The Lost Art of Euclidean Geometry

Moti Ben-Ari

Department of Science Teaching, Weizmann Institute of Science, Rehovot, 76100, Israel.

Corresponding author(s). E-mail(s): moti.ben.ari@gmail.com;

Abstract

This article surveys Kepler's discovery that planets have elliptical orbits and Newton's proof that if a body subject to a centripetal force follows an elliptical orbit, the force is subject to the inverse-square law. A proof is given for the very difficult Proposition XI, Problem VI of the *Principia*. The proof uses both Euclidean and analytic geometry, but Newton and even students throughout the nineteenth century used Euclidean geometry only. An outline of the complex proof of Proposition XI, Problem VI in Euclidean geometry is presented so that readers can appreciate the depth of Euclidean geometry. The final section of the article presents my views of the place of Euclidean geometry in mathematics education.

Keywords: Euclidean geometry, orbits, ellipses, gravity, centripetal force

1 Introduction

I find the following passage astonishing!

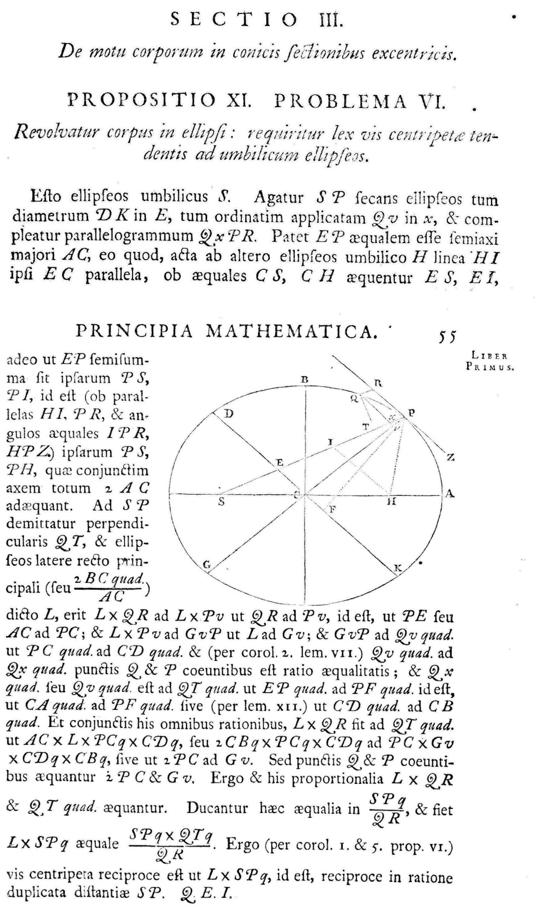
> In book 1, prop[osition] 10 (and notably in prop[osition] 11), Newton made use of a property of conics which he presents without proof, merely saying that the result in question comes from "the *Conics*." Here, as elsewhere in the *Principia*, Newton assumes the reader to be familiar with the principles of conics and of Euclid. *In the eighteenth and nineteenth centuries, when Newton's treatise was still being read in British universities*, authors of books on "conic sections"—for example, W. H. Besant[1], W. H. Drew[2], Isaac Milnes—supplied the proof of this theorem in order to help readers of the *Principia* who might be baffled by the problem of finding a proof [3, p. 330, my emphasis and references].

The development of analytic geometry and calculus notwithstanding, advanced Euclidean geometry remained the mainstay of mathematics education as late as the nineteenth century. In this article I wish to describe my journey from the work of Johannes Kepler and Isaac Newton on elliptical orbits into the richness of Euclidean geometry whose depth is no longer well-known. I conclude with a presentation of my opinion on the continued relevance of Euclidean geometry (Section 8).

The journey started with *Calculus in Context* [4] by Alexander J. Hahn: a textbook on introductory calculus that augments theory with applications in physics, astronomy, engineering and architecture. These are not just historical anecdotes but detailed computations. (Hahn also wrote an advanced book on orbits [5].)

Based on [4] I wrote a detailed presentation of the mathematics of planetary orbits through Newton's proof that the centripetal force on a planet in an elliptical orbit must follow the inverse square law [6]. This is the infamous Book I, Section III, Proposition XI, Problem VI of the *Principia*. To get an idea of the concise presentation that Newton gave, the reader is invited to look at Figure 1 from the *Principia*.

Hahn did not bring a complete proof because:


SECTIO III.

De motu corporum in conicis ſectionibus excentricis.

PROPOSITIO XI. PROBLEMA VI.

Revolvatur corpus in ellipſi: requiritur lex vis centripetæ tendentis ad umbilicum ellipſeos.

Eſto ellipſeos umbilicus *S*. Agatur *SP* ſecans ellipſeos tum diametrum *DK* in *E*, tum ordinatim applicatam *Qv* in *x*, & compleatur parallelogrammum *QxPR*. Patet *EP* æqualem eſſe ſemiaxi majori *AC*, eo quod, acta ab altero ellipſeos umbilico *H* linea *HI* ipſi *EC* parallela, ob æquales *CS*, *CH* æquentur *ES*, *EI*,

PRINCIPIA MATHEMATICA. 55

LIBER PRIMUS.

adeo ut *EP* ſemiſumma ſit ipſarum *PS*, *PI*, id eſt (ob parallelas *HI*, *PR*, & angulos æquales *IPR*, *HPZ*) ipſarum *PS*, *PH*, quæ conjunctim axem totum 2 *AC* adæquant. Ad *SP* demittatur perpendicularis *QT*, & ellipſeos latere recto principali (ſeu $\frac{2\,BC\,quad.}{AC}$) dicto *L*, erit *L* × *QR* ad *L* × *Pv* ut *QR* ad *Pv*, id eſt, ut *PE* ſeu *AC* ad *PC*; & *L* × *Pv* ad *GvP* ut *L* ad *Gv*; & *GvP* ad *Qv quad.* ut *PC quad.* ad *CD quad.* & (per corol. 2. lem. VII.) *Qv quad.* ad *Qx quad.* punctis *Q* & *P* coeuntibus eſt ratio æqualitatis; & *Qx quad.* ſeu *Qv quad.* eſt ad *QT quad.* ut *EP quad.* ad *PF quad.* id eſt, ut *CA quad.* ad *PF quad.* ſive (per lem. XII.) ut *CD quad.* ad *CB quad.* Et conjunctis his omnibus rationibus, *L* × *QR* fit ad *QT quad.* ut *AC* × *L* × *PCq* × *CDq*, ſeu 2 *CBq* × *PCq* × *CDq* ad *PC* × *Gv* × *CDq* × *CBq*, ſive ut 2 *PC* ad *Gv*. Sed punctis *Q* & *P* coeuntibus æquantur 2 *PC* & *Gv*. Ergo & his proportionalia *L* × *QR* & *QT quad.* æquantur. Ducantur hæc æqualia in $\frac{SPq}{QR}$, & fiet *L* × *SPq* æquale $\frac{SPq \times QTq}{QR}$. Ergo (per corol. 1. & 5. prop. VI.) vis centripeta reciproce eſt ut *L* × *SPq*, id eſt, reciproce in ratione duplicata diſtantiæ *SP*. *Q. E. I.*

Fig. 1 Proposition XI, Problem VI of the *Principia* (source: http://www.e-rara.ch/zut/wihibe/content/titleinfo/338026)

Newton's analysis of $\lim_{Q\to P} \frac{QR}{QT^2}$ in the case of an elliptical orbit is ingenious, but technical. It makes use of several subtle facts about the ellipse that are already developed in the *Conics* of Apollonius, but are no longer emphasized in today's textbooks [4, p. 242].

Section 2 shows how Kepler was able to deduce that orbits must be elliptical and Section 4 summarizes Newton's proof that the centripetal force must be inverse-square. A proof of Proposition XI, Problem VI based on [7] is given in Section 5. The required theorems on ellipses are proved in Sections 3 and 6, often using analytic geometry. The difficult proofs of properties of ellipses in Euclidean geometry are presented in Section 7 based upon [1].

My composition contains proofs of all the theorems and full-size diagrams [6]. In the interests of brevity this article states many theorems without proof, although proofs are given for Theorems 21 and 18 so that the reader can sample the depth of knowledge of Euclidean geometry of nineteenth century mathematics educators such as Besant and Drew.

2 Kepler and elliptical orbits

The Greeks knew that the Earth is round and Eratosthenes was able to measure the radius of the Earth. Using Eratosthenes's measurement, Aristarchus was able to measure and compute: the radius of the Moon, the radius of the Sun, the distance from the Earth to the Moon, the distance from the Earth to the Sun. While the computed values for the radii of the Earth and the Moon were reasonably accurate, the other values were not anywhere near the modern values.

Copernicus proposed a Sun-centered solar system with O, the center of the Earth's orbit, offset from S, the center of the Sun. Using observations of the orbit of Mars he calculated that the offset of the center of the orbit was $0.03r_e$, where r_e is the radius of the orbit Earth, at an angle 0.1129 radians, approximately 6.47°. From $r_e = 8,000,000$ km, he obtained that $OS = 240,000$, very far from the modern value $4,500,000$ km.

Towards the end of the sixteenth century, the astronomer Tycho Brahe carried out extremely precise observations. In 1600 he hired Johannes Kepler as his assistant and when Tycho died soon afterwards, Kepler was appointed to his position. Here we explain how Kepler was able to establish that planetary orbits are ellipses.

A Martian year is 687 days ≈ 1.88 Earth years. We know when Mars reaches a "new year" by observing its projection on the celestial sphere, but each time the position of the Earth in its orbit will be different. Figure 2 shows the orbit of the Earth—its center O offset from the Sun S—at four occasions when the position of Mars M at its new year was observed. Four triangles are created $\triangle OE_iM$. Tycho was able to show that the values for E_i are not equal:

$$
\begin{aligned}
E_1 &= 0.6682 \cdot OM\\
E_2 &= 0.6721 \cdot OM\\
E_3 &= 0.6785 \cdot OM\\
E_4 &= 0.6805 \cdot OM\,.
\end{aligned}
$$

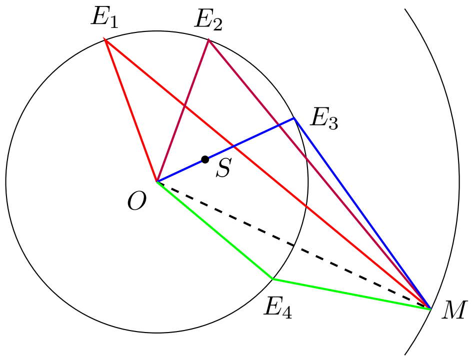


Fig. 2 Observations of the orbit of Mars from the Earth

If the Earth's orbit is circular, the only solution is to move the center of the orbit so that $\{E_1, E_2, E_3, E_4\}$ are all on the circle. While Kepler was able to modify the center of the orbit to be consistent with the observations, he was not able to adequately describe the orbit of Mars. After years of work, he concluded that its orbit must be oval like an egg. Oval, perhaps, but certainly not an ellipse, because he was certain that it would have been discovered by Archimedes [4, p. 94]! Figure 3 shows C, a position of Mars on a circular orbit, and an oval orbit (dashed), where M is the projection of C onto the oval orbit.

Kepler's computed that $(a-t)/t = 0.00429$ and $s/t = 1.00429$, so

$$\frac{a}{t} = 1 + 0.00429 = \frac{s}{t}\,,$$

and therefore $SM = s = a = AO$. The dashed oval is likely an ellipse, because in an ellipse $SM = AO$. Kepler then computed the projections of the four observations of Mars onto the x-axis and obtained for all of them that

$$\frac{M_iO_i}{C_iO_i} = \frac{t}{a} = \frac{1}{1.00429} = 0.99573\,.$$

By Theorem 2, since the ratio $MO/CO = b/a$ is constant for in an ellipse, Kepler was able to conclude that the orbit of Mars is an ellipse.

3 Interlude: Ellipses 1

Definition 1 (Ellipse). *Let S, H be two points in the plane (the* foci*), such that $SH = 2c \geq 0$ and let $2a > 2c$ (Figure 4). An* ellipse *is the locus of all points P such that $SP + PH = 2a$.*

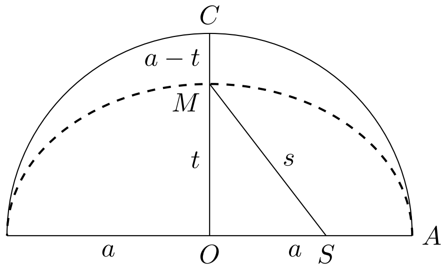


Fig. 3 The orbit of Mars as an oval "egg"

Construct a line through SH, where A, B (the vertices*) are the intersections of the line with the ellipse. AB is the* major axis *of the ellipse. Let O be the midpoint of SH; then AO, OB are the* semi-major axes *of the ellipse.*

Construct the perpendicular to AB at O and let C, D be its intersections with the ellipse. CD is the minor axis *of the ellipse and CO, OD are the* semi-minor axes *of the ellipse.*

Since A, B, C, D are all on the ellipse it follows that $AO = OB = SC = HC = a$ and $CO = OD$.

Are there any more points on the ellipse?

Theorem 1. *A point $P = (x, y)$ is on an ellipse if and only if satisfies the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{1}$$

Proof. Let $P = (x, y)$ be a point on the ellipse so that $SP + PH = 2a$. Since $S = (-c, 0)$, $H = (c, 0)$,

$$\begin{aligned} SP + PH &= \sqrt{(x-(-c))^2 + y^2} + \sqrt{(x-c)^2 + y^2} \\ &= 2a\,. \end{aligned}$$

Squaring twice results in

$$a^2 + \frac{c^2}{a^2}x^2 = x^2 + c^2 + y^2\,,$$

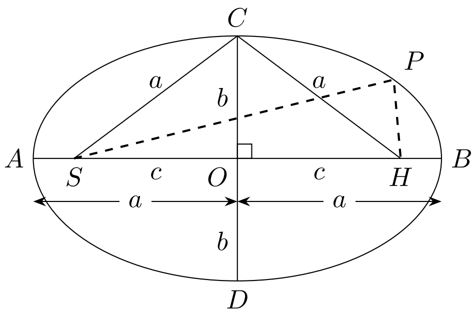


Fig. 4 The semi-major and semi-minor axes of an ellipse

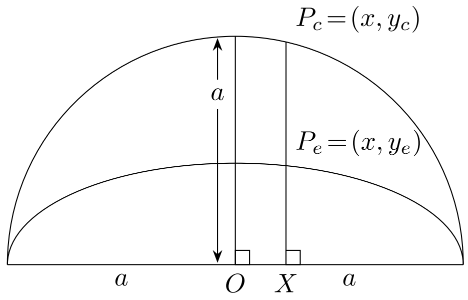


Fig. 5 A circle circumscribing an ellipse

and the result follows from $b^2 = a^2 - c^2$.

The proof of the reverse implication is similar.

□

Theorem 2. *The perpendicular to the major axis through a point $P_c = (x, y_c)$ on the circle circumscribing an ellipse intersects the ellipse at $P_e = (x, y_e) = (x, (b/a)y_c)$ (Figure 5).*

Definition 2. *Consider a line through a focus of an ellipse that is perpendicular the major axis. Let its intersections with the ellipse be L_1, L_2. $L = L_1L_2$ is a* latus rectum *of the ellipse (Figure 6).*

Theorem 3. *L, the length of the latus rectum of an ellipse, is $2b^2/a$.*

Theorem 4. *The area of an ellipse is πab.*

This follows easily from Theorem 2 and the formula πa^2 for the area of a circle, but, of course, the formula can only be proved as Archimedes did with a limit or as we do today with calculus.

4 Gravitation

Theorem 5. *If a planet subject to a centripetal force follows an elliptical orbit around the Sun, then the force decreases as the inverse square of the distance from the Sun.*

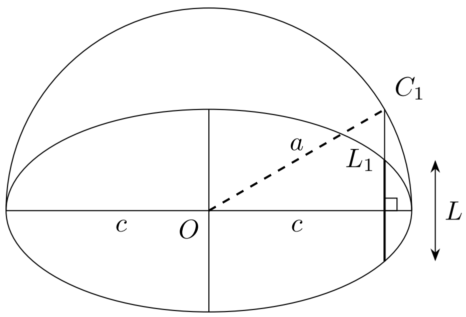


Fig. 6 The latus rectum of an ellipse

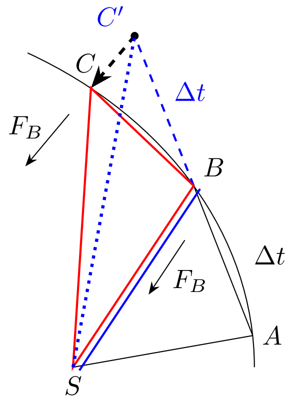


Fig. 7 Exerting force at discrete times

A centripetal force is a force directed to the Sun with no component in any other direction.

Newton's proof divides an area into very small sectors and then takes the limit. Consider three points A, B, C on the orbit that represent the positions of the planet at intervals of Δt (Figure 7). Newton assumed that the planet does not smoothly traverse the arcs, but rather it jumps in discrete steps from one point on the orbit to the next. The planet moves from A to B and we expect that the centripetal force at B will cause an acceleration that moves the planet to C. Instead, we "pretend" that the force is not applied at B, but, in the absence of an applied force, planet continues to move in the same direction and at the same speed. After another period of Δt has passed and the planet has reached point C', the force is now applied *in the same direction* as it would have been applied at B, moving the planet to C.

Newton proved that $A_{\triangle ASB}$, the area of $\triangle ASB$, is equal to $A_{\triangle BSC}$, the area of $\triangle BSC$. He did this by constructing the auxiliary line SC' and then showing that $A_{\triangle ASB} = A_{\triangle BSC'}$ and then that $A_{\triangle BSC'} = A_{\triangle BSC}$. As Δt is decreased, the area of each triangle, call it ΔA, approaches the area of the sector that contains it. For any sectors such as those shown in Figure 8,

$$\frac{A_{P_1SP_2}}{\Delta A} = \frac{t}{\Delta t} = \frac{A_{P_3SP_4}}{\Delta A},$$

from which Kepler's second law follows: $A_{P_1SP_2} = A_{P_3SP_4}$.

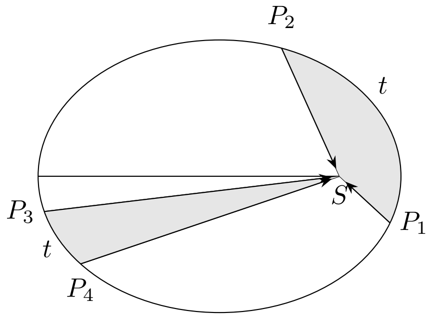


Fig. 8 Equal areas in equal times

Definition 3. *For a given elliptical orbit,* $\kappa = \dfrac{A}{t}$, *where* A *is the area of the ellipse and* t *is the period of the orbit, is called* Kepler's constant.

Newton's next step was to show that if the orbit of a planet is elliptical, the centripetal force must be proportional to the mass of the planet and inversely proportional to the square of its distance from the Sun. In Figure 9 P, Q are points on the orbit close to each other, PR is the tangent to the ellipse at P, R is chosen to that $QR \parallel SP$ and T is chosen so that $QT \perp SP$.

When a body at R of mass m is subject to an acceleration a for a period of Δt, its displacement $QR = \frac{1}{2}a(\Delta t)^2$. By Newton's second law, $a = F_P/m$ and

$$F_P = \frac{2mQR}{(\Delta t)^2}\,.$$

The area of the sector SPQ is approximately equal $A_{\triangle SPQ} = (1/2)QT \cdot SP$. Using Kepler's constant, $\Delta t = QT \cdot SP/2\kappa$, giving

$$F_P = 8\kappa^2 m \cdot \frac{QR}{QT^2} \cdot \frac{1}{SP^2}\,.$$

For a given planet m is constant and for a given elliptical orbit κ is constant, so the first factor does not depend on the distance. If QR/QT^2 is a constant as Δt approaches zero, F_P is subject to an inverse square law.

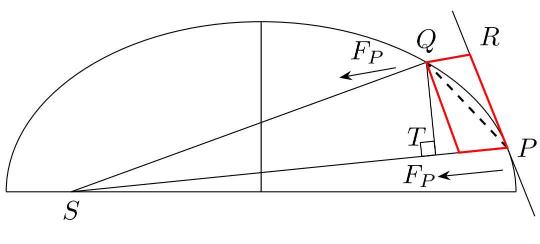


Fig. 9 The derivation of the inverse square law

Theorem 6. *In an elliptical orbit*

$$\lim_{\Delta t \to 0} \frac{QR}{QT^2} = \frac{1}{L}\,,$$

where L *is the length of the latus rectum of the ellipse.*

This theorem appears as Proposition XI, Problem VI of the *Principia*; Newton's proof is discussed in Section 5.

By Newton's third law, F_{SE}, the force that the Sun S exerts on the Earth E, equals F_{ES}, the force that the Earth exerts on the Sun. Let m be the mass of the Earth and M be the mass of the Sun; then:

$$\frac{8\kappa_E^2 m}{L_E} \cdot \frac{1}{r^2} = F_{SE} = F_{ES} = \frac{8\kappa_S^2 M}{L_S} \cdot \frac{1}{r^2}\,.$$

Let

$$C_E = \frac{8\kappa_E^2}{L_E}, \quad C_S = \frac{8\kappa_S^2 M}{L_S}\,,$$

so that $C_E m = C_S M$ and let $G = C_E/M = C_S/m$. Newton's law of universal gravitation follows.

$$F_{SE} = F_{ES} = G\frac{mM}{r^2}\,. \tag{2}$$

From universal gravitation, Kepler's third law follows easily.

Theorem 7. *Let* P_1, P_2 *be two planets whose elliptical orbits have semi-major axes* a_1, a_2 *and whose orbital periods around the Sun are* T_1 *and* T_2*. Then*

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2}\,.$$

5 A proof Proposition XI, Problem VI

This Section contains an outline of a proof of Theorem 6

$$\lim_{\Delta t \to 0} \frac{QR}{QT^2} = \frac{1}{L}\,.$$

To prove the theorem, Newton develops formulas for QR, QT^2, QR/QT^2 and then takes the limit as Δt approaches zero.

The proof of the formula for QR is based on Figure 10 (top). Let P, Q be two points on the

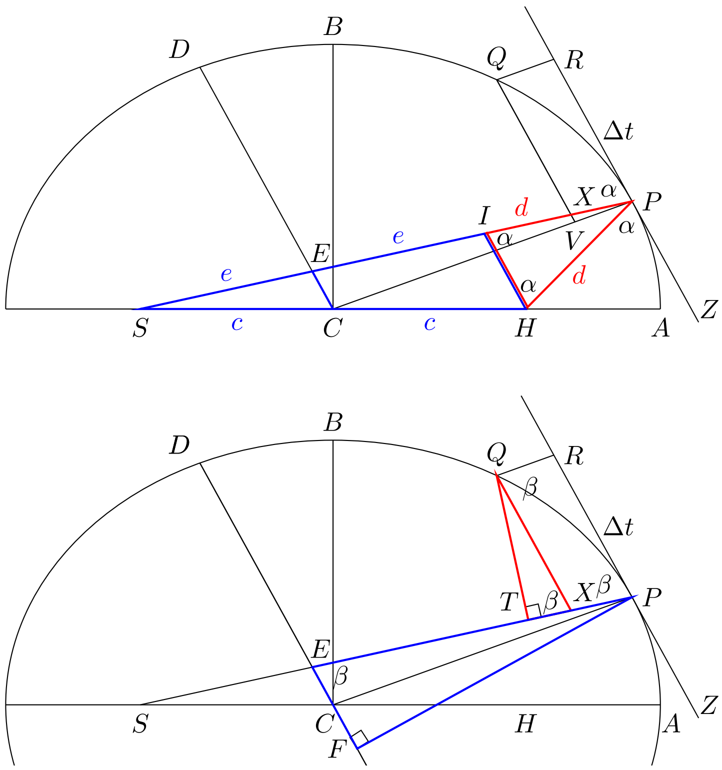


Fig. 10 Geometry of an elliptical orbit

ellipse that represent the movement of a body during a time interval Δt. Construct lines from P to the center C and to the foci S, H. Construct the tangent at P and choose R on the tangent such that the body would move from P to R if it continued for time Δt not subject to any force. Construct the parallelogram $PRQX$ and extend QX until it intersects PC at V. Construct a line parallel to RP through H and let I be its intersection with PS. Construct DC the conjugate diameter to PC (Definition 4), and let E be its intersection with PS.

Theorem 8. $QR = PV \cdot \dfrac{CA}{CP}$.

Now construct a perpendicular from P to DC and let its intersection with DC be F. Construct a perpendicular from Q to SP and let its intersection with SP be T (Figure 10, bottom).

Theorem 9.

$$QT = QX \cdot \frac{FP}{CA}\,.$$

Theorem 10.

$$\frac{QR}{QT^2} = \frac{CP \cdot CA}{CB^2} \cdot \frac{QV^2}{GV \cdot QX^2}\,. \tag{3}$$

The proof of this theorem uses Theorem 13 of the areas of parallelograms formed by tangents to conjugate diameters. Its proof in Euclidean geometry (Theorem 23) is particularly challenging.

6 Interlude: Ellipses 2

Theorem 11 (Focus-to-focus reflection property)**.** *Let P be a point on the ellipse whose foci are S, H. Let PU be the extension of SP such that $SU = AA' = 2a$. Let RQ be the bisector of $\angle HPU$. Then $\angle RPS = \angle QPH$ and RQ is the tangent to the ellipse at P (Figure 11).*

Definition 4. *There are two equivalent definitions of* conjugate diameters *(Figure 12).*

- *Let P be a point on an ellipse, PG a diameter and let t be the tangent to the ellipse at P. Diameter DK is a* conjugate diameter *if it is parallel to t.*

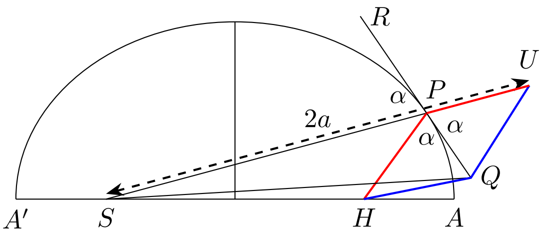


Fig. 11 Focus-to-focus reflection property

- *Two diameters PG and DK are* conjugate diameters *if the midpoint of any chord parallel to one diameter (DK) lies on the other diameter (PG).*

Theorem 12. *Let $P = (x, y)$ be a point on an ellipse (not on the major axis AA') and construct a perpendicular PV from P to the major axis (Figure 13). Then*

$$\frac{A'V \cdot AV}{PV^2} = \frac{AC^2}{BC^2}\,.$$

Theorem 13. *Let PG, DK be conjugate diameters of an ellipse and let Q be a point on the ellipse (Figure 14). Constrct the perpendicular QV from Q to the major axis, then*

$$PV = \frac{QV^2 \cdot CP^2}{GV \cdot CD^2}\,.$$

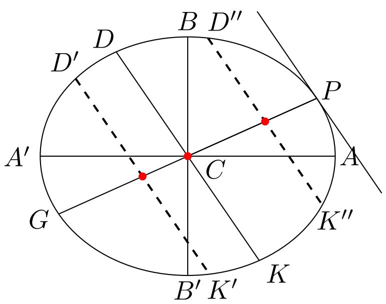


Fig. 12 Conjugate diameters

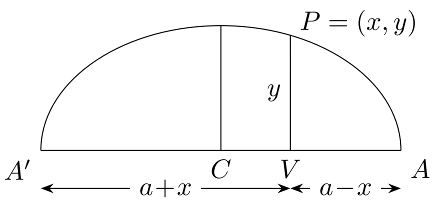


Fig. 13 Ratios on conjugate diameters

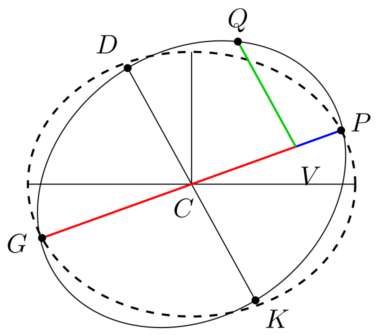


Fig. 14 Ratios on conjugate diameters

Figure 15 shows an ellipse and two circles: one whose radius is the length of the semi-major axis (dotted red) and one whose radius is the semi-minor axis (dashed blue). The figure shows the *parametric representation* of a point $P = (x, y)$ on the ellipse $(x, y) = (a \cos t,\ y = b \sin t)$.

Theorem 14. *The areas of the parallelograms formed by tangents to the intersections of any pair of conjugate diameters with the ellipse are equal (Figure 16).*

This theorem is the key to the proof of Problem VI. Note that AA' and BB' are conjugate diameters forming a parallelogram which is a rectangle so its area is trivial to compute.

Proof. By symmetry it suffices to prove that $A_{\triangle ACB} = A_{\triangle PCD}$. Let $P = (x_p, y_p) = (a \cos t, b \sin t)$ and $D = (x_d, y_d)$ be the parametric representations of the points on the ellipse. Conjugate diameters are perpendicular so $\angle DCP$ is a right angle and

$$\begin{aligned} D = (x_d, y_d) &= (a \cos(t + \pi/2), b \sin(t + \pi/2)) \\ &= (-a \sin t, b \cos t)\,. \end{aligned}$$

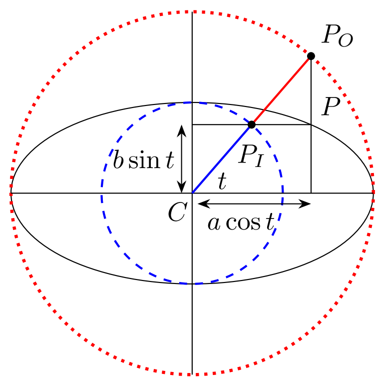


Fig. 15 Parametric representation of an ellipse

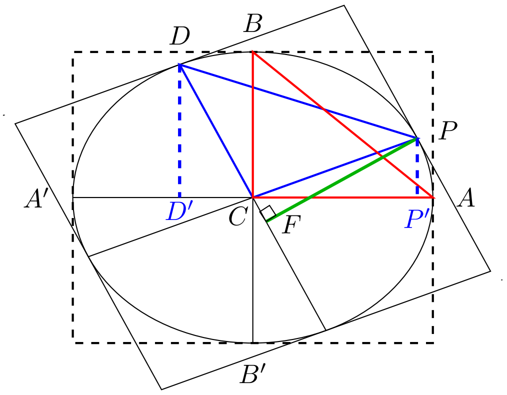


Fig. 16 Parallelograms formed by conjugate diameters

Construct $DD' = (x_d, 0)$ and $PP' = (x_p, 0)$ perpendicular to the major axis. Then $A_{\triangle PCD}$ can be computed as the area of the trapezoid $P'PDD'$ minus the $A_{\triangle D'DC}$ and $A_{\triangle P'PC}$, which is $A_{\triangle ABC}$. □

7 Ellipses in Euclidean geometry

We start with a different definition of ellipses.

Definition 5. *Let d be a line (the* directrix*) and S be a point (the* focus*) not on the directrix. Let $0<e<1$ be a number (the* eccentricity*). An* ellipse *is the locus of points P such that the ratio of PS to the distance of P from the directrix is e. Let X be the intersection of the perpendicular to the directrix from S. A on SX is a* vertex *of the ellipse if $SA/AX = e$.*

The definition is non-constructive because the only points known to be on the ellipse are the vertices. We now show how to construct arbitrary points on the ellipse (Figure 17).

Select an *arbitrary* point E on the directrix and construct lines from E through A and S. The line through S will make some angle α with SX. Construct a line from S at the *same angle* α from ES and let its intersection with EA be P. Construct the perpendicular from P to the directrix and let K be its intersection with the directrix. Let L be the intersection of KP with ES.

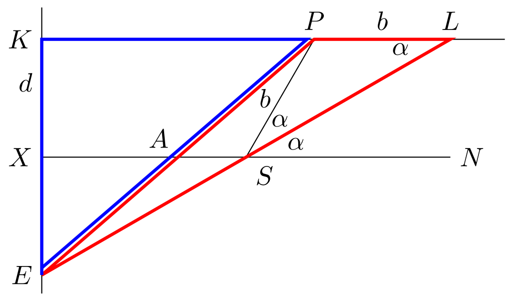


Fig. 17 Constructing points on the ellipse

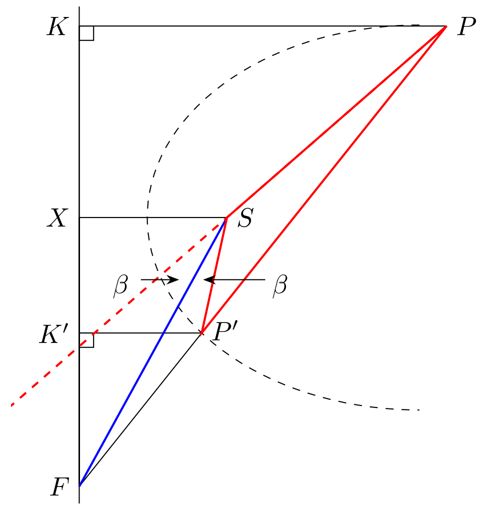


Fig. 18 Bisecting the angle at the focus

Theorem 15. *The point P is on the ellipse.*

By choosing different points E on the directrix, any point on the ellipse can be constructed.

The rest of this section is a sequence of the theorems that are required for the proof of Theorem 23, although this will not be apparent because only two of the proofs are given. Of course I started by studying the proof of Theorem 23 which cited the previous theorem and so on, leading to this complex sequence.

Theorem 16. *Let P, P' be points on the ellipse and let F be the intersection of PP' with the directrix. Then FS bisects the exterior angle of $\angle P'SP$ (Figure 18).*

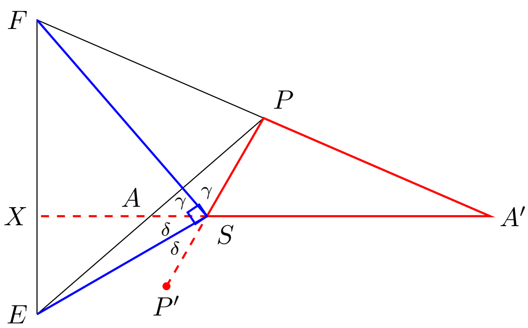


Fig. 19 The right angle at the focus

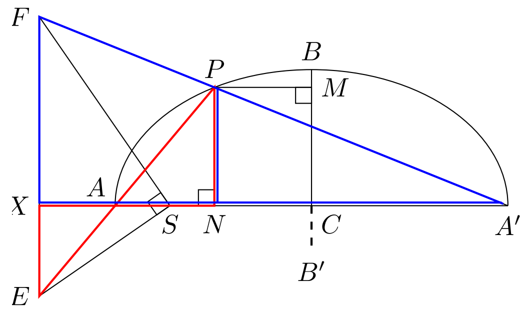


Fig. 20 Ratio of an ordinate

Theorem 17. *Let P be a point on the ellipse and construct lines PA, PA'. Let their intersections with the directrix be E and F, respectively. Then $\angle FSE$ is a right angle (Figure 19).*

Theorem 18. *Let P be a point on an ellipse not on the major axis and construct perpendiculars PN, PM from P to the major and minor axes, respectively (Figure 20). Then*

$$\frac{PN^2}{A'N \cdot NA} = \frac{BC^2}{AC^2} \tag{4}$$

$$\frac{PM^2}{B'M \cdot MA} = \frac{AC^2}{BC^2} \,. \tag{5}$$

Proof. (Equation 4) $\triangle AXE \sim \triangle ANP$ since they are right triangles and the vertical angles at A are equal (red). Therefore,

$$\frac{PN}{AN} = \frac{EX}{AX} \,. \tag{6}$$

$\triangle PA'N \sim \triangle FA'X$ (blue) so

$$\frac{PN}{A'N} = \frac{FX}{A'X} \,. \tag{7}$$

Multiplying Equations 6 and 7 gives

$$\frac{PN^2}{AN \cdot A'N} = \frac{EX \cdot FX}{AX \cdot A'X} \,.$$

By Theorem 17 $\triangle FSE$ is a right triangle, so SX^2, the square of the altitude to the hypotenuse, is equal to $EX \cdot FX$, the product of its segments:

$$\frac{PN^2}{AN \cdot A'N} = \frac{SX^2}{AX \cdot A'X} \,.$$

Since P was arbitrary this holds for any point on the ellipse, in particular, for B on the minor axis, where $PN = BC$ and $AN = AN' = AC$, so

$$\frac{BC^2}{AC \cdot AC} = \frac{SX^2}{AX \cdot A'X} \,,$$

which is a constant so

$$\frac{BC^2}{AC \cdot AC} = \frac{SX^2}{AX \cdot A'X} = \frac{PN^2}{AN \cdot A'N} \,,$$

for any point P on the ellipse. □

Proof. (Equation 5) Given the midpoint C of a line segment AA' and any other point N on the segment, it is easy to show that $AC^2 - CN^2 = AN \cdot NA'$. Therefore, since $CM = PN, PM = CN$, Equation 4 becomes

$$\frac{CM^2}{AC^2 - PM^2} = \frac{BC^2}{AC^2}$$

$$\frac{AC^2}{AC^2 - PM^2} = \frac{BC^2}{CM^2} \,. \tag{8}$$

By cross-multiplying Equation 8 it can be transformed into

$$\frac{AC^2}{PM^2} = \frac{BC^2}{BC^2 - CM^2} \,, \tag{9}$$

Using the theorem on dividing line segments, Equation 9 implies

$$\frac{AC^2}{BC^2} = \frac{PM^2}{BM \cdot MB'} \,.$$

□

The final and very difficult task is to prove that the area of the parallelogram formed by the tangents to conjugate diameters is equal to the parallelogram (rectangle) circumscribing the ellipse. Recall that the proof using analytic geometry is very simple using parametric coordinates. The proof in Euclidean geometry is divided into five, rather difficult, theorems. Here we state all five with diagrams but give the proof of just one.

Theorem 19. *Let Y be the intersection the perpendicular through the focus S to the tangent TT' at P, and let L be the intersection of $S'P$ and SY (Figure 21). Then Y is on the circumscribing circle and $CY \parallel S'L$.*

Theorem 20. *Let N be the intersection of the perpendicular through P to the major axis (Figure 22). Then $CN \cdot NT = AC^2 = AN \cdot NA'$.*

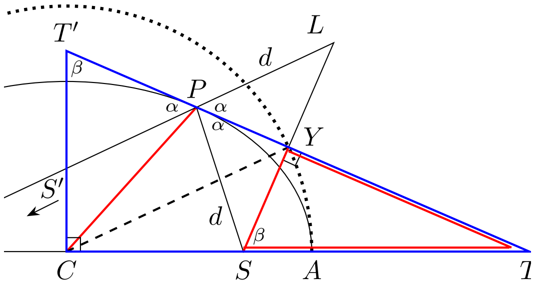


Fig. 21 The perpendicular from a focus to a tangent

Theorem 21. *Construct the normal to the tangent at P and let its intersection with the conjugate diameter DK be F and its intersection with the major axis be G. Construct a perpendicular from P to the major axis and let its intersection with the major axis be N. Let the intersection of the tangent with the minor axis be T and its intersection with the major axis be T' (Figure 23). Then*

$$PF \cdot PG = BC^2 \,.$$

Proof. $\triangle NPG \sim \triangle FPJ$ so $\angle PGN = \angle PJF = \alpha$ and

$$\begin{aligned} \frac{PF}{PN} &= \frac{PJ}{PG} \\ PF \cdot PG &= PJ \cdot PN \,. \end{aligned} \tag{10}$$

By vertical angles $\angle PGN = \angle CGF = \alpha$ so $\triangle NPG \sim \triangle FCG$ (red) and $\angle NPG = \angle FCG = \beta = 90° - \alpha$. $TPJC$ is a parallelogram, so $CT = PJ$ and $PF \cdot PG = CT \cdot PN$. The theorem will be proven if we can show that $CT \cdot PN = BC^2$.

$\triangle TT'C \sim \triangle PT'N$ (blue) so

$$\begin{aligned} \frac{CT}{CT'} &= \frac{PN}{NT'} \\ \frac{CT}{PN} &= \frac{CT'}{NT'} \,. \end{aligned} \tag{11}$$

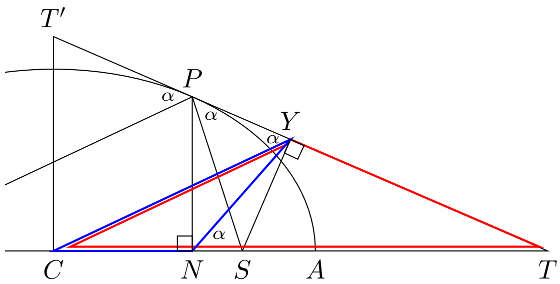


Fig. 22 Ratios of segments of the major axis

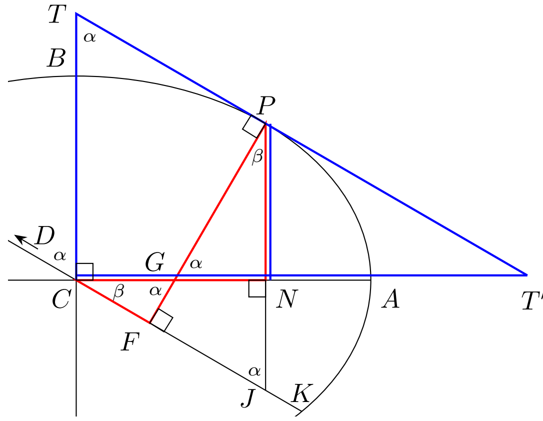


Fig. 23 Parallelograms formed by conjugate diameters

Multiply each side of Equation 11 by fractions equal to 1:

$$\frac{CT}{PN} \cdot \frac{PN}{PN} = \frac{CT'}{NT'} \cdot \frac{CN}{CN} \,.$$

By Theorem 20 $CN \cdot CT' = AC^2$ and $CN \cdot NT' = AN \cdot NA'$, so

$$\frac{CT \cdot PN}{PN^2} = \frac{AC^2}{AN \cdot NA'} \,.$$

Multiplying by PN^2/PN^2 and using Theorem 18 gives

$$\begin{aligned} \frac{CT \cdot PN}{PN^2} &= \frac{PN^2}{AN \cdot NA'} \cdot \frac{AC^2}{PN^2} \\ &= \frac{BC^2}{AC^2} \cdot \frac{AC^2}{PN^2} \\ CT \cdot PN &= BC^2 \,. \end{aligned} \tag{12}$$

□

Theorem 22. *In Figure 24,*

$$CN^2 = AM \cdot MA' \qquad CM^2 = AN \cdot NA'$$

$$\frac{DM}{CN} = \frac{BC}{AC} \qquad \frac{CM}{PN} = \frac{BC}{AC}$$

Theorem 23. *The area of the parallelogram formed by the tangents at the ends of the conjugate diameters PP', DK is equal to the area of the rectangle enclosing the ellipse at the ends of the axes (Figure 16).*

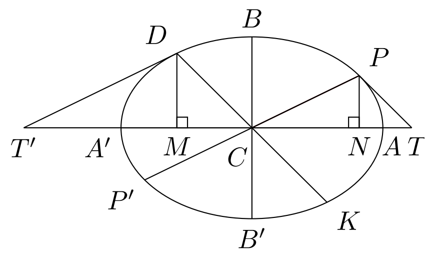


Fig. 24 Ratios of perpendiculars to the major axis

8 On Euclidean geometry

The ninth (!) edition of William H. Besant's *Conic Sections Treated Geometrically* [1] was published in 1895. The chapter on ellipses has 32 propositions and 110 exercises. A search of the text turns up no occurrences of "coordinate," "trigonometry" or "analytic." Who knew that Euclidean geometry was so rich? But who needs Euclidean geometry if Theorem 14 can be proved in a few lines of analytic geometry but the proof of the same Theorem 23 takes many pages of Euclidean geometry.

I claim that the teaching of Euclidean geometry should continue to be a major subject in mathematics curricula. At the secondary-school level, courses on algebra, analytic geometry and calculus tend to be heavy on calculation and light on proving theorems. This can deceive students into thinking that mathematicians do calculations and they will be disappointed when studying subjects like abstract algebra that are heavy on proving theorems and light on calculation. Conversely, students who are not good at calculation may decline further study of mathematics when they might actually be good at proving theorems. Euclidean geometry has the additional advantage that in spite of the richness of the theory, it deals with two-dimensional concrete objects. It therefore serves as an excellent first step towards learning about theorems and proofs. I am not proposing that we abandon analytic geometry and resurrect the Euclidean geometry of Newton and Besant, but let's grant it its rightful place in mathematics.

On the *Principia*

Isaac Newton published *Philosophiæ Naturalis Principia Mathematica* in Latin in 1687. Subsequent editions appeared in 1723 and 1726. The third edition was translated into English as *The Mathematical Principles of Natural Philosophy* by Andrew Motte in 1729. This translation has been modernized several times, but truly new translations have only appeared recently. The translation by I. Bernard Cohen is very useful because of his extensive *Guide* that precedes the translation [3]. Should you wish to study Newton's proof of Proposition XI, Problem VI, I recommend Section 10.9 of the *Guide* [3, pp. 324–329]. A comprehensive list of links to editions of the *Principia* can be found in the Wikipedia entry for *Philosophiæ Naturalis Principia Mathematica*.

Acknowledgments

References

[1] Besant, W.H.: Conic Sections, Treated Geometrically (Ninth Edition Revised and Enlarged). George Bell and Sons, London (1895). https://www.gutenberg.org/ebooks/29913 and https://archive.org/details/cu31924059322481 (Accessed 6 September 2023)

[2] Drew, W.H.: A Geometrical Treatise on Conic Sections (Second Edition). Macmillan, Cambridge (1862). https://archive.org/details/in.ernet.dli.2015.501433 (Accessed 6 September 2023)

[3] Cohen, I.B., Whitman, A., Budenz, J.: The Principia: Mathematical Principles of Natural Philosophy. University of California Press, Berkeley, CA (1999). Preceded by *A Guide to Newton's Principia* by I. Bernard Cohen

[4] Hahn, A.J.: Calculus in Context: Background, Basics, and Applications. Johns Hopkins University Press, Baltimore, MD (2017)

[5] Hahn, A.J.: Basic Calculus of Planetary Orbits and Interplanetary Flight: The Missions of the Voyagers, Cassini, And Juno. Springer, Cham, CH (2020)

[6] Ben-Ari, M.: The Geometry of Ellipses and Planetary Orbits. https://github.com/motib/orbits. (Accessed 6 September 2023) (2023)

[7] Griffiths, G.: The Inverse Square Law of Gravitation: An Alternative to Newton's Derivation. https://www.researchgate.net/publication/264978661_The_Inverse_Square_Law_of_Gravitation_An_Alternative_to_Newton's_Derivation. (Accessed 6 September 2023) (2009)