$. By construction $EC \parallel IH$ so $\triangle ESC \sim \triangle ISH$ (blue) and

$$\frac{SC}{SE} = \frac{SH}{SI}$$

$$SI = \frac{SH \cdot SE}{SC} = \frac{2c \cdot e}{c} = 2e\,.$$

By definition of an ellipse $SP + PH = SI + IP + PH = 2CA$ so $2e + d + d = 2CA$ and $EP = e + d = CA$.

$QV \parallel EC$ so $\triangle EPC \sim \triangle XPV$ and

$$\frac{PX}{PV} = \frac{EP}{PC} = \frac{CA}{PC}$$

$$PX = PV \cdot \frac{CA}{PC}\,.$$

Since $PRQX$ is a parallelogram $QR = PX$ and $QR = PV \cdot \dfrac{AC}{PC}$. ∎

6.2 A formula for QT

Construct a perpendicular from P to DC and label its intersection with DC by F. Construct a perpendicular from Q to SP and label its intersection with SP by T (Figure 6.2).

Theorem 6.3

$$QT = QX \cdot \frac{FP}{CA}\,.$$

Proof By construction, $QR \parallel PX$, so by alternate interior angles $\angle RQX = \angle QXT = \beta$. By construction, $RP \parallel QX \parallel DC$, so by alternate interior angles $\angle QXT = \angle PEF = \beta$. Since $\triangle PFE$ and $\triangle QTX$ are right triangles with an equal acute angle β, $\triangle PFE \sim \triangle QTX$. In

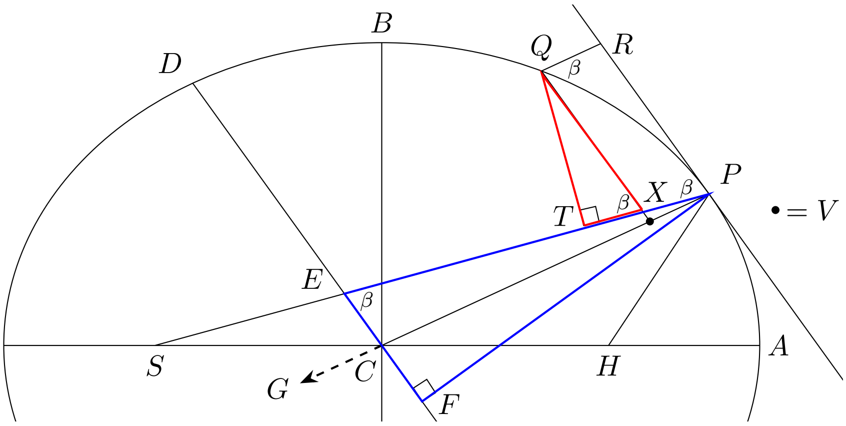


Figure 6.2: Geometry of an elliptical orbit (2)

the proof of Theorem 6.2 we showed that $EP = CA$ so

$$\frac{QT}{QX} = \frac{FP}{EP}$$

$$QT = QX \cdot \frac{FP}{EP} = QX \cdot \frac{FP}{CA} . \quad \blacksquare$$

6.3 A formula for QR/QT^2

Theorem 6.4

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2} . \tag{6.1}$$

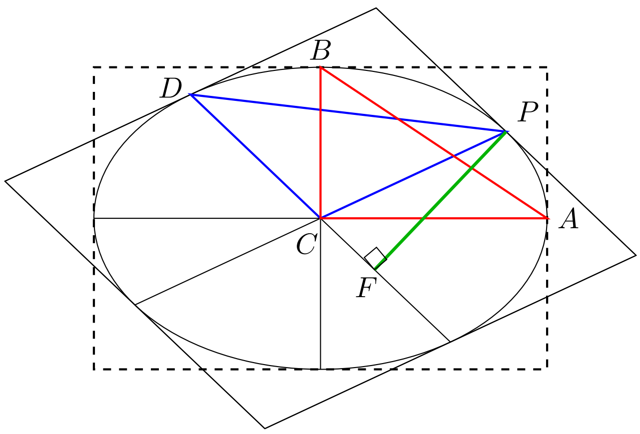


Figure 6.3: Parallelograms formed by conjugate diameters

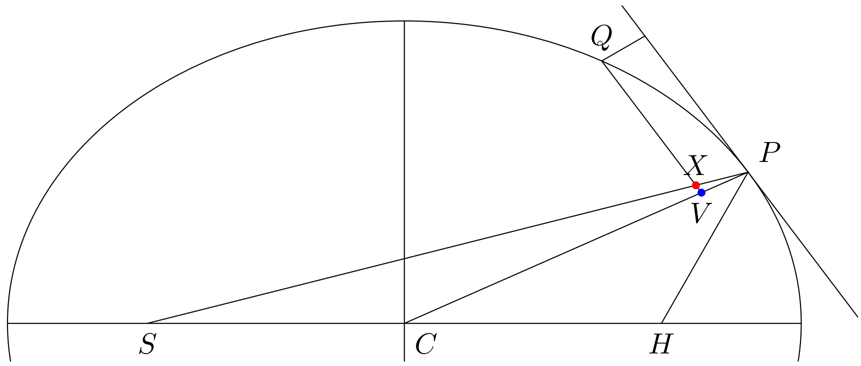


Figure 6.4: Geometry of an elliptical orbit (3)

Proof Let use combine the equations in Theorems 6.2 and 6.3 to get QR/QT^2.

$$\frac{QR}{QT^2} = \frac{PV \cdot \dfrac{CA}{CP}}{\left(QX \cdot \dfrac{FP}{CA}\right)^2} = \frac{PV \cdot CA^3}{QX^2 \cdot CP \cdot FP^2} \,. \tag{6.2}$$

DC and PC are conjugate diameters so Theorem 10.6 gives a formula for PV that we substitute into Equation 6.3.

$$\frac{QR}{QT^2} = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2} \cdot \frac{CA^3}{QX^2 \cdot CP \cdot FP^2} = \frac{CP \cdot CA^3}{CD^2 \cdot FP^2} \cdot \frac{QV^2}{GV \cdot QX^2} \,. \tag{6.3}$$

Next we show that $CD \cdot FP = CA \cdot CB$. By Theorem 10.7 the areas of the parallelograms formed by the tangents to conjugate diameters are equal. By symmetry the areas of the four small parallelograms are equal, as are the triangles formed by constructing diagonals. In Figure 6.3 the area of the $\triangle ABC$ (red), which is $(1/2)CA \cdot CB$, is equal to the area of $\triangle PCD$ (blue), which is $(1/2)CD \cdot FP$. Substituting $CA \cdot CB$ for $CD \cdot FP$ in Equation 6.3 gives

$$\frac{QR}{QT^2} = \frac{CP \cdot CA^3}{CB^2 \cdot CA^2} \cdot \frac{QV^2}{GV \cdot QX^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2} \,. \quad \blacksquare$$

Approaching the limit

Figure 6.4 is an enlarged diagram of part of Figure 6.2. As the time interval Δt gets smaller, $Q \to P$ which implies that (a) $X \to V$ so that $QX \to QV$, (b) $V \to P$ so that $CV \to CP$, and (c) $GV \to 2CP$. Substituting into Equation 6.1 and using Theorem 10.2 gives

$$\lim_{Q \to P} \frac{QR}{QT^2} = \lim_{Q \to P} \frac{CP \cdot CA}{CB^2} \cdot \frac{QX^2}{2CP \cdot QX^2} = \frac{CA}{2CB^2} = \frac{a}{2b^2} = \frac{1}{L} \,.$$

Chapter 7

The proofs by Feynman and Maxwell

Newton proved that if a planet is in an elliptical orbit around the Sun, as empirically determined by Kepler, it must be subject to the inverse-square law. Richard P. Feynman proved the converse: if a planet is subject to the inverse-square law then its orbit is elliptical. Using techniques similar to those later used by Feynman, James Clerk Maxwell proved Newton's theorem. Maxwell's proof was based on *hodographs* which were previous used by William Rowan Hamilton to prove Newton's theorem.

7.1 Dividing the orbit into sectors of equal angle

Consider Kepler's second law (Figure 7.1). If the planet traverses both the long arc $\widehat{P_1P_2}$ and the short arc $\widehat{P_3P_4}$ in the same period of time, its speed during $\widehat{P_1P_2}$ must be greater than the speed during $\widehat{P_3P_4}$. The speed will be greatest at the planets closest approach to the Sun at p (the *perihelion*) and least at a (the *aphelion*).[1]

Feynman's approach was to divide the orbit not into sectors traversed in equal times, but into sectors of equal angles (Figure 7.2). The planet will traverse the short arc $\widehat{P_1P_2}$ (near the perihelion) in a shorter time than long arc $\widehat{P_3P_4}$ (near the apehelion) so the areas of the sectors will not be equal.

[1]These terms refer only to orbits about the Sun; for an arbitary orbit, the terms are *periapsis* and *apoapsis*.

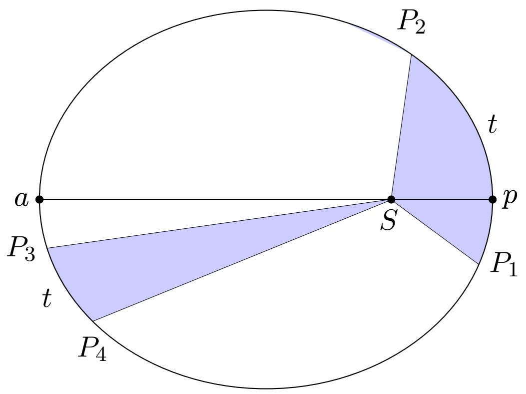


Figure 7.1: Equal areas in equal times

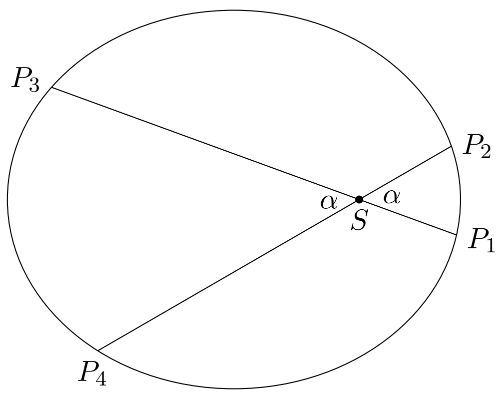


Figure 7.2: Equal angles

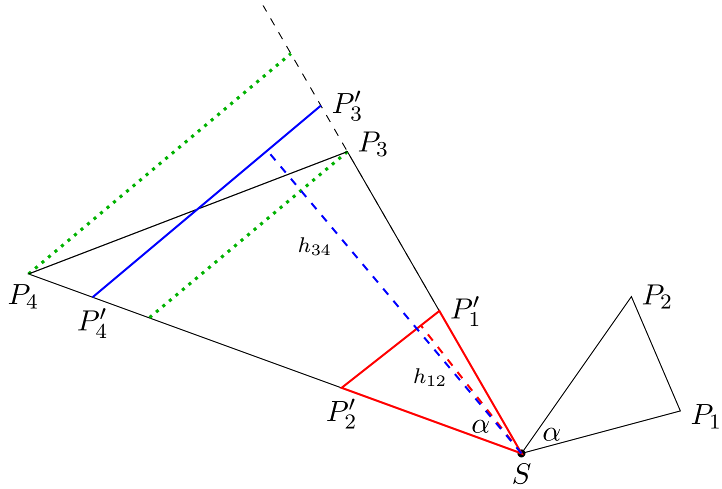


Figure 7.3: The area is proportional to the square of the distance

Theorem 7.1 *The area of a sector is proportional to the square of the distance of a planet from the Sun.*

Proof Figure 7.3 shows two sectors with the same central angle α approximated by triangles $\triangle P_1SP_2$ and $\triangle P_3SP_4$. Since the central angles are the same, we can "overlay" $\triangle P_1SP_2$ on $\triangle P_3SP_4$ to form $\triangle P_1'SP_2'$ (red).

Draw a line $P_3'P_4'$ parallel to $P_1'P_2'$ (blue) such that $A_{\triangle P_3'SP_4'}$, the area of $\triangle P_3'SP_4'$, equals $A_{\triangle P_3SP_4}$. The dotted lines (green) show that by continuity such a line must exist. Since $P_1'P_2' \parallel P_3'P_4'$, $\triangle P_1'SP_2' \sim \triangle P_3'SP_4'$ and the sides of the triangles are proportional, $P_3'P_4'/P_1'P_2' = k$, as are the heights $h_{34}/h_{12} = k$. It follows that $A_{\triangle P_3'SP_4'} = k^2 A_{\triangle P_1'SP_2'}$. As the angle approaches zero, the triangles approach the sectors and the heights of the triangles approach the distance from the Sun to a position on the orbit. ∎

Theorem 7.2 *For sectors whose angle is α, the change in velocity Δv is independent of r, the distance of the planet from the Sun!*

Proof We have the following proportionalities.

$\Delta A \sim \Delta t$	Kepler's second law
$\Delta A \sim r^2$	Theorem 7.1
$F \sim 1/r^2$	Inverse-square law
$F \sim \Delta v/\Delta t$	Newton's second law

Together we get

$$\Delta v \sim F\,\Delta t \sim F\,\Delta A \sim \frac{1}{r^2} \cdot r^2 = 1\,. \quad ∎$$

While Newton divided the orbit into sectors of equal time (Figure 5.6), Feynman divided the orbit into sectors of equal angle (Figure 7.4). By separating the motion along the orbit into unaccelerated motion continuing from the previous sector, followed by accelerated motion towards the Sun with the same direction as at the start of the sector, the diagrams look similar. By theorem Theorem 7.2, changes in velocity are of equal magnitude.

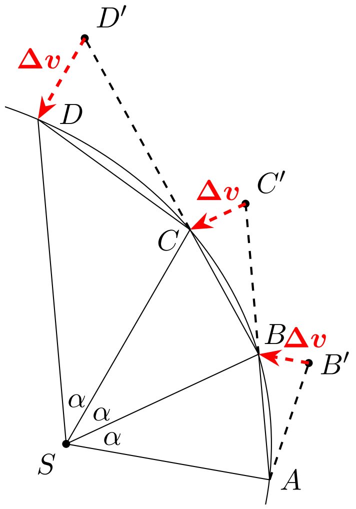


Figure 7.4: Orbit diagram

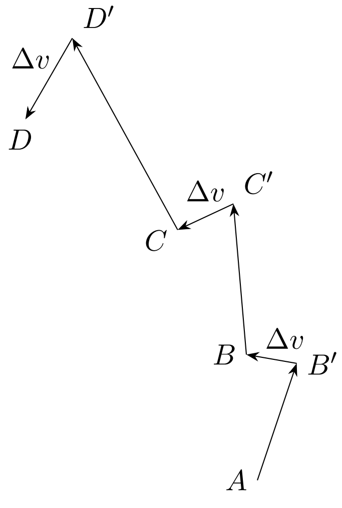


Figure 7.5: Velocity vectors

7.2 The velocity circle

We now extract the velocity vectors from the orbit diagram (Figure 7.5). Since vectors only have a direction and magnitude, not a position, we can reposition the vectors to have a common origin (Figure 7.6). The next theorem shows that the exterior angles are equal, from which we can deduce that Δv vectors (all of which are equal) form a regular polygon.

Theorem 7.3 *The exterior angles of the velocity diagram are equal.*

Proof Figure 7.7 is taken from the orbit diagram (Figure 7.4) with the Δv vectors extended. Let X be he intersection of CC' and DD'. By construction $CX \parallel BS$ and $DX \parallel CS$. Since the angles at S are equal (α), by alternate interior angles, the angles at C, D, X are equal. Since $\angle CXD = \alpha$, so is its vertical angle which is the exterior angle between $C'C$ and $D'D$ (examine Figure 7.6), and it follows that all exterior angles are equal. ∎

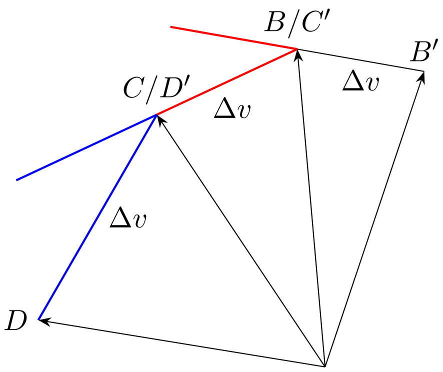


Figure 7.6: Exterior angles

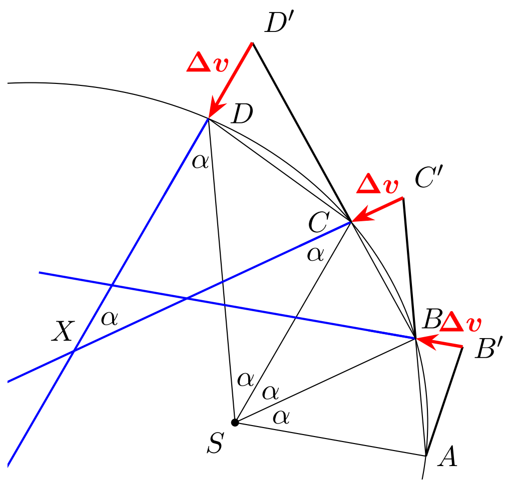


Figure 7.7: Equality of the exterior angles

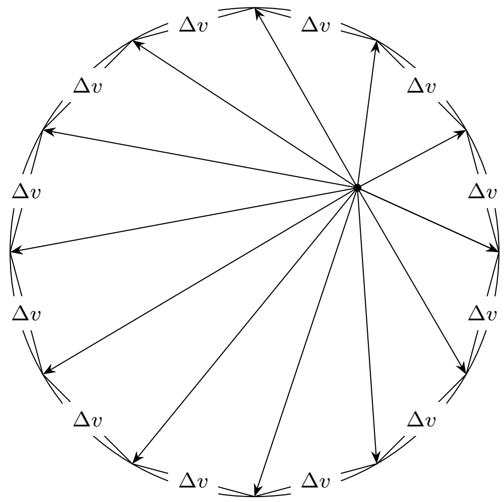


Figure 7.8: Velocity circle

Consider the polygon whose sides are formed by the Δv vectors for all sectors of the orbit. Since the exterior angles are equal, the polygon is regular, and as the number of sectors increases, the polygon approaches the "velocity circle."

We can now work with a smooth orbit and its velocity circle (Figures 7.9, 7.10). v_a is the (tangential) velocity at the perihelion, so it will be the longest line from origin of circle to its circumference and must pass through the center of the circle. The velocity vectors are tangents to the orbit, and as the planet moves in the orbit, they remain so. Since the orbit has been divided into sectors of equal angle, the sectors of the velocity circle are also equal, and therefore $\angle ASP = \angle aCp = \alpha$.

The final step is to show that as Cp rotates around the velocity circle, the orbit is an ellipse. In Figure 7.11 let p be an arbitrary point on the velocity circle with origin O and center C.[2] Construct the perpendicular bisector of Op at M and let its intersection with Cp be P.

[2]For convenience the circle is rotated 90° clockwise relative to the circle in Figure 7.10.

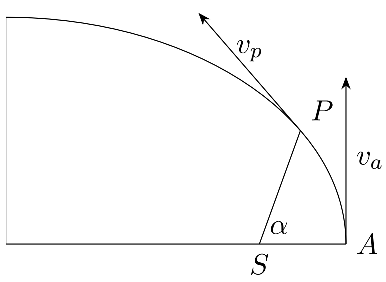


Figure 7.9: The velocity is tangent to the orbit

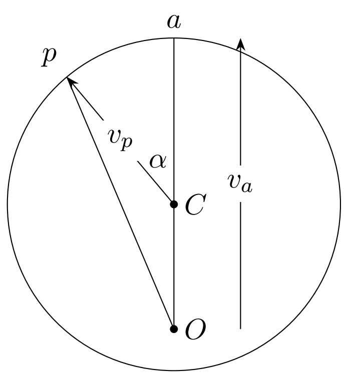


Figure 7.10: Vectors in the velocity circle

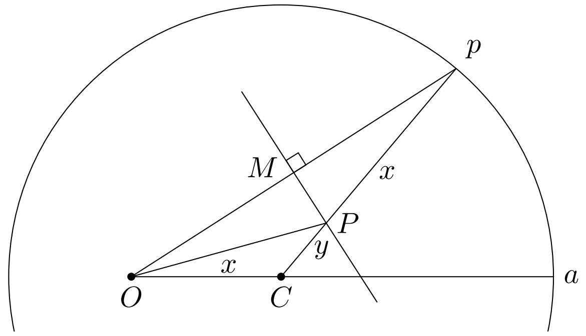


Figure 7.11: The derivation of the inverse square law

Theorem 7.4 *The locus of P as p moves around the velocity circle is an ellipse with foci O, C and $OP + PC$ equals the radius of the circle.*

Proof $\triangle POM \cong \triangle PpM$ by side-angle-side, so $OP = Pp = x$ and $CP + PO = CP + Pp = Cp$ which is the radius of the circle and therefore constant for any choice of p. ∎

Feynman found this to be the most difficult step to discover [6, p. 130] although similar constructions can be found in theorems on ellipses such as Theorem 10.4.

7.3 Maxwell's proof

Figure 7.12 shows a construction similar to the one used in the proof of Theorem 10.4. The foci are S, H, and P, Q are points on the ellipse close to each other. Extend SP to SU so that its length is the same as the length of the major axis AA'. Bisect HU at Z and draw ZP extended so that the perpendicular from S to the line intersects it at Y. Theorem 10.4 showed that ZP is a tangent to the ellipse and that $\angle HZP$ is a right angle.

The area swept out from P to Q is approximately that of the triangle $\triangle PSQ$ whose area is $\frac{1}{2}PQ \cdot SY$ since SY is the height of the triangle. The velocity at P is $v = PQ/\Delta t$ so

$$\kappa = \frac{\Delta A}{\Delta t} = \frac{\frac{1}{2}PQ \cdot SY}{\Delta t} = \frac{1}{2}vSY\,.$$

By Theorem 11.8, $SY \cdot HZ = BC^2$.

$$\begin{aligned} HU &= 2HZ = \frac{2BC^2}{SY} \\ &= \frac{2BC^2}{2\kappa/v} \\ v &= \frac{\kappa HU}{BC^2}\,. \end{aligned}$$

We conclude that HU is perpendicular to the velocity vector at P and proportional to the vector. Similarly, for HV, the line from H to V, the extension of SQ, and for any other point on the ellipse. Therefore, the lines $SU, SV, \ldots$ are all equal to AA', creating the velocity circle with center S and radius $r = AA'$.

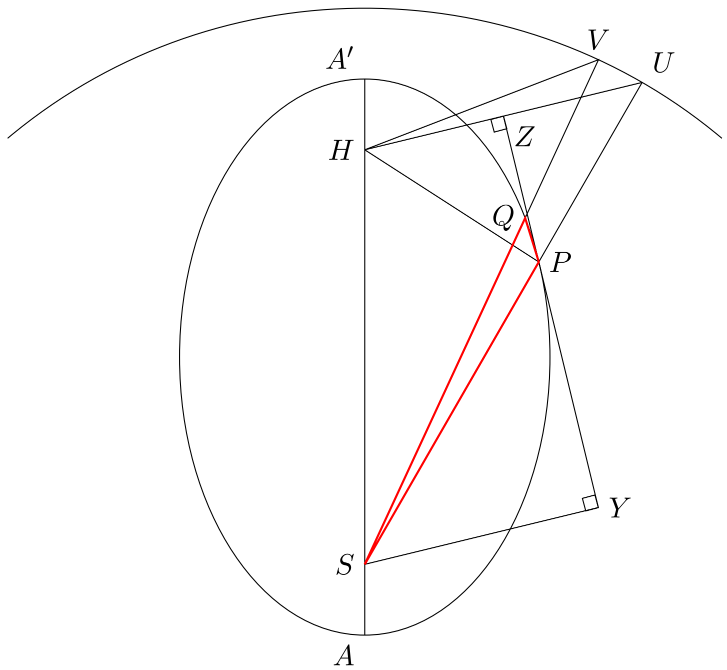


Figure 7.12: Maxwell's proof

By Theorem 7.1,

$$\begin{aligned}\kappa &= \frac{\Delta A}{\Delta t} \sim \frac{r^2}{\Delta t}\\ a &= \frac{\Delta v}{\Delta t} = \frac{\kappa \Delta v}{r^2}\,.\end{aligned}$$

By Theorem 7.2, Δv is independent of r, so the acceleration and hence the force to the focus S is proportional to the inverse square of the distance.

7.4 Hodographs

A *hodograph* is the curve traced out by the tips of velocity vectors of a trajectory or an orbit when their initial points are co-located. In Figure 7.6 the path $B'BCD$ is a hodograph as is the regular polygon in Figure 7.8. The circle containing U, V is a hodograph in Figure 7.12. This hodograph is instructive because it clearly shows how it is created by the velocity vectors starting at the focus S.

Hodographs were first proposed by William Rowan Hamilton who used them to prove Newton's theorem [10]. The advantage of hodographs is that velocity is a first-order derivative of position, while acceleration is a second-order derivative. Here we show that a hodograph can be used to easily obtain the horizontal distance traversed by a particle thrown at an angle and subject only to the force of gravity (Figure 7.13).

The initial velocity is v_0 and the final velocity is v_f, where $|v_0| = |v_f|$. The hodograph is shown in Figure 7.14. The horizontal component of the velocity is constant $v_h = v_0 \cos\theta$, while

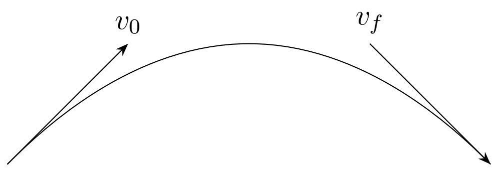


Figure 7.13: Path of a projectile

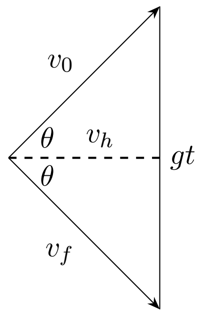


Figure 7.14: Hodograph of a projectile

the vertical velocity is gt, that is, it is proportional to the time until the particle hits the earth. Therefore, the area of the triangle,

$$\frac{1}{2}v_h \cdot gt = \frac{1}{2}v_0 \cos\theta \cdot gt$$

is one-half the distance traversed by the particle.

Chapter 8

Lagrange points

Consider a spacecraft orbiting the Sun and subject to the gravitational force of both the Earth and the Sun. Joseph-Louis Lagrange and Leonhard Euler discovered that there are five points where the spacecraft rotates with the same orbital period as the Earth and thus appears to maintain a fixed position as viewed from the Earth. These points are called the *Lagrange points* $L1, L2, L3, L4, L5$ and their positions are shown in Figure 8.1.[1]

In this section we present an approximate derivation of the locations of $L1, L2, L3$. The most significant approximation is that we assume that the spacecraft orbits around the center of the Sun, whereas it actually orbits around the barycenter of the Sun and the Earth. The derivation of the locations of $L4, L5$ is beyond the scope of this document. The final subsection describes the objects that exist at the Lagrange points.

8.1 Lagrange point $L1$

We assume that the Earth is in a circular orbit of radius r_E and period T_E around the Sun and that the masses of the two satisfy $m_E \ll m_S$. Let us suppose that we wish to place a space

[1] The orbits are clearly shown in the gif in the Wikipedia entry *Lagrange point*.

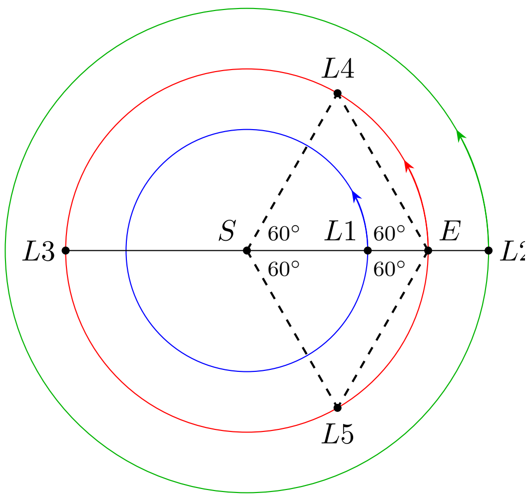


Figure 8.1: The five Lagrange points

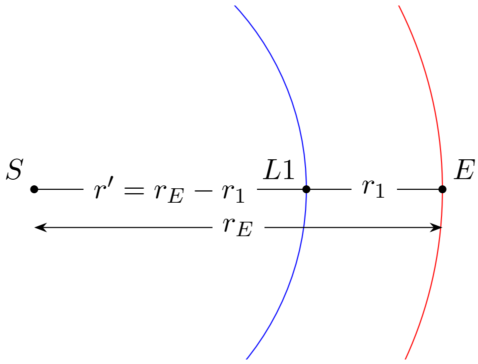


Figure 8.2: Lagrange point $L1$

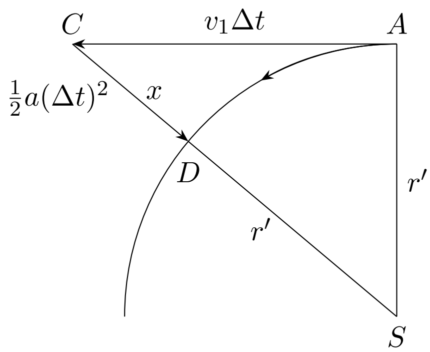


Figure 8.3: Tangential and central motion

telescope of mass $m_T \ll m_E$ in a circular orbit at L_1 which is at distance $r_1 \ll r_E$ from the Earth, such that its orbital period $T_1 = T_E$ (Figure 8.2). Is this possible?

If we ignore the gravitational force exerted by E on the telescope at $L1$, by Kepler's third law

$$\frac{A_E^3}{T_E^2} = \frac{A_1^3}{T_1^2} = \frac{A_1^3}{T_E^2},$$

then $A_1 = A_E$ and L_1 must be located at the center of the Earth.

You might think that L_1 should be chosen so that the gravitational force exerted by the Sun is exactly balanced by the gravitational force exerted by the Earth, but, of course, if there is no net force, by Newton's first law the telescope would simply move in a straight line off into space. Instead, we want the *net* centripetal force at L_1 to be

$$F = \frac{Gm_S m_T}{(r_e - r_1)^2} - \frac{Gm_E m_T}{r_1^2}, \tag{8.1}$$

so that the telescope moves in an orbit with period T_E. To simplify notation let $r' = r_E - r_1$. Since the length of the orbit at $L1$ is $2\pi r'$, the telescope's velocity is

$$v_1 = \frac{2\pi r'}{T_1}. \tag{8.2}$$

Consider Figure 8.3 where the motion from A to D along the orbit is separated into an unaccelerated tangential motion $v_1 \Delta t$ from A to C followed by an accelerated centripetal motion $\frac{1}{2}a(\Delta t)^2$ from C to S. By Pythagoras's theorem,

$$\begin{aligned} r'^2 + (v_1 \Delta t)^2 &= (r' + x)^2 = r'^2 + 2r'x + x^2 \\ (v_1 \Delta t)^2 &= x(2r' + x) \\ x &\approx \frac{v_1^2}{2r'}(\Delta t)^2, \end{aligned}$$

since Δt is assumed to be very small and since r' is close to r_E, $2r' + x \approx 2r'$. Therefore, the acceleration must be v_1^2/r' and from Equation 8.1, by Newton's second law the net centripetal force needed is

$$F = m_1 \cdot \frac{v_T^2}{r'} = \frac{Gm_S m_1}{r'^2} - \frac{Gm_E m_1}{r_1^2}.$$

Canceling m_1 gives

$$v_1^2 = \frac{Gm_S}{r'} - \frac{Gm_E r'}{r_1^2}\,.$$

The period of the desired orbit is $T_1 = T_E$ so by Equation 8.2,

$$\begin{aligned}\frac{4\pi^2 r'^2}{T_E^2} &= \frac{Gm_S}{r'} - \frac{Gm_E r'}{r_t^2}\\ \frac{4\pi^2}{T_E^2} &= \frac{Gm_S}{r'^3} - \frac{Gm_E}{r' r_t^2}\,.\end{aligned}$$

By Kepler's third law (Equation 5.4), where the elliptical semi-major axis a_i is the circular radius r_E,

$$\begin{aligned}\frac{4\pi^2 r_E^3 m_E}{T_E^2}\frac{1}{r_E^2} &= \frac{Gm_E m_S}{r_E^2}\\ \frac{Gm_S}{r'^3} - \frac{Gm_E}{r' r_1^2} &= \frac{Gm_S}{r_E^3}\\ \frac{1}{r_E^3} &= \frac{1}{r'^3} - \frac{m_E/m_S}{r' r_1^2}\,.\end{aligned} \tag{8.3}$$

Let $y = m_E/m_S$ and $z = r_1/r_E$ so $r' = r_E - r_1 = r_E(1-z)$. Multiply Equation 8.3 by r_E^3 and make the substitutions.

$$\begin{aligned}\frac{r_E^3}{r'^3} - \frac{m_E/m_S r_E^3}{r' r_1^2} &= 1\\ \frac{1}{(1-z)^3} - \frac{y r_E^3}{r_E(1-z) z^2 r_E^2} &= 1\\ \frac{1}{(1-z)^3} - \frac{y}{z^2(1-z)} &= 1\,.\end{aligned}$$

Since $z = r_1/r_E$ is very small, we get the following approximations from the Taylor series [8, Chapter 11.8].

$$\begin{aligned}\frac{1}{(1-z)} &= 1 + z + z^2 + \cdots \approx 1 + z\\ \frac{1}{(1-z)^3} &= 1 + 3z + 6z^2 + \cdots \approx 1 + 3z\,.\end{aligned}$$

Substituting these approximations gives

$$3z^3 \approx y(1+z) \approx y\,.$$

Let us plug in the numbers $m_S \approx 2 \times 10^{30}$ kg, $m_E \approx 6 \times 10^{24}$ kg and $r_E \approx 1.5 \times 10^8$ km.

$$\begin{aligned}\left(\frac{r_1}{1.5 \times 10^8}\right)^3 &\approx \frac{6 \times 10^{24}}{3 \times 2 \times 10^{30}} = 10^{-6}\\ r_1 &\approx 1.5 \times 10^8 \cdot \sqrt[3]{10^{-6}} \approx 1.5 \times 10^6\,.\end{aligned}$$

If an object is placed 1.5 million km from the Earth in the direction of the Sun, the period of its orbit around the Sun will be approximately one year. This is quite far—the Moon is less than $400,000$ km from the Earth—but still relatively far from the Sun which is 150 million km away.

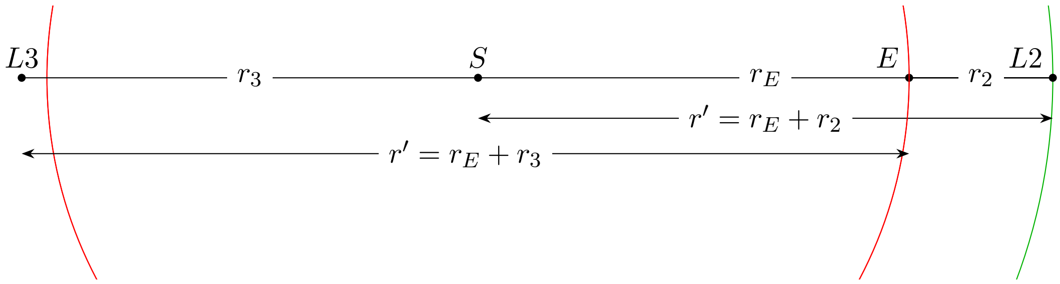


Figure 8.4: Lagrange points $L2$ and $L3$

8.2 Lagrange points $L2$ and $L3$

The computation for $L2$ is similar using $r' = r_E + r_2$ (Figure 8.4). With the appropriate modifications to Equation 8.1 we get

$$\begin{aligned}
F &= \frac{Gm_S m_1}{r'^2} + \frac{Gm_E m_1}{r_2^2} \\
\frac{1}{r_E^3} &= \frac{1}{r'^3} + \frac{m_E/m_S}{r' r_2^2} \\
1 &= \frac{1}{(1+z)^3} + \frac{y}{z^2(1+z)} \,.
\end{aligned}$$

The approximations based on the Taylor series are $(1+z)^{-3} \approx 1 - 3z$ and $(1+z)^{-1} \approx 1 - z$, leading to the same equation $3z^3 \approx y$. Therefore, $L2$ is the same distance from the Earth as $L1$ but on the opposite side of the Earth.

The Lagrange point $L3$ is on the other side of the Sun (Figure 8.4). The modifications to Equation 8.3 give

$$\begin{aligned}
\frac{1}{r_E^3} &= \frac{1}{r_3^3} + \frac{m_E/m_S}{r'^3} \\
1 &= \frac{1}{z^3} + \frac{y}{z^3(r_E + r_3)^3} \\
1 &= \frac{1}{z^3} + \frac{y}{(1+z)^3} \\
z^3 &= \frac{1}{1-y} \,,
\end{aligned}$$

since $z \ll 1$. But $y \approx 10^{-6}$ so $z^3 \approx 1$, $r_3 \approx r_E$ and T is approximately the same distance from the Sun as it is from the Earth.

8.3 Objects at the Lagrange points

Although the orbits of objects at $L1$, $L2$ and $L3$ are not stable, they are relatively stable so that a spacecraft can be placed into a small orbit around one of these points. Even when it drifts, the force required to return it to the Lagrange point is small, which means that the propellant in the spacecraft can maintain it on station for a long time.

The *Deep Space Climate Observatory (DSCOVR)* was placed at Lagrange point $L1$. It continually observes the Sun and the sunlit side of the Earth. The *James Webb Space Telescope* with its 6.5 meter diameter infrared telescope was placed at $L2$ in 2022. $L2$ is ideal for telescopes: if a sun shield is placed facing the Earth and the Sun, the spacecraft itself can remain at the very low temperature that its sensors require. Lagrange point $L3$ is not useful for spacecraft because the line-of-sight needed for communication with the Earth is blocked by the Sun.

The orbits of objects at $L4$ and $L5$ are stable. Asteroids that are stable at a Lagrange point are called *trojans* and most are located at the $L4$ and $L5$ points of Jupiter. There are two extremely small trojans at the Earth's $L4$ point.

Part II

Ellipses

Chapter 9

Definitions of ellipses

9.1 Four definitions of an ellipse

Ellipses can be defined analytically as the locus of points in the Cartesian plane satisfying a certain equation. Geometrically, an ellipse is defined by selecting two points, the foci, and defining the curve as the locus of points such that the sum of the distances to the foci is constant. Less well-known is the definition in terms of a focus and a directrix, but this was fundamental to the study of conic sections. The parametric representation of an ellipse that will be used in one of the proofs.

Analytic geometry

Definition 9.1 *Let a, b be positive real numbers. An* ellipse *is the locus of the points $P = (x, y)$ in the Cartesian plane that satisfy the equation*[1]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{9.1}$$

Two foci

Definition 9.2 *Let S and H be two points in the Cartesian plane such that $SH = 2c \geq 0$ and choose a such that $2a > 2c$ (Figure 9.1). An* ellipse *is the locus of points P such that $SP + PH = 2a$. If $c = 0$ the locus is a* circle.

Definition 9.3 *Let A, A' be the intersections of the line through the foci S, H with the ellipse. AA' is the* major axis *of the ellipse. Let O be the midpoint of SH. AO, OA' are the* semi-major axes *of the ellipse.*

Let B, B' be the intersections of the perpendicular to SH at O with the ellipse. BB' is the minor axis *of the ellipse and BO, OB' are the* semi-minor axes *of the ellipse.*

Theorem 9.4 *Using the notation in Figure 9.2, (1) $SB = HB = a$, (2) $AO = OA' = a$, (3) $BO = OB'$. Label $BO = OB'$ by b.*

[1] This equation holds for ellipses centered at the origin, whose axes are on the axes of the Cartesian plane.

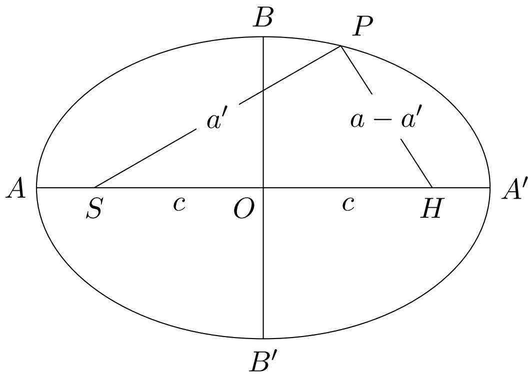


Figure 9.1: The foci S, H and the lengths a, c

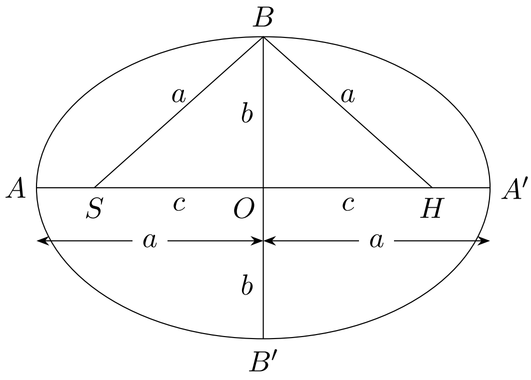


Figure 9.2: The axes of an ellipse

Proof

1. $\triangle SBO \cong \triangle HBO$ by side-angle-side so $SB = HB$. Since B is on the ellipse, $SB + HB = 2a$ and $SB = HB = a$ follows.

2. Since A is on the ellipse, $2a = AS + HA = (AO - c) + (AO + c) = 2AO$, so $AO = a$. Similarly, $OA' = a = AO$.

3. $BO = OB'$ follows from $\triangle SBO \cong \triangle SB'O$. ▮

Conic sections

Definition 9.5 *Let* d *be a line (the* directrix*) and* S *be a point (the* focus*) not on the directrix, where* d *is the distance from* S *to the directrix (Figure 9.3). Let* $0 < e < 1$ *be a number (the* eccentricity*). An* ellipse *is the locus of points* P *such that the ratio of* PS *to the distance of* P *to the directrix is* e*.*

Let X *be the intersection of the perpendicular to the directrix from* S*.* A *on* SX *is a* vertex *of the ellipse if* $SA/AX = e$*. In Figure 9.3 the eccentricity is the length of the red segments divided by the length of the blue segments.*

For other conic sections: $e = 1$ *for a parabola and* $e > 1$ *for a hyperbola.*

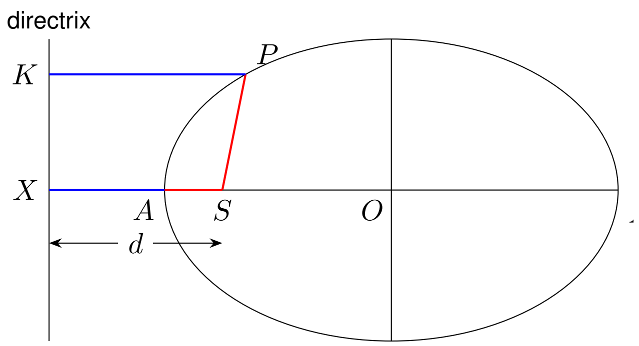


Figure 9.3: The focus and the directrix

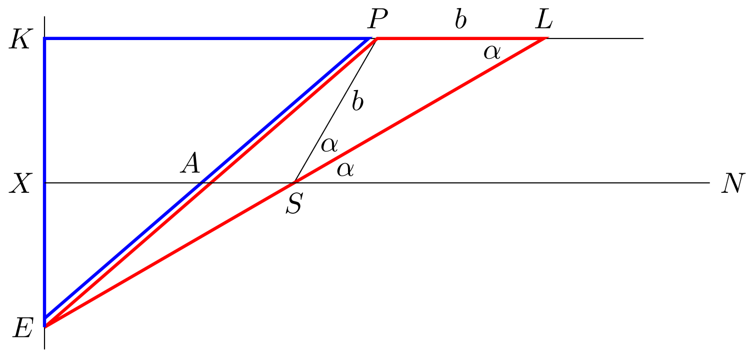


Figure 9.4: Constructing points on the ellipse

Generating the conic-section ellipse

Definition 9.5 is non-constructive. It states that the ellipse is the locus of points satisfying a certain property, but aside from the vertices we have not constructed any such points. The following construction shows how to construct arbitrary points on the ellipse.

Let E be a point on the directrix and construct lines from E through A and S. The line through S will make an angle α with XS. Construct a line from S at the *same angle* α from ES and let its intersection with EA be P. Construct the perpendicular from P to the directrix and let K be its intersection with the directrix. Let L be the intersection of KP with ES (Figure 9.4).

Theorem 9.6 *The point P is on the ellipse.*

Proof By construction $PK \parallel SX$ so $\triangle XEA \sim \triangle KEP$ and $\triangle AES \sim \triangle PEL$ are adjacent pairs of similar triangles and

$$\frac{PL}{PK} = \frac{SA}{AX} = e\,.$$

Now $\angle PLS = \angle LSN = \alpha$ by alternate interior angles, so $\triangle LPS$ is isosceles and $PL = SP$. Therefore, $SP/PK = PL/PK =$ and P is on the ellipse. By choosing different points E on the directrix, any point on the ellipse can be constructed. ∎

Parametric representation

Definition 9.7 *[Parametric representation] Consider two concentric circles, one of radius a (dotted red) and one of radius b (dashed blue) (Figure 9.5). An* ellipse *is the locus of points $P = (x, y)$ such that*

$$(x, y) = (a\cos t,\ y = b\sin t)\,,$$

for $0 \leq t < 2\pi$.

The parameter t is *not* the angle of P relative to the positive x-axis. Construct the perpendicular through P to the minor axis and let P_I be its intersection with the inner circle so that OP_I defines an angle t. Extend OP_I until it intersects the outer circle at P_O.

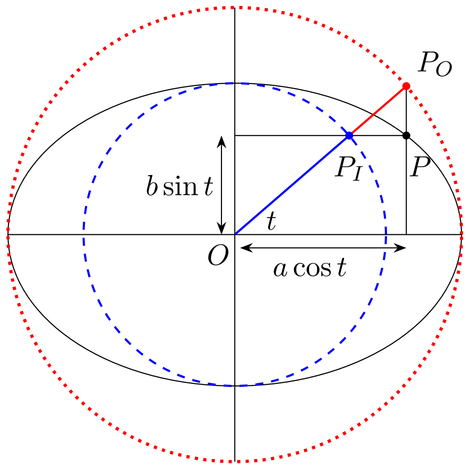


Figure 9.5: Parametric representation of an ellipse

9.2 The latus rectum and conjugate diameters

Definition 9.8 *Consider a line through a focus of an ellipse that is perpendicular the major axis. Let its intersections with the ellipse be L_1, L_2. Then $L = L_1L_2$ is a* latus rectum *of the ellipse (Figure 9.6).*

Definition 9.9 *There are two equivalent definitions of conjugate diameters (Figure 9.7).*

- *Let P be a point on an ellipse, PG a diameter and let t be the tangent to the ellipse at P. Diameter DK is a* conjugate diameter *if it is parallel to t.*
- *Two diameters PG and DK are* conjugate diameters *if the midpoints of chords ($D'K'$, $D''K''$) parallel to one diameter (DK) lie on another diameter (PG).*

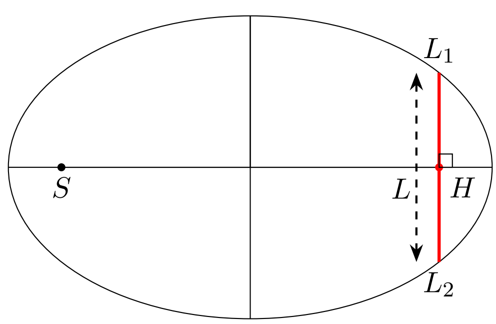


Figure 9.6: The circumscribed circle and the latus rectum of an ellipse

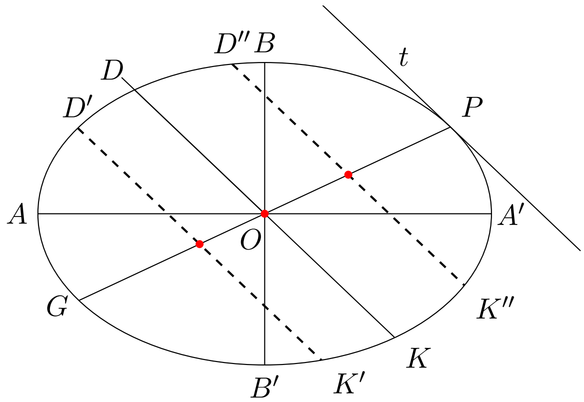


Figure 9.7: Conjugate diameters

9.3 Equivalence of the definitions

From the two-foci definition to the analytic equation

Theorem 9.10 *A point $P = (x, y)$ on an ellipse (Definition 9.2) satisfies Equation 9.1*

Proof Since $S = (-c, 0)$, $H = (c, 0)$ and $SP + PH = 2a$,

$$PS + PH = \sqrt{(x-(-c))^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a\,.$$

Move the second radical to the right-hand side of the equation and square the result.

$$\begin{aligned}
(x+c)^2 + y^2 &= \left(2a - \sqrt{(x-c)^2 + y^2}\right)^2 \\
4xc &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} \\
a - \frac{c}{a}x &= \sqrt{(x-c)^2 + y^2}\,.
\end{aligned}$$

Square again, simplify and divide by $a^2 - c^2$ to get

$$\begin{aligned}
a^2 + \frac{c^2}{a^2}x^2 &= x^2 + c^2 + y^2 \\
\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} &= \frac{a^2 - c^2}{a^2 - c^2} = 1\,.
\end{aligned}$$

By Theorem 9.4 and Pythagoras's theorem, $b^2 = a^2 - c^2$ so $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. ∎

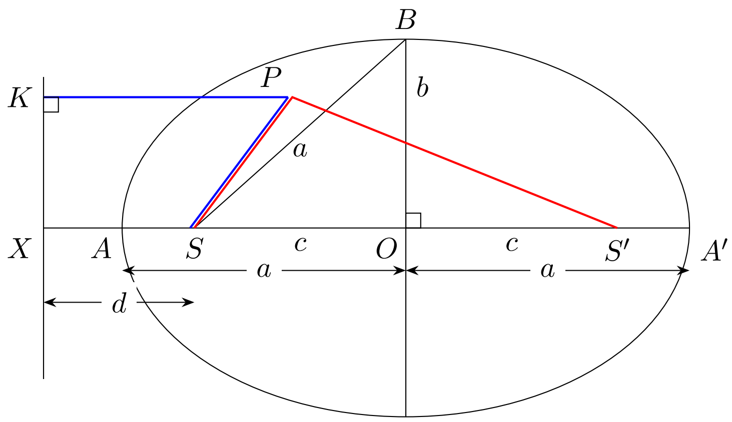


Figure 9.8: Two definitions of an ellipse

From the focus and directrix definition to the two-foci definition

Theorem 9.11 *(Figure 9.8) The parameters a and c of Definition 9.2 can be computed from d and e of Definition 9.5. Conversely, d can be computed from a and e.*

Proof The definition $e = SA/AX$ will be used implicitly in the proof.

We start by computing $AX, A'X$ from d, e.

$$\begin{aligned} SA + AX &= SX = d \\ SA &= d - \frac{SA}{e} = d \cdot \frac{e}{1+e} \qquad (9.2) \\ AX &= d \cdot \frac{1}{1+e} \\ A'X - SA' &= d \\ SA' &= \frac{SA'}{e} - d = d \cdot \frac{e}{1-e} \\ A'X &= d \cdot \frac{1}{1-e} \,. \end{aligned}$$

a can now be computed from $A'X, AX$.

$$\begin{aligned} a = \frac{AA'}{2} &= \frac{1}{2}(A'X - AX) = \frac{d}{2}\left(\frac{1}{1-e} - \frac{1}{1+e}\right) \\ &= \frac{d}{2} \cdot \frac{2e}{1-e^2} = d \cdot \frac{e}{1-e^2} \,. \qquad (9.3) \end{aligned}$$

c is $a - SA$ so by Equations 9.2, 9.3,

$$c = OS = a - SA = d \cdot \frac{e}{1-e^2} - d \cdot \frac{e}{1+e} = d \cdot \frac{e^2}{1-e^2} \,.$$

Finally, b can be computed as $\sqrt{a^2 - c^2}$.

$$b = d \cdot \sqrt{\frac{e^2}{(1-e^2)^2} - \frac{e^4}{(1-e^2)^2}} = d \cdot \frac{e^2}{1-e^2} \cdot \sqrt{1-e^2} = d \cdot \frac{e}{\sqrt{1-e^2}}\,.$$

Conversely, by Equation 9.3, d can be computed from a and e.

$$d = a \cdot \frac{1-e^2}{e}\,. \quad \blacksquare$$

Example: Compute the factors for $e = \sqrt{5}/3$ and then multiply it by various values of d.

$$\begin{aligned} a/d &= \frac{e}{1-e^2} = \frac{\sqrt{5}/3}{4/9} = \frac{3\sqrt{5}}{4} \\ c/d &= \frac{e^2}{1-e^2} = \frac{5/9}{4/9} = \frac{5}{4} \\ b/d &= \frac{e}{\sqrt{1-e^2}} = \frac{\sqrt{5}/3}{2/3} = \frac{\sqrt{5}}{2}\,. \end{aligned}$$

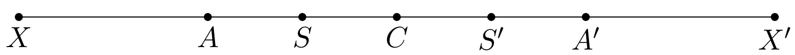
For $d = 4/\sqrt{5}$ we have $a = 3, b = 2, c = \sqrt{5}$.
Conversely,

$$d = a \cdot \frac{1-e^2}{e} = 3 \cdot \frac{4/9}{\sqrt{5}/3} = \frac{4}{\sqrt{5}}\,.$$

Figure 9.8 was drawn with $d = \sqrt{5}$ giving $a = 15/4 = 3.75, b = 5/2 = 2.5, c = 5\sqrt{5}/4 \approx 2.8$.

From the conic-section definition to the major axis

We wish to deduce $SP + PH = 2a$ from the conic-section definition. The proofs use the notation in the following diagram where the foci are named S, S'.



Theorem 9.12 *In an ellipse,*

$$\frac{SA}{AX} = \frac{AA'}{XX'}\,. \tag{9.4}$$

Proof From the diagram we have

$$\begin{aligned} \frac{AA'}{SA} - 1 &= \frac{AA'}{SA} - \frac{SA}{SA} = \frac{AA' - SA}{SA} = \frac{SA'}{SA} \\ \frac{XX'}{AX} - 1 &= \frac{XX'}{AX} - \frac{AX}{AX} = \frac{XX' - AX}{AX} = \frac{AX'}{AX}\,. \end{aligned}$$

But A, A' are both points on the ellipse so

$$\frac{SA'}{AX'} = \frac{SA}{AX} = e\,, \qquad \frac{SA'}{SA} = \frac{AX'}{AX}\,,$$

and Equation 9.4 follows from the previous two equation. $\blacksquare$

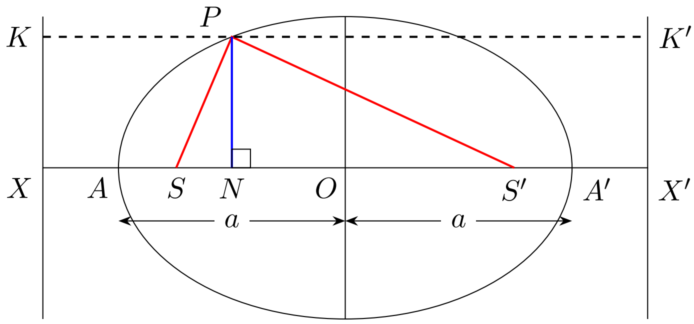


Figure 9.9: $SP + S'P = AA'$

Theorem 9.13 *In an ellipse,* $SP + S'P = AA'$.

Proof Let N be the perpendicular from P to the major axis (Figure 9.9). Since P is on the ellipse, $SP/PK = S'P/PK' = e$, but by construction $PK = NX$ and $PK' = NX'$, so $SP/NX = e$ or $S'P/NX' = e$.

$$\begin{aligned} \frac{SP}{NX} &= \frac{S'P}{NX'} \\ \frac{S'P}{SP} &= \frac{NX'}{NX} \\ \frac{S'P + SP}{SP} &= \frac{NX' + NX}{NX} = \frac{XX'}{NX} \,. \end{aligned}$$

By Equation 9.4,

$$\begin{aligned} \frac{S'P + SP}{XX'} &= \frac{SP}{NX} = \frac{SA}{AX} = \frac{AA'}{XX'} \\ S'P + SP &= AA' \,. \quad \blacksquare \end{aligned}$$

Chapter 10

Properties of ellipses

10.1 Geometric properties of an ellipse

Theorem 10.1 *The perpendicular to the major axis through a point $P_c = (x, y_c)$ on the circle circumscribing an ellipse intersects the ellipse at $P_e = (x, y_e) = \left(x, \frac{b}{a}y_c\right)$.*

Proof From Equation 9.1 and the formula $x^2 + y^2 = a^2$ for the circle,

$$y_e = \frac{b}{a}\sqrt{(a^2 - x^2)} = \frac{b}{a}y_c\,. \quad \blacksquare \tag{10.1}$$

Theorem 10.2 *L, the length of the latus rectum of an ellipse, is* $\frac{2b^2}{a}$.

Proof The latus rectum is the perpendicular at the focus $(c, 0)$. By Equation 10.1 and Pythagoras's theorem,

$$L = 2L_1 = 2 \cdot \frac{b}{a}\sqrt{a^2 - c^2} = \frac{2b^2}{a}\,. \quad \blacksquare \tag{10.2}$$

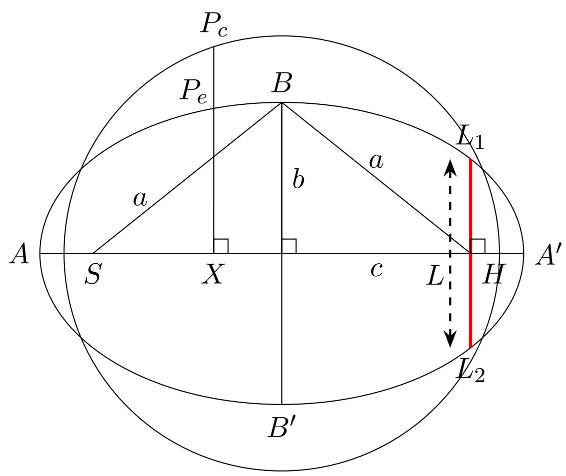


Figure 10.1: The circumscribed circle and the latus rectum of an ellipse

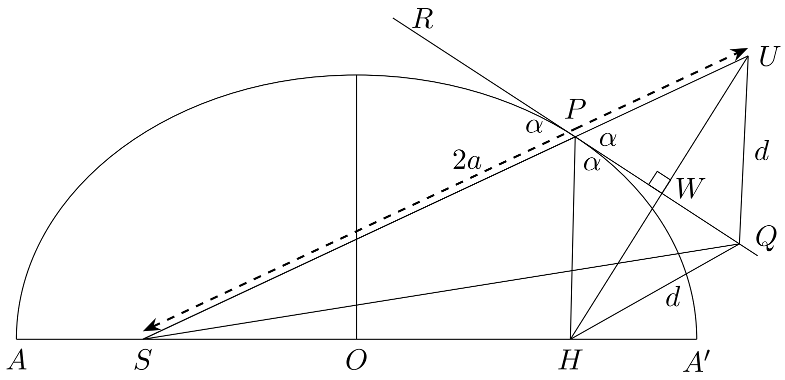


Figure 10.2: Angles at the tangent

Theorem 10.3 *The area of an ellipse is πab.*

Proof From Equation 10.1

$$
\begin{aligned}
y_e &= \frac{b}{a}\sqrt{a^2-x^2} \\
A_{ellipse} &= 2\int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\cdot 2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}A_{circle} = \pi ab\,. \quad \blacksquare
\end{aligned}
$$

10.2 The angles between a tangent and the lines to the foci

Theorem 10.4 *Let P be a point on the ellipse whose foci are S, H. Let PU be the extension of SP such that $SU = AA' = 2a$. Then the tangent RQ is the external angle bisector of $\angle HPU$ and $\angle RPS = \angle QPH$ (Figure 10.2).*

Proof We prove that the external angle bisector must be the tangent by showing that any point $Q \neq P$ on the bisector is not on the ellipse, so the bisector RQ has only one point of intersection with the ellipse and it must be the tangent at P. $\angle QPH = \angle QPU = \angle RPS = \alpha$ since RQ is the bisector and by vertical angles.

Construct the line HU to form the triangle $\triangle HPU$ which intersects PQ at W. By construction $PH = PU$ so $\triangle HPW \cong \triangle UPW$ by side-angle-side. Therefore, $\angle HWP = \angle UWP$ and they are right angles, so $\triangle HWQ = \triangle UWQ$ by side-angle-side and $UQ = HQ$. If Q is on the ellipse, $2a = SQ+QH = SQ+QU$, but by the triangle inequality $2a = SQ+QU > SU = 2a$, contradicting that Q is on the ellipse. $\blacksquare$

10.3 Conjugate diameters

Theorem 10.5 *Let $P = (x, y)$ be a point on an ellipse (not on the major axis AA') and construct a perpendicular PV from P to the major axis (Figure 10.3). Then*

$$\frac{A'V \cdot AV}{PV^2} = \frac{a^2}{b^2}\,.$$

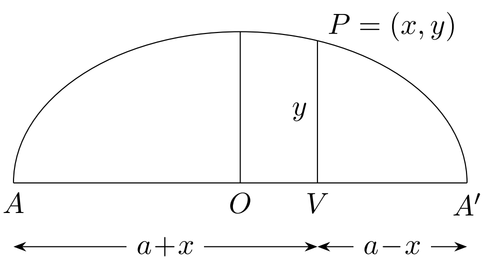


Figure 10.3: Ratios on conjugate diameters

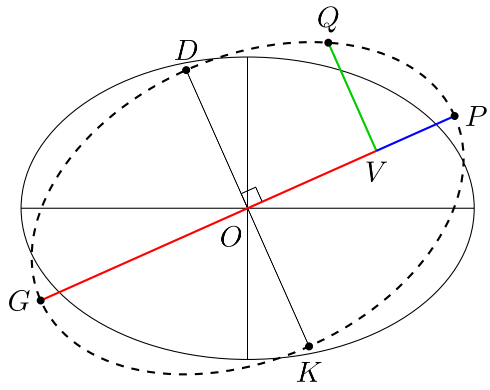


Figure 10.4: Rotating the ellipse

Proof By Equation 10.1,

$$y^2 = b^2 \cdot \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2(a^2 - x^2)}{a^2}$$

$$\frac{A'V \cdot AV}{PV^2} = \frac{(a+x)(a-x)}{y^2} = \frac{a^2(a^2 - x^2)}{b^2(a^2 - x^2)} = \frac{a^2}{b^2} . \quad \blacksquare$$

Theorem 10.6 *Let PG, DK be conjugate diameters of an ellipse and let Q be a point on the ellipse (Figure 10.4). Constrct the perpendicular QV from Q to the major axis, then*

$$PV = \frac{QV^2 \cdot OP^2}{GV \cdot OD^2} .$$

Proof Figure 10.4 shows a dashed ellipse which is the original ellipse rotated about the same center O, so that OP is the semi-major axis and OD is the semi-minor axis. By Theorem 10.5,

$$\frac{GV \cdot PV}{QV^2} = \frac{a^2}{b^2} = \frac{OP^2}{OD^2}$$

$$PV = \frac{QV^2 \cdot OP^2}{GV \cdot OD^2} . \quad \blacksquare$$

10.4 Areas of parallelograms

Theorem 10.7 *Let PG, DK be conjugate diameters of an ellipse. The tangents to the ellipse at P, G, D, K form a parallelogram $JKLM$ whose area is equal to the parallelogram (dashed rectangle) formed by the tangents to A, A', B, B' (Figure 10.5).*

Proof By symmetry it suffices to prove that the areas of one pair of quadrants of the parallelograms are equal: $A_{OA'QB} = A_{PODJ}$. Since diagonals bisect a parallelogram, it suffices to prove that that the area of $\triangle A'OB$ (red) equals the area of $\triangle POD$ (blue).

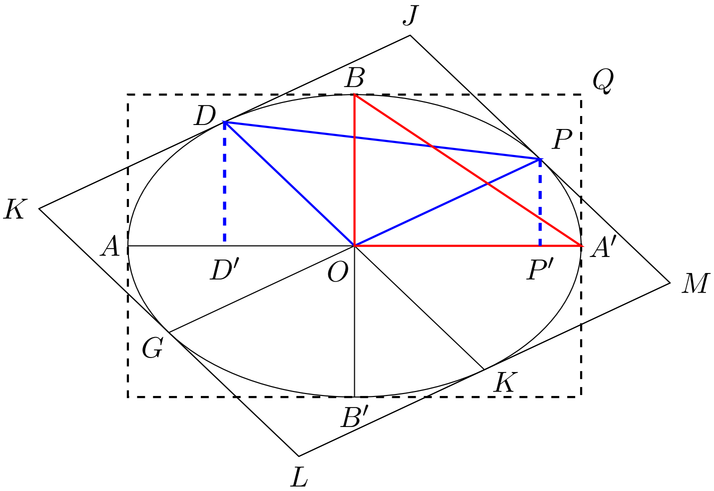


Figure 10.5: Parallelograms formed by conjugate diameters

Let $P = (x_p, y_p) = (a\cos t, b\sin t), D = (x_d, y_d)$ be the parametric representations of these points on the ellipse. Conjugate diameters are perpendicular so $\angle DOP$ is a right angle and

$$D = (x_d, y_d) = (a\cos(t + \pi/2), b\sin(t + \pi/2)) = (-a\sin t, b\cos t)\,.$$

Construct $DD' = (x_d, 0)$ and $PP' = (x_p, 0)$ perpendicular to the major axis. The area of $\triangle POD$ can be computed as the area of the trapezoid $P'PDD'$ minus the areas of the triangles $\triangle D'DO, \triangle P'PO$. Therefore,[1]

$$\begin{aligned}
\triangle POD &= \frac{y_p + y_d}{2}(x_p + (-x_d)) - \frac{1}{2}(-x_d)y_d - \frac{1}{2}x_p y_p \\
&= \frac{1}{2}\left(x_p y_d - x_d y_p\right) \\
&= \frac{1}{2}\left(a\cos t \cdot b\cos t - (-a)\sin t \cdot b\sin t\right) = \frac{1}{2}ab = \triangle A'OB\,. \quad \blacksquare
\end{aligned}$$

[1]The *length* between O and D' is $-x_d$.

Chapter 11

Ellipses in Euclidean geometry

The proofs of theorems about planetary orbits used analytic geometry and trigonometry, but for many years after the invention of analytic geometry, mathematicians continued to limit themselves to Euclidean geometry. This section contains proofs in Euclidean geometry of theorems that appeared in Chapter 10.[1] The goal is to prove Theorem 10.7 in Euclidean geometry.

To remain consistency with Besant, in this chapter the center of the ellipse will be denoted C instead of O.

11.1 A right angle at the focus of an ellipse

Theorem 11.1 *Let P, P' be points on the ellipse and let F be the intersection of PP' with the directrix. Then FS bisects the exterior angle of $\angle P'SP$(Figure 11.1).*

[1]Except for Theorems 10.4, 10.6 which were proved using Euclidean geometry and Theorem 10.3 which requires taking limits.

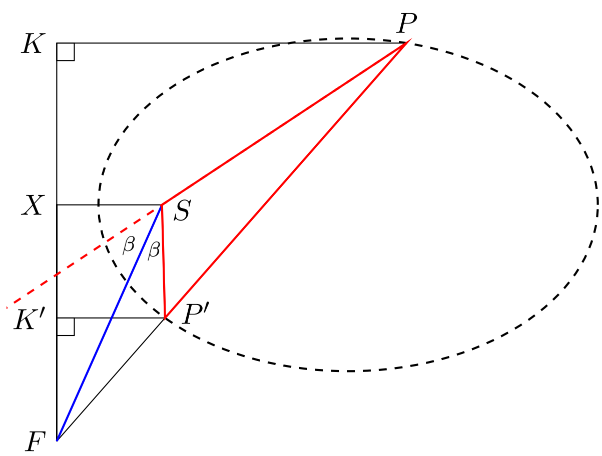


Figure 11.1: Bisecting the angle at the focus

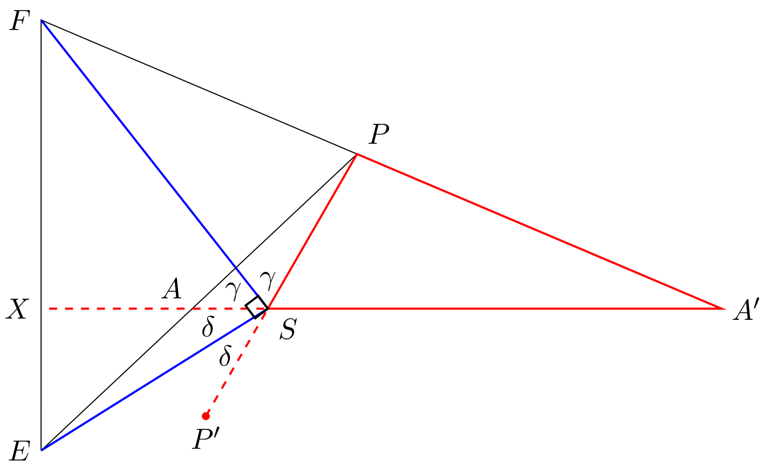


Figure 11.2: The right angle at the focus

Proof Since P, P' are on the ellipse

$$\frac{SP}{PK} = \frac{SP'}{P'K'} = e\,,$$

and since $\triangle PFK \sim P'FK'$,

$$\frac{SP}{SP'} = \frac{PK}{P'K'} = \frac{PF}{P'F}\,.$$

By the exterior angle bisector theorem (Theorem A.6), FS bisects the exterior angle of $\angle P'SP$. ∎

Theorem 11.2 *Let P be a point on the ellipse and construct lines PA, PA'. Label their intersections with the directrix by E and F, respectively. Then $\angle FSE$ is a right angle (Figure 11.2).*

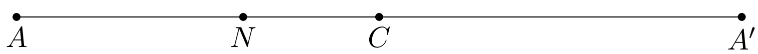
Proof P, A, A' are all points on the ellipse so Theorem 11.1 applies. FS bisects $\angle PSX = 2\gamma$ and ES bisects $\angle P'SX = 2\delta$, so $2\gamma + 2\delta = 180°$ and $\angle FSE = \gamma + \delta = 90°$. ∎

11.2 Ratios of perpendiculars to the axes

We start with a preliminary theorem.

Theorem 11.3 *Let AA' be a line segment whose midpoint is C. Then*

$$AC^2 - CN^2 = AN \cdot NA'\,.$$



Proof $AN = AC - CN$ and $NA' = A'C + CN = AC + CN$ since C is the midpoint of AA'. The result is obtaining by multiplying the two equations. ∎

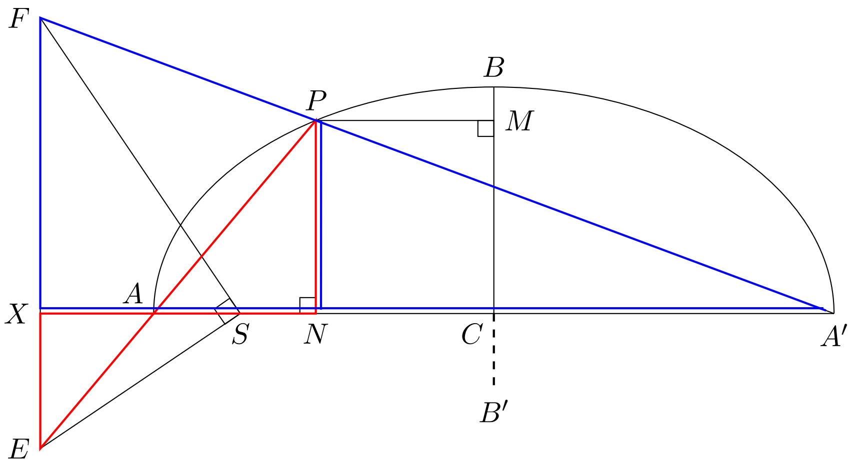


Figure 11.3: Ratio of an ordinate

Theorem 11.4 *(Theorem 10.5) Let P be a point on an ellipse not on the major axis and construct perpendiculars PN, PM from P to the major and minor axes, respectively (Figure 11.3). Then*

$$\frac{PN^2}{A'N \cdot NA} = \frac{BC^2}{AC^2} \tag{11.1}$$

$$\frac{PM^2}{B'N \cdot NA} = \frac{AC^2}{BC^2}\,. \tag{11.2}$$

Proof (Equation 11.1) $\triangle AXE \sim \triangle ANP$ (red) since they are right triangles and the vertical angles at A are equal. Therefore,

$$\frac{PN}{AN} = \frac{EX}{AX}\,. \tag{11.3}$$

$\triangle PA'N \sim \triangle FA'X$ (blue) so

$$\frac{PN}{A'N} = \frac{FX}{A'X}\,. \tag{11.4}$$

Multiplying Equations 11.3 and 11.4 gives

$$\frac{PN^2}{AN \cdot A'N} = \frac{EX \cdot FX}{AX \cdot A'X}\,.$$

By Theorem 11.2 $\triangle FSE$ is a right triangle so by Theorem A.2,

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X}\,.$$

Since P was arbitrary this holds for any point on the ellipse, in particular, for B on the minor axis, where $PN = BC$ and $AN = AN' = AC$. Therefore,

$$\frac{BC^2}{AC^2} = \frac{SX^2}{AX \cdot A'X}$$

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X} = \frac{BC^2}{AC^2}\,. \quad \blacksquare$$

Proof (Equation 11.2) Since $CM = PN, PM = CN$, by Theorem 11.3, Equation 11.1 becomes

$$\frac{CM^2}{AC^2 - PM^2} = \frac{BC^2}{AC^2}$$

$$\frac{AC^2}{AC^2 - PM^2} = \frac{BC^2}{CM^2}\,. \tag{11.5}$$

By inverting the ratios it can be shown that

$$\frac{AC^2}{PM^2} = \frac{BC^2}{BC^2 - CM^2}, \tag{11.6}$$

and then by Theorem 11.3 we have

$$\frac{PM^2}{BM \cdot MB'} = \frac{AC^2}{BC^2}\,. \quad \blacksquare$$

11.3 The circle circumscribing an ellipse

Theorem 11.5 *(Theorem 10.1) Consider circle circumscribed about an ellipse (Figure 11.4). Choose a point N on the major axis and construct a perpendicular through N. Let its intersections with the ellipse and the circle be P and Q, respectively. Then*

$$\frac{PN}{QN} = \frac{BC}{AC}\,.$$

Proof From Theorem 11.4,

$$\frac{PN^2}{AN \cdot NA'} = \frac{BC^2}{AC^2}\,,$$

and by Theorem A.2, $AN \cdot NA = QN^2$. $\blacksquare$

11.4 The latus rectum of an ellipse

The following theorem proves Theorem 10.2 in Euclidean geometry.

Theorem 11.6 *L, the length of the latus rectum of an ellipse, is $\dfrac{2BC^2}{AC}$ (Figure 11.4).*

Proof By Theorem 11.4,

$$\frac{HL_1^2}{AH \cdot HA'} = \frac{BC^2}{AC^2}\,.$$

By Theorem 7.2, $BH = AC$, so by Pythagoras's theorem,

$$BC^2 = BH^2 - HC^2 = AC^2 - HC^2 = (AC - HC)(AC + HC) = AS \cdot HA'\,.$$

Therefore, the length of one-half the latus rectum is

$$HL_1^2 = \frac{BC^4}{AC^2}\,. \quad \blacksquare$$

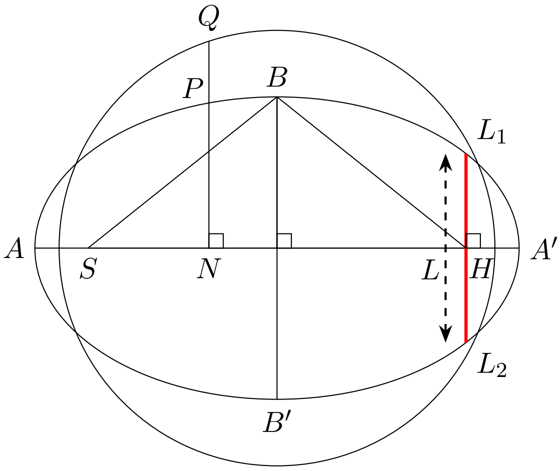


Figure 11.4: The circumscribed circle and the latus rectum of an ellipse

11.5 Areas of parallelograms

Theorem 11.7 *Let Y be the intersection the perpendicular from the focus S to the tangent TT' at P, and let L be the intersection of $S'P$ and SY (Figure 11.5). Then Y is on the circumscribing circle and $CY \parallel S'L$.*

Proof

$\triangle STY \sim \triangle T'TC$ since they are right triangles that share an acute angle, so $\angle CT'T = \angle YST = \beta$. By Theorem 10.4, $\angle SPY = \angle S'PT' = \alpha$ since they are the angles to the foci at the tangent. $\angle S'PT' = \angle YPL = \alpha$ are vertical angles, so $\angle SPY = \angle LPY = \alpha$.

Then $\triangle SPY \cong \triangle LPY$ since they are right triangles with an equal acute angle and a common side PY. Therefore, $PL = PS$ and $S'L = S'P + PL = S'P_PS = AA'$. Since $\triangle SPY \cong \triangle LPY$, $SY = YL$, and since S, S' are foci, $S'C = SC$. It follows that $\triangle CSY \sim \triangle S'SL$

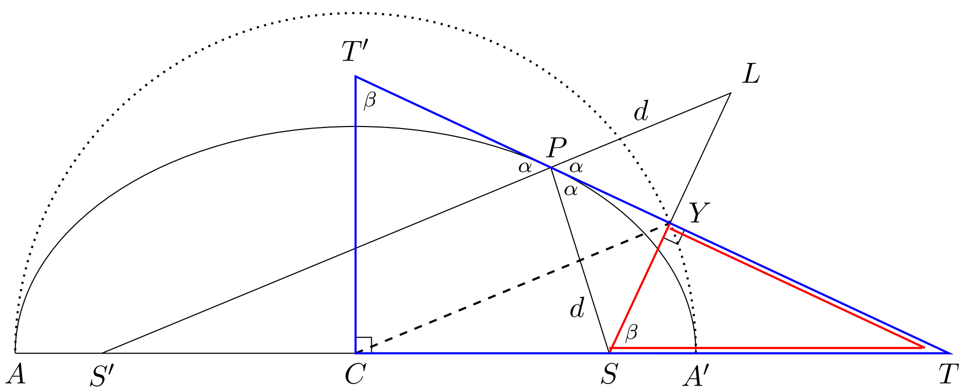


Figure 11.5: The perpendicular from a focus to a tangent

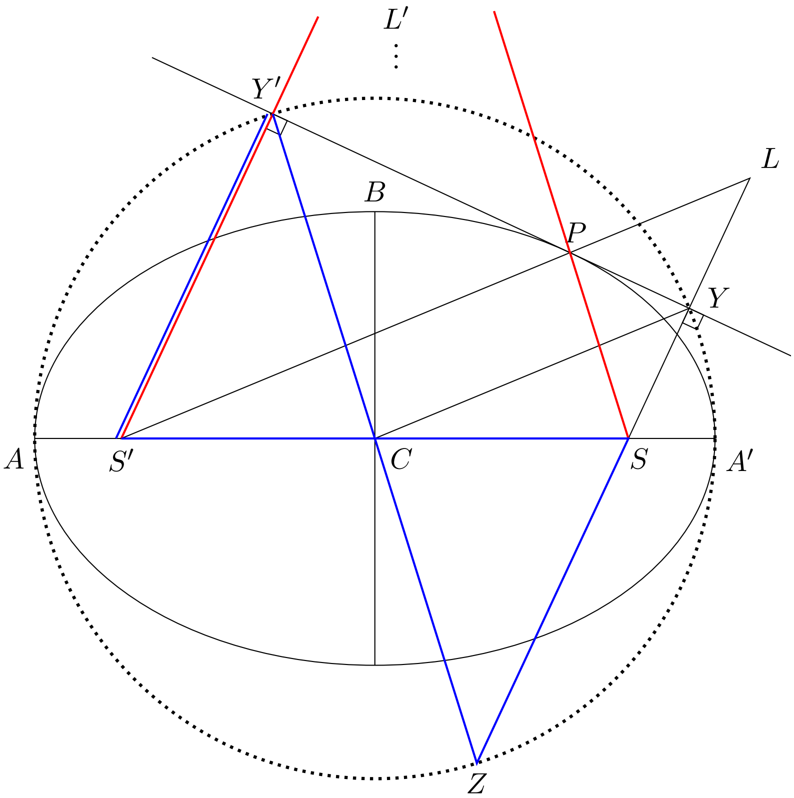


Figure 11.6: Perpendiculars from the foci to the tangent

and $CY \parallel S'L$. By similarity,

$$\frac{CY}{S'L} = \frac{CS}{S'S} = \frac{1}{2} \cdot \frac{CS}{CS} = \frac{1}{2}\,.$$

Therefore, $2CY = S'L = AA'$ so $CY = CA$ and Y is on the circumscribing circle. ∎

Figure 11.6 is based on Figure 11.5 with the addition the perpendicular from focus S' intersecting the tangent at Y' and intersecting the extension of SP at L' (not shown).

Theorem 11.8 $SY \cdot S'Y' = BC^2$.

Proof Theorem 11.7 showed that Y is on the circumscribing circle and the same proof shows that Y' is also on the circle. Extend YS to intersect the circle at Z and connect $Y'Z$. Since $\angle Y'YZ$ is a right angle, $Y'Z$ is a diameter and C is on the line. $\triangle SCZ \cong S'C'Y'$ by side-angle-side so $SY \cdot S'Y' = SY \cdot SZ$. But YZ and AA' are secants intersecting at S, so

$$SY \cdot SZ = AS \cdot SA' = AC^2 - CS^2 = BC^2\,,$$

by Theorems 11.3 and 9.4. ∎

Theorem 11.9 *Let N be the intersection of the perpendicular from P to the major axis (Figure 11.7). Then $CN \cdot NT = AC^2 = AN \cdot NA'$.*

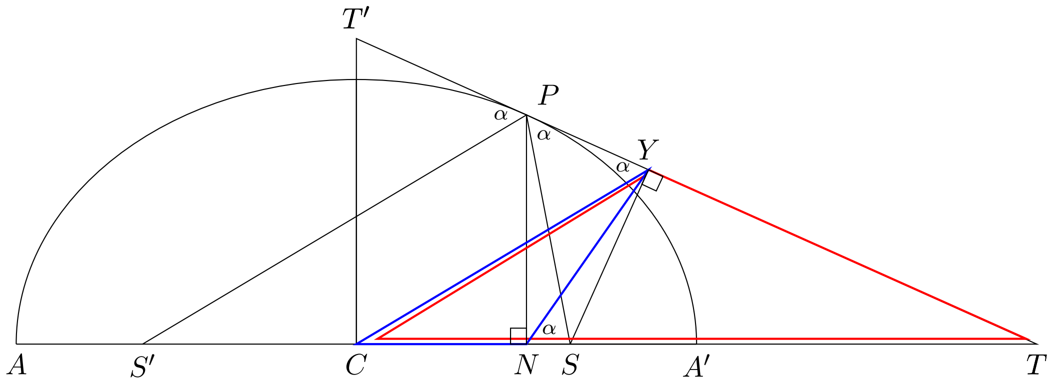


Figure 11.7: Ratios of segments of the major axis

Proof Continuing with the construction from Figure 11.5, we focus on the segments AN, NA' (Figure 11.7).

$CY \parallel S'P$ (Theorem 11.7) so $\angle CYP = \angle S'PT' = \angle SPY$ by corresponding angles and Theorem 10.4. $\angle SYP$ and $\angle SNP$ are right angles and therefore $SYPN$ is quadrilateral that can be circumscribed by a circle whose diameter is PS.[2] Therefore, $\angle SPY = \angle SNY$ since they are subtended by the same chord YS.

Since $\angle CYT \sim \angle CNY$ and $\angle YCT$ is a common angle, $\triangle CYT \sim \triangle CNY$ and

$$\frac{CN}{CY} = \frac{CY}{CT}$$

$$CN \cdot CT = CY^2 = AC^2 \,, \tag{11.7}$$

since the perpendicular to the tangent from a focus is on the circumscribing circle (Theorem 11.7). Since $NT = CT - CN$, we have $CN \cdot NT = CN \cdot CT - CN^2$ which equals $AC^2 - CN^2$ by Equation 11.7. This in turn equals $AN \cdot NA'$ by Theorem 11.3. ∎

Construct the normal to the tangent at P and let its intersection with the conjugate diameter DK be F and its intersection with the major axis be G. Construct a perpendicular from P to the major axis and let its intersection be N. Let the intersection of the tangent with the minor axis be T and its intersection with the major axis be T' (Figure 11.8).

Theorem 11.10 $PF \cdot PG = BC^2$.

Proof $\triangle NPG \sim \triangle FPJ$ so $\angle PGN = \angle PJF = \alpha$ and

$$\frac{PF}{PN} = \frac{PJ}{PG}$$

$$PF \cdot PG = PJ \cdot PN \,. \tag{11.8}$$

By vertical angles $\angle PGN = \angle CGF = \alpha$ so $\triangle NPG \sim \triangle FCG$ and $\angle NPG = \angle FCG = \beta = 90° - \alpha$. By adding β to the right angles $\angle BCN$ and $\angle TPF$, we get that $\angle TCJ = \angle TPJ$

[2] A quadrilateral whose opposite angle are supplementary can be circumscribed by a circle. If two opposite angles are right angles, they sum to $180°$ so the other two angles must also sum to $180°$.

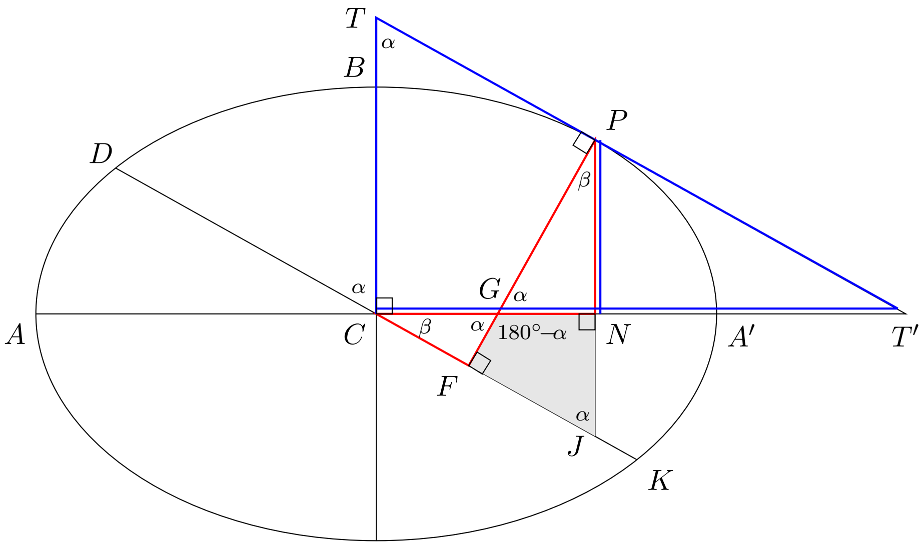


Figure 11.8: Parallelograms formed by conjugate diameters

and therefore $TPJC$ is a parallelogram, so $CT = PJ$ and $PF \cdot PG = PJ \cdot PN = CT \cdot PN$. By Equation 11.8, the theorem will be proven if we can show that $CT \cdot PN = BC^2$.

$\triangle TT'C \sim \triangle PT'N$ so

$$\frac{CT}{CT'} = \frac{PN}{NT'}$$
$$\frac{CT}{PN} = \frac{CT'}{NT'} .$$

Multiplying each side by fractions equal to 1 gives

$$\frac{CT \cdot PN}{PN^2} = \frac{CT' \cdot CN}{CN \cdot NT'} .$$

By Equation 11.7, $CN \cdot CT' = AC^2$, and by Theorem 11.9, $CN \cdot NT' = AN \cdot NA'$, so

$$\frac{CT \cdot PN}{PN^2} = \frac{AC^2}{AN \cdot NA'} .$$

Multiplying the right-hand side by PN^2/PN^2 and use Theorem 11.4 to get

$$\frac{CT \cdot PN}{PN^2} = \frac{PN^2}{AN \cdot NA'} \cdot \frac{AC^2}{PN^2} = \frac{BC^2}{AC^2} \cdot \frac{AC^2}{PN^2}$$
$$CT \cdot PN = BC^2 . \quad \blacksquare \tag{11.9}$$

Let PP', DK be conjugate diameters, let DT', PT be tangents to D, P, respectively, where T', T are their intersections with the major axis. Let DM, PN be perpendiculars to the major axis (Figure 11.9).

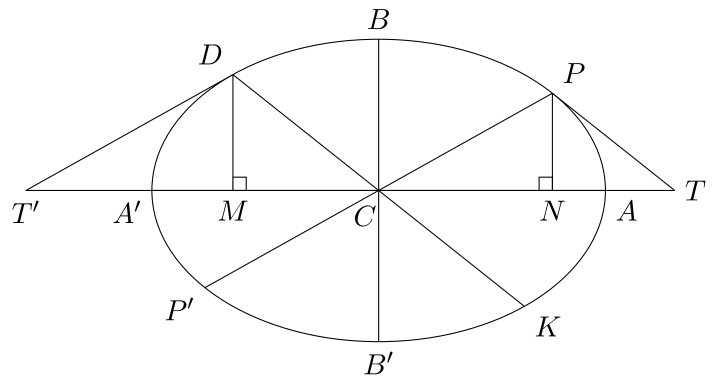


Figure 11.9: Ratios of perpendiculars to the major axis

Theorem 11.11

$$CN^2 = AM \cdot MA' \qquad CM^2 = AN \cdot NA'$$
$$\frac{DM}{CN} = \frac{BC}{AC} \qquad \frac{CM}{PN} = \frac{AC}{BC}\,.$$

Proof By Theorem 11.9,

$$CN \cdot CT = AC^2 = CM \cdot CT'$$
$$\frac{CM}{CN} = \frac{CT}{CT'}\,.$$

Since DK and PP' are conjugate diameters, $DT' \parallel PP'$ and $\triangle T'DC \sim \triangle CPT$, so

$$\frac{CM}{CN} = \frac{CT}{CT'} = \frac{CN}{MT'}$$
$$CN^2 = CM \cdot MT'\,,$$

Therefore,

$$CN^2 = CM \cdot MT' = AC^2 = AM \cdot MA' \tag{11.10}$$

by Theorem 11.9. By Theorem 11.4,

$$\frac{DM^2}{AM \cdot MA'} = \frac{BC^2}{AC^2}\,,$$

and by Equation 11.10,

$$\frac{DM^2}{CN^2} = \frac{BC^2}{AC^2}$$
$$\frac{DM}{CN} = \frac{BC}{AC}\,,$$

A symmetric argument shows that $CM^2 = AN \cdot NA'$ and $CM/PN = BC/AC$. ∎

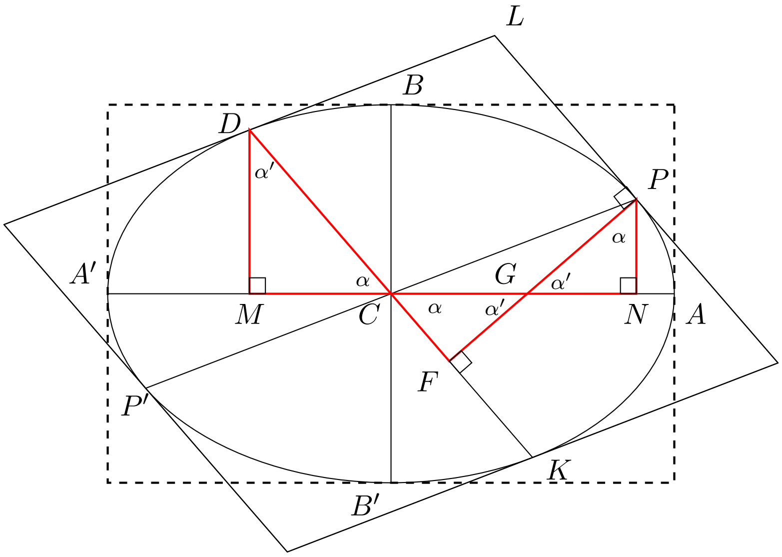


Figure 11.10: Areas of parallelograms ($\alpha' = 90° - \alpha$)

Theorem 11.12 *(Theorem 10.7) The area of the parallelogram formed by the tangents at the ends of the conjugate diameters PP', DK is equal to the area of the rectangle enclosing the ellipse at the ends of the axes (Figure 11.10).*

Proof By the definition of conjugate diameters, it is sufficient to show that the area of $PCDL$ is $AC \cdot BC$. The area of a parallelogram is width times height so we need to show that $CD \cdot PF = AC \cdot BC$.

By vertical angles $\angle DCM = \angle GCF$ and $\angle CGF = \angle PGN$, so $\triangle DCM \sim \triangle PGN$

$$\frac{PG}{CD} = \frac{PN}{CM} = \frac{BC}{AC},$$

by Theorem 11.11. From Theorem 11.10, $PF \cdot PG = BC^2$, giving

$$\frac{BC}{PF} = \frac{PG}{BC} = \frac{CD}{AC}. \quad \blacksquare \tag{11.11}$$

Chapter 12

Constructing an ellipse

There are many methods and tools for drawing ellipses. The simplest method is the *gardener's method* (Figure 12.1): given a, the length of the semi-major axis and two points S, H, the foci, attach a rope of length $2a$ to two stakes at the foci and pull taut with a pencil. As the pencil moves around the foci maintaining tension on the rope, it will trace an ellipse.

This Section presents several more accurate methods for drawing ellipses. First, we review two methods to construct *individual points* on an ellipse (Section 12.1). Section 12.2 describes a *roulette* called the *Tusi couple*. This is followed by four methods of constructing an ellipse with a *glissette*: Section 12.3 on the *Trammel of Archimedes*, Section 12.4 on two articulated glissettes by Frans van Schooten, and Section 12.5 on a triangle glissette. The final Section 12.6 describes how to construct a *confocal* ellipse.

12.1 Constructing individual points on an ellipse

From an arbitrary point on the directrix

Let us repeat without proof the construction for Theorem 9.6. Assume that we are given a, the length of the semi-major axis and c, the distance of a focus from the center. Let E be a point on the directrix and construct lines from E through A and S. The line through S will make an angle α with XS. Construct a line from S at the *same angle* α from ES and let its intersection with EA be P. Then P is a point on the ellipse (Figure 12.2).

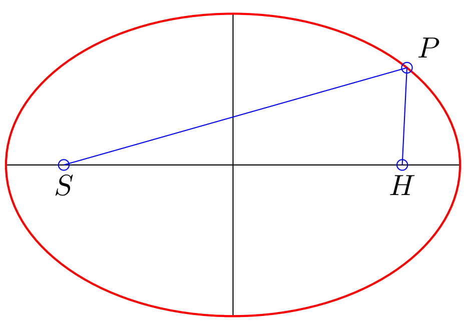


Figure 12.1: The gardener method for drawing an ellipse

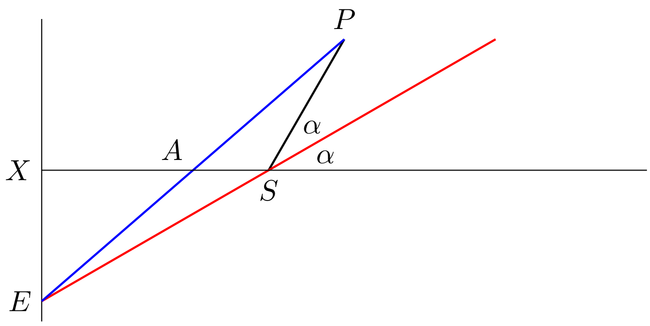


Figure 12.2: Constructing points on the ellipse

From an arbitrary angle to the circumscribed and inscribed circles

Recall the definition of the parametric representation given a and b, the lengths of the semi-axes (Definition 9.7): Construct two concentric circles, one of radius a (dotted red) and one of radius b (dashed blue) (Figure 12.3). For an arbitrary angle t, construct a ray that intersects the two circles at P_O and P_I. Draw a perpendicular from P_I to the y-axis and a perpendicular from P_O to the x-axis. Their intersection P is a point on the ellipse.

12.2 A roulette for drawing an ellipse—the Tusi couple

A *roulette* is a curve generated by one curve c_1 *rolling* on another curve c_2. An ellipse can be generated by a circle of radius r rotating within the circumference of a circle of radius $2r$.

Theorem 12.1 *Let P be an arbitrary point within with circle c_1 of the roulette. Then as c_1 rotates within c_2, the locus of P is an ellipse (Figure 12.4).*

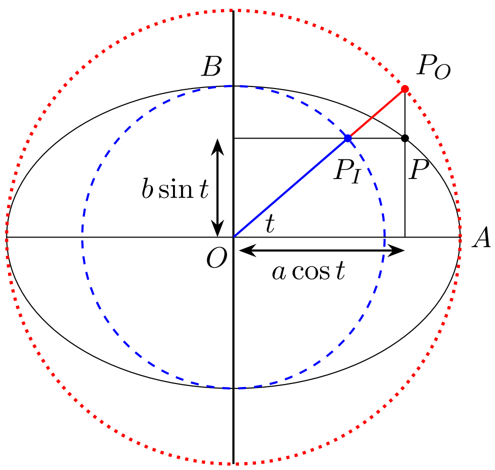


Figure 12.3: Parametric representation of an ellipse

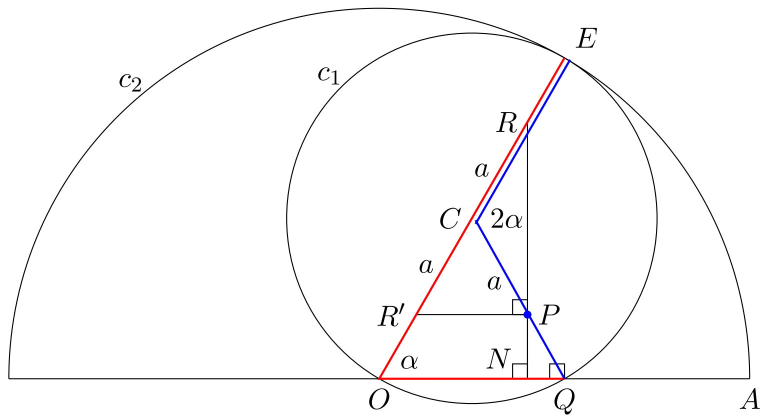


Figure 12.4: Constructing an ellipse from a roulette

Proof Let C be the center of c_1 and let O be the center of c_2. Let E be an arbitrary point on c_2 where it is contacted by c_1. The radius OE of length $2r$ is a chord of c_1 and since it equals twice the radius of c_1 it is a diameter. c_1 will intersect OA at some point Q and since OE is a diameter of c_1, $\angle EQO$ is a right angle.

$\angle ECQ = 2 \cdot \angle EOQ = 2\alpha$ because $\angle ECQ$ is a central angle of c_1 subtended by EQ which also subtends the inscribed angle $\angle EOQ$. Therefore the arc $\widehat{EQ}$ equals $2\alpha r$. But $\angle EOQ$ is the same angle as EOA and is therefore an inscribed angle of c_2. It follows that the arc $\widehat{EA}$ equals $2r\alpha$ and $\widehat{EA} = \widehat{EQ}$, so as c_1 rotates, Q is always on the diameter.

Let P be an arbitrary point on CQ and construct $RP \parallel EQ$ and $R'P \parallel OQ$. Let N be the intersection of RP with OA. Since $CE = CQ$ are radii of c_1, by $RP \parallel EQ$, $\triangle PCR \sim \triangle QCE$ so $\triangle PCR$ is isosceles and $CP = CR = a$. Similarly, $\triangle PCR'$ is isosceles and $CP = CR' = a$.

Now let c_1 rotate within c_2. Since P is fixed relative to C and E is fixed relative to O, $OR = r + a$ and $OR' = r - a$ are constant and their loci are circles. From $R'P \parallel ON$ we get

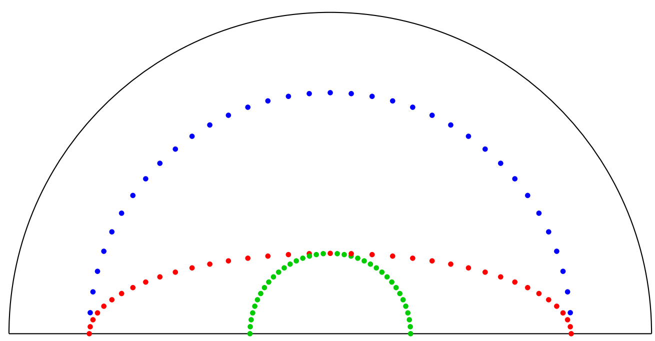


Figure 12.5: The loci of P (red), R (blue), R' (green)

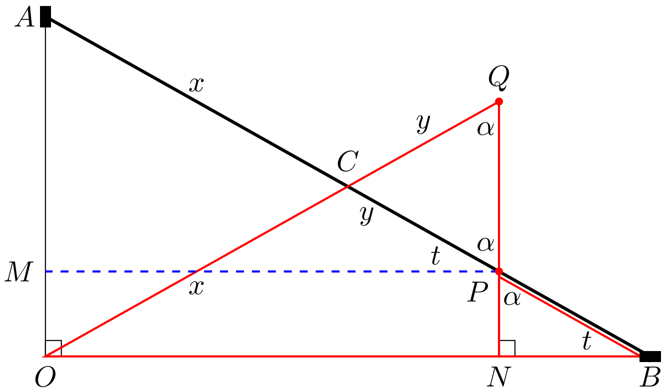


Figure 12.6: Constructing an ellipse from the glissette AB

$\triangle RPR' \sim \triangle RNO$ and

$$\frac{PN}{RN} = \frac{OR'}{OR} = \frac{PQ}{OR},$$

since $OR' = r - a = PQ$. By Theorem 11.5, the locus of P is an ellipse with OR the semi-major axis and $OR' = PQ$ the semi-minor axis. ∎

Figure 12.5 shows traces of P, R, R'.

12.3 A glissette for drawing an ellipse—the trammel of Archimedes

A *glissette* is a curve generated by one curve c_1 *sliding* on another curve c_2. In Figure 12.6, the line segment AB is constrained to move so that A slides on OA and B slides on OB where $OA \perp OB$. Let: P be an arbitrary point on AB, C be the bisector of AB, PN the perpendicular to OB, and Q intersection of PN with OC. As A slides on OA and B on OB, the locus of Q is a circle and the locus of P is an ellipse.

Since OC is the median to the hypotenuse of a right triangle, $AC = CB = OC = x$ and therefore $\angle COA = \angle CAO$. $\triangle ACO \sim \triangle PCQ$ since $AO \parallel QP$, so $CP = CQ = y, OQ = AP = x + y$. P is fixed (on AB) so the locus of Q is a circle.

Since $\triangle PCQ$ is isosceles, $\angle CPQ = \angle CQP = \alpha$ and $\angle CPQ = \angle BPN = \alpha$ by vertical angles. Therefore, $\triangle PBN \sim \triangle QON$ and

$$\frac{PN}{QN} = \frac{PB}{OQ} = \frac{PB}{AP}.$$

By Theorem 11.5, the locus of P is an ellipse with $AP = x + y$ the semi-major axis and $BP = x - y$ the semi-minor axis.

Figure 12.7 shows the loci of P and Q for $AB = 10$.

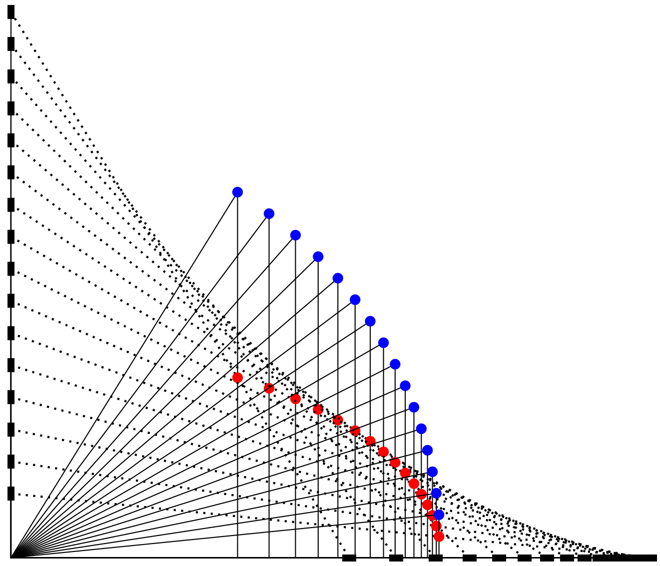
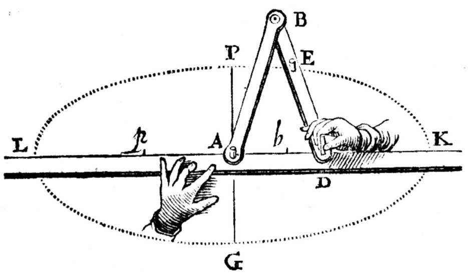
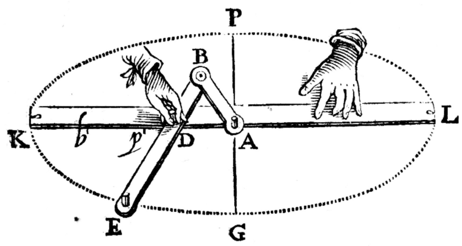


Figure 12.7: The circle (Q blue) and the ellipse (P red)

12.4 Articulated glissettes

The following images of the glissettes are taken from [12], a website devoted to the work of Frans van Schooten.





First glissette

Given a, b let ABD be an articulated arm of length $a + b$ with a rotating joint at its midpoint B. D can slide on the x-axis AK while A can rotate at the origin (Figure 12.8). Let E be at a fixed distance b from D. Using parametric equations, we have that the locus of E is an ellipse.

$$\begin{aligned}PE &= \left[\frac{a+b}{2} + \left(\frac{a+b}{2} - b\right)\right] \cdot \cos\alpha = a\cos\alpha \\ RE &= b\sin\alpha\,.\end{aligned}$$

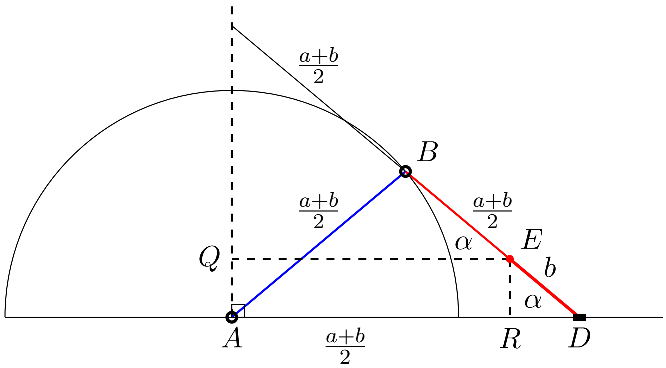


Figure 12.8: Constructing an ellipse with an articulated glissette (1)

Second glissette

Given a, b let ABE be an articulated arm of length $a+b$ with a rotating joint at B, where $AB = a, BE = b$. D, the intersection of BE with the x-axis, can slide on the x-axis while A can rotate at the origin (Figure 12.9). The locus of E is an ellipse. Since $AB = AD = a$, $\angle BAD = \angle BDA = \alpha$ and $\angle NDE = \alpha$ by vertical angles. Therefore, the coordinates of $E = (x, y)$ are

$$\begin{aligned} E &= (-2a\cos\alpha - (b-a)\cos\alpha, (b-a)\sin\alpha) \\ &= (-(a+b)\cos\alpha, b\sin\alpha)\,, \end{aligned}$$

and a simple computation shows that

$$\frac{x^2}{(a+b)^2} + \frac{y^2}{(b-a)^2} = 1\,,$$

so E is a point on an ellipse with semi-major axis $a+b$ and semi-minor axis $b-a$.

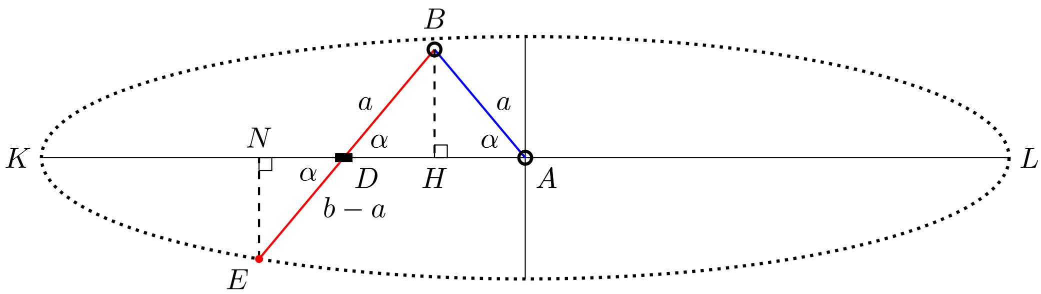


Figure 12.9: Constructing an ellipse with an articulated glissette (2)

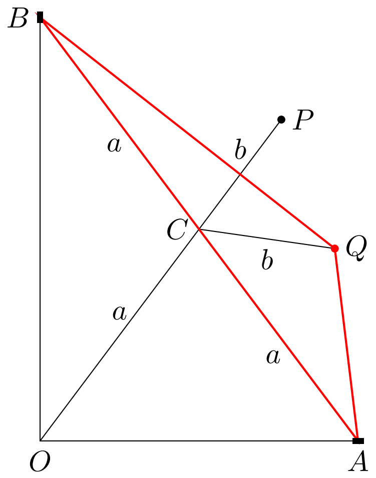


Figure 12.10: Constructing an ellipse from the glissette triangle (red)

12.5 A triangle glissette

An ellipse can be drawing by sliding a given triangle $\triangle AQB$ along perpendicular lines OA, OB (Figure 12.10). Let OC be the midpoint of AB and extend OC so that $CP = CQ = b$. Since $\triangle AOB$ is a right triangle, $OC = AC = BC = a$. The locus of P is a circle with radius $OC + CP = OC + CQ = a + b$.

Figure 12.11 shows that as B slides down and A slides right, CQ rotates upwards while OC rotates downwards. By considering the extremes (A near O and B near O), it is clear that OCQ will be "concave" up at one extreme and "concave" down at the other. Therefore, there must be a position of AB which OCQ is a line segment (green).

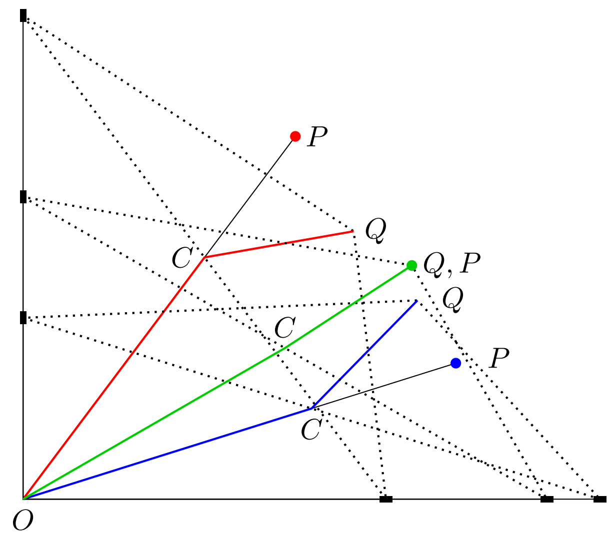


Figure 12.11: There are A, B such that OCQ is a line segment (green)

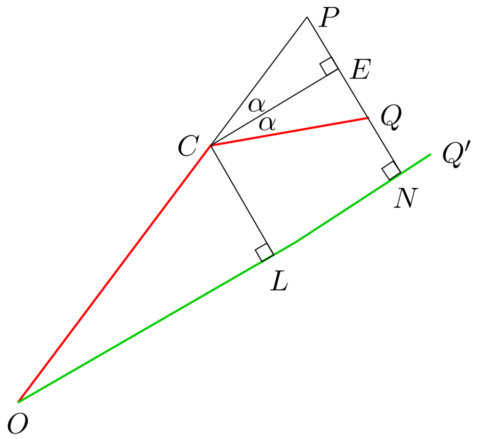


Figure 12.12: OQ' is the line segment and C, Q are for another arbitrary position of AB

Referring now to Figure 12.12, let OQ' is the straight line segment and let C, Q, P are points generated by another arbitrary position of AB. Construct $CE \parallel OQ'$ which bisects $\angle PCQ$ since $CP = CQ$, and construct $PN \perp OQ'$ and $CL \perp OQ'$.

From $\triangle OCL \sim \triangle CPE$ we have

$$\begin{aligned}
\frac{CL}{PE} &= \frac{OC}{CP} \\
\frac{CL}{PE} - \frac{PE}{PE} &= \frac{OC}{CP} - \frac{CP}{CP} \\
\frac{CL - PE}{PE} &= \frac{OC - CP}{CP},
\end{aligned}$$

and similarly,

$$\frac{CL + PE}{PE} = \frac{OC + CP}{CP}.$$

Now,

$$\begin{aligned}
QN &= EN - EQ = CL - PE \\
PN &= EN + PE = CL + PE \\
\frac{QN}{PN} &= \frac{CL - PE}{CL + PE} = \frac{OC - CP}{OC + CP}.
\end{aligned}$$

By Theorem 11.5, the locus of Q is an ellipse with $OC + CQ = a + b$ the semi-major axis and $OC - CQ = a - b$ the semi-minor axis.

12.6 Confocal ellipses

Confocal ellipses are ellipses with the same foci. Given an ellipse E, a confocal ellipse can be drawn using a modification of the gardener method. If the length of the major axis of E is $2a$, take a rope of length greater than $2a$, wrap it around E and pull taut with a pencil. As you move around E, the pencil will trace a confocal ellipse (Figure 12.13).

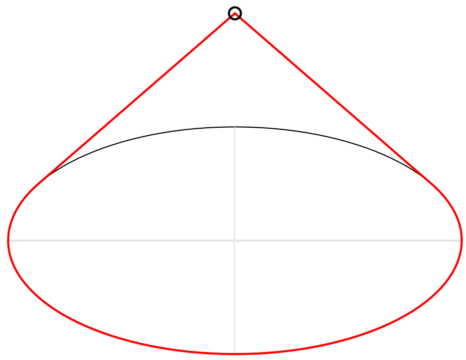


Figure 12.13: Gardner's method for constructing a confocal ellipse

To prove that the locus is an ellipse, let the tip of the pencil be at T and very soon after let it be at t.[1] The string forms two pairs of tangents PT, TQ and pt, tq. Now $PT \approx pt$ so

$$\begin{aligned} PT &\approx \widehat{Pp} + PV \approx \widehat{Pp} + pt - tV \\ tV &\approx \widehat{Pp} + pt - PT\,. \end{aligned}$$

Similarly,

$$\begin{aligned} tq &\approx \widehat{Qq} + QT - Ty \\ Ty &\approx \widehat{Qq} + QT - tq\,. \end{aligned}$$

Since the length of the string is fixed, $PT + TQ = pt + tq$ and $\widehat{Pp} \approx \widehat{Qq}$. Substituting into the above equations gives

$$tV = \widehat{Pp} + pt - PT \approx \widehat{Pp} + TQ - tq \approx \widehat{Qq} + QT - tq \approx Ty\,.$$

Consider now Figure 12.15 which is a detail from Firgure 12.14. The angles α are vertical angles and we showed that $TV \approx ty$ so $\triangle TVW \cong \triangle tyW$. Therefore, $\triangle TVt \cong tyT$ and

[1]The proof is from the solution for problem 46 in the 1890 edition of Besant `https://archive.org/details/solutionsexampl00besagoog`, with details supplied by "Intelligenti pauca" at `https://math.stackexchange.com/questions/4908340/besants-proof-of-graves-theorem`.

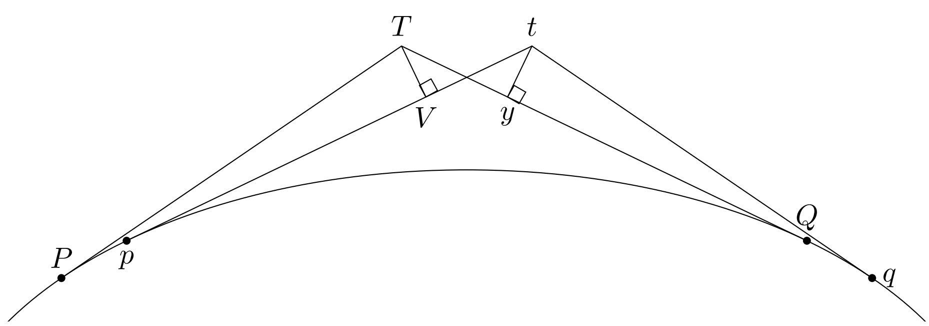


Figure 12.14: Constructing a confocal ellipse

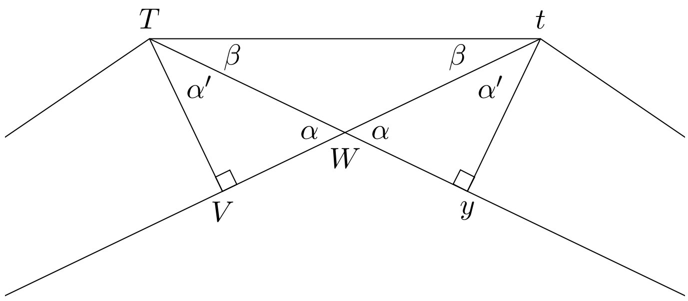


Figure 12.15: Detail of the proof for a confocal ellipse

$\angle TtW = \angle tTW = \beta$. What happens as t approaches T each other? tT becomes the “left” part of the tangent at T and Tt becomes the “right” part of the tangent. It follows by Theorem 10.4 that there is a tangent of an ellipse at T.

Chapter A

Theorems of Euclidean geometry

A.1 Constructing a circle from three points

Theorem A.1 *Given three non-collinear points a circle can be constructed that goes through all three points.*

Proof Three non-collinear points A, B, C define a triangle $\triangle ABC$ (Figure A.1). Construct the perpendicular bisectors of any two of its three sides, say, AC and BC. By definition the perpendicular bisector is the locus of points equidistant from the endpoints of the segment. Let O be the intersection of the two bisectors. Then the $AO = CO = BO$ is the radius of a circle centered at O that goes through A, B, C. ∎

Theorem A.2 *Let Q be a point on a circle whose diameter is AA' and construct a perpendicular QN to the diameter (Figure A.2). Then*

$$QN^2 = AN \cdot NA' .$$

The equation also holds if it is given *that $\triangle AQA'$ is a right triangle.*

Proof An angle that subtends a diameter is a right angle. Since the sum of the angles of a triangle is 180°, we can label the angles as shown in the Figure, from which follows that $\triangle QNA \sim \triangle A'NQ$. Therefore,

$$\frac{QN}{AN} = \frac{NA'}{QN} . \quad ∎$$

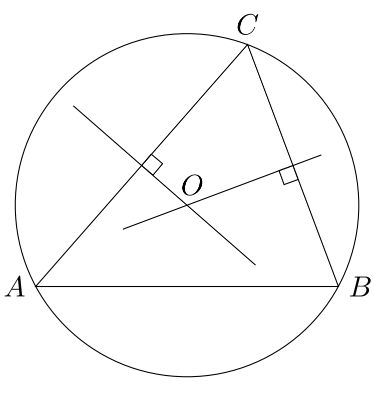


Figure A.1: A circle through three points

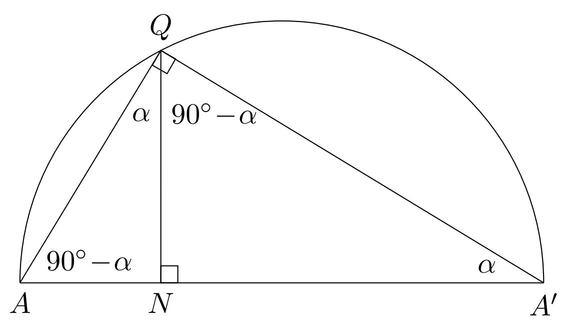


Figure A.2: Right triangle in a circle

A.2 Adjacent pairs of similar triangles

Definition A.3 *An* adjacent pair of similar triangles *is a pair of similar triangles that share sides. In Figure A.3, $\triangle BAC \sim \triangle EAF$ and $\triangle CAD \sim \triangle FAG$ are an adjacent pair of similar triangles.*

Theorem A.4 *For the adjacent pair of similar triangles in Figure A.3,*

$$\frac{AB}{AE} = \frac{AD}{AG}.$$

Proof By similar triangles,

$$\frac{AB}{AE} = \frac{AC}{AF} = \frac{AD}{AG}. \quad \blacksquare$$

Similar ratios hold between other sides of $\triangle BAC$ and $\triangle CAD$ by using an intermediate step with AC. We will use the term *by an adjacent pair of similar triangles* and leave it to the reader to make the intermediate step.

A.3 The angle bisector theorems

Theorem A.5 (Interior angle bisector theorem) *In $\triangle ABC$ let D be a point on BC (Figure A.4). Then AD bisects $\angle CAB$* if and only if

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Proof Suppose that AD bisects $\angle BAC$. Construct a line through C parallel to AB and let its intersection with AD be E. By alternate interior angles, $\angle BAD = \angle CED =$ and by vertical angles $\angle BDA = \angle CDE$. Therefore, $\triangle ABD \sim \triangle EDC$ so

$$\frac{BD}{CD} = \frac{AB}{CE}.$$

$\triangle ECA$ is isosceles so $CE = AC$ and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just "run" the proof backwards. $\blacksquare$

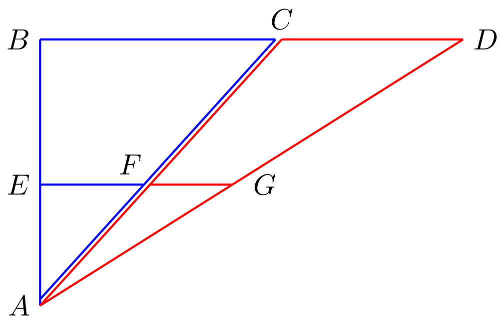


Figure A.3: Adjacent pairs of similar triangles

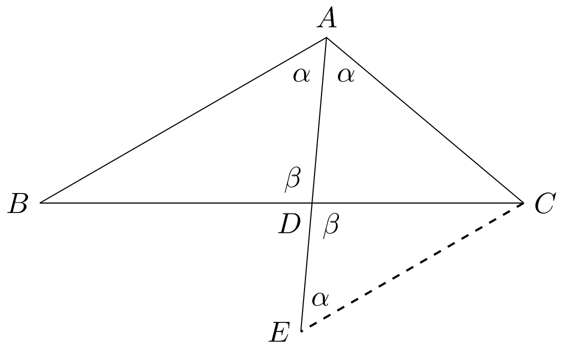


Figure A.4: The interior angle bisector theorem

Theorem A.6 (Exterior angle bisector theorem) *In $\triangle ABC$ let D be a point on the extension of CB (Figure A.5). Then AD bisects the exterior angle of $\angle BAC$ if and only if*

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Proof Suppose that AD bisects $\angle BAF$. Construct a line through B parallel to AD and let its intersection with AC be E. By alternate interior angles $\angle BAD = \angle ABE$ and by corresponding angles $\angle FAD = \angle AEB$. Therefore, $\triangle BCE \sim \triangle DCA$ so

$$\frac{BD}{CD} = \frac{AE}{AC}.$$

But $\triangle BAE$ is isosceles so $AE = AB$ and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just "run" the proof backwards. ∎

The exterior angle bisector theorem can be confusing to understand in a proof, because it can be hard to identify the components of a diagram. In the text the following color-coding is used: the triangle is red, the extension of one side is dashed red and the bisector is blue.

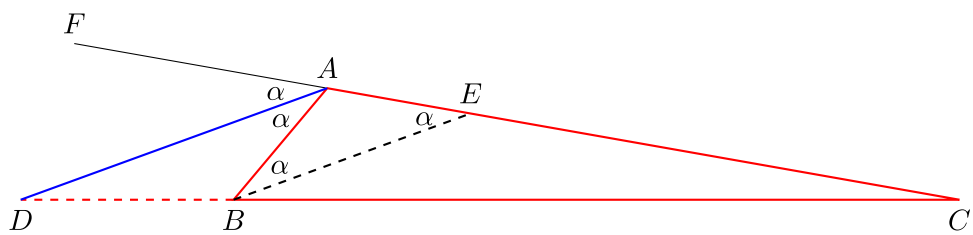


Figure A.5: The exterior angle bisector theorem

Sources and further reading

Sections 2–5 are based primarily on Hahn's book [8]. He has written a more advanced book on the orbits of planets and spacecraft [9]. The analytic proof in Chapter 6 is from [7]. The computations of the Lagrange points are from [17]. Other expositions of Newton's work on orbits can be found in [11, 16].

The presentation of the Euclidean geometry in Chapter 11 is based on Besant's textbook [3]. Drew's (shorter) textbook [5] is similar, while Smith's textbook [15] uses analytic geometry. Feynman's proof is found in [6] and Maxwell's is in [14, Article CXXXIII]. Hodographs are discussed in [1].

Chapter 12 is based on [3, Chapter X] and on [13]. Besant wrote an extended presentation on roulettes and glissettes [2].

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