Classical Adjoint (Adjugate) Matrix: Advanced Problems

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1 Adjugate and Matrix Transformations

Problem 1.1. *Similarity Transformation*

Question 1.1.1. *Prove that if* $B = P^{-1}AP$ *for an invertible matrix* P*, then* $\operatorname{adj}(B) = P^{-1}\operatorname{adj}(A)P$*[cite: 104].*

Hint 1.1.1.1. Start with the definition $B = P^{-1}AP$. Apply the adjugate operation to both sides: $\operatorname{adj}(B) = \operatorname{adj}(P^{-1}AP)$.

Hint 1.1.1.2. Recall the property $\operatorname{adj}(XYZ) = \operatorname{adj}(Z)\operatorname{adj}(Y)\operatorname{adj}(X)$ [derived by applying $\operatorname{adj}(MN) = \operatorname{adj}(N)\operatorname{adj}(M)$ twice, see cite: 49]. Apply this to $\operatorname{adj}(P^{-1}AP)$.

Hint 1.1.1.3. You should have $\operatorname{adj}(B) = \operatorname{adj}(P)\operatorname{adj}(A)\operatorname{adj}(P^{-1})$. Now we need to relate $\operatorname{adj}(P)$ and $\operatorname{adj}(P^{-1})$.

Hint 1.1.1.4. Recall the inverse formula: $M^{-1} = \frac{1}{\det(M)}\operatorname{adj}(M)$. This implies $\operatorname{adj}(M) = \det(M)M^{-1}$. Apply this to $M = P$ and $M = P^{-1}$.

Hint 1.1.1.5. Express $\operatorname{adj}(P)$ in terms of $\det(P)$ and P^{-1}. Express $\operatorname{adj}(P^{-1})$ in terms of $\det(P^{-1})$ and $(P^{-1})^{-1} = P$. Remember $\det(P^{-1}) = 1/\det(P)$.

Hint 1.1.1.6. Substitute the expressions for $\operatorname{adj}(P)$ and $\operatorname{adj}(P^{-1})$ from the previous hint back into the equation from Hint 3: $\operatorname{adj}(B) = (\det(P)P^{-1})\operatorname{adj}(A)((\det P)^{-1}P)$.

Hint 1.1.1.7. Simplify the expression by canceling the determinant factors. Does the result match the desired $P^{-1}\operatorname{adj}(A)P$?

2 Connections to Characteristic Polynomial and Eigenvectors

Problem 2.1. *Cayley-Hamilton Theorem and Adjugate*

Question 2.1.1. *Let* $p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ *be the characteristic polynomial of* A*. Show that* $\operatorname{adj}(A) = -(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_2A + c_1I)$*[cite: 148]. (Note: Requires careful handling of polynomial definition and potentially the identity* $\operatorname{adj}(\lambda I - A)(\lambda I - A) = p(\lambda)I$*).*

Hint 2.1.1.1. Recall the Cayley-Hamilton Theorem: $p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0$[cite: 147].

Hint 2.1.1.2. Consider the matrix polynomial $B(\lambda) = \operatorname{adj}(\lambda I - A)$. The entries of $B(\lambda)$ are cofactors of $\lambda I - A$, which are polynomials in λ of degree at most $n-1$. We can write $B(\lambda) = B_{n-1}\lambda^{n-1} + \cdots + B_1\lambda + B_0$, where B_k are matrix coefficients.

Hint 2.1.1.3. Use the fundamental identity: $(\lambda I - A)\operatorname{adj}(\lambda I - A) = \det(\lambda I - A)I = p(\lambda)I$. Substitute the polynomial forms: $(\lambda I - A)(B_{n-1}\lambda^{n-1} + \cdots + B_0) = (\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0)I$.

Hint 2.1.1.4. Expand the left side and collect terms by powers of λ. $\lambda^n B_{n-1} + \lambda^{n-1}(B_{n-2} - AB_{n-1}) + \cdots + \lambda(B_0 - AB_1) - AB_0$.

Hint 2.1.1.5. Equate the matrix coefficients of powers of λ on both sides of the identity from Hint 3. λ^n: $B_{n-1} = I$ λ^{n-1}: $B_{n-2} - AB_{n-1} = c_{n-1}I$ ... λ^1: $B_0 - AB_1 = c_1 I$ λ^0: $-AB_0 = c_0 I$

Hint 2.1.1.6. From the coefficient equations, express $B_{n-2}, B_{n-3}, \ldots, B_0$ recursively in terms of A and the coefficients c_k. $B_{n-1} = I$ $B_{n-2} = AB_{n-1} + c_{n-1}I = A + c_{n-1}I$ $B_{n-3} = AB_{n-2} + c_{n-2}I = A(A + c_{n-1}I) + c_{n-2}I = A^2 + c_{n-1}A + c_{n-2}I$ ... Continue this pattern. What is the general expression for B_k?

Hint 2.1.1.7. The constant term of the polynomial $B(\lambda) = \operatorname{adj}(\lambda I - A)$ is B_0. Find B_0 by setting $\lambda = 0$: $B(0) = \operatorname{adj}(-A)$.

Hint 2.1.1.8. We know $\operatorname{adj}(-A) = (-1)^{n-1}\operatorname{adj}(A)$. So, $B_0 = (-1)^{n-1}\operatorname{adj}(A)$.

Hint 2.1.1.9. From the recursive definition in Hint 6, what expression do you get for B_0? It should be $B_0 = A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I$. (Verify this also satisfies $-AB_0 = c_0 I$ using Cayley-Hamilton).

Hint 2.1.1.10. Equate the two expressions for B_0: $(-1)^{n-1}\operatorname{adj}(A) = A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I$. Does this match the target expression $\operatorname{adj}(A) = -(A^{n-1} + \cdots + c_1 I)$? It matches if n is even. If n is odd, $(-1)^{n-1} = 1$, giving $\operatorname{adj}(A) = A^{n-1} + \cdots + c_1 I$. There might be a sign convention difference in the polynomial definition or the source formula[cite: 148]. Let's re-check the source: it uses $-(\ldots)$. The standard derivation often leads to $(-1)^{n-1}(\ldots)$. Let's assume the source formula [cite: 148] is the target. How can we reconcile this? Check the relation $c_0 = \det(-A) = (-1)^n \det(A)$[cite: 149]. The equation $-AB_0 = c_0 I$ is key. If $B_0 = (-1)^{n-1}\operatorname{adj}(A)$, then $-A((-1)^{n-1}\operatorname{adj}(A)) = c_0 I$. $(-1)^n A \operatorname{adj}(A) = c_0 I$. $(-1)^n \det(A) I = c_0 I$. This matches $c_0 = (-1)^n \det(A)$. Now equate $B_0 = A^{n-1} + \cdots + c_1 I$ with $B_0 = (-1)^{n-1}\operatorname{adj}(A)$ to solve for $\operatorname{adj}(A)$. $\operatorname{adj}(A) = (-1)^{n-1}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I)$. Compare with[cite: 148]. The formula in [cite: 148] seems to be off by a factor of $(-1)^{n-1}$ based on this standard derivation unless their c_k definition differs. Assuming the derivation is correct, the result is $\operatorname{adj}(A) = (-1)^{n-1}(A^{n-1} + \cdots + c_1 I)$.

Question 2.1.2. *Let* $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. *Find the eigenvalues* λ *of* A. *For each eigenvalue, compute* $B = A - \lambda I$ *and find a non-zero column of* $\operatorname{adj}(B)$. *Verify that this column is an eigenvector of* A *corresponding to* λ.

Hint 2.1.2.1. First, find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$.

Hint 2.1.2.2. For the first eigenvalue λ_1, calculate the matrix $B_1 = A - \lambda_1 I$.

Hint 2.1.2.3. Calculate $\operatorname{adj}(B_1)$. Since $A - \lambda_1 I$ is singular, $\det(B_1) = 0$. What does this imply about $B_1 \operatorname{adj}(B_1)$?

Hint 2.1.2.4. Find $\operatorname{adj}(B_1)$ using the 2×2 shortcut. Is it the zero matrix? (It shouldn't be if the eigenvalue has geometric multiplicity 1, which is true here).

Hint 2.1.2.5. Let $\mathbf{c}_1$ be any non-zero column of $\operatorname{adj}(B_1)$. Verify that $B_1\mathbf{c}_1 = \mathbf{0}$. (This follows from $B_1 \operatorname{adj}(B_1) = 0$).

Hint 2.1.2.6. Since $B_1\mathbf{c}_1 = (A - \lambda_1 I)\mathbf{c}_1 = \mathbf{0}$, rewrite this as $A\mathbf{c}_1 = \lambda_1\mathbf{c}_1$. Does this confirm $\mathbf{c}_1$ is an eigenvector for λ_1?

Hint 2.1.2.7. Repeat steps 2-6 for the second eigenvalue λ_2.

3 Further Exploration of Identities and Properties

Problem 3.1. *Adjugate Iteration and Determinants*

Question 3.1.1. *Let A be an $n \times n$ matrix. Find an expression for* $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A)))$ *in terms of A and* $\det(A)$. *Assume $n \geq 2$.*

Hint 3.1.1.1. Let $B = \operatorname{adj}(A)$. We know $\operatorname{adj}(B) = \operatorname{adj}(\operatorname{adj}(A)) = (\det A)^{n-2}A$[cite: 67].

Hint 3.1.1.2. Let $C = \operatorname{adj}(B) = \operatorname{adj}(\operatorname{adj}(A))$. We want to find $\operatorname{adj}(C)$.

Hint 3.1.1.3. Use the formula $\operatorname{adj}(C) = \det(C)C^{-1}$ if C is invertible. When is $C = (\det A)^{n-2}A$ invertible? (Assume A is invertible for now).

Hint 3.1.1.4. Calculate $\det(C) = \det((\det A)^{n-2}A)$. Use $\det(kA) = k^n \det(A)$. $\det(C) = ((\det A)^{n-2})^n \det(A) = (\det A)^{n(n-2)+1}$.

Hint 3.1.1.5. Calculate $C^{-1} = ((\det A)^{n-2}A)^{-1}$. Use $(kA)^{-1} = k^{-1}A^{-1}$. $C^{-1} = ((\det A)^{n-2})^{-1}A^{-1} = (\det A)^{-(n-2)}A^{-1}$.

Hint 3.1.1.6. Substitute $\det(C)$ and C^{-1} into $\operatorname{adj}(C) = \det(C)C^{-1}$. Simplify the expression involving powers of $\det(A)$. Recall $A^{-1} = (1/\det A)\operatorname{adj}(A)$.

Hint 3.1.1.7. $\operatorname{adj}(C) = (\det A)^{n(n-2)+1}(\det A)^{-(n-2)}A^{-1} = (\det A)^{n^2-2n+1-n+2}A^{-1} = (\det A)^{n^2-3n+3}A^{-1}$. This is one form. Can we relate it back to A or $\operatorname{adj}(A)$?

Hint 3.1.1.8. Alternative approach: Apply the $\operatorname{adj}(\operatorname{adj}(X))$ formula to $X = \operatorname{adj}(A)$. $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A))) = (\det(\operatorname{adj} A))^{n-2} \operatorname{adj}(A)$.

Hint 3.1.1.9. Substitute $\det(\operatorname{adj} A) = (\det A)^{n-1}$. $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A))) = ((\det A)^{n-1})^{n-2} \operatorname{adj}(A) = (\det A)^{(n-1)(n-2)} \operatorname{adj}(A)$.

Hint 3.1.1.10. Does this formula hold even if A is singular? Consider the ranks. If $\operatorname{rank}(A) \leq n-2$, then $\operatorname{adj}(A) = 0$, $\operatorname{adj}(\operatorname{adj}(A)) = 0$, $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A))) = 0$. The formula gives $(\det A)^k \operatorname{adj}(A) = 0 \cdot 0 = 0$. OK. If $\operatorname{rank}(A) = n-1$, then $\det A = 0$. $\operatorname{adj}(A)$ has rank 1. $\operatorname{adj}(\operatorname{adj}(A)) = 0$ (if $n \geq 3$). Then $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A))) = \operatorname{adj}(0) = 0$ (if $n \geq 3$). The formula gives $0^k \operatorname{adj}(A) = 0$. OK for $n \geq 3$. Check $n = 2$: $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A))) = (\det A)^{(2-1)(2-2)} \operatorname{adj}(A) = (\det A)^0 \operatorname{adj}(A) = \operatorname{adj}(A)$. Let's verify. $\operatorname{adj}(A)$ is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. $\operatorname{adj}(\operatorname{adj}(A)) = A$. $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A))) = \operatorname{adj}(A)$. It holds. So the formula is $(\det A)^{(n-1)(n-2)} \operatorname{adj}(A)$.

Problem 3.2. *Rank Relationships Nuances*

Question 3.2.1. *Let A be an $n \times n$ matrix with $n \geq 3$. Suppose* $\text{rank}(A) = n - 1$. *We know* $\text{rank}(\text{adj}(A)) = 1$. *Let* $\mathbf{u}$ *span the 1D nullspace of* A^T *(left nullspace of A) and* $\mathbf{v}$ *span the 1D nullspace of A (right nullspace). Show that* $\text{adj}(A)$ *can be written in the form* $\text{adj}(A) = c\mathbf{u}\mathbf{v}^T$ *for some non-zero scalar c.*

Hint 3.2.1.1. Since $\text{rank}(\text{adj}(A)) = 1$, all columns of $\text{adj}(A)$ are scalar multiples of a single non-zero vector, say $\mathbf{w}_1$. Similarly, all rows of $\text{adj}(A)$ are scalar multiples of a single non-zero vector, say $\mathbf{w}_2^T$. This is a property of rank 1 matrices, which can always be written as an outer product $\mathbf{a}\mathbf{b}^T$. So $\text{adj}(A) = \mathbf{a}\mathbf{b}^T$.

Hint 3.2.1.2. We know that $A\,\text{adj}(A) = 0$. Substituting $\text{adj}(A) = \mathbf{a}\mathbf{b}^T$, we get $A(\mathbf{a}\mathbf{b}^T) = (A\mathbf{a})\mathbf{b}^T = 0$. Since $\mathbf{b}^T$ is non-zero (otherwise $\text{adj}(A) = 0$), this implies $A\mathbf{a} = \mathbf{0}$. What does this tell us about the vector $\mathbf{a}$ in relation to the nullspace of A?

Hint 3.2.1.3. Since the nullspace of A is spanned by $\mathbf{v}$, $\mathbf{a}$ must be a scalar multiple of $\mathbf{v}$. So, $\mathbf{a} = k_1\mathbf{v}$ for some $k_1 \neq 0$.

Hint 3.2.1.4. Now consider $\text{adj}(A)A = 0$. Substituting $\text{adj}(A) = \mathbf{a}\mathbf{b}^T$, we get $(\mathbf{a}\mathbf{b}^T)A = \mathbf{a}(\mathbf{b}^T A) = 0$. Since $\mathbf{a}$ is non-zero, this implies $\mathbf{b}^T A = \mathbf{0}^T$.

Hint 3.2.1.5. Taking the transpose, $A^T\mathbf{b} = \mathbf{0}$. What does this tell us about the vector $\mathbf{b}$ in relation to the nullspace of A^T?

Hint 3.2.1.6. Since the nullspace of A^T is spanned by $\mathbf{u}$, $\mathbf{b}$ must be a scalar multiple of $\mathbf{u}$. So, $\mathbf{b} = k_2\mathbf{u}$ for some $k_2 \neq 0$.

Hint 3.2.1.7. Substitute $\mathbf{a} = k_1\mathbf{v}$ and $\mathbf{b} = k_2\mathbf{u}$ back into $\text{adj}(A) = \mathbf{a}\mathbf{b}^T$. $\text{adj}(A) = (k_1\mathbf{v})(k_2\mathbf{u})^T = (k_1k_2)\mathbf{v}\mathbf{u}^T$. Wait, the target was $c\mathbf{u}\mathbf{v}^T$. Let's re-check the standard definition.

Hint 3.2.1.8. Rethink: The columns of $\text{adj}(A)$ are in the nullspace of A (spanned by $\mathbf{v}$). So $\text{adj}(A)$ must have the form $\mathbf{v}\mathbf{y}^T$ for some vector $\mathbf{y}$. The rows of $\text{adj}(A)$ are related to the nullspace of A^T. $(\text{adj}\,A)^T = \text{adj}(A^T)$. The columns of $\text{adj}(A^T)$ are in the nullspace of A^T (spanned by $\mathbf{u}$). So $\text{adj}(A^T) = \mathbf{u}\mathbf{z}^T$. Then $\text{adj}(A) = (\text{adj}(A^T))^T = (\mathbf{u}\mathbf{z}^T)^T = \mathbf{z}\mathbf{u}^T$.

Hint 3.2.1.9. Comparing the two forms: $\text{adj}(A) = \mathbf{v}\mathbf{y}^T = \mathbf{z}\mathbf{u}^T$. This implies $\mathbf{z}$ must be proportional to $\mathbf{v}$ and $\mathbf{y}$ must be proportional to $\mathbf{u}$. Let $\mathbf{z} = c\mathbf{v}$ and $\mathbf{y} = d\mathbf{u}$. Then $\text{adj}(A) = c\mathbf{v}\mathbf{u}^T$. This seems consistent. Where did the target $c\mathbf{u}\mathbf{v}^T$ come from? Let's re-read the question statement. Ah, it asks to show $\text{adj}(A) = c\mathbf{u}\mathbf{v}^T$. Let's test this form.

Hint 3.2.1.10. If $\text{adj}(A) = c\mathbf{u}\mathbf{v}^T$. Check $A\,\text{adj}(A) = A(c\mathbf{u}\mathbf{v}^T) = c(A\mathbf{u})\mathbf{v}^T$. Is $A\mathbf{u} = 0$? No, $\mathbf{u}$ is in the left nullspace ($A^T\mathbf{u} = 0$). Check $\text{adj}(A)A = (c\mathbf{u}\mathbf{v}^T)A = c\mathbf{u}(\mathbf{v}^T A)$. Is $\mathbf{v}^T A = \mathbf{0}^T$? Yes, since $A^T\mathbf{v} = \mathbf{0}$ is not necessarily true. We know $A\mathbf{v} = 0$. So $(\mathbf{v}^T A)^T = A^T(\mathbf{v}^T)^T = A^T\mathbf{v}$. No, this doesn't help. We know $A\mathbf{v} = 0$.

Hint 3.2.1.11. Let's reconsider $A\,\text{adj}(A) = 0$. Columns of $\text{adj}(A)$ must be in nullspace(A), spanned by $\mathbf{v}$. So $\text{adj}(A) = \mathbf{v}\mathbf{y}^T$. Let's reconsider $\text{adj}(A)A = 0$. Rows of $\text{adj}(A)$ must be in the left nullspace of A (nullspace of A^T), spanned by $\mathbf{u}^T$. So $\text{adj}(A) = \mathbf{z}\mathbf{u}^T$. Equating these gives $\mathbf{v}\mathbf{y}^T = \mathbf{z}\mathbf{u}^T$. This holds if $\mathbf{z} = k\mathbf{v}$ and $\mathbf{y}^T = l\mathbf{u}^T$. So $\text{adj}(A) = (k\mathbf{v})(l\mathbf{u})^T = (kl)\mathbf{v}\mathbf{u}^T$. This form seems correct. Let $c = kl$. $\text{adj}(A) = c\mathbf{v}\mathbf{u}^T$. The question might have swapped $\mathbf{u}$ and $\mathbf{v}$ roles or their definitions (row vs column). Assuming $\mathbf{u}$ spans $\text{Null}(A^T)$ and $\mathbf{v}$ spans $\text{Null}(A)$, the result is $\text{adj}(A) = c\mathbf{v}\mathbf{u}^T$. If the question intended $\mathbf{u}$ for $\text{Null}(A)$ and $\mathbf{v}$ for $\text{Null}(A^T)$, the result follows. Let's stick to the derivation $\text{adj}(A) = c\mathbf{v}\mathbf{u}^T$.

4 Block Matrices and Cramer's Rule

Problem 4.1. *Block Matrix Adjugate (Specific Entries)*

Question 4.1.1. *Let* $A = \begin{pmatrix} B & \mathbf{u} \\ \mathbf{v}^T & \alpha \end{pmatrix}$, *where* B *is* $(n-1) \times (n-1)$, $\mathbf{u}, \mathbf{v}$ *are column vectors, and* α *is a scalar. Find the entry in the bottom-right corner* (n, n) *of* $\operatorname{adj}(A)$, *and find the column vector in the last column, positions* 1 *to* $n-1$ *[see cite: 264-287 for context].*

Hint 4.1.1.1. The entry $(\operatorname{adj} A)_{nn}$ is the cofactor $C_{nn}(A)$. How is this cofactor defined?

Hint 4.1.1.2. $C_{nn}(A) = (-1)^{n+n} M_{nn}(A) = M_{nn}(A)$. What is the minor $M_{nn}(A)$? It's the determinant of the matrix obtained by removing row n and column n from A. What matrix block remains? Calculate its determinant.

Hint 4.1.1.3. Now consider the entries $(\operatorname{adj} A)_{in}$ for $i = 1, \ldots, n-1$. These are the cofactors $C_{ni}(A)$.

Hint 4.1.1.4. $C_{ni}(A) = (-1)^{n+i} M_{ni}(A)$. What is the minor $M_{ni}(A)$? It's the determinant of the matrix obtained by removing row n and column i from A. Let's call this submatrix S_{ni}.

Hint 4.1.1.5. Write down the structure of S_{ni}. Its rows are the first $n-1$ rows of A. Its columns are the first $n-1$ columns of A excluding column i, plus the last column of A. $S_{ni} = \begin{pmatrix} B_{\text{no col } i} & \mathbf{u} \end{pmatrix}$. This requires careful interpretation. Let's reconsider M_{ni}. Remove row n $(\mathbf{v}^T, \alpha)$ and column i (column i of B, and v_i). The remaining matrix is $\begin{pmatrix} B_{\text{no row } n, \text{ col } i} & \mathbf{u}_{\text{no row } n?} \\ \mathbf{v}^T_{\text{no col } i?} & ?? \end{pmatrix}$. No, this is getting confusing.

Hint 4.1.1.6. Let's use a known formula (e.g., from Wikipedia's "Adjugate matrix" page under block matrices, often derived alongside the block inverse formula). The block corresponding to $(\operatorname{adj} A)_{1:n-1,n}$ is given by $-\operatorname{adj}(B)\mathbf{u}$.

Hint 4.1.1.7. Let's try to verify this. Recall $A \operatorname{adj}(A) = \det(A) I$. The last column of this product should be $\det(A)\mathbf{e}_n$. This column is $A \cdot$ (last col of $\operatorname{adj} A$). Let the last column of $\operatorname{adj} A$ be $\begin{pmatrix} \mathbf{y} \\ \delta \end{pmatrix}$. We found $\delta = C_{nn} = \det(B)$. Let's assume $\mathbf{y} = -\operatorname{adj}(B)\mathbf{u}$. Then $A \begin{pmatrix} -\operatorname{adj}(B)\mathbf{u} \\ \det(B) \end{pmatrix} = \begin{pmatrix} B & \mathbf{u} \\ \mathbf{v}^T & \alpha \end{pmatrix} \begin{pmatrix} -\operatorname{adj}(B)\mathbf{u} \\ \det(B) \end{pmatrix} = \begin{pmatrix} -B \operatorname{adj}(B)\mathbf{u} + \mathbf{u} \det(B) \\ -\mathbf{v}^T \operatorname{adj}(B)\mathbf{u} + \alpha \det(B) \end{pmatrix}$.

Hint 4.1.1.8. Simplify the top block: $-\det(B) I \mathbf{u} + \det(B)\mathbf{u} = -\det(B)\mathbf{u} + \det(B)\mathbf{u} = \mathbf{0}$. This matches the top $n-1$ entries of $\det(A)\mathbf{e}_n$, which are zeros.

Hint 4.1.1.9. Simplify the bottom block: $\alpha \det(B) - \mathbf{v}^T \operatorname{adj}(B)\mathbf{u}$. Recall the formula for the determinant of a block matrix (Schur complement): $\det(A) = \alpha \det(B) - \mathbf{v}^T \operatorname{adj}(B)\mathbf{u}$. This matches the required (n, n) entry of $\det(A) I$.

Hint 4.1.1.10. So, the bottom-right entry is $\det(B)$ and the rest of the last column is the vector $-\operatorname{adj}(B)\mathbf{u}$.

Question 4.1.2. *Explain how Cramer's Rule* $x_i = \frac{\det(A_i)}{\det(A)}$ *for solving* $A\mathbf{x} = \mathbf{b}$ *follows directly from the adjugate inverse formula* $\mathbf{x} = \frac{1}{\det(A)} \operatorname{adj}(A)\mathbf{b}$.

Hint 4.1.2.1. Start with $\mathbf{x} = \frac{1}{\det(A)} \operatorname{adj}(A)\mathbf{b}$. Let's look at the i-th component, x_i.

Hint 4.1.2.2. $x_i = \frac{1}{\det(A)}(\text{row } i \text{ of } \operatorname{adj}(A)) \cdot \mathbf{b}$.

Hint 4.1.2.3. What are the entries of row i of $\operatorname{adj}(A)$? Remember $(\operatorname{adj} A)_{ij} = C_{ji}(A)$. So row i is $((\operatorname{adj} A)_{i1}, \ldots, (\operatorname{adj} A)_{in}) = (C_{1i}(A), C_{2i}(A), \ldots, C_{ni}(A))$.

Hint 4.1.2.4. Substitute this into the dot product: $x_i = \frac{1}{\det(A)} \sum_{j=1}^{n} (\operatorname{adj} A)_{ij} b_j = \frac{1}{\det(A)} \sum_{j=1}^{n} C_{ji}(A) b_j$.

Hint 4.1.2.5. Consider the sum $\sum_{j=1}^{n} b_j C_{ji}(A)$. Recall the Laplace (cofactor) expansion for a determinant along a column k: $\det(M) = \sum_{j=1}^{n} M_{jk} C_{jk}(M)$.

Hint 4.1.2.6. Let A_i be the matrix A with its i-th column replaced by the vector $\mathbf{b}$. Consider the Laplace expansion of $\det(A_i)$ along its i-th column. The entries of this column are $b_1, \ldots, b_n$. The cofactors $C_{ji}(A_i)$ for this column are calculated from the submatrices formed by removing row j and column i from A_i. Are these submatrices the same as those used to calculate $C_{ji}(A)$? Yes, because column i was removed.

Hint 4.1.2.7. Therefore, the Laplace expansion of $\det(A_i)$ along column i is $\det(A_i) = \sum_{j=1}^{n} (A_i)_{ji} C_{ji}(A_i) = \sum_{j=1}^{n} b_j C_{ji}(A)$.

Hint 4.1.2.8. Compare the sum in Hint 7 with the sum in Hint 4. They are identical.

Hint 4.1.2.9. Substitute $\sum_{j=1}^{n} C_{ji}(A) b_j = \det(A_i)$ back into the expression for x_i from Hint 4. Does this yield Cramer's Rule?