Mosteller's Challenging Problems in Probability

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Introduction

Frederick Mosteller

Frederick Mosteller (1916–2006) founded the Department of Statistics at Harvard University and served as its chairman from 1957 until 1971, retiring from the university in 2003. Mosteller was deeply interested in statistics education and wrote pioneering textbooks including [10] which emphasized the probabilistic approach to statistics, and [9] which was one of the first texts on data analysis. In an interview Mosteller described the development of his approach to statistics education [6].

This document

This document is a "reworking" of Mosteller's delightful book *Fifty Challenging Problems in Probability with Solutions* [8]. The problems and their solutions are presented as far as possible in a manner accessible to readers with an elementary knowledge of probability, and many of the problems are accessible to secondary-school students and teachers. The problems and solutions have been rewritten to include detailed calculations and additional explanations and diagrams. I have sometimes included additional solutions.

Many of the problems have been modified to make them accessible: they are simplified, divided into subproblems and hints are provided. As a personal preference I have rephrased the problems in a more abstract way than Mosteller does and I have not given units like inches or currencies like dollars. The numbering and titles of the problems have been retained to facilitate comparison with Mosteller's book.

Modern scientific calculators, including applications for smartphones, can perform the computations with no difficultly, although we use Stirling's approximation in two problems so that readers can see how it is used.

Basic concepts of probability are reviewed in the final section which is based on [11]. Since students may not be familiar with random variables and expectation, those concepts are presented in more detail.

Problems that are more difficult are annotated with D. Even a problem not marked D can be difficult so do not be discouraged if you cannot solve it. However, it is worthwhile attempting to solve all the problems because any progress you make will be encouraging.

Source code

 LaTeX and Python source code is available at:

`https://github.com/motib/probability-mosteller`

Acknowledgments

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Simulations

Monte Carlo simulations (named after the famous casino in Monaco) written in the Python 3 programming language are provided for most problems. A computer program "performs an experiment," such as "tossing a pair of dice" or "flipping a coin," a very large number of times and computes averages which are displayed. The random number generators built into Python, `random.random()` and `random.randint()`, are used to obtain random outcomes for each experiment.

The programs run each simulation 10000 times and the results are displayed to four decimal places. A simulated result will almost certainly not be exactly the same as that obtained from computing the expectation or the probability.

The files are named `N-name.py` where `N` is the problem number and `name` is the problem title.

For each simulation two results are displayed:

- The theoretical value which is either a *probability* or an *expectation*. In general, rather than copy the values from the text they are calculated from the formulas.
- The result of the simulation is either a *proportion* of successes relative to the number of trials corresponding to a probability, or the *average* number of successes corresponding to an expectation.

It is important to understand that "probability" and "expectation" are theoretical concepts. The *law of large numbers* ensures that the outcomes of many trials are very close to the theoretical values, but they won't be exactly the same. For example, the probability of obtaining a 6 when a fair die is thrown is $1/6 \approx 0.1667$. Running a simulation for 10000 throws resulted in a range of values: $0.1684, 0.1693, 0.1687, 0.1665, 0.1656$.

The simulation programs are very short and straightforward once a problem is understood. I suggest running the simulations for various numbers of trials and other parameters to help understand their sensitivity to these parameters.

For Problem 36, *The Gambler's Ruin*, I have written a Python program that plots the results of each step. You can use this as a template to write other plotting programs.

Problems and solutions

1. The sock drawer

A drawer contains both red socks and black socks. If two socks are drawn at random (without replacement) the probability that both are red is $\frac{1}{2}$.

Question 1: How small can the number of black socks in the drawer be? What is the corresponding number of red socks?

Question 2: How small can the number of black socks in the drawer be if the number of black socks is *even*? What is the corresponding number of red socks?

Solution 1

Answer 1: Let r be the number of red socks in the drawer and let b the number of black socks. $r \geq 2$ since two red socks are drawn, and $b \geq 1$ since otherwise the probability of drawing two red socks would be 1. Multiplying the probabilities for the two selections gives:

$$P(\textsf{both red}) = \frac{r}{r+b} \cdot \frac{(r-1)}{(r-1)+b} = \frac{1}{2}\,. \tag{1}$$

Simplifying results in a quadratic equation in the variable r:

$$r^2 - r(2b+1) - (b^2 - b) = 0\,. \tag{2}$$

Since r, b are positive integers the discriminant must be the square of an integer:

$$(2b+1)^2 + 4(b^2 - b) = 8b^2 + 1 \tag{3}$$

The discriminant is a square when $b = 1$ (its smallest value). From Equation 2, $r = 3$ where we reject the solution $r = 0$ because $r \geq 2$. The total number of socks is 4.

Check: $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

Answer 2: Check even positive integer values of b to find the smallest one for which the discriminant is a square:

b	$8b^2+1$	$\sqrt{8b^2+1}$
2	33	5.74
4	129	11.36
6	**289**	**17**

For $b = 6$ the corresponding value for r is 15 obtained by solving Equation 2.

Check: $\frac{15}{21} \cdot \frac{14}{20} = \frac{1}{2}$.

Solution 2

Answer 1: Is the following inequality is true?

$$\frac{r}{r+b} \overset{?}{>} \frac{r-1}{(r-1)+b}\,. \tag{4}$$

$r \geq 2, b \geq 1$, so both denominators are positive and we can multiply the two sides:

$$\begin{aligned} r(r-1+b) &\overset{?}{>} (r-1)(r+b) \\ r^2 - r + rb &\overset{?}{>} r^2 - r + rb - b \\ b &\overset{?}{>} 0\,. \end{aligned}$$

$b > 1$ so Equation 4 is true.

By Equations 1, 4:

$$\left(\frac{r}{r+b}\right)^2 = \frac{r}{r+b} \cdot \frac{r}{r+b} > \frac{r}{r+b} \cdot \frac{r-1}{(r-1)+b} = \frac{1}{2}\,, \tag{5}$$

and similarly:

$$\left(\frac{r-1}{(r-1)+b}\right)^2 = \frac{r-1}{(r-1)+b} \cdot \frac{r-1}{(r-1)+b} < \frac{r}{r+b} \cdot \frac{r-1}{(r-1)+b} = \frac{1}{2}\,. \tag{6}$$

The denominator $r+b$ is non-zero so we can take the square root and simplify Equation 5:

$$\begin{aligned} \frac{r}{r+b} &> \sqrt{\frac{1}{2}} \\ r &> \frac{b}{\sqrt{2}-1} \\ r &> \frac{b}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} \\ r &> b(\sqrt{2}+1)\,. \end{aligned}$$

Similarly for Equation 6:

$$\begin{aligned} \frac{r-1}{(r-1)+b} &< \sqrt{\frac{1}{2}} \\ r-1 &< \frac{b}{\sqrt{2}-1} \\ r-1 &< b(\sqrt{2}+1)\,. \end{aligned}$$

Combining both equations we get:

$$r - 1 < (\sqrt{2}+1)b < r\,. \tag{7}$$

For $b = 1$ we have $2.141 < r < 3.141$ and $b = 1, r = 3$ is a solution.

Answer 2: Checking even numbers for b:

b	$(\sqrt{2}+1)b$	$< r <$	$(\sqrt{2}+1)b+1$	r	$P(\text{two reds})$
2	4.8	$< r <$	5.8	5	0.4762
4	9.7	$< r <$	10.7	10	0.4945
6	14.5	$< r <$	15.5	15	0.5000

Mosteller mentions a connection between this problem and advanced number theory, and gives another solution: $b = 35, r = 85$.

Simulation

```
Expectation of both red  = 0.5000
Average of both red for (red =  3, black =  1) = 0.5053
Average of both red for (red = 15, black =  6) = 0.5013
Average of both red for (red = 85, black = 35) = 0.4961
```

Comment

In both solutions we don't actually prove a *sufficient* condition for the values of r, b. In Solution 1 we derive a necessary condition—by Equation 3 the discriminant must be an integer—and start searching for values of b that satisfy this requirement. In Solution 2 the necessary condition is that r, b must satisfy the inequalities in Equation 7 and then we search for values that satisfy the requirement that the probability of two reds must be 0.5. I wrote a short program to search for solutions in the range $[1, 50]$. The output for values near 35 are:

```
32 78  90.52 0.500917
33 80  93.34 0.499368
34 83  96.17 0.501474
35 85  99.00 0.500000
36 87 101.83 0.498601
37 90 104.66 0.500562
```

where the columns are the number of black socks, the number of red socks, the square root of the discriminant (Equation 3), the probability of drawing two red socks.

Using a computer program I found the following solutions where the number of black socks is less than one million:

black	red
1	3
6	15
35	85
204	493
1189	2871
6930	16731
40391	97513
235416	568345

2. Successive wins

You play a sequence of three games alternately against two players and you win the sequence if you win at least two of the three games. The probability that you will win a game against player P_1 is p_1 and the probability that you will win a game against player P_2 is p_2. It is given that $p_1 > p2$. Which of these scenarios gives you a better chance of winning the sequence?

- You play against P_1, P_2, P_1 in that order.
- You play against P_2, P_1, P_2 in that order.

Solution 1

You win if: (a) you win the first two games and lose the last game, (b) you lose the first game and win the last two games, or (c) you win all three games.

Let p_{121} and p_{212} be the probabilities that you win the sequence in the two scenarios:

$$\begin{aligned} p_{121} &= p_1 p_2 (1-p_1) + (1-p_1) p_2 p_1 + p_1 p_2 p_1 \\ p_{212} &= p_2 p_1 (1-p_2) + (1-p_2) p_1 p_2 + p_2 p_1 p_2 \,. \end{aligned}$$

You have a better chance of winning the sequence in the first scenario if $p_{121} > p_{212}$, that is, if:

$$\begin{aligned} p_1 p_2 (1-p_1) + (1-p_1) p_2 p_1 + p_1 p_2 p_1 &\overset{?}{>} p_2 p_1 (1-p_2) + (1-p_2) p_1 p_2 + p_2 p_1 p_2 \\ -p_1 p_2 p_1 &\overset{?}{>} -p_2 p_1 p_2 \\ p_1 &\overset{?}{<} p_2 \,. \end{aligned}$$

By assumption $p_1 > p_2$ so you should choose the second scenario.

Solution 2

The result is counter-intuitive. Intuitively, you should choose to play two games with P_1 and one game with P_2 because more likely to win games against P_1. However, the only way that you can win the sequence is by winning the *middle* game, and, therefore, you should play the middle set against P_1, the player you are more likely to defeat.

Simulation

```
For p1 = 0.6, p2 = 0.5
Proportion of P121 wins = 0.4166
Proportion of P212 wins = 0.4473

For p1 = 0.6, p2 = 0.4
Proportion of P121 wins = 0.3300
Proportion of P212 wins = 0.3869

For p1 = 0.6, p2 = 0.2
Proportion of P121 wins = 0.1625
Proportion of P212 wins = 0.2141
```

Explain why the proportions don't add up to 1.

3. The flippant juror

There are two options to reach a decision: (a) A three-person panel consisting of two members who independently make the correct decision with probability p and one member who makes

the correct decision with probability 1/2. The final decision is determined by a majority vote.
(b) A one-person panel whose only member has probability p of making the correct decision.
Which option has the higher probability of making the correct decision?

Solution

The three-person panel makes the correct decision if all three members make the correct decision or if any subset of two members makes the correct decision. The probability is:

$$\overbrace{\left(p \cdot p \cdot \frac{1}{2}\right)}^{\text{all three correct}} + \overbrace{\left(p(1-p) \cdot \frac{1}{2} + (1-p)p \cdot \frac{1}{2} + p \cdot p \cdot \frac{1}{2}\right)}^{\text{two out of three correct}} = p\,,$$

so there is no difference between the two options.

Simulation

```
Prediction: probabilities of (a) and (b) are equal
For p = 0.25, proportion correct of (a) = 0.5019, (b) = 0.5046
For p = 0.50, proportion correct of (a) = 0.5072, (b) = 0.4970
For p = 0.75, proportion correct of (a) = 0.5062, (b) = 0.5040
```

4. Trials until first success

What is the expectation of the number of throws of a die until a 6 appears?

Solution 1

The probability that the ith throw will be the first occurrence of 6 is the probability of $i-1$ throws of one of the other five numbers times the probability that the ith throw will give 6. To simplify the notation we use p for 1/6:

$$P(\text{first } 6 \text{ on } i\text{th throw}) = (1-p)^{i-1}p\,,$$

The number of throws is unbounded.

Let $E = E(\text{first throw of } 6)$. Then:

$$E = 1p(1-p)^0 + 2p(1-p)^1 + 3p(1-p)^2 + 4p(1-p)^3 + \cdots = \sum_{i=1}^{\infty} ip(1-p)^{i-1}\,. \quad (8)$$

Without the i the sum would be the probability of eventually throwing a 6:

$$P(\text{eventually throwing a } 6) = \sum_{i=1}^{\infty} p(1-p)^{i-1} = p \cdot \frac{1}{1-(1-p)} = 1\,. \quad (9)$$

This is not a surprising result.

The calculation of the expectation can be performed as follows:

$$\begin{array}{rcccccccc} E & = & p(1-p)^0 & + & p(1-p)^1 & + & p(1-p)^2 & + & p(1-p)^3 & + \cdots \\ & & & & p(1-p)^1 & + & p(1-p)^2 & + & p(1-p)^3 & + \cdots \\ & & & & & & p(1-p)^2 & + & p(1-p)^3 & + \cdots \\ & & & & & & & & p(1-p)^3 & + \cdots \end{array}$$

The first row is the sum of the geometric series from Equation 9 which is 1. The second row is the same infinite geometric series except that the first element is $p(1-p)$ so its sum is:

$$\frac{p(1-p)}{1-(1-p)} = 1-p\,.$$

Similarly, the sum of the third row will be $(1-p)^2$ and the sum of the ith row will be $(1-p)^{i-1}$. Therefore, the expectation is the sum of the infinite geometric series:

$$E = 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots = \frac{1}{1-(1-p)} = \frac{1}{p} = 6\,.$$

Solution 2

Multiply Equation 8 by $1-p$ and subtract the result from that equation. The result is the geometric series in Equation 9:

$$\begin{array}{rcll} E & = & p(1-p)^0 & + \quad 2p(1-p)^1 + 3p(1-p)^2 + 4p(1-p)^3 + \cdots \\ E\cdot(1-p) & = & & \qquad p(1-p)^1 + 2p(1-p)^2 + 3p(1-p)^3 + \cdots \\ E\cdot(1-(1-p)) & = & p & + \quad p(1-p)^1 + p(1-p)^2 + p(1-p)^3 + \cdots \\ & = & 1 & \\ E & = & 1/p\,. & \end{array}$$

Since $p = 1/6$ the expectation of the number of throws until a 6 appears is 6.

Solution 3

Consider the first throw separately from the rest of the throws. If the first throw is a 6 (probability p) then one throw is sufficient. Otherwise, if the first throw is not a 6 (probability $1-p$), then the remaining throws form a sequence identical to the original one so the expectation of this sequence is E. The expectation is therefore:

$$\begin{aligned} E &= 1p + (E+1)(1-p) \\ E &= \frac{1}{p} = 6\,. \end{aligned}$$

Simulation

```
Expectation of first success = 6
Average of first success     = 6.0161
```

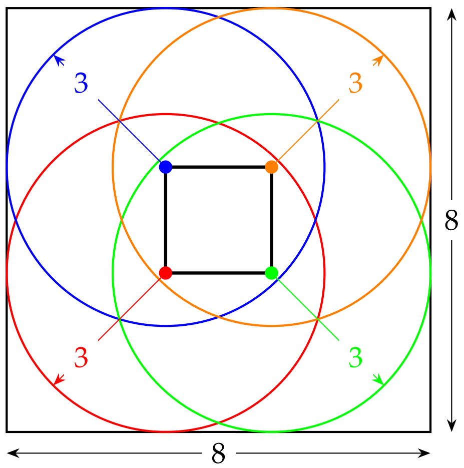


Figure 1: Boundaries for coins not intersecting a square

5. Coin in a square

Question 1: Given a square of side 8 and a coin of radius 3, throw the coin onto the square. The location of the center of the coin is uniformly distributed within the square. What is the probability that the coin lands entirely within the square?

Question 2: For each throw you win 5 if the coin lands within the square and lose 1 if it touches the square. What is the expectation of your winnings for each throw?

Question 3: Develop a formula for the probability of the coin landing within the square if the side of the square is a and the radius of the coin is $r < a/4$.

Solution

Answer 1: Figure 1 shows a square of side 8 and four circles of radius 3 inscribed within the corners of the square. The centers of the circles form an inner square of side 2. Any coin whose center is outside the inner square will touch an edge of the outer square. Since the center of the coin is uniformly distributed, the probability that the coin lands entirely within the square is the ratio of the area of the inner square to the area of the outer square:

$$P(\text{coin lands within the square}) = \frac{2 \cdot 2}{8 \cdot 8} = \frac{1}{16} = 0.0625\,.$$

Answer 2: The expectation is negative:

$$E(\text{winnings per throw}) = 5 \cdot \frac{1}{16} + (-1) \cdot \frac{15}{16} = -\frac{10}{16} = -0.625\,.$$

Answer 3: Figure 2 shows four circles inscribed in the corners of the square. The side of the inner square is $a - 2r$ so:

$$P(\text{coin lands within the square}) = \frac{(a-2r)^2}{a^2}\,.$$

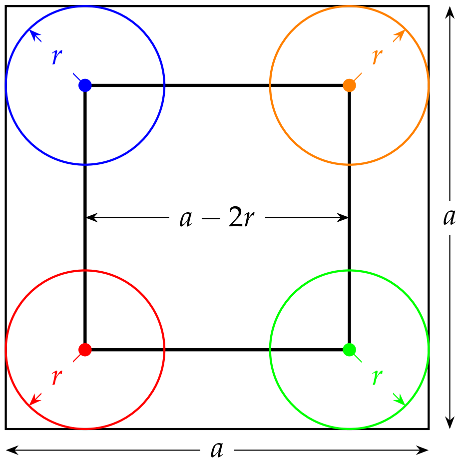


Figure 2: Coins in a large square

Simulation

```
For side = 8, radius = 1:
Probability of landing within the square = 0.5625
Proportion landing within the square     = 0.5704
For side = 8, radius = 2:
Probability of landing within the square = 0.2500
Proportion landing within the square     = 0.2481
For side = 8, radius = 3:
Probability of landing within the square = 0.0625
Proportion landing within the square     = 0.0639
For side = 8, radius = 4:
Probability of landing within the square = 0.0000
Proportion landing within the square     = 0.0000
```

6. Chuck-a-luck

Choose number n between 1 and 6. Throw three dice. If n does not appear on any of the dice you lose 1; if n appears on one die you win 1; if n appears on two dice you win 2; if n appears on three dice you win 3. What is the expectation of your winnings?

Solution

Let $P(k)$ be the probability of n appearing on k dice. Then:

$$E(\textsf{winnings per throw}) = -1P(0) + 1P(1) + 2P(2) + 3P(3)\,.$$

The throws of the three dice are independent so each of these probabilities is given by the binomial distribution with $p = 1/6$, the probability that n appears on a die:

$$\begin{aligned}
E(\text{winnings per throw}) &= -1\binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 + 1\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 + \\
&\quad 2\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 + 3\binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 \\
&= \frac{1}{216}(-125 + 75 + 30 + 3) \\
&= -\frac{17}{216} \approx -0.0787\,.
\end{aligned}$$

Simulation

```
Expectation of winnings = -0.0787
Average winnings        = -0.0724
```

7. Curing the compulsive gambler

Roulette is a game played with a wheel having 38 numbered pockets: 18 red, 18 black and 2 green.[1] The wheel is spun and a ball lands in one of the pockets. The casino wins if the ball lands in a green pocket; otherwise, you win 36 for each 1 bet on the number of the (red or black) pocket where the ball lands. You play 36 rounds of roulette, betting 1 in each round.

Question 1: What is the expectation your winnings?

Question 2: Your friend offers to bet you 20 that after 36 rounds you will have *lost* money. What is the expectation of your winnings, taking into account the money won or lost from both the game and your friend's bet?

Solution

Answer 1: The probability of winning a single round is $1/38$ so:

$$\begin{aligned}
E(\text{winning one round}) &= 35 \cdot \frac{1}{38} + (-1) \cdot \frac{37}{38} = -\frac{2}{38} \approx -0.0526 \\
E(\text{winning 36 rounds}) &= 36 \cdot -0.05266 = -1.8947\,.
\end{aligned}$$

(Your net win is 35 because the 36 you receive includes your bet of 1 which is returned.)

Answer 2: Consider the four outcomes of playing roulette for 36 rounds:

- If you lose all the rounds, you lose 36.
- If you win one round, you win 35 and lose 35 on the other rounds no money is won or lost.

[1]There are two green pockets in American roulette and one green pocket in European roulette.

- If you win two rounds, you win 70 and lose 34 on the other rounds for a net win of 36.
- If you win k rounds for $2 < k \leq 36$, your net win is $35k - (36 - k) > 0$.

Therefore, you lose the bet only if you lose all rounds:

$$P(\text{losing 36 rounds}) = \left(\frac{37}{38}\right)^{36} \approx 0.3829\,.$$

The probability of not losing all rounds is $1 - 0.3829 = 0.6171$. Therefore:

$$\overbrace{-1.8947}^{\text{E of all rounds}} + \overbrace{-20 \cdot 0.3829}^{\text{lose bet}} + \overbrace{20 \cdot 0.6171}^{\text{win bet}} \approx 2.7904\,.$$

Clearly you should take the bet!

Simulation

```
Expectation of winning a round = -0.0526
Average winnings for a round   = -0.0593
```

The simulation showed a large variance which was reduced by running one million trials.

8. Perfect bridge hand

Randomly select 13 cards from a deck. What is the probability that they will all be of the same suit?

Solution 1

Since there are 13 cards of each suit there are $\binom{52}{13}$ ways of selecting 13 cards of a single suit, say hearts. Only one of them consists of 13 hearts so:

$$P(\text{selecting 13 hearts}) = \frac{1}{\binom{52}{13}} = \frac{13!39!}{52!} = \approx 1.5747 \times 10^{-12}\,.$$

There are four suits so:

$$P(\text{selecting 13 cards of the same suit}) = 4 \cdot \frac{13!39!}{52!} \approx 6.2991 \times 10^{-12}\,.$$

Solution 2

There are 52 ways of selecting the first card. Then there are 12 ways of selecting the second card of the same suit from the remaining 51 cards, 11 ways of selecting a third card, and so on. Therefore:

$$P(\text{selecting 13 cards of the same suit}) = \frac{52}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdots \frac{1}{40} = \frac{12!}{51!/39!} \approx 6.2991 \times 10^{-12}\,.$$

Simulation There is no point in running a simulation with 52 cards because the result would almost certainly be zero. A simulation was run with a deck of 16 cards and 4 suits.

```
Probability of perfect hand = 0.0022
Proportion perfect hand     = 0.0020
```

9. Craps ^{D}

Craps is a played with a pair of dice. On the first throw you win if the sum of the numbers is 7 or 11 and you lose if the sum is 2, 3 or 12. If the sum on the first throw is $n = 4, 5, 6, 8, 9, 10$ (called a *point*), continue to throw the dice until the sum is the point n (a win) or 7 (a loss).

Question 1: What are the probabilities of the events on the first throw: winning, losing, neither?

Question 2: What is the probability of a win?

Solution 1

Answer 1: The probability of any outcome in a throw of a die is uniformly distributed and is equal to $1/6$. Since the outcomes of a throw of a pair of dice are independent, the probability of any outcome is $1/36$. The number of ways of obtaining each of the events (the sums of a pair of dice) $2, \ldots, 12$ is:

Sum	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	4	5	6	5	4	3	2	1

On the first throw there are 8 ways of throwing 7 or 11 for a probability of $8/36$ for winning, and 4 ways of throwing $2, 3, 12$ for a probability of $4/36$ for losing. The probability of neither winning nor losing on the first throw is:

$$1 - \frac{8}{36} - \frac{4}{36} = \frac{24}{36}\,.$$

Answer 2: Consider two cases referring to the table above:

- The point is 4. The probability of winning on the second throw (a 4) is $3/36$ and the probability of losing (a 7) is $6/36$. The probability of neither winning nor losing is $1 - (3/36) - (6/36) = 27/36$.
- The point is 8. The probability of winning on the second throw (an 8) is $5/36$ and the probability of losing (a 7) is $6/36$. The probability of neither winning nor losing is $1 - (5/36) - (6/36) = 25/36$.

We see that the probability of winning must be computed separately for each of the points $4, 5, 6, 8, 9, 10$. We develop a general formula for the probability.

After throwing a *point* of the first throw, let P_n be the probability of winning by throwing the point n on a thrown and let Q_n the probability of neither winning nor losing on a throw. W_n, the probability of winning by *eventually* throwing the point n after the first throw, is computed by adding:

- The probability of throwing the point on the second throw.
- The probability of neither winning nor losing on the second throw and throwing the point on the third throw.
- The probability of neither winning nor losing on the second and third throws and throwing the point on the fourth throw,

and so on.

$$
\begin{aligned}
W_n &= P_n + Q_n P_n + Q_n^2 P_n + Q_n^3 P_n + \cdots \\
&= P_n \left(1 + Q_n^1 + Q_n^2 + Q_n^3 + \cdots\right) \\
&= P_n \left(\frac{1}{1 - Q_n}\right) .
\end{aligned}
$$

You lose the game on any throw after the first if you throw a 7 with probability $6/36$ so:

$$
\begin{aligned}
Q_n &= 1 - P_n - (6/36) \\
W_n &= \frac{P_n}{P_n + (6/36)} .
\end{aligned}
$$

W_n for the six points are:

n	4	5	6	8	9	10
P_n	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$
W_n	$\frac{3}{9}$	$\frac{4}{10}$	$\frac{5}{11}$	$\frac{5}{11}$	$\frac{4}{10}$	$\frac{3}{9}$

W, the probability of winning, can be computed by adding the probability of winning on the first throw to the sum of the probabilities for the six wins on points each multiplied by the probability of throwing *that point* on the first throw:

$$
W = \frac{8}{36} + \sum_{n \in \{4,5,6,8,9,10\}} P_n W_n \approx 0.4929 . \tag{10}
$$

The casino's probability of winning a single game of craps is only $0.5 - 0.4949 \approx 0.5\%$ but the law of large numbers ensures that they will eventually win and you will eventually lose!

Solution 2

Answer 2: Consider the following sequences of throws where in all sequences the point is 4:

4	8	9	9	9	8	8	8	9	8	4
4	8	9	9	9	8	8	8	9	8	7
4	9	9	9	8	8	4				

The games only terminates if a 4 is thrown (win) or a 7 is thrown (loss), so an appearance of an 8 or a 9 doesn't affect the result! Therefore, once a point has been thrown, the probability of winning is the conditional probability that a 4 is thrown given that a 4 or a 7 is thrown. Let f be the event that a 4 is thrown and s be the event that a 7 is thrown. Then:

$$P(f|f \cup s) = \frac{P(f) \cap P(f \cup s)}{P(f \cup s)} = \frac{P(f)}{P(f \cup s)} = \frac{3/36}{(3+6)/36} = \frac{3}{9},$$

exactly the result W_4 in the table above. Equation 10 can now be used to compute W.

Conditional probability is implicitly used in the first solution because W_n is a probability that is conditional on the first throw resulting in the point n.

Simulation

```
Probability of winning = 0.4929
Proportion of wins     = 0.4948
```

13. The prisoner's dilemma ^{D}

Three prisoners A, B, C are eligible for parole. The parole board will release two of them with equal probability for $\{A,B\}, \{A,C\}, \{B,C\}$, so the probability that A will be released is $2/3$. Prisoner A is told the name of one of the other prisoners who will be released. If he is told that prisoner B will be released, what is the probability that A too will be released?

For a comprehensive article on the prisoner's dilemma problem and the related Monty Hall problem see [2].

Solution 1

Let $P(A), P(B), P(C)$ be the probabilities that A, B, C are released. A is interested in the conditional probability $P(A|B)$ of his being released if B will be released:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{2/3} = \frac{1}{2}\,.$$

But that is *not* the correct conditional probability! Let R_{AB} be the event that A is *told* that B will be released. The probability that must be computed is $P(A|R_{AB})$:

$$P(A|R_{AB}) = \frac{P(A \cap R_{AB})}{P(R_{AB})}\,.$$

We assume that the report of B's release is true so:

$$P(A \cap R_{AB}) = P(\{A,B\}) = \frac{1}{3}\,.$$

Now:

$$P(R_{AB}) = P(\{A,B\}) + P(\{B,C\}) = \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}\,.$$

If $\{B,C\}$ are to be released, A could be told that either B or C is to be released, hence the factor of $1/2$. Therefore:

$$P(A|R_{AB}) = \frac{P(A \cap R_{AB})}{P(R_{AB})} = \frac{1/3}{1/2} = \frac{2}{3}\,,$$

so if A is told that B will be released the probability that he will be released does not change.

Solution 2

There are four possible events:

e_1: A is told that B will be released and $\{A,B\}$ are released.

e_2: A is told that C will be released and $\{A,C\}$ are released.

e_3: A is told that B will be released and $\{B,C\}$ are released.

e_4**:** A is told that C will be released and $\{B, C\}$ are released.

Each pair of prisoners has equal probability of being released so:

$$P(e_1) = P(e_2) = P(e_3 \cup e_4) = \frac{1}{3}.$$

We assume that if $\{B, C\}$ are to be released, A is told B or C with equal probability, so $P(e_3) = P(e_4) = 1/6$. Therefore the probability that A will be released given the event $R_{AB} = e_1 \cup e_3$ that he is told that B will be released is:

$$P(A|R_{AB}) = \frac{P(e_1 \cap (e_1 \cup e_3))}{P(e_1 \cup e_3)} = \frac{P(e_1)}{P(e_1 \cup e_3)} = \frac{1/3}{(1/3) + (1/6)} = \frac{2}{3}.$$

Solution 3

A riddle attributed to Abraham Lincoln asks: "If you call the tail of a dog a leg, how many legs does the dog have?" The answer is that calling a tail a leg doesn't make it a leg, so the dog still has four legs. Clearly, whether A knows B's future doesn't change his chances of being released.

Simulation

There is no simulation: the problem asks if the probability changes as the result of some knowledge but we have shown that it does not change.

14. Collecting coupons

Given a sequence of boxes each of which contains coupons numbered 1 to 5. You randomly draw one coupon from each box one after another.

Question 1: What is the expectation of the number of coupons drawn until you have all five of the numbers?

Question 2: Develop a formula for the expectation for n numbers.

Hint: Use the solution to Problem 4 on page 10 and the approximation for the sum of harmonic numbers (page 84).

Solution

Answer 1: What is the expectation of the number of draws until you get a number that is *different from* all the previous ones? By Problem 4 this is $1/p$ where p is the probability of drawing a different number. For the first draw the probability is 1 so the expectation is also 1, for the second draw the probability is 4/5 so expectation is 5/4, and so on. Therefore:

$$E(\textsf{all five numbers}) = \frac{5}{5} + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + \frac{5}{1} = \; = \frac{1370}{120} \approx 11.4167.$$

Answer 2: Use the same method and the approximation for H_n, the nth harmonic number (page 84):

$$E(\text{all } n \text{ numbers}) = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots \frac{1}{2} + \frac{1}{1}\right) = nH_n \approx n\left(\ln n + \frac{1}{2n} + 0.5772\right).$$

For $n = 5$ this gives:

$$E(\text{all five numbers}) = 5H_5 \approx 5(\ln 5 + \frac{1}{10} + 0.5772) \approx 11.4332\,.$$

Simulation

```
For  5 coupons:
Expectation of draws = 11.4332
Average draws        = 11.3506
For 10 coupons:
Expectation of draws = 29.2979
Average draws        = 29.4428
For 20 coupons:
Expectation of draws = 71.9586
Average draws        = 72.2191
```

15. The theater row

Arrange eight even numbers and seven odd numbers randomly in a row, for example:

$$10 \quad 12 \quad 3 \quad 2 \quad 9 \quad 6 \quad 1 \quad 13 \quad 7 \quad 10 \quad 3 \quad 8 \quad 8 \quad 5 \quad 20\,,$$

which we can write as follows since the specific numbers are not important:

$$E \quad E \quad O \quad E \quad O \quad E \quad O \quad O \quad O \quad E \quad O \quad E \quad E \quad O \quad E\,.$$

What is the expectation of the number of even-odd or odd-even adjacent pairs?

In the example there are 10 EO or OE adjacent pairs.

Hint: Consider each adjacent pair of separately. What is the probability that they are different?

Solution

The following table shows the ten possible arrangements of three even and two odd numbers. The total number of different adjacent pairs is 24 and the average is $24/10 = 2.4$.

Arrangement	Pairs	Arrangement	Pairs
$EEEOO$	1	$EEOEO$	3
$EEOOE$	2	$EOEOE$	4
$EOEEO$	3	$EOOEE$	2
$OEEOE$	3	$OEEEO$	2
$OEOEE$	3	$OOEEE$	1

Return to the example with 15 numbers. Let P_d be the probability that a given pair in an arrangement is EO or OE. Then:

$$P_d = P(EO) + P(OE) = \frac{8}{15} \cdot \frac{7}{14} + \frac{7}{15} \cdot \frac{8}{14} = 2 \cdot \frac{8}{15} \cdot \frac{7}{14} = \frac{8}{15}\,.$$

Let E_d be the expectation of the number of pairs in an arrangement that are EO or OE. Since an (EO, OE) pair contributes 1 to the count of different pairs and an (EE, OO) pair contributes 0:

$$E_d = \sum_{\text{pairs}} 1 \cdot P_d = 14 \cdot \frac{8}{15} \approx 7.4667\,.$$

For ten numbers:

$$\begin{aligned} P_d &= P(EO) + P(OE) = \frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5} \\ E_d &= 4 \cdot \frac{3}{5} = \frac{12}{5} = 2.4\,. \end{aligned}$$

Simulation

```
For  5 places:
Expectation of different pairs = 2.4000
Average different pairs        = 2.3855
For 15 places:
Expectation of different pairs = 7.4667
Average different pairs        = 7.4566
For 27 places:
Expectation of different pairs = 13.4815
Average different pairs        = 13.4835
For 49 places:
Expectation of different pairs = 24.4898
Average different pairs        = 24.4725
```

16. Will the second-best be runner-up?

Eight players in a tournament $\{a_1, \ldots, a_8\}$ are randomly assigned to play games $\{g_1, \ldots, g_8\}$ in a schedule such that player a_{k_i} initially plays in position g_{k_i} (Figure 3). The players are ranked from the best a_1 to the worst a_8 and the better player will *always* defeat her opponent. Clearly a_1 will win the tournament.

Question 1: What is the probability that a_2 will be the runner-up by playing a_1 in the final round and losing to her?

Question 2: If there are 2^n players what is the probability that a_2 will be the runner-up by playing a_1 in the final round and losing to her?

Solution

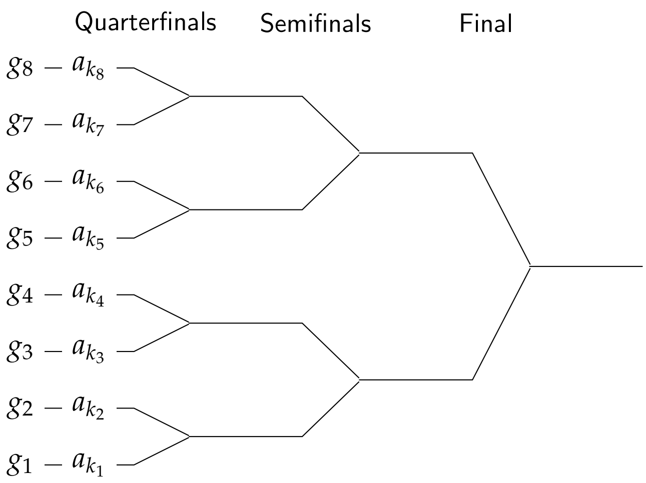


Figure 3: A tournament schedule

Answer 1: If a_1 is assigned to one of the games $\{g_1, g_2, g_3, g_4\}$ none of the other players assigned to these games will reach the final, so a_2 must be assigned to one of $\{g_5, g_6, g_7, g_8\}$. The temptation is to conclude that the probability of a_2 being the runner-up is $1/2$ since he must be assigned to one of the four games $\{g_1, g_2, g_3, g_4\}$. However, whatever game a_1 is assigned to, a_2 will be the runner up only if he is assigned to one of four of the remaining seven games so the probability is $4/7$.

Answer 2: Similarly, of the $2^n - 1$ games that a_1 is not assigned to, a_2 must be assigned to one of the 2^{n-1} games not in the same half as a_1. Therefore:

$$P(a_1, a_2 \text{ playing each other in the final}) = \frac{2^{n-1}}{2^n - 1}.$$

Simulation

```
For   8 players:
Probability a2 is runner-up                = 0.5714
Proportion of games where a2 is runner-up  = 0.5707
For  32 players:
Probability a2 is runner-up                = 0.5161
Proportion of games where a2 is runner-up  = 0.5184
For 128 players:
Probability a2 is runner-up                = 0.5039
Proportion of games where a2 is runner-up  = 0.5060
```

17. Twin knights ^{D}

Eight players in a tournament $\{a_1, \ldots, a_8\}$ are randomly assigned to play games $\{g_1, \ldots, g_8\}$ in a schedule such that a_{k_i} initially plays in position g_{k_i} (Figure 3). For all i, j, the probability that a_i wins in a game against a_j is $1/2$ as is the probability that a_j wins against a_i.

Question 1: What is the probability that a_1, a_2 play each other?

Question 2: If there are 2^n players, what is the probability that a_1, a_2 play each other?

Solution

Answer 1: Without loss of generality assign a_1 to g_1. Consider the different possibilities that a_1, a_2 play each other. With probability $1/7$, a_2 is assigned to g_2. With probability $2/7$, a_2 is assigned to g_3 or g_4, but he doesn't play a_1 unless *both* of them win their first game, so we need to multiply the probability of this assignment by $1/4$. With probability $4/7$, a_2 is assigned to g_5, g_6, g_7, g_8, but he doesn't play a_1 unless *both* of them win their first two games, so we need to multiply the probability of this assignment by $1/16$. Therefore:

$$P(a_1, a_2 \text{ play each other}) = \frac{1}{7} + \frac{1}{4} \cdot \frac{2}{7} + \frac{1}{16} \cdot \frac{4}{7} = \frac{1}{4}.$$

Answer 2: Let P_n be the probability that in a tournament with 2^n players, a_1 and a_2 play each other. We have shown that $P_3 = 1/4$. What about P_4? Using the same approach:

$$\begin{aligned} P_4 &= \frac{1}{15} + \frac{1}{4} \cdot \frac{2}{15} + \frac{1}{16} \cdot \frac{4}{15} + \frac{1}{64} \cdot \frac{8}{15} \\ &= \frac{1}{15} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{8}. \end{aligned}$$

It is reasonable to conjecture that $P_n = 1/2^{n-1}$.

Proof 1: Using the same approach and the formula for the sum of a geometric series:

$$\begin{aligned} P_n &= \frac{1}{2^n - 1} \sum_{i=0}^{n-1} 2^i \cdot \left(\frac{1}{2}\right)^{2i} \\ &= \frac{1}{2^n - 1} \sum_{i=0}^{n-1} 2^{-i} \\ &= \frac{1}{2^n - 1} \left(\frac{1 - \left(\frac{1}{2}\right)^{(n-1)+1}}{1 - \frac{1}{2}} \right) = \frac{1}{2^{n-1}}. \end{aligned}$$

Proof 2: By induction. The base case is $P_3 = 1/4 = 1/2^{3-1}$.

There are two inductive steps:

Case 1: a_1 and a_2 are assigned to different halves of the tournament:

$$P(a_1, a_2 \text{ assigned to different halves}) = \frac{2^{n-1}}{2^n - 1}.$$

They can only meet in the final game and therefore both must win all of their $n-1$ games:

$$P(a_1, a_2 \text{ meet if assigned to different halves}) = \frac{2^{n-1}}{2^n - 1} \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right)^{n-1} = \frac{2^{-(n-1)}}{2^n - 1}. \quad (11)$$

Case 2: a_1 and a_2 are assigned to the same half of the tournament:

$$P(a_1, a_2 \text{ assigned to the same half}) = \frac{2^{n-1}-1}{2^n-1}\,.$$

Since both players are in the same half the problem has been reduced to determining P_{n-1}. By the inductive hypothesis:

$$P(a_1, a_2 \text{ meet if assigned to the same half}) = \frac{2^{n-1}-1}{2^n-1} \cdot \frac{1}{2^{n-2}} = \frac{2^1 - 2^{-(n-2)}}{2^n-1}\,. \qquad (12)$$

Combining Equations 11, 12 gives:

$$\begin{aligned}
P_n &= \frac{2^{-(n-1)}}{2^n-1} + \frac{2^1 - 2^{-(n-2)}}{2^n-1} \\
&= \frac{2^{n-1}}{2^{n-1}} \cdot \frac{2^{-(n-1)} + 2^1 - 2^{-(n-2)}}{2^n-1} \\
&= \frac{1}{2^{n-1}} \cdot \frac{2^0 + 2^n - 2^1}{2^n-1} = \frac{1}{2^{n-1}}\,.
\end{aligned}$$

Simulation

```
For   8 players:
Probability that a1, a2 meet = 0.2500
Proportion a1, a2 meet       = 0.2475
For  32 players:
Probability that a1, a2 meet = 0.0625
Proportion a1, a2 meet       = 0.0644
For 128 players:
Probability that a1, a2 meet = 0.0156
Proportion a1, a2 meet       = 0.0137
```

18. An even split at coin tossing

Question 1: If you toss a fair coin 20 times, what is the probability of obtaining exactly 10 heads?

Question 2: If you toss a fair coin 40 times, what is the probability of obtaining exactly 20 heads?

Solution

Answer 1: Since the coin is fair the probability of obtaining 10 heads in 20 tosses is given by the binomial coefficient:

$$P(10 \text{ heads in 20 tosses}) = \binom{20}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10} \approx 0.1762\,.$$

Answer 2: You might expect the probability to be the same as in the previous question, but:

$$P(\text{20 heads in 40 tosses}) = \binom{40}{20}\left(\frac{1}{2}\right)^{20}\left(\frac{1}{2}\right)^{20} \approx 0.1254\,.$$

By the law of large numbers the numbers of heads and tails will be "roughly" equal [11, Section 8.4], but they are unlikely to be exactly the same, and this event becomes less likely as the number of tosses increases.

Simulation

```
Probability of 10 heads for  20 tosses = 0.1762
Proportion  of 10 heads for  20 tosses = 0.1790
Probability of 20 heads for  40 tosses = 0.1254
Proportion  of 20 heads for  40 tosses = 0.1212
Probability of 50 heads for 100 tosses = 0.0796
Proportion  of 50 heads for 100 tosses = 0.0785
```

19. Isaac Newton helps Samuel Pepys

Question 1: What is the probability of obtaining *at least one* 6 when 6 dice are thrown?

Question 2: What is the probability of obtaining *at least two* 6s when 12 dice are thrown?

Question 3: Develop a formula for the probability of obtaining at least n 6s when $6n$ dice are thrown.

Solution

Answer 1: The probability is the complement of the probability of obtain zero 6s in 6 throws, which is the product of obtaining a number different from 6 in all throws:

$$P(\text{at least one 6}) = 1 - \left(\frac{5}{6}\right)^{6} \approx 0.6651\,.$$

Answer 2: The probability is the complement of the probability of obtain zero or one 6s in 12 throws:

$$P(\text{at least two 6s}) = 1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{11} \approx 0.6187\,.$$

This event is less probable than the previous one.

Answer 3: The probability is the complement of the probability of obtain less than n 6s in $6n$ throws:

$$\begin{aligned} P(\text{at least } n \text{ 6s}) &= 1 - \binom{6n}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{6n-0} - \binom{6n}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{6n-1} - \cdots \\ &= 1 - \sum_{i=0}^{n-1}\binom{6n}{i}\left(\frac{1}{6}\right)^{i}\left(\frac{5}{6}\right)^{6n-i}\,. \end{aligned}$$

Simulation

```
For   6 dice to throw  1 sixes:
Probability = 0.6651
Proportion  = 0.6566
For  12 dice to throw  2 sixes:
Probability = 0.6187
Proportion  = 0.6220
For  18 dice to throw  3 sixes:
Probability = 0.5973
Proportion  = 0.5949
For  96 dice to throw 16 sixes:
Probability = 0.5424
Proportion  = 0.5425
For 360 dice to throw 60 sixes:
Probability = 0.5219
Proportion  = 0.5250
```

20. The three-cornered duel

A, B, C fight a sequence of duels. Each of them has a fixed probability of winning a duel regardless of who the opponent is:

$$P(A) = 0.3, \quad P(B) = 1, \quad P(C) = 0.5\,.$$

A person who is hit no longer participates in the duels. The shots are fired one at a time sequentially in the order A, B, C. If two opponents are still standing the shooter can decide whom to fire at. Assume that each person always makes an optimal decision.

Question 1: What is A's optimal strategy?

Question 2: Suppose that A fires the first shot into the air. Is this a better strategy?

Hint: Compute the conditional probabilities of A winning depending on whether he chooses to shoot first at B or C.

Solution

Notation: $X \xrightarrow{H} Y$ denotes that X shoots at Y and hits. $X \xrightarrow{M} Y$ denotes that X shoots at Y and misses.

Answer 1: Compute the conditional probabilities of A winning.

Case 1: A chooses to shoot first at C.

If $A \xrightarrow{M} C$ (probability 0.7) then $B \xrightarrow{H} C$ since C is more dangerous than A. A now shoots again at B with probability 0.3 of hitting, but if A misses then $B \xrightarrow{H} A$ with probability 1 and A loses.

If $A \xrightarrow{H} C$ (probability 0.3) then $B \xrightarrow{H} A$ with probability 1 and A loses.

Compute the expectation with 1 when A wins and 0 when A loses:

$$E(A \text{ wins } | A \text{ chooses to shoot first at } C) =$$

$$\overbrace{1 \cdot (0.7 \cdot 0.3)}^{A \xrightarrow{M} C, A \xrightarrow{H} B} + \overbrace{0 \cdot (0.7 \cdot 0.7 \cdot 1)}^{A \xrightarrow{M} C, A \xrightarrow{M} B, B \xrightarrow{H} A} + \overbrace{0 \cdot (0.3 \cdot 1)}^{A \xrightarrow{M} C, B \xrightarrow{H} A} = 0.2100\,.$$

Case 2: A chooses to shoot first at B.
If $A \xrightarrow{M} B$ (probability 0.7) then as before $B \xrightarrow{H} C$ and A has one more chance to hit B (probability 0.3), otherwise $B \xrightarrow{H} A$ with probability 1 and A loses.

If $A \xrightarrow{H} B$ (probability 0.3) then A, C alternately shoot at each other until one is hit. The possibilities are:

$$\begin{array}{ll}
(1) & C \xrightarrow{H} A \\
(2) & C \xrightarrow{M} A \xrightarrow{H} C \\
(3) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{H} A \\
(4) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{H} C \\
(5) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{H} A \\
(6) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{H} C \\
 & \cdots
\end{array}$$

The probability of A wins by eventually hitting C is the sum of the probabilities of the even-numbered scenarios in the list:

$$\begin{aligned}
P(A \text{ wins } | A \text{ hits } B) &= (0.5 \cdot 0.3) + \\
&\quad (0.5 \cdot 0.7)(0.5 \cdot 0.3) + \\
&\quad (0.5 \cdot 0.7)(0.5 \cdot 0.7)(0.5 \cdot 0.3) + \cdots \\
&= 0.15 \sum_{i=0}^{\infty} 0.35^i = \frac{0.15}{1-0.35} = \frac{3}{13} \approx 0.2308\,.
\end{aligned}$$

Similarly, the probability of C winning is $\dfrac{0.5}{1-0.35} = \dfrac{1}{13} \approx 0.0760$.

The expectation is:

$$E(A \text{ wins}) = E(A \text{ wins } | \ A \text{ misses } B) + E(A \text{ wins } | \ A \text{ hits } B) =$$

$$\overbrace{1 \cdot (0.7 \cdot 1 \cdot 0.3)}^{A \xrightarrow{M} B, B \xrightarrow{H} C, A \xrightarrow{H} B} + \overbrace{0 \cdot (0.7 \cdot 1 \cdot 0.7 \cdot 1)}^{A \xrightarrow{M} B, B \xrightarrow{H} C, A \xrightarrow{M} B, B \xrightarrow{H} A} + \overbrace{1 \cdot (0.2308)}^{A \xrightarrow{H} B, C \xleftrightarrow{H} {*} A, C \xrightarrow{H} A} + \overbrace{0 \cdot (0.3 \cdot (0.0769))}^{A \xrightarrow{H} B, C \xleftrightarrow{H} A, C \xrightarrow{H} A} \approx$$

$$0.2792\,,$$

which is higher than the expectation of winning by shooting at C first.

Answer 2: If A shoots into the air not hitting anyone then $B \xrightarrow{H} C$ with probability 1 and A can try to hit B with probability 0.3. The expectation is:

$$E(A \text{ wins} | A \text{ shoots in the air}) = 1 \cdot (0.3) + 0 \cdot (0.7) = 0.3\,,$$

which is better than the expectation for the other two strategies!

Simulation

```
For A fires first at C:
Expectation of wins = 0.2100
Average wins        = 0.2138
For A fires first at B:
Expectation of wins = 0.2792
Average wins        = 0.2754
For A fires in the air:
Expectation of wins = 0.3000
Average wins        = 0.3000
```

21. Should you sample with or without replacement? ^{D}

Urn A contains 2 red balls and 1 green ball, and urn B contains 101 red balls and 100 green balls. An urn is chosen at random and two balls are randomly drawn from the selected urn. You win if you correctly identify whether the selected urn was A or B.

Compute the probabilities of winning for each of the following rules and decide which gives you the highest probability of winning.

Question 1: The first ball is replaced before the second drawing.

Question 2: The first ball is not replaced before the second drawing.

Question 3: After the first ball is drawn you can decide whether it will be replaced or not.

Hint: When computing probabilities:

$$\frac{101}{201} \approx \frac{100}{201} \approx \frac{100}{200} \approx \frac{1}{2}\,.$$

Solution

There are four outcomes which we denote by RR, RG, GR, GG. For each rule compute the conditional probabilities of the four outcomes given that urn A or urn B was selected initially. These probabilities are multiplied by $1/2$ to take into account the random selection of a urn.

Answer 1: Drawing with replacement:

$$\begin{array}{lcccc}
P(RR|A) & = & \frac{2}{3}\cdot\frac{2}{3} & = & \frac{4}{9} \\
P(RR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(RG|A) & = & \frac{2}{3}\cdot\frac{1}{3} & = & \frac{2}{9} \\
P(RG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GR|A) & = & \frac{1}{3}\cdot\frac{2}{3} & = & \frac{2}{9} \\
P(GR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GG|A) & = & \frac{1}{3}\cdot\frac{1}{3} & = & \frac{1}{9} \\
P(GG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4}
\end{array}$$

If the outcome is RR there is a higher probability that urn A was selected $(4/9)$ than that urn B was selected $(1/4)$; otherwise, there is a higher probability that urn B was selected. Therefore:

$$P(\text{winning}) = \frac{1}{2}\left(\frac{4}{9}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) = \frac{43}{72} \approx 0.5972\,.$$

Answer 2: Drawing without replacement:

$$\begin{array}{lcccc}
P(RR|A) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(RG|A) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GR|A) & = & \frac{1}{3}\cdot 1 & = & \frac{1}{3} \\
P(GR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GG|A) & = & \frac{1}{3}\cdot 0 & = & 0 \\
P(GG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4}
\end{array}$$

If the outcome is GG there is (of course!) a higher probability that urn B was selected than that urn A was selected; otherwise, there is a higher probability that urn A was selected. Therefore:

$$P(\text{winning}) = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}\right) = \frac{5}{8} = 0.6250\,,$$

which is greater than the probability of winning when sampling with replacement.

Answer 3: The decision is based on the outcome of the first draw.

If the first drawing is from urn A the probabilities must be conditioned on the decision to sample with or without replacement. Drawing first from urn B does not affect the probabilities because of the approximation in the hint.

$$\begin{array}{lcccc}
P(RR|A,w) & = & \frac{2}{3}\cdot\frac{2}{3} & = & \frac{4}{9} \\
P(RR|A,w/o) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(RG|A,w) & = & \frac{2}{3}\cdot\frac{1}{3} & = & \frac{2}{9} \\
P(RG|A,w/o) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GR|A,w) & = & \frac{1}{3}\cdot\frac{2}{3} & = & \frac{2}{9} \\
P(GR|A,w/o) & = & \frac{1}{3}\cdot 1 & = & \frac{1}{3} \\
P(GR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GG|A,w) & = & \frac{1}{3}\cdot\frac{1}{3} & = & \frac{1}{9} \\
P(GG|A,w/o) & = & \frac{1}{3}\cdot 0 & = & 0 \\
P(GG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4}
\end{array}$$

If a red ball is drawn first then $\frac{4}{9} > \frac{1}{4}$ and $\frac{2}{9} < \frac{1}{4}$ with replacement, whereas $\frac{1}{3} > \frac{1}{4}$ and $\frac{1}{3} > \frac{1}{4}$ without replacement, so the second ball can help identify the urn only if the drawing is done *with* replacement: urn A if red, urn B if green. Choose the draw with replacement and:

$$P(\textsf{winning if red first}) = \frac{1}{2}\left(\frac{4}{9}+\frac{1}{4}\right) = \frac{25}{72}\,.$$

If a green ball is drawn first then $\frac{2}{9} < \frac{1}{4}$ and $\frac{1}{9} < \frac{1}{4}$ with replacement, whereas $\frac{1}{3} > \frac{1}{4}$ and $0 < \frac{1}{4}$ without replacement, so the second ball can help identify the urn only if the drawing is done *without* replacement: urn A if red, urn B if green. Choose the draw without replacement and:

$$P(\textsf{winning if green first}) = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{24}\,.$$

The probability of winning is:

$$P(\textsf{winning}) = \frac{25}{72}+\frac{7}{24} = \frac{23}{36} \approx 0.6389\,.$$

The highest probability of winning is obtained when the decision to draw with or without replacement depends on the result of the first draw.

Simulation

```
With replacement:
Expectation of winning = 0.5972
Average wins           = 0.5976
Without replacement:
Expectation of winning = 0.6250
Average wins           = 0.6207
Decide after first draw:
Expectation of winning = 0.6389
Average wins           = 0.6379
```

22. The ballot box

In an election there are two candidates A and B. A receives a votes and B receives b votes, $a > b$. The votes are counted one-by-one and the running totals $(a_i, b_i), 1 \le i \le a+b$ are updated as each vote is counted. What is the probability that for at least one i, $a_i = b_i$?

Question 1: Solve for $a = 3, b = 2$ by listing (a_i, b_i) for $1 \le i \le 5$.

Question 2: Solve for all $a > b$.

Hint 1: What can you say about the identity of the candidate who leads until the *first* tie occurs?

Hint 2: What is the significance of the first vote counted?

Solution

Answer 1: The number of arrangements of running totals is $\binom{5}{2} = \binom{5}{3} = 10$, because the positions of the votes for one candidate determine the positions of the votes for the other candidate. The following table lists the possible arrangements of the votes and of running totals with first ties emphasized:

A	A	A	B	B	$(1,0)$	$(2,0)$	$(3,0)$	$(3,1)$	$(3,2)$
A	A	B	A	B	$(1,0)$	$(2,0)$	$(2,1)$	$(3,1)$	$(3,2)$
A	B	A	A	B	$(1,0)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(3,1)$	$(3,2)$
B	A	A	A	B	$(0,1)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(3,1)$	$(3,2)$
A	A	B	B	A	$(1,0)$	$(2,0)$	$(2,1)$	$(\mathbf{2},\mathbf{2})$	$(3,2)$
A	B	A	B	A	$(1,0)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(2,2)$	$(3,2)$
B	A	A	B	A	$(0,1)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(2,2)$	$(3,2)$
A	B	B	A	A	$(1,0)$	$(\mathbf{1},\mathbf{1})$	$(1,2)$	$(2,2)$	$(3,2)$
B	A	B	A	A	$(0,1)$	$(\mathbf{1},\mathbf{1})$	$(1,2)$	$(2,2)$	$(3,2)$
B	B	A	A	A	$(0,1)$	$(0,2)$	$(1,2)$	$(\mathbf{2},\mathbf{2})$	$(3,2)$

There are ties in all the arrangements except for the first two so:

$$P(\text{tie occurs with } (3,2) \text{ votes}) = \frac{8}{10} = \frac{4}{5}\,.$$

Answer 2: We begin the solution with a discussion on how to approach the second question. Here are some arrangements for $(a,b) = (3,2)$ votes until the *first tie* occurs:

A leads until tie	B leads until tie
A B	B A
A A B B	B B A A

For every arrangement where A leads until the first tie, there is a mirror image arrangement where B leads until the first tie. The mirror image is obtained by exchanging all A's and B's.

Before the first tie one of the candidates must be leading. If the first vote counted is for B there must be a tie later in the count since $a > b$. The probability that the first vote is for B is:

$$P(\text{first vote for } B) = \frac{b}{a+b}\,.$$

By mirroring the positions of the votes, the number of sequences resulting in a tie that begin with a vote for A is the same as the number of sequences resulting in a tie that begin with a vote for B. Therefore:

$$P(\text{tie occurs}) = 2 \cdot \frac{b}{a+b}\,.$$

Check:

$$P(\text{tie occurs with } (3,2) \text{ votes}) = 2 \cdot \frac{2}{2+3} = \frac{4}{5}\,.$$

Simulation

```
For a =  3, b =  2:
Probability of a tie = 0.8000
Proportion of ties   = 0.8118
For a = 10, b =  8:
Probability of a tie = 0.8889
Proportion of ties   = 0.8977
For a = 20, b = 18:
Probability of a tie = 0.9474
Proportion of ties   = 0.9354
```

23. Ties in matching pennies ^{D}

Toss a pair of fair coins N times, N even, and keep count of how many times the parity is even (heads-heads or tails-tails) and how many times the parity is odd (heads-tails or tails-heads). What is the probability of obtaining a tie (not counting the $0-0$ tie at the start)?

Question 1: Solve for $N = 4$ by writing out all the possible outcomes.

Question 2: Solve for $N = 6$ by developing a formula for the probability.

Question 3: Develop a formula for arbitrary even N.

Question 4: Explain why the probability for the odd number $N + 1$ is the same as the probability for the even number N.

Hint: Use the solution of Problem 22.

Solution

Answer 1: Denote tosses with even parity by E and tosses with odd parity by O. Ten out of the sixteen arrangements of tosses have ties (emphasized):

EEEE	EEEO	EEOE	**EEOO**	**EOEE**	**EOEO**	**EOOE**	**EOOO**
OEEE	**OEEO**	**OEOE**	**OEOO**	**OOEE**	OOEO	OOOE	OOOO

Answer 2: By Problem 22:

$$P(\text{tie on toss } i) = \begin{cases} 2i/N & \text{if } i \leq N/2 \\ 2(N-i)/N & \text{if } i \geq N/2\,, \end{cases} \tag{13}$$

since the ballot box problem showed that the smaller value determines the probability.

The following computations are quite complex so we justify each step in detail.

The probability of i evens is given by the binomial distribution:

$$P(i \text{ evens}) = \binom{N}{i}\left(\frac{1}{2}\right)^{i}\left(\frac{1}{2}\right)^{N-i} = \binom{N}{i}\left(\frac{1}{2}\right)^{N} = 2^{-N}\binom{N}{i}\,. \tag{14}$$

The probability of a tie is the sum over i of the probability of obtaining i evens times the probability of a tie on the ith toss (Equation 13). For $N = 6$:

$$P(\text{ties}) = 2 \cdot 2^{-6} \left[\frac{0}{6}\binom{6}{0} + \frac{1}{6}\binom{6}{1} + \frac{2}{6}\binom{6}{2} + \frac{3}{6}\binom{6}{3} + \frac{2}{6}\binom{6}{4} + \frac{1}{6}\binom{6}{5} + \frac{0}{6}\binom{6}{6} \right] . \tag{15}$$

Equation 16 follows from Equation 15 by deleting the two zero terms, expressing the combinations as factorials, canceling $1/6$ from $6!$:

$$P(\text{ties}) = 2^{-5} \left[1 \cdot \frac{5!}{1!5!} + 2 \cdot \frac{5!}{2!4!} + 3 \cdot \frac{5!}{3!3!} + 2 \cdot \frac{5!}{4!2!} + 1 \cdot \frac{5!}{5!1!} \right] . \tag{16}$$

Equation 17 is obtained by canceling i from $i!$:

$$P(\text{ties}) = 2^{-5} \left[\frac{5!}{1!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} + \frac{5!}{4!1!} + \frac{5!}{5!1!} \right] . \tag{17}$$

To obtain Equation 18 from Equation 17 add and subtract $\frac{5!}{3!2!}$:

$$P(\text{ties}) = 2^{-5} \left[\left(\frac{5!}{1!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} + \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!1!} \right) - \frac{5!}{3!2!} \right] . \tag{18}$$

Equation 19 results from replacing $1!$ by $0!$:

$$P(\text{ties}) = 2^{-5} \left[\left(\frac{5!}{0!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} + \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!} \right) - \frac{5!}{3!2!} \right] . \tag{19}$$

By expressing the factorials back as combinations we obtain Equation 20:

$$P(\text{ties}) = 2^{-5} \left[\left(\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} - \binom{5}{3} \right) \right] . \tag{20}$$

Finally, Equation 21 results from the binomial theorem:

$$P(\text{ties}) = 2^{-5} \left(2^5 - 10 \right) = \frac{11}{16} \approx 0.6875 . \tag{21}$$

Answer 3: Perform the same calculations as in **Answer 2** but using arbitrary N. The result is:

$$P(\text{ties}) = 2^{-N+1} \left[2^{N-1} - \binom{N-1}{N/2} \right] = \left[1 - \binom{N-1}{N/2} \Big/ 2^{N-1} \right] .$$

Answer 4: The first tie on the $N + 1$'st toss occurs only if the counts are nearly equal after the Nth toss:

$$((N/2) - 1, (N/2) + 1)$$
$$((N/2), (N/2))$$
$$((N/2) + 1, (N/2) - 1)$$

but whatever the outcome of the final toss the counts will not be equal.

Simulation

```
For  4 tosses:
Probability of ties = 0.6250
Proportion of ties  = 0.6192
For  6 tosses:
Probability of ties = 0.6875
Proportion of ties  = 0.6900
For  7 tosses:
Probability of ties = 0.6875
Proportion of ties  = 0.6811
For 10 tosses:
Probability of ties = 0.7539
Proportion of ties  = 0.7559
For 20 tosses:
Probability of ties = 0.8238
Proportion of ties  = 0.8255
```

25. Lengths of random chords

Select a random chord in the unit circle. What is the probability that the length of the chord is greater than 1?

To solve the problem you first have to decide what "select a random chord" means. Solve the problem for each of the following possibilities:

Question 1: The distance of the chord from the center is uniformly distributed.

Question 2: The midpoint of the chord is uniformly distributed within the circle.

Question 3: The endpoints of the chord are uniformly distributed on the circumference of the circle.

Solution

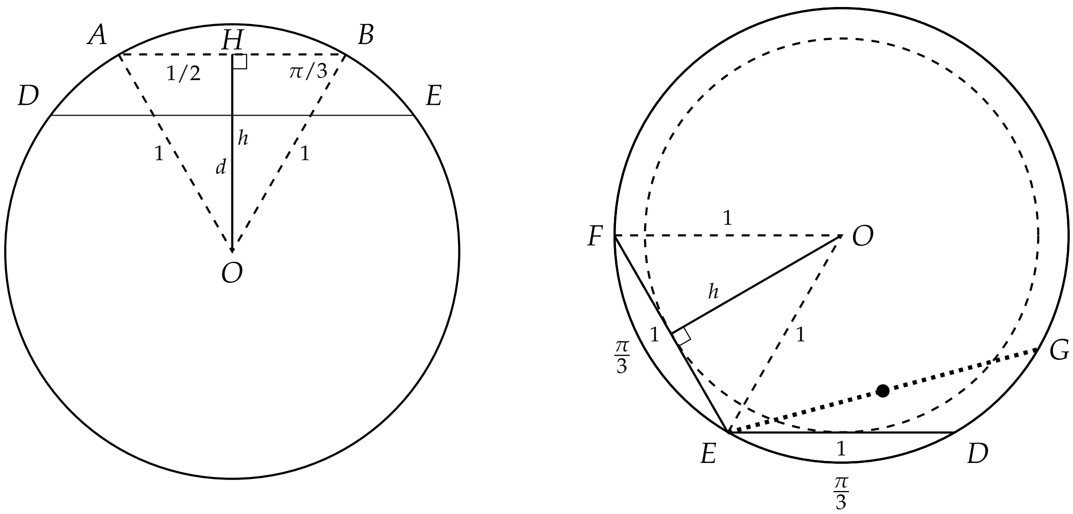
Answer 1: A chord is larger than the radius if it is closer to the center than a chord of length 1. Let $\overline{AB}$ be a chord of length 1 and construct the altitude $\overline{OH}$ from the center O to the chord (Figure 4a). Since $\triangle AOB$ is equilateral, $\triangle OHB$ is a right triangle and the length of the altitude is:

$$h = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\,.$$

Let d be the distance of a chord $\overline{DE}$ from the center. By assumption d is uniformly distributed in $(0,1)$ so:

$$P(\overline{DE} > 1) = P(d < h) = \frac{h}{1} = \frac{\sqrt{3}}{2} \approx 0.866\,.$$

Answer 2: Construct a circle with center O and radius h, where h is the length of the altitude to a chord of length 1. A tangent to any point on this circle will be a chord $\overline{FE}$ whose length



(a) Distance of chord from center uniformly distributed in $(0, 1)$

(b) Midpoint of chord uniformly distributed within the circle and endpoints of chord uniformly distributed on the circumference

is 1. Any chord $\overline{EG}$ whose midpoint is within this circle will have a length greater than 1 (Figure 4b). The probability that the length of the chord is greater than 1 is therefore the ratio of the areas of the two circles:

$$P(\overline{EG} > 1) = \frac{\pi \cdot h^2}{\pi \cdot 1^2} = h^2 = \frac{3}{4}.$$

This is the square of the probability computed in the previous question.

Answer 3: Choose an arbitrary point on the circumference of the unit circle (E in Figure 4b). Any other point on the circumference (such as G in the Figure) determines a chord whose length is greater than one unless that point falls on the arcs $\widehat{EF}$ or $\widehat{ED}$. The probability is therefore the ratio of the arc $\widehat{FD}$ to the circumference of the unit circle:

$$P(\overline{EG} > 1) = \frac{(2\pi - (2\pi/3)) \cdot 1}{2\pi \cdot 1} = \frac{2}{3}.$$

Simulation

The simulation is for choosing two random points on the circumference.

```
Probability of long chords = 0.6667
Proportion of long chords  = 0.6627
```

26. The hurried duelers

A and *B* arrive at a meeting point at a random time with uniform distribution within a one-hour period. If *A* arrives first and *B* does not arrive within 5 minutes, *A* leaves. Similarly if *B* arrives first and *A* does not arrive within 5 minutes, *B* leaves. What is the probability that they meet?

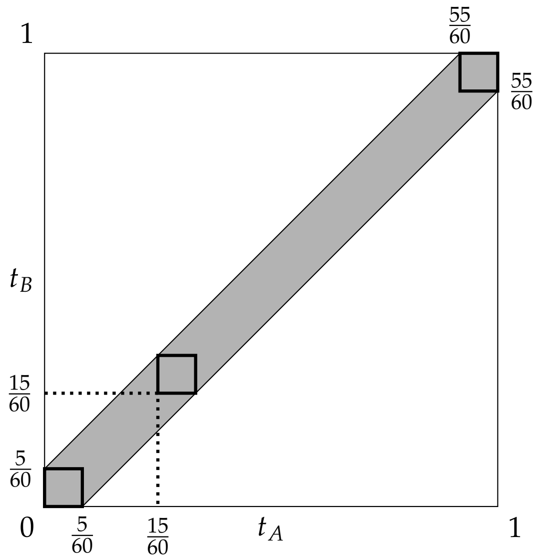


Figure 5: Times that ensure a meeting between A and B

Time within the one-hour period is *continuous* in the range $[0, 1]$, that is, you cannot *count* a discrete number of minutes or seconds to compute probabilities. You can compute the probabilities of *durations*.

Hint: Draw a graph with A's time of arrival as the the x-axis and B's time of arrival as the y-axis.

Solution

Without loss of generality assume that A arrives first. If A arrives at $t_A = 0$ and if B arrives before $t_B = 5/60$ they meet, otherwise they do not. This is shown in Figure 5 by the small square at the origin. If A arrives later then B also has to arrive later by the same amount; for example, if A arrives at $t_A = 15$, B must arrive between $t_B = 15$ and $t_B = 20$. Therefore, the meeting will take place during a square of time obtained by moving the square by 15 from $(0, 0)$ to $(15/60, 15/60)$.

The probability of a meeting is the ratio of the area of the graph colored gray to the area of the large square. It is easier to compute the complement which is the ratio of the area of the two white triangles to the area of the large square:

$$
\begin{aligned}
P(A, B \text{ meet}) &= 1 - P(A, B \text{ don't meet}) \\
&= 1 - 2 \cdot \left(\frac{1}{2} \cdot \frac{55}{60} \cdot \frac{55}{60}\right) = \frac{23}{144} \approx 0.1597\,.
\end{aligned}
$$

Simulation

```
Probability of meeting   = 0.1597
Proportion of meetings   = 0.1549
```

27. Catching the cautious counterfeiter

There are n boxes each with n coins one of which is counterfeit. Draw one coin from each box and test it to determine whether it is counterfeit or genuine. What is the probability that all the coins that are drawn are are real?

Question 1: Solve for $n = 10$.

Question 2: Solve for $n = 100$.

Question 3: Solve for arbitrary n.

Question 4: Develop a formula for limit of the probability as n tends to infinity.

Solution

The draws are independent so the probability is the product of the probabilities for each draw.

Answer 1:

$$P(\textsf{all coins are genuine}) = \left(\frac{9}{10}\right)^{10} = 0.3487\,.$$

Answer 2:

$$P(\textsf{all coins are genuine}) = \left(\frac{99}{100}\right)^{100} = 0.3660\,.$$

Answer 3:

$$P(\textsf{all coins are genuine}) = \left(\frac{n-1}{n}\right)^{n}\,.$$

Answer 4:

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = \frac{1}{e} \approx 0.3679\,. \tag{22}$$

This limit can be proved using differential calculus. First we compute the limit of the natural logarithm of the lefthand side of Equation 22:

$$\lim_{n\to\infty}\ln\left(1-\frac{1}{n}\right)^{n} = \lim_{n\to\infty} n\ln\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)}{1/n}\,.$$

Taking the limit gives $(ln\ 1)/0 = 0/0$ but by l'Hôpital's rule we can replace expression by the quotient of the derivatives:

$$\begin{aligned}\lim_{n\to\infty}\ln\left(1-\frac{1}{n}\right)^{n} &= \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)^{-1}\left(-(-n^{-2})\right)}{-n^{-2}} = -1\\ \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} &= e^{-1}\,.\end{aligned}$$

Simulation

```
For  10 boxes:
Probability of all real = 0.3487
Proportion all real     = 0.3480
For 100 boxes:
Probability of all real = 0.3660
Proportion all real     = 0.3730
For 200 boxes:
Probability of all real = 0.3670
Proportion all real     = 0.3690
```

28. Catching the greedy counterfeiter

There are n boxes each with n coins m of which are counterfeit. Draw one coin from each box and test it to determine whether it is counterfeit or not. What is the probability $P(n,m,r)$ that r of the coins that are drawn are counterfeit?

Question 1: Develop a formula for $P(n,m,r)$.

Question 2: Compute $P(20,10,2), P(20,10,8), P(20,5,2), P(20,5,4)$.

Solution

Answer 1: There are $\binom{n}{r}$ choices of boxes from which the counterfeit coins can be drawn. From the binomial distribution:

$$P(n,m,r) = \binom{n}{r}\left(\frac{m}{n}\right)^r\left(\frac{n-m}{n}\right)^{n-r}.$$

Answer 2:

$$\begin{aligned}
P(20,10,2) &= \binom{20}{2}\left(\frac{10}{20}\right)^2\left(\frac{10}{20}\right)^{18} \approx 0.0002\\
P(20,10,8) &= \binom{20}{8}\left(\frac{10}{20}\right)^8\left(\frac{10}{20}\right)^{12} \approx 0.1201\\
P(20,5,2) &= \binom{20}{2}\left(\frac{5}{20}\right)^2\left(\frac{15}{20}\right)^{18} \approx 0.0669\\
P(20,5,4) &= \binom{20}{4}\left(\frac{5}{20}\right)^4\left(\frac{15}{20}\right)^{12} \approx 0.1952\,.
\end{aligned}$$

Mosteller shows that for given m, r, as n tends to infinity:

$$\lim_{n\to\infty} P(n,m,r) = \frac{e^{-m}m^r}{r!}\,. \tag{23}$$

Simulation

```
For 10 bad coins,  2 draws:
Probability of counterfeit  = 0.0002
Proportion counterfeit      = 0.0002
For 10 bad coins,  8 draws:
Probability of counterfeit  = 0.1201
Proportion counterfeit      = 0.1181
For  5 bad coins,  2 draws:
Probability of counterfeit  = 0.0669
Proportion counterfeit      = 0.0688
For  5 bad coins,  4 draws:
Probability of counterfeit  = 0.1897
Proportion counterfeit      = 0.1905
```

29. Moldy gelatin

A rectangular plate is divided into n small squares. There are an average of r microbes in each square.

Question 1: Develop a formula for probability that there are exactly r microbes in the n squares.

Question 2: Compute the probability for $n = 100, r = 3$.

Hint: This problem is similar the Problem 28.

Solution

Answer 1: Let p be the probability that a single square contains a microbe. (Ignore the possibility that a microbe is partially contained within two or more squares.) m, the average number of microbes per square, is the number of squares n times the probability p that a square contains a microbe. $P(n, m, r)$, the probability that there are exactly r microbes in the n squares is given by the binomial distribution:

$$P(n,m,r) = \binom{n}{r} \left(\frac{m}{n}\right)^r \left(\frac{n-m}{n}\right)^{n-r} .$$

Answer 2:

$$P(10,3,3) = \binom{100}{3} \left(\frac{3}{100}\right)^3 \left(\frac{97}{100}\right)^{97} \approx 0.2275 .$$

Equation 23 also applies here:

$$\lim_{n\to\infty} P(n,3,3) = \frac{e^{-3} \cdot 3^3}{3!} \approx 0.2240 .$$

Simulation

```
For  20 squares:
Probability of exactly  3 microbes  = 0.2428
Proportion of exactly   3 microbes  = 0.2436
Probability of exactly  5 microbes  = 0.2023
Proportion of exactly   5 microbes  = 0.1954
For 100 squares:
Probability of exactly  3 microbes  = 0.2275
Proportion of exactly   3 microbes  = 0.2247
Probability of exactly  5 microbes  = 0.1800
Proportion of exactly   5 microbes  = 0.1851
```

31. Birthday pairings

Randomly select 23 people and ask them what their birthdays are. Assume a uniform distribution of 365 different birthdays (no one was born on February 29). Show that the probability that at least two of them have the same birthday is greater than 0.5.

Solution

Compute the probability that none of the 23 people have the same birthday and show that it is less than 0.5. Select the first birthday arbitrarily, then the next birthday from the remaining days, then the next birthday from the remaining days, and so on:

$$\begin{aligned}
P(\textsf{no birthday pair}) &= \overbrace{\frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365}\cdot\;\cdots\;\cdot\frac{344}{365}\cdot\frac{343}{365}}^{\textsf{23 fractions}} \\
&= \frac{365!}{365^{23}\cdot 342!} \approx 0.4927\,.
\end{aligned}$$

Most people guess that more than 23 people are needed to find two with the same birthday!

A modern calculator can compute the probability, but it is worthwhile computing it using Stirling's approximation $\ln n! \approx n\ln n - n$:

$$\begin{aligned}
\ln P(\textsf{no birthday pair}) &= \ln\left(\frac{365!}{342!\cdot 365^{23}}\right) = \ln 365! - \ln 342! - 23\ln 365 \\
&\approx (365\ln 365 - 365) - (342\ln 342 - 342) - 23\ \ln 365 \\
&\approx -0.7404 \\
P(\textsf{no birthday pair}) &\approx e^{-0.7404} = 0.4769\,.
\end{aligned}$$

The reader is invited to compute the probability with the following better approximation:

$$\ln n! \approx n\ln n - n + \frac{1}{6}\left(8n^3 + 4n^2 + n + \frac{1}{30}\right) + \frac{1}{2}\ln\pi\,.$$

Simulation

```
For 21 people:
Expectation of no pairs = 0.5563
Average no pairs        = 0.5497
For 22 people:
Expectation of no pairs = 0.5243
Average no pairs        = 0.5237
For 23 people:
Expectation of no pairs = 0.4927
```

```
Average no pairs        = 0.4933
For 24 people:
Expectation of no pairs = 0.4617
Average no pairs        = 0.4576
For 25 people:
Expectation of no pairs = 0.4313
Average no pairs        = 0.4345
```

32. Finding your birthmate

Your *birthmate* is a person with the same birthday as yours.

Why is finding a birthmate a different problem than finding a birthday pairing?

Question 1: How many people do you have to ask until the probability of finding your brithmate becomes greater than 0.5?

Question 2: Solve the problem by using the approximation in Equation 22 (page 39).

Solution

Many people could have the same birthday which is considered a success for find a birthday pairing, but not for finding a birthmate unless that birthday is the same as yours.

Answer 1: Find the smallest number of people for which the probability that none of them are birthmates is less than 0.5. The probability that the first person you ask is not a birthmate is $364/365$, but that is also the probability that the second, third, ..., person is not a birthmate. The solution is the smallest k such that:

$$P(\textsf{birthmate not found}) = \left(\frac{364}{365}\right)^k < \frac{1}{2},$$

which is $k = 253$:

$$\left(\frac{364}{365}\right)^{253} \approx 0.4995\,.$$

Answer 2: Equation 22 is:

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n = \frac{1}{e},$$

which can be used to approximate the probability:

$$\begin{aligned}
P(\textsf{birthmate not found}) &= \left(\frac{365-1}{365}\right)^k = \left[\left(\frac{364}{365}\right)^{365}\right]^{k/365}\\
&\approx e^{-k/365}\\
e^{-253/365} &\approx 0.5000\,.
\end{aligned}$$

Simulation

```
For 251 people:
Probability of no match = 0.5023
Proportion no match     = 0.5120
For 252 people:
Probability of no match = 0.5009
Proportion no match     = 0.5055
For 253 people:
Probability of no match = 0.4995
Proportion no match     = 0.4984
For 254 people:
Probability of no match = 0.4982
Proportion no match     = 0.4987
For 255 people:
Probability of no match = 0.4968
Proportion no match     = 0.5078
```

33. Relating the birthday pairings and the birthmate problems

Let $P_{\textsf{pair}}(r)$ be the probability that out of r people two are a birthday pair (Problem 31) and let $P_{\textsf{mate}}(n)$ be the probability that out of n people at least one is your birthmate (Problem 32). Given r, for what n does $P_{\textsf{pair}}(r) \approx P_{\textsf{mate}}(n)$?

Solution 1

The solution is based on [7].

Using the notation $P_{\textsf{no pair}}(r)$ for the complement of $P_{\textsf{pair}}(r)$, from the solution to Problem 31 we have:

$$\begin{aligned}
P_{\textsf{no pair}}(r) &= \frac{365}{365} \cdot \frac{365-1}{365} \cdot \frac{365-2}{365} \cdot \cdots \cdot \frac{365-(r-1)}{365} \\
&= 1\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right) \cdot \cdots \cdot \left(1-\frac{r-1}{365}\right) \\
&\approx 1-\frac{1}{365}-\frac{2}{365}-\cdots-\frac{r-1}{365} \\
&= 1-\frac{1+2+3+\cdots+(r-1)}{365} \\
&= 1-\frac{r(r-1)/2}{365},
\end{aligned}$$

where the approximation in the third equation results from deleting powers of $1/365$ greater than one because they are too small to significantly affect the result.

Using the notation $P_{\textsf{no mate}}(n)$ for the complement and the same approximation, from the solution to Problem 32 we have:

$$\begin{aligned}
P_{\textsf{no mate}}(n) &= \overbrace{\left(1-\frac{1}{365}\right)\left(1-\frac{1}{365}\right)\cdots\left(1-\frac{1}{365}\right)}^{n} \\
&\approx 1-\overbrace{\frac{1}{365}-\frac{1}{365}\cdots-\frac{1}{365}}^{n} \\
&\approx 1-\frac{n}{365}
\end{aligned}$$

Therefore $P_{\textsf{no pair}}(r) \approx P_{\textsf{no mate}}(n)$ when:

$$n = \frac{r(r-1)}{2} .$$

For $r = 23$, $n = (23 \cdot 22)/2 = 253$.

Solution 2

Mosteller [7, p. 322] gives the following intuitive solution:

> In comparing the birthday and birthmate problems, one observes that for r people in the birthday problem, there are $r(r-1)/2$ pairs or *opportunities* for like birthdays; whereas, if n people are questioned in the birthmate problem, there are only n opportunities for me to find one or more birthmates.

From this he concludes that $n \approx r(r-1)/2$.

This reasoning can be understood as follows: For the birthday problem choose an arbitrary date and ask if two people out of r have *that birthday*. There are:

$$\binom{r}{2} = \frac{r!}{2!(r-2)!} = \frac{r(r-1)}{2}$$

ways of doing so. For the birthmate problem your own birthday is given. Any of the n people can have the same birthday. By equating the two expressions we have the n such that $P_{\textsf{pair}}(r) \approx P_{\textsf{mate}}(n)$.

Simulation

You can run the simulations using the programs for Problems 31, 32 and check this result.

34. Birthday holidays ^{D}

A factory is closed whenever one of its workers has a birthday. There are no other holidays.

Question 1: How many workers should be employed in order to maximize the number of work-days in one year?

Question 2: What is the expectation of the ratio of the maximum work-days to 365^2, the number of possible work-days if each one of 365 workers worked every day?

Hint: Prove that there must be a maximum by considering extreme cases. Then develop a formula for the expectation of the number of work-days for a single day.

Solution

Answer 1: At one extreme if there is only one worker there are 364 work-days. If there are two workers there are $363 + 363 = 726$ workers days (ignoring the very small possibility that both workers have the same birthday). At the other extreme if there are one million workers the number of work-days will almost certainly be zero. Since the number of work-days rises initially and then returns to zero, there must be a maximum in between the extremes.

To simplify the notation we will denote the number of days in a year by N and the number of workers by n.

For any given day the probability that it is a work-day is the probability that each worker has a birthday on some other day:

$$P(\textsf{a given day is a work-day}) = \overbrace{\frac{N-1}{N} \cdot \;\cdots\; \cdot \frac{N-1}{N}}^{n} = \left(1 - \frac{1}{N}\right)^{n} .$$

Denote $\left(1 - \frac{1}{N}\right)$ by p. Then:

$$E(\textsf{work-days for a given day}) = n \cdot p^n + 0 \cdot (1 - p^n) = np^n .$$

All the days in the year have this same expectation, so we just multiply by N to get the expectation for a year:

$$E(\textsf{work-days for a year}) = Nnp^n . \tag{24}$$

To find the maximum we take the derivative of Equation 24 with respect to n and use $(p^n)' = p^n \ln p$ which can be proved using the chain rule:

$$(p^n)' = ((e^{\ln p})^n)' = (e^{n \ln p})' = e^{n \ln p}(n \ln p)' = (e^{\ln p})^n \ln p = p^n \ln p .$$

The derivative of Equation 24 is therefore:

$$(Nnp^n)' = N(p^n + n(p^n)') = N(p^n + np^n \ln p) ,$$

which is 0 when:

$$n = -\frac{1}{\ln p} .$$

For $N = 365$ this gives $n = 364.5$. Since n is a positive integer the maximum is achieved at $n = 364$ or $n = 365$ which give the same expectation of the number of work-days:

$$\begin{aligned} E(\textsf{work-days for a year}) &= Nnp^n \\ &= 365 \cdot 364 \cdot \left(\frac{364}{365}\right)^{364} \\ &= 365 \cdot 364 \cdot \frac{365}{365}\left(\frac{364}{365}\right)^{364} \\ &= 365 \cdot 365 \cdot \left(\frac{364}{365}\right)^{365} \\ &= 48944 . \end{aligned}$$

Answer 2: The expectation of the ratio is:

$$E(\text{max work-days/possible work-days}) = \frac{365 \cdot 365 \cdot \left(\frac{364}{365}\right)^{365}}{365 \cdot 365} = \left(\frac{364}{365}\right)^{365} \approx 0.3674\,.$$

By Equation 22:

$$\lim_{n\to\infty} E(\text{max work-days/possible work-days}) = \lim_{N\to\infty} \left(1 - \frac{1}{N}\right) = \frac{1}{e}\,.$$

Simulation

```
For 100 people
Expectation work-days    = 27742
Average work days        = 27743
Ratio work-days / 365**2 = 0.2082
For 250 people
Expectation work-days    = 45958
Average work days        = 45939
Ratio work-days / 365**2 = 0.3450
For 364 people
Expectation work-days    = 48944
Average work days        = 48936
Ratio work-days / 365**2 = 0.3674
For 365 people
Expectation work-days    = 48944
Average work days        = 48917
Ratio work-days / 365**2 = 0.3674
```

35. The cliff-hanger

A particle is initially placed on the x-axis at position 1. At any position on the x-axis it can move right with probability $2/3$ and left with probability $1/3$ (Figure 6).

Question 1: What is the probability that the particle will eventually be at position 0?

Question 2: If the probability of moving right is p and the probability of moving left is $1 - p$, what is the probability that the particle will eventually be at position 0? Analyze the result for various values of p.

Hint: Use conditional probabilities after the first move.

Solution

It is no more difficult to compute the probabilty for arbitrary p as it is for $p = 2/3$, so we give solutions for both questions together.

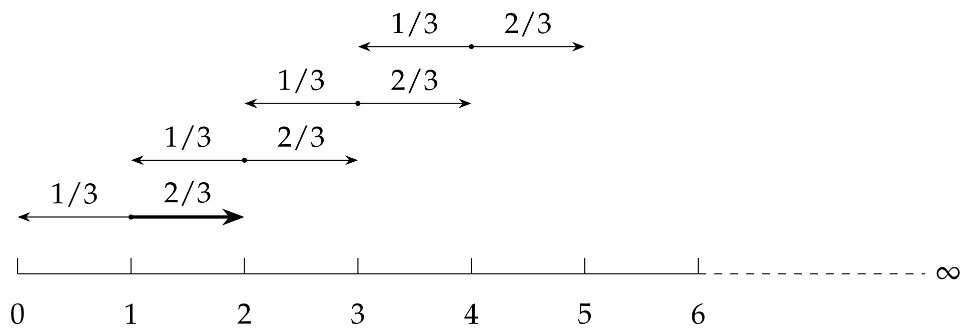


Figure 6: Can the particle return to 0 (axis is infinite to the right)?

Answer 1,2: Let us try to compute the probability directly. Denote a move left by L and a move right by R. The particle can reach 0 directly by moving L with probability $\frac{1}{3}$, or by moving RLL with probability $\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$, or by moving $RRLLL$ with probability $\left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^3, \ldots$. This computations seems to be a straightforward geometric progression, but it ignores possibilities such as $RLRLL$.

Compute the probability that the particle reaches 0 from 1 conditioned on the first step:

$$\begin{aligned} P(\textsf{reaches 0 from 1}) &= P(\textsf{reaches 0 from 1}|\textsf{first move left}) + \\ &\quad\; P(\textsf{reaches 0 from 1}|\textsf{first move right}) \\ &= (1-p) \cdot 1 + pP(\textsf{reaches 1 from 2})P(\textsf{reaches 0 from 1})\,. \end{aligned}$$

But the probability of reaching 1 from 2 is exactly the same as the probability of reaching 0 from 1. Denoting $P(\textsf{reaches 0 from 1})$ by P we have:

$$\begin{aligned} P &= (1-p) + pP^2 \\ pP^2 - P + (1-p) &= 0 \\ P &= \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} \\ P &= 1,\ (1-p)/p\,. \end{aligned}$$

If $p \leq 1/2$ then $(1-p)/p \geq 1$, so $P = 1$ is the only solution and it is certain that the particle will reach 0.

If $p = 1$ then $P = 0$ since if the particle always moves to the right it cannot return to 0.

Suppose $P = 1$ for $1/2 < p < 1$, that is, *P does not depend on p*. But P cannot suddenly "jump" from 1 to 0 as p approaches 1: in Figure 7 the dashed red line and the red dot at $(1, 0)$. Therefore, for $p > 1/2$ the only solution is $P = (1-p)/p < 1$.[2]

For $p = 2/3, P = 1/2$ and for $p = 1/2, P = 1$. This is a surprising result because one would not expect that the particle would always return to 0 if the direction of the moves were determined by flipping a fair coin! You have to have a very unfair coin (probability of heads is $2/3$) to even the chances or returning to 0 or not.

[2]Mosteller writes that this follows by continuity but does not give a proof.

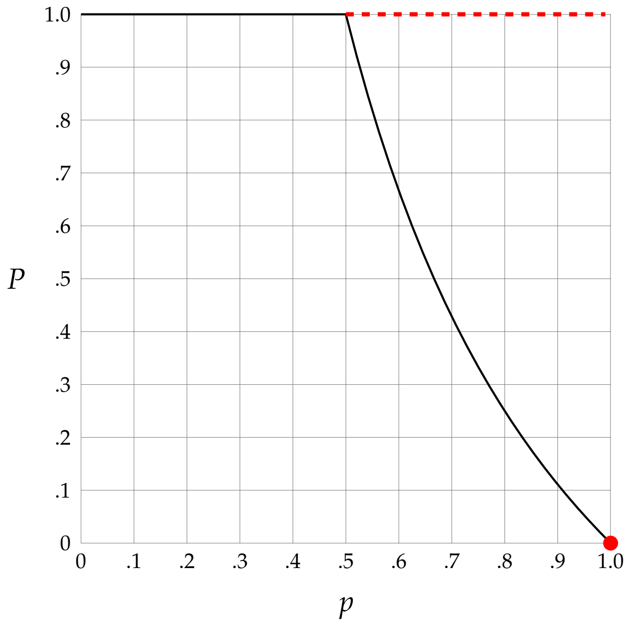


Figure 7: Graph of $P = \min(p/(1-p), 1)$ for $p \in [0, 1]$

Simulation

```
For probability = 0.2500:
Probability of reaching 0 = 1.0000
Proportion reaching 0     = 1.0000
For probability = 0.5000:
Probability of reaching 0 = 1.0000
Proportion reaching 0     = 0.9612
For probability = 0.6667:
Probability of reaching 0 = 0.5000
Proportion reaching 0     = 0.5043
For probability = 0.7500:
Probability of reaching 0 = 0.3333
Proportion reaching 0     = 0.3316
For probability = 0.8000:
Probability of reaching 0 = 0.2500
Proportion reaching 0     = 0.2502
```

36. Gambler's ruin ^{D}

A particle is initially placed on the x-axis at position $m \geq 1$. At any position on the x-axis it can move right with probability $p > 1/2$ and left with probability $1 - p$.

Question 1: What is the probability that the particle will eventually be at position 0?

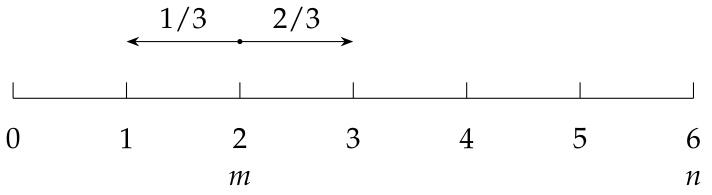


Figure 8: Can the particle return to 0 (axis is finite)?

Question 2: Let $n > m$. If the particle reaches position 0 or position n its stops moving (Figure 8). What is the probability that the particle will eventually be at position 0? What is the probability that the particle will eventually be at position n?

Note: Problem 35 is represents a gambler with a finite amount of money playing against a casino with unlimited money. The problem asks for the probabiliy that the gambler loses all his money. This problem represents one gambler who starts with m playing against a second gambler who starts with $n - m$. The problem asks for the probabilities that *one* of the gamblers loses all her money to the other one.

Solution

The solution is based on [11, Chapter 2, Example 4m].

Answer 1: The solution to Problem 35 showed that for $p > 1/2$ (here it is given), if a particle is at position 1 the probability of its reaching position 0 is $r = (1-p)/p$. Notation: let $P(i,j)$ be the probability of reaching i from j. Since the probability of a particle reaching one position from another does not depend on the absolute position:

$$P(0,m) = P(m-1,m)P(m-2,m-1)\cdots P(1,2)P(0,1) = r^m\,. \tag{25}$$

Answer 2: Let $P_i = P(n,i)$ and compute it using conditional probability:

$$\begin{aligned}
P_i &= pP_{i+1} + (1-p)P_{i-1}\\
pP_{i+1} &= 1\cdot P_i - (1-p)P_{i-1}\\
pP_{i+1} &= (p + (1-p))P_i - (1-p)P_{i-1}\\
p(P_{i+1} - P_i) &= (1-p)(P_i - P_{i-1})\\
P_{i+1} - P_i &= r(P_i - P_{i-1})\,.
\end{aligned}$$

$P_0 = 0$ since if the particle is at 0 it does not move. Therefore:

$$\begin{aligned}
P_2 - P_1 &= r(P_1 - P_0) = rP_1\\
P_3 - P_2 &= r(P_2 - P_1) = r^2P_1\\
\cdots &= \cdots\\
P_i - P_{i-1} &= r(P_{i-1} - P_{i-2}) = r^{i-1}P_1\,.
\end{aligned}$$

Most of the terms on the lefthand sides cancel when we add the equations:

$$\begin{aligned}
P_i - P_1 &= P_1 \sum_{j=2}^{i} r^{j-1} \\
&= P_1 + P_1 \sum_{j=2}^{i} r^{j-1} - P_1 \\
P_i &= P_1 \sum_{j=0}^{i-1} r^{j} = P_1 \left(\frac{1-r^i}{1-r} \right) .
\end{aligned}$$

If the particle is at n then it is already at n so $P_n = 1$:

$$\begin{aligned}
1 &= P_1 \left(\frac{1-r^n}{1-r} \right) \\
P_1 &= \left(\frac{1-r}{1-r^n} \right) ,
\end{aligned}$$

and therefore (using a symmetrical argument exchanging p and $1-p$):

$$P(n,i) = \left(\frac{1-r^i}{1-r^n} \right) \tag{26}$$

$$P(0,i) = \left(\frac{1-(1/r)^{n-i}}{1-(1/r)^n} \right) . \tag{27}$$

We leave it to the reader to show that the sum of Eqs. 26, 27 is 1 meaning that one of the players will certainly win and one will lose. For $m = 1, n = 3, p = 2/3$:

$$\begin{aligned}
P(0,1) &= \left(\frac{1-\left(\frac{1}{2}\right)^1}{1-\left(\frac{1}{2}\right)^3} \right) = \frac{4}{7} \\
P(3,1) &= \left(\frac{1-2^2}{1-2^3} \right) = \frac{3}{7} .
\end{aligned}$$

Simulation

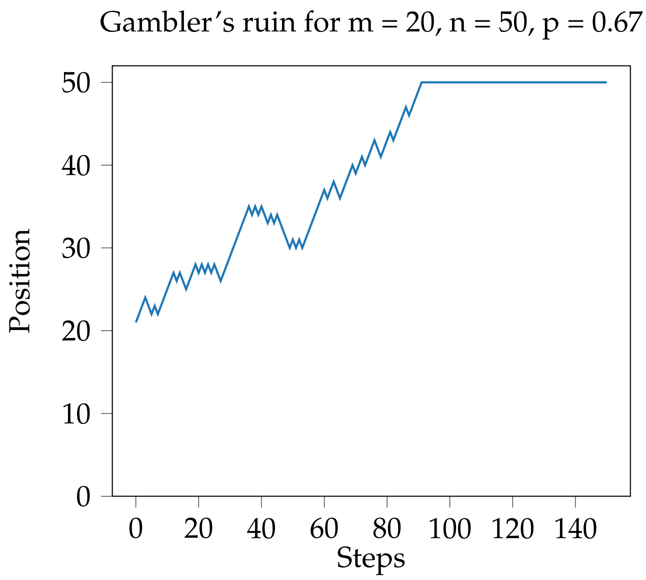
```
For probability = 0.6667:
Probability of reaching (0,10) from 1 = (0.4995,0.5005)
Proportion reaching     (0,10) from 1 = (0.5056,0.4944)
Probability of reaching (0,10) from 4 = (0.0616,0.9384)
Proportion reaching     (0,10) from 4 = (0.0643,0.9357)
Probability of reaching (0,10) from 6 = (0.0147,0.9853)
Proportion reaching     (0,10) from 6 = (0.0123,0.9877)

For probability = 0.7500:
Probability of reaching (0,10) from 1 = (0.3333,0.6667)
Proportion reaching     (0,10) from 1 = (0.3395,0.6605)
Probability of reaching (0,10) from 4 = (0.0123,0.9877)
Proportion reaching     (0,10) from 4 = (0.0115,0.9885)
Probability of reaching (0,10) from 6 = (0.0014,0.9986)
Proportion reaching     (0,10) from 6 = (0.0015,0.9985)
```

The greater the amount of money that the left player has and the greater his probability of winning each bet, the higher his probability of winning.

Plot of steps

This plot was generated by the Python library `matplotlib`; the source code appears in the file `36-gamblers-ruin-plot.py`.



37. Bold play vs. cautious play

In roulette you can bet that the ball will fall into a pocket with an even number. The probability is $18/38$ since there are 18 even numbers, 18 odd numbers and 2 green numbers where the casino wins.

Which of the following strategies is better?

1. Bold play: betting 20 in one round.
2. Cautious play: bettin 1 per round until you win or lose 20.

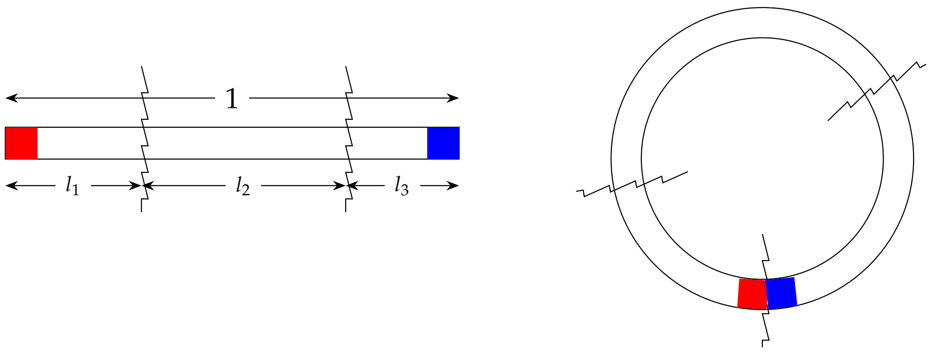
Hint: Use the results of Problem 36.

Solution

The probability of winning with bold play is $18/38 \approx 0.4737$.

The probability of winning with cautious play is (Equation 26):

$$\begin{aligned} r &= \frac{20}{38}\Big/\frac{18}{38} = \frac{20}{18} \\ P(40, 20) &= \frac{1-(20/18)^{20}}{1-(20/18)^{40}} \approx 0.1084\,. \end{aligned}$$



(a) Breaking a rod into three pieces (b) Breaking a ring into three pieces

Clearly, bold play is preferable to cautious play.

Mosteller writes that the intuitive explanation for this result is that betting in more rounds exposes the player to the probability of $2/38$ that the casino wins.

Simulation

```
Probability of bold wins     = 0.4737
Proportion bold wins         = 0.4677
Probability of cautious wins = 0.1084
Proportion cautious wins     = 0.1094
```

39. The clumsy chemist

You have a large number of glass rods of length 1 with one end colored red and the other colored blue. When you drop them on the floor they each break into three pieces with a uniform distribution of the length of the pieces (Figure 9a). What is the expectation of the length of the piece whose end is colored blue?

Hint: Instead of straight rods suppose that you are given (unmarked) glass rings that also break into three pieces (Figure 9b).

Solution 1

The rods are not symmetric because the end pieces are different from the center piece. However, the ring is symmetric so the distributions of all three pieces must be uniform with expectation $1/3$. By choosing and coloring one of breaks as shown in Figure 9b, the problem is now the same as that of the rods so the distributions remain the same. Therefore the expectation of the lengths of the pieces is also $1/3$.

Solution 2

Here is an elegant solution from [4].

						a						
	9	9	8	7	6	5	4	3	2	1	0	
	8	8	7	6	5	4	3	2	1	0	1	
	7	7	6	5	4	3	2	1	0	1	2	
a	6	6	5	4	3	2	1	0	1	2	3	
	5	5	4	3	2	1	0	1	2	3	4	
	4	4	3	2	1	0	1	2	3	4	5	
y	3	3	2	1	0	1	2	3	4	5	6	a
	2	2	1	0	1	2	3	4	5	6	7	
	1	1	0	1	2	3	4	5	6	7	8	
	0	0	1	2	3	4	5	6	7	8	9	
		0	1	2	3	4	5	6	7	8	9	
						x		a				

Table 1: Distribution of breaks on $(0,1) \times (0,1)$

Assume that the rod represents the line segment $(0,1)$. The rod is broken in two places which are represented as two uniform independent random variables $X, Y \in (0,1)$. Let us compute the probability $P(|X-Y| > a)$.

Table 1 shows points (x,y), where $x, y \in \{0.0, 0.1, 0.2, \ldots, 0.9\}$ and the decimal point is omitted. The values that appear in the table are $|X-Y|$. For $a = 0.6$ the entries in the upper left corner $(0,6)$–$(6,9)$ and above, and the entries in the lower right corner $(6,0)$–$(9,6)$ and below, are those outcomes that define $P(|X-Y| > a)$:

$$P(|X-Y| > a) = 2 \cdot \frac{1}{2}(1-a)(1-a) = (1-a)^2 \,.$$

For $a = 0.6$, $P(|X-Y| > 0.6) = (0.4)^2 = 0.16$.

Taking the complement gives:

$$P(|X-Y| < a) = 1 - (1-a)^2 \,.$$

This is the cumulative probability distribution (CPD) for the interval $(0,1)$. The probability density function (PDF) can be obtained by differentiating the CDP:

$$P(|X-Y| = a) = \frac{d}{da} P(|X-Y| < a) = \frac{d}{da}(1 - (1-a)^2) = 2(1-a) \,.$$

The expectation is the integral of the PDF multiplied by the value:

$$E(|X-Y|) = \int_0^1 a \cdot 2(1-a)\, da = 2\left(\frac{a^2}{2} - \frac{a^3}{3}\right)\Big|_0^1 = \frac{1}{3} \,.$$

Simulation

```
Expectation of length of right piece = 0.3333
Average length of right piece        = 0.3359
```

40. The first ace

Deal cards from a well-shuffled deck of cards until an ace appears. What is the expectation of the number of cards that must be dealt?

Hint: Consider the deck of cards without the aces to be laid out in a line.

Solution

The cards form a "rod" of length 48 which is "broken" by the 4 aces into 5 "pieces." The solution of Problem 39 applies and the expectation of the length of a piece is $48/5 = 9.6$.

Simulation

```
Expectation of first ace = 9.6000
Average first ace        = 9.5805
```

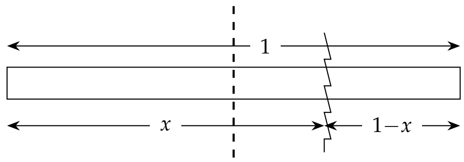


Figure 10: Breaking a stick into two pieces

42. The little end of the stick

A large number of glass rods each of length 1 are broken into two pieces each. The breaking point is uniformly distributed along the lengths of the rods.

Question 1: What is the expectation of the length of the *smaller* piece?

Question 2: What is the expectation of the ratio of the length of the smaller piece to the larger piece?

Solution

Answer 1: The probability that the break is on the left half of a rod is $1/2$ as is the probability that it is on the right half. The smaller piece is on the same side as the break so the expectation of its position is halfway between the end and the middle:

$$E(\textsf{length of smaller piece}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}\,.$$

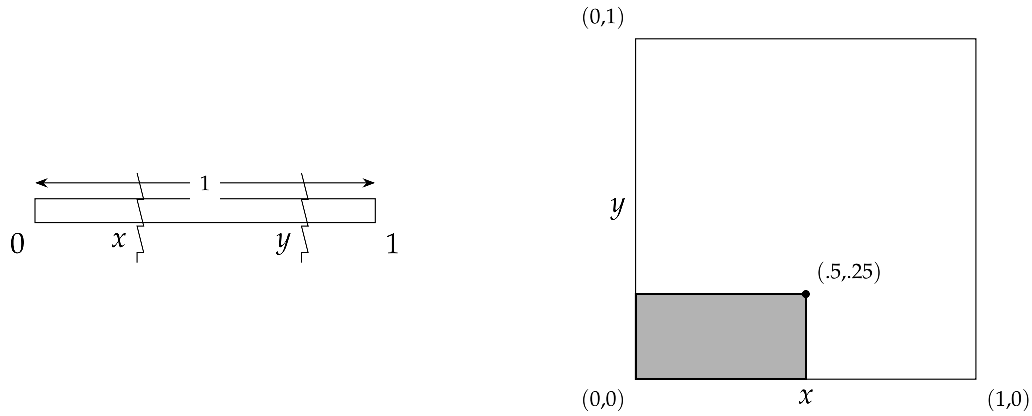
Answer 2: Without loss of generality assume that the break occurred in the right half of the rod (Figure 10). The ratio of the smaller piece to the larger piece is $(1-x)/x$ and the length of the larger piece x is uniformly distributed in $(1/2, 1)$. Therefore:

$$\begin{aligned}
E(\textsf{ratio of smaller to larger}) &= \left(\frac{1}{1-(1/2)}\right) \int_{1/2}^{1} \frac{1-x}{x}\,dx \\
&= 2\int_{1/2}^{1} \left(\frac{1}{x} - 1\right) dx \\
&= 2\,(\ln|x| - x)\big|_{1/2}^{1} = 2\ln 2 - 1 \approx 0.3863\,.
\end{aligned}$$

Simulation

```
Expectation of length of smaller = 0.2500
Average length of smaller        = 0.2490
Expectation of smaller/larger    = 0.3863
Average smaller/larger           = 0.3845
```

43. The broken bar ^{D}



(a) Break a rod into two pieces

(b) Representation of the lengths in the unit square

A large number of glass rods of length 1 are broken in two places (Figure 11a).

Question 1: What is the expectaton of the length of the shortest piece?

Question 2: What is the expectaton of the length of the longest piece?

Hint: x, y are independent random variables with a uniform distribution from $(0, 1)$. Each pair (x, y) can be represented as a point in the unit square $(0, 1) \times (0, 1)$ (Figure 11b). What is the probability that $(x, y) < (.5, .25)$?

Hint: For **Question 1:** assume that left piece is the shortest one and for **Question 2:** assume that the left piece is the longest one.

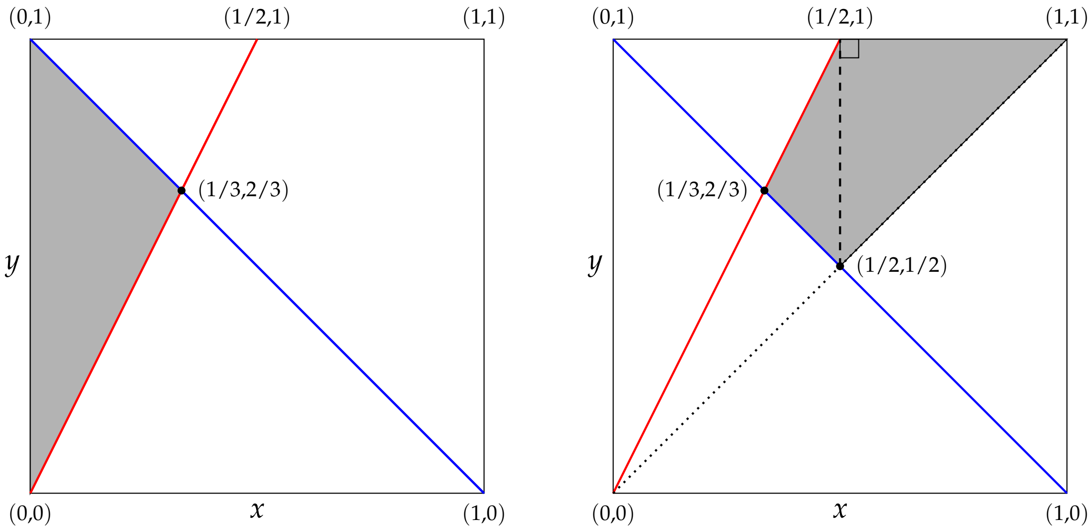
Solution

Answer 1: Without loss of generality assume that the left piece of length x is the shortest. Then $x < y - x$ and $x < 1 - y$, from which we have $2x < y$ and $x + y < 1$.

Figure 12a shows the lines $y = 2x$ (red) and $y = 1 - x$ (blue). For the inequalities to be true, (x, y) must be in the shaded region left of the two lines. The point of intersection $(1/2, 2/3)$ can be computed by solving the two equations.

While the values of (x, y) are in the square $(0, 1) \times (0, 1)$, the expectation is computed over the shaded subset of the square so the integral must be divided by the area of the shaded region: $\frac{1}{2}(\frac{1}{3} \cdot 1) = \frac{1}{6}$:

$$
\begin{aligned}
E(x) &= \frac{1}{1/6} \int_0^{1/3} x[(1 - x) - 2x]\, dx \\
&= \int_0^{1/3} (6x - 18x^2)\, dx \\
&= (3x^2 - 6x^3)\big|_0^{1/3} = \frac{2}{18} \approx 0.1111\,.
\end{aligned}
$$



(a) Shaded area for shortest bar

(b) Shaded area for longest bar

Answer 2: For the left piece to be the longest, $x > y - x$ and $x > 1 - y$, so (x, y) must lie to the right of $y = 2x$ (red) and to the right of $y = 1 - x$ (blue) (Figure 12b). Furthermore, by the assumption that x is to the left of y, (x, y) must lie to the left of $y = x$ (dotted).

For convenience we divide the shaded region into two triangles (dashed line) and compute the expectations separately. The area of the shaded region is the sum of the areas of the triangles $1/24 + 1/8 = 1/6$. Then:

$$\begin{aligned}
E(x \text{ in left triangle}) &= 6 \int_{1/3}^{1/2} x[2x - (1 - x)]\, dx \\
&= \int_{1/3}^{1/2} \left(18x^2 - 6x\right)\, dx \\
&= \left(6x^3 - 3x^2\right)\Big|_{1/3}^{1/2} = \frac{1}{9} \\
E(x \text{ in right triangle}) &= 6 \int_{1/2}^{1} x[1 - x]\, dx \\
&= \int_{1/2}^{1} (6x - 6x^2)\, dx \\
&= \left(3x^2 - 2x^3\right)\Big|_{1/2}^{1} = \frac{1}{2} \\
E(x) &= \frac{1}{9} + \frac{1}{2} = \frac{11}{18} \approx 0.6111\,.
\end{aligned}$$

The expectation of the length of the middle-sized piece is $1 - \frac{2}{18} - \frac{11}{18} = \frac{5}{18} \approx 0.2778$.

Simulation

```
Expectations: shortest = 0.1111, middle = 0.2778, longest = 0.6111
Averages:     shortest = 0.1115, middle = 0.2783, longest = 0.6102
```

44. Winning an unfair game ^{D}

You are given an unfair coin whose probability of heads is $1/3 < p < 1/2$. Toss a coin an even number of times $N = 2n$. You win if and only *more* than half of the tosses are heads.

Question 1: Develop a formula for the probability P_N of winning and develop a formula for the probability T_N of a tie occurring.

Question 2: Develop a formula for the N that gives the highest probability of winning.

Hint: If N tosses gives the highest probability of winning then $P_{N-2} \le P_N$ and $P_N \ge P_{N+2}$.

Solution

Answer 1: To win, heads needs to appear in $i \in \{n+1, n+2, \ldots, 2n-1, 2n = N\}$ tosses. From the binomial distribution:

$$\begin{aligned} P_N &= \sum_{i=n+1}^{2n} \binom{2n}{i} p^i (1-p)^{2n-i} \\ T_N &= \binom{2n}{n} p^n (1-p)^n . \end{aligned}$$

Answer 2: For $N = 2n$ to give the highest probability of winning we must have:

$$P_{2n-2} \le P_{2n} \quad \text{and} \quad P_{2n} \ge P_{2n+2} .$$

When is $P_{2n-2} \neq P_{2n}$?

Case 1: After toss $2n-2$, heads has appeared n times and tails $n-2$ times (so you would have won if you stop here), but tails appears in the next two tosses. You now have n heads and n tails, and therefore you lose. The probability is:

$$\binom{2n-2}{n} p^n (1-p)^{n-2} (1-p)^2 .$$

Case 2: After toss $2n-2$, heads has appeared $n-1$ times and tails $n-1$ times (so you would have lost if you stop here), but heads appears in the next two tosses. You now have $n+1$ heads and $n-1$ tails and therefore you win. The probability is:

$$\binom{2n-2}{n-1} p^{n-1} (1-p)^{n-1} p^2 .$$

For $P_{2n-2} \le P_{2n}$ to hold P_{2n-2} cannot increase while P_{2n} remains the same (Case 1), although P_{2n} can become greater than P_{2n-2} (Case 2). Therefore:

$$\begin{aligned} \binom{2n-2}{n} p^n (1-p)^{n-2} (1-p)^2 &\le \binom{2n-2}{n-1} p^{n-1} (1-p)^{n-1} p^2 \\ \frac{1}{n}(1-p) &\le \frac{1}{n-1} p \\ (n-1)(1-p) &\le np \\ n &\le \frac{1-p}{1-2p} \\ 2n &\le \frac{1}{1-2p} + 1 . \end{aligned}$$

Similarly, for $P_{2n} \geq P_{2n+2}$ to hold it must be true that:

$$\begin{aligned}
\binom{2n}{n+1} p^{n+1}(1-p)^{n-1}(1-p)^2 &\geq \binom{2n}{n} p^n (1-p)^n p^2 \\
\frac{1}{n+1}(1-p) &\geq \frac{1}{n} p \\
n(1-p) &\geq (n+1)p \\
n &\geq \frac{p}{1-2p} \\
2n &\geq \frac{1}{1-2p} - 1\,.
\end{aligned}$$

Therefore, value for $N = 2n$ that givs the highest probability for winning is the nearest even integer to $1/(1-2p)$. We leave to the reader to show that if $1/(1-2p)$ is odd then $P_{2n} = P_{2n+2}$.

Simulation

```
For probability             = 0.3700
Optimal games to be played  = 4
For  2 games, average won   = 0.1372
For  4 games, average won   = 0.1445
For  6 games, average won   = 0.1431

For probability             = 0.4000
Optimal games to be played  = 6
For  4 games, average won   = 0.1820
For  6 games, average won   = 0.1845
For  8 games, average won   = 0.1680

For probability             = 0.4500
Optimal games to be played  = 10
For  8 games, average won   = 0.2671
For 10 games, average won   = 0.2646
For 12 games, average won   = 0.2640
```

45. Average number of matches

Lay out a deck of cards in a row in the standard order and then lay out a second deck in a row in a random order below the first row (Figure 13). What is the expectation of the number of matches of a card in the first row with the card below it?

Solution

The distribution is uniform because each card in the second row has the same probability of being matched with the card above it. Therefore:

$$E(\text{number of matches}) = 52 \cdot \frac{1}{52} = 1\,.$$

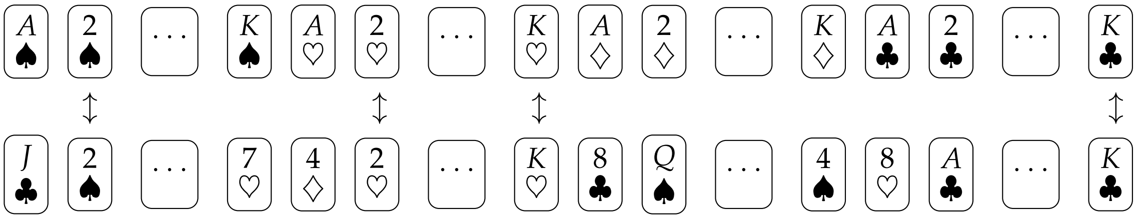


Figure 13: Matching two decks of cards

```
Expectation of matches = 1.00
Average of matches     = 1.01
```

46. Probabilities of matches

Lay out a deck of n cards in a row in the standard order and then lay out a second deck in a row in a random order below the first (Figure 13). Develop a formula for $P(n,r)$, the probability that there will be exactly r matches of a card in the first row with the card below? Assume that $P(k,0)$ is given for $0 \leq k \leq n$.

Solution

At first glance this problem seems to be related to Problem 28 but there is a major difference. The drawings from the boxes are independent, whereas here the matchings are not independent. For example, if the first match occurs on the first card (with probability $1/n$), the probability that the second card matches is $1/(n-1)$.

The probability that any *given* set of r cards match is:

$$\frac{1}{n} \cdot \frac{1}{n-1} \cdot \cdots \cdot \frac{1}{n+r-1}\,. \tag{28}$$

To obtain exactly r matches Equation 28 must be multiplied by $P(n-r,0)$, the probability that there are no matches in the remaining $n-r$ cards. Finally, there are $\binom{n}{r}$ ways of choosing the r matches. Therefore:

$$\begin{aligned}
P(n,r) &= \binom{n}{r} \frac{1}{n(n-1)(n+r-1)} P(n-r,0) \\
&= \frac{n!}{r!(n-r)!} \cdot \frac{1}{n!/(n-r)!} P(n-r,0) \\
&= \frac{1}{r!} P(n-r,0)\,,
\end{aligned}$$

which solves the problem since $P(k,0)$ is given.

Mosteller develops a closed formula and a limit for $P(n, r)$:

$$P(n,k) = \frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!} \tag{29}$$

$$\lim_{n-r\to\infty} P(n,k) \approx \frac{1}{k!}e^{-1}\,. \tag{30}$$

Simulation

The simulation was run for $n = 52$ cards and the probability computed from Equation 30.

```
Probability of 1 matches = 0.3679
Proportion 1 matches     = 0.3710
Probability of 2 matches = 0.1839
Proportion 2 matches     = 0.1828
Probability of 3 matches = 0.0613
Proportion 3 matches     = 0.0569
Probability of 4 matches = 0.0153
Proportion 4 matches     = 0.0168
```

47. Choosing the largest dowry ^{D}

Place n cards in a row face down. There is a positive integer written on the face of each card but you have no knowledge as to their distribution. Turn the cards over one-by-one and look at the numbers. After turning over each card you can declare that it is the largest number. If you are correct you win the game, otherwise you lose. For example, let the sequence of cards be $(47, 23, 55, 4)$. You win only if you choose the third card.

Consider the following strategy: for some fixed r reject the first $r - 1$ cards and select the first card whose number is greater than all the $r - 1$ cards.

Question 1: For $n = 4$ and $r = 3$ check all permutations and determine the number of permutations where you win.

Question 2: Develop a formula for the probability of a win for arbitrary n, r.

Question 3: Find an approximation for the probability when $n, r \to \infty$.

Hint: Given r in what positions can the largest number m be and in what positions are numbers less or equal to m?

Solution

Answer 1: To simplify notation we write the rank of the numbers as $1, 2, 3, 4$, although the actual numbers are not known, say they are $4, 23, 47, 55$. If you uncover cards $1, 2, 3$ (which are actually $4, 23, 47$), you do not know whether to accept 47 or to reject it and select the last card.

There are 24 permutations of the four numbers. By the strategy you reject the first two cards and select either the third card or the fourth card, so you lose if the permutation has 4 in the first position. What about the permutation $(1,2,3,4)$? You reject $1,2$ and select 3 since it greater than $1,2$ but this is not the largest card so you lose. What about the permutation $(1,3,2,4)$? Again, $1,3$ are rejected by the strategy, but 2 is also rejected because it is *not* larger than $1,3$. Now you select 4 and win. Carry out this reasoning for all the permutations and check that permutations with boxed 4s are wins:

$$
\begin{array}{cc|cc@{\qquad}cc|cc@{\qquad}cc|cc@{\qquad}cc|cc@{\qquad}cc|cc@{\qquad}cc|cc}
1 & 2 & 3 & 4 & 1 & 2 & \boxed{4} & 3 & 1 & 3 & 2 & \boxed{4} & 1 & 3 & \boxed{4} & 2 & 1 & 4 & 2 & 3 & 1 & 4 & 3 & 2 \\
2 & 1 & 3 & 4 & 2 & 1 & \boxed{4} & 3 & 2 & 3 & 1 & \boxed{4} & 2 & 3 & \boxed{4} & 1 & 2 & 4 & 1 & 3 & 2 & 4 & 3 & 1 \\
3 & 1 & 2 & \boxed{4} & 3 & 1 & \boxed{4} & 2 & 3 & 2 & 1 & \boxed{4} & 3 & 2 & \boxed{4} & 1 & 3 & 4 & 2 & 1 & 3 & 4 & 2 & 1 \\
4 & 1 & 2 & 3 & 4 & 1 & 3 & 2 & 4 & 2 & 1 & 3 & 4 & 2 & 3 & 1 & 4 & 3 & 1 & 2 & 4 & 3 & 2 & 1
\end{array}
$$

The probability of winning is $10/24$.

Answer 2: If the largest number is in one the of positions $1,\ldots,r-1$ you lose. Therefore, in order to win the largest number must be in the mth position for $r \leq m \leq n$:

$$
1 \quad 2 \quad \cdots \quad r-2 \quad r-1 \quad \overbrace{r \quad r+1 \quad \cdots \quad m-1 \quad m \quad m+1 \quad \cdots \quad n}^{\textsf{largest number must be here}}\,.
$$

By the strategy you reject the first $r-1$ cards. You will choose position m only if *all* the numbers in $(r,\ldots,m-1)$ are less than *all* the numbers in $(1,\ldots,r)$. In other words, the largest card in the sequence $(1,\ldots,m-1)$ is *not* in the second part of the sequence $(r,\ldots m-1)$ but in the first part $(1,\ldots,r-1)$. The probability is:

$$
P(\textsf{largest card in } (1,\ldots,m-1) \textsf{ is in } (1,\ldots,r-1)) = \frac{r-1}{m-1}\,.
$$

Since the probability that the largest card is at m is $1/n$:

$$
P(\textsf{win}) = \sum_{m=r}^{n} \frac{1}{n} \cdot \frac{r-1}{m-1} = \frac{r-1}{n} \sum_{m=r}^{n} \frac{1}{m-1}\,. \tag{31}
$$

For $n=4, r=3$, $P(\textsf{win}) = 5/12 = 10/24$.

Equation 31 is not defined for $r=1$ but the probability of winning when choosing the first number is $1/n$. A larger r will have a higher probability of winning as shown in the example.

Answer 3: Rewrite Equation 31 as:

$$
P(\textsf{win}) = \frac{r-1}{n} \left(\sum_{m=2}^{n} \frac{1}{m-1} - \sum_{m=2}^{r-1} \frac{1}{m-1} \right). \tag{32}
$$

For large n, r Equation 32 can be approximated by:

$$
P(\textsf{win}) = \frac{r}{n}(\ln n - \ln r) = \frac{r}{n} \ln \frac{n}{r} = -\frac{r}{n} \ln \frac{r}{n}\,.
$$

Denote $x = r/n$ and find the maximum by taking derivatives:

$$
\begin{aligned}
(-x \ln x)' &= -x \cdot \frac{1}{x} + (-1) \ln x = 0 \\
\ln x &= -1 \\
x &= 1/e\,.
\end{aligned}
$$

To maximize that probability of winning choose $r \approx n/e$.

Simulation The simulation was run with 100 cards and values of r near $100/e$:

```
Reject cards before r = 36:
Probability of wins   = 0.3674
Proportion wins       = 0.3641
Reject cards before r = 37:
Probability of wins   = 0.3678
Proportion wins       = 0.3759
Reject cards before r = 38:
Probability of wins   = 0.3679
Proportion wins       = 0.3548
Reject cards before r = 30:
Probability of wins   = 0.3590
Proportion wins       = 0.3601
```

48. Choosing the largest random number ^{D}

Place a sequence of n cards face down. On the face of each card is a real number with uniform distribution in $0.0 \leq x < 1.0$. Turn the cards over one-by-one and look at the numbers. After turning over each card you can declare that it is the largest number in the sequence. If you are correct you win the game, otherwise you lose.

Use the strategy of Problem 37: decide upon a value r such that you reject the first $r-1$ cards and then select the first card that is larger than the largest value in the first $r-1$ cards.

Definition: d, the *indifference value*, is the value below which you decide to reject the card and above which you decide to select the card.

Question 1: Compute d for $n = 1$ and compute the probability of winning.

Question 1: Compute d for $n = 2$ and compute the probability of winning.

Question 3: Compute d for $n = 3$. Do not try to compute the probability of winning!

Note: In Problem 37 the values could be $100, 200, 300$ or $100, 50, 20$ so uncovering the first number gives no information about the other numbers. In this problem, since the distribution is uniform, if the first number is 0.2, the second number has probability 0.8 of being larger, and if the first number is 0.8 the second number has a probability of 0.2 of being larger.

Solution

Let v_1, v_2, v_3 be the values of the three cards.

Answer 1: You have no choice but to select the first card since there are no other cards. There is no indifference value. Since v_1 is the "largest" number $P(win) = 1$.

Answer 2: If you select the first card $P(\text{win}) = v_1$ which is the probability that the second card has a smaller value. If you reject the first card, $P(\text{win}) = 1 - v_1$, which is the probability that $v_2 > v_1$. Therefore, if $v_1 < 0.5$ select the second card because $1 - v_1 > 0.5$ and if $v_1 > 0.5$ select the first card because $1 - v_1 < 0.5$. It follows that $d = 0.5$.

Here is the formula for the probability of winning:

$$P(\text{win}) = p(\text{win} \mid v_1 < 0.5)\, p(v_1 < 0.5) + p(\text{win} \mid v_1 > 0.5)\, p(v_1 > 0.5)\,.$$

$p(v_1 < 0.5) = 0.5$ follows by the uniform distribution. What about $p(\text{win} \mid v_1 < 0.5)$? By the strategy you win if $0.5 < v_2 < 1$, but you also win if $v_1 < v_2 < 0.5$. Since v_1 is uniformly distributed in $(0, 0.5)$:

$$p(\text{win} \mid v_1 < 0.5) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}\,.$$

A similar computation holds for $v_1 > 0.5$. Putting the partial computations together gives:

$$P(\text{win}) = \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{4}\,.$$

Answer 3:

If you select the first card $P(\text{win}) = v_1^2$ because the second and third cards must be smaller than the first.

If you reject the first card and select the second because $v_2 > v_1$ then:

- $P(\text{win}) = (1 - v_1)v_1$ if $v_2 > v_1$ and $v_3 < v_1$.
- $P(\text{win}) = v_1(1 - v_1)$ if $v_2 < v_1$ and $v_3 > v_1$.
- $P(\text{win}) = \frac{1}{2}(1 - v_1)^2$ if $v_2 > v_1$ and $v_3 > v_1$, since winning depends on the order: $(0.55, 0.75, 0.65)$ wins and $(0.55, 0.65, 0.75)$ loses.

The indifference value d is the value such that the probability of winning by selecting the first card equals the probability of winning by rejecting the first card:

$$\begin{aligned}
d^2 &= 2d(1 - d) + \frac{1}{2}(1 - d)^2 \\
5d^2 - 2d - 1 &= 0 \\
d &= \frac{1 + \sqrt{6}}{5} \approx 0.6899\,.
\end{aligned}$$

Gilbert and Mosteller [3, page 55] show that for $n = 3$:

$$P(\text{win}) = \frac{1}{3} + \frac{d}{2} + \frac{d^2}{1} - \frac{3d^3}{2} \approx 0.6617\,.$$

Simulation:

```
For  3 cards:
Indifference value = 0.6000
Probability of win = 0.6693
```

```
Proportion of wins = 0.6628
Indifference value = 0.6899
Probability of win = 0.6617
Proportion of wins = 0.6711
Indifference value = 0.7200
Probability of win = 0.6519
Proportion of wins = 0.6473
```

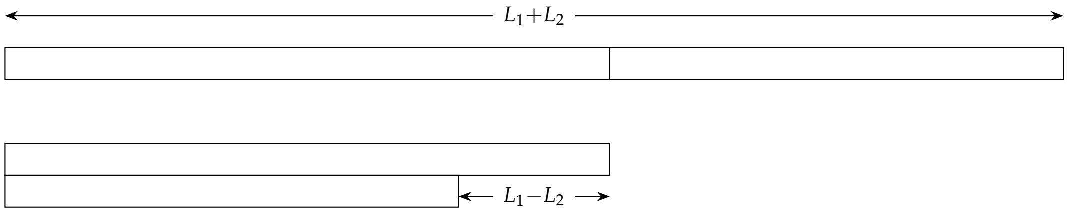


Figure 14: Measuring the lengths of two rods

49. Doubling your accuracy

You are given two rods of lengths $L_1 > L_2$ and a length-measuring instrument whose possible error is given by a normal distribution with mean 0 and variance σ^2. The lengths of the two rods can be measured by measuring each one separately. Is there a more accurate method?

Solution Place the rods end-to-end and measure $L_s = L_1 + L_2$ and then place the rods side-by-side and measure $L_d = L_2 - L_1$ (Figure 14). Compute L_1, L_2:

$$\begin{aligned}\tfrac{1}{2}(L_s + L_d) &= \tfrac{1}{2}((L_1 + L_2) + (L_1 - L_2)) = L_1\\ \tfrac{1}{2}(L_s - L_d) &= \tfrac{1}{2}((L_1 + L_2) - (L_1 - L_2)) = L_2\,.\end{aligned}$$

The errors in the measurements of the sum and the difference are e_s, e_d, respectively, so the errors in the results are:

$$\begin{aligned}\tfrac{1}{2}((L_s + e_s) + (L_d + e_d)) &= L_1 + \tfrac{1}{2}(e_s + e_d)\\ \tfrac{1}{2}((L_s + e_s) - (L_d + e_d)) &= L_2 + \tfrac{1}{2}(e_s - e_d)\,.\end{aligned}$$

Since the mean of the measurement instrument is 0, the mean of the errors of the two measurements is also 0. The variance is reduced to half its previous value:[3]

$$\begin{aligned}\mathrm{Var}\left(\tfrac{1}{2}\left(L_s + L_d\right)\right) &= \tfrac{1}{4}(\sigma^2 + \sigma^2) = \tfrac{1}{2}\sigma^2\\ \mathrm{Var}\left(\tfrac{1}{2}(L_s - L_d)\right) &= \tfrac{1}{4}(\sigma^2 + (-1)^2\sigma^2) = \tfrac{1}{2}\sigma^2\,.\end{aligned}$$

Simulation

In the simulation values calculated for L_1, L_2 from the sum and difference of two measurements are denoted by `sd`. The errors of the mean remain close to zero but the variances are reduced by one half.

```
For L1 = 10, L2 = 6, variance = 1.00:
Errors of means:
    L1 = 0.0076, L1_sd = 0.0005
    L2 = 0.0001, L2_sd = 0.0057
Variances:
    L1 = 1.0273, L1_sd = 0.5102
    L2 = 1.0009, L2_sd = 0.5012
```

[3] The measurements are independent so the covariance is 0.

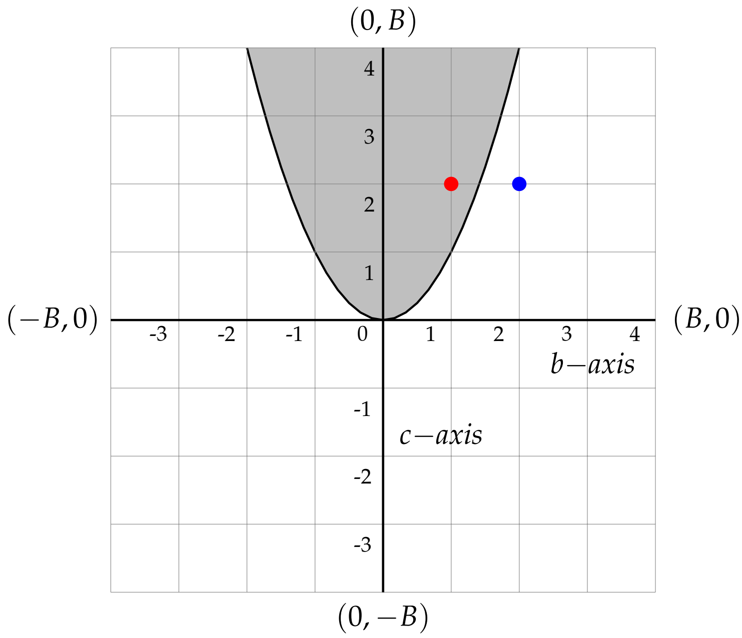


Figure 15: For (b, c) in the shaded area the roots of $c = b^2$ are complex

50. Random quadratic equations

Consider the quadratic equation $x^2 + 2bx + c = 0$ defined on $[-B, B] \times [-B, B]$ for $B \geq 1$.

Question 1: What is the probability that the roots are real?

Question 2: As $B \to \infty$ what is the probability that the roots are real?

Solution

Answer 1: The roots will be real if the discriminant is non-negative $4b^2 - 4c \geq 0$. Figure 15 shows a plot of the parabola $c = b^2$ where the complex roots are within the shaded area. For example, for $(b, c) = (1, 2)$, $x^2 + 2x + 2$ has complex roots (red dot) while for $(b, c) = (2, 2)$, $x^2 + 4x + 2$ has real roots (blue dot).

The shaded area can be computed by integration:

$$\int_{-\sqrt{B}}^{\sqrt{B}} (B - b^2)\, db = Bb - \frac{b^3}{3}\bigg|_{-\sqrt{B}}^{\sqrt{B}} = \left(B^{3/2} - \frac{B^{3/2}}{3}\right) - \left(-B^{3/2} + \frac{B^{3/2}}{3}\right) = \frac{4}{3}B^{3/2}.$$

The total area of the range $[-B, B] \times [-B, B]$ is $4B^2$ so:

$$P(\text{complex roots}) = \frac{\frac{4}{3}B^{3/2}}{4B^2} = \frac{1}{3\sqrt{B}}$$
$$P(\text{real roots}) = 1 - \frac{1}{3\sqrt{B}}.$$

Answer 2:

$$\lim_{B\to\infty} P(\text{real roots}) = \lim_{B\to\infty}\left(1-\frac{1}{3\sqrt{B}}\right) = 1\,.$$

Simulation

```
For B =  4:
Probability of real roots = 0.8333
Proportion real roots     = 0.8271
For B = 16:
Probability of real roots = 0.9167
Proportion real roots     = 0.9205
For B = 64:
Probability of real roots = 0.9583
Proportion real roots     = 0.9582
```

51. Two-dimensional random walk

A particle is placed at the origin of a two-dimensional coordinate system. The particle moves left or right on the x-axis with probabilities $1/2$ for each direction and *simultaneouly* up or down the y-axis with probabilities $1/2$ for each direction. Figure 16 shows a random walk of 22 steps starting at and returning to the origin.

Question 1: What is the probability that the particle returns to the origin in 2 moves?

Question 2: Develop a formula for the probability that the particle returns (one or more times) to the origin.

Question 3: Use Stirling's approximation to obtain an estimate of the probability for large n.

Solution

Answer 1: The dots in Figure 17 show the possible positions of the particle after two moves:

- The green path shows how to move to $(\pm 2, \pm 2)$ by taking two moves in the same direction. The probability is $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$.
- The red path shows how to move to $(\pm 2, 0)$ or $(0, \pm 2)$. There are two possible paths for each one so the probability is $2 \cdot \left(\frac{1}{4}\right)^2 = \frac{2}{16}$.
- The blue path shows how to move to $(\pm 1, \pm 1)$ and back to the origin. The probability is $1/16$. Since there are four paths that return to the origin the probability is $\frac{4}{16}$.

The blue paths are the only ones that return to the origin so:

$$P(\text{return to origin in two moves}) = \frac{4}{16}\,.$$

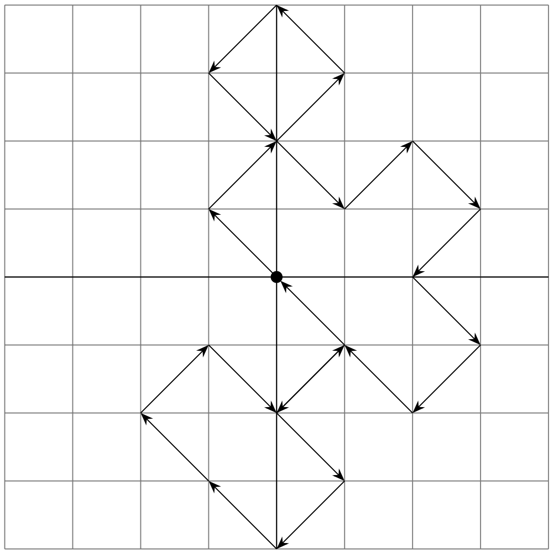


Figure 16: Two-dimensional random walk

Answer 3: The choice of direction for both axes are independent so for $2n$ moves:

$$P_{2n}(\text{return to origin}) = P_{2n}(\text{return to } x=0)\, P_{2n}(\text{return to } y=0)\,. \tag{33}$$

The particle will return to the origin if and only if for both axes the number of $+1$ moves equals the number of -1 moves. There are $\binom{2n}{n}$ ways to arrange $+1$s and -1s so:

$$P_{2n}(\text{return to } x=0) = P_{2n}(\text{return to } y=0) = \binom{2n}{n}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{n} \tag{34}$$

$$P_{2n}(\text{return to origin}) = \left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^2 \tag{35}$$

$$P(\text{return to origin}) = \sum_{n=1}^{\infty} P_{2n}(\text{return to origin}) = \sum_{n=1}^{\infty}\left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^2 . \tag{36}$$

Answer 3: By Stirling's approximation $n! \approx \sqrt{2\pi n}\,(n/e)^n$:

$$\begin{aligned}
P_{2n}(\text{return to origin}) &= \left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^2 \\
&= \left[\frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}\right]^2 \\
&\approx \left(\frac{1}{2}\right)^{4n}\frac{(\sqrt{2\pi\cdot 2n})^2\,(2n/e)^{4n}}{(\sqrt{2\pi n})^4\,(n/e)^{4n}} \\
&= \left(\frac{1}{2}\right)^{4n}\frac{4\pi n}{4\pi^2 n^2}\cdot\frac{(n/e)^{4n}\cdot 2^{4n}}{(n/e)^{4n}} \\
&= \frac{1}{\pi n}
\end{aligned}$$

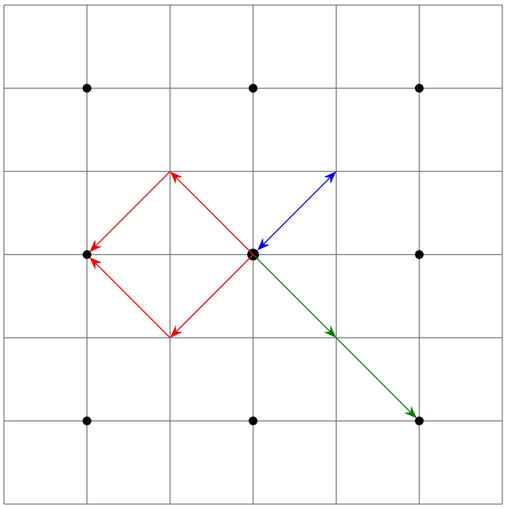


Figure 17: Two moves of the random walk

$$P(\text{return to origin}) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n},$$

which is the *harmonic series* that diverges, therefore, with probability 1 the particle returns to the origin!

Simulation The simulation was run one million times instead of ten thousand times but the result doesn't show that the particle must return to the origin.

```
Proportion returned to origin       = 0.8700
```

52. Three-dimensional random walk ^{D}

A particle is placed at the origin of a three-dimensional coordinate system. The particle moves left or right on the x-axis with probabilities $1/2$ *and* up or down the y-axis with probabilities $1/2$ *and* in or out on the z-axis with probabilities $1/2$.

Question 1: What is the expectation of the number of times that the particle returns to the origin?

Hint: Compute the probability and then use an indicator variable.

Question 2: What is the probability that the particle will return to the origin at least once?

Hint: Use the technique from Problem 4.

Solution

P_{2n}, probability of returning to the origin after $2n$ steps, is given by generalizing Equation 34 to three dimensions:

$$P_{2n} = P_{2n}(\text{return to } x = 0)\, P_{2n}(\text{return to } y = 0\, P_{2n}(\text{return to } z = 0)\,.$$

P_r, the probability of returning to the origin one or more times, is given by the analogue of Equation 36:

$$P_r = \sum_{n=1}^{\infty} P_{2n} = \sum_{n=1}^{\infty} \left[\binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \right]^3 .$$

From Stirling's approximation:[4]

$$\begin{aligned}
P_{2n} &= \left[\frac{(2n)!}{(n!)^2} \left(\frac{1}{2}\right)^{2n} \right]^3 \\
&\approx \left(\frac{1}{2}\right)^{6n} \frac{(\sqrt{2\pi \cdot 2n})^3 \, (2n/e)^{6n}}{(\sqrt{2\pi n})^6 \, (n/e)^{6n}} \\
&= \frac{(4\pi n)^{3/2}}{(2\pi n)^3} = \frac{1}{(\pi n)^{3/2}} \\
P_r &= \sum_{n=1}^{\infty} \frac{1}{(\pi n)^{3/2}} \approx 0.3772 .
\end{aligned}$$

Let I_k be the indicator variable for a return to the origin on step k:

$$I_k = \begin{cases} 1, & \text{if the particle returns to origin on step } k \\ 0, & \text{if the particle does not returns to origin on step } k . \end{cases} \tag{37}$$

Then:

$$E(\text{number of returns to the origin}) = \sum_{n=1}^{\infty} P_{2n} \, I_{2n} = P_r \approx 0.3772 ,$$

so the expectation of the number of returns is equal to the probability.

Question 2: Let P_1 be the probability that the particle returns to the origin *at least once*. From Problem 4 we know that the expectation of the number of trials until the first one where the particle *does not* return to the origin is $1/(1 - P_1)$. Therefore, the expectation of the number of trials for which the particle does return to the origin is one less, because the particle can return to the origin many times until finally it does not [5].

Let $E_r = E(\text{number of returns to the origin})$, then:

$$\begin{aligned}
E_r &= \frac{1}{1 - P_1} - 1 \\
P_1 &= \frac{E_r}{1 + E_r} .
\end{aligned}$$

In **Answer 1** we computed that $E_r \approx 0.3772$ so:

$$P_1 \approx 1 - \frac{1}{1 + 0.3772} \approx 0.2739 .$$

Simulation

[4]Mosteller used 18 terms in his computation and obtained 0.315. My program used 500 terms to obtain 0.3772.

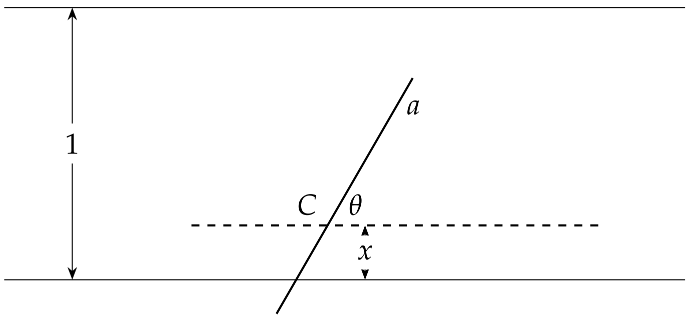


Figure 18: Buffon's needle

```
Expectation of reaching origin = 0.3772
Average times reached origin   = 0.3630
Probability of reaching origin = 0.2739
Proportion reached origin      = 0.2790
```

53. Buffon's needle ^{D,S}

Consider a needle of length $a \leq 1$ and a surface ruled with parallel lines 1 apart. Throw the needle onto the surface. What is the probability that the needle crosses a line?[5]

Hint: There are two independent random variables (Figure 18): x, the position of the center of the needle relative to the closest line which is uniformly distributed in the range $[0, 1/2]$, and θ, the angle between the needle and the parallel lines which is uniformly distributed in the range $[0, \pi/2]$.

Solution 1

Let $p(a)$ be the probability that a needle of length a crosses a line and define the indicator variable:

$$I_{\textsf{crosses}} = \begin{cases} 1, & \textsf{if needle of length } a \textsf{ crosses a line} \\ 0, & \textsf{if needle of length } a \textsf{ does not cross a line}\,. \end{cases}$$

Then:

$$E(I_{\textsf{crosses}}) = 1 \cdot p(a) + 0 \cdot (1 - p(a)) = p(a)\,, \tag{38}$$

and the probability can be computed by computing the expectation.

Let m be a line perpendicular to the parallel lines that passes through the center of the needle and let θ be the angle between the needle and a parallel line. Project the needle onto m to give the line segment $\overline{CD}$. The probability that the needle will cross a line is:

$$P(\textsf{needle of length } a, \textsf{ angle } \theta \textsf{ crosses line}) = \frac{\overline{CD}/2}{1/2} = \frac{(a/2)\sin\theta}{1/2} = a \sin\theta\,. \tag{39}$$

[5]The problem has been simplified by specifying the distance between the parallel lines as 1. We ignore the possibility that the needle lies completely along the line or just touches two lines since the probability of these events is zero.

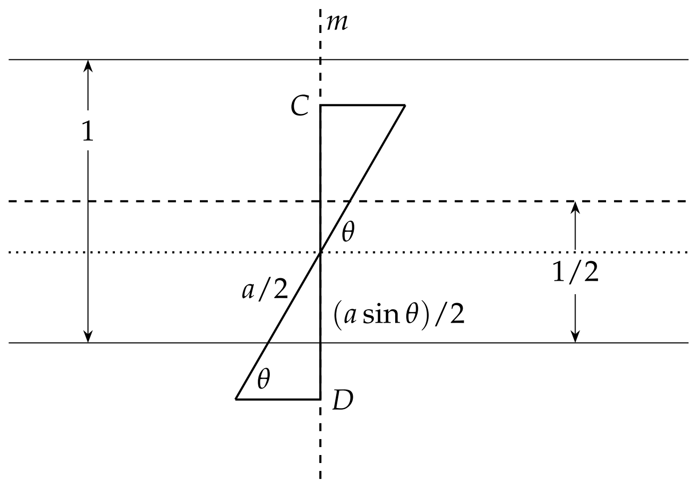


Figure 19: Right triangle for solving Buffon's needle problem

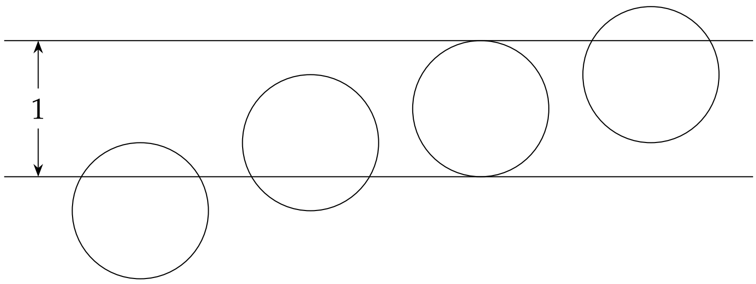


Figure 20: Solving Buffon's needle with circles

The expectation of the number of lines crossed is given by integrating over possible angles:

$$E(\textsf{lines crossed}) = \frac{1}{(\pi/2) - 0} \int_0^{\pi/2} a \sin\theta \, d\theta = \frac{2}{\pi} \cdot a(-\cos\theta) \Big|_0^{\pi/2} = \frac{2a}{\pi} . \tag{40}$$

Solution 2 This solution is based upon [1, Chapter 26].

Let $E(x)$ be the expectation of the number of parallel lines crossed by a line of length x. Consider a line formed into a circle C of *diameter* 1 and circumference π. If the circle is thrown onto the surface it will cross lines *exactly* twice (Figure 20) so:

$$E(C) = 2 . \tag{41}$$

Inscribe a regular polygon Q_n (red) within c (green) and circumscribe a regular polygon R_n (blue) around c (Figure 21). Any line that Q_n crosses (red) must also cross the circle and any line that crosses the circle (blue) must also cross R_n. Therefore:

$$E(Q_n) \leq E(C) \leq E(R_n) . \tag{42}$$

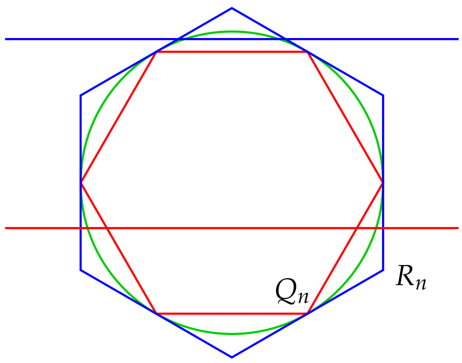


Figure 21: Polygons approximate a circle

Let a_Q, a_R be the sums of the lengths of the sides of Q_n, R_n, respectively. By the linearity of expectation:

$$E(Q_n) = \sum_{i=1}^{n} E(\text{sides of } a_Q) = a_Q E(1) \tag{43}$$

$$E(R_n) = \sum_{i=1}^{n} E(\text{sides of } a_R) = a_R E(1)\,. \tag{44}$$

As $n \to \infty$ both polygons approximate the circle so:

$$\lim_{n\to\infty} a_Q = \lim_{n\to\infty} a_R = \pi\,, \tag{45}$$

the circumference of the circle. From Equations 43–45 we have:

$$\begin{aligned}
&\lim_{n\to\infty} E(Q_n) = E(C) = \lim_{n\to\infty} E(R_n)\\
&E(C) = aE(1) = \pi E(1) = 2\\
&E(1) = \frac{2}{\pi}\\
&E(a) = aE(1) = \frac{2a}{\pi}\,.
\end{aligned}$$

Simulation

Since $\pi = 2a/E$ you can obtain an approximation of its value by running the simulation or throwing needles on a table!

```
For length = 0.2:
Expectation of crossings = 0.1273
Average crossings        = 0.1308
Empirical value for pi   = 3.0581

For length = 0.5:
Expectation of crossings = 0.3183
```

```
Average crossings          = 0.3227
Empirical value for pi     = 3.0989

For length = 1.0:
Expectation of crossings = 0.6366
Average crossings          = 0.6333
Empirical value for pi     = 3.1581
```

54. Buffon's needle with horizontal and vertical rulings

Solve Buffon's needle problem for a surface that is covered by a grid with squares of size 1×1. A needle can cross a vertical line (green), a horizontal line (blue), both (red) or neither (orange) (Figure 22).

Hint: Are the numbers of crossings of the horizontal and vertical lines independent?

Solution

The numbers of crossings of the horizontal and vertical lines are independent and by the linearity of expectation:

$$\begin{aligned} E(\text{lines crossed by a needle of length } a) &= E(\text{vertical lines crossed by a needle of length } a + \\ &\qquad \text{horizontal lines crossed by a needle of length } a) \\ &= E(\text{vertical lines crossed by a needle of length } a) + \\ &\quad E(\text{horizontal lines crossed by a needle of length } a) \\ &= \frac{2a}{\pi} + \frac{2a}{\pi} = \frac{4a}{\pi}\,. \end{aligned}$$

Simulation

```
For length = 0.2:
```

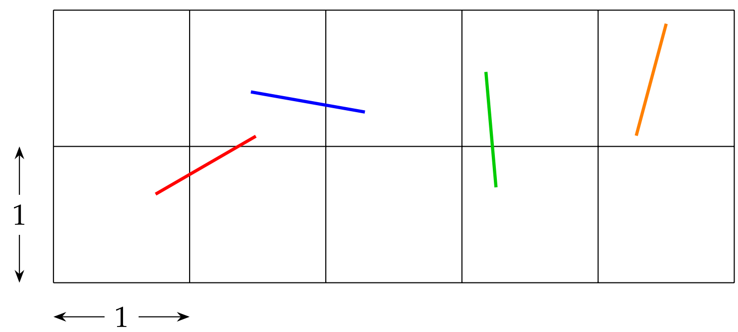


Figure 22: Buffon's needle with horizonal and vertical lines

```
Expectation of crossings = 0.2546
Average crossings        = 0.2532
For length = 0.5:
Expectation of crossings = 0.6366
Average crossings        = 0.6355
For length = 1.0:
Expectation of crossings = 1.2732
Average crossings        = 1.2736
```

55. Long needles ^{D}

Let the length of the needle in Buffon's problem be $a > 1$.

Question 1: What is the expectation of the *number of crossings*?

Question 2: What is the probability that there is *at least one crossing*?

Hint: For what angles θ is the probability a crossing 1?

Solution

Answer 1: Break the needle into pieces of lengths $\{a_1, a_2, \ldots, a_n\}$, $a_i < 1$, such that $\sum_{i=1}^n a_i = a$. By the linearity of expectation and the solution of Problem 53:

$$E(a) = \sum_{i=1}^{n} E(a_i) = \frac{2a}{\pi}\,.$$

Answer 2: This solution is based on [12] and [1, Chapter 26].

By Equation 39 the probability that the needle will cross a line is $a \sin\theta$ *if* $a \sin\theta \le 1$, that is, if $0 \le \theta \le \sin^{-1}(1/a)$. However, if $a\sin\theta > 1$ then the probability is 1 (Figure 23). Let us generalize Equation 40 for arbitrary $a > 0$. The integral is computed in two parts, one for $\theta < \sin^{-1}(1/a))$ and one for $\theta > \sin^{-1}(1/a))$:

$$\begin{aligned}
E(a) &= \frac{2}{\pi}\left(\int_0^{\sin^{-1}(1/a)} a\sin\theta\, d\theta + \int_{\sin^{-1}(1/a)}^{\pi/2} 1\, d\theta\right)\\
&= \frac{2}{\pi}\left(a(-\cos\theta)\big|_0^{\sin^{-1}(1/a)} + \left(\frac{\pi}{2} - \sin^{-1}(1/a)\right)\right)\\
&= 1 + \frac{2}{\pi}\left(a\left(1 - \sqrt{1 - \frac{1}{a^2}}\right) - \sin^{-1}(1/a)\right)\,.
\end{aligned}$$

Simulation

```
For length = 1.5:
Expectation of crossings = 0.7786
```

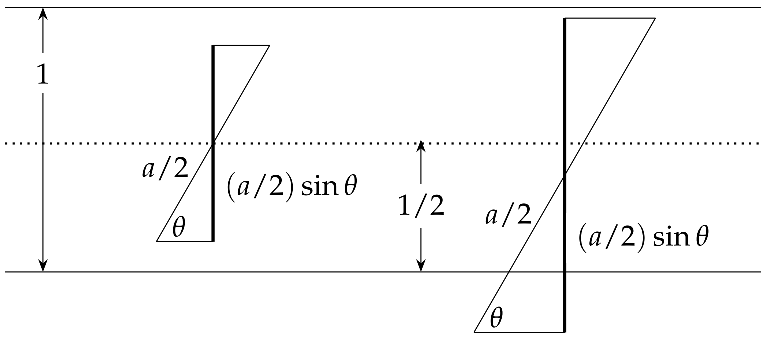


Figure 23: Long needles

```
Average crossings        = 0.7780
For length = 2.0:
Expectation of crossings = 0.8372
Average crossings        = 0.8383
For length = 3.0:
Expectation of crossings = 0.8929
Average crossings        = 0.8897
```

56. Molina's urns

Let U_1, U_2 be two urns containing the same number of balls. U_1 has w_1 white balls and b_1 black balls, while U_2 has w_2 white balls and b_2 black balls. n balls are drawn *with replacement* from each urn.

Question 1: For different values of $n > 1$ find w_1, b_1, w_2, b_2 such that:

$$P(\textsf{balls drawn from } U_1 \textsf{ are all white}) = P(\textsf{balls drawn from } U_2 \textsf{ are all white or all black})\,.$$

Solution

Answer 1: For $n = 2$ the equation that must be solved is:

$$\left(\frac{w_1}{m}\right)^2 = \left(\frac{w_2}{m}\right)^2 + \left(\frac{b_2}{m}\right)^2$$
$$w_1^2 = w_2^2 + b_2^2\,.$$

By Fermat's Last Theorem, proved in 1995 by Andrew Wiles, there are no solutions to $w_1^n = w_2^n + b_2^n$ for $n \geq 3$.

Simulation

The simulation was run for $n = 2$ and several Pythagorean triples.

```
For w1 = 17, w2 = 8, b2 =15:
Proportion of two whites in urn 1          = 0.5523
```

```
Proportion of two whites or black in urn 2 = 0.5387
For w1 = 29, w2 = 20, b2 =21:
Proportion of two whites in urn 1          = 0.5003
Proportion of two whites or black in urn 2 = 0.5026
For w1 = 65, w2 = 33, b2 =56:
Proportion of two whites in urn 1          = 0.5381
Proportion of two whites or black in urn 2 = 0.5384
```

Review of Probability

This section reviews concepts of probability. An example of each concept is given using the activity of throwing fair six-sided dice.

Experiment This is an undefined primitive concept, the intention being an action that has possible results. An experiment is also called a *trial*. Throwing a die is an experiment.[6]

Outcome The result of an experiment. If you throw a die one outcome is 4.

Sample space The set of all possible outcomes of an experiment. The set $S = \{1, 2, 3, 4, 5, 6\}$ is the sample space of the outcomes of throwing a die.

Event A subset of the sample space. The subset $e = \{2, 4, 6\} \subseteq S$ is the event of a die showing an even number.

Random variable A function from a sample space to (real) numbers. Let T be the sample space of throwing a pair of dice:

$$T = \{(a, b) | a, b \in \{1, 2, 3, 4, 5, 6\}\} .$$

Define the random variable X as the function $X : T \mapsto \{2, 3, \ldots, 11, 12\}$ which maps the outcomes of throwing a pair of dice to the sum of the numbers on the dice:

$$X((a, b)) = a + b . \tag{46}$$

Union, intersection, complement Since events are sets these concepts take on their usual set-theoretical meaning. Let $e_1 = \{2, 4, 6\}$ and $e_2 = \{1, 2, 3\}$. Then:

$$e_1 \cup e_2 = \{1, 2, 3, 4, 6\} \quad e_1 \cap e_2 = \{2\} \quad \overline{e_1} = S \setminus e_1 = \{1, 3, 5\} .$$

The intersection is the set of even numbers among the first three elements of the sample space. The complement is the set of odd numbers among the elements of the sample space.

Mutually exclusive Two or more events are mutually exclusive if their intersection is the empty set. $e_1 = \{2, 4, 6\}$ and $e_2 = \{1, 3, 5\}$ are mutually exclusive since $e_1 \cap e_2 = \emptyset$, that is, there are no outcomes which are numbers that are both even and odd.

Probability Probability is the limiting relative frequency of an event. Let e be an event and let n_e be the number of times that e occurs in n repetitions of the trial. $P(e)$, the probability of the event e, is:

$$P(e) = \lim_{n \to \infty} \frac{n_e}{n} .$$

This definition is problematic because we don't actually know that the limit exists. The definition also depends on "repetitions of an event" but we want to define probability without reference to a specific sequence of trials.

Modern probability theory is based on a set of three axioms, but we won't develop this theory, although the importance of two of the axioms is clear:

$$\begin{aligned} P(e) &\geq 0 \\ P(S) &= 1 . \end{aligned}$$

[6] *Die* is the singular of the more familiar plural noun *dice*.

Any event either occurs with some non-zero probability or it doesn't occur, and the outcome space is by definition all the possible outcomes.

The *law of large numbers* ensure that our intuitive concept of probability as relative frequency is very similar to what happens when an trial is repeated many times.

Uniformly distributed If all outcomes in the sample space have equal probability, the probability is said to be uniformly distributed. If S is a finite set with probability that is uniformly distributed then:

$$P(e) = \frac{|e|}{|S|}\,.$$

If you throw a *fair* die the probability of the outcomes is uniformly distributed, so for $e = \{2, 4, 6\}$:

$$P(e) = \frac{|e|}{|S|} = \frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{2}\,.$$

Conditional probability Let e_1, e_2 be events. $P(e_1|e_2)$, the conditional probability that e_1 occurs given that e_2 occurs, is given by:

$$P(e_1|e_2) = \frac{P(e_1 \cap e_2)}{P(e_2)}\,.$$

Let $e_1 = \{1, 2, 3\}$ be the event that a die shows a number less than or equal to 3 and let $e_2 = \{2, 4, 6\}$ be the event that the die shows an even number. Then:

$$P(e_2|e_1) = \frac{P(E_2 \cap E_1)}{P(e_1)} = \frac{P(\{2\})}{P(\{2,4,6\})} = \frac{1/6}{1/2} = \frac{1}{3}\,.$$

This makes sense since if you know that a number is less than or equal to 3, only one out of the three outcomes is an even number.

Independence Two events are independent if the probability of their intersection is the product of their individual probabilities:

$$P(e_1 \cap e_2) = P(e_1)\,P(e_2)\,.$$

In terms of conditional probability:

$$P(e_1|e_2) = \frac{P(e_1) \cap P(e_2)}{P(e_2)} = \frac{P(e_1)\,P(e_2)}{P(e_2)} = P(e_1)\,.$$

For independent events e_1, e_2, if you know the probability of e_2 it gives you no information as to the probability of e_1. Three throws of a fair die are independent so the probability of all of them showing an even number is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

Average Let $S = \{a_1, \ldots, a_n\}$ be a set of values. Then:

$$Average(S) = \frac{\sum_{i=1}^{n} a_i}{n}\,.$$

An average is computed over a set of values but the average may not be an element of the set. If there are 1000 families in a town and they have 3426 children, the average number of children per family is 3.426 although clearly no family has 3.426 children. If you throw a die six times and receive the numbers $\{2,2,4,4,5,6\}$ the average is:

$$\frac{2+2+4+4+5+6}{6} = \frac{23}{6} \approx 3.8\,,$$

again, a value not in the set.

Expectation The expectation of a random variable is the sum of the probability of each outcome times the value of random variable for that outcome. For a fair die each outcome has the same probability so:

$$E(\textsf{value of a die}) = 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6} = 3.5\,.$$

Consider the random variable defined by the function X (Equation 46) that maps the numbers appearing in a pair of dice to the sum of the numbers. The probability of each pair is $1/36$, but since the pairs $(2,5)$ and $(5,2)$ have the same sum they belong to the same outcome. The values of the random variable are $\{2,\ldots,12\}$ and that the number of ways of obtaining each one is:

Sum	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	4	5	6	5	4	3	2	1

The expectation is the average of the values of the random variable *weighted* by the probability of each outcome. Let E_s be the expectation of the sum of the values appearing when two dice are thrown. Then:

$$E_s = 2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+\cdots+10\cdot\frac{3}{36}+11\cdot\frac{2}{36}+12\cdot\frac{1}{36} = 7\,. \tag{47}$$

For an arbitrary set of events $\{e_1,\ldots,e_n\}$ the expectation is:

$$E = \sum_{i=1}^{n} e_i P(e_i)\,.$$

Linearity of expectation Consider again at the expectation of the sum of a pair of dice (Equation 47). Let $E(e_6)$ be the expectation of the event of obtaining a 6. Then:

$$E(e_6) = X(e_6)P(e_6) = 6\cdot\frac{5}{36}\,,$$

because 5 out of the possible 36 combinations sum to 6: $(1,5),(2,4),(3,3),(4,2),(5,1)$. But the expectation can also be computed as follows where $P(i,j)$ is the probability of obtaining the pair (i,j) and $E(i,j)$ is the expectation of $i+j$:

$$\begin{aligned}
E(X(e_6)) &= 6\cdot P(1,5)+6\cdot P(2,4)+6\cdot P(3,3)+6\cdot P(4,2)+6\cdot P(5,1)\\
&= (1+5)\cdot\frac{1}{36}+(2+4)\cdot\frac{1}{36}+(3+3)\cdot\frac{1}{36}+(4+2)\cdot\frac{1}{36}+(5+1)\frac{1}{36}\\
&= E(1,5)+E(2,4)+E(3,3)+E(4,2)+E(5,1)\\
E(X(k=i+j=6)) &= \sum_{k,k=i+j=6} E(k)\,.
\end{aligned}$$

The computation is dependent on the fact that the events are mutually exclusive. Obviously, the events $(2,4)$ and $(3,3)$ cannot both appear in the same trial.

By performing a similar computation for all pairs:

$$E(X(\textsf{all pairs})) = \sum_{k=\textsf{sum of a pair}} E(k)\,.$$

The same method can be used to prove the generalization [11, Section 4.9]:

$$E\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} a_i E(e_i)\,.$$

This is called the linearity of expectation. In the special case of two random variables $E(ae_1 + be_2) = aE(e_1) + bE(e_2)$.

Indicator variable Let e be an event whose probability is $P(e)$. Define I_e, an indicator variable for e, as follows [11, Chapter 4, Example 3b]:

$$I_e = \begin{cases} 1, & \textsf{if } e \textsf{ occurs} \\ 0, & \textsf{if } e \textsf{ does not occur}\,. \end{cases}$$

Then $E(I_e) = 1 \cdot P(e) + 0 \cdot (1 - P(e)) = P(e)$.

Mathematical formulas

Binomial theorem If p is the probability of event e then the probability that a sequence of n independent trials results in *exactly* k events e is given by the *binomial coefficient*:

$$\binom{n}{k} p^k (1-p)^{n-k}\,.$$

By the binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\,.$$

For $p, 1-p$ the equation gives $(p + (1-p))^n = 1$, as expected, since one of the outcomes must occur.

Sum of a geometric series For $0 < r < 1$:

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1-r}, \qquad \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}\,.$$

Sum of a harmonic series For positive integer n the harmonic series is:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \ln n + \frac{1}{2n} + \gamma\,,$$

where $\gamma \approx 0.5772$ is *Euler's constant*. As n approaches infinity the series diverges:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty\,,$$

because $\ln n$ is unbounded.

Stirling's approximation Computing $n!$ for large n is very difficult. It is convenient to use one of the formulas of *Stirling's approximation*:

$$\begin{aligned}
n! &\approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \\
\ln(n!) &\approx n\ln n - n \\
\ln(n!) &\approx n\ln n - n + \frac{1}{6}\left(8n^3 + 4n^2 + n + \frac{1}{30}\right) + \frac{1}{2}\ln\pi\,.
\end{aligned}$$

Continuous probability distribution

Continuous probability distributions do not appear very often in the book but for readers with the appropriate background we review the basic concepts.

Probabilities can be defined over continuous random variables. A *probability density function (PDF)* $f(x) : \mathcal{R} \rightarrow \mathcal{R}$ maps an outcome x to the value of the function, thus defining:

$$P(x) = f(x)\,.$$

The reason for this terminology is that each *individual* real number has zero probability of occurring, so the proper interpretation is to assign probabilities to neighborhoods of points.

The *cumulative probability distribution (CPD)* for the interval $[-\infty, a]$ is obtained by integrating the PDF:

$$P(x < a) = \int_{-\infty}^{a} f(x)\,dx\,.$$

This is also $P(x \leq a)$ since $P(a) = 0$.

Like all probabilities, for a PDF $P(x) \geq 0$ for all x, and:

$$\int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1\,.$$

If the integral does not evaluate to 1 a *normalization constant* must be used. If a PDF is uniformly distributed in the range $[a, b]$ then:

$$P(a \leq x \leq b) = \int_{a}^{b} 1\,dx = (b - a)\,,$$

and therefore we must define:

$$P(a \leq x \leq b) = \frac{1}{b-a}\int_{a}^{b} 1\,dx = \frac{1}{b-a}\cdot(b-a) = 1\,.$$

The expectation can be obtained by integrating the PDF $f(x)$ multiplied by x:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx\,.$$

The PDF can be obtained by differentiating the CPD:

$$P(x < a) = \frac{d}{da}CDP(x < a)\,.$$

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