Mosteller's Challenging Problems in Probability

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Contents

Introduction

Frederick Mosteller

Frederick Mosteller (1916–2006) founded the Department of Statistics at Harvard University and served as its chairman from 1957 until 1971, retiring from the university in 2003. Mosteller was deeply interested in statistics education and wrote pioneering textbooks including [10] which emphasized the probabilistic approach to statistics, and [9] which was one of the first texts on data analysis. In an interview Mosteller described the development of his approach to statistics education [6].

This document

This document is a "reworking" of Mosteller's delightful book *Fifty Challenging Problems in Probability with Solutions* [8]. The problems and their solutions are presented as far as possible in a manner accessible to readers with an elementary knowledge of probability, and many of the problems are accessible to secondary-school students and teachers. The problems and solutions have been rewritten to include detailed calculations and additional explanations and diagrams. I have sometimes included additional solutions.

Many of the problems have been modified to make them accessible: they are simplified, divided into subproblems and hints are provided. As a personal preference I have rephrased the problems in a more abstract way than Mosteller does.

The numbering and titles of the problems have been retained to facilitate comparison with Mosteller's book.

Modern scientific calculators, including applications for smartphones, can perform the computations with no difficultly.

Simulations written in the Python programming language are given for most problems.

Basic concepts of probability are reviewed in the final section which is based on [11].

Problems are annotated as follows:

- Problems annotated with D are more difficult.
- Problems for which a simulation is available are annotated with S.

Even a problem not marked D can be difficult so do not be discouraged if you cannot solve it. However, it is worthwhile attempting to solve all the problems because any progress you make will be encouraging.

Simulations

Monte Carlo simulations (named after the famous casino in Monaco) were written in the Python 3 programming language. A computer program "performs an experiment," such as "tossing a pair of dice" or "flipping a coin," a very large number of times and computes averages which are displayed. The random number generators built into Python, `random.random()` and `random.randint()`, are used to obtain random outcomes for each experiment.

The programs run each simulation 10000 times and the results are displayed to four decimal places. A simulated result will almost certainly not be exactly the same as that obtained from computing the expectation or the probability. You can run the program many times to see how the results vary.

The Python source code is available at `...github...`. The files are named `N-name.py` where `N` is the problem number and `name` is the problem title.

For each simulation two results are displayed:

- The theoretical value which is either a *probability* or an *expectation*. In general, rather than copy the values from the text they are calculated from the formulas.
- The result of the simulation is either a *proportion* of successes, corresponding to a probability, or an *average*, corresponding to an expectation.

It is important to understand that "probability" and "expectation" are theoretical concepts. The *laws of large numbers* ensure that the outcomes of many trials are very close to the theoretical values, but they won't be exactly the same. For example, the probability of obtaining a 6 when a fair die is thrown is $1/6 \approx 0.1667$. Running a simulation for 10000 throws resulted in a range of values: $0.1684, 0.1693, 0.1687, 0.1665, 0.1656$.

Problems and solutions

1. The sock drawer ^{S}

A drawer contains both red socks and black socks. If two socks are drawn at random (without replacement) the probability that both are red is $\frac{1}{2}$.

Q1: How small can the number of black socks in the drawer be? What is the corresponding number of red socks?

Q2: How small can the number of black socks in the drawer be if the number of black socks is *even*? What is the corresponding number of red socks?

Solution 1

A1: Let r be the number of red socks in the drawer and let b the number of black socks. $r \geq 2$ since two red socks are drawn, and $b \geq 1$ since otherwise the probability of drawing two red socks would be 1. Multiplying the probabilities for the two selections gives:

$$\frac{r}{r+b} \cdot \frac{(r-1)}{(r-1)+b} = \frac{1}{2}\,. \tag{1}$$

Multiplying out and simplifying results in a quadratic equation in the variable r:

$$r^2 - r(2b+1) - (b^2 - b) = 0\,. \tag{2}$$

Since r, b are positive integers the discriminant:

$$(2b+1)^2 + 4(b^2 - b) = 8b^2 + 1$$

must be the square of an integer.

The discriminant is a square when $b = 1$ (its smallest value). Then from Equation 2, $r = 3$ and we reject the solution $r = 0$ because $r \geq 2$. The total number of socks is 4.

Check: $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

A2: Check even positive integer values of b to find the smallest one for which the discriminant is a square:

b	$8b^2+1$	$\sqrt{8b^2+1}$
2	33	5.74
4	129	11.36
6	**289**	**17**

For $b = 6$ the corresponding value for r is 15.

Check: $\frac{15}{21} \cdot \frac{14}{20} = \frac{1}{2}$.

Solution 2

A1: Is the following inequality is true?

$$\frac{r}{r+b} \overset{?}{>} \frac{r-1}{(r-1)+b}\,. \tag{3}$$

$r \geq 2, b \geq 1$, so both denominators are positive and we can multiply the two sides:

$$\begin{aligned} r(r-1+b) &\overset{?}{>} (r-1)(r+b) \\ r^2 - r + rb &\overset{?}{>} r^2 - r + rb - b \\ b &\overset{?}{>} 0\,. \end{aligned}$$

$b > 1$ so Equation 3 is true.

By Equations 1, 3:

$$\left(\frac{r}{r+b}\right)^2 = \frac{r}{r+b} \cdot \frac{r}{r+b} > \frac{r}{r+b} \cdot \frac{r-1}{(r-1)+b} = \frac{1}{2}\,, \tag{4}$$

and similarly:

$$\left(\frac{r-1}{(r-1)+b}\right)^2 = \frac{r-1}{(r-1)+b} \cdot \frac{r-1}{(r-1)+b} < \frac{r}{r+b} \cdot \frac{r-1}{(r-1)+b} = \frac{1}{2}\,. \tag{5}$$

The denominator $r+b$ is non-zero so we can take the square root and simplify Equation 4:

$$\begin{aligned} \frac{r}{r+b} &> \sqrt{\frac{1}{2}} \\ r &> \frac{b}{\sqrt{2}-1} \\ r &> \frac{b}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} \\ r &> b(\sqrt{2}+1)\,. \end{aligned}$$

Similarly for Equation 5:

$$\begin{aligned} \frac{r-1}{(r-1)+b} &< \sqrt{\frac{1}{2}} \\ r-1 &< \frac{b}{\sqrt{2}-1} \\ r-1 &< b(\sqrt{2}+1)\,. \end{aligned}$$

Combining both equations we get:

$$r - 1 < (\sqrt{2}+1)b < r\,.$$

For $b = 1$ we have $2.141 < r < 3.141$ and $b = 1, r = 3$ is a solution.

A2: Checking even numbers for b:

b	$(\sqrt{2}+1)b$	$< r <$	$(\sqrt{2}+1)b+1$	r
2	4.8	$< r <$	5.8	5
4	9.7	$< r <$	10.7	10
6	**14.5**	$< \mathbf{r} <$	**15.5**	**15**

Mosteller mentions a connection between this problem and advanced number theory, and gives another solution: $b = 35, r = 85$ (check!).

Simulation

```
Expectation of both red  = 0.5000
Average of both red for (red =  3, black =  1) = 0.5053
Average of both red for (red = 15, black =  6) = 0.5013
Average of both red for (red = 85, black = 35) = 0.4961
```

2. Successive wins ^{S}

You are playing a series of three games against two players and you win the series if you win at least two of the three games. The probability that you will win a game against player P_1 is p_1 and the probability that you will win a game against player P_2 is p_2. It is known that $p_1 > p2$. Which of the following scenarios gives you a better chance of winning the series?

- You play against P_1, P_2, P_1 in that order.
- You play against P_2, P_1, P_2 in that order.

Solution 1

You win if: (a) you win the first two games and lose the last game, (b) you lose the first game and win the last two games, or (c) you win all three games.

Let p_{121} and p_{212} be the probabilities that you win the series in the two scenarios:

$$\begin{aligned} p_{121} &= p_1p_2(1-p_1) + (1-p_1)p_2p_1 + p_1p_2p_1 \\ p_{212} &= p_2p_1(1-p_2) + (1-p_2)p_1p_2 + p_2p_1p_2\,. \end{aligned}$$

You have a better chance of winning the series in the first scenario if $p_{121} > p_{212}$, that is, if:

$$\begin{aligned} p_1p_2(1-p_1) + (1-p_1)p_2p_1 + p_1p_2p_1 &\overset{?}{>} p_2p_1(1-p_2) + (1-p_2)p_1p_2 + p_2p_1p_2 \\ -p_1p_2p_1 &\overset{?}{>} -p_2p_1p_2 \\ p_1 &\overset{?}{<} p_2\,. \end{aligned}$$

By assumption $p_1 > p_2$ so you should choose the second scenario.

Solution 2

The result is counter-intuitive. Intuitively, you should choose to play two games with P_1 and one game with P_2 because more likely to win games against P_1. However, the only way that you can win the series is by winning the *middle* game, and, therefore, you should play the middle set against P_1, the player you are more likely to defeat.

Simulation

```
For p1 = 0.6, p2 = 0.5
Proportion of P_121 wins = 0.4172
Proportion of P_212 wins = 0.4422

For p1 = 0.6, p2 = 0.4
Proportion of P_121 wins = 0.3405
Proportion of P_212 wins = 0.3881

For p1 = 0.6, p2 = 0.2
Proportion of P_121 wins = 0.1699
Proportion of P_212 wins = 0.2187
```

Explain why the proportions don't add up to 1.

3. The flippant juror ^{S}

There are two options to reach a decision: (a) A three-person panel consisting of two members who independently make the correct decision with probability p and one member who makes the correct decision with probability $1/2$. The final decision is determined by a majority vote. (b) A one-person panel whose only member has probability p of making the correct decision. Which option has the higher probability of making the correct decision?

Solution

The three-person panel makes the correct decision if all three members make the correct decision or if any subset of two members makes the correct decision. The probability is:

$$\overbrace{\left(p \cdot p \cdot \frac{1}{2}\right)}^{\textsf{all three correct}} \quad + \quad \overbrace{\left(p(1-p) \cdot \frac{1}{2} + (1-p)p \cdot \frac{1}{2} + p \cdot p \cdot \frac{1}{2}\right)}^{\textsf{two out of three correct}} = p\,,$$

so there is no difference between the two options.

Simulation

```
Prediction: probabilities of (a) and (b) are equal
For p = 0.25, proportion correct of (a) = 0.5019, (b) = 0.5046
For p = 0.50, proportion correct of (a) = 0.5072, (b) = 0.4970
For p = 0.75, proportion correct of (a) = 0.5062, (b) = 0.5040
```

4. Trials until first success ^{S}

What is the expectation of the number of throws of a die until a 6 appears?

Solution 1

The probability that the ith throw will be the first occurrence of 6 is the probability of $i-1$ throws of one of the other five numbers times the probability that the ith throw will give 6. To simplify the notation we use p for $1/6$:

$$P(\text{first 6 on } i\text{th throw}) = (1-p)^{i-1}p\,,$$

With a run of bad "luck" the number of throws until a 6 appears could be very large so it is unbounded.

Let $E = E(\text{first throw of } 6)$. Then:

$$E = 1p(1-p)^0 + 2p(1-p)^1 + 3p(1-p)^2 + 4p(1-p)^3 + \cdots = \sum_{i=1}^{\infty} ip(1-p)^{i-1}\,. \tag{6}$$

Without the i the sum would be the probability of eventually throwing a 6:

$$P(\text{eventually throwing a 6}) = \sum_{i=1}^{\infty} p(1-p)^{i-1} = p \cdot \frac{1}{1-(1-p)} = 1\,. \tag{7}$$

This is not a surprising result.

The calculation of the expectation can be performed as follows:

$$\begin{array}{rcccccccc} E & = & p(1-p)^0 & + & p(1-p)^1 & + & p(1-p)^2 & + & p(1-p)^3 & + \cdots \\ & & & & p(1-p)^1 & + & p(1-p)^2 & + & p(1-p)^3 & + \cdots \\ & & & & & & p(1-p)^2 & + & p(1-p)^3 & + \cdots \\ & & & & & & & & p(1-p)^3 & + \cdots \end{array}$$

The first row is the sum of the geometric series from Equation 7 which is 1. The second row is the same infinite geometric series except that the first element is $p(1-p)$ so its sum is:

$$\frac{p(1-p)}{1-(1-p)} = 1-p\,.$$

Similarly, the sum of the third row will be $(1-p)^2$ and the sum of the ith row will be $(1-p)^{i-1}$. Therefore, the expectation is the sum of the infinite geometric series:

$$E = 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots = \frac{1}{1-(1-p)} = \frac{1}{p} = 6\,.$$

Solution 2

Multiply Equation 6 by $1-p$ and subtract the result from that equation. The result is the geometric series in Equation 7:

$$\begin{aligned}
E &= p(1-p)^0 + 2p(1-p)^1 + 3p(1-p)^2 + 4p(1-p)^3 + \cdots \\
E\cdot(1-p) &= \phantom{p(1-p)^0 + {}} p(1-p)^1 + 2p(1-p)^2 + 3p(1-p)^3 + \cdots \\
E\cdot(1-(1-p)) &= p + p(1-p)^1 + p(1-p)^2 + p(1-p)^3 + \cdots \\
&= 1 \\
E &= 1/p\,.
\end{aligned}$$

Since $p = 1/6$ the expectation of the number of throws until a 6 appears is 6.

Solution 3

Consider the first throw separately from the rest of the throws. If the first throw is a 6 (probability p) then one throw is sufficient. Otherwise, if the first throw is not a 6 (probability $1-p$), then the remaining throws form a sequence identical to the original one so the expectation of this sequence is E. The expectation is therefore:

$$\begin{aligned}
E &= 1p + (E+1)(1-p) \\
E &= \frac{1}{p} = 6\,.
\end{aligned}$$

Simulation

```
Expectation of first success = 6
Average of first success     = 6.0161
```

5. Coin in a square ^{S}

Q1: Given a square of side 8 and a coin of radius 3, throw the coin onto the square. The location of the center of the coin is uniformly distributed within the square. What is the probability that the coin lands entirely within the square?

Q2: For each throw you win 5 if the coin lands within the square and lose 1 if it touches the square. What is the expectation of your winnings for each throw?

Q3: Develop a formula for the probability of the coin landing within the square if the side of the square is a and the radius of the coin is $r < a/4$.

Solution

A1: Figure 1 shows a square of side 8 and four circles of radius 3 inscribed within the corners of the square. The centers of the circles form an inner square of side 2. Any coin whose center is outside the inner square will touch an edge of the outer square. Since the center of the coin

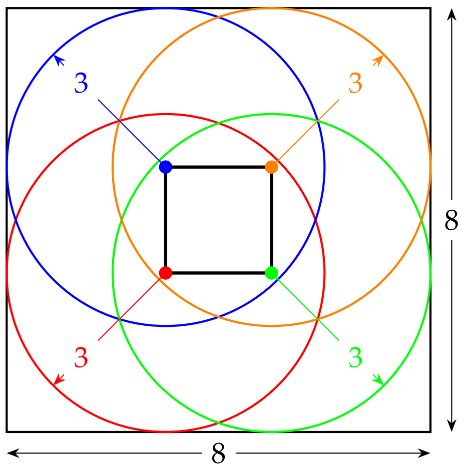


Figure 1: Boundaries for coins not intersecting a square

is uniformly distributed, the probability that the coin lands entirely within the square is the ratio of the area of the inner square to the area of the outer square:

$$P(\textsf{coin lands within the square}) = \frac{2 \cdot 2}{8 \cdot 8} = \frac{1}{16} = 0.0625\,.$$

A2: The expectation is negative:

$$E(\textsf{winnings per throw}) = 5 \cdot \frac{1}{16} + (-1) \cdot \frac{15}{16} = -\frac{10}{16} = -0.625\,.$$

A3: Figure 2 shows four circles inscribed in the corners of the square. The side of the inner square is $a - 2r$ so the probability of landing in the inner square is:

$$P(\textsf{coin lands within the square}) = \frac{(a-2r)^2}{a^2}\,.$$

Simulation

```
For side = 8, radius = 1:
Probability of landing within the square = 0.5625
Proportion landing within the square     = 0.5704
For side = 8, radius = 2:
Probability of landing within the square = 0.2500
Proportion landing within the square     = 0.2481
For side = 8, radius = 3:
Probability of landing within the square = 0.0625
Proportion landing within the square     = 0.0639
For side = 8, radius = 4:
Probability of landing within the square = 0.0000
Proportion landing within the square     = 0.0000
```

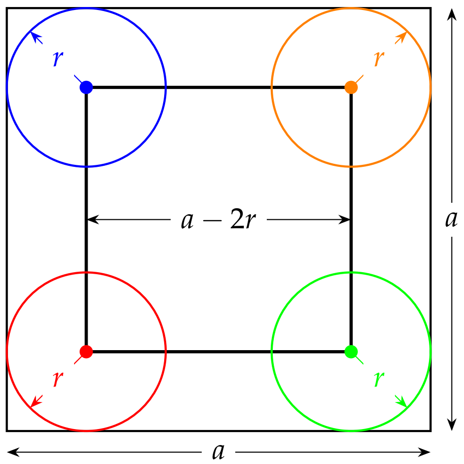


Figure 2: Coins in a large square

6. Chuck-a-luck ^{S}

Choose number n between 1 and 6. Throw three dice. If n does not appear on any of the dice you lose 1; if n appears on one die you win 1; if n appears on two dice you win 2; if n appears on three dice you win 3. What is the expectation of your winnings?

Solution

Let $P(k)$ be the probability of n appearing on k dice. The expectation of your winnings is:

$$E(\text{winnings per throw}) = -1 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3)\,.$$

The throws of the three dice are independent so each of these probabilities is given by the binomial distribution with $p = 1/6$, the probability that n appears on a die:

$$\begin{aligned}
E(\text{winnings per throw}) &= (-1)\cdot\binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 + 1\cdot\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 + \\
&\quad 2\cdot\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 + 3\cdot\binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 \\
&= \frac{1}{216}(-125 + 75 + 30 + 3) \\
&= -\frac{17}{216} \approx -0.0787\,.
\end{aligned}$$

Simulation

```
Expectation of winnings = -0.0787
Average winnings        = -0.0724
```

7. Curing the compulsive gambler ^{S}

Roulette is a game played with a wheel having 38 numbered pockets: 18 red, 18 black and 2 green.[1] The wheel is spun and a ball lands in one of the pockets. The casino wins if the ball lands in a green pocket; otherwise, you win 36 for each 1 bet on the number of the (red or black) pocket where the ball lands. You play 36 rounds of roulette, betting 1 in each round.

Q1: What is the expectation your winnings?

Q2: Your friend offers to bet you 20 that after 36 rounds you will have *lost* money. What is the expectation of your winnings, taking into account the money won or lost from both the game and your friend's bet?

Solution

A1: The probability of winning a single round is $1/38$ and the expectations are:

$$\begin{aligned} E(\text{winning one round}) &= 35 \cdot \frac{1}{38} + (-1) \cdot \frac{37}{38} = -\frac{2}{38} \approx -0.0526 \\ E(\text{winning 36 rounds}) &= 36 \cdot -0.05266 = -1.8947 . \end{aligned}$$

(Your net win is 35 because the 36 you receive includes your bet of 1 which is returned.)

A2: Consider the four outcomes of playing roulette for 36 rounds:

- If you lose all the rounds, you *lose* 36.
- If you win one round, you win 35 and lose 35 on the other rounds for a net win of 0.
- If you win two rounds, you win 70 and lose 34 on the other rounds for a net win of 36.
- If you win k rounds for $2 < k \leq 36$, your net win is $35k - (36 - k) > 0$.

Therefore, you lose the bet only if you lose all rounds:

$$P(\text{losing 36 rounds}) = \left(\frac{37}{38}\right)^{36} \approx 0.3829 .$$

The probability of not losing is $1 - 0.3829 = 0.6171$. Therefore the expectation of the playing 36 rounds together with bet is:

$$\overbrace{-1.8947}^{\text{E of all rounds}} + \overbrace{-20 \cdot 0.3829}^{\text{lose bet}} + \overbrace{20 \cdot 0.6171}^{\text{win bet}} \approx 2.7904 .$$

Clearly you should take the bet!

Simulation

```
Expectation of winning a round = -0.0526
Average winnings for a round   = -0.0593
```

[1]There are two green pockets in American roulette and one green pocket in European roulette.

The simulation showed a large variance which was reduced by running one million trials.

8. Perfect bridge hand

Randomly select 13 cards from a deck. What is the probability that they will all be of the same suit?

Solution 1

Since there are 13 cards of each suit there are $\binom{52}{13}$ ways of selecting 13 cards of a single suit, say hearts. Only one of them consists of 13 hearts so:

$$P(\text{selecting 13 hearts}) = \frac{1}{\binom{52}{13}} = \frac{13!39!}{52!} =\approx 1.5747 \times 10^{-12}\,.$$

There are four suits so:

$$P(\text{selecting 13 cards of the same suit}) = 4 \cdot \frac{13!39!}{52!} \approx 6.2991 \times 10^{-12}\,.$$

Solution 2

There are 52 ways of selecting the first card from the 52 cards in the deck. Then there are 12 ways of selecting the second card of the same suit from the remaining 51 cards, 11 ways of selecting a third card, and so on. Therefore:

$$P(\text{selecting 13 cards of the same suit}) = \frac{52}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdots \frac{1}{40} = \frac{12!}{51!/39!} \approx 6.2991 \times 10^{-12}\,.$$

There is no point in running a simulation because the result would almost certainly be zero.

9. Craps ^{D,S}

Craps is a played with a pair of dice. On the first throw you win if the sum of the numbers is 7 or 11 and you lose if the sum is 2, 3 or 12. If the sum on the first throw is $n = 4, 5, 6, 8, 9, 10$ (called a *point*), continue to throw the dice until the sum is n (a win) or 7 (a loss).

Q1: What are the probabilities of the events on the first throw: winning, losing, neither?

Q2: What is the probability of a win?

Solution 1

A1: The probability of an outcome in a throw of a die is uniformly distributed and equal to $1/6$. Since the outcomes of a throw of a pair of dice are independent, the probability of any outcome is $1/36$. The number of ways of obtaining each of the outcomes (sums) $2, \dots, 12$ is:

Sum	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	4	5	6	5	4	3	2	1

On the first throw there are 8 ways of throwing 7 or 11 for a probability of $8/36$ for winning and 4 ways of throwing $2, 3, 12$ for a probability of $4/36$ for losing. The probability of neither winning nor losing on the first throw is:

$$1 - \frac{8}{36} - \frac{4}{36} = \frac{24}{36}\,.$$

A2: Consider two cases referring to the table above:

- The point is 4. The probability of winning on the second throw (a 4) is $3/36$ and the probability of losing (a 7) is $6/36$. The probability of neither winning nor losing is $1 - (3/36) - (6/36) = 27/36$.
- The point is 8. The probability of winning on the second throw (an 8) is $5/36$ and the probability of losing (a 7) is $6/36$. The probability of neither winning nor losing is $1 - (5/36) - (6/36) = 25/36$.

The probability of winning must be computed separately for each of the points $4, 5, 6, 8, 9, 10$. We develop a general formula for the probability.

Starting with the *second throw* after obtaining a point on the first throw, let P_n be the probability of winning by throwing the point n and let Q_n the probability of neither winning nor losing. W_n, the probability of winning *by eventually throwing the point n*, is computed by adding:

- The probability of throwing the point on the second throw.
- The probability of neither winning nor losing on the second throw and throwing the point on the third throw.
- The probability of neither winning nor losing on the second and third throws and throwing the point on the fourth throw.
- ....

$$\begin{aligned}
W_n &= P_n + Q_n P_n + Q_n^2 P_n + Q_n^3 P_n + \cdots \\
&= P_n \left(1 + Q_n^1 + Q_n^2 + Q_n^3 + \cdots\right) \\
&= P_n \left(\frac{1}{1 - Q_n}\right).
\end{aligned}$$

You lose the game on any throw after the first if you throw a 7 with probability $6/36$, so Q_n, the probability of neither winning nor losing, is:

$$Q_n = 1 - P_n - (6/36)\,,$$

and:

$$W_n = \frac{P_n}{P_n + (6/36)}\,.$$

W_n for the six points are:

n	4	5	6	8	9	10
P_n	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$
W_n	$\frac{3}{9}$	$\frac{4}{10}$	$\frac{5}{11}$	$\frac{5}{11}$	$\frac{4}{10}$	$\frac{3}{9}$

W, the probability of winning, can be computed by adding the probability of winning on the first throw to the sum of the probabilities for the six wins on points each multiplied by the probability of throwing *that point* on the first throw:

$$W = \frac{8}{36} + \sum_{n\in\{4,5,6,8,9,10\}} P_n W_n \approx 0.4929\,. \tag{8}$$

The casino's probability of winning a single round of craps is only $0.5 - 0.4949 \approx 0.5\%$ but the law of large numbers ensures that they will eventually win and you will eventually lose!

Solution 2

A2: Consider the following sequences of throws, where in all sequences the point is 4.

```
4 8 9 9 9 8 8 8 9 8 4
4 8 9 9 9 8 8 8 9 8 7
4 9 9 9 8 8 4
```

The games only terminates if a 4 is thrown (win) or a 7 is thrown (loss), so throwing 8s or 9s doesn't affect the result! Therefore, once a point has been thrown, the chance of winning is simply the conditional probability that a 4 is thrown given that a 4 or a 7 is thrown. Let f be the event that a 4 is thrown and s be the event that a 7 is thrown. Then:

$$P(f|f\cup s) = \frac{f\cap(f\cup s)}{f\cup s} = \frac{f}{f\cup s} = \frac{3/36}{(3+6)/36} = \frac{3}{9}\,,$$

exactly the result W_4 in the table above. Equation 8 can now be used to compute W.

Conditional probability is implicitly used in the first solution because W_n is conditional on the first throw resulting in the point n.

Simulation

```
Probability of winning = 0.4929
Proportion of wins     = 0.4948
```

13. The prisoner's dilemma ^{D}

Three prisoners A, B, C are eligible for parole. The parole board will release two of them with equal probability for $\{A, B\}, \{A, C\}, \{B, C\}$, so the probability that A will be released is $2/3$. Prisoner A is told the name of one of the other prisoners who will be released. If he is told that prisoner B will be released, what is the probability that A will be released?

For a comprehensive article on the Prisoner's Dilemma Problem and the related Monty Hall Problem see [2].

Solution 1

Let $P(A), P(B), P(C)$ be the probabilities that A, B, C are released. A is interested in the conditional probability $P(A|B)$ of his being released if B will be released. The following computation seems to be what we want:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{2/3} = \frac{1}{2}\,.$$

Let $P(R)$ be the probability that A will be told that B will be released. Then:

$$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{1/3}{1/2} = \frac{2}{3}\,.$$

Here is the justification of this computation. We know that $P(A) = 2/3$. $P(R) = P(R|B)P(B)$, where $P(B) = 2/3$. $P(R|B) = 1/2$ since even if B is released, the probability is only $1/2$ that A will be told that B will be released, because if the prisoners to be released are $\{B, C\}$, A could be told C instead.

Being told that B will be released does not change the probability that A will be released.

Solution 2 There are *four* outcomes in the sample space:

e_1**:** The guard says B and $\{A, B\}$ released.

e_2**:** The guard says C and $\{A, C\}$ released.

e_3**:** The guard says B and $\{B, C\}$ released.

e_4**:** The guard says C and $\{B, C\}$ released.

Each pair of prisoners has equal probability of being released so:

$$P(e_1) = P(e_2) = P(e_3 \cup e_4) = \frac{1}{3}\,.$$

We assume that if $\{B, C\}$ are to be released the guard answers B or C with equal probability, so $P(e_3) = P(e_4) = 1/6$. Therefore, $P(A|B)$, the probability that A will be released given that the guard says that B will be released, is:

$$P(A|B) = \frac{P(e_1 \cap (e_1 \cup e_3))}{P(e_1 \cup e_3)} = \frac{P(e_1)}{P(e_1 \cup e_3)} = \frac{1/3}{(1/3) + (1/6)} = \frac{2}{3},$$

since e_1 and e_2 are mutually exclusive.

Solution 3

A riddle attributed to Abraham Lincoln asks: "If you call the tail of a dog a leg, how many legs does it have?" The answer is that calling a tail a leg doesn't make it a leg, so the dog still has four legs. Clearly, whether A knows B's future doesn't change his chances of being released so the probability is still $2/3$.

14. Collecting coupons ^{S}

Given a sequence of boxes each of which contains coupons numbered 1 to 5. You randomly draw one coupon from each box.

Q1: What is the expectation of the number of coupons drawn until you have all five of the numbers?

Q2: Develop a formula for the expectation for n numbers.

Hint: Use the solution to Problem 4 on page 10 and the approximation of harmonic numbers (page 82).

Solution

A1: What is the expectation of the number of draws until you get a number that is *different from* the previous ones? By Problem 4 this is $1/p$ where p is the probability of drawing a different number. For the first draw the probability is 1 so the expectation is also 1, for the second draw the probability is $4/5$ so expectation is $5/4$, and so on. Therefore:

$$E(\text{all five numbers}) = \frac{5}{5} + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + \frac{5}{1} = \; = \frac{1370}{120} \approx 11.4167\,.$$

A2: Use the same method and the approximation for H_n, the nth harmonic number (page 82):

$$E(\text{all } n \text{ numbers}) = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots \frac{1}{2} + \frac{1}{1}\right) = nH_n \approx n\left(\ln n + \frac{1}{2n} + 0.5772\right)\,.$$

For $n = 5$ this gives:

$$E(\text{all five numbers}) = 5H_5 \approx 5(\ln 5 + \frac{1}{10} + 0.5772) \approx 11.4332\,.$$

Simulation

```
For  5 coupons:
Expectation of draws = 11.9332
Average draws        = 11.4272
For 10 coupons:
Expectation of draws = 29.7979
Average draws        = 29.2929
For 20 coupons:
Expectation of draws = 72.4586
Average draws        = 72.2136
```

15. The theater row ^{S}

Arrange eight even numbers and seven odd numbers randomly in a row, for example:

$$10 \quad 12 \quad 3 \quad 2 \quad 9 \quad 6 \quad 1 \quad 13 \quad 7 \quad 10 \quad 3 \quad 8 \quad 8 \quad 5 \quad 20\,,$$

which we can write as follows since the specific numbers are not important:

$$E \quad E \quad O \quad E \quad O \quad E \quad O \quad O \quad O \quad E \quad O \quad E \quad E \quad O \quad E\,.$$

What is the expectation of the number of even-odd or odd-even adjacent pairs?

Hint: Consider each adjacent pair of separately. What is the probability that they are different?

Solution

Example: The following table shows the ten possible arrangements of three even numbers and two odd numbers. The total number of different adjacent pairs is 24 and the expectation is $24/10 = 2.4$.

Arrangement	Pairs	Arrangement	Pairs
EEEOO	1	*EEOEO*	3
EEOOE	2	*EOEOE*	4
EOEEO	3	*EOOEE*	1
OEEOE	3	*OEEEO*	2
OEOEE	3	*OOEEE*	1

We return to the problem with fifteen numbers. An adjacent pair can be EE, EO, OE, OO, so:

$$P(\textsf{one pair different}) = P(EO) + P(OE) = \frac{8}{15} \cdot \frac{7}{14} + \frac{7}{15} \cdot \frac{8}{14} = 2 \cdot \frac{8}{15} \cdot \frac{7}{14} = \frac{8}{15}\,,$$

and the number of equal adjacent pairs is $1 - (8/15) = 7/15$.

Since (EO, OE) pair contributes 1 to the count of different pairs and (EE, OO) contributes 0 to the count, the expectation of the number of different pairs in an arrangement is:

$$E(\textsf{one pair different}) = 1 \cdot \frac{8}{15} + 0 \cdot \frac{7}{15} = \frac{8}{15}\,.$$

For all fourteen pairs:

$$\begin{aligned}
E(\text{different pairs}) &= E\left(\sum_{\substack{\text{number}\\\text{of pairs}}} 1\cdot P(\text{one pair different})\right)\\
&= \sum_{\substack{\text{number}\\\text{of pairs}}} E(\text{one pair different}) = 14\cdot\frac{8}{15}\approx 7.4667\,,
\end{aligned}$$

by the linearity of expectation (page 81).

For ten numbers:

$$\begin{aligned}
P(\text{one pair different}) &= P(EO)+P(OE) = \frac{3}{5}\cdot\frac{2}{4}+\frac{2}{5}\cdot\frac{3}{4} = \frac{3}{5}\\
E(\text{different pairs}) &= 4\cdot\frac{3}{5} = \frac{12}{5} = 2.4\,,
\end{aligned}$$

the same result found by counting the pairs in the arrangements.

Simulation

```
For  5 places:
Expectation of different pairs = 2.4000
Average different pairs        = 2.3855
For 15 places:
Expectation of different pairs = 7.4667
Average different pairs        = 7.4566
For 27 places:
Expectation of different pairs = 13.4815
Average different pairs        = 13.4835
For 49 places:
Expectation of different pairs = 24.4898
Average different pairs        = 24.4725
```

16. Will the second-best be runner-up? ^{S}

Eight players in a tournament $\{a_1,\ldots,a_8\}$ are randomly assigned to play games $\{g_1,\ldots,g_8\}$ in a schedule such that a_{k_i} initially plays in position g_{k_i} (Figure 3). The players are ranked from the best a_1 to the worst a_8 and the better player will *always* defeat her opponent. Clearly a_1 will win the tournament.

Q1: What is the probability that a_2 will be the runner-up by playing a_1 in the final round and losing to her?

Q2: If there are 2^n players what is the probability that a_2 will be the runner-up by playing a_1 in the final round and losing to her?

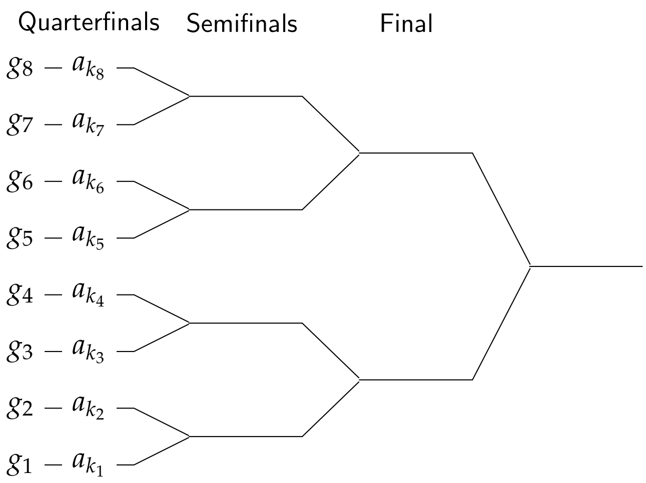


Figure 3: A tournament schedule

Solution

A1: If a_1 is assigned to one of the games $\{g_1, g_2, g_3, g_4\}$ none of the other players assigned to these games will reach the final, so a_2 must be assigned to one of $\{g_5, g_6, g_7, g_8\}$. The temptation is to conclude that the probability of a_2 being the runner-up is $1/2$ since he must be assigned to one of the 4 games $\{g_1, g_2, g_3, g_4\}$. However, if a_1 is first assigned a game, there are only 7 remaining games of which 4 will ensure that a_2 is runner up, so the probability is $4/7$. Similarly, if a_2 is first assigned a game, a_1 must be assigned one of the 4 out of 7 games not in a_2's group.

A2: Without loss of generality let a_1 be assigned to g_1. There remain $2^n - 1$ games and a_2 must be assigned to one of the 2^{n-1} games not in the same half as a_1. Therefore:

$$P(a_1, a_2 \text{ playing each other in the final}) = \frac{2^{n-1}}{2^n - 1}\,.$$

Simulation

```
For   8 players:
Probability a2 is runner-up                 = 0.5714
Proportion of games where a2 is runner-up  = 0.5707
For  32 players:
Probability a2 is runner-up                 = 0.5161
Proportion of games where a2 is runner-up  = 0.5184
For 128 players:
Probability a2 is runner-up                 = 0.5039
Proportion of games where a2 is runner-up  = 0.5060
```

17. Twin knights ^{D,S}

Eight players in a tournament $\{a_1, \ldots, a_8\}$ are randomly assigned to play games $\{g_1, \ldots, g_8\}$ in a schedule such that a_{k_i} initially plays in position g_{k_i} (Figure 3). For all i, j, the probability that a_i wins is $1/2$ as is the probability that a_j wins.

Q1: What is the probability that a_1, a_2 play each other?

Q2: If there are 2^n players, what is the probability that a_1, a_2 play each other?

Solution

A1: Arbitrarily assign a_1 to g_1. Consider the different possibilities that a_1, a_2 can play each other. With probability $1/7$, a_2 is assigned to g_2. With probability $2/7$, a_2 is assigned to g_3 or g_4, but he doesn't play a_1 unless *both* of them win their first game, so we need to multiply the probability of this assignment by $1/4$. With probability $4/7$, a_2 is assigned to g_5, g_6, g_7, g_8, but he doesn't play a_1 unless *both* of them win their first two games, so we need to multiply the probability of this assignment by $1/16$. Therefore:

$$P(a_1, a_2 \text{ play each other}) = \frac{1}{7} + \frac{1}{4} \cdot \frac{2}{7} + \frac{1}{16} \cdot \frac{4}{7} = \frac{1}{4}\,.$$

A2: Let P_n be the probability that in a 2^n tournament a_1 and a_2 play each other. We have shown that $P_3 = 1/4$. What about P_4? Using the same approach:

$$\begin{aligned} P_4 &= \frac{1}{15} + \frac{1}{4} \cdot \frac{2}{15} + \frac{1}{16} \cdot \frac{4}{15} + \frac{1}{64} \cdot \frac{8}{15} \\ &= \frac{1}{15}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{8}\,. \end{aligned}$$

It is reasonable to conjecture that $P_n = 1/2^{n-1}$.

Proof 1: Using the same approach and the formula for the sum of a geometric series:

$$\begin{aligned} P_n &= \frac{1}{2^n - 1} \sum_{i=0}^{n-1} 2^i \cdot \left(\frac{1}{2}\right)^{2i} \\ &= \frac{1}{2^n - 1} \sum_{i=0}^{n-1} 2^{-i} \\ &= \frac{1}{2^n - 1} \left(\frac{1 - \left(\frac{1}{2}\right)^{(n-1)+1}}{1 - \frac{1}{2}}\right) = \frac{1}{2^{n-1}}\,. \end{aligned}$$

Proof 2: By induction. The base case is $P_3 = 1/4 = 1/2^{3-1}$.

There are two inductive steps:

Case 1: a_1 and a_2 are assigned to different halves of the tournament:

$$P(a_1, a_2 \text{ assigned to different halves}) = \frac{2^{n-1}}{2^n - 1}\,.$$

They can only meet in the final game and therefore both must win all of their $n-1$ games:

$$P(a_1, a_2 \text{ meet if assigned to different halves}) = \frac{2^{n-1}}{2^n-1}\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right)^{n-1} = \frac{2^{-(n-1)}}{2^n-1}. \quad (9)$$

Case 2: a_1 and a_2 are assigned to the same half of the tournament:

$$P(a_1, a_2 \text{ assigned to the same half}) = \frac{2^{n-1}-1}{2^n-1}.$$

Since both players are in the same half the problem has been reduced to determining P_{n-1}. By the inductive hypothesis:

$$P(a_1, a_2 \text{ meet if assigned to the same half}) = \frac{2^{n-1}-1}{2^n-1}\cdot\frac{1}{2^{n-2}} = \frac{2^1-2^{-(n-2)}}{2^n-1}. \quad (10)$$

Combining Equations 9, 10 gives:

$$\begin{aligned} P_n &= \frac{2^{-(n-1)}}{2^n-1}+\frac{2^1-2^{-(n-2)}}{2^n-1} \\ &= \frac{2^{n-1}}{2^{n-1}}\cdot\frac{2^{-(n-1)}+2^1-2^{-(n-2)}}{2^n-1} \\ &= \frac{1}{2^{n-1}}\cdot\frac{2^0+2^n-2^1}{2^n-1} = \frac{1}{2^{n-1}}. \end{aligned}$$

Simulation

```
For   8 players:
Probability that a1, a2 meet = 0.2500
Proportion a1, a2 meet       = 0.2475
For  32 players:
Probability that a1, a2 meet = 0.0625
Proportion a1, a2 meet       = 0.0644
For 128 players:
Probability that a1, a2 meet = 0.0156
Proportion a1, a2 meet       = 0.0137
```

18. An even split at coin tossing ^{S}

Q1: If you toss a fair coin 20 times, what is the probability of obtaining exactly 10 heads?

Q2: If you toss a fair coin 40 times, what is the probability of obtaining exactly 20 heads?

Solution

A1: The probabilities of obtaining heads and tails are both $1/2$. The probability of obtaining an arrangement with 10 heads is:

$$P(\textsf{arrangement with 10 heads}) = \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{20} = \frac{1}{1048576}\,.$$

Using the binomial coefficient for the number of arrangements with 10 heads, the probability of obtaining 10 heads is:

$$P(\textsf{10 heads in 20 tosses}) = \binom{20}{10} \left(\frac{1}{2}\right)^{20} = \frac{184756}{1048576} \approx 0.1762\,.$$

A2: You might expect the probability to be the same as in the previous question, but:

$$P(\textsf{20 heads in 40 tosses}) = \binom{40}{20} \left(\frac{1}{2}\right)^{40} \approx 0.1254\,.$$

By the law of large numbers the numbers of heads and tails will be "roughly" equal [11, Section 8.4], but they are unlikely to be exactly the same, and this event becomes less likely as the number of tosses increases.

Simulation

```
Probability of 10 heads for  20 tosses = 0.1762
Proportion  of 10 heads for  20 tosses = 0.1790
Probability of 20 heads for  40 tosses = 0.1254
Proportion  of 20 heads for  40 tosses = 0.1212
Probability of 50 heads for 100 tosses = 0.0796
Proportion  of 50 heads for 100 tosses = 0.0785
```

19. Isaac Newton helps Samuel Pepys ^{S}

Q1: What is the probability of obtaining *at least one* 6 when 6 dice are thrown?

Q2: What is the probability of obtaining *at least two* 6s when 12 dice are thrown?

Q3: Develop a formula for the probability of obtaining at least n 6s when $6n$ dice are thrown.

Solution

A1: The probability is the complement of the probability of obtain zero 6s in 6 throws, which is the product of obtaining a number different from 6 in all throws:

$$P(\textsf{at least one 6}) = 1 - \left(\frac{5}{6}\right)^{6} \approx 0.6651\,.$$

A2: The probability is the complement of the probability of obtain zero or one 6s in 12 throws:

$$P(\textsf{at least two 6s}) = 1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1} \left(\frac{1}{6}\right)^{1} \left(\frac{5}{6}\right)^{11} \approx 0.6187\,.$$

This event is less probable than the previous one.

A3: The probability is the complement of the probability of obtain less than n 6s in $6n$ throws:

$$\begin{aligned}
P(\text{at least } n \text{ 6s}) &= 1 - \binom{6n}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{6n-0} - \binom{6n}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{6n-1} - \cdots \\
&= 1 - \sum_{i=0}^{n-1}\binom{6n}{i}\left(\frac{1}{6}\right)^{i}\left(\frac{5}{6}\right)^{6n-i}.
\end{aligned}$$

Simulation

```
For   6 dice to throw  1 sixes:
Probability = 0.6651
Proportion  = 0.6566
For  12 dice to throw  2 sixes:
Probability = 0.6187
Proportion  = 0.6220
For  18 dice to throw  3 sixes:
Probability = 0.5973
Proportion  = 0.5949
For  96 dice to throw 16 sixes:
Probability = 0.5424
Proportion  = 0.5425
For 360 dice to throw 60 sixes:
Probability = 0.5219
Proportion  = 0.5250
```

20. The three-cornered duel ^{S}

A, B, C fight a sequence of duels. Each of them has a fixed probability of winning a duel regardless of who the opponent is:

$$P(A) = 0.3, \quad P(B) = 1, \quad P(C) = 0.5\,.$$

A person who is hit no longer participates in the duels. The shots are fired one at a time sequentially in the order A, B, C. If two opponents are still standing the shooter can decide whom to fire at. *Assume that each person always makes an optimal decision.*

Q1: What is A's optimal strategy?

Q2: Suppose that A fires the first shot into the air. Is this a better strategy?

Hint: Compute the conditional probabilities of A winning depending on whether he chooses to shoot first at B or C.

Solution

Notation: $X \xrightarrow{H} Y$ denotes that X shoots at Y and hits. $X \xrightarrow{M} Y$ denotes that X shoots at Y and misses.

A1: Compute the conditional probabilities of A winning.

Case 1: A chooses to shoot first at C.

If $A \xrightarrow{M} C$ (probability 0.7) then $B \xrightarrow{H} C$ since C is more dangerous than A. A now shoots again at B with probability 0.3 of hitting, but if A misses then $B \xrightarrow{H} A$ with probability 1 and A loses.

If $A \xrightarrow{H} C$ (probability 0.3) then $B \xrightarrow{H} A$ with probability 1 and A loses.

Compute the expectation with 1 when A wins and 0 when A:

$$E(A \text{ wins } | A \text{ chooses to shoot first at } C) =$$

$$\overbrace{1 \cdot (0.7 \cdot 0.3)}^{A \xrightarrow{M} C, A \xrightarrow{H} B} + \overbrace{0 \cdot (0.7 \cdot 0.7 \cdot 1)}^{A \xrightarrow{M} C, A \xrightarrow{M} B, B \xrightarrow{H} A} + \overbrace{0 \cdot (0.3 \cdot 1)}^{A \xrightarrow{M} C, B \xrightarrow{H} A} = 0.2100\,.$$

Case 2: A chooses to shoot first at B.

If $A \xrightarrow{M} B$ (probability 0.7) then as before $B \xrightarrow{H} C$ and A has one more chance to hit B (probability 0.3), otherwise $B \xrightarrow{H} A$ with probability 1 and A loses.

If $A \xrightarrow{H} B$ (probability 0.3) then A, C alternately shoot at each other until one is hit. The possibilities are:

$$\begin{array}{ll}
(1) & C \xrightarrow{H} A \\
(2) & C \xrightarrow{M} A \xrightarrow{H} C \\
(3) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{H} A \\
(4) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{H} C \\
(5) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{H} A \\
(6) & C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{M} C \xrightarrow{M} A \xrightarrow{H} C \\
 & \cdots
\end{array}$$

The probability of A wins by eventually hitting C is the sum of the probabilities of the even-numbered scenarios in this list:

$$\begin{aligned}
P(A \text{ wins } | A \text{ hits } B) &= (0.5 \cdot 0.3) + \\
&\quad (0.5 \cdot 0.7)(0.5 \cdot 0.3) + \\
&\quad (0.5 \cdot 0.7)(0.5 \cdot 0.7)(0.5 \cdot 0.3) + \cdots \\
&= 0.15 \sum_{i=0}^{\infty} 0.35^i = \frac{0.15}{1 - 0.35} = \frac{3}{13} \approx 0.2308\,.
\end{aligned}$$

Similarly, the probability of C winning is $\dfrac{0.5}{1 - 0.35} = \dfrac{1}{13} \approx 0.0760$.

The expectation is:

$$E(A \text{ wins}) = E(A \text{ wins} \mid A \text{ misses } B) + E(A \text{ wins} \mid A \text{ hits } B) =$$

$$\overbrace{1 \cdot (0.7 \cdot 1 \cdot 0.3)}^{A \xrightarrow{M} B, B \xrightarrow{H} C, A \xrightarrow{H} B} + \overbrace{0 \cdot (0.7 \cdot 1 \cdot 0.7 \cdot 1)}^{A \xrightarrow{M} B, B \xrightarrow{H} C, A \xrightarrow{M} B, B \xrightarrow{H} A} + \overbrace{1 \cdot (0.2308)}^{A \xrightarrow{H} B, C \xleftrightarrow{H} * A, C \xrightarrow{H} A} + \overbrace{0 \cdot (0.3 \cdot (0.0769))}^{A \xrightarrow{H} B, C \xleftrightarrow{H} A, C \xrightarrow{H} A} \approx$$

$$0.2792\,,$$

which is higher than the expectation of winning by shooting at C first.

A2: If A shoots into the air not hitting anyone then $B \xrightarrow{H} C$ with probability 1 and A can try to hit B with probability 0.3. The expectation is:

$$E(A \text{ wins} | A \text{ shoots in the air}) = 1 \cdot (0.3) + 0 \cdot (0.7) = 0.3\,,$$

which is better than the expectation for the other two strategies!

Simulation

```
For A fires first at C:
Expectation of wins = 0.2100
Average wins        = 0.2138
For A fires first at B:
Expectation of wins = 0.2792
Average wins        = 0.2754
For A fires in the air:
Expectation of wins = 0.3000
Average wins        = 0.3000
```

21. Should you sample with or without replacement? ^{D,S}

Urn A contains 2 red balls and 1 green ball, and urn B contains 101 red balls and 100 green balls. An urn is chosen at random and two balls are randomly drawn *from the selected urn*. You win if you correctly identify whether the selected urn was A or B.

Which of the following rules gives you the highest probability of winning?

Q1: The first ball is replaced before the second drawing.

Q2: The first ball is not replaced before the second drawing.

Q3: After the first ball is drawn you can decide whether it will be replaced or not.

Hint: When computing probabilities:

$$\frac{101}{201} \approx \frac{100}{201} \approx \frac{100}{200} \approx \frac{1}{2}\,.$$

Solution

There are four outcomes which we denote by RR, RG, GR, GG. For each rule compute the conditional probabilities of the four outcomes given that urn A or urn B was selected initially. These multiply by $1/2$ to take into account the random selection of the urn.

A1: Drawing with replacement:

$$\begin{array}{lcccc}
P(RR|A) & = & \frac{2}{3}\cdot\frac{2}{3} & = & \frac{4}{9} \\
P(RR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(RG|A) & = & \frac{2}{3}\cdot\frac{1}{3} & = & \frac{2}{9} \\
P(RG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GR|A) & = & \frac{1}{3}\cdot\frac{2}{3} & = & \frac{2}{9} \\
P(GR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GG|A) & = & \frac{1}{3}\cdot\frac{1}{3} & = & \frac{1}{9} \\
P(GG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4}
\end{array}$$

If the outcome is RR there is a higher probability that urn A was selected ($4/9$) than that urn B was selected ($1/4$); otherwise, there is a higher probability that urn B was selected:

$$P(\mathsf{winning}) = \frac{1}{2}\left(\frac{4}{9}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) = \frac{43}{72} \approx 0.5972\,.$$

A2: Drawing without replacement:

$$\begin{array}{lcccc}
P(RR|A) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(RG|A) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GR|A) & = & \frac{1}{3}\cdot 1 & = & \frac{1}{3} \\
P(GR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GG|A) & = & \frac{1}{3}\cdot 0 & = & 0 \\
P(GG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4}
\end{array}$$

If the outcome is GG there is (of course!) a higher probability that urn B was selected than that urn A was selected; otherwise, there is a higher probability that urn A was selected. The probability of winning is:

$$P(\mathsf{win}) = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}\right) = \frac{5}{8} = 0.6250\,,$$

which is greater than the probability of winning when sampling with replacement.

A3: The decision is based on the outcome of the first draw.

If the first drawing is from urn A the probabilities must be conditioned on the decision to sample with or without replacement. Drawing first from urn B does not affect the probabilities because of the approximation in the hint.

$$\begin{array}{lcccc}
P(RR|A,w) & = & \frac{2}{3}\cdot\frac{2}{3} & = & \frac{4}{9} \\
P(RR|A,w/o) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(RG|A,w) & = & \frac{2}{3}\cdot\frac{1}{3} & = & \frac{2}{9} \\
P(RG|A,w/o) & = & \frac{2}{3}\cdot\frac{1}{2} & = & \frac{1}{3} \\
P(RG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GR|A,w) & = & \frac{1}{3}\cdot\frac{2}{3} & = & \frac{2}{9} \\
P(GR|A,w/o) & = & \frac{1}{3}\cdot 1 & = & \frac{1}{3} \\
P(GR|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4} \\
\hline
P(GG|A,w) & = & \frac{1}{3}\cdot\frac{1}{3} & = & \frac{1}{9} \\
P(GG|A,w/o) & = & \frac{1}{3}\cdot 0 & = & 0 \\
P(GG|B) & = & \frac{1}{2}\cdot\frac{1}{2} & = & \frac{1}{4}
\end{array}$$

If a red ball is drawn first then $\frac{4}{9} > \frac{1}{4}$ and $\frac{2}{9} < \frac{1}{4}$ with replacement, whereas $\frac{1}{3} > \frac{1}{4}$ and $\frac{1}{3} > \frac{1}{4}$ without replacement, so the second ball can help identify the urn only if the drawing is done *with* replacement: urn A if red, urn B if green. Choose the draw with replacement:

$$P(\textsf{winning if red first}) = \frac{1}{2}\left(\frac{4}{9} + \frac{1}{4}\right) = \frac{25}{72}\,.$$

If a green ball is drawn first then $\frac{2}{9} < \frac{1}{4}$ and $\frac{1}{9} < \frac{1}{4}$ with replacement, whereas $\frac{1}{3} > \frac{1}{4}$ and $0 < \frac{1}{4}$ without replacement, so the second ball can help identify the urn only if the drawing is done *without* replacement: urn A if red, urn B if green. Choose the draw without replacement:

$$P(\textsf{winning if green first}) = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{24}\,.$$

The total probability of winning is:

$$P(\textsf{winning}) = \frac{25}{72} + \frac{7}{24} = \frac{23}{36} \approx 0.6389\,.$$

The highest probability of winning is obtained when the decision to draw with or without replacement depends on the result of the first draw.

Simulation

```
With replacement:
Expectation of winning = 0.5972
Average wins           = 0.5976
Without replacement:
Expectation of winning = 0.6250
Average wins           = 0.6207
Decide after first draw:
Expectation of winning = 0.6389
Average wins           = 0.6379
```

22. The ballot box ^{S}

In an election there are two candidates A and B. A receives a votes and B receives b votes, $a > b$. The votes are counted one-by-one and the running totals $(a_i, b_i), 1 \leq i \leq a + b$ are updated as each vote is counted. What is the probability that for at least one i, $a_i = b_i$?

Q1: Solve for $a = 3, b = 2$ by listing (a_i, b_i) for $1 \leq i \leq 5$.

Q2: Solve the problem for all $a > b$.

Hint 1: What can you say about which candidate leads until the *first* tie occurs?

Hint 2: What is the significance of the first vote counted?

Solution

A1: The number of arrangements of running totals is $\binom{5}{2} = \binom{5}{3} = 10$, because the positions of the votes for one candidate determine the positions of the votes for the other candidate. The following table lists the possible arrangements of the votes and of running totals with first ties emphasized:

A	A	A	B	B	$(1,0)$	$(2,0)$	$(3,0)$	$(3,1)$	$(3,2)$
A	A	B	A	B	$(1,0)$	$(2,0)$	$(2,1)$	$(3,1)$	$(3,2)$
A	B	A	A	B	$(1,0)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(3,1)$	$(3,2)$
B	A	A	A	B	$(0,1)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(3,1)$	$(3,2)$
A	A	B	B	A	$(1,0)$	$(2,0)$	$(2,1)$	$(\mathbf{2},\mathbf{2})$	$(3,2)$
A	B	A	B	A	$(1,0)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(2,2)$	$(3,2)$
B	A	A	B	A	$(0,1)$	$(\mathbf{1},\mathbf{1})$	$(2,1)$	$(2,2)$	$(3,2)$
A	B	B	A	A	$(1,0)$	$(\mathbf{1},\mathbf{1})$	$(1,2)$	$(2,2)$	$(3,2)$
B	A	B	A	A	$(0,1)$	$(\mathbf{1},\mathbf{1})$	$(1,2)$	$(2,2)$	$(3,2)$
B	B	A	A	A	$(0,1)$	$(0,2)$	$(1,2)$	$(\mathbf{2},\mathbf{2})$	$(3,2)$

There are ties in all the arrangements except for the first two so:

$$P(\text{tie occurs with } (3,2) \text{ votes}) = \frac{8}{10} = \frac{4}{5}\,.$$

A2: The following discussion indicates how to approach the second question. List arrangements for A, B with $(3,2)$ votes until the *first tie* occurs:

A leads until tie	B leads until tie
A B	B A
A A B B	B B A A

For every arrangement where A leads until the first tie there is a mirror image arrangement where B leads until the first tie which is obtained by exchanging A's and B's.

Before the first tie one of the candidates must be leading. If the first vote counted is for B there must be a tie since $a > b$. The probability that the first vote is for b is:

$$P(\text{first vote for } B) = \frac{b}{a+b}\,.$$

By mirroring the positions of the votes, the number of sequences resulting in a tie that begin with a vote for A is the same as the number of sequences resulting in a tie that begin with a vote for B. But we just computed the latter probability so the probability of a tie is:

$$P(\text{tie occurs}) = 2 \cdot \frac{b}{a+b}\,.$$

Check:

$$P(\text{tie occurs with } (3,2) \text{ votes}) = 2 \cdot \frac{2}{2+3} = \frac{4}{5}\,.$$

Simulation

```
For a =  3, b =  2:
Probability of a tie = 0.8000
Proportion of ties   = 0.8118
For a = 10, b =  8:
Probability of a tie = 0.8889
Proportion of ties   = 0.8977
For a = 20, b = 18:
Probability of a tie = 0.9474
Proportion of ties   = 0.9354
```

23. Ties in matching pennies ^{D,S}

Toss a pair of fair coins N times, N even, and keep count of how many times the parity is even (heads-heads, tails-tails) and how many times the parity is odd (heads-tails, tails-heads). What is the probability of obtaining a tie (not counting the $0-0$ tie at the start)?

Q1: Solve for $N = 4$ by writing out all the possible outcomes.

Q2: Solve for $N = 6$ by developing a formula for the probability.

Q3: Develop a formula for arbitrary even N.

Q4: Explain why the probability for the odd number $N + 1$ is the same as the probability for the even number N.

Hint: Use the solution of Problem 22.

Solution

A1: Denote tosses with even parity by E and tosses with odd parity by O. Ten out of the sixteen arrangements of tosses have ties (emphasized):

EEEE	EEEO	EEOE	**EEOO**	**EOEE**	**EOEO**	**EOOE**	**EOOO**
OEEE	**OEEO**	**OEOE**	**OEOO**	**OOEE**	OOEO	OOOE	OOOO

A2: By Problem 22:

$$P(\text{tie on toss } i) = \begin{cases} 2i/N & \text{if } i \leq N/2 \\ 2(N-i)/N & \text{if } i \geq N/2\,, \end{cases} \tag{11}$$

since the ballot box problem showed that the smaller value determines the probability.

The following computations are quite complex so we justify each step in detail.

The probability of i evens is given by the binomial coefficient:

$$P(i \text{ evens}) = \binom{N}{i}\left(\frac{1}{2}\right)^{i}\left(\frac{1}{2}\right)^{N-i} = \binom{N}{i}\left(\frac{1}{2}\right)^{N} = 2^{-N}\binom{N}{i}\,. \tag{12}$$

The probability of a tie is the sum over i of the probability of obtaining i evens times the probability of a tie on the ith toss (Equation 11). For $N = 6$:

$$P(\text{ties}) = 2 \cdot 2^{-6} \left[\frac{0}{6}\binom{6}{0} + \frac{1}{6}\binom{6}{1} + \frac{2}{6}\binom{6}{2} + \frac{3}{6}\binom{6}{3} + \frac{2}{6}\binom{6}{4} + \frac{1}{6}\binom{6}{5} + \frac{0}{6}\binom{6}{6} \right]. \tag{13}$$

Equation 14 follows from Equation 13 by deleting the two zero terms, expressing the combinations as factorials, canceling $\frac{1}{6}$ from 6!:

$$P(\text{ties}) = 2^{-5} \left[1 \cdot \frac{5!}{1!5!} + 2 \cdot \frac{5!}{2!4!} + 3 \cdot \frac{5!}{3!3!} + 2 \cdot \frac{5!}{4!2!} + 1 \cdot \frac{5!}{5!1!} \right]. \tag{14}$$

Equation 15 is obtained by canceling i from $i!$:

$$P(\text{ties}) = 2^{-5} \left[\frac{5!}{1!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} + \frac{5!}{4!1!} + \frac{5!}{5!1!} \right]. \tag{15}$$

To obtain Equation 16 from Equation 15 add and subtract $\frac{5!}{3!2!}$:

$$P(\text{ties}) = 2^{-5} \left[\left(\frac{5!}{1!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} + \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!1!} \right) - \frac{5!}{3!2!} \right]. \tag{16}$$

Equation 17 results from replacing 1! by 0!:

$$P(\text{ties}) = 2^{-5} \left[\left(\frac{5!}{0!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} + \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!} \right) - \frac{5!}{3!2!} \right]. \tag{17}$$

By expressing the factorials back as combinations we obtain Equation 18:

$$P(\text{ties}) = 2^{-5} \left[\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} - \binom{5}{3} \right]. \tag{18}$$

Finally, Equation 19 results from the binomial theorem:

$$P(\text{ties}) = 2^{-5} \left(2^5 - 10 \right) = \frac{11}{16} \approx 0.6875. \tag{19}$$

A3: Perform the same calculations as in **A2:** but using arbitrary N. The result is:

$$P(\text{ties}) = 2^{-N+1} \left[2^{N-1} - \binom{N-1}{N/2} \right] = \left[1 - \binom{N-1}{N/2} \Big/ 2^{N-1} \right].$$

A4: The first tie on the $N + 1$'st toss occurs only if the counts are nearly equal after the Nth toss:

$$\begin{array}{l} ((N/2) - 1, (N/2) + 1) \\ ((N/2), (N/2)) \\ ((N/2) + 1, (N/2) - 1) \end{array}$$

but whatever the outcome of the final toss the counts will not be equal.

Simulation

```
For  4 tosses:
Probability of ties = 0.6250
Proportion of ties  = 0.6192
For  6 tosses:
Probability of ties = 0.6875
Proportion of ties  = 0.6900
For  7 tosses:
Probability of ties = 0.6875
Proportion of ties  = 0.6811
For 10 tosses:
Probability of ties = 0.7539
Proportion of ties  = 0.7559
For 20 tosses:
Probability of ties = 0.8238
Proportion of ties  = 0.8255
```

25. Lengths of random chords ^{S}

Select a random chord in the unit circle. What is the probability that the length of the chord is greater than 1?

To solve the problem you first have to decide what "select a random chord" means. Solve the problem for each of the following possibilities:

Q1: The distance of the chord from the center is uniformly distributed in the range $(0, 1)$.

Q2: The midpoint of the chord is uniformly distributed within the circle.

Q3: The endpoints of the chord are uniformly distributed on the circumference of the circle.

Solution

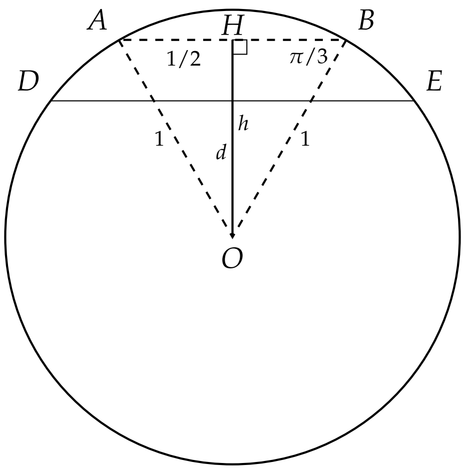
A1: A chord is larger than the radius if it is closer to the center than a chord of length 1. Let $\overline{AB}$ be a chord of length 1 and construct the altitude $\overline{OH}$ from O to the chord (Figure 4a). Since $\triangle AOB$ is equilateral, $\triangle OHB$ is a right triangle and the length of the altitude is:

$$h = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\,.$$

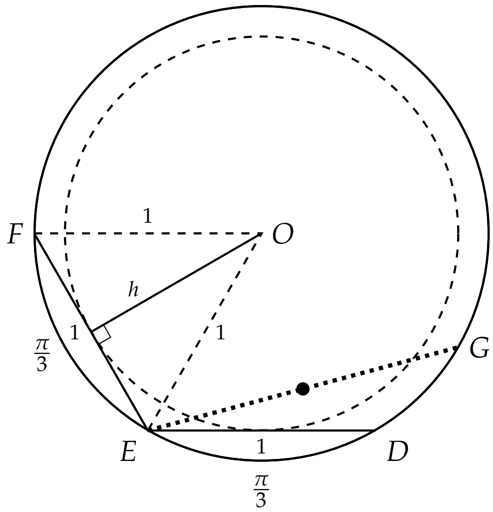
Let d be the distance of a chord $\overline{DE}$ from the center and by assumption d is uniformly distributed in $(0, 1)$. Then:

$$P(\overline{DE} > 1) = P(d < h) = \frac{h}{1} = \frac{\sqrt{3}}{2} \approx 0.866\,.$$

A2: Consider any point on a circle of radius h, the altitude to a chord of length 1. A tangent to this point will be a chord $\overline{FE}$ whose length is 1. Any chord $\overline{EG}$ whose midpoint is within



(a) Distance of chord from center uniformly distributed in $(0, 1)$



(b) Midpoint of chord uniformly distributed within the circle and endpoints of chord uniformly distributed on the circumference

this circle will have a length greater than 1 (Figure 4b). The probability is therefore the ratio of the areas of the two circles:

$$P(\overline{EG} > 1) = \frac{\pi \cdot h^2}{\pi \cdot 1^2} = h^2 = \frac{3}{4},$$

which happens to be the square of the probability computed in the first answer.

A3: To select the two endpoints of a chord, first arbitrarily choose one point (E in Figure 4b). Any other point determines a chord whose length is greater than one unless that point falls on the arcs $\widehat{EF}$ or $\widehat{ED}$. The probability is therefore the ratio of the arc $\widehat{FD}$ to the circumference of the unit circle:

$$P(\overline{EG} > 1) = \frac{(2\pi - (2\pi/3)) \cdot 1}{2\pi \cdot 1} = \frac{2}{3}.$$

Simulation

The simulation is for choosing two random points on the circumference.

```
Probability of long chords = 0.6667
Proportion of long chords  = 0.6627
```

26. The hurried duelers ^{S}

A and B arrive at a meeting point at a random time with uniform distribution within a one-hour period. If A arrives first and B does not arrive within 5 minutes, A leaves. Similarly, if B arrives first and A does not arrive within 5 minutes, B leaves. What is the probability that they meet?

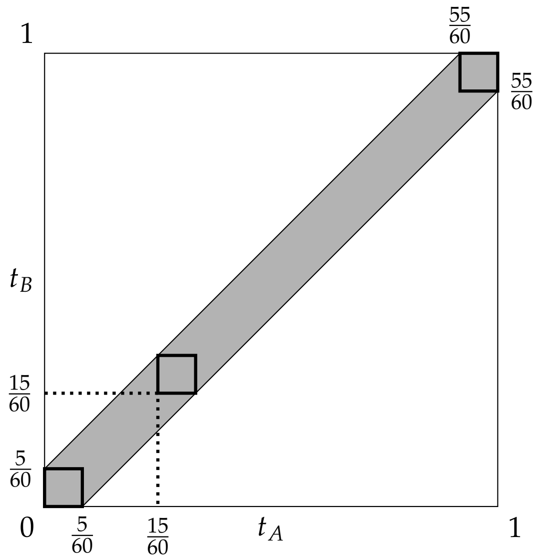


Figure 5: Times that ensure a meeting between A and B

Time within the one-hour period is *continuous* in the range $[0,1]$, that is, you cannot *count* a discrete number of minutes or seconds to compute probabilities. You can compute the probabilities of *durations*.

Hint: Draw a graph with A's time of arrival as the the x-coordinate and B's time of arrival as the y-coordinate.

Solution

Without loss of generality A arrives first. If A arrives at $t = 0$ and if B arrives before $t = 5/60$ they meet, otherwise they do not. This is shown in Figure 5 by the small square at the origin. If A arrives later then B also has to arrive later by the same amount; for example, if A arrives at 15, B must arrive between 15 and 20. Therefore, the meeting will take place during a square of time obtained by moving the small square by 15 from $(0,0)$ to $(15/60, 15/60)$.

The probability that a meeting will occur is the ratio of the area of the graph colored gray to the area of the large square. It is easier to compute the complement which is the area of the white triangles in the upper left and lower right:

$$\begin{aligned} P(A, B \text{ meet}) &= 1 - P(A, B \text{ don't meet}) \\ &= 1 - 2 \cdot \left(\frac{1}{2} \cdot \frac{55}{60} \cdot \frac{55}{60}\right) = \frac{23}{144} \approx 0.1597\,. \end{aligned}$$

Simulation

```
Probability of meeting   = 0.1597
Proportion of meetings   = 0.1549
```

27. Catching the cautious counterfeiter ^{S}

There are n boxes each with n coins and one coin in each box is counterfeit. Draw one coin from each box and test it to determine whether it is counterfeit or not. What is the probability that all the coins that are drawn are are real?

Q1: Solve for $n = 10$.

Q2: Solve for $n = 100$.

Q3: Solve for arbitrary n.

Q4: What is the limit, if it exists, of the probability as n tends to infinity?

Solution

The draws are independent so the probability is the product of the probabilities for each draw.

A1:

$$P(\text{all real}) = \left(\frac{9}{10}\right)^{10} = 0.3487\,.$$

A2:

$$P(\text{all real}) = \left(\frac{99}{100}\right)^{100} = 0.3660\,.$$

A3:

$$P(\text{all real}) = \left(\frac{n-1}{n}\right)^{n}\,.$$

A4:

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = \frac{1}{e} \approx 0.3679\,. \tag{20}$$

This limit can be proved using differential calculus. First we compute the limit of the natural logarithm of the lefthand side of Equation 20:

$$\lim_{n\to\infty}\ln\left(1-\frac{1}{n}\right)^{n} = \lim_{n\to\infty} n\ln\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)}{1/n}\,.$$

Taking the limit gives $\ln(1)/0 = 0/0$ but by l'Hôpital's rule we can replace it by the quotient of the derivatives:

$$\begin{aligned}\lim_{n\to\infty}\ln\left(1-\frac{1}{n}\right)^{n} &= \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)^{-1}(-(-n^{-2}))}{-n^{-2}} = -1\\ \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} &= e^{-1}\,.\end{aligned}$$

Simulation

```
For  10 boxes:
Probability of all real = 0.3487
Proportion all real     = 0.3480
For 100 boxes:
Probability of all real = 0.3660
Proportion all real     = 0.3730
For 200 boxes:
Probability of all real = 0.3670
Proportion all real     = 0.3690
```

28. Catching the greedy counterfeiter ^{S}

There are n boxes each with n coins and m coins in each box are counterfeit. Draw one coin from each box and test it to determine whether it is counterfeit or not. What is the probability $P(n,m,r)$ that r of the coins that are drawn are *counterfeit*?

Q1: Develop a formula for $P(n,m,r)$.

Q2: Compute $P(20,10,2), P(20,10,8), P(20,5,2), P(20,5,4)$.

Solution

A1: There are $\binom{n}{r}$ choices of boxes from which the counterfeit coins can be drawn. From the binomial distribution:

$$P(n,m,r) = \binom{n}{r}\left(\frac{m}{n}\right)^{r}\left(\frac{n-m}{n}\right)^{n-r}.$$

A2:

$$\begin{aligned}
P(20,10,2) &= \binom{20}{2}\left(\frac{10}{20}\right)^{2}\left(\frac{10}{20}\right)^{18} \approx 0.0002\\
P(20,10,8) &= \binom{20}{8}\left(\frac{10}{20}\right)^{8}\left(\frac{10}{20}\right)^{12} \approx 0.1201\\
P(20,5,2) &= \binom{20}{2}\left(\frac{5}{20}\right)^{2}\left(\frac{15}{20}\right)^{18} \approx 0.0669\\
P(20,5,4) &= \binom{20}{4}\left(\frac{5}{20}\right)^{4}\left(\frac{15}{20}\right)^{12} \approx 0.1952\,.
\end{aligned}$$

Mosteller shows that in as n tends to infinity, for fixed m, r the probability is:

$$\lim_{n\to\infty} P(n,m,r) = \frac{e^{-m}m^{r}}{r!}\,. \tag{21}$$

Simulation

```
For 10 bad coins,  2 draws:
Probability of counterfeit  = 0.0002
Proportion counterfeit      = 0.0002
For 10 bad coins,  8 draws:
Probability of counterfeit  = 0.1201
Proportion counterfeit      = 0.1181
For  5 bad coins,  2 draws:
Probability of counterfeit  = 0.0669
Proportion counterfeit      = 0.0688
For  5 bad coins,  4 draws:
Probability of counterfeit  = 0.1897
Proportion counterfeit      = 0.1905
```

29. Moldy gelatin ^{S}

A rectangular plate is divided into n small squares. There are an average of r microbes in each square.

Q1: Develop a formula for probability that there are exactly r microbes in the n squares.

Q2: Compute the probability for $n = 100, r = 3$.

Hint: This problem is similar the Problem 28.

Solution

A1: Let p be the probability that a single square contains a microbe. (Ignore the possibility that a microbe is partially contained within two or more squares.) m, the average number of microbes per square, is the number of squares n times the probability p that a square contains a microbe. $P(n, m, r)$, the probability that there are exactly r microbes in the n squares is given by the binomial distribution:

$$P(n,m,r) = \binom{n}{r} \left(\frac{m}{n}\right)^r \left(\frac{n-m}{n}\right)^{n-r} .$$

A2:

$$P(10,3,3) = \binom{100}{3} \left(\frac{3}{100}\right)^3 \left(\frac{97}{100}\right)^{97} \approx 0.2275 .$$

Equation 21 also applies here:

$$\lim_{n\to\infty} P(n,3,3) = \frac{e^{-3} \cdot 3^3}{3!} \approx 0.2240 .$$

Simulation

```
For  20 squares:
Probability of exactly  3 microbes  = 0.2428
Proportion of exactly   3 microbes  = 0.2436
Probability of exactly  5 microbes  = 0.2023
Proportion of exactly   5 microbes  = 0.1954
For 100 squares:
Probability of exactly  3 microbes  = 0.2275
Proportion of exactly   3 microbes  = 0.2247
Probability of exactly  5 microbes  = 0.1800
Proportion of exactly   5 microbes  = 0.1851
```

31. Birthday pairings ^{S}

Randomly select 23 people and ask them what their birthdays are. Assume a uniform distribution of 365 different birthdays (no one was born on February 29). Show that the probability that at least two of them have the same birthday is greater than 0.5.

Solution

Compute the probability that *none* of the 23 people have the same birthday and show that it is less than 0.5. Select the first birthday arbitrarily, then the next birthday from the remaining days, then the next birthday from the remaining days, and so on:

$$\begin{aligned}
P(\textsf{no birthday pair}) &= \overbrace{\frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365}\cdot\;\cdots\;\cdot\frac{344}{365}\cdot\frac{343}{365}}^{\textsf{23 fractions}}\\
&= \frac{365!}{365^{23}\cdot 342!}\approx 0.4927\,.
\end{aligned}$$

Most people guess that more than 23 people are needed to find two with the same birthday!

A modern calculator can compute the probability, but it is worthwhile computing it using Stirling's approximation $\ln n! \approx n\ln n - n$:

$$\begin{aligned}
\ln P(\textsf{no birthday pair}) &= \ln\left(\frac{365!}{342!\cdot 365^{23}}\right) = \ln 365! - \ln 342! - 23\ln 365\\
&\approx (365\ln 365 - 365) - (342\ln 342 - 342) - 23\,\ln 365\\
&\approx -0.7404\\
P(\textsf{no birthday pair}) &\approx e^{-0.7404} = 0.4769\,.
\end{aligned}$$

The reader is invited to compute the probability with the following better approximation:

$$\ln n! \approx n\ln n - n + \frac{1}{6}\left(8n^3+4n^2+n+\frac{1}{30}\right) + \frac{1}{2}\ln\pi\,.$$

Simulation

```
For 21 people:
Expectation of no pairs = 0.5563
Average no pairs        = 0.5497
For 22 people:
Expectation of no pairs = 0.5243
Average no pairs        = 0.5237
For 23 people:
Expectation of no pairs = 0.4927
```

```
Average no pairs         = 0.4933
For 24 people:
Expectation of no pairs = 0.4617
Average no pairs         = 0.4576
For 25 people:
Expectation of no pairs = 0.4313
Average no pairs         = 0.4345
```

32. Finding your birthmate ^{S}

Your *birthmate* is a person with the same birthday as yours.

Why is finding a birthmate different from finding a birthday pairing?

Q1: How many people do you have to ask until the probability of finding your brithmate is greater than 0.5?

Q2: Solve by using the approximation in Equation 20 on page 38.

Solution

Many people could have the same birthday which is considered a success for find a birthday pairing, but not for finding a birthmate unless that birthday is the same as yours.

A1: Find the smallest number of people for which the probability that *none* of theme are birthmates is less than 0.5. The probability that the first person you ask is not a birthmate is 364/365, but that is also the probability that the second, third, . . . , person is not a birthmate. The solution is the smallest k such that:

$$P(\text{not a birthmate}) = \left(\frac{364}{365}\right)^k < \frac{1}{2},$$

which is $k = 253$:

$$\left(\frac{364}{365}\right)^{253} \approx 0.4995\,.$$

A2: Equation 20 is:

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n = \frac{1}{e},$$

which can be used to approximate the probability:

$$\begin{aligned}
P(\text{not a birthmate}) &= \left(\frac{365-1}{365}\right)^k = \left[\left(\frac{364}{365}\right)^{365}\right]^{k/365} \\
&\approx e^{-k/365} \\
e^{-253/365} &\approx 0.5000\,.
\end{aligned}$$

Simulation

```
For 251 people:
Probability of no match = 0.5023
Proportion no match     = 0.5120
For 252 people:
Probability of no match = 0.5009
Proportion no match     = 0.5055
For 253 people:
Probability of no match = 0.4995
Proportion no match     = 0.4984
For 254 people:
Probability of no match = 0.4982
Proportion no match     = 0.4987
For 255 people:
Probability of no match = 0.4968
Proportion no match     = 0.5078
```

33. Relating the birthday pairings and the birthmate problems

Let $P_{\mathsf{pair}}(r)$ be the probability that two people out of r are a birthday pair (Problem 31) and let $P_{\mathsf{mate}}(n)$ be the probability that two people out of n are birthmates (Problem 32). Given r for what n does $P_{\mathsf{pair}}(r) \approx P_{\mathsf{mate}}(n)$?

Solution 1

The solution is based on [7].

Using the notation $P_{\mathsf{no\ pair}}(r)$ for the complement, from the solution to Problem 31 we have:

$$\begin{aligned}
P_{\mathsf{no\ pair}}(r) &= \frac{365}{365} \cdot \frac{365-1}{365} \cdot \frac{365-2}{365} \cdot \cdots \cdot \frac{365-(r-1)}{365} \\
&= 1\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right) \cdot \cdots \cdot \left(1-\frac{r-1}{365}\right) \\
&\approx 1-\frac{1}{365}-\frac{2}{365}-\cdots-\frac{r-1}{365} \\
&= 1-\frac{1+2+3+\cdots+(r-1)}{365} \\
&= 1-\frac{r(r-1)/2}{365},
\end{aligned}$$

where the approximation in the third equation results from deleting powers of $1/365$ greater than one because they are too small to significantly affect the result.

Using the notation $P_{\text{no mate}}(n)$ for the complement and the same approximation, from the solution to Problem 32 we have:

$$\begin{aligned}
P_{\text{no mate}}(n) &= \overbrace{\left(1-\frac{1}{365}\right)\left(1-\frac{1}{365}\right)\cdots\left(1-\frac{1}{365}\right)}^{n} \\
&\approx 1-\overbrace{\frac{1}{365}-\frac{1}{365}\cdots-\frac{1}{365}}^{n} \\
&\approx 1-\frac{n}{365}
\end{aligned}$$

Therefore $P_{\text{no pair}}(r) \approx P_{\text{no mate}}(n)$ when:

$$n = \frac{r(r-1)}{2}\,.$$

For $r = 23$, $n = (23 \cdot 22)/2 = 253$.

Solution 2

Mosteller [7, p. 322] gives the following intuitive solution:

> In comparing the birthday and birthmate problems, one observes that for r people in the birthday problem, there are $r(r-1)/2$ pairs or *opportunities* for like birthdays; whereas, if n people are questioned in the birthmate problem, there are only n opportunities for me to find one or more birthmates.

From this he concludes that $n \approx r(r-1)/2$.

This reasoning can be understood as follows. For the birthday problem choose an arbitrary date and ask if two people out of r have *that* birthday. There are:

$$\binom{r}{2} = \frac{r!}{2!(r-2)!} = \frac{r(r-1)}{2}$$

ways of doing so. For the birthmate problem your own birthday is given. Any of the n people can have the same birthday. By equating the two we have the n such that $P_{\text{pair}}(r) \approx P_{\text{mate}}(n)$.

You can run the simulations for Problems 31, 32 and check this result.

34. Birthday holidays ^{D,S}

A factory is closed whenever one of its workers has a birthday. There are no other holidays.

Q1: How many workers should be employed in order to maximize the number of work-days in a year?

Q2: What is the expectation of the ratio of the maximum work-days to 365^2, the number of possible work-days if each one of 365 workers worked every day?

Hint: Prove that there must be a maximum by considering extreme cases. Then develop a formula for the expectation of the number of work-days for a single day.

Solution

A1: At one extreme if there is only one worker there are 364 work-days. If there are two workers there are $363 + 363 = 726$ workers days (ignoring the very small possibility that both workers have the same birthday). At the other extreme if there are one million workers the number of work-days will almost certainly be zero. Since the number of work-days rises initially and then returns to zero, there must be a maximum in between.

To simplify the notation we will denote the number of days in a year by N and the number of workers by n.

For any given day the probability that it is a work-day is the probability that each worker has a birthday on some other day:

$$P(\text{a given day is a work-day}) = \overbrace{\frac{N-1}{N} \cdot \cdots \cdot \frac{N-1}{N}}^{n} = \left(1 - \frac{1}{N}\right)^{n}.$$

Denote $\left(1 - \frac{1}{N}\right)$ by p.

The expectation of the number of work-days for a given day is:

$$E(\text{work-days for a given day}) = n \cdot p^n + 0 \cdot (1 - p^n) = np^n .$$

All the days in the year have this same expectation, so we just multiply by N to get the expectation for a year:

$$E(\text{work-days for a year}) = Nnp^n . \tag{22}$$

To find the maximum we take the derivative of Equation 22 with respect to n and use $(p^n)' = p^n \ln p$ which can be proved using the chain rule:

$$(p^n)' = ((e^{\ln p})^n)' = (e^{n \ln p})' = e^{n \ln p}(n \ln p)' = (e^{\ln p})^n \ln p = p^n \ln p .$$

The derivative of Equation 22 is therefore:

$$(Nnp^n)' = N(p^n + n(p^n)') = N(p^n + np^n \ln p) ,$$

which is 0 when:

$$n = -\frac{1}{\ln p} .$$

For $N = 365$ this gives $n = 364.5$. Since n is a positive integer the maximum is achieved at $n = 364$ or $n = 365$ which give the same expectation of the number of work-days:

$$\begin{aligned} E(\text{work-days for a year}) &= Nnp^n \\ &= 365 \cdot 364 \cdot \left(\frac{364}{365}\right)^{364} \end{aligned}$$

$$
\begin{aligned}
&= 365 \cdot 364 \cdot \frac{365}{365}\left(\frac{364}{365}\right)^{364} \\
&= 365 \cdot 365 \cdot \left(\frac{364}{365}\right)^{365} \\
&= 48944\,.
\end{aligned}
$$

A2: The expectation of the ratio is:

$$
E(\mathsf{max\ work\text{-}days/possible\ work\text{-}days}) = \frac{365 \cdot 365 \cdot \left(\frac{364}{365}\right)^{365}}{365 \cdot 365} = \left(\frac{364}{365}\right)^{365} \approx 0.3674\,.
$$

By Equation 20:

$$
\lim_{n\to\infty} E(\mathsf{max\ work\text{-}days/possible\ work\text{-}days}) = \lim_{N\to\infty}\left(1-\frac{1}{N}\right) = \frac{1}{e}\,.
$$

Simulation

```
For 100 people
Expectation work-days    = 27742
Average work days        = 27743
Ratio work-days / 365**2 = 0.2082
For 250 people
Expectation work-days    = 45958
Average work days        = 45939
Ratio work-days / 365**2 = 0.3450
For 364 people
Expectation work-days    = 48944
Average work days        = 48936
Ratio work-days / 365**2 = 0.3674
For 365 people
Expectation work-days    = 48944
Average work days        = 48917
Ratio work-days / 365**2 = 0.3674
```

35. The cliff-hanger ^{S}

A particle is initially placed on the x-axis at position 1. At any position on the x-axis it can move right with probability $2/3$ and left with probability $1/3$ (Figure 6).

Q1: What is the probability that the particle will eventually be at position 0?

Q2: If the probability of moving right is p and the probability of moving left is $1-p$, what is the probability that the particle will eventually be at position 0? Analyze the result for various values of p.

Hint: Use conditional probabilities after the first move.

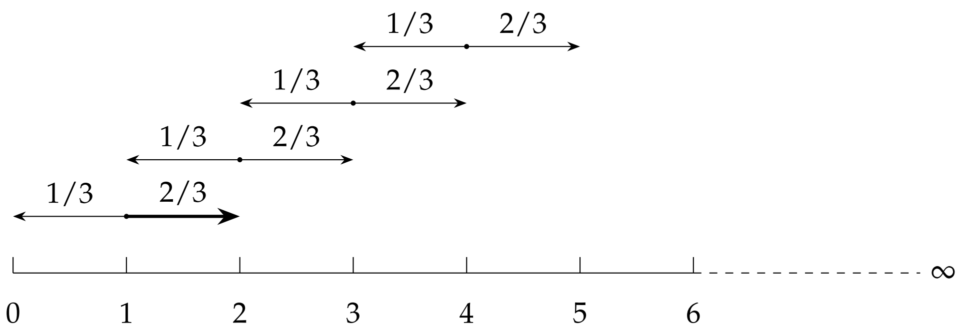


Figure 6: Can the particle return to 0 (infinite line)?

Solution

It is just as easy to compute the probabilty for arbitrary p as it is for $p = 2/3$.

A1,2: Let us try to compute the probability directly. Denote a move left by L and a move right by R. The particle can reach 0 directly by moving L with probability $\frac{1}{3}$, or by moving RLL with probability $\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$, or by moving $RRLLL$ with probability $\left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^3, \ldots$. This seems to be a straightforward geometric progression, but it ignores possibilities such as $RLRLL$.

Compute the probability that the particle reaches 0 from 1 conditioned on the first step:

$$\begin{aligned} P(\textsf{reaches 0 from 1}) &= P(\textsf{reaches 0 from 1}|\textsf{first move left}) + \\ &\quad P(\textsf{reaches 0 from 1}|\textsf{first move right}) \\ &= (1-p) \cdot 1 + pP(\textsf{reaches 1 from 2})P(\textsf{reaches 0 from 1})\,. \end{aligned}$$

But the probability of reaching 1 from 2 is exactly the same as the probability of reaching 0 from 1. Abbreviating $P(\textsf{reaches 0 from 1})$ as P we have:

$$\begin{aligned} P &= (1-p) + pP^2 \\ pP^2 - P + (1-p) &= 0 \\ P &= \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} \\ P &= 1,\ (1-p)/p\,. \end{aligned}$$

If $p \leq 1/2$ then $(1-p)/p \geq 1$, so $P = 1$ is the only solution and it is certain that the particle will reach 0.

If $p = 1$ then $P = 0$ since if the particle always moves to the right it cannot return to 0.

Suppose $P = 1$ for $1/2 < p < 1$, that is, *P does not depend on p*. Then P cannot suddenly "jump" from 1 to 0 as p approaches 1: in Figure 7 the dashed red line and the red dot at $(1,0)$. Therefore, for $p > 1/2$ the only solution is $P = (1-p)/p < 1$.[2]

For $p = 2/3, P = 1/2$ and for $p = 1/2, P = 1$. This is a surprising result because one would not expect that the particle would always return to 0 if the direction of the moves were

[2]Mosteller writes that this follows by continuity but does not give a proof.

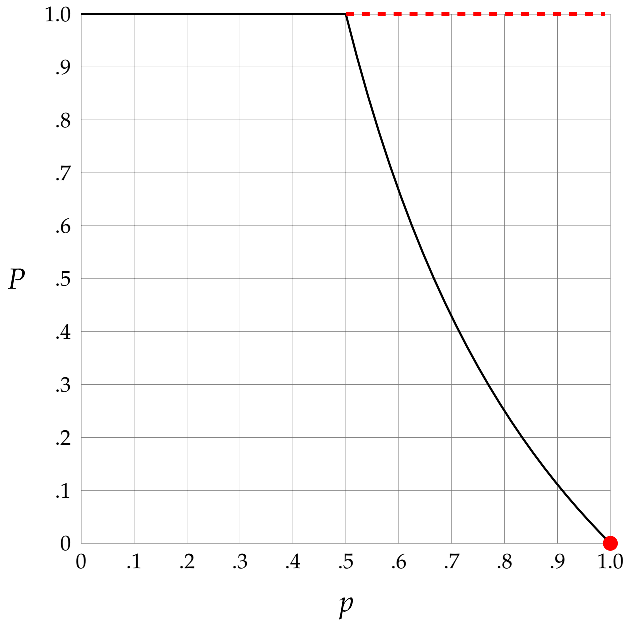


Figure 7: Graph of $P = \min(p/(1-p), 1)$ for $p \in [0, 1]$

determined by flipping a fair coin! You have to have a very unfair coin (probability of heads is $2/3$) to even the chances or returning to 0 or not.

Simulation

```
For probability = 0.2500:
Probability of reaching 0 = 1.0000
Proportion reaching 0     = 1.0000
For probability = 0.5000:
Probability of reaching 0 = 1.0000
Proportion reaching 0     = 0.9612
For probability = 0.6667:
Probability of reaching 0 = 0.5000
Proportion reaching 0     = 0.5043
For probability = 0.7500:
Probability of reaching 0 = 0.3333
Proportion reaching 0     = 0.3316
For probability = 0.8000:
Probability of reaching 0 = 0.2500
Proportion reaching 0     = 0.2502
```

36. Gambler's ruin ^{D,S}

A particle is initially placed on the x-axis at position $m \geq 1$. At any position on the x-axis it can move right with probability $p > 1/2$ and left with probability $1 - p$.

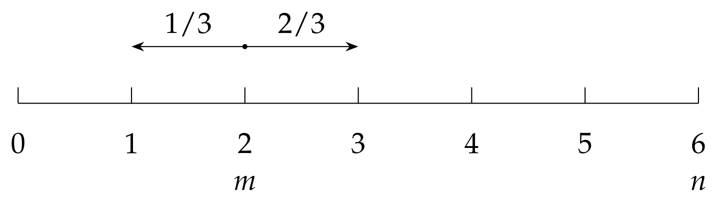


Figure 8: Can the particle return to 0 (finite line)?

Q1: What is the probability that the particle will eventually be at position 0?

Q2: Let $n > m$. If the particle reaches position 0 or position n its stops moving. What is the probability that the particle will eventually be at position 0? What is the probability that the particle will eventually be at position n?

Note: Problem 35 is represents a player with a finite amount of money betting against a casino with unlimited money. The problem asks for the probabiliy that the player loses all his money. Of course the casino never runs out of money. This problem represents one player who starts with m betting against a second player who starts with $n - m$. The problem asks for the probabilities that *one* of the players loses all her money to the other player.

Solution

The solution is based on [11, Chapter 2, Example 4m].

A1: The solution to Problem 35 showed that for $p > 1/2$ (which is true here by assumption), if a particle is at position 1 the probability of its reaching position 0 is $r = (1-p)/p$. Notation: let $P(i, j)$ be the probability of reaching i from j. Since the claim does not depend on the actual position of the particle, the probability of a particle reaching 0 from position m is:

$$P(0, m) = P(m-1, m)P(m-2, m-1) \cdots P(1, 2)P(0, 1) = r^m . \tag{23}$$

A2: Let $P_i = P(n, i)$ and compute it using conditional probability:

$$\begin{aligned} P_i &= pP_{i+1} + (1-p)P_{i-1} \\ pP_{i+1} &= 1 \cdot P_i - (1-p)P_{i-1} \\ pP_{i+1} &= (p + (1-p))P_i - (1-p)P_{i-1} \\ p(P_{i+1} - P_i) &= (1-p)(P_i - P_{i-1}) \\ P_{i+1} - P_i &= r(P_i - P_{i-1}) . \end{aligned}$$

$P_0 = 0$ since if the particle is at 0 it does not move. Therefore:

$$\begin{aligned} P_2 - P_1 &= r(P_1 - P_0) = rP_1 \\ P_3 - P_2 &= r(P_2 - P_1) = r^2 P_1 \\ \cdots &= \cdots \\ P_i - P_{i-1} &= r(P_{i-1} - P_{i-2}) = r^{i-1} P_1 . \end{aligned}$$

Most of the terms on the lefthand sides cancel when we add the equations:

$$\begin{aligned}
P_i - P_1 &= P_1 \sum_{j=2}^{i} r^{j-1} \\
&= P_1 + P_1 \sum_{j=2}^{i} r^{j-1} - P_1 \\
P_i &= P_1 \sum_{j=0}^{i-1} r^j = P_1 \left(\frac{1-r^i}{1-r} \right) .
\end{aligned}$$

If the particle is at n then it is already at n so $P_n = 1$:

$$\begin{aligned}
1 &= P_1 \left(\frac{1-r^n}{1-r} \right) \\
P_1 &= \left(\frac{1-r}{1-r^n} \right) ,
\end{aligned}$$

and therefore (using a symmetrical argument exchanging p and $1-p$):

$$P(n,i) = \left(\frac{1-r^i}{1-r^n} \right) \tag{24}$$

$$P(0,i) = \left(\frac{1-(1/r)^{n-i}}{1-(1/r)^n} \right) . \tag{25}$$

We leave it to the reader to show that the sum of Eqs. 24, 25 is 1 meaning that one of the players will certainly win and one will lose.

For $m = 1, n = 3, p = 2/3$:

$$\begin{aligned}
P(0,1) &= \left(\frac{1-\left(\frac{1}{2}\right)^1}{1-\left(\frac{1}{2}\right)^3} \right) = \frac{4}{7} \\
P(3,1) &= \left(\frac{1-2^2}{1-2^3} \right) = \frac{3}{7} .
\end{aligned}$$

Simulation

```
For probability = 0.6667:
Probability of reaching (0,10) from 1 = (0.4995,0.5005)
Proportion reaching     (0,10) from 1 = (0.5056,0.4944)
Probability of reaching (0,10) from 4 = (0.0616,0.9384)
Proportion reaching     (0,10) from 4 = (0.0643,0.9357)
Probability of reaching (0,10) from 6 = (0.0147,0.9853)
Proportion reaching     (0,10) from 6 = (0.0123,0.9877)

For probability = 0.7500:
Probability of reaching (0,10) from 1 = (0.3333,0.6667)
Proportion reaching     (0,10) from 1 = (0.3395,0.6605)
```

```
Probability of reaching (0,10) from 4 = (0.0123,0.9877)
Proportion reaching     (0,10) from 4 = (0.0115,0.9885)
Probability of reaching (0,10) from 6 = (0.0014,0.9986)
Proportion reaching     (0,10) from 6 = (0.0015,0.9985)
```

The greater the amount of money that the left player has and the greater his probability of winning each bet, the higher his probability of winning.

37. Bold play vs. cautious play ^{S}

In roulette you can bet that the ball will fall into a pocket with an even number. The probability is $18/38$ since there are 18 even numbers, 18 odd numbers and 2 green numbers where the casino wins.

Which of the following strategies is better?

1. Bold play: betting 20 in one round.

2. Cautious play: 1 per round until you win or lose 20.

Hint: Use the results of Problem 36.

Solution

The probability of winning with bold play is $18/38 \approx 0.4737$.

The probability of winning with cautious play is $P(40, 20)$ with $r = \frac{20}{38} / \frac{18}{38} = \frac{20}{18}$ (Equation 24):

$$P(\text{reaches 40 from 20}) = \left(\frac{1 - (20/18)^{20}}{1 - (20/18)^{40}}\right) \approx 0.1084\,.$$

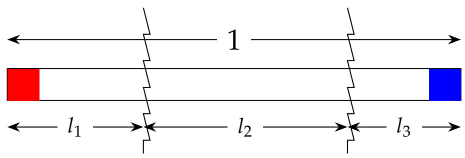
Clearly, bold play is preferable to cautious play.

Mosteller writes that the intuitive explanation for this result is that betting in more rounds exposes the player to the probability of $2/38$ that the casino wins.

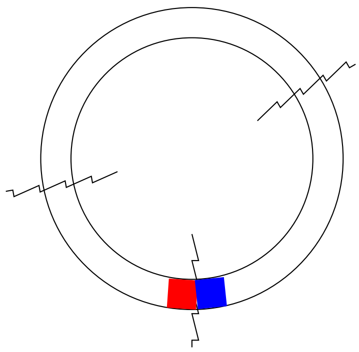
Simulation

```
Probability of bold wins     = 0.4737
Proportion bold wins         = 0.4677
Probability of cautious wins = 0.1084
Proportion cautious wins     = 0.1094
```

39. The clumsy chemist ^{S}



(a) Breaking a rod into three pieces



(b) Breaking a ring into three pieces

You have a large number of glass rods of length 1 with one end colored red and the other colored blue. When you drop them on the floor they each break into three pieces with a uniform distribution of the length of the pieces (Figure 9a). What is the expectation of the length of the piece whose end is colored blue?

Hint: Instead of straight rods suppose that you are given (unmarked) glass rings that also break into three pieces (Figure 9b).

Solution 1

The rods are not symmetric because the end pieces are different from the center piece. However, the ring is symmetric so the distributions of all three pieces must be uniform with expectation $1/3$. By choosing and coloring one of breaks, the problem is now the same as that of the rods so the distributions remain the same. Therefore the expectation of the breaks in the rod is also $1/3$.

Solution 2

Here is an elegant solution from [4].

Assume that the rod represents the line segment $(0, 1)$. The rod is broken in two places which are represented as two uniform independent random variables $X, Y \in (0, 1)$. Let us compute the probability $P(|X - Y| > a)$.

Table 1 shows points (x, y), where $x, y \in \{0.0, 0.1, 0.2, \ldots, 0.9\}$ and the decimal point is omitted. The values that appear in the table are $|X - Y|$. In the table, for $a = 0.6$ the points in the upper left and lower right corners of $[0, 1] \times [0, 1]$: $(0, 6)$–$(6, 9)$ and above, $(6, 0)$–$(9, 6)$ and below, are those outcomes that define $P(|X - Y| > a)$. The area of one corner is $\frac{1}{2}(1 - a)(1 - a)$, so:

$$P(|X - Y| > a) = 2 \cdot \frac{1}{2}(1 - a)(1 - a) = (1 - a)^2 .$$

For $a = 0.6$, $P(|X - Y| > 0.6) = (0.4)^2 = 0.16$.

a

	9	9	8	7	6	5	4	3	2	1	0	
	8	8	7	6	5	4	3	2	1	0	1	
	7	7	6	5	4	3	2	1	0	1	2	
a	6	6	5	4	3	2	1	0	1	2	3	
	5	5	4	3	2	1	0	1	2	3	4	
	4	4	3	2	1	0	1	2	3	4	5	
y	3	3	2	1	0	1	2	3	4	5	6	a
	2	2	1	0	1	2	3	4	5	6	7	
	1	1	0	1	2	3	4	5	6	7	8	
	0	0	1	2	3	4	5	6	7	8	9	
		0	1	2	3	4	5	6	7	8	9	

x a

Table 1: Distribution of breaks on $(0,1) \times (0,1)$

Taking the complement gives:

$$P(|X - Y| < a) = 1 - (1 - a)^2 .$$

This is the cumulative probability distribution (CPD) for the interval $(0, 1)$. The probability density function (CDF) can be obtained by differentiating the CDP:

$$P(|X - Y| = a) = \frac{d}{da} P(|X - Y| < a) = \frac{d}{da}(1 - (1 - a)^2) = 2(1 - a) .$$

The expectation is the integral of the probability density function multiplied by the value:

$$E(|X - Y|) = \int_0^1 a \cdot 2(1 - a)\, da = 2 \left(\frac{a^2}{2} - \frac{a^3}{3} \right) \Big|_0^1 = \frac{1}{3} .$$

Simulation

```
Expectation of length of right piece = 0.3333
Average length of right piece        = 0.3359
```

40. The first ace ^{S}

Deal cards from a well-shuffled deck of cards until an ace appears. What is the expectation of the number of cards that must be dealt?

Hint: Consider the deck of cards without the aces to be laid out in a line.

Solution

The cards form a “rod” of length 48 which is “broken” by the 4 aces into 5 “pieces.” The solution of Problem 39 applies and the expectation of the length of a piece is $48/5 = 9.6$.

Simulation

```
Expectation of first ace = 9.6000
Average first ace        = 9.5805
```

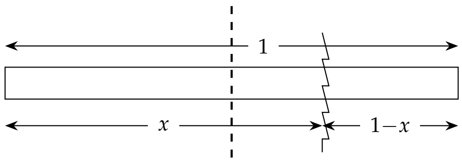


Figure 10: Breaking a stick into two pieces

42. The little end of the stick ^{S}

A large number of glass rods each of length 1 are broken into two pieces each.

Q1: What is the expectation of the length of the *smaller* piece?

Q2: What is the expectation of the ratio of the length of the smaller piece to the larger piece?

Solution

A1: The probability that the break is on the left half of a rod is $1/2$ as is the probability that it is on the right half. The smaller piece is on the same side as the break and since the break occurs at a random place, the expectation of its position is halfway between the end and the middle. Therefore:

$$E(\text{length of smaller piece}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}\,.$$

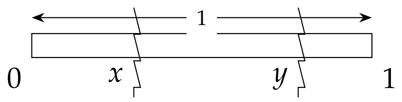
A2: Without loss of generality assume that the break occurred in the right half of the rod (Figure 10). The ratio of the smaller piece to the larger piece is $(1-x)/x$ and the length of the larger piece x is uniformly distributed in $(1/2, 1)$. The expectation of the ratio is:

$$\begin{aligned} E(\text{ratio}) &= \left(\frac{1}{1-(1/2)}\right)\int_{1/2}^{1} \frac{1-x}{x}\,dx \\ &= 2\int_{1/2}^{1}\left(\frac{1}{x}-1\right)dx \\ &= 2\,(\ln|x|-x)\big|_{1/2}^{1} = 2\ln 2 - 1 \approx 0.3863\,. \end{aligned}$$

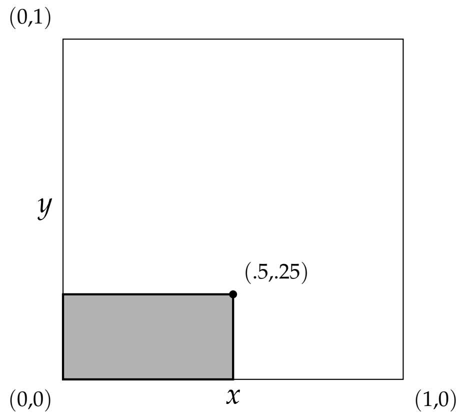
Simulation

```
Expectation of length of smaller = 0.2500
Average length of smaller        = 0.2490
Expectation of smaller/larger    = 0.3863
Average smaller/larger           = 0.3845
```

43. The broken bar ^{D,S}



(a) Break a rod into two pieces



(b) Representation of the lengths in the unit square

A large number of glass rods of length 1 are broken in two places (Figure 11a).

Q1: What is the expectaton of the length of the shortest bar?

Q2: What is the expectaton of the length of the longest bar?

Hint: x, y are selected independently in a uniform distribution from $(0, 1)$. Each pair (x, y) can be represented as a point in the unit square $(0, 1) \times (0, 1)$ (Figure 11b). What is the probability that $(x, y) < (.5, .25)$?

Hint: For **Q1:** assume that left piece is the shortest one and for **Q2:** assume that the left piece is the longest one.

Solution

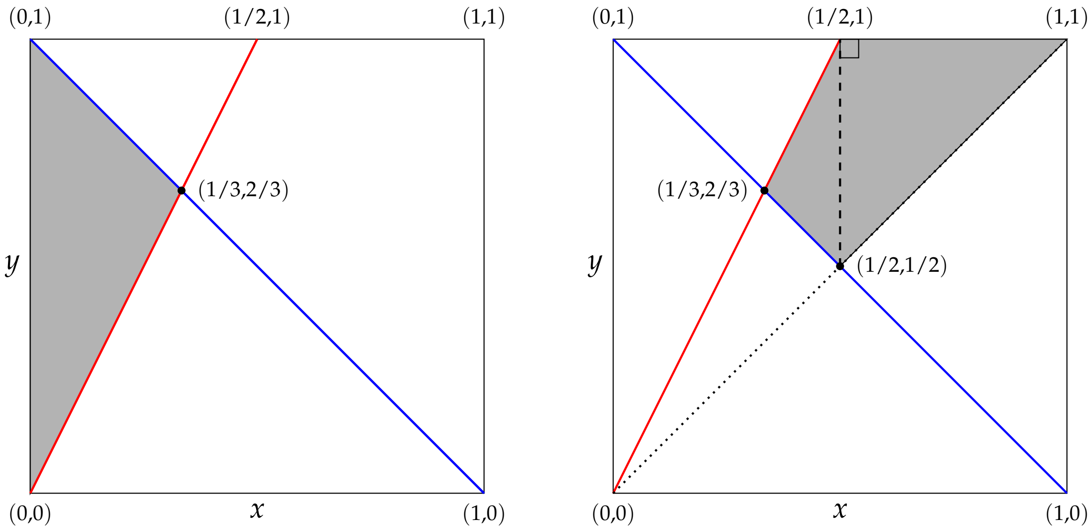
A1: Without loss of generality assume that the left piece of length x is the shortest. Then $x < y - x$ and $x < 1 - y$, from which we have $2x < y$ and $x + y < 1$.

Figure 12a shows the lines $y = 2x$ (red) and $y = 1 - x$ (blue). For the inequalities to be true, (x, y) must be in the shaded region left of the two lines. The point of intersection $(1/2, 2/3)$ can be computed by solving the two equations.

While the values of (x, y) are in the range $(0, 1) \times (0, 1)$, the expectation is computed over the subset of $(0, 1) \times (0, 1)$ denoted by the shaded region of the unit square. Therefore, the expectation must be divided by the area of the shaded region , which is $\frac{1}{2}(\frac{1}{3} \cdot 1) = \frac{1}{6}$.

The expectation of the value of x is:

$$\begin{aligned} E(x) &= \frac{1}{1/6} \int_0^{1/3} x[(1-x) - 2x]\, dx \\ &= \int_0^{1/3} (6x - 18x^2)\, dx \\ &= (3x^2 - 6x^3)\big|_0^{1/3} = \frac{2}{18} \approx 0.1111\,. \end{aligned}$$



(a) Shaded area for shortest bar

(b) Shaded area for longest bar

A2: For the left piece to be the longest, $x > y - x$ and $x > 1 - y$, so (x, y) must lie to the right of $y = 2x$ (red) and to the right of $y = 1 - x$ (blue) (Figure 12b). Furthermore, by the assumption that x is to the left of y, (x, y) must lie to the left of $y = x$ (dotted).

For convenience we divide the shaded region into two triangles (dashed line) and compute the expectations separately. The area of the shaded region is the sum of the areas of the triangles $1/24 + 1/8 = 1/6$. Then:

$$\begin{aligned}
E(x \text{ in left triangle}) &= 6\int_{1/3}^{1/2} x[2x - (1-x)]\,dx \\
&= \int_{1/3}^{1/2} \left(18x^2 - 6x\right)\,dx \\
&= \left(6x^3 - 3x^2\right)\Big|_{1/3}^{1/2} = \frac{1}{9} \\
E(x \text{ in right triangle}) &= 6\int_{1/2}^{1} x[1-x]\,dx \\
&= \int_{1/2}^{1} (6x - 6x^2)\,dx \\
&= \left(3x^2 - 2x^3\right)\Big|_{1/2}^{1} = \frac{1}{2} \\
E(x) &= \frac{1}{9} + \frac{1}{2} = \frac{11}{18} \approx 0.6111\,.
\end{aligned}$$

The expectation of the length of the middle-sized piece is $1 - \frac{2}{18} - \frac{11}{18} = \frac{5}{18} \approx 0.2778$.

Simulation

```
Expectations: shortest = 0.1111, middle = 0.2778, longest = 0.6111
Averages:     shortest = 0.1115, middle = 0.2783, longest = 0.6102
```

44. Winning an unfair game ^{D,S}

You are given an unfair coin whose probability of heads is $1/3 < p < 1/2$. Toss a coin an even number of times $N = 2n$. You win if and only *more* than half of the tosses are heads.

Q1: Develop a formula for the probability P_N of winning the game and develop a formula for the probability T_N of a tie occurring.

Q2: Develop a formula for the N that gives the highest probability of winning.

Hint: If N tosses gives the highest probability of winning then $P_{N-2} \leq P_N$ and $P_N \geq P_{N+2}$.

Solution

A1: To win the game head needs to appear in $i \in \{n+1, n+2, \ldots, 2n-1, 2n = N\}$ tosses. From the binomial distribution:

$$\begin{aligned} P_N &= \sum_{i=n+1}^{2n} \binom{2n}{i} p^i (1-p)^{2n-i} \\ T_N &= \binom{2n}{n} p^n (1-p)^n . \end{aligned}$$

A2: For $N = 2n$ to give the highest probability of winning we must have:

$$P_{2n-2} \leq P_{2n} \quad \text{and} \quad P_{2n} \geq P_{2n+2} .$$

When is $P_{2n-2} \neq P_{2n}$?

Case 1: After toss $2n-2$ heads has appeared n times and tails $n-2$ times (so you would have won if you stop here), but tails appears in the next two tosses. You now have n heads and n tails, and therefore lose the game. The probability is:

$$\binom{2n-2}{n} p^n (1-p)^{n-2} (1-p)^2 .$$

Case 2: After toss $2n-2$ heads has appeared $n-1$ times and tails $n-1$ times (so you would have lost if you stop here), but heads appears in the next two tosses. You now have $n+1$ heads and $n-1$ tails and therefore win the game. The probability is:

$$\binom{2n-2}{n-1} p^{n-1} (1-p)^{n-1} p^2 .$$

For $P_{2n-2} \leq P_{2n}$ to hold P_{2n-2} cannot increase while P_{2n} remains the same (Case 1), although P_{2n} can become greater than P_{2n-2} (Case 2). Therefore:

$$\begin{aligned} \binom{2n-2}{n} p^n (1-p)^{n-2} (1-p)^2 &\leq \binom{2n-2}{n-1} p^{n-1} (1-p)^{n-1} p^2 \\ \frac{1}{n}(1-p) &\leq \frac{1}{n-1} p \\ (n-1)(1-p) &\leq np \\ n &\leq \frac{1-p}{1-2p} \\ 2n &\leq \frac{1}{1-2p} + 1 . \end{aligned}$$

Similarly, for $P_{2n} \geq P_{2n+2}$ to hold it must be true that:

$$
\begin{aligned}
\binom{2n}{n+1} p^{n+1}(1-p)^{n-1}(1-p)^2 &\geq \binom{2n}{n} p^n (1-p)^n p^2 \\
\frac{1}{n+1}(1-p) &\geq \frac{1}{n} p \\
n(1-p) &\geq (n+1)p \\
n &\geq \frac{p}{1-2p} \\
2n &\geq \frac{1}{1-2p} - 1\,.
\end{aligned}
$$

Therefore, the value for $N = 2n$ that gives the highest probability for winning is the nearest even integer to $1/(1-2p)$. We leave to the reader to show that if $1/(1-2p)$ is odd then $P_{2n} = P_{2n+2}$.

Simulation

```
For probability            = 0.3700
Optimal games to be played  = 4
For  2 games, average won   = 0.1372
For  4 games, average won   = 0.1445
For  6 games, average won   = 0.1431

For probability            = 0.4000
Optimal games to be played  = 6
For  4 games, average won   = 0.1820
For  6 games, average won   = 0.1845
For  8 games, average won   = 0.1680

For probability            = 0.4500
Optimal games to be played  = 10
For  8 games, average won   = 0.2671
For 10 games, average won   = 0.2646
For 12 games, average won   = 0.2640
```

45. Average number of matches ^{S}

Lay out a deck of cards in a row in the standard order and then lay out a second deck in a row in a random order below the first (Figure 13). What is the expectation of the number of matches of a card in the first row with the card below it?

Solution

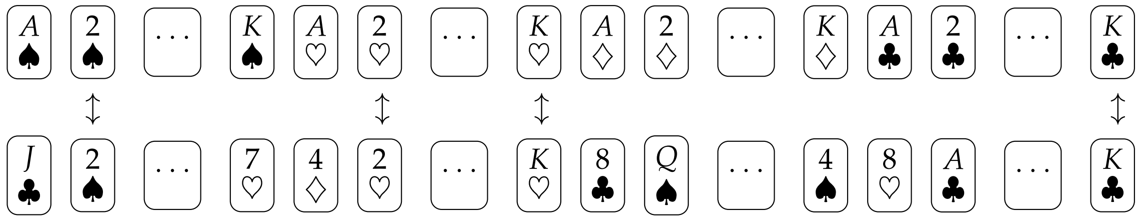


Figure 13: Matching two decks of cards

The distribution is uniform because each card has the same probability of being matched with the card above it. Therefore:

$$E(\text{number of matches}) = 52 \cdot \frac{1}{52} = 1\,.$$

```
Expectation of matches = 1.00
Average of matches     = 1.01
```

46. Probabilities of matches ^{S}

Lay out a deck of n cards in a row in the standard order and then lay out a second deck in a row in a random order below the first (Figure 13). Develop a formula for $P(n,r)$, the probability that there will be exactly r matches of a card in the first row with the card below? Assume that $P(k,0)$ is given for all $0 \leq k \leq n$.

Solution

At first glance this problem seems to be related to Problem 28 but there is a major difference. The drawings from the boxes of counterfeit coins are independent, whereas here they are not. For example, if the first match occurs on the first card (with probability $1/n$), the probability that the second card matches is $1/(n-1)$.

The probability that any *given* r cards match is:

$$\frac{1}{n} \cdot \frac{1}{n-1} \cdot \cdots \cdot \frac{1}{n+r-1}\,. \tag{26}$$

To obtain exactly r matches Equation 26 must be multiplied by $P(n-r,0)$, the probability that there are no matches in the remaining $n-r$ cards. Finally, there are $\binom{n}{r}$ ways of choosing the r matches. Therefore:

$$\begin{aligned}
P(n,r) &= \binom{n}{r} \frac{1}{n(n-1)(n+r-1)} P(n-r,0) \\
&= \frac{n!}{r!(n-r)!} \cdot \frac{1}{n!/(n-r)!} P(n-r,0) \\
&= \frac{1}{r!} P(n-r,0)\,,
\end{aligned}$$

which solves the problem since $P(k,0)$ is given.

Mosteller develops a closed formula and a limit for $P(n,r)$:

$$P(n,k) = \frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!} \tag{27}$$

$$\lim_{n-r\to\infty} P(n,k) \approx \frac{1}{k!}e^{-1}\,. \tag{28}$$

Simulation

The simulation was run for $n = 52$ cards and the probability computed from Equation 28.

```
Probability of 1 matches = 0.3679
Proportion 1 matches     = 0.3710
Probability of 2 matches = 0.1839
Proportion 2 matches     = 0.1828
Probability of 3 matches = 0.0613
Proportion 3 matches     = 0.0569
Probability of 4 matches = 0.0153
Proportion 4 matches     = 0.0168
```

47. Choosing the largest dowry ^{D,S}

Place a sequence of n cards face down. There is a positive integer written on the face of each card but you have no knowledge as to their distribution. Turn the cards over one-by-one and look at the numbers. After turning over each card you can declare that it is the largest number. If you are correct you win the game, otherwise you lose. For example, let the sequence of cards be $(47, 23, 55, 4)$. You win only if you choose the third card.

Consider the following strategy: for some fixed r reject the first $r-1$ cards and select the first card whose number is greater than all the $r-1$ cards.

Q1: For $n = 4$ and $r = 3$ check all permutations to determine how many games you will win.

Q2: Develop a formula for the probability of a win for arbitrary n, r.

Q3: Find an approximation for the probability when $n, r \to \infty$.

Hint: Given r where must the largest number m and the numbers less or equal to m be?

Solution

A1: To simplify notation we write the rank of the numbers $1, 2, 3, 4$, although the actual numbers are not known, say they are $4, 23, 47, 55$. If you uncover cards $1, 2, 3$ $(4, 23, 47)$ you do not know whether to accept 47 or to reject it and select the last card.

There are 24 permutations of the four numbers. By the strategy you select either the third card or the fourth card, so you lose if the permutation has 4 in the first position. What about the permutation $(1,2,3,4)$? You ignore $1,2$ and select 3 since it greater than $1,2$ but this is not the largest card so you lose. What about the permutation $(1,3,2,4)$? Again, $1,3$ are ignored by the strategy, but 2 is also rejected because it is *not* larger than $1,3$. Now you select 4 and win. Carry out this reasoning for all the permutations and check that permutations with boxed 4s are wins:

1 2	3 4	1 2	$\boxed{4}$ 3	1 3	2 $\boxed{4}$	1 3	$\boxed{4}$ 2	1 4	2 3	1 4	3 2
2 1	3 4	2 1	$\boxed{4}$ 3	2 3	1 $\boxed{4}$	2 3	$\boxed{4}$ 1	2 4	1 3	2 4	3 1
3 1	2 $\boxed{4}$	3 1	$\boxed{4}$ 2	3 2	1 $\boxed{4}$	3 2	$\boxed{4}$ 1	3 4	2 1	3 4	2 1
4 1	2 3	4 1	3 2	4 2	1 3	4 2	3 1	4 3	1 2	4 3	2 1

The probability of winning is $10/24$.

A2: If the largest number is in one the of positions $1,\ldots,r-1$ you lose. Therefore, in order to win the largest number must be in the mth position for $r \leq m \leq n$:

$$1 \quad 2 \quad \cdots \quad r-2 \quad r-1 \quad \overbrace{r \quad r+1 \quad \cdots \quad m-1 \quad m \quad m+1 \quad \cdots \quad n}^{\textsf{largest number must be here}}\,.$$

By the strategy you reject the first $r-1$ cards. You will choose position m only if *all* the numbers in $(r,\ldots,m-1)$ are less than *all* the numbers in $(1,\ldots,r)$. In other words, the largest card in the entire sequence $(1,\ldots,m-1)$, the sequence up until m, is *not* in the second part of the sequence $(r,\ldots m-1)$ but in the first part $(1,\ldots,r-1)$. The probability is:

$$P(\textsf{largest card in } (1,\ldots,m-1) \textsf{ is in } (1,\ldots,r-1)) = \frac{r-1}{m-1}\,.$$

Since the probability that the largest card is at m is $1/n$:

$$P(\textsf{win}) = \sum_{m=r}^{n} \frac{1}{n} \cdot \frac{r-1}{m-1} = \frac{r-1}{n} \sum_{m=r}^{n} \frac{1}{m-1}\,. \tag{29}$$

For $n=4, r=3$, $P(\textsf{win}) = 5/12 = 10/24$.

Equation 29 is not defined for $r=1$ but the probability of winning when choosing the first number is $1/n$. A larger r will have a higher probability of winning as shown in the example.

A3: Rewrite Equation 29 as:

$$P(\textsf{win}) = \frac{r-1}{n} \left(\sum_{m=2}^{n} \frac{1}{m-1} - \sum_{m=2}^{r-1} \frac{1}{m-1} \right)\,. \tag{30}$$

For large n, r Equation 30 can be approximated by:

$$P(\textsf{win}) = \frac{r}{n}(\ln n - \ln r) = \frac{r}{n} \ln \frac{n}{r} = -\frac{r}{n} \ln \frac{r}{n}\,.$$

Denote $x = r/n$ and find the maximum by taking derivatives:

$$\begin{aligned}
(-x \ln x)' &= -x \cdot \frac{1}{x} + (-1) \ln x = 0 \\
\ln x &= -1 \\
x &= 1/e\,.
\end{aligned}$$

To maximize that probability of winning choose $r \approx n/e$.

Simulation The simulation was run with 100 cards and values of r near $100/e$:

```
Reject cards before r = 36:
Probability of wins   = 0.3674
Proportion wins       = 0.3641
Reject cards before r = 37:
Probability of wins   = 0.3678
Proportion wins       = 0.3759
Reject cards before r = 38:
Probability of wins   = 0.3679
Proportion wins       = 0.3548
Reject cards before r = 30:
Probability of wins   = 0.3590
Proportion wins       = 0.3601
```

48. Choosing the largest random number ^{D,S}

Place a sequence of n cards face down. On the face of each card is a real number with uniform distribution in $0.0 \leq x < 1.0$. Turn the cards over one-by-one and look at the numbers. After turning over each card you can declare that it is the largest number. If you are correct you win the game, otherwise you lose.

Use the strategy of Problem 37: decide upon a value r such that you reject the first $r-1$ cards and then select the first card that is larger than the largest value in the first $r-1$ cards.

Definition: d, the *indifference value*, is the value below which you decide to reject the card and above which you decide to select the card.

Q1: Compute d for $n = 1$ and compute the probability of winning.

Q1: Compute d for $n = 2$ and compute the probability of winning.

Q3: Compute d for $n = 3$. Do not try to compute the probability of winning!

Note: In Problem 37 the values could be $100, 200, 300$ or $100, 50, 20$ so uncovering the first number gives no information about the other numbers. In this problem, since the distribution is uniform, if the first number is 0.2, the second number has probability 0.8 of being larger, and if the first number is 0.8 the second number has a probability of 0.2 of being larger.

Solution

Let v_1, v_2, v_3 be the values of the three cards.

A1: You have no choice but to select the first card since there are no other cards. There is no indifference value. Since v_1 is the "largest" number $P(win) = 1$.

A2: If you select the first card $P(\text{win}) = v_1$ which is the probability that the second card has a smaller value. If you reject the first card, $P(\text{win}) = 1 - v_1$, which is the probability that $v_2 > v_1$. Therefore, if $v_1 < 0.5$ select the second card because $1 - v_1 > 0.5$ and if $v_1 > 0.5$ select the first card because $1 - v_1 < 0.5$. It follows that $d = 0.5$.

Here is the formula for the probability of winning:

$$P(\text{win with two cards}) = p(\text{win} \mid v_1 < 0.5)\, p(v_1 < 0.5) + p(\text{win} \mid v_1 > 0.5)\, p(v_1 > 0.5)\,.$$

$p(v_1 < 0.5) = 0.5$ is immediate by uniform distribution. What about $p(\text{win} \mid v_1 < 0.5)$? By the strategy you win if $0.5 < v_2 < 1$, but you also win if $v_1 < v_2 < 0.5$. Since v_1 is uniformly distributed in $(0, 0.5)$:

$$p(\text{win} \mid v_1 < 0.5) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}\,.$$

A similar computation holds for $v_1 > 0.5$. Putting this together gives:

$$P(\text{win with two cards}) = \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{4}\,.$$

A3:

If you select the first card $P(\text{win}) = v_1^2$ because the second and third cards must be smaller than the first.

If you reject the first card and select the second because $v_2 > v_1$:

- $P(\text{win}) = (1 - v_1)v_1$ if $v_2 > v_1$ and $v_3 < v_1$.
- $P(\text{win}) = v_1(1 - v_1)$ if $v_2 < v_1$ and $v_3 > v_1$.
- $P(\text{win}) = \frac{1}{2}(1 - v_1)^2$ if $v_2 > v_1$ and $v_3 > v_1$, since winning depends on the order: $(0.55, 0.75, 0.65)$ wins and $(0.55, 0.65, 0.75)$ loses.

The indifference value d is the value such that the probability of winning by selecting the first card equals the probability of winning by rejecting the first card:

$$\begin{aligned} d^2 &= 2d(1-d) + \frac{1}{2}(1-d)^2 \\ 5d^2 - 2d - 1 &= 0 \\ d &= \frac{1 + \sqrt{6}}{5} \approx 0.6899\,. \end{aligned}$$

Gilbert and Mosteller [3, page 55] show that for $n = 3$:

$$P(\text{win}) = \frac{1}{3} + \frac{d}{2} + \frac{d^2}{1} - \frac{3d^3}{2} \approx 0.6617\,.$$

Simulation:

```
For  3 cards:
Indifference value = 0.6000
Probability of win = 0.6693
```

```
Proportion of wins = 0.6628
Indifference value = 0.6899
Probability of win = 0.6617
Proportion of wins = 0.6711
Indifference value = 0.7200
Probability of win = 0.6519
Proportion of wins = 0.6473
```

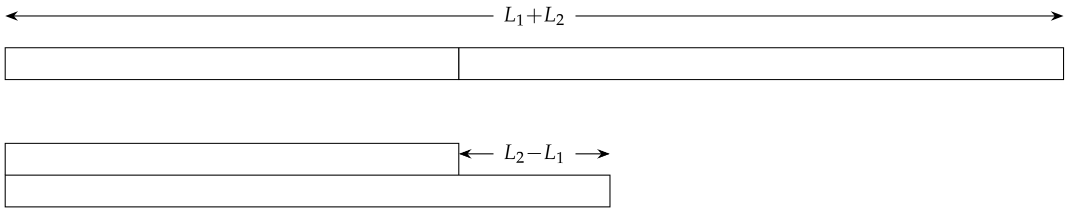


Figure 14: Measuring the lengths of two rods

49. Doubling you accuracy

You are given two rods of lengths $L_1 < L_2$ and a measuring instrument whose possible error is given by a normal distribution with mean 0 and variance σ^2. The lengths of the two rods can be measured by measuring each one separately. Is there a more accurate method?

Solution

Place the rods end-to-end and measure $L_s = L_1 + L_2$ and then place the rods side-by-side and measure $L_d = L_2 - L_1$ (Figure 14). Compute L_1, L_2:

$$\begin{aligned}
\tfrac{1}{2}(L_s - L_d) = \tfrac{1}{2}((L_1 + L_2) - (L_2 - L_1)) &= L_1 \\
\tfrac{1}{2}(L_s + L_d) = \tfrac{1}{2}((L_1 + L_2) + (L_2 - L_1)) &= L_2 \,.
\end{aligned}$$

The errors in the measurements are e_s, e_d so the errors in the results are:

$$\begin{aligned}
\tfrac{1}{2}((L_s + e_s) - (L_d + e_d)) &= L_1 + \tfrac{1}{2}(e_s - e_d) \\
\tfrac{1}{2}((L_s + e_s) + (L_d + e_d)) &= L_2 + \tfrac{1}{2}(e_s + e_d)\,.
\end{aligned}$$

Since the mean of the measurement instrument is 0, the mean of the errors of these measurements is also zero. The variance is reduced to half its previous value:[3]

$$\begin{aligned}
\mathrm{Var}\left(\tfrac{1}{2}\left(e_s - e_d\right)\right) &= \tfrac{1}{4}(\sigma^2 + (-1)^2\sigma^2) = \tfrac{1}{2}\sigma^2 \\
\mathrm{Var}\left(\tfrac{1}{2}(e_s + e_d)\right) &= \tfrac{1}{4}(\sigma^2 + \sigma^2) = \tfrac{1}{2}\sigma^2\,.
\end{aligned}$$

50. Random quadratic equations ^{S}

Consider the quadratic equation $x^2 + 2bx + c = 0$ defined on $[-B, B] \times [-B, B]$ for $B \geq 1$.

Q1: What is the probability that the roots are real?

Q2: As $B \to \infty$ what is the probability that the roots are real?

Solution

[3]We use the fact that the measurements are independent so the covariance is zero.

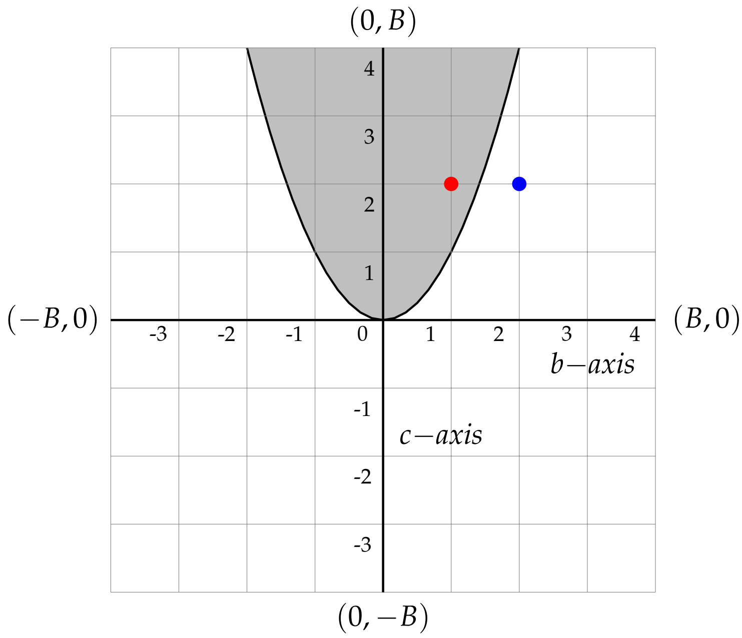


Figure 15: For (b, c) in the shaded area the roots of $c = b^2$ are complex

A1: The roots will be real if the discriminant is non-negative $4b^2 - 4c \geq 0$. Figure 15 shows a plot of the parabola $c = b^2$ where the complex roots are within the shaded area. For example, for $(b, c) = (1, 2)$, $x^2 + 2x + 2$ has complex roots (red dot) while for $(b, c) = (2, 2)$, $x^2 + 4x + 2$ has real roots (blue dot).

The shaded area can be computed by integration:

$$\int_{-\sqrt{B}}^{\sqrt{B}} (B - b^2)\, db = Bb - \frac{b^3}{3}\bigg|_{-\sqrt{B}}^{\sqrt{B}} = \left(B^{3/2} - \frac{B^{3/2}}{3}\right) - \left(-B^{3/2} + \frac{B^{3/2}}{3}\right) = \frac{4}{3}B^{3/2}.$$

The total area of the range $[-B, B] \times [-B, B]$ is $4B^2$ so:

$$\begin{aligned} P(\text{complex roots}) &= \frac{\frac{4}{3}B^{3/2}}{4B^2} = \frac{1}{3\sqrt{B}} \\ P(\text{real roots}) &= 1 - \frac{1}{3\sqrt{B}}. \end{aligned}$$

A2:

$$\lim_{B\to\infty} P(\text{real roots}) = \lim_{B\to\infty} \left(1 - \frac{1}{3\sqrt{B}}\right) = 1.$$

Simulation

```
For B =  4:
Probability of real roots = 0.8333
Proportion real roots     = 0.8271
For B = 16:
```

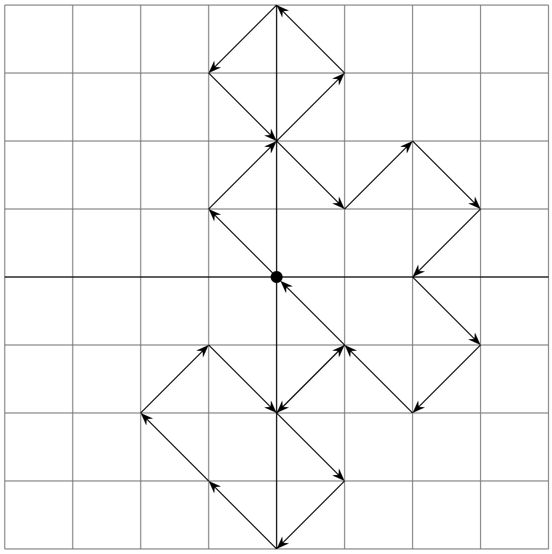


Figure 16: Two-dimensional random walk

```
Probability of real roots = 0.9167
Proportion real roots     = 0.9205
For B = 64:
Probability of real roots = 0.9583
Proportion real roots     = 0.9582
```

51. Two-dimensional random walk ^{S}

A particle is placed at the origin of a two-dimensional coordinate system. The particle moves left or right on the x-axis with probabilities $1/2$ *and* up or down the y-axis with probabilities $1/2$. Figure 16 shows a random walk of 22 steps starting at and returning to the origin.

Q1: What is the probability of returning to the origin in 2 moves?

Q2: What is the probability that the particle returns (one or more times) to the origin?

Q3: Use Stirling's approximation to obtain an estimate of the probability for large n.

Solution

A1: The dots in Figure 17 show the possible positions of the particle after two moves:

- The green path shows how to move to $(\pm 2, \pm 2)$ by taking two moves in the same direction. The probability is $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$.
- The red path shows how to move to $(\pm 2, 0)$ or $(0, \pm 2)$. There are two possible paths for each one so the probability is $2 \cdot \left(\frac{1}{4}\right)^2 = \frac{2}{16}$.

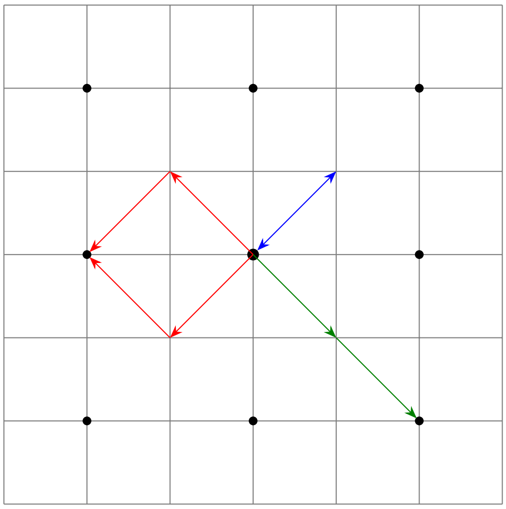


Figure 17: Two moves of the random walk

- The blue path shows how to move to $(\pm 1, \pm 1)$ and back to the origin. The probability is $1/16$. Since there are four paths that return to the origin the probability is $\frac{4}{16}$.

The blue paths are the only ones that return to the origin so:

$$P(\text{return to origin in two moves}) = \frac{4}{16}\,.$$

A3: The choices of direction for x and y are independent so for $2n$ moves:

$$P_{2n}(\text{return to origin}) = P_{2n}(\text{return to } x=0)\, P_{2n}(\text{return to } y=0)\,. \tag{31}$$

The particle will return to the origin if and only if for both axes the number of $+1$ moves equals the number of -1 moves. There are $\binom{2n}{n}$ ways to arrange $+1$s and -1s so:

$$P_{2n}(\text{return to } x=0) = P_{2n}(\text{return to } y=0) = \binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n \tag{32}$$

$$P_{2n}(\text{return to origin}) = \left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^2 \tag{33}$$

$$P(\text{return to origin}) = \sum_{n=1}^{\infty} P_{2n}(\text{return to origin}) = \sum_{n=1}^{\infty}\left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^2\,. \tag{34}$$

A3: By Stirling's approximation $n! \approx \sqrt{2\pi n}\,(n/e)^n$:

$$\begin{aligned}
P_{2n}(\text{return to origin}) &= \left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^2 \\
&= \left[\frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}\right]^2
\end{aligned}$$

$$
\begin{aligned}
&\approx \left(\frac{1}{2}\right)^{4n} \frac{(\sqrt{2\pi\cdot 2n})^2\,(2n/e)^{4n}}{(\sqrt{2\pi n})^4\,(n/e)^{4n}}\\
&= \left(\frac{1}{2}\right)^{4n} \frac{4\pi n}{4\pi^2 n^2}\cdot\frac{(n/e)^{4n}\cdot 2^{4n}}{(n/e)^{4n}}\\
&= \frac{1}{\pi n}\\
P(\text{return to origin}) &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,,
\end{aligned}
$$

which is the *harmonic series* that diverges. This means that with probability 1 the particle returns to the origin.

Simulation The simulation was run one million times instead of ten thousand times but still there is no certainty of returning to the origin:

```
Proportion returned to origin      = 0.8700
```

52. Three-dimensional random walk ^{D,S}

A particle is placed at the origin of a three-dimensional coordinate system. The particle moves left or right on the x-axis with probabilities $1/2$ *and* up or down the y-axis with probabilities $1/2$ *and* in or out on the z-axis with probabilities $1/2$.

Q1: What is the expectation of the number of times that the particle returns to the origin?

Hint: Compute the probability and then use an indicator variable.

Q2: What is the probability that the particle will return to the origin (at least once)?

Hint: Use the technique from Problem 4.

Solution

P_{2n}, probability of returning to the origin after $2n$ steps, is given by the analogue of Equation 31:

$$P_{2n} = P_{2n}(\text{return to } x=0)\,P_{2n}(\text{return to } y=0\,P_{2n}(\text{return to } z=0)\,.$$

P_r, the probability of returning to the origin one or more times, is given by the analogue of Equation 34:

$$P_r = \sum_{n=1}^{\infty} P_{2n} = \sum_{n=1}^{\infty}\left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^3 .$$

From Stirling's approximation:[4]

$$P_{2n} = \left[\frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}\right]^3$$

[4]Mosteller used 18 terms in his computation and obtained 0.315. My program used 500 terms to obtain 0.3772.

$$
\begin{aligned}
&\approx \left(\frac{1}{2}\right)^{6n} \frac{(\sqrt{2\pi\cdot 2n})^3\,(2n/e)^{6n}}{(\sqrt{2\pi n})^6\,(n/e)^{6n}}\\
&= \frac{(4\pi n)^{3/2}}{(2\pi n)^3} = \frac{1}{(\pi n)^{3/2}}\\
P_r &= \sum_{n=1}^{\infty} \frac{1}{(\pi n)^{3/2}} \approx 0.3772\,.
\end{aligned}
$$

Let I_k be the indicator variable for a return to the origin on step k:

$$
I_k = \begin{cases} 1, & \text{if particle returns to origin on step } k \\ 0, & \text{if particle does not returns to origin on step } k\,. \end{cases} \tag{35}
$$

Then:

$$
E(\text{number of returns to the origin}) = \sum_{n=1}^{\infty} P_{2n}\, I_{2n} = P_r \approx 0.3772\,,
$$

so the expectation of the number of returns is equal to the probability.

Q2: Let P_1 be the probability that the particle returns to the origin *at least once*. From Problem 4 we know that the expectation of the number of trials until the first one where the particle *does not* return to the origin is $1/(1-P_1)$. Therefore, the expectation of the number of trials for which the particle does return to the origin is one less, because the particle can return to the origin many times until finally it does not.[5]

Let $E_r = E(\text{number of returns to the origin})$, then:

$$
\begin{aligned}
E_r &= \frac{1}{1-P_1} - 1\\
P_1 &= \frac{E_r}{1+E_r}\,.
\end{aligned}
$$

In **A1:** we computed that $E_r \approx 0.3772$ so: Then:

$$
P_1 \approx 1 - \frac{1}{1+0.3772} \approx 0.2739\,.
$$

Simulation

```
Expectation of reaching origin = 0.3772
Average times reached origin   = 0.3630
Probability of reaching origin = 0.2739
Proportion reached origin      = 0.2790
```

53. Buffon's needle ^{D,S}

Consider a needle of length $a \leq 1$ and a surface ruled with parallel lines 1 apart. Throw the needle onto the surface. What is the probability that the needle crosses a line?[6]

[5] Mosteller's presentation is not easy to follow. I would like to thank Aaron Montgomery for clarification [5].

[6] The problem has been simplified by specifying the distance between the parallel lines as 1. We ignore the possibility that the needle lies completely along the line or just touches two lines since the probability of these events is zero.

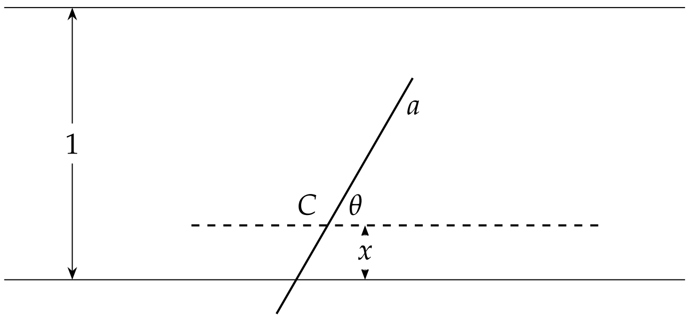


Figure 18: Buffon's needle

Hint: There are two independent random variables (Figure 18): x, the position of the center of the needle relative to the closest line which is uniformly distributed in the range $[0, 1]$, and θ, the angle formed by the needle relative to the parallel lines which is uniformly distributed in the range $[0, \pi/2]$.

Solution 1

Let $p(a)$ be the probability that a needle of length a crosses a line and define the indicator variable:

$$I_{\text{crosses}} = \begin{cases} 1, & \text{if needle of length } a \text{ crosses a line} \\ 0, & \text{if needle of length } a \text{ does not cross a line}\,. \end{cases}$$

Then:

$$E(I_{\text{crosses}}) = 1 \cdot p(a) + 0 \cdot (1 - p(a)) = p(a)\,, \tag{36}$$

and the probability can be computed by computing the expectation.

Let m be a line perpendicular to the parallel lines that passes through the center of the needle and let θ be the angle between the needle and a parallel line. Project the needle onto m to give the line segment $\overline{CD}$. The probability that the needle will cross a line is:

$$P(\text{needle of length } a, \text{ angle } \theta \text{ crosses line}) = \frac{\overline{CD}/2}{1/2} = \frac{(a/2)\sin\theta}{1/2} = a\sin\theta\,. \tag{37}$$

The expectation of the number of lines crossed is given by integrating over possible angles:

$$E(\text{lines crossed}) = \frac{1}{(\pi/2) - 0}\int_0^{\pi/2} a\sin\theta\, d\theta = \frac{2}{\pi}\cdot a(-\cos\theta)\Big|_0^{\pi/2} = \frac{2a}{\pi}\,. \tag{38}$$

Solution 2 This solution is taken from [1, Chapter 26].

Let $E(x)$ be the expectation of the number of parallel lines crossed by a line x. Consider a line formed into a circle C of *diameter* 1 and circumference π. If the circle is thrown onto the surface it will have *exactly* two intersections with the lines (Figure 20), that is:

$$E(C) = 2\,. \tag{39}$$

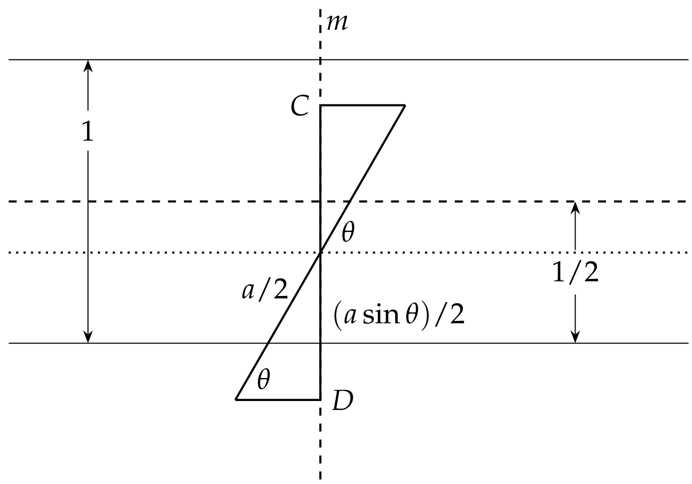


Figure 19: Right triangle for solving Buffon's needle problem

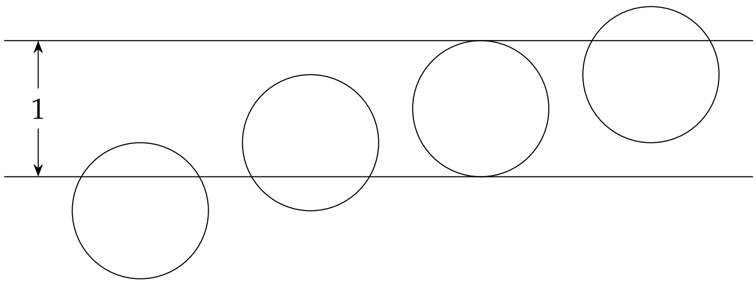


Figure 20: Solving Buffon's needle with circles

Inscribe a regular polygon Q_n (red) within c (green) and circumscribe a regular polygon R_n (blue) around c (Figure 21). Any line (red) that Q_n crosses must also cross the circle and any line (blue) that crosses the circle must also cross R_n. Therefore:

$$E(Q_n) \leq E(C) \leq E(R_n)\,. \tag{40}$$

Let a_Q, a_R be the sums of the lengths of the sides of Q_n, R_n, respectively. By the linearity of expectation:

$$E(Q_n) = \sum_{i=1}^{n} E(\text{segments of } a_Q) = a_Q E(1) \tag{41}$$

$$E(R_n) = \sum_{i=1}^{n} E(\text{segments of } a_R) = a_R E(1)\,. \tag{42}$$

As $n \to \infty$ both polygons approximate the circle so:

$$\lim_{n\to\infty} a_Q = \lim_{n\to\infty} a_R = \pi\,, \tag{43}$$

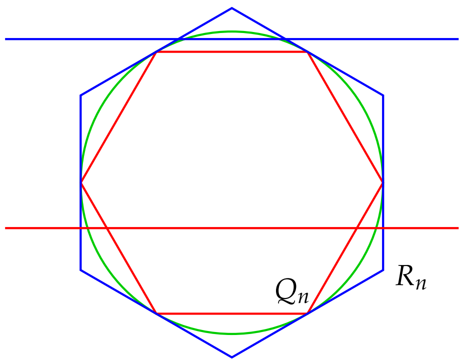


Figure 21: Polygons approximate a circle

the circumference of the circle. From Equations 39–43 we have:

$$\begin{aligned}
&\lim_{n\to\infty} E(Q_n) = E(C) = \lim_{n\to\infty} E(R_n)\\
&E(C) = aE(1) = \pi E(1) = 2\\
&E(1) = \frac{2}{\pi}\\
&E(a) = aE(1) = \frac{2a}{\pi}\,.
\end{aligned}$$

Simulation

Since $\pi = 2a/E$ the simulation (or actually throwing needles on a table!) can be used to obtain an approximation of π.

```
For length = 0.2:
Expectation of crossings = 0.1273
Average crossings        = 0.1308
Empirical value for pi   = 3.0581

For length = 0.5:
Expectation of crossings = 0.3183
Average crossings        = 0.3227
Empirical value for pi   = 3.0989

For length = 1.0:
Expectation of crossings = 0.6366
Average crossings        = 0.6333
Empirical value for pi   = 3.1581
```

54. Buffon's needle with horizontal and vertical rulings

Solve Buffon's needle problem for a surface that is covered by a grid with squares of size 1×1. A needle can cross a vertical line (green), a horizontal line (blue), both (red) or neither (orange) (Figure 22).

Hint: Are the numbers of crossings of the horizontal and vertical lines independent?

Solution

The numbers of crossings of the horizontal and vertical lines are independent and expectation is linear so:

$$\begin{aligned} E(\text{lines crossed by } a) &= E(\text{vertical lines crossed by } a + \text{horizontal lines crossed by } a) \\ &= E(\text{vertical lines crossed by } a) + E(\text{horizontal lines crossed by } a) \\ &= \frac{2a}{\pi} + \frac{2a}{\pi} = \frac{4a}{\pi}\,. \end{aligned}$$

55. Long needles ^{D,S}

Let the length of the needle in Buffon's problem be $a > 1$.

Q1: What is the expectation of the *number of crossings*?

Q2: What is the probability that there is *at least one crossing*?

Hint: For what angles θ is the probability a crossing 1?

Solution

A1: Break the needle into pieces of lengths $\{a_1, a_2, \ldots, a_n\}$, $a_i < 1$, such that $\sum_{i=1}^{n} a_i = a$. By the linearity of expectation and the solution of Problem 53:

$$E(a) = \sum_{i=1}^{n} E(a_i) = \frac{2a}{\pi}\,.$$

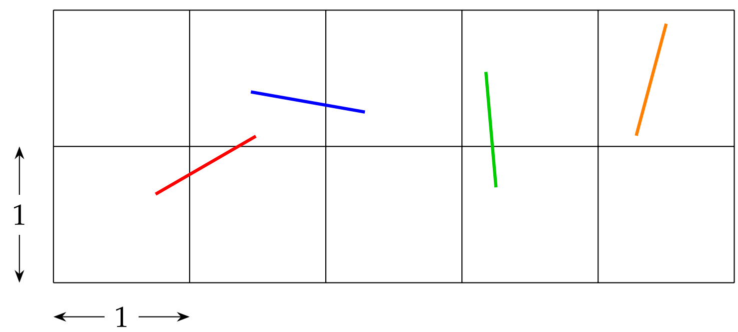


Figure 22: Buffon's needle with horizonal and vertical crossings

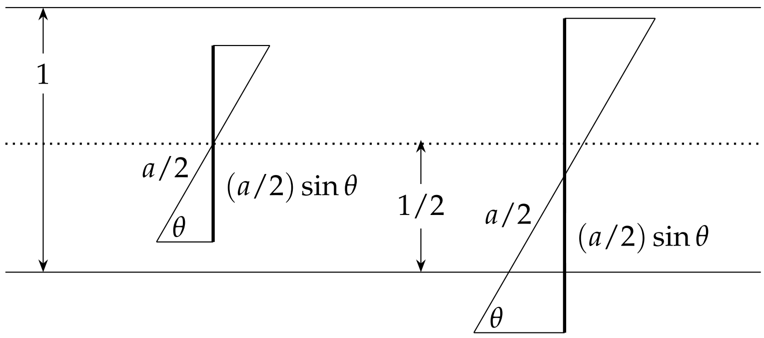


Figure 23: Long needles

A2: This solution is based on [12] and [1, Chapter 26].

By Equation 37 the probability that a needle will cross a line is $a \sin\theta$ *if* $a\sin\theta \leq 1$, that is, if $0 \leq \theta \leq \sin^{-1}(1/a)$. However, if $a\sin\theta > 1$ then the probability is 1 (Figure 23). Let us generalize Equation 38 for arbitrary $a > 0$. The integral is divided into two, one for $\theta < \sin^{-1}(1/a))$ and one for $\theta > \sin^{-1}(1/a))$:

$$\begin{aligned}
E(a) &= \frac{2}{\pi}\left(\int_0^{\sin^{-1}(1/a)} a\sin\theta\, d\theta + \int_{\sin^{-1}(1/a)}^{\pi/2} 1\, d\theta\right)\\
&= \frac{2}{\pi}\left(a(-\cos\theta)|_0^{\sin^{-1}(1/a)} + \left(\frac{\pi}{2} - \sin^{-1}(1/a)\right)\right)\\
&= 1 + \frac{2}{\pi}\left(a\left(1-\sqrt{1-\frac{1}{a^2}}\right) - \sin^{-1}(1/a)\right).
\end{aligned}$$

Simulation

```
For length = 1.5:
Expectation of crossings = 0.7786
Average crossings        = 0.7780
For length = 2.0:
Expectation of crossings = 0.8372
Average crossings        = 0.8383
For length = 3.0:
Expectation of crossings = 0.8929
Average crossings        = 0.8897
```

56. Molina's urns

Let U_1, U_2 be two urns containing m balls each. U_1 has w_1 white balls and b_1 blacks while U_2 has w_2 white balls and b_2 blacks. n balls are drawn *with replacement* from each urn. Find w_1, b_1, w_2, b_2 such that:

$$p(\text{balls drawn from } U_1 \text{ are all white}) = p(\text{balls drawn from } U_2 \text{ are all white or all black}).$$

Q1: Find values of w_1, b_1, w_2, b_2 for $n = 2$.

Q2: Explain why the problem cannot be solved for $n \geq 3$.

Solution

A1: The equation that must be solved is:

$$\begin{aligned}\left(\frac{w_1}{m}\right)^2 &= \left(\frac{w_2}{m}\right)^2 + \left(\frac{b_2}{m}\right)^2 \\ w_1^2 &= w_2^2 + b_2^2 .\end{aligned}$$

One solution is $w_1 = 10, b_1 = 4, w_2 = 6, b_2 = 8$.

A2: By Fermat's Last Theorem, proved in 1995 by Andrew Wiles, there are no solutions to $w_1^n = w_2^n + b_2^n$ for $n \geq 3$.

Review of Probability

This section reviews concepts of probability. An example of each concept is given using the activity of throwing fair six-sided dice.

Experiment This is an undefined primitive concept, the intention being an action that has a possible result. An experiment is also called a *trial*. Throwing a die is an experiment.[7]

Outcome The result of an experiment. If you throw a die one outcome is 4.

Sample space The set of all possible outcomes of an experiment. The set $S = \{1, 2, 3, 4, 5, 6\}$ is the sample space of the outcomes of throwing a die.

Event A subset of the sample space. The subset $e = \{2, 4, 6\} \subseteq S$ is the event of a die showing an even number.

Random variable A function from a sample space to (real) numbers. Let T be the sample space of throwing two dice:

$$T = \{(a, b) | a, b \in \{1, 2, 3, 4, 5, 6\}\} .$$

Define the random variable X as the function $X : T \mapsto \{2, 3, \ldots, 11, 12\}$ which gives the sum of the numbers on the two dice:

$$X((a, b)) = a + b . \tag{44}$$

Union, intersection, complement Since events are sets these concepts take on their normal set-theoretical meaning. Let $e_1 = \{2, 4, 6\}$ and $e_2 = \{1, 2, 3\}$. Then:

$$e_1 \cup E_2 = \{1, 2, 3, 4, 6\} \quad e_1 \cap e_2 = \{2\} \quad \overline{e_1} = S \setminus e_1 = \{1, 3, 5\} .$$

The intersection is the set of even numbers among the first three outcomes in the sample space. The complement is the set of odd outcomes in the sample space.

Mutually exclusive Two or more events are mutually exclusive if their intersection is the empty set. $e_1 = \{2, 4, 6\}$ and $e_2 = \{1, 3, 5\}$ are mutually exclusive since $e_1 \cap e_2 = \emptyset$, that is, there are no outcomes which are both even and odd.

Probability Probability is the limiting relative frequency of an event. Let e be an event and let n_e be the number of times that e occurs in n repetitions of the event. Then $P(e)$, the probability of the event e, is:

$$P(e) = \lim_{n \to \infty} \frac{n_e}{n} .$$

This is not a very good definition because we don't actually know that the limit exists. The definition also depends on "repetitions of an event" but we want to define probability without reference to a specific sequence of events.

Modern probability theory is based on a set of three axioms, but we won't develop this theory, though two of the axioms are clearly seen to be fundamental:

$$\begin{aligned} P(e) &\geq 0 \\ P(S) &= 1 . \end{aligned}$$

[7] *Die* is the singular of the more familiar plural noun *dice*.

Any event either occurs with some non-zero probability or it doesn't occur, and the outcome space is by definition all the possible outcomes.

The *laws of large numbers* ensure that our intuitive concept of probability as relative frequency is very similar to what happens when an event is repeated many times.

Uniformly distributed If all outcomes in the sample space have equal probability (are equally likely to occur), the probability is said to be uniformly distributed. If S is finite and the probability is uniformly distributed then:

$$P(e) = \frac{|e|}{|S|}.$$

For example, if you throw a *fair* die the probability of the outcomes is uniformly distributed, so for $e = \{2,4,6\}$:

$$P(e) = \frac{|e|}{|S|} = \frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{2}.$$

Conditional probability Let e_1, e_2 be events. $P(e_1|e_2)$, the conditional probability that e_1 occurs given that e_2 occurs, is given by:

$$P(e_1|e_2) = \frac{P(e_1 \cap e_2)}{P(e_2)}.$$

Let $e_1 = \{1,2,3\}$ be the event that a die shows a number less than or equal to 3 and let $e_2 = \{2,4,6\}$ be the event that the die shows an even number. Then:

$$P(e_2|e_1) = \frac{P(E_2 \cap E_1)}{P(e_1)} = \frac{P(\{2\})}{P(\{2,4,6\})} = \frac{1/6}{1/2} = \frac{1}{3}.$$

This makes sense since if you know that a number less than or equal to 3 is thrown, only one out of the three outcomes is an even number.

Independence Two events are independent if the probability of their intersection is the product of their individual probabilities:

$$P(e_1 \cap e_2) = P(e_1)\, P(e_2).$$

In terms of conditional probability:

$$P(e_1|e_2) = \frac{P(e_1) \cap P(e_2)}{P(e_2)} = \frac{P(e_1)\, P(e_2)}{P(e_2)} = P(e_1).$$

For independent events e_1, e_2, if you know the probability of e_2 it gives you no information as to the probability of e_1. Three throws of a fair die are independent so the probability of all of them showing an even number is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

Average Let $S = \{a_1, \ldots, a_n\}$ be a set of values. Then:

$$Average(S) = \frac{\sum_{i=1}^{n} a_i}{n}.$$

An average is computed over a set of values but the average may not be an element of the set. If there are 1000 families in a town and 3426 children, the average number of children per family is 3.426 although clearly no family has 3.426 children. If you throw a die six times and receive the numbers $\{2,2,4,4,5,6\}$. The average is:

$$\frac{2+2+4+4+5+6}{6} = \frac{23}{6} \approx 3.8\,,$$

again, a value not in the set.

Expectation The expectation of a random variable is the sum of the probability of each outcome times the value of random variable for that outcome. For a fair die each outcome has the same probability:

$$E(\textsf{value of a die}) = 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6} = 3.5\,.$$

Consider the random variable defined by the function X (Equation 44) that maps the numbers appearing in a pair of dice to the sum of the numbers. The probability of each pair is $1/36$, but since the pairs $(2,5)$ and $(5,2)$ have the same sum they belong to the same outcome. The values of the random variable are $\{2,\ldots,12\}$ and that the number of ways of obtaining each one is:

Sum	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	4	5	6	5	4	3	2	1

The expectation is the average of the values of the random variable *weighted* by the probability of each outcome:

$$E(\textsf{sum of two dice}) = 2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+\cdots+10\cdot\frac{3}{36}+11\cdot\frac{2}{36}+12\cdot\frac{1}{36} = 7\,.$$

For an arbitrary set of events $\{e_1,\ldots,e_n\}$ the expectation is:

$$E = \sum_{i=1}^{n} e_i P(e_i)\,.$$

Linearity of expectation Expectation is a linear function $E(ae_1+be_2) = aE(e_1)+bE(e_2)$ and for an arbitrary linear expression:

$$E\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} a_i E(e_i)\,.$$

For a proof see [11, Section 4.9].

Indicator variable Let e be an event whose probability is $P(e)$. Define I_e, an indicator variable for e, as follows [11, Chapter 4, Example 3b]:

$$I_e = \begin{cases} 1, & \textsf{if } e \textsf{ occurs} \\ 0, & \textsf{if } e \textsf{ does not occur}\,. \end{cases}$$

Then $E(I_e) = 1\cdot P(e) + 0\cdot(1-P(e)) = P(e)$.

Mathematical formulas

Binomial theorem If the probability of an event e is p then the probability that a sequence of n independent trials results in *exactly* k events e is given by the *binomial coefficient*:

$$\binom{n}{k} p^k (1-p)^{n-k} .$$

By the *binomial theorem*:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} .$$

For $p, 1-p$ the is $(p+(1-p))^n = 1$, as expected, since one of the outcomes must occur.

Sum of a geometric series For $0 < r < 1$:

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}, \qquad \sum_{i=0}^{\infty} r^i = \frac{1}{1-r} .$$

Sum of a harmonic series For positive integer n the harmonic series is:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \ln n + \frac{1}{2n} + \gamma ,$$

where $\gamma \approx 0.5772$ is *Euler's constant*. As n approaches infinity the series diverges:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty ,$$

because $\ln n$ is unbounded.

Stirling's approximation Computing $n!$ for large n is very difficult. It is convenient to use one of the formulas of *Stirling's approximation*:

$$\begin{aligned}
n! &\approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \\
\ln(n!) &\approx n \ln n - n \\
\ln(n!) &\approx n \ln n - n + \frac{1}{6}\left(8n^3 + 4n^2 + n + \frac{1}{30}\right) + \frac{1}{2}\ln \pi .
\end{aligned}$$

Continuous probability distribution

A beginning student may not have learned continuous probability distributions, but they do not appear very often in the book. For readers with the appropriate background, we review the basic concepts.

Probabilities can be defined over continuous random variables. A *probability density function (PDF)* $f(x) : \mathcal{R} \rightarrow \mathcal{R}$ maps an outcome x to the value of the function, thus defining:

$$P(x) = f(x) .$$

The reason for this terminology is that each *individual* real number has zero probability of occurring, so the proper interpretation is to assign probabilities to neighborhoods of points.

The *cumulative probability distribution (CPD)* for the interval $[-\infty, a]$ is obtained by integrating the PDF:

$$P(x < a) = \int_{-\infty}^{a} f(x)\, dx\,.$$

Of course this is also $P(x \leq a)$ since $P(a) = 0$.

Like probabilities, for a PDF, $P(x) \geq 0$ for all x, and:

$$\int_{-\infty}^{\infty} P(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1\,.$$

If the integral does not evaluate to 1 a *normalization constant* must be used. For example, if a PDF is uniformly distributed in the range $[a, b]$ then:

$$P(a \leq x \leq b) = \int_{a}^{b} 1\, dx = (b - a)\,,$$

and therefore we must define:

$$P(a \leq x \leq b) = \frac{1}{b-a}\int_{a}^{b} 1\, dx = \frac{1}{b-a} \cdot (b - a) = 1\,.$$

The expectation can be obtained by integrating the PDF $f(x)$ multiplied by x:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx\,.$$

The PDF can be obtained by differentiating the CPD:

$$P(x < a) = \frac{d}{da} CDP(x < a)\,.$$

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