

# The Mathematics of Planetary Orbits

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# 1 Introduction

Everyone “knows” that Kepler discovered that the orbits of the planets are ellipses and that Newton showed that an elliptical orbit implies that the force of gravity must be inversely proportional to the square of the distance from the Sun. Although I knew these facts, I had never seen them demonstrated.

*Calculus in Context* [4] by Alexander J. Hahn is a comprehensive textbook on introductory calculus that augments theory with applications in physics and astronomy, such as the work of Kepler, Newton and Galileo, as well as applications in engineering such as building bridges and domed structures. These are not just historical anecdotes but detailed computations.

This document works out one topic based on the presentation in the book: the determination of orbits by Aristarchus, Copernicus, Kepler and Newton. The presentation is mathematical, since the historical and astronomical aspects are thoroughly described in [4], as well as in other works. The document is intended to enrich the learning of mathematics by secondary-school students and students in introductory university courses.

Section 2 presents the measurements of the radii and distances of the Earth, the Moon and the Sun by Eratosthenes and Aristarchus. Section 3 describes the construction of a model of a Sun-centered system by Copernicus. Section 4 shows how Kepler developed his three laws of planetary motion and Section 5 presents Newton’s derivation of the inverse-square law of gravitation from of Kepler’s laws. One step of Newton’s derivation requires a theorem whose proof is very long, so it is split off into Section 6. Appendix A contains more than you ever wanted to know about the mathematics of ellipses, but these theorems are necessary. I suggest that you look up each theorem (and its proof) as needed, rather than trying to study them all at once.

The computations in Hahn’s book are faithful to the historical record, for example, measuring distances in units such as the stadia of the Greeks. Here, the computations are fully modernized and use modern units such as kilometers.

It is easy to measure angles. We are familiar with the use of a protractor<sup>1</sup> in school and these can be scaled-up to obtain more accurate measurements. Measuring long distances was impossible until the recent inventions of radar and lasers. At most one could pace-off distances with low accuracy. The only *measured* distance used here is the estimate of 800 km by Eratosthenes for the distance between two places in Egypt (Section 2.1).

This documents uses many diagrams to facilitate understanding each step in the geometrical proofs, more than appear in other sources. For example, Newton proved his difficult theorem (Section 6) using only a single diagram. The diagrams are not to scale and are often distorted. Otherwise, it would be impossible to draw the Earth, Moon and Sun on a piece of paper, since the Sun is almost 400 times farther away from the Earth than the Moon. If we drew the Moon one centimeter from the Earth, we would have to draw the Sun four meters away!

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<sup>1</sup>More precisely, a device for measuring angles is called a *goniometer*.

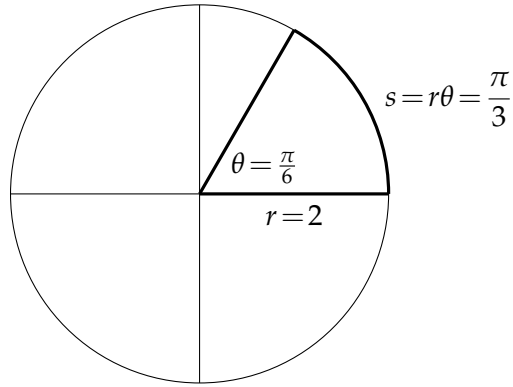


Figure 1: Angles and arcs

The prerequisites for reading the document are a good knowledge of Euclidean geometry, some trigonometry, a bit calculus and Newton's laws of motions. In particular you must know that if a sector of a circle subtends an angle of  $\theta$  radians, the length of the sector is  $r\theta$ , where  $r$  is the radius of the circle (Figure 1).

### Notation

The following shortcuts facilitate a less verbose presentation:

- $a$  and  $b$  always denote the semi-major and semi-minor axes of an ellipse.
- $AB$  denotes both a line segment and its length.
- $\triangle ABC$  denotes both a triangle and its area.

### Acknowledgments

I am grateful to Alex Hahn and Graham Griffiths for their guidance.

## 2 The sizes of the Earth, Moon and Sun

### 2.1 Eratosthenes' measurement of the radius of the earth

The ancient Greeks knew that the Earth is round and Eratosthenes was able to measure the radius of the Earth (Figure 2). Choose two points  $A, B$  on the same longitude and measure the distance  $d$  between them. Plant a vertical stick (red) in the ground at  $A$  and another (blue) at  $B$ . On a day in the year when the stick at  $A$  produces no shadow at noon, at the same time the stick at  $B$  produces a shadow whose angle is  $\alpha$ . The sun is so far away from the Earth that over the relatively short distance  $d$ , the rays of the Sun are essentially parallel. By alternate interior angles, the angle between the two sticks as measured from the center of the Earth is also  $\alpha$ .

The angle that Eratosthenes measured at the blue stick was

$$\alpha = 7.5^\circ \cdot \frac{2\pi}{360} \approx 0.131 \text{ radians},$$

and the distance  $d$  between  $A$  and  $B$  was known to be approximately 800 km. The arc  $\widehat{AB}$  subtends the angle  $\alpha = d/r_e$  where  $r_e$  is the radius of the Earth, so

$$r_e = \frac{d}{\alpha} = \frac{800}{0.131} \approx 6107 \text{ km}. \quad (1)$$

This value is quite close to the modern measurement of 6370 km.

### 2.2 Aristarchus' measurements

Using  $r_e$ , Eratosthenes' measurement of the radius of the Earth, Aristarchus was able to measure and compute the following values:

- $r_m$ : the radius of the Moon,
- $r_s$ : the radius of the Sun,
- $d_m$ : the distance from the Earth to the Moon,
- $d_s$ : the distance from the Earth to the Sun.

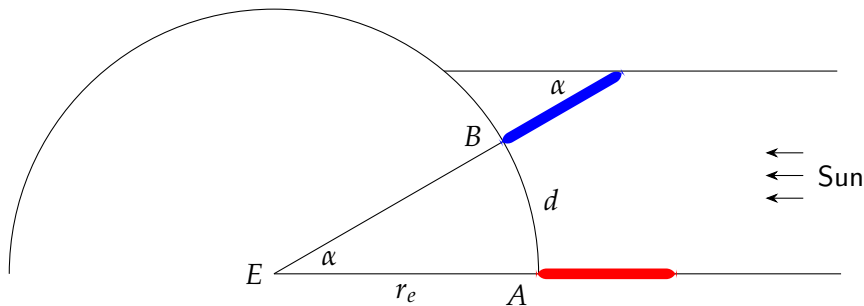


Figure 2: Eratosthenes' measurement of the radius of the earth

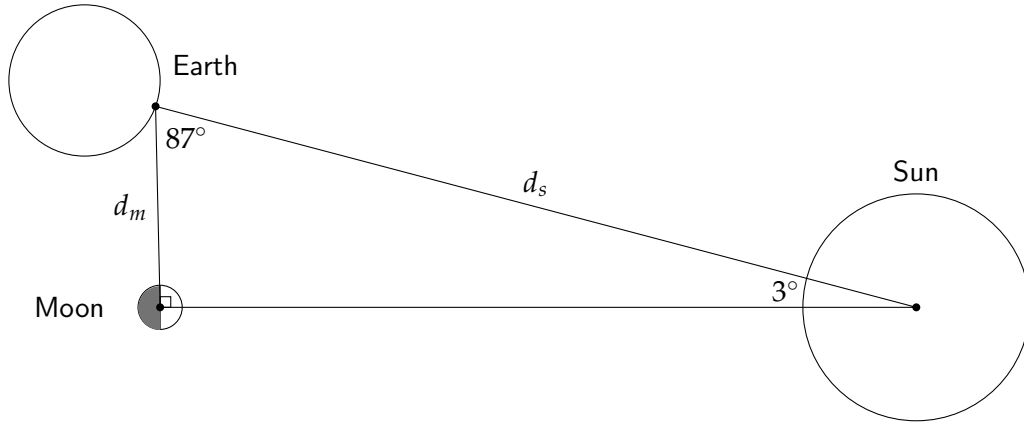


Figure 3: Observing a first quarter moon

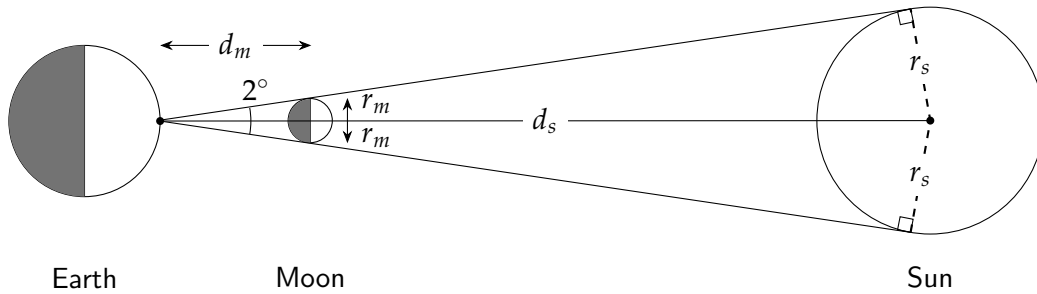


Figure 4: A solar eclipse

### Computing $d_s/d_m$

An observer on Earth can follow the phases of the Moon as it revolves around the Earth. At one point in the month the phase will be first quarter, meaning that the one half of the moon is illuminated while the other half is not (Figure 3). The angle between the Sun and the Moon will be  $87^\circ$ . Since exactly half of the moon is illuminated, we know that the angle  $\angle$  Earth-Moon-Sun is a right-angle so

$$\begin{aligned}\cos 87^\circ &= \frac{d_m}{d_s} \\ \frac{d_s}{d_m} &= \frac{1}{\cos 87^\circ} \approx 19.\end{aligned}\tag{2}$$

### Computing $r_s/r_m$ and $d_m/r_m$

The Moon is much, much smaller than the Sun, but it is also much, much closer to the Earth. When the Moon is precisely positioned between the Earth and the Sun, its “disk” exactly covers the “disk” of the Sun, causing a total solar eclipse (Figure 4).

The angle subtended by the Moon is  $2^\circ$  degrees. Bisecting the angle creates two right triangles with an acute angle of  $1^\circ$ , where the right angles are the tangents to Moon and the Sun. By similar triangles, Equation 2 and Figure 4,

$$\frac{r_s}{r_m} = \frac{d_s}{d_m} = 19 \quad (3)$$

$$\frac{d_m}{r_m} = \frac{1}{r_m/d_m} = \frac{1}{\sin 1^\circ} \approx 57. \quad (4)$$

### Computing the radii and distances

Figure 5 shows a lunar eclipse. Unlike a solar eclipse where the Moon exactly covers the Sun, the Earth more than covers the Moon and its shadow is four times the Moon's radius.

Figure 6 show a lunar eclipse annotated with the distances  $d_m, d_s$  and the radii  $r_m, r_e, r_s$ . The ray from the top of the Sun is tangent to both the Sun and the Earth, so it forms right angles with their radii, as well as with the extension of the Moon's radius. The thick horizontal lines are constructed parallel to the line connecting the centers, forming two right triangles. The triangles are similar so using Equation 3,

$$\frac{r_s - r_e}{r_e - 2r_m} = \frac{d_s}{d_m} = \frac{r_s}{r_m}$$

$$r_s r_e + r_m r_e = 3r_s r_m.$$

Again from Equation 3,  $r_s = 19r_m$ , so

$$r_m = \frac{20}{57} r_e.$$

By Equation 1,  $r_e \approx 6107$  km, by Equation 4,  $d_m = 57r_m$ , and by Equation 2,  $d_s = 19d_m$ , so we can compute the radii and distances:

$$r_m = \frac{20}{57} r_e \approx 2143 \text{ km}$$

$$r_s = 19r_m \approx 40,713 \text{ km}$$

$$d_m = 57r_m \approx 122,140 \text{ km}$$

$$d_s = 19d_m \approx 2,320,660 \text{ km}.$$

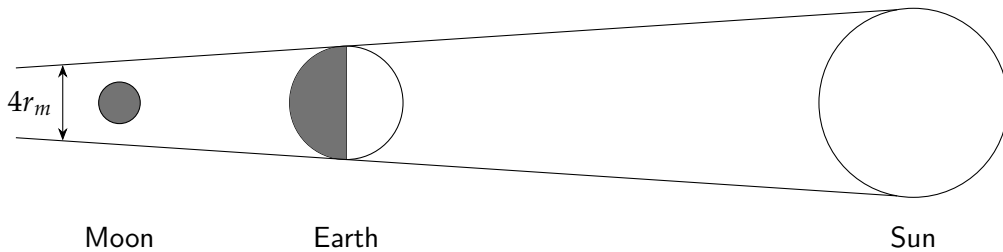


Figure 5: A lunar eclipse



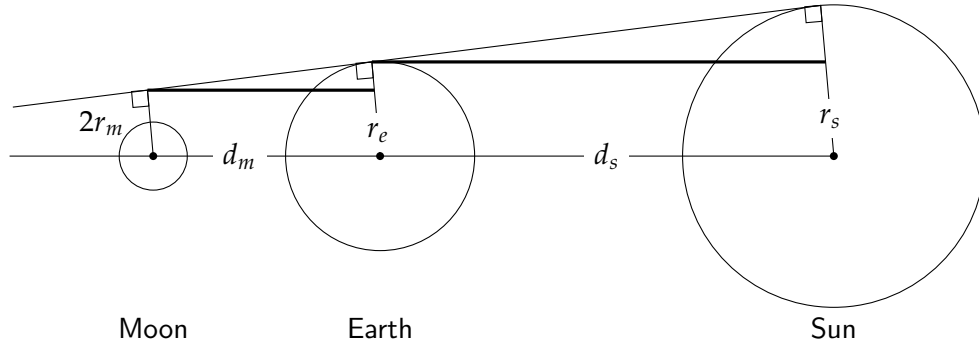


Figure 6: Detail of a lunar eclipse

Table 1 summarizes these data together with the modern values [4, Table 1.3]. While the computed values for the radii of the Earth and the Moon are not far off from the modern values, the other computed values are not near the modern values. Nevertheless, they do show that the Greeks understood the immense size of the solar system.

		Computed (km)	Modern (km)
$r_e$	radius of Earth	6107	6370
$r_m$	radius of Moon	2143	1740
$r_s$	radius of Sun	40,713	695,500
$d_m$	distance Earth-Moon	122,140	384,570
$d_s$	distance Earth-Sun	2,320,660	150,000,000

Table 1: Values computed by Aristarchus' method compared with modern values

### 3 The Sun-centered solar system

As everyone living far from equator knows, the time between sunrise and sunset varies with the seasons. The reason is that the axis of the rotation of the Earth is offset by  $23.5^\circ$  relative to the orbit of the Earth. The plane of the orbit of the Earth around the Sun is called the *ecliptic*. Measuring the length of the day as the time from sunrise to sunset, there is a day in June, called the *summer solstice*, when the length of the day is longest. Similarly, there is a day in December, called the *winter solstice*, when the length of the day is shortest.<sup>2</sup> There are also two days when the length of the day equals the length of the night: the *autumn equinox* in September and the *spring equinox* in March.

Today we know that the universe is immensely large and that the stars are moving at extremely high speeds, but an observer on Earth sees them as if their positions are fixed on a sphere around the earth, called the *celestial sphere*. This solstices and equinoxes can be associated with the projection of the Sun on the celestial sphere as seen from the Earth. The details of the Earth's orbit can be found in books on astronomy, as well as in the Wikipedia articles on *Equinox* and *Solstice*.

#### 3.1 The length of the year and the length of the seasons

Let us assume that the Earth orbits the Sun in a circle, such that the center of the orbit  $O$  is the center of the Sun  $S$ . In Figure 7 the inner circle is orbit of the Earth and the outer circle is the celestial sphere. The orbit can be divided into four quadrants called *seasons*: spring, summer, autumn, winter.

The length of a year is approximately  $365\frac{1}{4}$  days. The extra  $\frac{1}{4}$  day is accounted for by adding a day in leap years.<sup>3</sup> The length of each season as determined by the equinoxes and the solstices is  $365.25/4 = 91.3125$  days. However, measurements by the Greek astronomer Hipparchus showed that the actual lengths of the seasons differed from this number (Table 2) and a model of the solar system must be able to explain these differences.

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<sup>2</sup>This holds for the northern hemisphere; in the southern hemisphere the opposite holds.

<sup>3</sup>The length of a year is actually 365.2425. In the sixteenth century, the *Gregorian calendar* accounted for the difference by removing three leap years in every four hundred years.

Season	Days	%
Spring	$94\frac{1}{4}$	25.8
Summer	$92\frac{1}{2}$	25.3
Autumn	$88\frac{1}{8}$	24.1
Winter	$90\frac{1}{8}$	24.7

Table 2: The lengths of the seasons and their percentages of a year

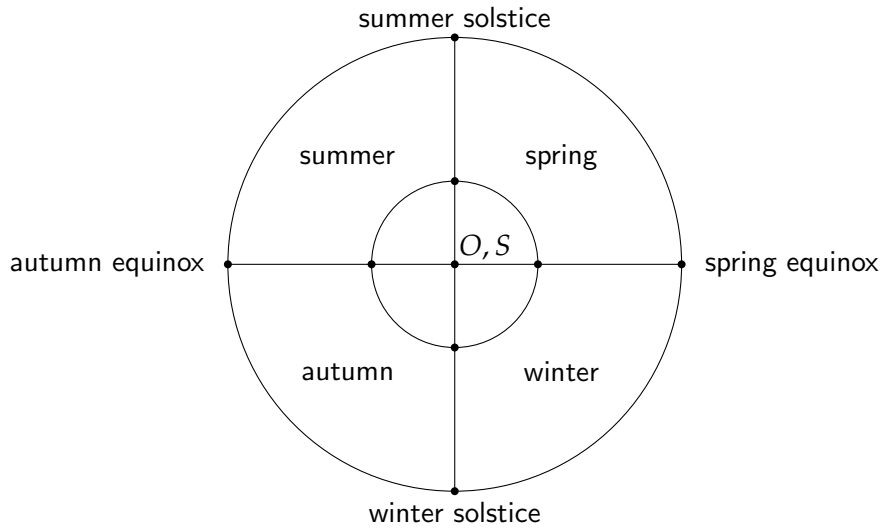


Figure 7: The orbit of the Earth and the seasons

In his Earth-centered solar system, Hipparchus proposed that the center of the Earth's orbit be offset from the center of the Sun. Copernicus used the same idea in his Sun-centered solar system (Figure 8). If the center of the Earth's orbit is in the upper-left quadrant of the coordinate system defined by the equinoxes and the solstices, the angles for spring and summer are obtuse, so the seasons are longer than one-fourth of a year, whereas the angles for the autumn and winter are acute, so the they are shorter than one-fourth of a year.

Figure 9 shows a magnified and distorted view of Figure 8. It has been annotated with

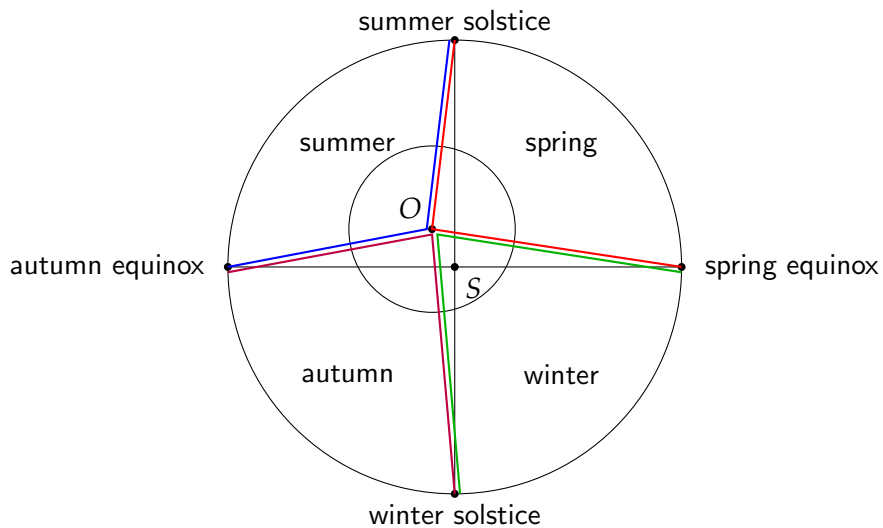


Figure 8: The lengths of the seasons are not equal

additional lines and labels that will facilitate the demonstration of Copernicus' computation. The axes  $A'C'$  and  $B'D'$  have their origin  $O$  at the center of the Earth's orbit and are parallel to the axes in the ecliptic. The dashed lines from  $O$  are all radii of the Earth's orbit that will be denoted  $r$ . The dotted lines form two right triangles which will be used in the computation.

### 3.2 The location of the center of the Earth's orbit

Copernicus' task was to locate the position of the center of the Earth's orbit  $O$  relative to the center of the Sun  $S$ . This will be given in polar coordinates  $OS = c$  and  $\angle FSB = \lambda$  (the label is on the large circle of the ecliptic). The strategy of the computation is as follows:

- Initially, we compute the angles of the arcs in radians; the lengths of the arc can then be obtained by multiplying by the radius  $r$ .
- We use the lengths of the seasons that Copernicus used: summer is  $93\frac{14.5}{60}$  days and spring is  $92\frac{51}{60}$ .
- The angle of the arc  $\widehat{AC}$  can be computed from the combined length of spring and summer and the angle of the arc  $\widehat{AB}$  can be computed from the length of spring.
- From  $\widehat{AC}$  and  $\widehat{AB}$ , the angle  $\alpha$  subtended by  $\widehat{AA'}$  and the angle  $\beta$  subtended by  $\widehat{BB'}$  can be computed.
- Since the Earth is very close to the Sun relative to the radius of its orbit,  $r\widehat{AA'}$  and  $r\widehat{BB'}$  approximate the line segments  $a$  and  $b$ . From these  $c$  and  $\lambda$  can be computed.

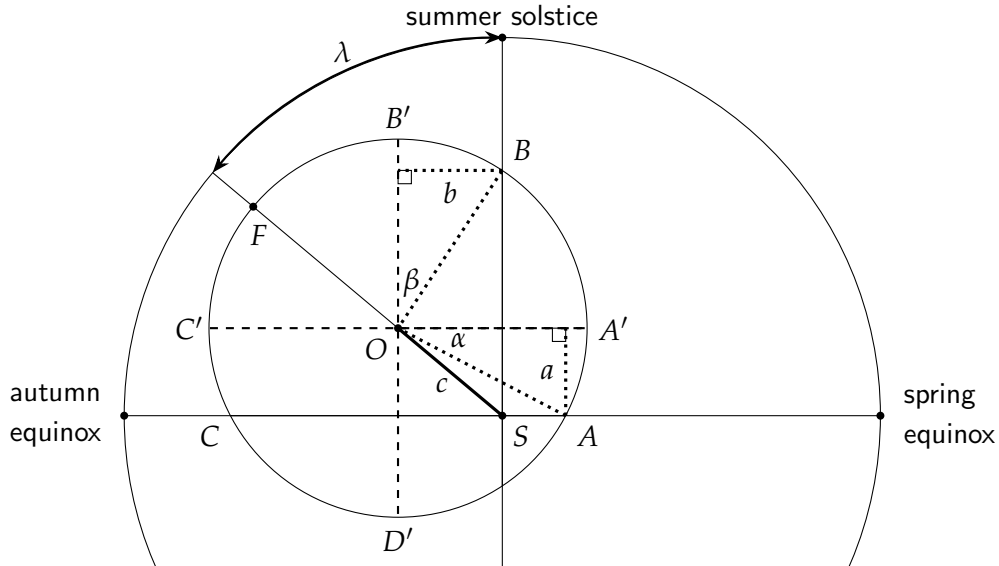


Figure 9: Computing the center of the Earth

### Computing the angles of the arcs $\widehat{AB}, \widehat{AC}$

The arcs  $\widehat{AB}, \widehat{AC}$  are sectors of the Earth's orbit and their angles are their proportions of a full year times  $2\pi$  radians.

$$\widehat{AB} = 2\pi \cdot \frac{92\frac{51}{60}}{365.25} = 2\pi \cdot \frac{92.85}{365.25} = 1.5972 \text{ radians}$$

$$\widehat{AC} = 2\pi \cdot \frac{92\frac{51}{60} + 93\frac{14.5}{60}}{365.25} = 2\pi \cdot \frac{186.09}{365.25} = 3.2012 \text{ radians}.$$

### Computing the angles of the arcs $\widehat{AA'}, \widehat{BB'}$

Let us express the arcs  $\widehat{AC}$  and  $\widehat{AB}$  in terms of the arcs that comprise them. Since  $AC$  is parallel to  $A'C'$ ,  $\widehat{AA'} = \widehat{C'C}$ . Compute  $\widehat{AA'}$ :

$$\begin{aligned}\widehat{AC} &= \widehat{AA'} + \widehat{A'C'} + \widehat{C'C} = 2\widehat{AA'} + \pi \\ \widehat{AA'} &= \frac{1}{2}(3.2012 - \pi) = 0.0298 \text{ radians}.\end{aligned}$$

Now that we have computed  $\widehat{AB}$  and  $\widehat{AA'}$  we can compute  $\widehat{BB'}$ :

$$\begin{aligned}\widehat{AB} &= \widehat{AA'} + \widehat{A'B'} - \widehat{BB'} \\ \widehat{BB'} &= 0.0298 + \frac{\pi}{2} - 1.5927 = 0.0034 \text{ radians}.\end{aligned}$$

### Computing the lengths of the arcs $\widehat{AA'}, \widehat{BB'}$

$OA$  and  $OB$  are radii of the Earth's orbit so

$$\begin{aligned}\sin \alpha &= \frac{a}{r} \approx \alpha \\ a &\approx r\alpha = r\widehat{AA'} = 0.0298r \\ \sin \beta &= \frac{b}{r} \approx \beta \\ b &\approx r\beta = r\widehat{BB'} = 0.0034r,\end{aligned}$$

where we have used the assumption that  $O$ , the center of the Earth's orbit, is very close to the Sun  $S$  so that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

### Computing the position of $O$ relative to $S$

Figure 10 shows a magnified diagram of a portion of Figure 9. In the dotted triangles, we have already computed the lengths  $a$  and  $b$ . Since  $OT$  is parallel to  $A'A$  and  $TS$  is parallel

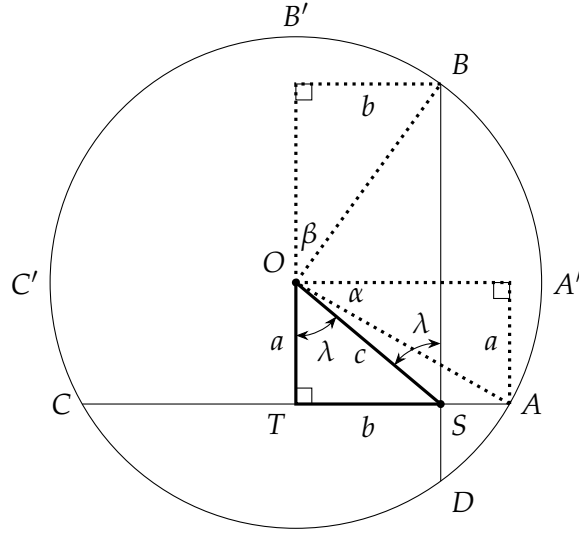


Figure 10: Three triangles

to  $BB'$ , we can label  $OT$  by  $a$  and  $TS$  by  $b$ . The emphasized triangle is a right triangle and  $c$ , the distance of  $O$  from  $S$ , can be obtained from Pythagoras' theorem:

$$c = \sqrt{a^2 + b^2} = r\sqrt{(0.0298)^2 + (0.0034)^2} = 0.03r.$$

$\lambda$  can be obtained from trigonometry:

$$\lambda = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{0.0034}{0.03} = 0.1129 \text{ radians} \approx 6.47^\circ.$$

The distance  $0.03r$  is shown in Table 3 using the values of  $r$  from Table 1.

	Aristarchus (km)	Copernicus (km)	Modern (km)
radius of Earth's orbit	2,320,660	8,000,000	150,000,000
distance of $O$ from $S$	69,620	240,000	4,500,000

Table 3: Values computed by Aristarchus' method compared with modern values

## 4 Elliptical orbits

Towards the end of the sixteenth century, the astronomer Tycho Brahe carried out extremely precise observations. In 1600 he hired Johannes Kepler as his assistant and when Tycho died soon afterwards, Kepler was appointed to his position. Here we explain how Kepler was able to establish that planetary orbits are ellipses.

### 4.1 Determining the radius of the Earth's orbit

A Martian year is 687 days, that is, it equals  $\frac{687}{365.25} = 1.88$  Earth years. We know when Mars reaches a new “year” by observing its projection on the celestial sphere, but each time the position of the Earth in its orbit will be different. Figure 11 shows the orbit of the Earth—its center  $O$  offset from the Sun  $S$  as Copernicus showed—at four occasions when the position of Mars  $M$  at its new year was observed. Four triangles are created  $\triangle OE_iM$ .

Figure 12 shows one of the triangles with the angles labeled. Using the law of sines,

$$\frac{OE_i}{\sin \beta} = \frac{OM}{\sin \alpha}$$

$$OE_i = OM \frac{\sin \beta}{\sin \alpha}.$$

Tycho Brahe was able to measure all three angles (Table 4)<sup>4</sup>. The values of  $OE$  are given in the fourth column of the table and they are not equal. Assuming (as everyone did at that time) that the Earth's orbit is circular, the only solution was to move the center of the orbit so that  $\{E_1, E_2, E_3, E_4\}$  were all on the circle.

<sup>4</sup>The values are rounded to one decimal point.

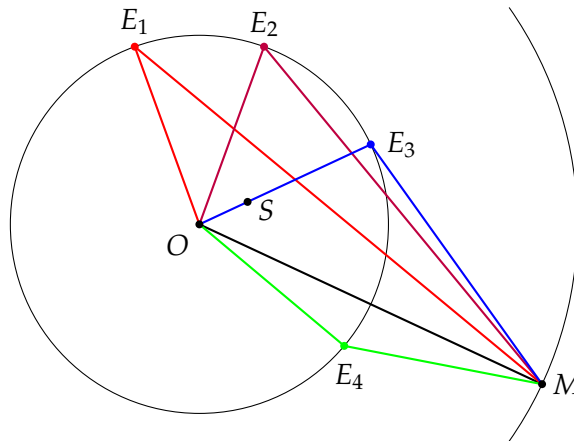


Figure 11: Observations of the orbit of Mars from the Earth

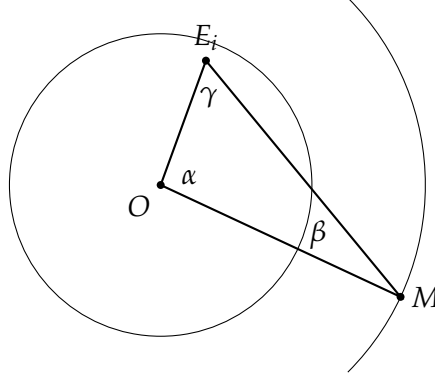


Figure 12: One triangle Earth-Sun-Moon

## 4.2 Measuring the angles in the triangle Sun-Earth-Mars

How can the angles  $\alpha, \beta, \gamma$  be measured? Since  $\triangle E_i OM$  is a triangle, it is sufficient to measure two of the angles.  $\gamma$  is easily measured by observing Mars and the Sun at the same time. However, neither  $\alpha$  nor  $\beta$  can be measured directly since they are not accessible to an observer on Earth.

Tycho's measurement used the known periods of the orbits to compute the angles. The Earth moves counterclockwise around Sun. Given any point  $E$ , after  $t$  days the Earth will eventually move to  $E'$  and Mars will move to  $M'$  so that they are in *opposition*, that is, Mars will be in the continuation of the Earth-Sun line (Figure 13). Since the Earth completes an orbit in close to half the time that Mars takes, this will occur before either has completed a full orbit. The angles  $\theta_E$  and  $\theta_M$  are fractions of a circular orbit of  $360^\circ$ , so

$$\begin{aligned}\frac{t}{365.25} &= \frac{\theta_E}{360} \\ \theta_E &= \frac{360 t}{365.35} \\ \frac{t}{687} &= \frac{\theta_M}{360}\end{aligned}$$

	$\alpha$	$\beta$	$\gamma$	$OE_i (\times OM)$
$E_1$	127.1	20.8	32.1	0.6682
$E_2$	84.2	35.8	60.5	0.6721
$E_3$	41.3	42.4	96.4	0.6785
$E_4$	1.6	3.4	175.0	0.6805

Table 4: Tycho's measurements of the distance of the Earth from the center



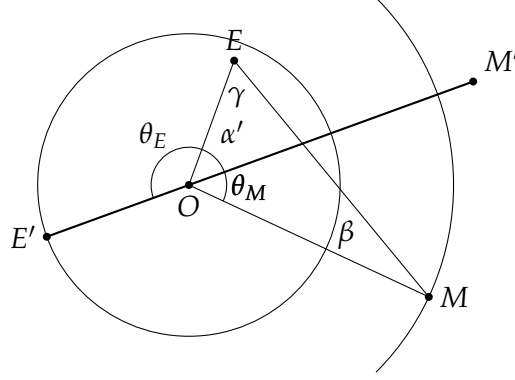


Figure 13: The Earth and Mars in opposition

$$\theta_M = \frac{360 t}{687}.$$

This gives values for  $\theta_E$  and  $\theta_M$ . From geometry we have

$$\begin{aligned}\alpha' &= 180 - \theta_E = \alpha - \theta_M \\ \alpha &= 180 - \theta_E + \theta_M,\end{aligned}$$

and the values of  $OE_i$  in Table 4 can be computed.

#### 4.3 A new location for the center of the Earth's orbit

Kepler's next task was to obtain a new value  $O'$  for the center of the Earth's orbit such that the  $E_i$  in Table 4 are on the orbit. We need the following theorem in geometry.

**Theorem 4.1** *Given three non-collinear points,<sup>5</sup> a circle can be constructed that goes through all three points.*

**Proof** Three non-collinear points  $A, B, C$  define a triangle  $\triangle ABC$  (Figure 14). Construct the perpendicular bisectors of any two of its three sides, say,  $AC$  and  $BC$ . By definition the perpendicular bisector is the geometric locus of points equidistant from the endpoints of the segment. Let  $O$  be the intersection of the two bisectors. Then the  $AO = CO = BO$  is the radius of a circle centered at  $O$  that goes through  $A, B, C$ . ■

Given the new locations of the Earth  $\{E_1, E_2, E_3, E_4\}$ , a circle centered at  $O'$  can be constructed that goes through  $\{E_1, E_2, E_3\}$  (Figure 15). To verify that this is the correct orbit, check that  $E_4$  is on the circle.

<sup>5</sup>Collinear means that the points all on the same line.

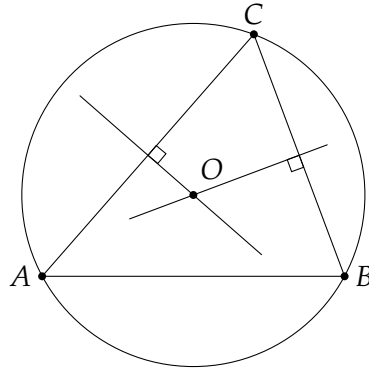


Figure 14: A circle through three arbitrary points

#### 4.4 Orbits are ellipses

While Kepler was able to modify the center of the orbit of the Earth to be consistent with the observations, he was not able to adequately describe the orbit of Mars. After years of work, he came to the conclusion that the orbit must be oval like an egg. Oval, perhaps, but certainly not an ellipse, because he was certain that it would have been discovered by Archimedes! Figure 16 shows  $C$ , a position on a circular orbit, and an oval orbit (dashed), where  $M$  is the position of Mars on the oval corresponding to  $C$ . The radius of the circular orbit is labeled  $a$  and the unknown distances to  $M$  are labeled  $s = SM$  and  $t = OM$ .

Kepler's computed that  $\frac{a-t}{t} = 0.00429$  and  $\frac{s}{t} = 1.00429$ , so that

$$\begin{aligned}\frac{a-t}{t} &= \frac{s}{t} - 1 \\ a-t &= s-t,\end{aligned}$$

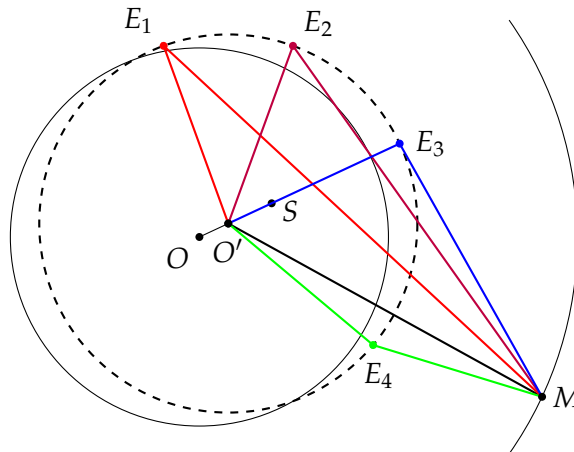


Figure 15: Observations of the orbit of Mars from the new Earth's orbit

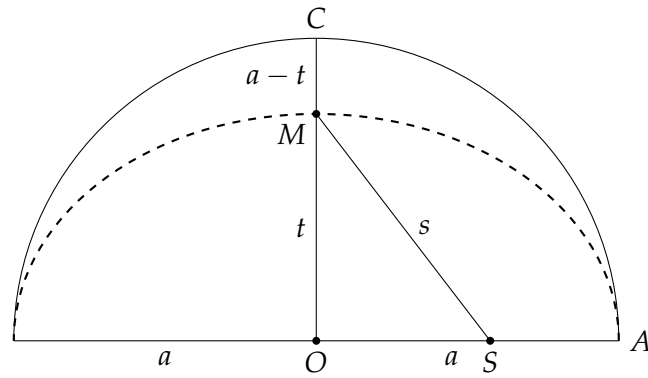


Figure 16: The orbit of Mars as an oval “egg”

and therefore  $SM = s = a = AO = CO$ . If the dashed oval is, in fact, an ellipse, because in an ellipse  $SM = AO$  (by Theorem A.2). Kepler then computed the projections of the observations of Mars on the  $x$ -axis (Figure 17) and obtained for all of them that

$$\frac{M_i O_i}{C_i O_i} = \frac{1}{1.00429} = 0.99573.$$

By Theorem A.4, since the ratio  $MO/CO = b/a$  is constant for in an ellipse, Kepler was able to conclude that the orbit of Mars is an ellipse.

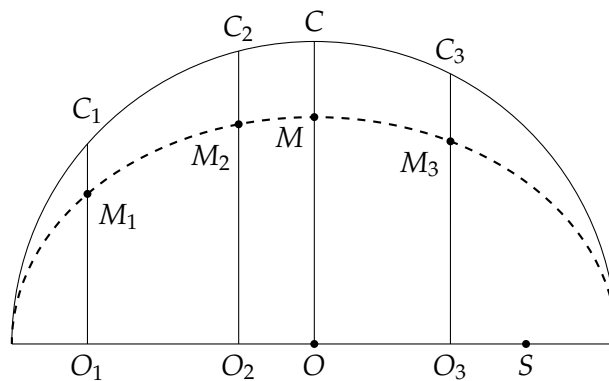


Figure 17: The orbit of Mars as an ellipse

## 5 Gravitation

In this section we present Isaac Newton's demonstrations that the inverse-square law can be deduced from Kepler's observation that the orbits of planets are ellipses. After a review of Newton's Laws of force and motion, we show that Kepler's second law must hold in any system subject to a centripetal force. The next step is to show that the inverse square law for gravitation must hold in an elliptical orbit. Then it is a small step to universal gravitation and Kepler's third law. For readers who wish to study the *Principia* the labels of the points in the diagrams are the same as those used by Newton.

### 5.1 Newton's laws of motion

1. A body in uniform motion (including a body at rest) continues with the same motion unless a force is applied.
2. A force  $F$  applied to a body causes an acceleration  $a$  in the direction of the force whose magnitude  $a = F/m$ , where  $m$ , the constant of proportionality, is called the *mass* of the body.
3. If one body exerts a force on a second body, the second body exerts a force on the first of equal magnitude but in the opposite direction.

Forces are denoted by vectors, where the direction of the vector represents the direction of the force and the length of the vector represents the magnitude of the force. Forces can be decomposed into perpendicular components (Figure 18), or into components in any directions (Figure 19). The components form a parallelogram whose diagonal is the resultant force.

Newton was interested in *centripetal force* which is a force exerted by a single body on another, in particular, the gravitational force exerted by the Sun on a planet (Figure 20). Since the only force is that directed towards the Sun, the planet does not move "up" or "down" so its orbit is in a plane.

Kepler's second law states that a planet in orbit sweeps out equal area in intervals of equal duration, if it takes the planet time  $t$  to move from  $P_1$  to  $P_2$  and also  $t$  to move

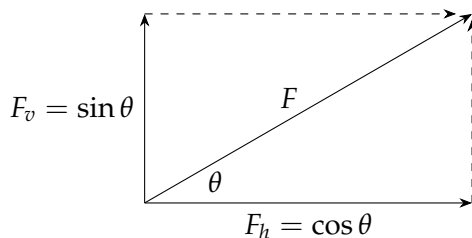


Figure 18: Perpendicular components of a force

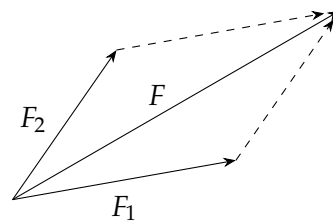


Figure 19: Arbitrary components of a force

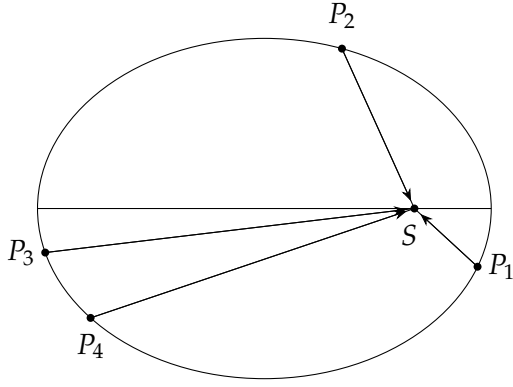


Figure 20: Centripetal force

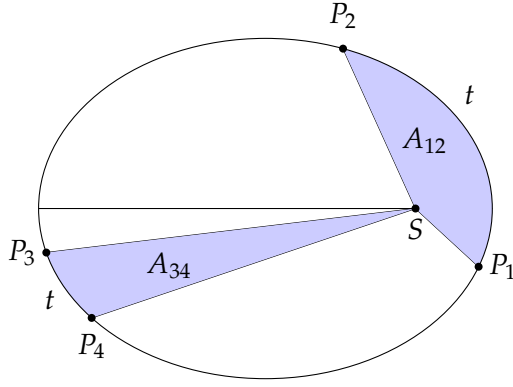


Figure 21: Equal areas in equal times

from  $P_3$  to  $P_4$ , then  $A_{12}$ , the area of the sector  $P_1SP_2$ , is equal to  $A_{34}$ , the area of  $P_3SP_4$  (Figure 21). Obviously, this means that the speed of the planet must vary as it traverses its orbit  $v_{P_1P_2} \gg v_{P_3P_4}$ . Newton proved that this must be true in any system where a body is subject to a centripetal force from another body.

The proof is based on dividing an area into very small sectors and then taking the limit. Consider three points  $A, B, C$  on the orbit (Figure 22) that represent the positions of the planet at intervals of  $\Delta t$ . For clarity we have drawn them spaced out, but the intention is that they are very close together. Newton assumed that the planet does not smoothly traverse the arcs, but rather that it every  $\Delta t$  it jumps in discrete steps from one point on the orbit to the next.

Figure 23 shows how the force is exerted in discrete steps. The planet moves from  $A$  to  $B$  and we expect that the centripetal force at  $B$  will cause an acceleration that moves the planet to  $C$ , the next point on the orbit. Instead, we “pretend” that the force is not applied at  $B$ , but, in the absence of an applied force, planet continues to move in the same direction and at the same speed. After another period of  $\Delta t$  as passed and the planet has reached point  $C'$ , the force is now applied *in the same direction* as it would have been applied at  $B$ , moving the planet to  $C$ .

**Theorem 5.1** *The area of  $\triangle ASB$  is equal to the area of  $\triangle BSC$ .*

**Proof** This will be done in two steps by showing that  $\triangle ASB = \triangle BSC'$  and  $\triangle BSC' = \triangle BSC$ . In Figure 24,  $\triangle ASB$  is shown in blue and  $\triangle BSC'$  is shown in red. It is assumed that  $AB = BC'$  (the planet moves from  $B$  to  $C'$  during the same interval  $\Delta t$ ), so since  $SH$  is the height of both triangles, their areas are equal. In Figure 25,  $\triangle BSC$  is shown in blue and  $\triangle BSC'$  is shown in red. It is assumed  $CC'$  is parallel to  $SB$  (the planet is subject to the centripetal force at  $C'$  in the *same* direction as the force at  $B$ , so the heights of both triangles to the common side  $SB$  are equal and their areas are equal. It follows that  $\triangle ASB = \triangle BSC' = \triangle BSC$ , which we denote by  $\Delta A$ . ■

We assume that the sectors of the orbit are each divided up into small sectors of uniform

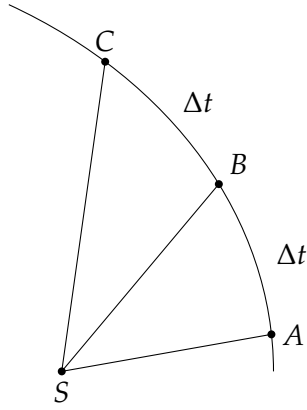


Figure 22: “Small” sectors of an orbit

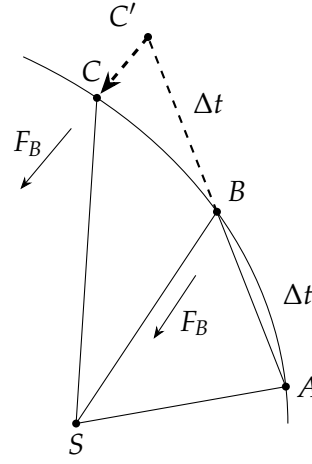


Figure 23: Exerting force at discrete times

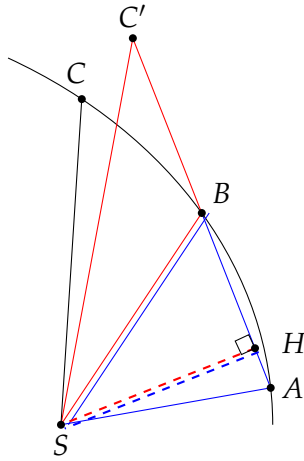


Figure 24:  $\triangle ASB = \triangle BSC'$

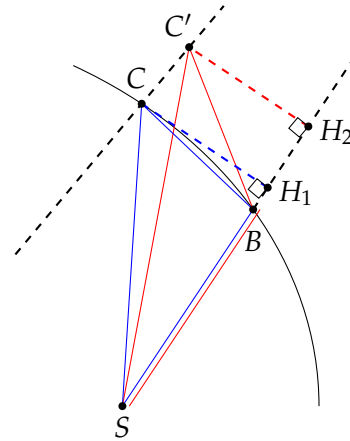


Figure 25:  $\triangle BSC' = \triangle BSC$

duration  $\Delta t$ . By the theorem, each sector has the same area  $\Delta A$ . Therefore,

$$\frac{A_{12}}{\Delta A} = \frac{t}{\Delta t} = \frac{A_{34}}{\Delta A},$$

from which Kepler's second law follows:

$$\begin{aligned} \frac{A_{12}}{t} &= \frac{A_{34}}{t} \\ A_{12} &= A_{34}. \end{aligned}$$

The proof used two approximations:

- $\Delta A$  is an approximation of the area of each sector.
- The force at  $C'$  is an approximation to the force at  $B$ .

**Definition 5.2** For a given elliptical orbit,  $\kappa = \frac{A}{t}$ , where  $A$  is the area of the ellipse and  $t$  is the period of the orbit, is called Kepler's constant.

Newton's next step was to show that if the orbit of a planet is elliptical, the centripetal force must be proportional to the mass of the planet and inversely proportional to the square of its distance from the Sun. In Figure 26,  $S$  is the Sun, and  $P$  and  $Q$  are points on the orbit that are very close to each other.  $PR$  is a tangent to the ellipse at  $P$  and  $R$  is a point such that  $QR$  is parallel to  $SP$ .  $QT$  is constructed perpendicular to  $SP$ . Denote the lengths  $r_p = SP$ ,  $h = QT$ ,  $q = QR$  and the time interval from  $P$  to  $Q$  by  $\Delta t$ .

$$q = \frac{1}{2} \frac{F_P}{m} (\Delta t)^2$$

$$F_P = \frac{2mq}{(\Delta t)^2}.$$
$$\Delta t = \frac{\Delta A}{\kappa} = \frac{\hbar r_P}{2\kappa}$$

$$F_P = 2mq \cdot \frac{4\kappa^2}{(\hbar r_P)^2} = 8\kappa^2 m \cdot \frac{q}{\hbar^2} \cdot \frac{1}{r_P^2}.$$
[illegible]

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so the first factor does not depend on the distance. What about the second factor  $q/h^2$ , in particular, what value does it have if  $\Delta t$  approaches zero?

**Theorem 5.3** *In an elliptical orbit*

$$\lim_{\Delta t \rightarrow 0} \frac{q}{h^2} = \frac{1}{L},$$

where  $L$  is the length of the latus rectum of the ellipse (Definition A.5).

Newton's proof is very long and is presented separately in Section 6.

Since  $L$  is constant for any given ellipse, the inverse square law can be written

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2}.$$

**A simplified formula for the force** The formula can be re-written so that the constant values appearing are more familiar:  $a$ , the semi-major axis and  $T$ , the period of the orbit. By Theorem A.6,  $L=2b^2/a$  and by Theorem A.7,  $\kappa = A_{\text{ellipse}}/T=\pi ab/T$ , so

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2} = \frac{8(\pi ab)^2 m}{T^2} \cdot \frac{a}{2b^2} \cdot \frac{1}{r_P^2} = \frac{4\pi^2 a^3 m}{T^2} \cdot \frac{1}{r_P^2}. \quad (5)$$

**Applicability of the inverse square law** Newton was able to show that

- The inverse square law applies to all conic sections including a parabola and a hyperbola, not just to an ellipse and, of course, a circle. Some comets have hyperbolic orbits and orbit the Sun only once.
- The converse holds: if a planet is subject to an inverse-square centripetal force then the orbit must be an ellipse (or other conic section).
- The proof assumes that a planet is a very small point, but the result holds even for large planets as long as the density of the planet is radially symmetric, that is, for a given distance from the center the density is constant.

### 5.3 Universal gravitation

Recall Newton's third law: If one body exerts a force on a second body, the second body exerts a force on the first that is equal in magnitude but in the opposite direction. Therefore, we can equate the force that the Sun  $S$  exerts on a planet  $S$  with the force that the planet exerts on the Sun. Let  $m$  be the mass of the planet and  $M$  be the mass of the Sun. Let  $F_{S \leftarrow E}$  be the force exerted by the Sun on the Earth and let  $F_{E \leftarrow S}$  be the force exerted by the Earth on the Sun. Using Equation 5,

$$F_{S \leftarrow E} = \frac{8\kappa_E^2 m}{L_E} \cdot \frac{1}{r^2} = \frac{C_E m}{r^2}$$



$$F_{E \leftarrow S} = \frac{8\kappa_S^2 M}{L_S} \cdot \frac{1}{r^2} = \frac{C_S M}{r^2}$$

$$\frac{C_E}{M} \cdot \frac{1}{r^2} = \frac{C_S}{m} \cdot \frac{1}{r^2},$$

from some constants  $C_E, C_S$ .

Why are the constants different? The Earth and the Sun both rotate around their center of mass called the *barycenter*, which is very close to the center of the Sun since the Sun is so much more massive than the Earth. The ellipse of the Sun's orbit is very small relative to the Earth so  $A$  and  $L$  are smaller, and the Sun's period is large so  $T$  is larger. The different values for  $8\kappa^2/L$  are encapsulated into the constants  $C_E, C_S$ . Let  $G = \frac{C_E}{M} = \frac{C_S}{m}$  so that

$$F_{S \leftarrow E} = F_{E \leftarrow S} = G \frac{mM}{r^2}. \quad (6)$$

This is Newton's law of universal gravitation. It is not specific to planetary orbits but holds between any two bodies with masses  $m, M$ .

## 5.4 Kepler's third law

Newton proved that Kepler's third law follows from the law of universal gravitation.

**Theorem 5.4 (Kepler's third law)** *Let  $P_1, P_2$  be two planets whose elliptical orbits have semi-major axes  $a_1, a_2$  and whose orbital periods around the Sun are  $T_1$  and  $T_2$ . Then*

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2}.$$

**Proof** By Equations 5 and 6,

$$F = \frac{4\pi^2 a_i^3 m}{T_i^2} \frac{1}{r_i^2} = \frac{GmM}{r_i^2}.$$

After canceling  $m$  and  $r_i$  we get

$$\frac{a_i^3}{T_i^2} = \frac{GM}{4\pi^2}.$$

$\frac{GM}{4\pi^2}$  is a constant that depends only on the mass of the Sun and the gravitational constant, so  $\frac{a_i^3}{T_i^2}$  is constant for all planets rotating around the Sun. ■

## 6 The proof of Theorem 5.3

Theorem 5.3 is Section III, Proposition XI, Problem VI of the *Principia*.

Study Figure 27:<sup>6</sup>

- Given an ellipse whose center is  $C$  and whose foci are  $S$  and  $H$ , let  $P, Q$  be two points on the ellipse. The points represent the movement of a body in an elliptical orbit separated by a time interval  $\Delta t$ .
- Construct lines from  $P$  to the center and the foci.
- Let  $RPZ$  be the tangent at  $P$  such that the line segment  $PR$  is the path that the body would transverse if it continued moving for time  $\Delta t$ , but was not subject to any force.
- Construct the parallelogram  $PRQX$ .
- Extend  $QX$  until it intersects  $PC$  at  $V$ .
- Construct a line parallel to  $RP$  through  $H$  and let  $I$  be its intersection with  $PS$ .
- Construct  $DC$  the conjugate diameter to  $PC$ (Definition A.11). Let  $E$  be its intersection with  $PS$ .

<sup>6</sup>To save space the bottom half of the ellipse has been clipped, but we still refer to lines from the center to points on the ellipse as diameters.

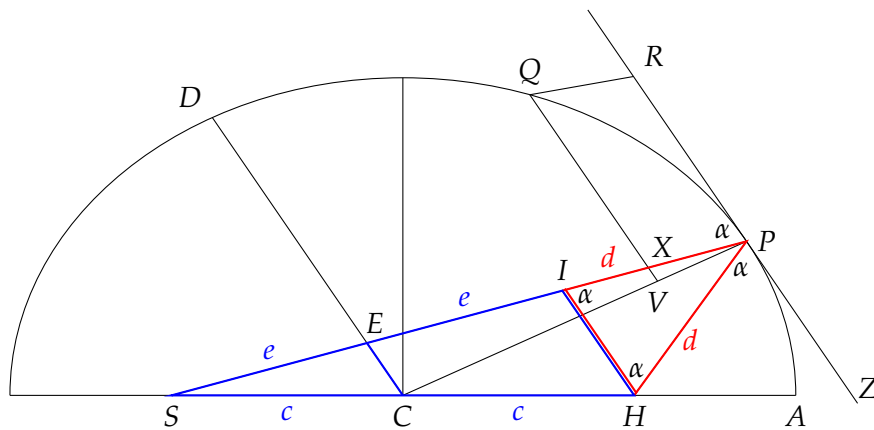


Figure 27: Geometry of an elliptical orbit (1)

**Theorem 6.1**  $QR = PV \cdot \frac{CA}{CP}$ .

$$\angle PHI = \angle ZPH = \alpha = \angle RPX = \angle PIH,$$

$SC = CH = c$  are equal because they are the distances of the foci from the center of the ellipse. By construction  $EC \parallel IH$  so  $\triangle ESC \sim \triangle ISH$  (blue) and

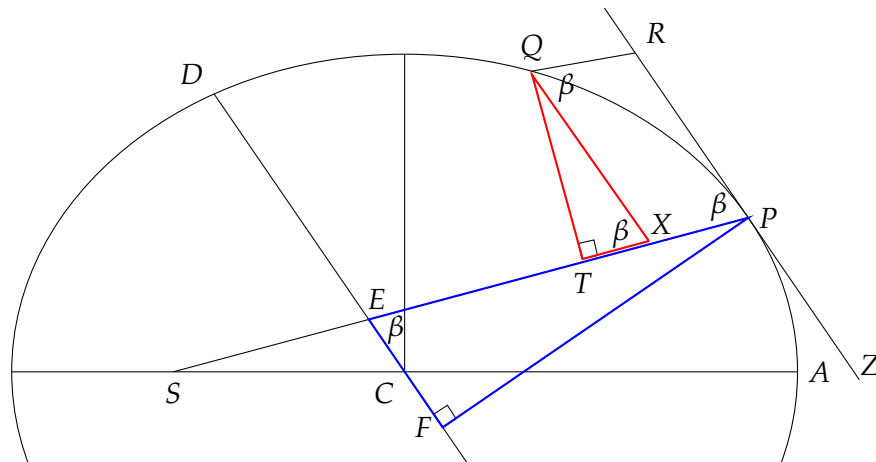
$$\frac{SC}{SE} = \frac{SH}{SI}$$

$$\frac{c}{e} = \frac{2c}{SI}.$$

$QV \parallel EC$  so  $\triangle EPC \sim \triangle XPV$  and

$$\frac{PX}{PV} = \frac{EP}{PC} = \frac{CA}{PC}$$

$$PX = PV \cdot \frac{CA}{PC}.$$



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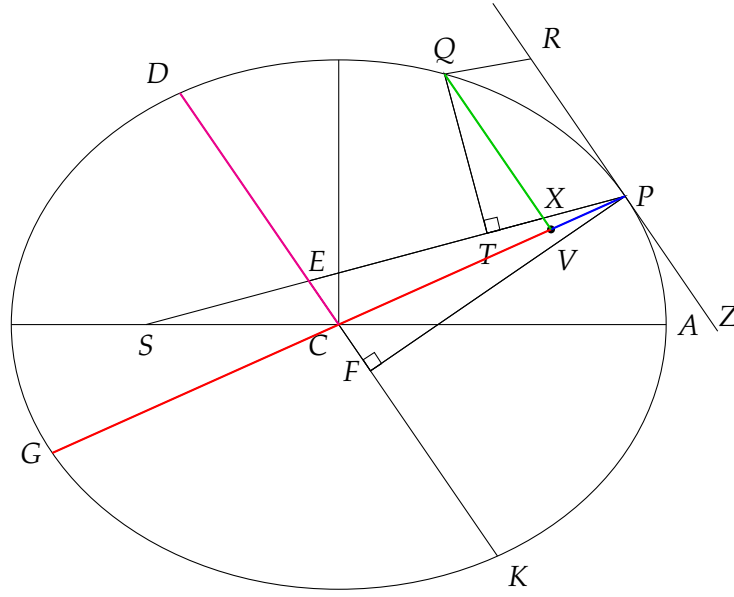


Figure 29: Ratios on conjugate diameters

## 6.2 A formula for $QT$

Add a perpendicular from  $P$  to  $DC$  to the diagram and label its intersection with  $DC$  by  $F$  (Figure 28).

### Theorem 6.2

$$QT = QX \cdot \frac{FP}{CA}.$$

**Proof** By construction,  $QR \parallel PX$ , so by alternate interior angles  $\angle RQX = \angle QXT = \beta$ . By construction,  $QX \parallel DC$ , so by alternate interior angles  $\angle QXT = \angle PEF = \beta$ . Since  $\triangle PFE$  and  $\triangle QTX$  are right triangles with an equal acute angle  $\beta$ ,  $\triangle PFE \sim \triangle QTX$ . In the proof of Theorem 6.1 we showed that  $EP = CA$  so

$$\begin{aligned} \frac{QT}{QX} &= \frac{FP}{EP} \\ QT &= QX \cdot \frac{FP}{EP} = \frac{FP}{CA}, \end{aligned}$$

■

### 6.3 A formula for $QR/QT^2$

**Theorem 6.3**

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}.$$

Let use combine the equations in Theorems 6.1 and 6.2 to get  $QR/QT^2$ .

$$\frac{QR}{QT^2} = \frac{PV \cdot \frac{CA}{CP}}{\left(QX \cdot \frac{FP}{CA}\right)^2} = \frac{PV \cdot CA^3}{QX^2 \cdot CP \cdot FP^2}. \quad (7)$$

Theorem A.14 gives a formula for  $PV$  that we substitute into Equation 7.

$$\frac{QR}{QT^2} = \frac{QV^2 \cdot CP^2}{GV \cdot PV \cdot CD^2} \cdot \frac{CA^3}{QX^2 \cdot CP \cdot FP^2} = \frac{CP}{CD^2} \cdot \frac{CA^3}{FP^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \quad (8)$$

Next we show that  $CD \cdot FP = CA \cdot CB$  (Figure 29). Theorem A.12 proved that the areas of the parallelograms formed by the tangents conjugate diameters are equal. By symmetry the areas of the four small parallelograms are equal, as are the triangles formed by constructing corresponding diagonals. In Figure 30 the area of the  $\triangle ABC'$  (red), which is  $AC \cdot AB$ , is equal to the area of  $\triangle PCD$  (blue), which is  $FP \cdot CD$ .

Substituting for  $CD \cdot FP$  in Equation 8 gives

$$\frac{QR}{QT^2} = \frac{CP}{CD^2} \cdot \frac{CA^3}{FP^2} \cdot \frac{QV^2}{GV \cdot QX^2}$$

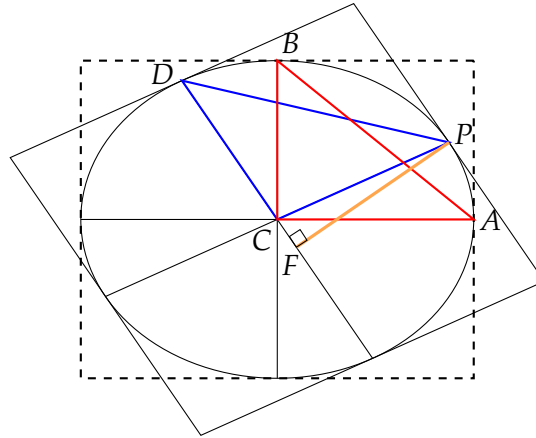


Figure 30: Parallelograms formed by conjugate diameters

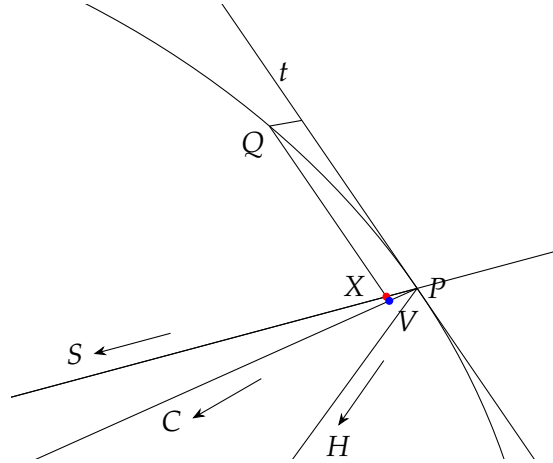


Figure 31: Geometry of an elliptical orbit (3)

$$\begin{aligned}
 &= \frac{CP \cdot CA^3}{CB^2 \cdot CA^2} \cdot \frac{QV^2}{GV \cdot QX^2} \\
 &= \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}.
 \end{aligned} \tag{9}$$

■

#### 6.4 Approaching the limit

Figure 31 is an enlarged diagram of part of Figure 29. Compare the two diagrams and you will see that as the time interval  $\Delta t$  gets smaller,  $Q \rightarrow P$  which implies that

- $X \rightarrow V$  so that  $QX \rightarrow QV$ .
- $V \rightarrow P$  so that  $CV \rightarrow CP$  and hence  $GV \rightarrow 2CP$ .
- In the limit  $QX = QV$  and  $GV = 2CP$ . Substituting into Equation 9 gives

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QX^2}{2CP \cdot QX^2} = \frac{CA}{2CB^2} = \frac{a}{2b^2} = \frac{1}{L},$$

using the result of Theorem A.6 for the length of the latus rectum.

## A Ellipses

### A.1 Fundamental properties

#### Definition A.1 (Ellipse)

- Let  $F_1$  and  $F_2$  be two points in the plane such that  $F_1F_2 = 2c \geq 0$  and choose  $a$  such that  $2a > 2c$  (Figure 32). An ellipse is the geometric locus of all points  $P$  such that  $F_1P + F_2P = 2a$ . If  $c = 0$  the geometric locus is a circle.
- Construct  $AB$  through  $F_1F_2$ , where  $A, B$  are the intersections of the line with the ellipse.  $AB$  is the major axis of the ellipse. Let  $O$  be the midpoint of  $F_1F_2$ .  $AO$  and  $OB$  are the semi-major axes of the ellipse.
- Construct the perpendicular to  $AB$  at  $O$  and let  $C, D$  be its intersections with the ellipse.  $CD$  is the minor axis of the ellipse and  $CO$  and  $OD$  are the semi-minor axes of the ellipse.

#### Theorem A.2 (Figure 33)

1.  $F_1C = F_2C = a$ .
2.  $AO = OB = a$ .
3.  $CO = OD$ . (Label  $CO = OD$  by  $b$ .)

#### Proof

1.  $\triangle F_1CO$  is congruent to  $\triangle F_2CO$  by side-angle-side ( $F_1O = F_2O$ ,  $CO$  is a common side,  $\angle COF_i$  are right angles), so  $F_1C = F_2C$ . Since  $C$  is on the ellipse,  $F_1C + F_2C = 2a$  and  $F_1C = F_2C = a$  follows.

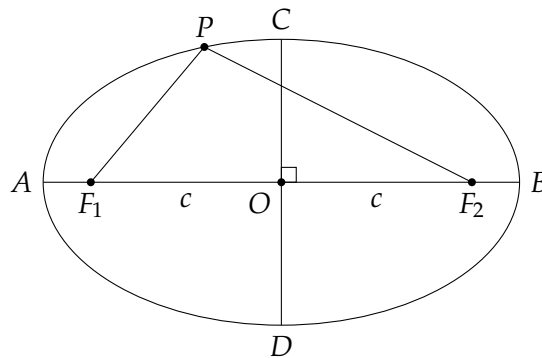


Figure 32: The definition of an ellipse

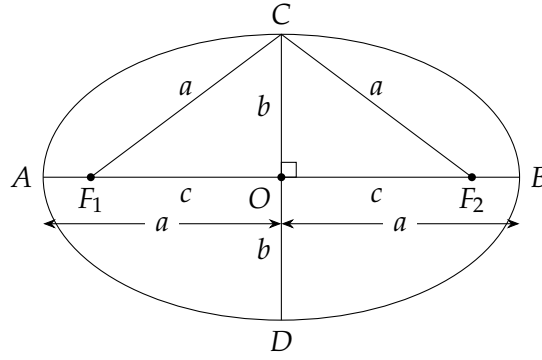


Figure 33: The semi-major and semi-minor axes of an ellipse

2. Since  $A$  is on the ellipse,

$$2a = AF_1 + AF_2 = (AO - c) + (AO + c) = 2AO,$$

so  $AO = a$ .  $OB = a = AO$  follows by symmetry.

3.  $CO = OD$  follows since  $\triangle F_1CO$  is congruent to  $\triangle F_1DO$ . ■

$a$  and  $b$  will always denote the semi-major and semi-minor axes of an ellipse.

**Theorem A.3** A point  $P = (x, y)$  on an ellipse satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

**Proof** Since  $F_1 = (-c, 0)$ ,  $F_2 = (c, 0)$  and  $PF_1 + PF_2 = 2a$ , from the formula for length in the Cartesian plane we have

$$PF_1 + PF_2 = \sqrt{(x - (-c))^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

Squaring twice results in

$$(x + c)^2 + y^2 = \left(2a - \sqrt{(x - c)^2 + y^2}\right)^2$$

$$4xc = 4a^2 - 4a\sqrt{(x - c)^2 + y^2}$$

$$a - \frac{c}{a}x = \sqrt{(x - c)^2 + y^2}$$

$$a^2 + \frac{c^2}{a^2}x^2 = x^2 + c^2 + y^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{c^2 - a^2} = \frac{c^2 - a^2}{c^2 - a^2} = 1.$$



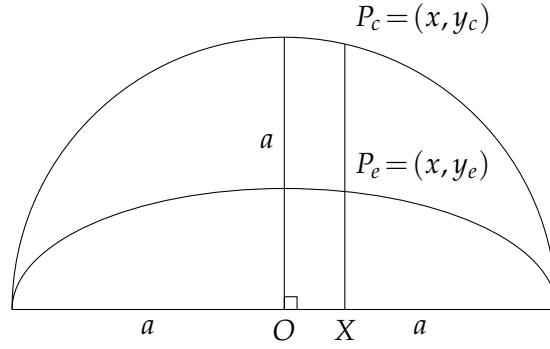


Figure 34: A circle circumscribing an ellipse

By Theorem A.2 and Pythagoras' theorem,  $b^2 = a^2 - c^2$  so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

■

## A.2 A circle circumscribing an ellipse

Consider a circle of radius  $a$  with the same center as an ellipse (Figure 34). Construct a vertical line at an arbitrary point  $X = (x, 0)$  on the  $x$ -axis, and let its intersections with the ellipse and the circle be  $P_e = (x, y_e)$  and  $P_c = (x, y_c)$ , respectively.

**Theorem A.4** *The vertical projection onto an ellipse of a point  $P_c = (x, y_c)$  that is on a circle circumscribing the ellipse is  $P_e = (x, y_e) = \left(x, \frac{b}{a}y_c\right)$ .*

**Proof** Using the formulas for points on an ellipse and on a circle we have

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y_e^2}{b^2} &= 1 \\ y_e^2 &= b^2 \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2 - x^2) \end{aligned} \tag{10}$$

$$\begin{aligned} x^2 + y_c^2 &= a^2 \\ y_e &= \frac{b}{a} \sqrt{a^2 - x^2} = \frac{b}{a} y_c. \end{aligned} \tag{11}$$

■

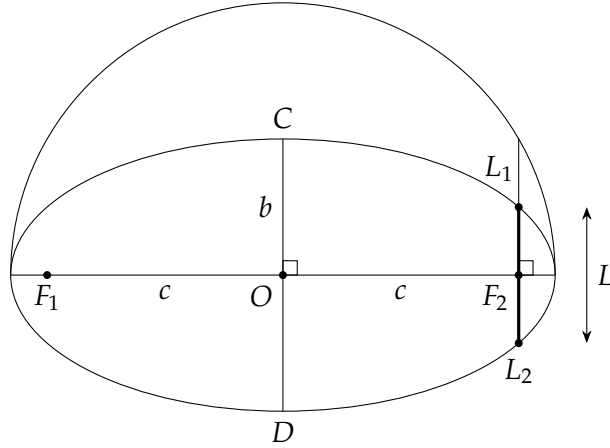


Figure 35: The latus rectum of an ellipse

### A.3 The latus rectum of an ellipse

**Definition A.5** A latus rectum of an ellipse is a vertical line segment through a focus that intersects the ellipse (Figure 35).

**Theorem A.6**  $L$ ,<sup>7</sup> the length of the latus rectum of an ellipse, is  $\frac{2b^2}{a}$ .

**Proof** By Equation 11 and Pythagoras' theorem,

$$L_1 = \left( c, \frac{b}{a} \sqrt{a^2 - c^2} \right) = \left( c, \frac{b^2}{a} \right)$$

$$L = L_1 L_2 = \frac{2b^2}{a}.$$

■

### A.4 The area of an ellipse

**Theorem A.7** The area of an ellipse is  $\pi ab$ .

**Proof** From Equation 11

$$y_e = \frac{b}{a} \sqrt{a^2 - x^2},$$

so the area of an ellipse is

$$A_e = 2 \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} dx = 2 \frac{b}{a} \int_{-a}^a \sqrt{a^2 - x^2} dx = \frac{b}{a} A_c.$$

If we can show that the area of a circle is  $\pi a^2$  the theorem follows.

■

<sup>7</sup>Normally, points are denoted by upper-case letters and line segments or lengths by lower-case letters, but  $L$  for the latus rectum is the standard notation.

**Theorem A.8** *The area of a circle is  $\pi a^2$ .*

**Proof** The proof uses polar coordinates, where  $x = a \cos \theta$  and  $y = a \sin \theta$ . First, we derive the formula for the integral of  $\sin^2 \theta$  using the double-angle identity.

$$\begin{aligned}\cos 2\theta &= 1 - 2 \sin^2 \theta \\ \sin^2 \theta &= \frac{1 - \cos 2\theta}{2} \\ \int \sin^2 \theta \, d\theta &= \int \frac{1 - \cos 2\theta}{2} \, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C.\end{aligned}$$

Now we can compute the area of a circle as twice the area of a semicircle by changing from Cartesian to polar coordinates and integrating.

$$\begin{aligned}A_c &= 2 \int_{-a}^a \sqrt{a^2 - x^2} \, dx \\ &= 2 \int_{-\pi}^0 \sqrt{a^2 - (a \cos \theta)^2} \, d(a \cos \theta) \\ &= 2 \cdot a \cdot a \int_{-\pi}^0 \sin \theta (-\sin \theta) \, d\theta = -2a^2 \int_{-\pi}^0 \sin^2 \theta \, d\theta \\ &= -2a^2 \left( \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C \right) \Big|_{-\pi}^0 = \pi a^2.\end{aligned}$$

■

## A.5 The angles between a tangent and the lines to the foci

We will need the following theorem:

**Theorem A.9 (Angle bisector theorem)** *In  $\triangle ABC$  let the angle bisector of  $\angle BAC$  intersect  $BC$  at  $D$  (Figure 36). Then:*

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

**Proof** We prove the theorem by computing the areas of two triangles in two ways: the base and the height, and the base, the angle and a side.

$$\begin{aligned}\triangle ABD &= \frac{1}{2} BD h = \frac{1}{2} AB AD \sin \alpha \\ \frac{BD}{AB} &= \frac{AD \sin \alpha}{h} \\ \triangle ACD &= \frac{1}{2} CD h = \frac{1}{2} AC AD \sin \alpha \\ \frac{CD}{AC} &= \frac{AD \sin \alpha}{h} \\ \frac{BD}{CD} &= \frac{AB}{AC}.\end{aligned}$$



10

By construction  $SU = 2a$  so  $PU = PH$  and by the angle bisector theorem,

$\triangle UWQ$  and  $\triangle HWQ$  are congruent by side-angle-side ( $UW = HW$ , the right angle and the common side  $QW$ ),  $UQ = HQ$ . Suppose that  $Q$  is on the ellipse. Then  $2a = SQ + HQ = SQ + QU$ . By the triangle inequality  $2a = SQ + QU > SU = 2a$ , contradicting that  $Q$  is on the ellipse.  $\blacksquare$

36

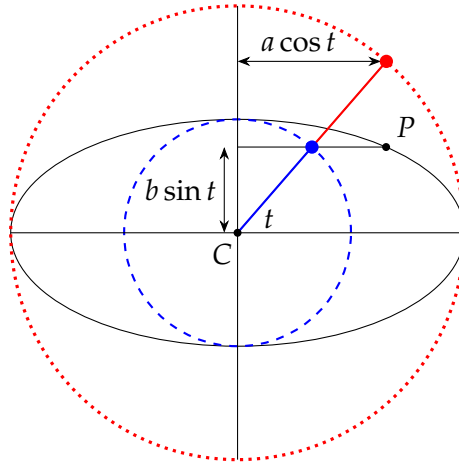


Figure 38: Parametric representation of an ellipse

## A.6 The parametric representation of an ellipse

Figure 38 shows an ellipse and two circles: one whose radius is the length of the semi-major axis (red) and one whose radius is the semi-minor axis (blue). The Figure shows the *parametric representation* of a point  $P = (x, y)$  on the ellipse:

$$(x, y) = (a \cos t, y = b \sin t).$$

The parameter  $t$  is *not* the angle of  $P$  relative to the positive  $x$ -axis. It is constructed by projecting  $P$  onto the minor axis. Then the line at angle  $t$  is constructed from  $C$  to the intersection of the projection with the inner circle (blue dot). The line is extended to intersect the outer circle (red dot). The parametric representation of  $P$  is computed by multiplying the lengths of the axes by the appropriate trigonometry function.

## A.7 Parallelograms formed by tangents to conjugate diameters

There are two equivalent definitions of conjugate diameters.

### Definition A.11

- Let  $P$  be a point on an ellipse and  $t$  the tangent to the ellipse at  $P$ . Then the diameter  $DK$  is a conjugate diameter if it is parallel to  $t$  (Figure 39).
- Two diameters  $PG$  and  $DK$  are conjugate diameters if the midpoints of chords  $(D'K', D''K'')$  parallel to on diameter  $(DK)$  lie on another diameter  $(PG)$ .

**Theorem A.12** All parallelograms formed by tangents to the intersections of conjugate diameters with the ellipse are equal (Figure 42).

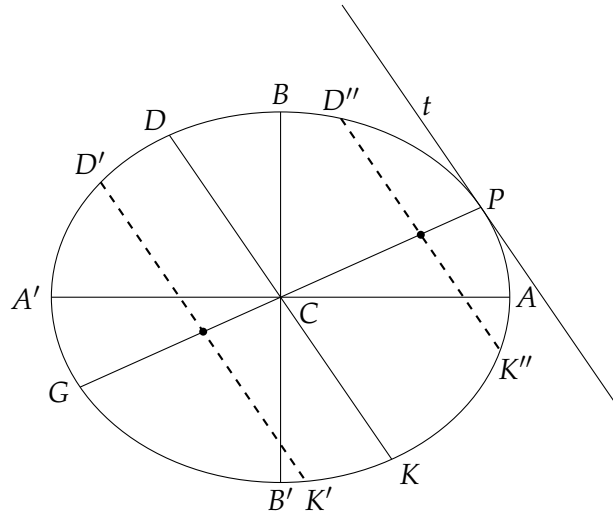


Figure 39: A conjugate diameter

**Proof** We will show that the area of the parallelogram  $JKLM$  is equal to the area of the parallelogram formed by tangents to the major and minor axes (dashed rectangle). By symmetry it suffices to prove that the areas of one of the four quadrants of those parallelograms are equal:  $A_{ACBC'} = A_{PCDJ}$ . Furthermore, since the diagonals bisect a

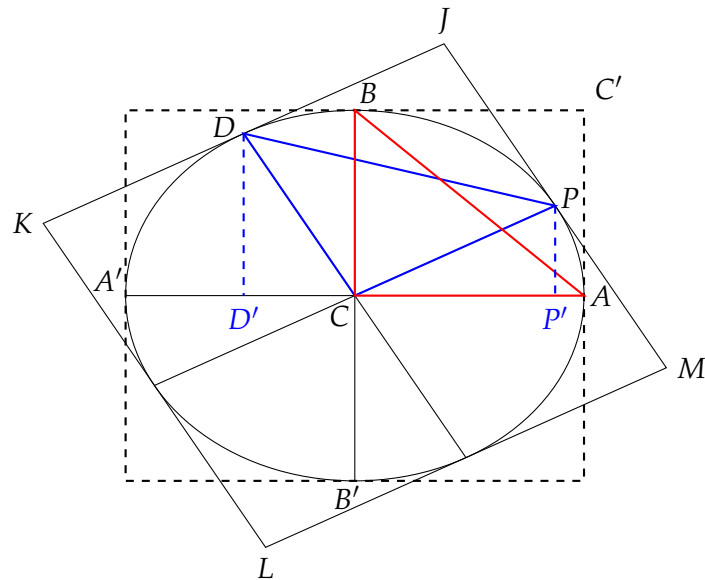


Figure 40: Parallelograms formed by conjugate diameters

parallelogram, it suffices to prove that the area of  $\triangle ABC$  (red) equals the area of  $\triangle PCD$  (blue).

Let  $P = (x_p, y_p) = (a \cos t, b \sin t)$  in parametric representation. Since conjugate diameters are perpendicular,  $\angle DCP$  is a right angle and

$$D = (x_d, y_d) = (a \cos(t + \pi/2), b \sin(t + \pi/2)) = (-a \sin t, b \cos t).$$

Construct  $DD' = (x_d, 0)$  and  $PP' = (x_p, 0)$  perpendicular to the major axis. The area of  $\triangle PCD$  can be computed as the area of the trapezoid  $P'PDD'$  minus the areas of the triangles  $\triangle D'DC$ ,  $\triangle P'PC$ .

$$\begin{aligned} \triangle PCD &= P'PDD' - \triangle D'DC - \triangle P'PC \\ &= \frac{y_p + y_d}{2}(x_p - x_d) - \frac{1}{2}x_d y_d - \frac{1}{2}x_p y_p \\ &= \frac{1}{2}(x_p y_d - x_d y_p) \\ &= \frac{1}{2}(a \cos t \cdot b \cos t - (-a) \sin t \cdot b \sin t) = \frac{1}{2}ab = \triangle ABC. \end{aligned}$$

■

## A.8 Ratios

**Theorem A.13** Let  $P = (x, y)$  be a point on an ellipse (not on the major axis) and drop a perpendicular  $PV$  from  $P$  to the major axis (Figure 40). Then

$$\frac{A'V \cdot AV}{PV^2} = \frac{a^2}{b^2}.$$

**Proof** Using the formula for a point on the ellipse (Equation 10) we get

$$\begin{aligned} y^2 &= b^2 \cdot \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2(a^2 - x^2)}{a^2} \\ \frac{A'V \cdot AV}{PV^2} &= \frac{(a+x)(a-x)}{y^2} = \frac{a^2(a^2 - x^2)}{b^2(a^2 - x^2)} = \frac{a^2}{b^2}. \end{aligned}$$

■

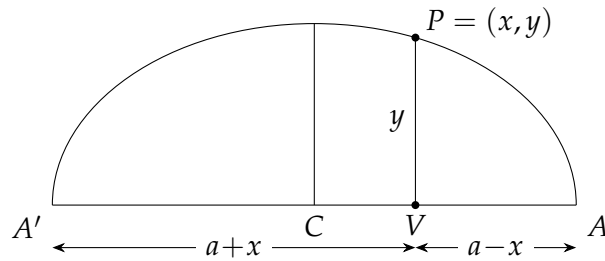


Figure 41: Ratios on conjugate diameters

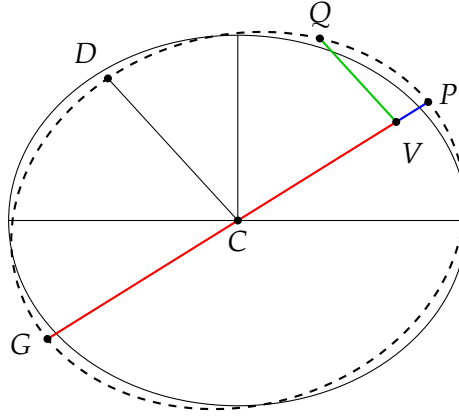


Figure 42: Ratios on conjugate diameters

## A.9 Ratios on conjugate diameters

**Theorem A.14** (Figure 42)

$$\frac{GV \cdot PV}{QV^2} = \frac{CP^2}{CD^2}$$

$$PV = \frac{QV^2 \cdot CP^2}{GV \cdot PV \cdot CD^2}.$$

**Proof** Figure 42 shows a dashed ellipse which is the original ellipse rotated about the same center  $C$ , so that  $CP$  is the semi-major axis and  $CD$  is the semi-minor axis. By Theorem A.13

$$\frac{GV \cdot PV}{QV^2} = \frac{a'^2}{b'^2}.$$

The ellipses are different so  $a' \neq a$  and  $b' \neq b$ , but by construction  $a' = CP$  and  $b' = CD$  so the theorem follows. ■



## Further reading

This document is based primarily on Hahn's book [4]. He has written a more advanced book on the orbits of planets and spacecraft [5]. Section 6 is based on [3]. Two additional articles present this aspect of Newton's work [6, 7]. Should you wish to read Newton's work, [2] is an up-to-date translation into English. Cohen's length Guide will enable you to follow Newton's often-terse presentation. The relevant sections of the Guide are 10.8–10.10. The proofs of the theorems on ellipses is based on [8]. Currently, there seems to be less interest in the geometry of conic sections; a comprehensive book on the subject is [1], although the language and notation can be unfamiliar.

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