The Geometry of Ellipses and Planetary Orbits

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Contents

1 Introduction

Everyone "knows" that Kepler discovered that the orbits of the planets are ellipses and that Newton showed that an elliptical orbit implies that the force of gravity must be inversely proportional to the square of the distance from the Sun. Although I knew these facts, I had never seen them demonstrated.

Calculus in Context [5] by Alexander J. Hahn is a comprehensive textbook on introductory calculus that augments theory with applications in physics and astronomy, such as the work of Kepler, Newton and Galileo, as well as applications in engineering such as building bridges and domed structures. These are not just historical anecdotes but detailed computations.

This document contains a detailed explanation of one topic from Hahn's book: the determination of orbits by Aristarchus, Copernicus, Kepler and Newton. The presentation is mathematical, since the historical and astronomical aspects are thoroughly described in [5], as well as in other works. The document is intended to enrich the learning of mathematics by secondary-school students and students in introductory university courses. The prerequisites are a very good knowledge of Euclidean geometry along with some trigonometry, a bit calculus and Newton's laws of motions.

Section 2 presents the measurements of the radii and distances of the Earth, Moon and Sun by Eratosthenes and Aristarchus. Section 3 describes the construction of a model of a Sun-centered system by Nicolaus Copernicus. Section 4 shows how Johannes Kepler developed his three laws of planetary motion and Section 5 presents Isaac Newton's derivation of the inverse-square law of gravitation from of Kepler's laws. One step of Newton's derivation requires a theorem whose proof is very long, so it is split off into Section 6. Section 7 contains more than you ever wanted to know about the mathematics of ellipses, but these theorems are necessary. I suggest that you look up each theorem (and its proof) as needed, rather than trying to study them all at once. Section 8 is discussed in the next paragraph. Theorems of Euclidean geometry that are used but do not concern ellipses are collected in Appendix A.

Euclidean geometry

It is easy to measure angles. We are familiar with the use of a protractor in school and these can be scaled-up to obtain more accurate measurements. Measuring long distances was impossible until the recent inventions of radar and lasers. At most one could pace-off distances with low accuracy. The only *measured* distance used here is the estimate of 800 km by Eratosthenes for the distance between two places in Egypt (Section 2.1). For this reason, the mathematics used is primarily Euclidean geometry, in particular, similar triangles and ratios of their sides.

The final steps in Newton's derivation requires the use of limits, which had been used already by Archimedes to compute the circumference and area of a circle by approximating the circle. Newton (along with his contemporary Gottfried Wilhelm Leibniz) developed the calculus from the concept of limits. However, Newton's *Principia* uses Euclidean geometry almost exclusively, although analytic geometry had been developed by René Descartes and Pierre de Fermat even before Newton was born.

Newton expected his readers to have an extensive knowledge of geometry. This expectation continued until relatively recently:

> In book 1, prop[osition] 10 (and notably in prop[osition] 11), Newton made use of a property of conics which he presents without proof, merely saying that the result in question comes from "the *Conics*." Here, as elsewhere in the *Principia*, Newton assumes the reader to be familiar with the principles of conics and of Euclid. In the eighteenth and nineteenth centuries, when Newton's treatise was still being read in British universities, authors of books on "conic sections"—for example, W. H. Besant, W. H. Drew, Isaac Milnes—supplied the proof of this theorem in order to help readers of the *Principia* who might be baffled by the problem of finding a proof. They even chose letters to designate points on the diagrams so that the final result would appear in exactly the same form as in the *Principia* [2, p. 330].

To keep this spirit alive, Section 8 brings the necessary proofs of ellipses using Euclidean geometry. The reader is forewarned, however, that as Euclid said, "There is no royal road to geometry!"

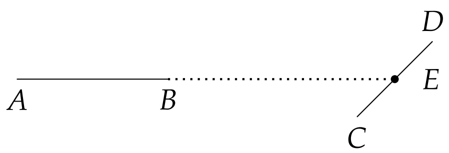
Style and notation

The computations in Hahn's book are faithful to the historical record, for example, measuring distances in units such as the stadia of the Greeks. Here, the computations are fully modernized and use modern units such as kilometers.

Diagrams are used to facilitate understanding each step in the geometrical proofs, more than appear in other sources. For example, Newton proved his difficult theorem (Section 6) using only a single diagram. The diagrams are not to scale and are often distorted. Otherwise, it would be impossible to draw planetary orbits on a piece of paper.

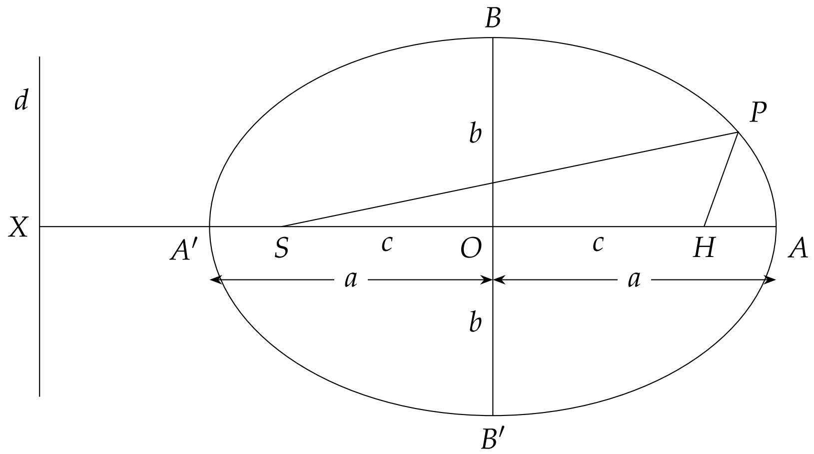
The following shortcuts facilitate a less verbose presentation:

- AB denotes both a line segment and its length.
- $\triangle ABC$ denotes both a triangle and its area.
- The phrase "AB intersects CD" is used even if AB needs to be *extended*[1] until it intersects CD:



[1] The term used by Newton is *produced*.

- The notation in the following diagram will be used consistently to refer to elements of an ellipse. The diagram is for reference since the terms will only be defined when needed.
  - O is the center of the ellipse and a, b are the semi-major and semi-minor axes of an ellipse. A, A' are the vertices on the major axis and B, B' are the vertices on the minor axes.
  - S and H are the foci and c is the distance from a focus to the center.
  - X is the intersection of the major axis with the directrix d. (There is a second directrix on the right side of the ellipse.)



2 The sizes of the Earth, Moon and Sun

2.1 Eratosthenes's measurement of the radius of the earth

The ancient Greeks knew that the Earth is round and Eratosthenes was able to measure the radius of the Earth (Figure 1). Choose two points A, B on the same longitude and measure the distance d between them. Plant a vertical stick (red) in the ground at A and another (blue) at B. On a day in the year when the stick at A produces no shadow at noon, at the same time the stick at B produces a shadow whose angle is α. The sun is so far away from the Earth that over the relatively short distance d, the rays of the Sun are essentially parallel. By alternate interior angles, the angle between the two sticks as measured from the center of the Earth is also α.

The angle that Eratosthenes measured at the blue stick was

$$\alpha = 7.5^\circ \cdot \frac{2\pi}{360} \approx 0.131 \text{ radians},$$

and the distance d between A and B was known to be approximately 800 km. The arc $\widehat{AB}$ subtends the angle $\alpha = d/r_e$ where r_e is the radius of the Earth, so

$$r_e = \frac{d}{\alpha} = \frac{800}{0.131} \approx 6107 \text{ km}. \tag{1}$$

This value is quite close to the modern measurement of 6370 km.

2.2 Aristarchus's measurements

Using r_e, Eratosthenes's measurement of the radius of the Earth, Aristarchus was able to measure and compute the following values:

- r_m: the radius of the Moon,
- r_s: the radius of the Sun,
- d_m: the distance from the Earth to the Moon,
- d_s: the distance from the Earth to the Sun.

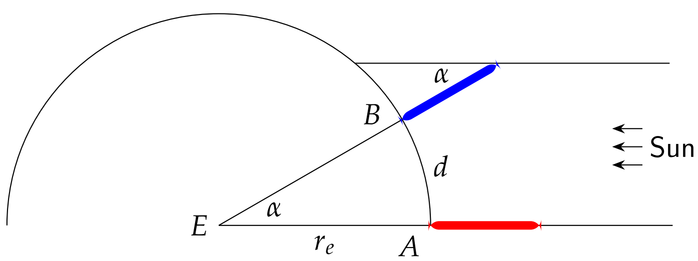


Figure 1: Eratosthenes's measurement of the radius of the earth

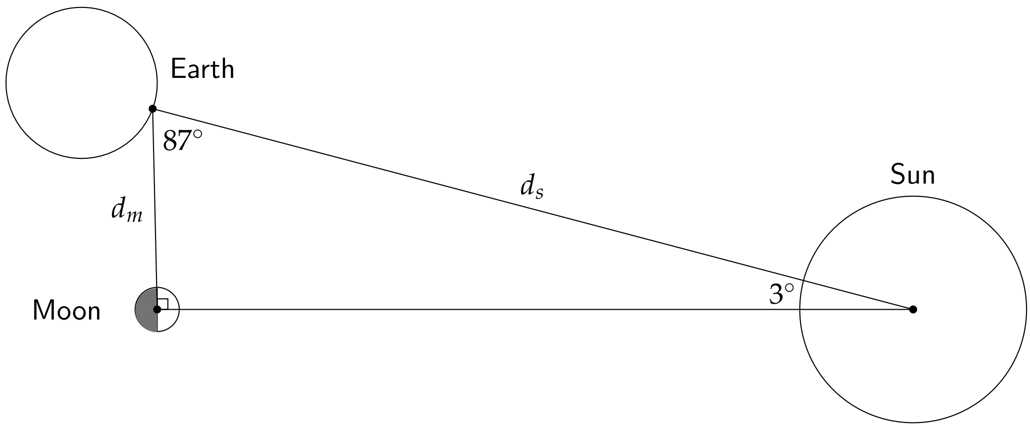


Figure 2: Observing a first quarter moon

Computing d_s/d_m

An observer on Earth can follow the phases of the Moon as it revolves around the Earth. At one point in the month the phase will be first quarter, meaning that the one half of the moon is illuminated while the other half is not (Figure 2). The angle between the Sun and the Moon will be 87°. Since exactly half of the moon is illuminated, we know that the angle $\angle$ Earth-Moon-Sun is a right-angle so

$$\begin{aligned} \cos 87^\circ &= \frac{d_m}{d_s} \\ \frac{d_s}{d_m} &= \frac{1}{\cos 87^\circ} \approx 19\,. \end{aligned} \tag{2}$$

Computing r_s/r_m and d_m/r_m

The Moon is much, much smaller than the Sun, but it is also much, much closer to the Earth. When the Moon is precisely positioned between the Earth and the Sun, its "disk" exactly covers the "disk" of the Sun, causing a total solar eclipse (Figure 3).

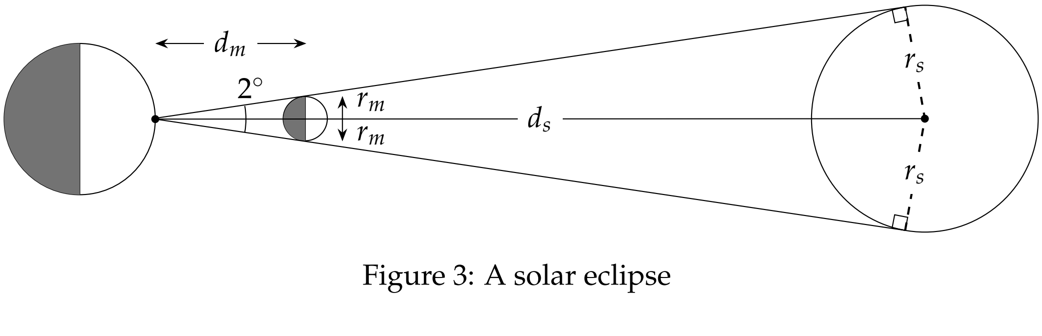


Figure 3: A solar eclipse

The angle subtended by the Moon is 2° degrees. Bisecting the angle creates two right triangles with an acute angle of 1°, where the right angles are the tangents to Moon and the Sun. By similar triangles, Equation 2 and Figure 3,

$$\frac{r_s}{r_m} = \frac{d_s}{d_m} = 19 \tag{3}$$

$$\frac{d_m}{r_m} = \frac{1}{r_m/d_m} = \frac{1}{\sin 1^\circ} \approx 57\,. \tag{4}$$

Computing the radii and distances

Figure 4 shows a lunar eclipse. Unlike a solar eclipse where the Moon exactly covers the Sun, the Earth more than covers the Moon and its shadow is four times the Moon's radius.

Figure 5 show a lunar eclipse annotated with the distances d_m, d_s and the radii r_m, r_e, r_s. The ray from the top of the Sun is tangent to both the Sun and the Earth, so it forms right angles with their radii, as well as with the extension of the Moon's radius. The thick horizontal lines are constructed parallel to the line connecting the centers, forming two similar right triangles, so using Equation 3,

$$\frac{r_s - r_e}{r_e - 2r_m} = \frac{d_s}{d_m} = \frac{r_s}{r_m}$$

$$r_s r_e + r_m r_e = 3 r_s r_m\,.$$

Again from Equation 3, $r_s = 19r_m$, so

$$r_m = \frac{20}{57} r_e\,.$$

By Equation 1, $r_e \approx 6107$ km, by Equation 4, $d_m = 57r_m$, and by Equation 2, $d_s = 19d_m$, so we can compute the radii and distances:

$$\begin{aligned} r_m &= \frac{20}{57} r_e \approx 2143 \text{ km} \\ r_s &= 19r_m \approx 40,713 \text{ km} \\ d_m &= 57r_m \approx 122,140 \text{ km} \\ d_s &= 19d_m \approx 2,320,660 \text{ km}\,. \end{aligned}$$

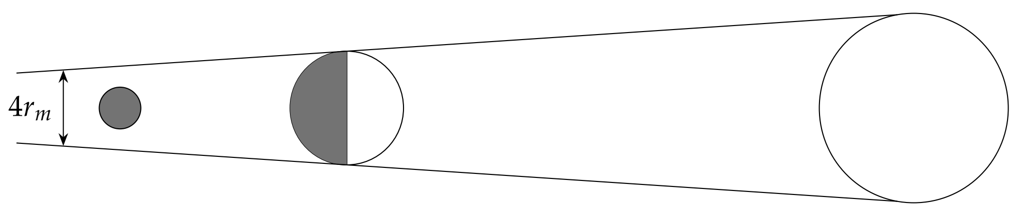


Figure 4: A lunar eclipse

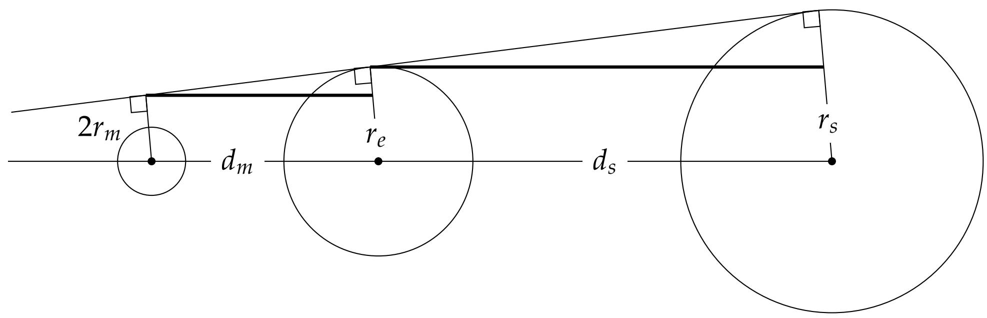


Figure 5: Detail of a lunar eclipse

The following table summarizes these data together with the modern values [5, Table 1.3]. While the computed values for the radii of the Earth and the Moon are not far off from the modern values, the other computed values are not anywhere near the modern values. Nevertheless, they do show that the Greeks understood the immense size of the solar system.

		Computed (km)	Modern (km)
r_e	radius of Earth	6107	6370
r_m	radius of Moon	2143	1740
r_s	radius of Sun	$40,713$	$695,500$
d_m	distance Earth-Moon	$122,140$	$384,570$
d_s	distance Earth-Sun	$2,320,660$	$150,000,000$

3 The Sun-centered solar system

As everyone living far from equator knows, the time between sunrise and sunset varies with the seasons. The reason is that the axis of the rotation of the Earth is offset by $23.5°$ relative to the orbit of the Earth. The plane of the orbit of the Earth around the Sun is called the *ecliptic*. Measuring the length of the day as the time from sunrise to sunset, there is a day in June, called the *summer solstice*, when the length of the day is longest. Similarly, there is a day in December, called the *winter solstice*, when the length of the day is shortest.[2] There are also two days when the length of the day equals the length of the night: the *autumn equinox* in September and the *spring equinox* in March.

Today we know that the universe is immensely large and that the stars are moving at extremely high speeds, but an observer on Earth sees them as if their positions are fixed on a sphere around the earth, called the *celestial sphere*. This solstices and equinoxes can be associated with the projection of the Sun on the celestial sphere as seen from the Earth. The details of the Earth's orbit can be found in books on astronomy, as well as in the Wikipedia articles on *Equinox* and *Solstice*.

3.1 The length of the year and the length of the seasons

Let us assume that the Earth orbits the Sun in a circle, such that the center of the orbit O is the center of the Sun S. In Figure 6 the inner circle is orbit of the Earth and the outer circle is the celestial sphere. The orbit can be divided into four quadrants called *seasons*: spring, summer, autumn, winter.

[2]This holds for the northern hemisphere; in the southern hemisphere the opposite holds.

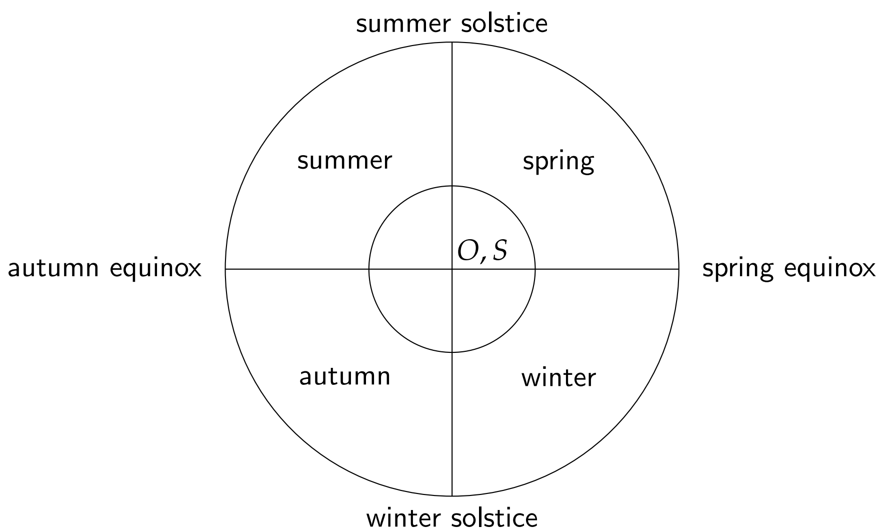


Figure 6: The orbit of the Earth and the seasons

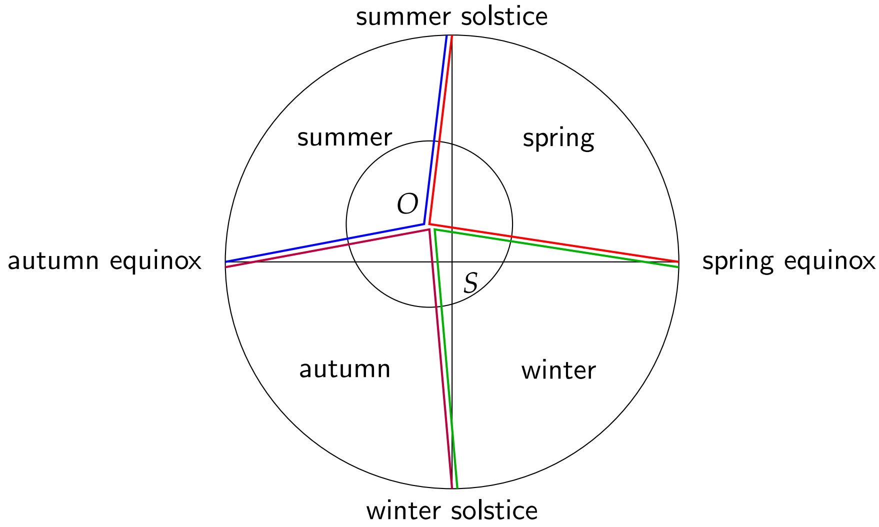


Figure 7: The lengths of the seasons are not equal

The length of a year is approximately $365\frac{1}{4}$ days. The extra $\frac{1}{4}$ day is accounted for by adding a day in leap years.[3] The length of each season as determined by the equinoxes and the solstices is $365.25/4 = 91.3125$ days. However, measurements by the Greek astronomer Hipparchus showed that the actual lengths of the seasons differed from this number and a model of the solar system must be able to explain these differences:

Season	Days	%
Spring	$94\frac{1}{4}$	25.8
Summer	$92\frac{1}{2}$	25.3
Autumn	$88\frac{1}{8}$	24.1
Winter	$90\frac{1}{8}$	24.7

In his Earth-centered solar system, Hipparchus proposed that the center of the Earth's orbit is offset from the center of the Sun. Copernicus used the same idea in his Sun-centered solar system (Figure 7). If the center of the Earth's orbit is in the upper-left quadrant of the coordinate system defined by the equinoxes and the solstices, the angles for spring and summer are obtuse, so the seasons are longer than one-fourth of a year, whereas the angles for the autumn and winter are acute, so they are shorter than one-fourth of a year.

Figure 8 shows a magnified and distorted view of Figure 7. It has been annotated with additional lines and labels that will facilitate the demonstration of Copernicus's computation. The axes $A'C'$ and $B'D'$ have their origin O at the center of the Earth's orbit and are parallel to the axes in the ecliptic. The dashed lines from O are all radii of the Earth's orbit that will be denoted r. The dotted right triangles will be used in the computation.

[3] The length of a year is actually 365.2425. In the sixteenth century, the *Gregorian calendar* accounted for the difference by removing three leap years in every four hundred years.

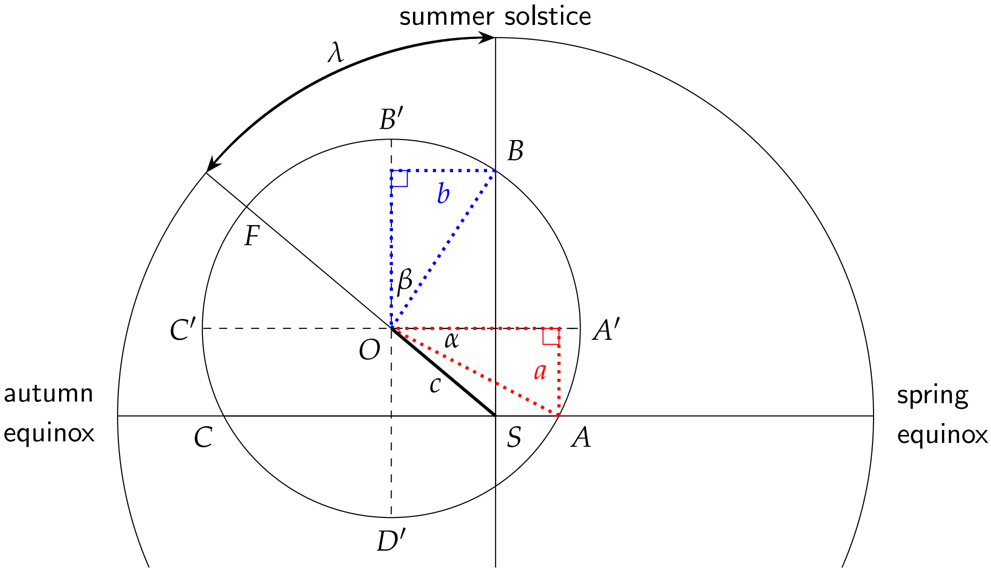


Figure 8: Computing the center of the Earth

3.2 The location of the center of the Earth's orbit

Copernicus's task was to locate the position of the center of the Earth's orbit O relative to the center of the Sun S. This will be given in polar coordinates $OS = c$ and $\angle FSB = \lambda$ (the label is on the large circle of the ecliptic). The strategy of the computation is as follows:

- Initially, we compute the angles of the arcs in radians; the lengths of the arc can then be obtained by multiplying by the radius r.
- We use the lengths of the seasons that Copernicus used: summer is $93\frac{14.5}{60}$ days and spring is $92\frac{51}{60}$.
- The angle of the arc $\widehat{AC}$ can be computed from the combined length of spring and summer and the angle of the arc $\widehat{AB}$ can be computed from the length of spring.
- From $\widehat{AC}$ and $\widehat{AB}$, the angle α subtended by $\widehat{AA'}$ and the angle β subtended by $\widehat{BB'}$ can be computed.
- Since the Earth is very close to the Sun relative to the radius of its orbit, $r\,\widehat{AA'}$ and $r\,\widehat{BB'}$ approximate the line segments a and b. From these c and λ can be computed.

Computing the angles of the arcs $\widehat{AB}, \widehat{AC}$

The arcs $\widehat{AB}, \widehat{AC}$ are sectors of the Earth's orbit and their angles are their proportions of a full year times 2π radians.

$$\widehat{AB} = 2\pi \cdot \frac{92\frac{51}{60}}{365.25} = 2\pi \cdot \frac{92.85}{365.25} = 1.5972 \text{ radians}$$

$$\widehat{AC} = 2\pi \cdot \frac{92\frac{51}{60} + 93\frac{14.5}{60}}{365.25} = 2\pi \cdot \frac{186.09}{365.25} = 3.2012 \text{ radians}\,.$$

Computing the angles of the arcs $\widehat{AA'}, \widehat{BB'}$

Let us express the arcs $\widehat{AC}$ and $\widehat{AB}$ in terms of the arcs that comprise them. Since AC is parallel to $A'C'$, $\widehat{AA'} = \widehat{C'C}$. Compute $\widehat{AA'}$:

$$\begin{aligned}\widehat{AC} &= \widehat{AA'} + \widehat{A'C'} + \widehat{C'C} = 2\widehat{AA'} + \pi \\ \widehat{AA'} &= \frac{1}{2}(3.2012 - \pi) = 0.0298 \text{ radians}\,.\end{aligned}$$

Now that we have computed $\widehat{AB}$ and $\widehat{AA'}$ we can compute $\widehat{BB'}$:

$$\begin{aligned}\widehat{AB} &= \widehat{AA'} + \widehat{A'B'} - \widehat{BB'} \\ \widehat{BB'} &= 0.0298 + \frac{\pi}{2} - 1.5927 = 0.0034 \text{ radians}\,.\end{aligned}$$

Computing the lengths of the arcs $\widehat{AA'}, \widehat{BB'}$

OA and OB are radii of the Earth's orbit so

$$\begin{aligned}\sin\alpha &= \frac{a}{r} \approx \alpha \\ a &\approx r\alpha = r\widehat{AA'} = 0.0298r \\ \sin\beta &= \frac{b}{r} \approx \beta \\ b &\approx r\beta = r\widehat{BB'} = 0.0034r\,,\end{aligned}$$

where we have used the assumption that O, the center of the Earth's orbit, is very close to the Sun S so that $\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = \lim_{\beta\to 0} \frac{\sin\beta}{\beta} = 1$.

Computing the position of O relative to S

Figure 9 shows a magnified diagram of a portion of Figure 8. In the dotted triangles, we have already computed the lengths a and b. Since OT is parallel to $A'A$ and TS is parallel to BB', we can label OT by a and TS by b. The emphasized triangle is a right triangle and c, the distance of O from S, can be obtained from Pythagoras's theorem:

$$c = \sqrt{a^2 + b^2} = r\sqrt{(0.0298)^2 + (0.0034)^2} = 0.03r\,.$$

λ can be obtained from trigonometry:

$$\lambda = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0.0034}{0.03} = 0.1129 \text{ radians} \approx 6.47^\circ\,.$$

The distance $0.03r$ is shown in the following table using the values of r from the table on page 7.

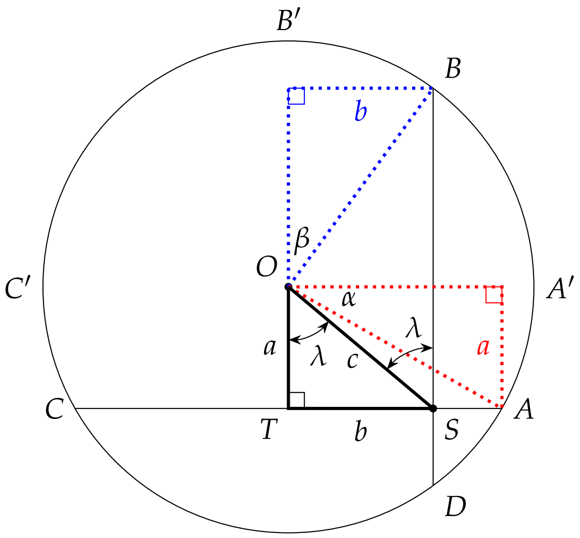


Figure 9: Three triangles

	Aristarchus (km)	Copernicus (km)	Modern (km)
radius of Earth's orbit	$2,320,660$	$8,000,000$	$150,000,000$
distance of O from S	$69,620$	$240,000$	$4,500,000$

4 Elliptical orbits

Towards the end of the sixteenth century, the astronomer Tycho Brahe carried out extremely precise observations. In 1600 he hired Johannes Kepler as his assistant and when Tycho died soon afterwards, Kepler was appointed to his position. Here we explain how Kepler was able to establish that planetary orbits are ellipses.

4.1 Determining the radius of the Earth's orbit

A Martian year is 687 days, that is, it equals $\frac{687}{365.25} = 1.88$ Earth years. We know when Mars reaches a "new year" by observing its projection on the celestial sphere, but each time the position of the Earth in its orbit will be different. Figure 10 shows the orbit of the Earth—its center O offset from the Sun S as Copernicus showed—at four occasions when the position of Mars M at its new year was observed. Four triangles are created $\triangle OE_iM$. Figure 11 shows one of the triangles with the angles labeled. Using the law of sines,

$$\frac{OE_i}{\sin\beta} = \frac{OM}{\sin\alpha}$$

$$OE_i = OM\,\frac{\sin\beta}{\sin\alpha}\,.$$

Tycho Brahe was able to measure all three angles:[4]

	α	β	γ	OE_i
E_1	127.1	20.8	32.1	$0.6682 \cdot OM$
E_2	84.2	35.8	60.5	$0.6721 \cdot OM$
E_3	41.3	42.4	96.4	$0.6785 \cdot OM$
E_4	1.6	3.4	175.0	$0.6805 \cdot OM$

[4]The values are rounded to one decimal point.

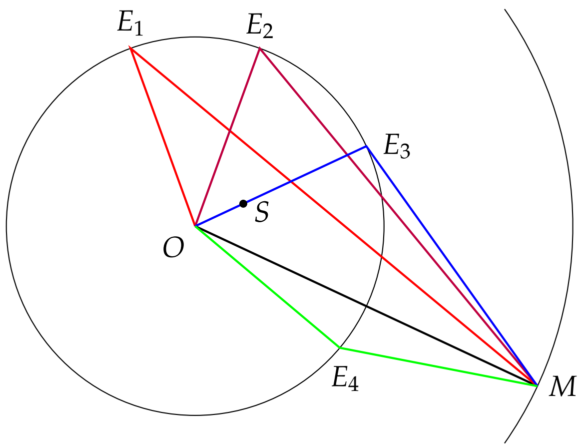


Figure 10: Observations of the orbit of Mars from the Earth

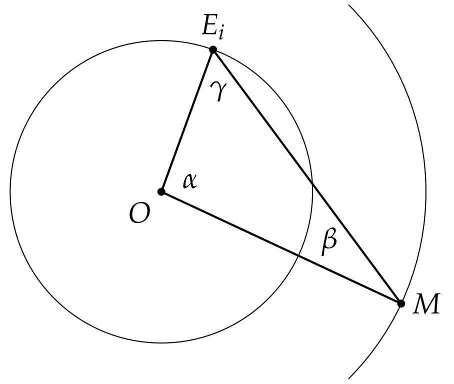


Figure 11: One triangle Earth-Sun-Moon

The values of OE_i given in the fourth column of the table are not equal. Assuming (as everyone did at that time) that the Earth' orbit is circular, the only solution was to move the center of the orbit so that $\{E_1, E_2, E_3, E_4\}$ were all on the circle.

4.2 Measuring the angles in the triangle Sun-Earth-Mars

How can the angles α, β, γ be measured? Since $\triangle E_iOM$ is a triangle, it is sufficient to measure two of the angles. γ is easily measured by observing Mars and the Sun at the same time. However, neither α nor β can be measured directly since they are not accessible to an observer on Earth.

Tycho's measurement used the known periods of the orbits to compute the angles. The Earth moves counterclockwise around Sun. Given any point E, for some t, t days later the Earth will have moved to E' and Mars will have moved to M', so that they are in *opposition*, that is, Mars will be on the continuation of the Earth-Sun line (Figure 12). Since the Earth completes an orbit in about half the time that Mars takes to complete an orbit, t will be such that neither the Earth nor Mars has completed a full orbit. The angles θ_E and θ_M are fractions of a circular orbit of 360°, so

$$
\begin{aligned}
\frac{t}{365.25} &= \frac{\theta_E}{360} \\
\theta_E &= \frac{360}{365.35}\, t \\
\frac{t}{687} &= \frac{\theta_M}{360} \\
\theta_M &= \frac{360}{687}\, t\,.
\end{aligned}
$$

This gives values for θ_E and θ_M. Since $E'M'$ is a straight line, we have that $\alpha - \theta_M = 180° - \theta_E$, so that $\alpha = 180 - \theta_E + \theta_M$, and the values of OE_i can be computed.

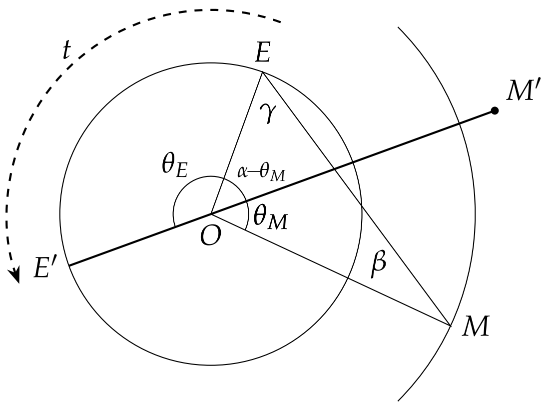


Figure 12: The Earth and Mars in opposition

4.3 A new location for the center of the Earth's orbit

Kepler's next task was to obtain a new value O' for the center of the Earth's orbit such that the E_i's are on the orbit. Given the new locations of the Earth $\{E_1, E_2, E_3, E_4\}$, by Theorem A.1 a circle centered at O' can be constructed that goes through $\{E_1, E_2, E_3\}$ (Figure 13). To verify that this is the correct orbit, check that E_4 is on the circle.

4.4 Orbits are ellipses

While Kepler was able to modify the center of the orbit of the Earth to be consistent with the observations, he was not able to adequately describe the orbit of Mars. After years of work, he came to the conclusion that the orbit must be oval like an egg. Oval, perhaps, but certainly not an ellipse, because he was certain that it would have been discovered by

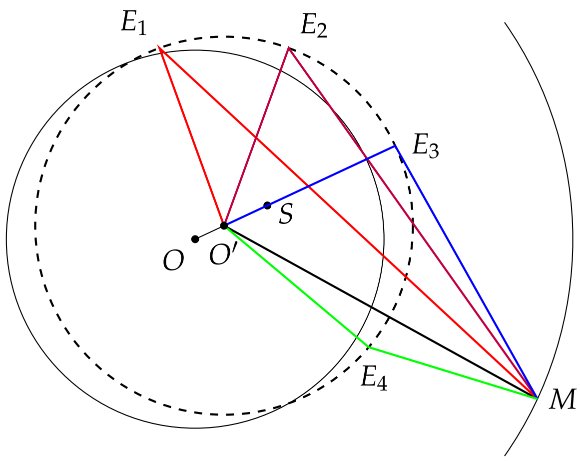


Figure 13: Observations of the orbit of Mars from the new Earth's orbit

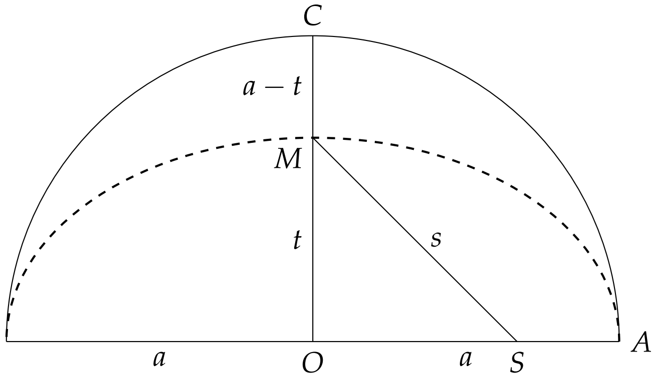


Figure 14: The orbit of Mars as an oval "egg"

Archimedes! Figure 14 shows C, a position on a circular orbit, and an oval orbit (dashed), where M is the position of Mars on the oval corresponding to C. The radius of the circular orbit is labeled a and the unknown distances to M are labeled $s = SM$ and $t = OM$. Kepler's computed that $\dfrac{a-t}{t} = 0.00429$ and $\dfrac{s}{t} = 1.00429$, so that

$$\begin{aligned}\frac{a-t}{t} &= \frac{s}{t} - 1\\ a - t &= s - t\,,\end{aligned}$$

and therefore $SM = s = a = AO = CO$. The dashed oval is likely an ellipse, because in an ellipse $SM = AO$ (by Theorem 7.2). Kepler then computed the projections of the observations of Mars on the x-axis (Figure 15) and obtained for all of them that

$$\frac{M_iO_i}{C_iO_i} = \frac{1}{1.00429} = 0.99573\,.$$

By Theorem 7.4, since the ratio $MO/CO = b/a$ is constant for in an ellipse, Kepler was able to conclude that the orbit of Mars is an ellipse.

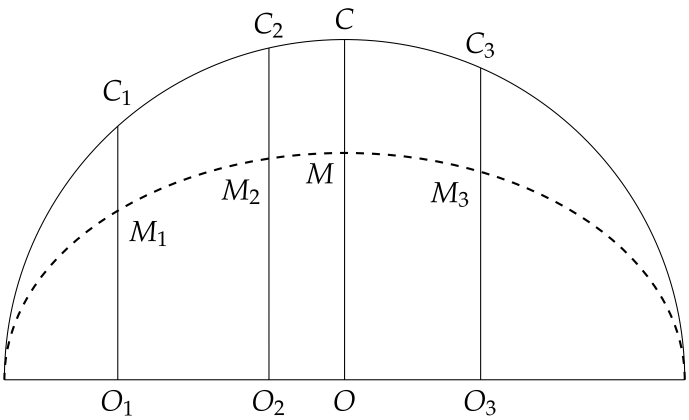


Figure 15: The orbit of Mars as an ellipse

5 Gravitation

In the *Principia* Isaac Newton proved the following theorem led to the theory of universal gravitation.

> **Theorem 5.1** *If a planet subject to a centripetal force follows an elliptical orbit around the Sun, then the force decreases as the inverse square of the distance from the Sun.*

After a review of Newton's Laws of force and motion, we show that Kepler's second law must hold in *any* system subject to a centripetal force. The next step is to show the inverse square law and then it is a small step to universal gravitation and Kepler's third law.

5.1 Newton's laws of motion

1. A body in uniform motion (including a body at rest) continues with the same motion unless a force is applied.
2. A force F applied to a body causes an acceleration a in the direction of the force whose magnitude $a = F/m$, where m, the constant of proportionality, is called the *mass* of the body.
3. If one body exerts a force on a second body, the second body exerts a force on the first of equal magnitude but in the opposite direction.

Forces are denoted by vectors, where the direction of the vector represents the direction of the force and the length of the vector represents the magnitude of the force. Forces can be decomposed into perpendicular components (Figure 16), or into components in any directions (Figure 17). The components form a parallelogram whose diagonal is the resultant force.

Newton was interested in *centripetal force* which is a force exerted by a single body on another, in particular, the gravitational force exerted by the Sun on a planet (Figure 18). Since the only force is that directed towards the Sun, the planet does not move "up" or "down" so its orbit is in a plane.

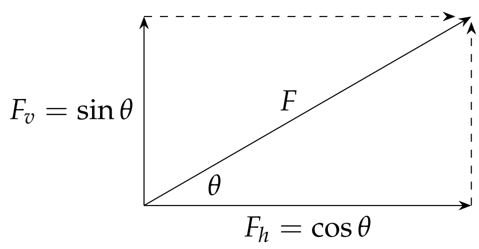


Figure 16: Perpendicular components of a force

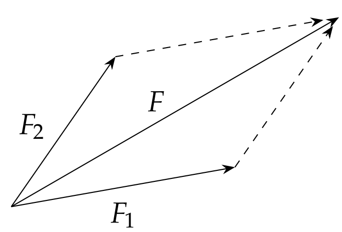


Figure 17: Arbitrary components of a force

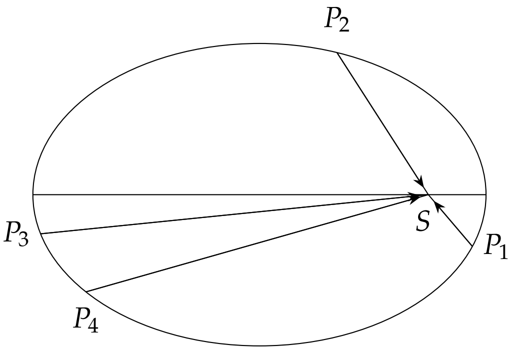


Figure 18: Centripetal force

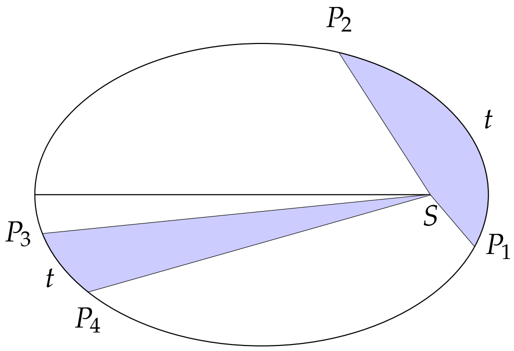


Figure 19: Equal areas in equal times

Kepler's second law states that a planet in orbit sweeps out equal area in intervals of equal duration, that is, if it takes the planet time t to move from P_1 to P_2 and also t to move from P_3 to P_4, then the area of the sector P_1SP_2 is equal to the area of P_3SP_4 (Figure 19). (Obviously, this means that the speed of the planet must vary as it traverses its orbit $v_{P_1P_2} \gg v_{P_3P_4}$.) Newton proved that this must be true in any system where a body is subject to a centripetal force from another body.

The proof is based on dividing an area into very small sectors and then taking the limit. Consider three points A, B, C on the orbit (Figure 20) that represent the positions of the planet at intervals of Δt. For clarity we have drawn them spaced out, but the intention is that they are very close together. Newton assumed that the planet does not smoothly traverse the arcs, but rather that it every Δt it jumps in discrete steps from one point on the orbit to the next.

Figure 21 shows how the force is exerted in discrete steps. The planet moves from A to B and we expect that the centripetal force at B will cause an acceleration that moves the planet to C, the next point on the orbit. Instead, we "pretend" that the force is not applied at B, but, in the absence of an applied force, planet continues to move in the same direction and at the same speed. After another period of Δt as passed and the planet has reached point C', the force is now applied *in the same direction* as it would have been applied at B, moving the planet to C.

Theorem 5.2 *The area of $\triangle ASB$ is equal to the area of $\triangle BSC$.*

Proof The proof will be done in two steps by showing that $\triangle ASB = \triangle BSC'$ and then that $\triangle BSC' = \triangle BSC$.

- In Figure 22, $\triangle ASB$ is shown in blue and $\triangle BSC'$ is shown in red. It is assumed that $AB = BC'$ (the planet moves from B to C' during the same interval Δt), so since SH is the height of both triangles, their areas are equal.

- In Figure 23, $\triangle BSC$ is shown in blue and $\triangle BSC'$ is shown in red. It is assumed CC' is parallel to SB (the planet is subject to the centripetal force at C' in the *same*

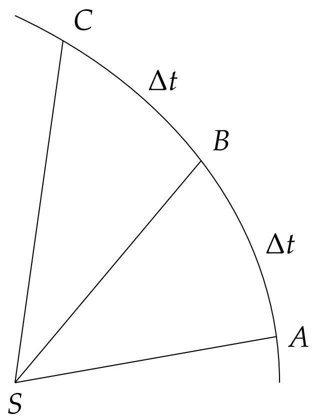


Figure 20: "Small" sectors of an orbit

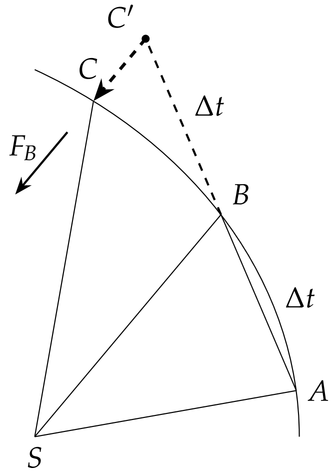


Figure 21: Exerting force at discrete times

direction as the force at B), so the heights of both triangles to the common side SB are equal and their areas are equal. It follows that $\triangle ASB = \triangle BSC' = \triangle BSC$, which we denote by ΔA. ▮

We assume that the sectors of the orbit are each divided up into small sectors of uniform duration Δt. By the theorem, each sector has the same area ΔA. Therefore (see Figure 19),

$$\frac{A_{P_1SP_2}}{\Delta A} = \frac{t}{\Delta t} = \frac{A_{P_3SP_4}}{\Delta A},$$

from which Kepler's second law follows: $A_{P_1SP_2} = A_{P_3SP_4}$.

The proof used two approximations:

- ΔA is an approximation of the area of each sector.
- The force at C' is an approximation to the force at B.

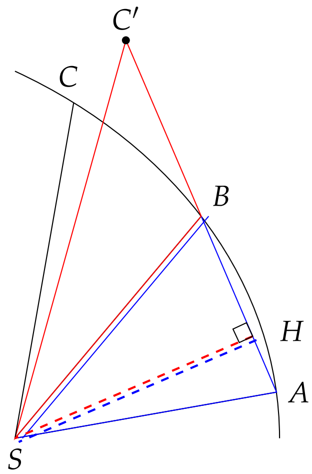


Figure 22: $\triangle ASB = \triangle BSC'$

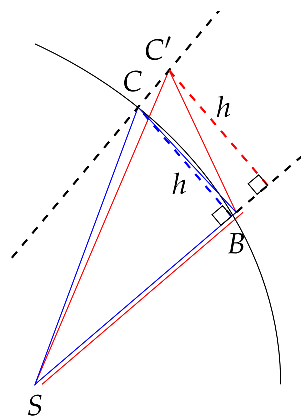


Figure 23: $\triangle BSC' = \triangle BSC$

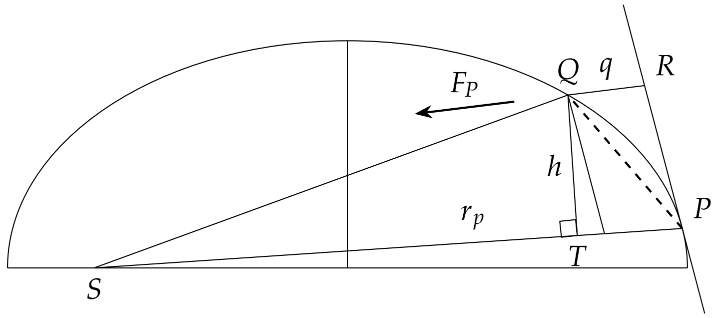


Figure 24: The derivation of the inverse square law

In the limit as the size of the sectors decreases, the errors become negligible.

Definition 5.3 *For a given elliptical orbit, $\kappa = \dfrac{A}{t}$, where A is the area of the ellipse and t is the period of the orbit, is called* Kepler's constant.

5.2 The inverse square law for gravitation

Newton's next step was to show that if the orbit of a planet is elliptical, the centripetal force must be proportional to the mass of the planet and inversely proportional to the square of its distance from the Sun. In Figure 24, S is the Sun, and P and Q are points on the orbit that are very close to each other. PR is the tangent to the ellipse at P, and R is chosen to that QR is parallel to SP. QT is constructed perpendicular to SP. Denote the lengths $r_P = SP$, $h = QT$, $q = QR$ and the time interval from P to Q by Δt.

When a point is subject to an acceleration a for a period of Δt, its displacement is $\frac{1}{2}a(\Delta t)^2$. From Newton's second law we know that at point R, the planet is subjected to an acceleration of F_P/m, so

$$q = \frac{1}{2}\frac{F_P}{m}(\Delta t)^2$$

$$F_P = \frac{2mq}{(\Delta t)^2}\,.$$

Now we compute the area of $\Delta A_{PSQ} \approx \triangle SPQ = (1/2)hr_p$ and use Kepler's constant:

$$\Delta t = \frac{\Delta A_{PSQ}}{\kappa} = \frac{hr_P}{2\kappa}$$

$$F_P = 2mq \cdot \frac{4\kappa^2}{(hr_P)^2} = 8\kappa^2 m \cdot \frac{q}{h^2} \cdot \frac{1}{r_P^2}\,.$$

To obtain an inverse-square law for the force, the first two factors have to be independent of the distance. For a given planet m is constant and for a given elliptical orbit κ is constant, so the first factor does not depend on the distance. What about the second factor q/h^2, in particular, what value does it have as Δt approaches zero?

Theorem 5.4 *In an elliptical orbit*

$$\lim_{\Delta t \to 0} \frac{q}{h^2} = \frac{1}{L},$$

where L is the length of the latus rectum of the ellipse (Definition 7.5).

Newton's proof is very complex and is presented separately in Section 6.

Since L is constant for any given ellipse, the inverse square law can be written

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2}. \tag{5}$$

The formula can be re-written so that the constant values appearing are more familiar: a, the semi-major axis and T, the period of the orbit. By Theorem 7.6, $L{=}2b^2/a$ and by Theorem 7.7, $\kappa = A_{ellipse}/T{=}\pi ab/T$, so

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{r_P^2} = \frac{8(\pi ab)^2 m}{T^2} \cdot \frac{a}{2b^2} \cdot \frac{1}{r_P^2} = \frac{4\pi^2 a^3 m}{T^2} \cdot \frac{1}{r_P^2}. \tag{6}$$

Newton was able to show that:

- The inverse square law applies to all conic sections including a parabola and a hyperbola, not just to an ellipse and, of course, to a circle. Some comets have hyperbolic orbits and orbit the Sun only once.
- The converse holds: if a planet is subject to an inverse-square centripetal force then the orbit must be an ellipse (or another conic section).
- The proof assumes that a planet is a very small point, but the result holds even for large planets as long as the density of the planet is radially symmetric, that is, for a given distance from the center the density is constant.

5.3 Universal gravitation

By Newton's third law, we can equate the force $F_{S\leftarrow E}$ that the Sun S exerts on the Earth E with the force $F_{E\leftarrow S}$ that the Earth exerts on the Sun. Let m be the mass of the Earth and M be the mass of the Sun, then by Equation 5,

$$\begin{aligned}
F_{S\leftarrow E} &= \frac{8\kappa_E^2 m}{L_E} \cdot \frac{1}{r^2} = \frac{C_E m}{r^2} \\
F_{E\leftarrow S} &= \frac{8\kappa_S^2 M}{L_S} \cdot \frac{1}{r^2} = \frac{C_S M}{r^2} \\
\frac{C_E}{M} \cdot \frac{1}{r^2} &= \frac{C_S}{m} \cdot \frac{1}{r^2},
\end{aligned}$$

from some constants C_E, C_S.

Why are the constants different? The Earth and the Sun both rotate around their center of mass called the *barycenter*, which is very close to the center of the Sun since the Sun is so much more massive than the Earth. The ellipse of the Sun's orbit is very small relative to the Earth so A and L are smaller, and the Sun's period is large so T is larger. The different values for $8\kappa^2/L$ are encapsulated into the constants C_E, C_S. Let $G = \dfrac{C_E}{M} = \dfrac{C_S}{m}$ so that

$$F_{S \leftarrow E} = F_{E \leftarrow S} = G\frac{mM}{r^2}\,. \tag{7}$$

This is Newton's law of universal gravitation. It is not specific to planetary orbits but holds between any two bodies with masses m, M.

5.4 Kepler's third law

Theorem 5.5 (Kepler's third law) *Let P_1, P_2 be two planets whose elliptical orbits have semi-major axes a_1, a_2 and whose orbital periods around the Sun are T_1 and T_2. Then*

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2}\,.$$

Proof By Equations 6 and 7,

$$F = \frac{4\pi^2 a_i^3 m}{T_i^2}\frac{1}{r_i^2} = \frac{GmM}{r_i^2}\,. \tag{8}$$

After canceling m and r_i we get

$$\frac{a_i^3}{T_i^2} = \frac{GM}{4\pi^2}\,.$$

$GM/4\pi^2$ is a constant that depends only on the mass of the Sun and the gravitational constant, so a_i^3/T_i^2 is constant for all planets rotating around the Sun. ∎

6 A proof Proposition XI, Problem VI

Theorem 5.4 is Book I, Section III, Proposition XI, Problem VI of the *Principia*.

Study Figure 25:[5]

- Let P, Q be two points on the ellipse that represent the movement of a body in an elliptical orbit separated by a time interval Δt. Construct lines from P to the center C and the foci S, H.
- Construct the tangent at P and choose R on the tangent such that the body would move from P to R if it continued for time Δt not subject to any force. Construct the parallelogram $PRQX$ and extend QX until it intersects PC at V.
- Construct a line parallel to RP through H and let I be its intersection with PS.
- Construct DC the conjugate diameter to PC (Definition 7.9), and let E be its intersection with PS.

6.1 A formula for QR

Theorem 6.1 $QR = PV \cdot \dfrac{CA}{CP}$.

Proof By Theorem 7.8, $\angle RPX = \angle ZPH = \alpha$ and by alternate interior angles,

$$\angle PHI = \angle ZPH = \alpha = \angle RPX = \angle PIH\,,$$

so $\triangle IPH$ (red) is isosceles and $PI = PH = d$.

[5]The bottom half of the ellipse is not shown, but we still refer to lines DC, PC as diameters.

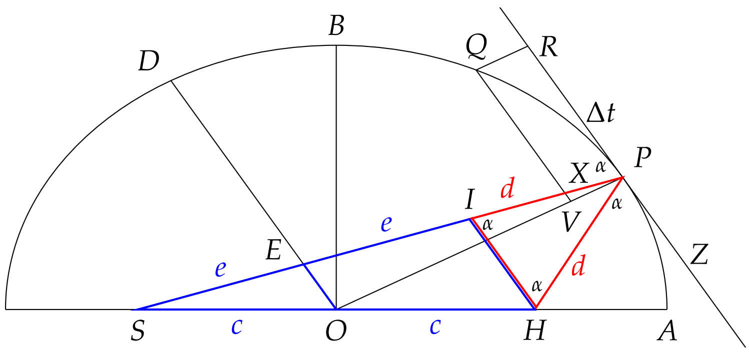


Figure 25: Geometry of an elliptical orbit (1)

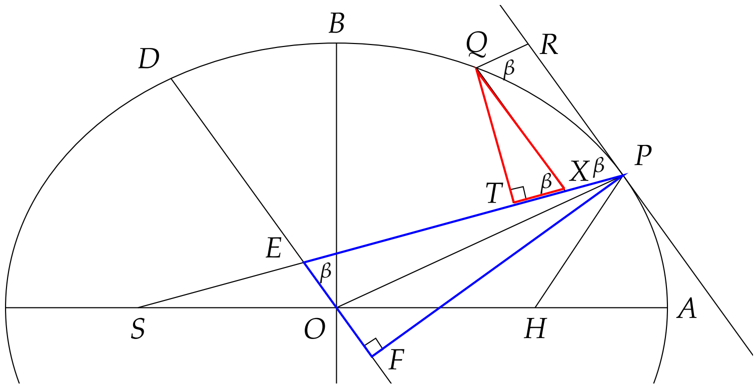


Figure 26: Geometry of an elliptical orbit (2)

$SC = CH = c$ are equal because they are the distances of the foci from the center of the ellipse. Let $SE = e$. By construction $EC \parallel IH$ so $\triangle ESC \sim \triangle ISH$ (blue) and

$$\frac{SC}{SE} = \frac{SH}{SI}$$
$$SI = \frac{SH \cdot SE}{SC} = \frac{2c \cdot e}{c} = 2e\,.$$

By definition of an ellipse $SP + PH = SI + IP + PH = 2CA$ so $2e + d + d = 2CA$ and $EP = e + d = CA$.

$QV \parallel EC$ so $\triangle EPC \sim \triangle XPV$ and

$$\frac{PX}{PV} = \frac{EP}{PC} = \frac{CA}{PC}$$
$$PX = PV \cdot \frac{CA}{PC}\,.$$

Since $PRQX$ is a parallelogram $QR = PX$, $QR = PV \cdot \dfrac{AC}{PC}$. ▮

6.2 A formula for QT

Construct a perpendicular from P to DC and label its intersection with DC by F. Construct a perpendicular from Q to SP and label its intersection with SP by T (Figure 26).

Theorem 6.2

$$QT = QX \cdot \frac{FP}{CA}\,.$$

Proof By construction, $QR \parallel PX$, so by alternate interior angles $\angle RQX = \angle QXT = \beta$. By construction, $QX \parallel DC$, so by alternate interior angles $\angle QXT = \angle PEF = \beta$. Since $\triangle PFE$

and $\triangle QTX$ are right triangles with an equal acute angle β, $\triangle PFE \sim \triangle QTX$. In the proof of Theorem 6.1 we showed that $EP = CA$ so

$$\frac{QT}{QX} = \frac{FP}{EP}$$
$$QT = QX \cdot \frac{FP}{EP} = QX \cdot \frac{FP}{CA}. \quad \blacksquare$$

6.3 A formula for QR/QT^2

Theorem 6.3

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \tag{9}$$

Proof Let use combine the equations in Theorems 6.1 and 6.2 to get QR/QT^2.

$$\frac{QR}{QT^2} = \frac{PV \cdot \dfrac{CA}{CP}}{\left(QX \cdot \dfrac{FP}{CA}\right)^2} = \frac{PV \cdot CA^3}{QX^2 \cdot CP \cdot FP^2}. \tag{10}$$

DC and PC are conjugate diameters so Theorem 7.11 gives a formula for PV that we substitute into Equation 11.

$$\frac{QR}{QT^2} = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2} \cdot \frac{CA^3}{QX^2 \cdot CP \cdot FP^2} = \frac{CP \cdot CA^3}{CD^2 \cdot FP^2} \cdot \frac{QV^2}{GV \cdot QX^2}. \tag{11}$$

Next we show that $CD \cdot FP = CA \cdot CB$. By Theorem 7.12 the areas of the parallelograms formed by the tangents to conjugate diameters are equal. By symmetry the areas of the four small parallelograms are equal, as are the triangles formed by constructing diagonals.

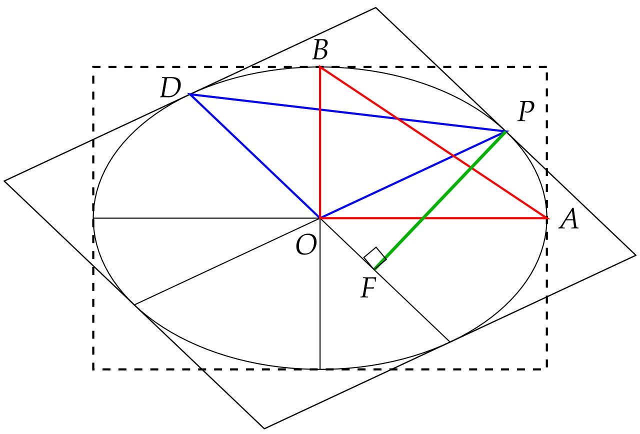


Figure 27: Parallelograms formed by conjugate diameters

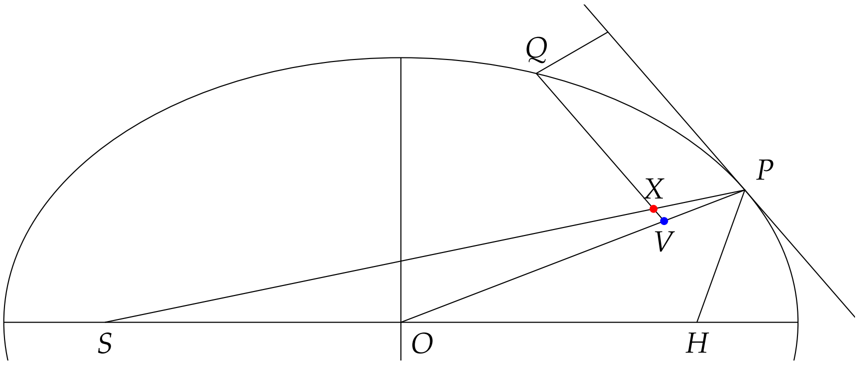


Figure 28: Geometry of an elliptical orbit (3)

In Figure 27 the area of the $\triangle ABC$ (red), which is $(1/2)CA \cdot CB$, is equal to the area of $\triangle PCD$ (blue), which is $(1/2)CD \cdot FP$. Substituting for $CD \cdot FP$ in Equation 11 gives

$$\frac{QR}{QT^2} = \frac{CP \cdot CA^3}{CB^2 \cdot CA^2} \cdot \frac{QV^2}{GV \cdot QX^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2} . \quad \blacksquare$$

6.4 Approaching the limit

Figure 28 is an enlarged diagram of part of Figure 26. As the time interval Δt gets smaller, $Q \to P$ which implies that

- $X \to V$ so that $QX \to QV$.
- $V \to P$ so that $CV \to CP$ and hence $GV \to 2CP$.
- In the limit $QX = QV$ and $GV = 2CP$. Substituting into Equation 9 gives

$$\lim_{Q \to P} \frac{QR}{QT^2} = \lim_{Q \to P} \frac{CP \cdot CA}{CB^2} \cdot \frac{QX^2}{2CP \cdot QX^2} = \frac{CA}{2CB^2} = \frac{a}{2b^2} = \frac{1}{L},$$

using the result of Theorem 7.6 for the length of the latus rectum.

7 Ellipses

7.1 Fundamental properties

Definition 7.1 (Ellipse)

- *Let S and H be two points in the plane such that $SH = 2c \geq 0$ and choose a such that $2a > 2c$ (Figure 29). An* ellipse *is the geometric locus of all points P such that $SP + PH = 2a$. If $c = 0$ the geometric locus is a* circle*.*
- *Construct AA' through SH, where A, B are the intersections of the line with the ellipse. AA' is the* major axis *of the ellipse. Let O be the midpoint of SH. AO, OA' are the* semi-major axes *of the ellipse.*
- *Construct the perpendicular to AA' at O and let B, B' be its intersections with the ellipse. BB' is the* minor axis *of the ellipse and BO, OB' are the* semi-minor axes *of the ellipse.*

Theorem 7.2 *(Figure 30)*

1. *$SB = HB = a$.*
2. *$AO = OA' = a$.*
3. *$BO = OB'$. (Label $BO = OB'$ by b.)*

Proof

1. $\triangle SBO \cong \triangle HBO$ by side-angle-side so $SB = HB$. Since B is on the ellipse, $SB + HB = 2a$ and $SB = HB = a$ follows.

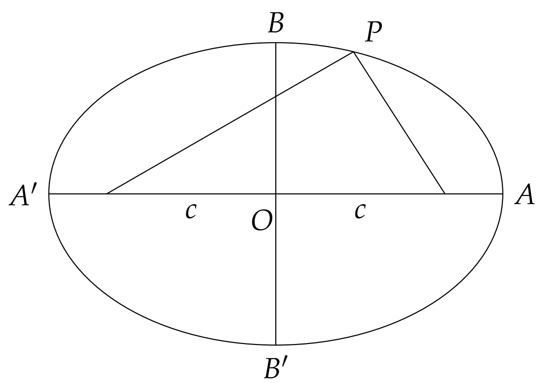


Figure 29: The definition of an ellipse

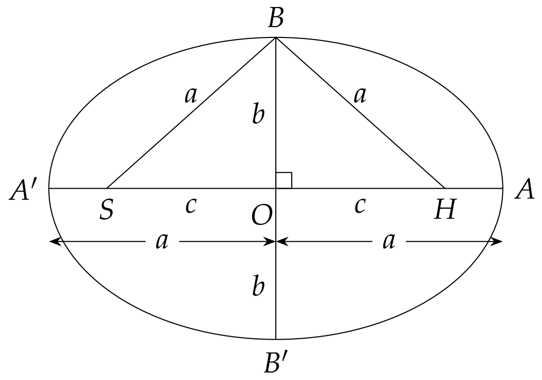


Figure 30: The semi-major and semi-minor axes of an ellipse

2. Since A is on the ellipse,

$$2a = AS + HA = (AO - c) + (AO + c) = 2AO\,,$$

so $AO = a$. $OA' = a = AO$ follows by symmetry.

3. $BO = OB'$ follows from $\triangle SBO \cong \triangle SB'O$. ∎

Theorem 7.3 *A point $P = (x, y)$ on an ellipse satisfies the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{12}$$

Proof Since $S = (-c, 0)$, $H = (c, 0)$ and $SP + PH = 2a$,

$$PS + PH = \sqrt{(x - (-c))^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a\,.$$

Squaring twice results in

$$\begin{aligned}
(x + c)^2 + y^2 &= \left(2a - \sqrt{(x - c)^2 + y^2}\right)^2 \\
4xc &= 4a^2 - 4a\sqrt{(x - c)^2 + y^2} \\
a - \frac{c}{a}x &= \sqrt{(x - c)^2 + y^2} \\
a^2 + \frac{c^2}{a^2}x^2 &= x^2 + c^2 + y^2 \\
\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} &= \frac{a^2 - c^2}{a^2 - c^2} = 1\,.
\end{aligned}$$

By Theorem 7.2 and Pythagoras's theorem, $b^2 = a^2 - c^2$ so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \quad ∎$$

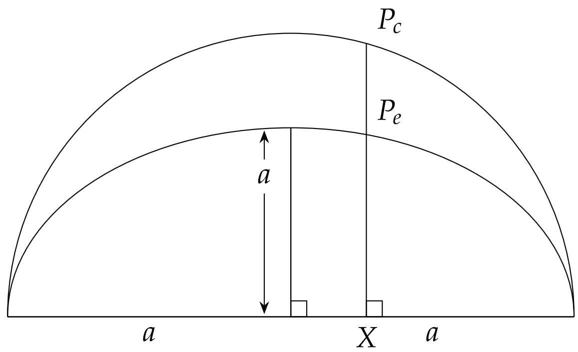


Figure 31: A circle circumscribing an ellipse

7.2 A circle circumscribing an ellipse

Consider a circle of radius a with the same center as an ellipse (Figure 31). Choose a point X on the major axis and construct a perpendicular through $X = (x, 0)$. Let its intersections with the ellipse and the circle be $P_e = (x, y_e)$ and $P_c = (x, y_c)$, respectively.

Theorem 7.4 *The perpendicular to the major axis through a point $P_c = (x, y_c)$ on the circle circumscribing an ellipse intersects the ellipse at $P_e = (x, y_e) = \left(x, \frac{b}{a} y_c\right)$.*

Proof From Equation 12 and the formula $x^2 + y^2 = a^2$ for the circle,

$$y_e = \frac{b}{a}\sqrt{(a^2 - x^2)} = \frac{b}{a} y_c \,. \quad \blacksquare \tag{13}$$

7.3 The latus rectum of an ellipse

Definition 7.5 *Consider a line through a focus of an ellipse that is perpendicular the major axis. Let its intersections with the ellipse be L_1, L_2. Then $L = L_1L_2$ is a* latus rectum *of an ellipse (Figure 32).*[6]

Theorem 7.6 *L, the length of the latus rectum of an ellipse, is $\frac{2b^2}{a}$.*

Proof By Equation 13 and Pythagoras's theorem,

$$L = 2L_1 = 2 \cdot \frac{b}{a}\sqrt{a^2 - c^2} = \frac{2b^2}{a} \,. \quad \blacksquare$$

[6]Usually, lines are denoted by lower-case letters, but L for the latus rectum is the standard notation.

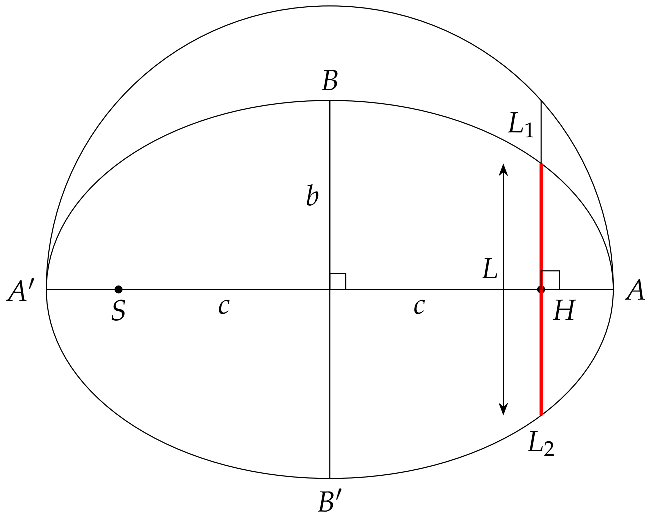


Figure 32: The latus rectum of an ellipse

7.4 The area of an ellipse

Theorem 7.7 *The area of an ellipse is πab.*

Proof From Equation 13

$$y_e = \frac{b}{a}\sqrt{a^2 - x^2}\,,$$

so the area of an ellipse is

$$A_e = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\;dx = \frac{b}{a}\cdot 2\int_{-a}^{a}\sqrt{a^2 - x^2}\;dx = \frac{b}{a}A_c\,.$$

If we can show that the area of a circle is πa^2 the theorem follows.

The proof uses polar coordinates, where $x = a\cos\theta$ and $y = a\sin\theta$. First, we derive the formula for the integral of $\sin^2\theta$ using the double-angle identity.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\int \sin^2\theta\; d\theta = \int \frac{1-\cos 2\theta}{2}\; d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C\,.$$

Now we can compute the area of a circle as twice the area of a semicircle by changing from Cartesian to polar coordinates and integrating.

$$\begin{aligned}
A_c &= 2\int_{-a}^{a}\sqrt{a^2 - x^2}\;dx = 2\int_{-\pi}^{0}\sqrt{a^2 - (a\cos\theta)^2}\;\; d(a\cos\theta)\\
&= 2\cdot a\cdot a\int_{-\pi}^{0}\sin\theta(-\sin\theta)\;d\theta = -2a^2\int_{-\pi}^{0}\sin^2\theta\;d\theta\\
&= -2a^2\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} + C\right)\Bigg|_{-\pi}^{0} = \pi a^2\,. \quad \blacksquare
\end{aligned}$$

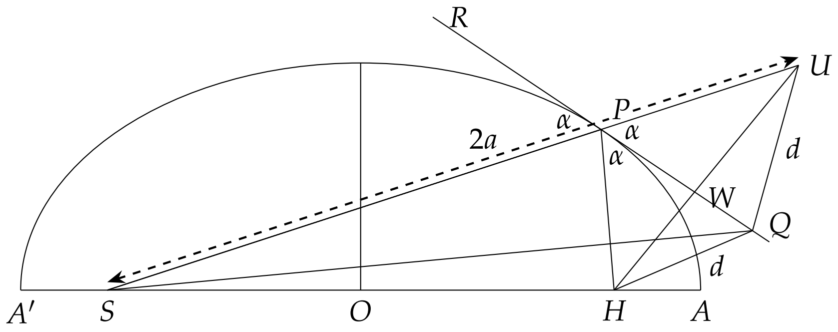


Figure 33: Angles at the tangent

7.5 The angles between a tangent and the lines to the foci

Theorem 7.8 *Let P be a point on the ellipse whose foci are S, H. Let PU be the extension of SP such that $SU = AA' = 2a$. Let RQ be the bisector of $\angle HPU$. Then $\angle RPS = \angle QPH$ and RQ is the tangent to the ellipse at P (Figure 33).*

Proof We prove that any point $Q \neq P$ on the bisector is not on the ellipse, so the bisector RQ has only one point of intersection with the ellipse and it must be the tangent at P. Since RQ is the angle bisector of $\angle HPU$ (the exterior angle of $\angle SPH$), $\angle QPH = \angle QPU = \alpha$, and by vertical angles $\angle QPU = \angle RPS = \alpha$.

Construct the line HU to form the triangle $\triangle HPU$ which intersects PQ at W. By construction $PH = PU$ so $\triangle HPW \cong \triangle UPW$ by side-angle-side and $UW = HW$. But HU is a straight line, therefore, if $\angle HWQ = \angle UWQ$, then they are both right angles and $\triangle HWQ = \triangle UWQ$ by side-angle-side, so $UQ = HQ = d$. Suppose that Q is on the ellipse so that $2a = SQ + QH = SQ + QU$. By the triangle inequality $2a = SQ + QU > SU = 2a$, contradicting that Q is on the ellipse. ∎

7.6 Conjugate diameters

Definition 7.9 *There are two equivalent definitions of conjugate diameters.*

- *Let P be a point on an ellipse, PG a diameter and let t be the tangent to the ellipse at P. Diameter DK is a* conjugate diameter *if it is parallel to t (Figure 34).*
- *Two diameters PG and DK are* conjugate diameters *if the midpoints of chords ($D'K'$, $D''K''$) parallel to one diameter (DK) lie on another diameter (PG).*

Theorem 7.10 *Let $P = (x, y)$ be a point on an ellipse (not on the major axis AA') and construct a perpendicular PV from P to the major axis (Figure 35). Then*

$$\frac{A'V \cdot AV}{PV^2} = \frac{a^2}{b^2}.$$

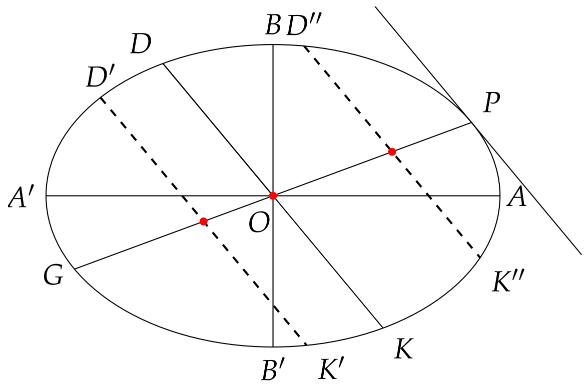


Figure 34: Conjugate diameters

Proof By Equation 13,

$$y^2 = b^2 \cdot \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2(a^2 - x^2)}{a^2}$$

$$\frac{A'V \cdot AV}{PV^2} = \frac{(a+x)(a-x)}{y^2} = \frac{a^2(a^2-x^2)}{b^2(a^2-x^2)} = \frac{a^2}{b^2}. \quad \blacksquare$$

Theorem 7.11 *Let PG, DK be conjugate diameters of an ellipse and let Q be a point on the ellipse (Figure 36). Constrct the perpendicular QV from Q to the major axis, then*

$$PV = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2}.$$

Proof Figure 36 shows a dashed ellipse which is the original ellipse rotated about the same center O, so that OP is the semi-major axis and OD is the semi-minor axis. By Theorem 7.10,

$$\frac{GV \cdot PV}{QV^2} = \frac{a'^2}{b'^2},$$

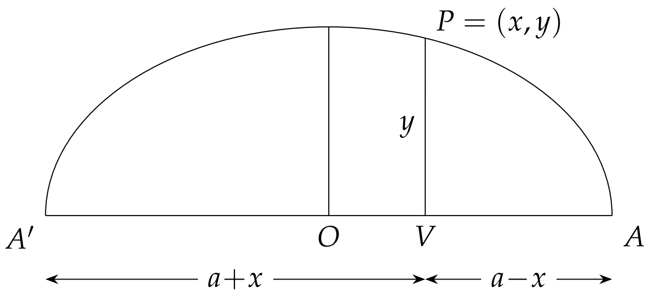


Figure 35: Ratios on conjugate diameters

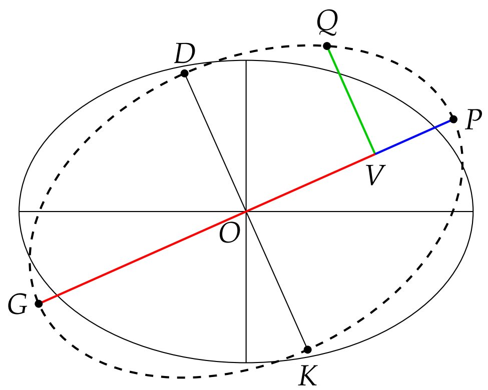


Figure 36: Ratios on conjugate diameters

where a', b' are the lengths of the semi-major and semi-minor axes of the rotated ellipse. By construction $a' = OP$ and $b' = OD$ so

$$\frac{GV \cdot PV}{QV^2} = \frac{CP^2}{OD^2}$$
$$PV = \frac{QV^2 \cdot CP^2}{GV \cdot OD^2} . \quad \blacksquare$$

7.7 The parametric representation of an ellipse

Figure 37 shows an ellipse and two circles: one whose radius is the length of the semi-major axis (dotted red) and one whose radius is the semi-minor axis (dashed blue). The figure shows the *parametric representation* of a point $P = (x, y)$ on the ellipse:

$$(x, y) = (a \cos t,\ y = b \sin t) .$$

The parameter t is *not* the angle of P relative to the positive x-axis. Construct the perpendicular through P to the minor axis and let P_I be its intersection with the inner circle so that CP_I defines an angle t. Extend C_I until it intersects the outer circle at P_O. The parametric representation of P is computed from the lengths of the axes and trigonometry functions of t.

7.8 Areas of parallelograms

Theorem 7.12 *The areas of the parallelograms formed by tangents to the intersections of and pair of conjugate diameters with the ellipse are equal (Figure 38).*

Proof We show that the area of the parallelogram $JKLM$ is equal to the area of the parallelogram formed by tangents to the major and minor axes (dashed). By symmetry it suffices to prove that the areas of one of the quadrants of those parallelograms are equal:

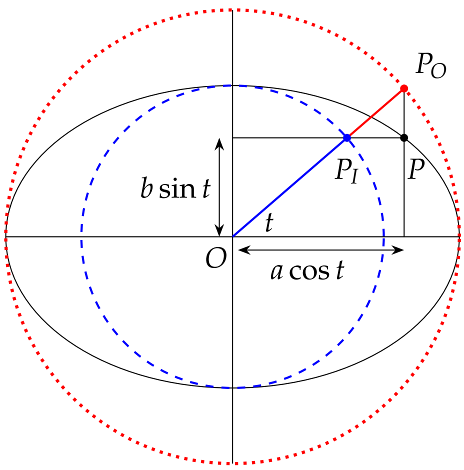


Figure 37: Parametric representation of an ellipse

$A_{ACBC'} = A_{PCDJ}$. Since the diagonals bisect a parallelogram, it suffices to prove that that the area of $\triangle ABC$ (red) equals the area of $\triangle PCD$ (blue).

Let $P = (x_p, y_p) = (a\cos t, b\sin t), D = (x_d, y_d)$ be the parametric representations of the points on the ellipse. Conjugate diameters are perpendicular so $\angle DCP$ is a right angle and

$$D = (x_d, y_d) = (a\cos(t + \pi/2), b\sin(t + \pi/2)) = (-a\sin t, b\cos t)\,.$$

Construct $DD' = (x_d, 0)$ and $PP' = (x_p, 0)$ perpendicular to the major axis. The area of $\triangle PCD$ can be computed as the area of the trapezoid $P'PDD'$ minus the areas of the

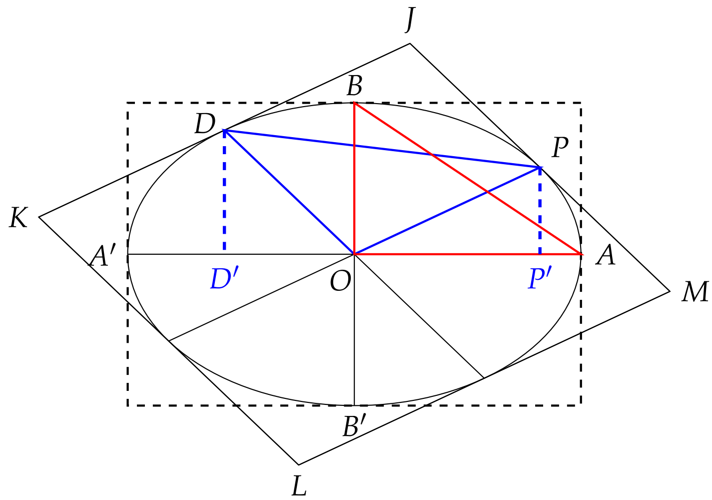


Figure 38: Parallelograms formed by conjugate diameters

triangles $\triangle D'DC, \triangle P'PC$. Therefore,

$$\begin{aligned}\triangle PCD &= \frac{y_p + y_d}{2}(x_p - x_d) - \frac{1}{2}x_d y_d - \frac{1}{2}x_p y_p = \frac{1}{2}\left(x_p y_d - x_d y_p\right) \\ &= \frac{1}{2}\left(a\cos t \cdot b\cos t - (-a)\sin t \cdot b\sin t\right) = \frac{1}{2}ab = \triangle ABC\,. \quad \blacksquare\end{aligned}$$

8 Ellipses in Euclidean geometry

The proofs of theorems about planetary freely used analytic geometry and trigonometry, but for many years after the invention of analytic geometry, mathematicians continued to limit themselves to Euclidean geometry. In this section, I present proofs in Euclidean geometry of theorems that appeared in Section 7.[7] The proofs are based on the definition of ellipses in terms of the geometric concepts of focus and directrix instead of the familiar analytic definition (Equation 12).

8.1 The definition of an ellipse using the focus and the directrix

Definition 8.1 *Let d be a line (the* directrix*) and S be a point (the* focus*) not on the directrix. Let $0 < e < 1$ be a number (the* eccentricity*). An* ellipse *is the locus of points P such that the ratio of PS to the distance of P to the directrix is e.*

All the conic sections (parabolas, ellipses and hyperbolas) are defined the same way and are distinguished by their eccentricity.

Definition 8.2 *Let X be the intersection of the perpendicular to the directrix from S. A on SX is a* vertex *of the ellipse if $SA/AX = e$ (in Figure 39, $e = 1/2$).*

Given $e = SA/AX$ we can locate A as follows:

$$\begin{aligned} SA + AX &= SX \\ SA &= SX - \frac{SA}{e} = SX \cdot \frac{e}{1+e} \\ AX &= SX \cdot \frac{1}{1+e}. \end{aligned}$$

[7]Theorems 7.8, 7.11 were already proved using Euclidean geometry and Theorem 7.7 requires taking limits.

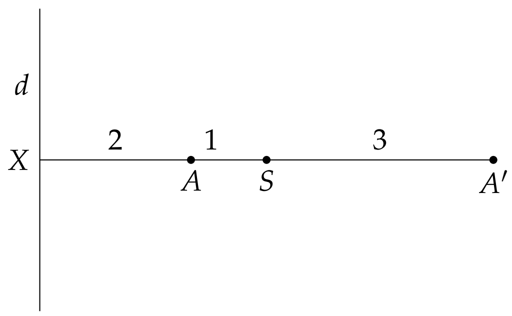


Figure 39: The elements of the definition of an ellipse

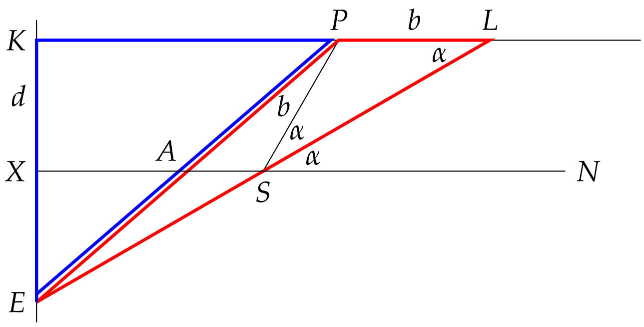


Figure 40: Constructing points on the ellipse

Similarly, there is a second vertex A' where

$$\begin{aligned} A'X - SA' &= SX \\ SA' &= SX + \frac{SA'}{e} = SX \cdot \frac{e}{1-e} \\ A'X &= SX \cdot \frac{1}{1-e}\,. \end{aligned}$$

Definition 8.1 is non-constructive. It states that the ellipse is the locus of points satisfying a certain property, but aside from the vertices we have not constructed any such points. Here we show how to construct any of the points on the ellipse.

Select an *arbitrary* point E on the directrix and construct lines from E through A and S. The line through S will make an angle α with SX. Construct a line from S at the *same angle* α from ES and let its intersection with EA be P. Construct the perpendicular from P to the directrix and let K be its intersection with the directrix. Let L be the intersection of KP with ES (Figure 40).

Theorem 8.3 *The point P is on the ellipse.*

Proof $\angle PLS = \angle LSN = \alpha$ by alternate interior angles, so $\triangle LPS$ is isosceles and $PL = SP$. Since $PK \parallel SX$, $\triangle XEA \sim \triangle KEP$ and $\triangle AES \sim \triangle PEL$ are adjacent pairs of similar triangles, so

$$\frac{PS}{PK} = \frac{PL}{PK} = \frac{SA}{AX} = e\,.$$

Therefore, P is on the ellipse. ∎

By choosing different points E on the directrix, any point on the ellipse can be constructed.

8.2 A right angle at the focus of an ellipse

Theorem 8.4 *Let P, P' be points on the ellipse and let F be the intersection of PP' with the directrix. Then FS bisects the exterior angle of $\angle P'SP$(Figure 41).*

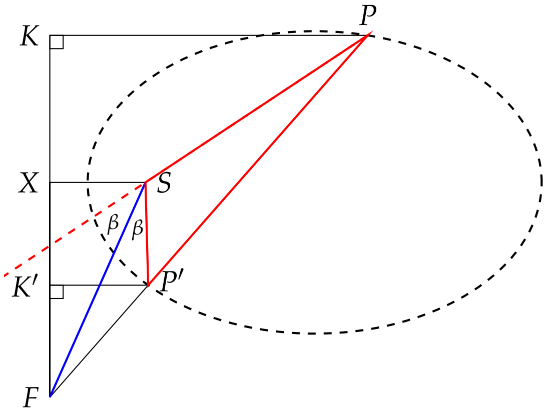


Figure 41: Bisecting the angle at the focus

Proof Since P, P' are on the ellipse

$$\frac{SP}{PK} = \frac{SP'}{P'K'} = e\,,$$

and since $\triangle PFK \sim P'FK'$,

$$\frac{SP}{SP'} = \frac{PK}{P'K'} = \frac{PF}{P'F}\,.$$

By the exterior angle bisector theorem (Theorem A.7), FS bisects the exterior angle of $\angle P'SP$. ∎

Theorem 8.5 *Let P be a point on the ellipse and construct lines PA, PA'. Label their intersections with the directrix by E and F, respectively. Then $\angle FSE$ is a right angle (Figure 42).*

Proof P, A, A' are all points on the ellipse so Theorem 8.4 applies. FS bisects $\angle PSX = 2\gamma$ and ES bisects $\angle P'SX = 2\delta$, so $2\gamma + 2\delta = 180^\circ$ and $\angle FSE = \gamma + \delta = 90^\circ$. ∎

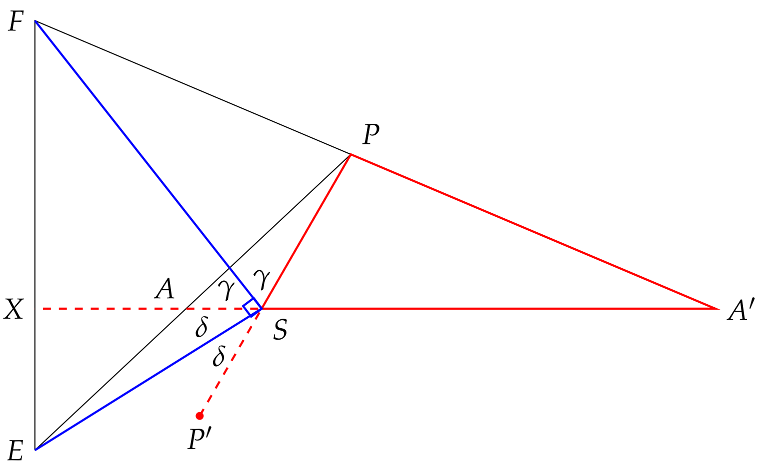


Figure 42: The right angle at the focus

8.3 Ratios of perpendiculars to the axes

This theorem proves Theorem 7.10 using Euclidean geometry.

Theorem 8.6 *Let P be a point on an ellipse not on the major axis and construct perpendiculars PN, PM from P to the major and minor axes, respectively (Figure 43). Then*

$$\frac{PN^2}{A'N \cdot NA} = \frac{BC^2}{AC^2} = \frac{b^2}{a^2} \tag{14}$$

$$\frac{PM^2}{B'N \cdot NA} = \frac{AC^2}{BC^2} = \frac{a^2}{b^2}\,. \tag{15}$$

Proof (Equation 14) $\triangle AXE \sim \triangle ANP$ since they are right triangles and the vertical angles at A are equal (red). Therefore,

$$\frac{PN}{AN} = \frac{EX}{AX}\,. \tag{16}$$

$\triangle PA'N \sim \triangle FA'X$ (blue) so

$$\frac{PN}{A'N} = \frac{FX}{A'X}\,. \tag{17}$$

Multiplying Equations 16 and 17 gives

$$\frac{PN^2}{AN \cdot A'N} = \frac{EX \cdot FX}{AX \cdot A'X}\,.$$

By Theorem 8.5 $\triangle FSE$ is a right triangle so by Theorem A.3,

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X}\,.$$

Since P was arbitrary this holds for any point on the ellipse, in particular, for B on the minor axis, where $PN = BC$ and $AN = AN' = AC$. Therefore,

$$\frac{BC^2}{AC^2} = \frac{SX^2}{AX \cdot A'X}$$

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X} = \frac{BC^2}{AC^2} = \frac{b^2}{a^2}\,. \quad \blacksquare$$

Proof (Equation 15) Since $CM = PN, PM = CN$, by Theorem A.2, Equation 14 becomes

$$\frac{CM^2}{AC^2 - PM^2} = \frac{BC^2}{AC^2}$$

$$\frac{AC^2}{AC^2 - PM^2} = \frac{BC^2}{CM^2}\,. \tag{18}$$

It follows that

$$\frac{AC^2}{PM^2} = \frac{BC^2}{BC^2 - CM^2}, \tag{19}$$

By cross-multiplying the Equations 18 and 19. By Theorem A.2, Equation 19 implies

$$\frac{PM^2}{BM \cdot MB'} = \frac{AC^2}{BC^2}\,. \quad \blacksquare$$

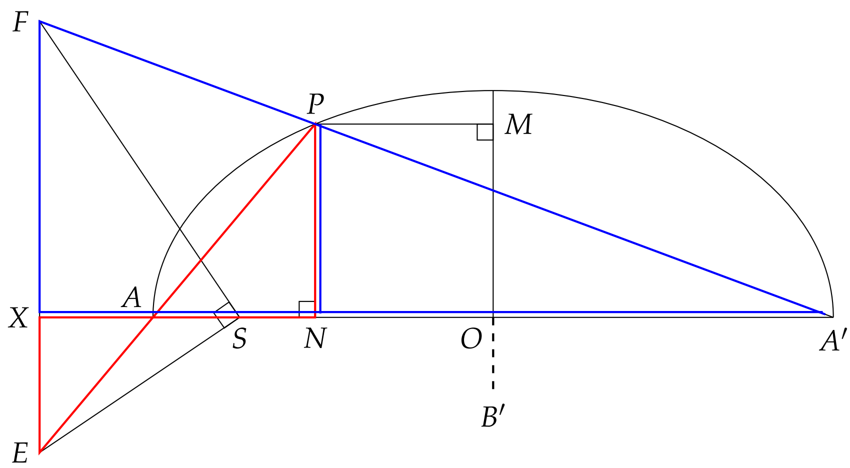


Figure 43: Ratio of an ordinate

8.4 A circle circumscribing an ellipse

This theorem proves Theorem 7.4 in Euclidean geometry.

Consider a circle of radius a with the same center as an ellipse (Figure 44). Choose a point N on the major axis and construct a perpendicular through N. Let its intersections with the ellipse and the circle be P and Q, respectively.

Theorem 8.7

$$\frac{PN}{QN} = \frac{BC}{AC} = \frac{b}{a}.$$

Proof From Theorem 8.6,

$$\frac{PN^2}{AN \cdot NA'} = \frac{BC^2}{AC^2},$$

and by Theorem A.3, $AN \cdot NA = QN^2$. ∎

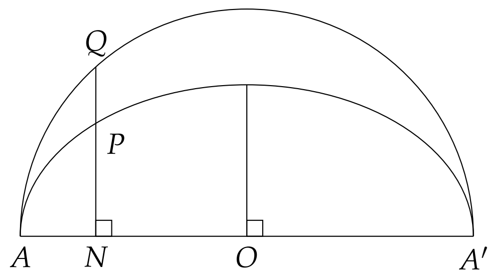


Figure 44: A circle circumscribing an ellipse

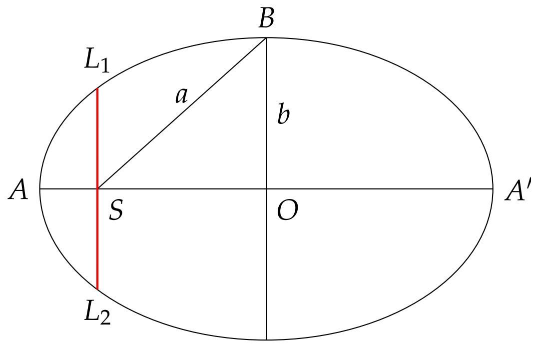


Figure 45: The latus rectum of an ellipse

8.5 The latus rectum of an ellipse

This theorem proves Theorem 7.6 in Euclidean geometry.

Theorem 8.8 *L, the length of the latus rectum of an ellipse (Definition 7.5), is* $\dfrac{2b^2}{a}$.

Proof By Theorem 8.6,

$$\frac{SL_1^2}{AS \cdot SA'} = \frac{BC^2}{AC^2}\,.$$

By Theorem 7.2, $SB = AC = a$, so by Pythagoras's theorem,

$$BC^2 = BS^2 - SC^2 = AC^2 - SC^2 = (AC - SC)(AC + SC) = AS \cdot SA'\,.$$

Therefore, the length of one-half the latus rectum is

$$SL_1^2 = \frac{BC^4}{AC^2}$$

$$SL_1 = \frac{BC^2}{AC} = \frac{b^2}{a}\,. \quad \blacksquare$$

8.6 Areas of parallelograms

Theorem 8.9 *Let Y be the intersection the perpendicular through the focus S to the tangent TT' at P, and let L be the intersection of $S'P$ and SY (Figure 46). Then Y is on the circumscribing circle and $CY \parallel S'L$.*

Proof

$\triangle STY \sim \triangle T'TC$ since they are right triangles that share the acute angle $\angle CTT' = \angle YTS$, so $\angle CT'T = \angle YST = \beta$.

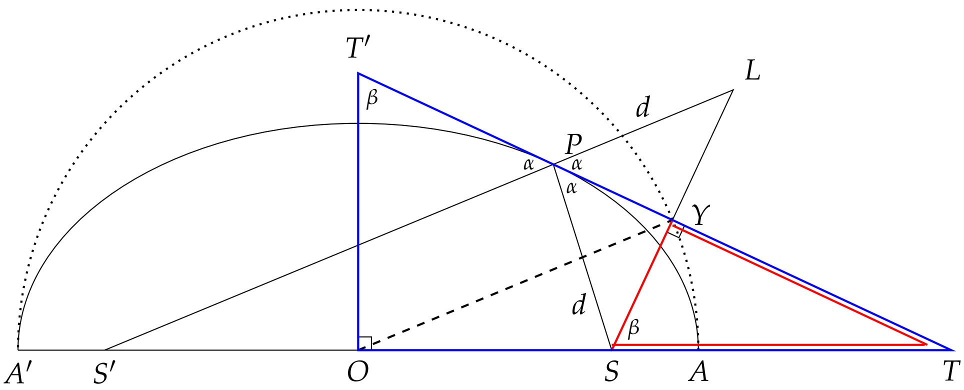


Figure 46: The perpendicular from a focus to a tangent

By Theorem 7.8, $\angle SPY = \angle S'PT' = \alpha$ since they are the angles to the foci at the tangent. $\angle S'PT' = \angle YPL = \alpha$ are vertical angles, so $\angle SPY = \angle LPY = \alpha$.

Then $\triangle SPY \cong \triangle LPY$ since they are right triangles with an equal acute angle and a common side PY. Therefore, $PL = PS$ and $S'L = S'P + PL = AA' = 2a$.

Since $\triangle SPY \cong \triangle LPY$, $SY = YL$, and since S, S' are foci, $S'C = SC$. It follows that $\triangle CSY \sim \triangle S'SL$ and $CY \parallel S'L$. By similarity,

$$\frac{CY}{S'L} = \frac{CS}{S'S} = \frac{CS}{2CS}\,.$$

Therefore, $2CY = S'L = 2a$ so $CY = a$ and Y is on the circumscribing circle of radius a. ∎

Theorem 8.10 *Let N be the intersection of the perpendicular through P to the major axis (Figure 47). Then $CN \cdot NT = AC^2 = AN \cdot NA'$.*

Proof Continuing with the construction from Figure 46, we focus on the segments AN, NA' (Figure 47). We can deduce the angles that are shown.

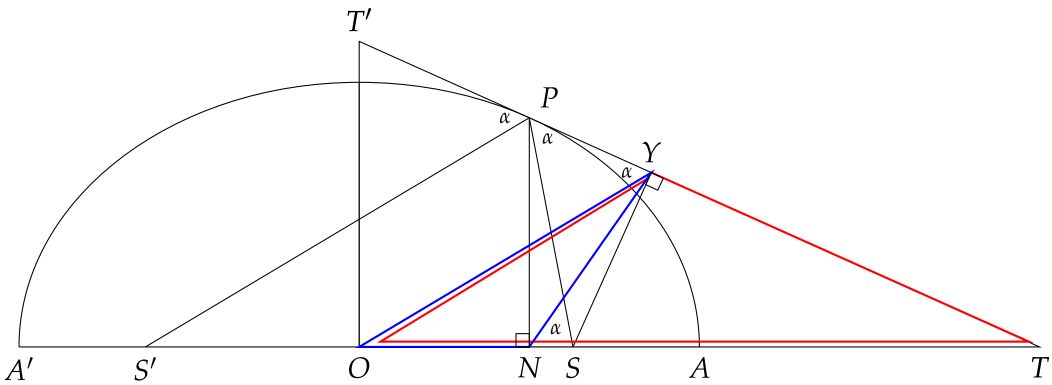


Figure 47: Ratios of segments of the major axis

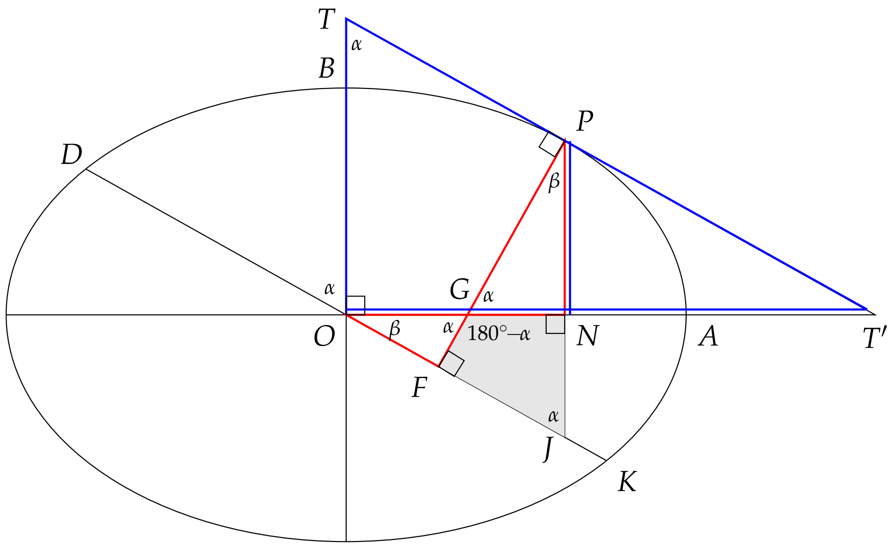


Figure 48: Parallelograms formed by conjugate diameters

- $CY \parallel S'P$ (Theorem 8.9) so $\angle CYP = \angle S'PT'$ by corresponding angles.
- $\angle S'PT' = \angle SPY$ by Theorem 7.8 since they are the angles to the foci at the tangent.
- $\angle SYP$ and $\angle SNP$ are right angles and therefore $SYPN$ is quadrilateral that can be circumscribed by a circle whose diameter is PS.[8] Therefore, $\angle SPY = \angle SNY$ since they are subtended by the same chord YS.

Since $\angle CYT \sim \angle CNY$ and $\angle YCT$ is a common angle, $\triangle CYT \sim \triangle CNY$ and

$$\frac{CN}{CY} = \frac{CY}{CT}$$

$$CN \cdot CT = CY^2 = AC^2\,, \tag{20}$$

since the perpendicular to the tangent from a focus is on the circumscribing circle (Theorem 8.9). Since $NT = CT - CN$, we have $CN \cdot NT = CN \cdot CT - CN^2$ which equals $AC^2 - CN^2$ by Equation 20. This in turn equals $AN \cdot NA'$ by Theorem A.2. ∎

Construct the normal to the tangent at P and let its intersection with the conjugate diameter DK be F and its intersection with the major axis be G. Construct a perpendicular from P to the major axis and let its intersection be N. Let the intersection of the tangent with the minor axis be T and its intersection with the major axis be T' (Figure 48).

Theorem 8.11

$$PF \cdot PG = BC^2\,.$$

[8]It can be proven that a quadrilateral whose opposite angle are supplementary can be circumscribed by a circle. If two opposite angles are right angles that sum to 180°, then the other two angles must also sum to 180°.

Proof $\triangle NPG \sim \triangle FPJ$ (rotate $\triangle NPG$ to see this) so $\angle PGN = \angle PJF = \alpha$ and

$$\begin{aligned} \frac{PF}{PN} &= \frac{PJ}{PG} \\ PF \cdot PG &= PJ \cdot PN\,. \end{aligned} \tag{21}$$

By vertical angles $\angle PGN = \angle CGF = \alpha$ so $\triangle NPG \sim \triangle FCG$ and $\angle NPG = \angle FCG = \beta = 90^\circ - \alpha$. By adding β to the right angles $\angle BCN$ and $\angle TPF$, we get that $\angle TCJ = \angle TPJ$ and therefore $TPJC$ is a parallelogram, so $CT = PJ$ and $PF \cdot PG = CT \cdot PN$. By Equation 21, the theorem will be proven if we can show that $CT \cdot PN = BC^2$.

$\triangle TT'C \sim \triangle PT'N$ so

$$\begin{aligned} \frac{CT}{CT'} &= \frac{PN}{NT'} \\ \frac{CT}{PN} &= \frac{CT'}{NT'}\,. \end{aligned}$$

Multiplying each side by fractions equal to 1 gives

$$\frac{CT \cdot PN}{PN^2} = \frac{CT' \cdot CN}{CN \cdot NT'}\,.$$

By Equation 20, $CN \cdot CT' = AC^2$, and by Theorem 8.10, $CN \cdot NT' = AN \cdot NA'$, so

$$\frac{CT \cdot PN}{PN^2} = \frac{AC^2}{AN \cdot NA'}\,.$$

Multiplying by PN^2/PN^2 and using Theorem 8.6 gives

$$\begin{aligned} \frac{CT \cdot PN}{PN^2} &= \frac{PN^2}{AN \cdot NA'} \cdot \frac{AC^2}{PN^2} = \frac{BC^2}{AC^2} \cdot \frac{AC^2}{PN^2} \\ CT \cdot PN &= BC^2\,. \quad \blacksquare \end{aligned} \tag{22}$$

Theorem 8.12 *In Figure 49,*

$$\begin{aligned} CN^2 &= AM \cdot MA', \quad CM^2 = AN \cdot NA' \\ \frac{DM}{CN} &= \frac{BC}{AC} \\ \frac{CM}{PN} &= \frac{BC}{AC} \end{aligned}$$

Proof By Theorem 8.10,

$$\begin{aligned} CN \cdot CT &= AC^2 = CM \cdot CT' \\ \frac{CM}{CN} &= \frac{CT}{CT'}\,. \end{aligned}$$

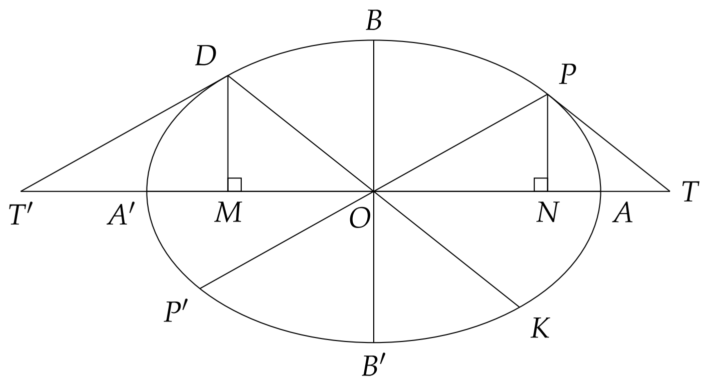


Figure 49: Ratios of perpendiculars to the major axis

Since DK and PP' are conjugate diameters, $DT' \parallel PP'$ and $\triangle T'DC \sim \triangle CPT$, so

$$\frac{CM}{CN} = \frac{CT}{CT'} = \frac{CN}{MT'}$$
$$CN^2 = CM \cdot MT',$$

Therefore,

$$CN^2 = CM \cdot MT' = AC^2 = AM \cdot MA' \tag{23}$$

by Theorem 8.10. By Theorem 8.6,

$$\frac{DM^2}{AM \cdot MA'} = \frac{BC^2}{AC^2},$$

and by Equation 23,

$$\frac{DM^2}{CN^2} = \frac{BC^2}{AC^2}$$
$$\frac{DM}{CN} = \frac{BC}{AC},$$

A symmetric argument shows that

$$CM^2 = AN \cdot NA'$$
$$\frac{CM}{PN} = \frac{BC}{AC} \quad \blacksquare.$$

This theorem proves Theorem 7.12 in Euclidean geometry.

Theorem 8.13 *The area of the parallelogram formed by the tangents at the ends of the conjugate diameters PP', DK is equal to the area of the rectangle enclosing the ellipse at the ends of the axes (Figure 50).*

Proof By the definition of conjugate diameters, it is sufficient to show that the area of $PCDL$ is $AC \cdot BC$. The area of a parallelogram is width times height so it is $CD \cdot PF$.

By vertical angles $\angle DCM = \angle GCF$ and $\angle CGF = \angle PGN$, so $\triangle DCM \sim \triangle PGN$

$$\frac{PG}{CD} = \frac{PN}{CM},$$

and by Theorem 8.12

$$\frac{PG}{CD} = \frac{AC}{BC}$$

$$\frac{CD}{AC} = \frac{PG}{BC}. \tag{24}$$

By Theorem 8.11,

$$\frac{PG}{BC} = \frac{BC}{PF}. \tag{25}$$

From Equations 24 and 25 gives $CD \cdot PF = AC \cdot BC$. ∎

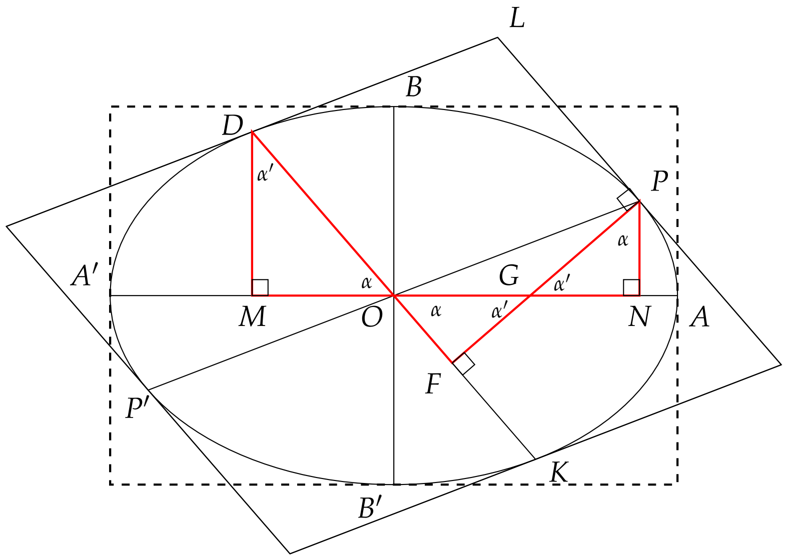


Figure 50: Areas of parallelograms ($\alpha' = 90° - \alpha$)

9 Lagrange points

Consider a spacecraft orbiting the Sun and subject to the gravitational force of both the Earth and the Sun. Joseph-Louis Lagrange and Leonhard Euler discovered that there are five points where the spacecraft rotates with the same orbital period as the Earth and thus appears to maintain a fixed position as viewed from the Earth. These points are called the *Lagrange points* $L1, L2, L3, L4, L5$ and their positions are shown in Figure 51.[9]

In this section we present an approximate derivation of the locations of $L1, L2, L3$. The most significant approximation is that we assume that the spacecraft orbits around the center of the Sun, whereas it actually orbits around the barycenter of the Sun and the Earth. The derivation of the locations of $L4, L5$ is beyond the scope of this document. The final subsection describes the objects that exist at the Lagrange points.

9.1 Lagrange point $L1$

We assume that the Earth is in a circular orbit of radius r_E around the Sun and that the masses of the two satisfy $m_S \gg m_E$. Let us suppose that we wish to place a space telescope T of mass $m_T \ll m_E$ in a circular orbit at distance $r_1 \ll r_E$ from the Earth (Figure 52). Furthermore, let us suppose that we want T to have the same orbital period as the Earth (one year) so that it will be possible to easily receive data at ground stations. Is this possible?

[9] The orbits are clearly shown in the gif in the Wikipedia entry *Lagrange point*.

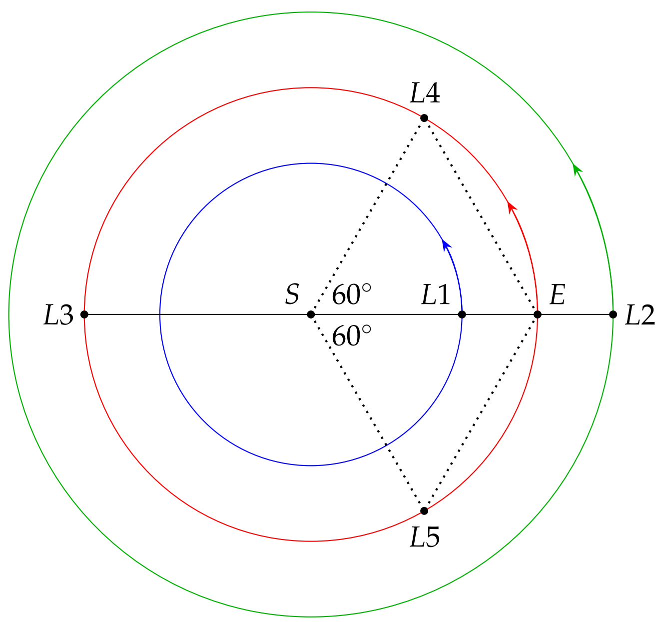


Figure 51: The five Lagrange points

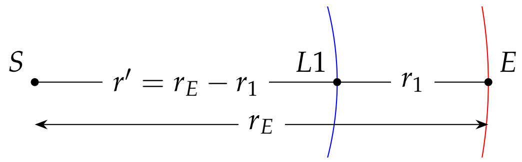


Figure 52: Lagrange point $L1$

If we ignore the gravitational force exerted by E on T at $L1$, obviously not, because by Kepler's third law

$$\frac{A_E^3}{T_E^2} = \frac{A_1^3}{T_1^2} = \frac{A_1^3}{T_E^2},$$

so $A_1 = A_E$ and T must be located at the center of the Earth. Is is possible that the gravitational force exerted by E on T can cause $T_1 = T_E$ while r_1 is greater than the radius of the Earth?

You might think that T should be placed so that the gravitational force exerted by the Sun is exactly balanced by the gravitational force exerted by the Earth, but, of course, if there is no net force on T, by Newton's first law T would simply move in a straight line of into space. Instead, we want the net centripetal force on T to be

$$F = \frac{Gm_S m_T}{r'^2} - \frac{Gm_E m_T}{r_1^2}, \tag{26}$$

so that it moves in an orbit with period T_E. To simplify notation we let $r' = r_E - r_1$.

Since the length of the orbit of T at $L1$ is $2\pi r'$, T's velocity is

$$v_1 = \frac{2\pi r'}{T_1}. \tag{27}$$

As Newton did in his investigation of elliptical orbits, in Figure 53 the motion from A to D is separated into the tangential motion AC where there is no force on T, followed by motion towards the center with a centripetal force from C to S. The distances indicated are those of the non-accelerated $v_t \Delta t$ and the accelerated motion $\frac{1}{2}a(\Delta t)^2$ during a very small time period Δt. We need to find the acceleration a that will cause T to reach D. Applying Pythagoras's theorem we get

$$\begin{aligned} r'^2 + (v_1 \Delta t)^2 &= (r' + x)^2 = r'^2 + 2r'x + x^2 \\ (v_1 \Delta t)^2 &= x(2r' + x). \end{aligned}$$

Since Δt is assumed to be a very small time interval and since r_T is close to r_E, $2r' + x \approx 2r'$ so using the definition of x, from Newton's second law we have

$$\begin{aligned} \frac{1}{2}a(\Delta t)^2 &= \frac{1}{2}\frac{v_1^2}{r'}(\Delta t)^2 \\ F &= m_T a = \frac{m_T v_1^2}{r'}. \end{aligned}$$

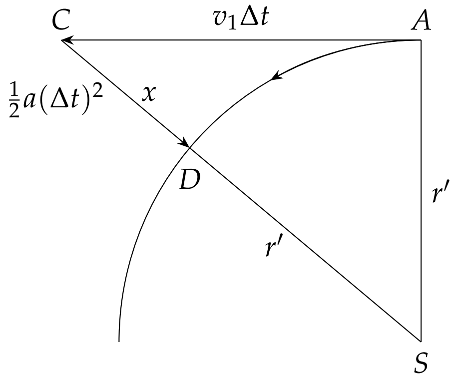


Figure 53: Non-accelerated and accelerated motion

By Equation 26 the force needed to keep T in the desired orbit is

$$F = \frac{m_T v_T^2}{r'} = \frac{Gm_S m_T}{r'^2} - \frac{Gm_E m_T}{r_1^2}\,,$$

so the velocity of T at $L1$ must satisfy

$$v_1^2 = \frac{Gm_S}{r'} - \frac{Gm_E r'}{r_1^2}\,.$$

The period of the desired orbit is $T_1 = T_E$ so using Equation 27, we get

$$\begin{aligned}\frac{4\pi^2 r'^2}{T_E^2} &= \frac{Gm_S}{r'} - \frac{Gm_E r'}{r_t^2}\\ \frac{4\pi^2}{T_E^2} &= \frac{Gm_S}{r'^3} - \frac{Gm_E}{r' r_t^2}\,.\end{aligned}$$

But by Kepler's third law (Equation 8), where the elliptical semi-major axis a_i is the circular radius r_E,

$$\begin{aligned}\frac{4\pi^2 r_E^3 m}{T_E^2}\frac{1}{r_E^2} &= \frac{GmM}{r_E^2}\\ \frac{Gm_S}{r_E^3} &= \frac{Gm_S}{r'^3} - \frac{Gm_E}{r' r_1^2}\\ \frac{1}{r_E^3} &= \frac{1}{r'^3} - \frac{m_E/m_S}{r' r_1^2}\,. \qquad (28)\end{aligned}$$

Let $y = m_E/m_S$ and $z = r_1/r_E$ so $r' = r_E - r_1 = r_E(1-z)$. Multiply by r_E^3 and make the substitutions.

$$\begin{aligned}\frac{r_E^3}{r'^3} - \frac{m_E/m_S r_E^3}{r' r_1^2} &= 1\\ \frac{1}{(1-z)^3} - \frac{y r_E^3}{r_E(1-z)z^2 r_E^2} &= 1\\ \frac{1}{(1-z)^3} - \frac{y}{z^2(1-z)} &= 1\,.\end{aligned}$$

Since $z = r_1/r_E$ is very small, we get the following approximations from the Taylor series [5, Section 11.8]:

$$\begin{aligned}
\frac{1}{(1-z)} &= 1 + z + z^2 + \cdots \approx 1 + z \\
\frac{1}{(1-z)^3} &= 1 + 3z + 6z^2 + \cdots \approx 1 + 3z \\
1 + 3z - \frac{y}{z^2}(1+z) &\approx 1 \\
3z^3 &\approx y(1+z) \approx y\,.
\end{aligned}$$

Let us plug in the numbers $m_S \approx 2 \times 10^{30}$ kg, $m_E \approx 6 \times 10^{24}$ kg, $r_E \approx 1.5 \times 10^8$ km.

$$\begin{aligned}
\left(\frac{r_1}{1.5 \times 10^8}\right)^3 &\approx \frac{6 \times 10^{24}}{3 \times 2 \times 10^{30}} \approx 10^{-6} \\
r_1 &\approx 1.5 \times 10^8 \cdot \sqrt[3]{10^{-6}} \approx 1.5 \times 10^6\,.
\end{aligned}$$

If an object is placed 1.5 million km from the Earth, the period of its orbit around the Sun will be approximately one year. This is quite far—the Moon is less than 400,000 km from the Earth—but still relatively far from the Sun which is 150 million km away.

9.2 Lagrange point $L2$

The computation for $L2$ is similar using $r' = r_E + r_2$ (Figure 54). With the appropriate modifications to Equation 26 we get

$$\begin{aligned}
F &= \frac{G m_S m_T}{r'^2} + \frac{G m_E m_T}{r_2^2} \\
\frac{1}{r_E^3} &= \frac{1}{r'^3} + \frac{m_E/m_S}{r' r_2^2} \\
1 &= \frac{1}{(1+z)^3} + \frac{y}{z^2(1+z)}\,.
\end{aligned}$$

The approximations based on the Taylor series are $(1+z)^{-3} \approx 1 - 3z$ and $(1+z)^{-1} \approx 1 - z$, leading to the same equation $3z^3 \approx y$. Therefore, $L2$ is the same distance from the Earth as $L1$ but on the opposite side of the Earth.

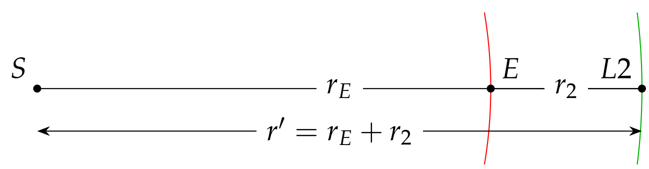


Figure 54: Lagrange point $L2$

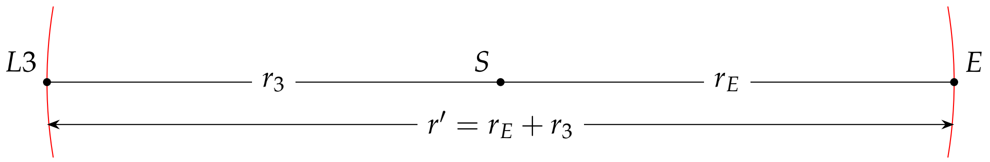


Figure 55: Lagrange point $L3$

9.3 Lagrange point $L3$

The Lagrange point $L3$ is on the other side of the Sun (Figure 55). The modifications to Equation 28 give

$$\begin{aligned}
\frac{1}{r_E^3} &= \frac{1}{r_3^3} + \frac{m_E/m_S}{r'^3} \\
1 &= \frac{1}{z^3} + \frac{y}{z^3(r_E + r_3)^3} \\
1 &= \frac{1}{z^3} + \frac{y}{(1+z)^3} \\
z^3 &= \frac{1}{1-y},
\end{aligned}$$

since $z \ll 1$. But $y \approx 10^{-6}$ so $z^3 \approx 1$, $r_3 \approx r_E$ and T is approximately the same distance from the Sun as it is from the Earth.

9.4 Objects at the Lagrange points

Objects at the $L1$, $L2$ and $L3$ points are not stable so there are no natural bodies at those points. However, they are relatively stable so a spacecraft can be placed into a small orbit around one of these Lagrange points. Even when it drifts, the force required to return it to the Lagrange point is very small, which means that the propellant in the spacecraft can maintain it on station for a very long time.

The *Deep Space Climate Observatory (DSCOVR)* was placed at Lagrange point $L1$. It continually observes the Sun and the sunlit side of the Earth.

The $L2$ point is ideal for telescopes that observe the solar system and outer space. If a sun shield is placed facing the Earth and the Sun, the spacecraft itself can remain at the very low temperature that its sensors require. The *James Webb Space Telescope* with its 6.5 meter diameter infrared telescope was placed at $L2$ in 2022.

Lagrange point $L3$ is not useful for spacecraft because the line-of-sight to the Earth is blocked by the Sun.

The orbits of objects at $L4$ and $L5$ are stable. Asteroids that are stable at a Lagrange point are called *trojans* since most are located at the $L4$ and $L5$ points of Jupiter from Greek mythology. There are two extremely small trojans at the Earth's $L4$ point.

A Theorems of Euclidean Geometry

A.1 Constructing a circle from three points

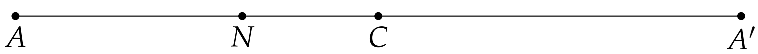
Theorem A.1 *Given three non-collinear points,*[10] *a circle can be constructed that goes through all three points.*

Proof Three non-collinear points A, B, C define a triangle $\triangle ABC$ (Figure 56). Construct the perpendicular bisectors of any two of its three sides, say, AC and BC. By definition the perpendicular bisector is the geometric locus of points equidistant from the endpoints of the segment. Let O be the intersection of the two bisectors. Then the $AO = CO = BO$ is the radius of a circle centered at O that goes through A, B, C. ∎

A.2 The product of two subsegments

Theorem A.2 *Let AA' be a line segment whose midpoint is C. Then*

$$AC^2 - CN^2 = AN \cdot NA' .$$



Proof $AN = AC - CN$ and $NA' = A'C + CN = AC + CN$ since C is the midpoint of AA'. The result is obtaining by multiplying the two equations. ∎

Theorem A.3 *Let Q be a point on a circle whose diameter is AA' and construct a perpendicular QN to the diameter (Figure 57). Then*

$$QN^2 = AN \cdot NA' .$$

The equation also holds if it is given *that $\triangle AQA'$ is a right triangle.*

[10]*Collinear* means that the points all on the same line.

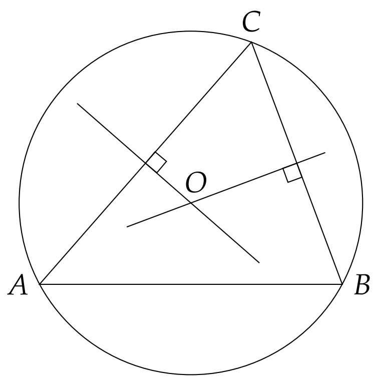


Figure 56: A circle through three arbitrary points

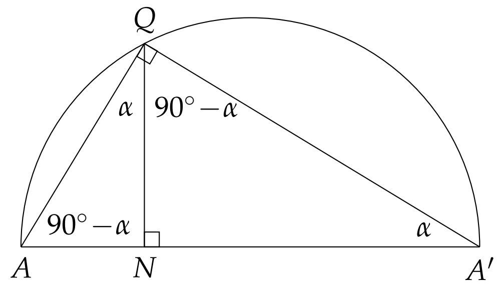


Figure 57: Right triangle in a circle at the diameter

Proof An angle that subtends a diameter is a right angle. Since the sum of the angles of a triangle is 180°, we can label the angles as shown in the Figure, from which follows that $\triangle QNA \sim \triangle A'NQ$. Therefore,

$$\frac{QN}{AN} = \frac{NA'}{QN} . \quad \blacksquare$$

A.3 Adjacent pairs of similar triangles

I have not encountered the following definition before but it explicitly expresses a relation among similar triangles that would have been obvious to geometers.

Definition A.4 *An* adjacent pair of similar triangles *is a pair of (a pair of) similar triangles that share sides. In Figure 58, $\triangle BAC \sim \triangle EAF$ and $\triangle CAD \sim \triangle FAG$ are an adjacent pair of similar triangles.*

Theorem A.5 *For the adjacent pair of similar triangles in Figure 58,*

$$\frac{AB}{AE} = \frac{AD}{AG} .$$

Proof By similar triangles,

$$\frac{AB}{AE} = \frac{AC}{AF}$$

$$\frac{AC}{AF} = \frac{AD}{AG}$$

$$\frac{AB}{AE} = \frac{AD}{AG} . \quad \blacksquare$$

Similar ratios hold between other sides of $\triangle BAC$ and $\triangle CAD$ by using an intermediate step with AC. We will use the term *by an adjacent pair of similar triangles* and leave it to the reader to make the intermediate step.

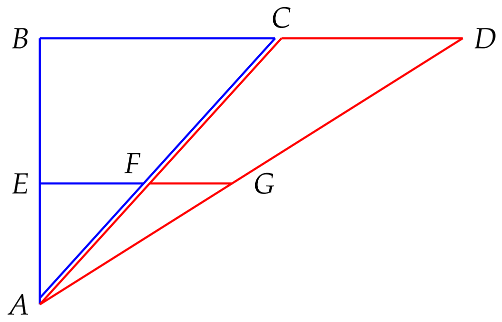


Figure 58: Adjacent pairs of similar triangles

A.4 The angle bisector theorems

Theorem A.6 (Interior angle bisector theorem) *In* $\triangle ABC$ *let* D *be a point on* BC *(Figure 59). Then* AD *bisects* $\angle CAB$ if and only if

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Proof Suppose that AD bisects $\angle BAC$. Construct a line through C parallel to AB and let its intersection with AD be E. By alternate interior angles, $\angle BAD = \angle CED =$ and by vertical angles $\angle BDA = \angle CDE$. Therefore, $\triangle ABD \sim \triangle EDC$ so

$$\frac{BD}{CD} = \frac{AB}{CE}.$$

$\triangle ECA$ is isosceles so $CE = AC$ and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just "run" the proof backwards. ∎

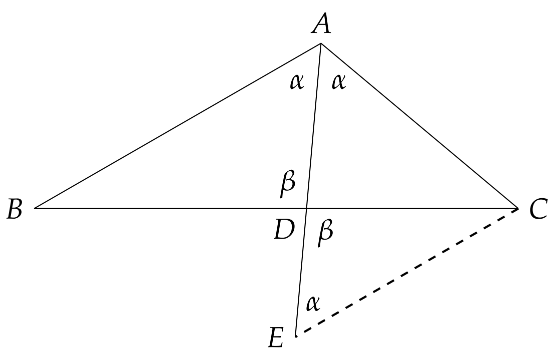


Figure 59: The interior angle bisector theorem

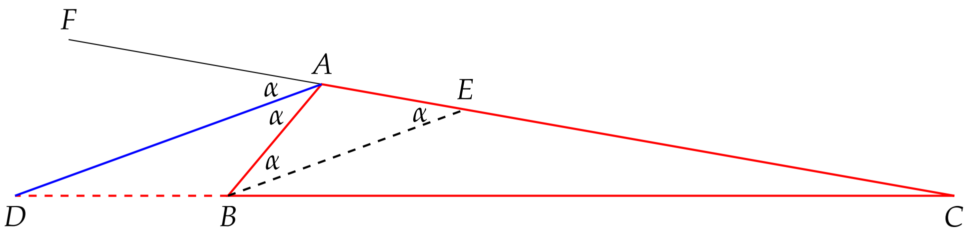


Figure 60: The exterior angle bisector theorem

Theorem A.7 (Exterior angle bisector theorem) *In $\triangle ABC$ let D be a point on the extension of CB outside the triangle (Figure 60). Then AD bisects the exterior angle of $\angle BAC$ if and only if*

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

Proof Suppose that AD bisects $\angle BAF$. Construct a line through B parallel to AD and let its intersection with AC be E. By alternate interior angles $\angle BAD = \angle ABE$ and by corresponding angles $\angle FAD = \angle AEB$. Therefore, $\triangle BCE \sim \triangle DCA$ so

$$\frac{BD}{CD} = \frac{AE}{AC}.$$

But $\triangle BAE$ is isosceles so $AE = AB$ and

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

To prove the converse just "run" the proof backwards. ∎

The exterior angle bisector theoren confusing to understand in a proof, because it can be hard to identify the components of a diagram. In the text the following color-coding is used: the triangle is red, the extension of one side is dashed red and the bisector is blue.

Sources and further reading

Sections 2–5 are primarily on Hahn's book [5]. He has written a more advanced book on the orbits of planets and spacecraft [6]. The proof in Section 6 is from [4]. Two additional articles present this aspect of Newton's work [7, 9]. Should you wish to read Newton's work, [2] is an up-to-date translation into English, and Cohen's lengthy *Guide* will facilitate understanding Newton's often terse presentation. The relevant sections of the Guide are 10.8–10.10. The Wikipedia entry on ellipses is very comprehensive.

Section 8 is based on Besant's textbook that uses Euclidean geometry exclusively to prove theorems on conic sections [1]. Drew wrote a similar textbook, though shorter and more elementary [3]. Compare these textbooks one by Smith that uses analytic geometry [8].[11] The computations of Section 9 are from [10].

References

[1] W. H. Besant. *Conic Sections, Treated Geometrically (Ninth Edition Revised and Enlarged)*. George Bell and Sons, London, 1895. `https://www.gutenberg.org/ebooks/29913` and `https://archive.org/details/cu31924059322481` (Accessed 6 September 2023).

[2] I. Bernard Cohen, Anne Whitman, and Julia Budenz. *The Principia: Mathematical Principles of Natural Philosophy*. University of California Press, 1999. Preceded by *A Guide to Newton's Principia* by I. Bernard Cohen.

[3] W. H. Drew. *A Geometrical Treatise on Conic Sections (Second Edition)*. Macmillan, Cambridge, 1862. `https://archive.org/details/in.ernet.dli.2015.501433` (Accessed 6 September 2023).

[4] Graham Griffiths. The inverse square law of gravitation: An alternative to Newton's derivation. `https://www.researchgate.net/publication/264978661_The_Inverse_Square_Law_of_Gravitation_An_Alternative_to_Newton's_Derivation`, 2009. (Accessed 6 September 2023).

[5] Alexander J. Hahn. *Calculus in Context: Background, Basics, and Applications*. Johns Hopkins University Press, 2017.

[6] Alexander J. Hahn. *Basic Calculus of Planetary Orbits and Interplanetary Flight: The Missions of the Voyagers, Cassini, and Juno*. Springer, 2020.

[7] Kai Hauser and Reinhard Lang. On the geometrical and physical meaning of Newton's solution to Kepler's problem. *The Mathematical Intelligencer*, 25(4):35–44, 2003.

[8] Charles Smith. *An Elementary Treatise on Conic Sections*. Macmillan, New York, 1904. `https://archive.org/details/elementarytreati1905smit` (Accessed 6 September 2023).

[9] S. K. Stein. Exactly how did Newton deal with his planets? *The Mathematical Intelligencer*, 18(2):6–11, 1996.

[10] David P. Stern. From stargazers to starships. `http://www.phy6.org/stargaze/Sintro.htm`, 2014. (Accessed 10 October 2023).

[11] The dates given for all three books are for the editions that I used.