Classical Adjoint (Adjugate) Matrix: Problem Set

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1 Introduction: Defining the Classical Adjoint

1.1 Formal Definition and Notation

Problem 1.1. *Understanding Minors and Cofactors Let* $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

Question 1.1.1. *Calculate the minor* M_{23} *and the cofactor* C_{23} *of* A*[cite: 4, 5, 6].*

Hint 1.1.1.1. Recall that the minor M_{ij} is the determinant of the submatrix formed by removing the i-th row and j-th column of A[cite: 5]. Which row and column should you remove for M_{23}?

Hint 1.1.1.2. After removing the 2nd row and 3rd column, what 2×2 matrix remains? Calculate its determinant. This is M_{23}[cite: 5].

Hint 1.1.1.3. The cofactor C_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$[cite: 6]. Use the value of M_{23} you found and the appropriate sign based on $i = 2, j = 3$ to find C_{23}[cite: 6, 11].

Question 1.1.2. *Explain why the entry in the* i*-th row and* j*-th column of* $\operatorname{adj}(A)$ *is* C_{ji}*, not* C_{ij}*[cite: 8, 9, 10].*

Hint 1.1.2.1. Start with the definition of the cofactor matrix C, whose (i, j)-th entry is C_{ij}[cite: 7].

Hint 1.1.2.2. How is the adjugate matrix $\operatorname{adj}(A)$ formally defined in terms of the cofactor matrix C? [cite: 8]

Hint 1.1.2.3. What operation transforms a matrix M such that its (i, j)-th entry becomes the (j, i)-th entry of the original matrix? How does this apply to the relationship between C and $\operatorname{adj}(A)$? [cite: 8, 9]

Hint 1.1.2.4. Consider the fundamental identity $A \cdot \operatorname{adj}(A) = \det(A) I$[cite: 29]. Think about the dot product of the i-th row of A with the j-th column of $\operatorname{adj}(A)$. What elements from the definition of $\operatorname{adj}(A)$ does this involve? [cite: 30, 31, 32] Does using C_{ji} in the (i, j) position of $\operatorname{adj}(A)$ lead to the correct identity?

2 Basic Computation of the Adjoint

2.1 Adjoint of a 2x2 Matrix

Problem 2.1. *Computing 2x2 Adjoints*

Question 2.1.1. *Let* $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. *Compute* $\operatorname{adj}(A)$*[cite: 12, 15, 16]. What does the result represent geometrically?*

Hint 2.1.1.1. Recall the shortcut for finding the adjugate of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$[cite: 16]. What happens to the diagonal elements? What happens to the off-diagonal elements?

Hint 2.1.1.2. Apply the shortcut to the given matrix A with $a = \cos\theta$, $b = -\sin\theta$, $c = \sin\theta$, $d = \cos\theta$[cite: 15, 16].

Hint 2.1.1.3. Compare the resulting matrix $\operatorname{adj}(A)$ with the original matrix A and its transpose A^T. What transformation does A represent? What transformation does $\operatorname{adj}(A)$ represent? (Think about rotations).

Hint 2.1.1.4. Consider the determinant of A. How does $\operatorname{adj}(A)$ relate to A^{-1} when A represents a rotation? [cite: 38, 130]

Question 2.1.2. *Verify the formula* $\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ *for* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *by explicitly calculating the cofactor matrix* C *and transposing it[cite: 13, 14, 15].*

Hint 2.1.2.1. Calculate the four minors $M_{11}, M_{12}, M_{21}, M_{22}$[cite: 13]. Remember M_{ij} is the determinant of the 1×1 matrix obtained by removing row i and column j. The determinant of (x) is just x.

Hint 2.1.2.2. Calculate the four cofactors $C_{11}, C_{12}, C_{21}, C_{22}$ using $C_{ij} = (-1)^{i+j} M_{ij}$ and form the cofactor matrix C[cite: 6, 14]. Pay attention to the signs!

Hint 2.1.2.3. Find the transpose of the cofactor matrix C. Does it match the shortcut formula for $\operatorname{adj}(A)$? [cite: 8, 15]

2.2 Adjoint of a 3x3 Matrix

Problem 2.2. *Computing a 3x3 Adjoint*

Question 2.2.1. *Calculate* $\operatorname{adj}(A)$ *for* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ *[cite: 20, 21, 22, 23, 24].*

Hint 2.2.1.1. First, find the 3×3 matrix of minors. For each element A_{ij}, calculate M_{ij} by finding the determinant of the 2×2 submatrix obtained by removing row i and column j[cite: 20, 21]. For example, $M_{11} = \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Calculate all nine minors[cite: 22].

Hint 2.2.1.2. Next, find the cofactor matrix C. Apply the checkerboard pattern of signs $+-+, -+-, +-+$ to the matrix of minors, or equivalently, compute $C_{ij} = (-1)^{i+j} M_{ij}$ for each entry[cite: 22, 23].

Hint 2.2.1.3. Finally, the adjugate matrix $\text{adj}(A)$ is the transpose of the cofactor matrix C[cite: 24]. Swap the rows and columns of C.

Hint 2.2.1.4. Double-check your calculation by computing $A \cdot \text{adj}(A)$. Does it equal $\det(A)I_3$? [cite: 29] First compute $\det(A)$.

Problem 2.3. *Adjoint of a Singular Matrix*

Question 2.3.1. *Let* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ *[cite: 25]. Calculate* $\text{adj}(A)$. *What do you notice about the columns (or rows) of* $\text{adj}(A)$*? Relate this to the determinant and rank of* A.

Hint 2.3.1.1. Follow the three steps: Calculate the matrix of minors, apply signs to get the cofactor matrix, and transpose to get the adjugate[cite: 20, 22, 24].

Hint 2.3.1.2. Notice that calculating the minors might yield some zeros or simple relationships. For instance, $M_{11} = 5 \times 9 - 6 \times 8 = 45 - 48 = -3$. $M_{12} = 4 \times 9 - 6 \times 7 = 36 - 42 = -6$. $M_{13} = 4 \times 8 - 5 \times 7 = 32 - 35 = -3$. Continue for all nine minors.

Hint 2.3.1.3. After finding $\text{adj}(A)$, observe its columns. Are they related? Are they scalar multiples of each other? [cite: 85]

Hint 2.3.1.4. Calculate $\det(A)$. You can use cofactor expansion or notice that the columns (or rows) are linearly dependent (e.g., Col2 - Col1 = Col3 - Col2). What does $\det(A) = 0$ imply about $A \cdot \text{adj}(A)$? [cite: 29, 39]

Hint 2.3.1.5. What is the rank of A? (It's not 3 since $\det(A) = 0$. Is it 2 or 1?)[cite: 81]. How does the rank of A relate to the rank of $\text{adj}(A)$? Does your calculated $\text{adj}(A)$ have the expected rank? [cite: 80, 81, 82]

3 Fundamental Properties and Theorems

3.1 The Fundamental Identity

Problem 3.1. *Applying the Core Identity*

Question 3.1.1. *Let* A *be a* 3×3 *matrix with* $\det(A) = 5$. *Without calculating* A *or* $\text{adj}(A)$ *explicitly, find the matrix product* $A \cdot \text{adj}(A)$.

Hint 3.1.1.1. Recall the fundamental identity relating A, $\text{adj}(A)$, and $\det(A)$[cite: 29].

Hint 3.1.1.2. The identity is $A \cdot \text{adj}(A) = \det(A)I_n$[cite: 29]. What is n in this case? What is I_n?

Hint 3.1.1.3. Substitute the given value of $\det(A)$ and the appropriate identity matrix I_n into the formula.

Question 3.1.2. *Suppose A is an $n \times n$ matrix such that $A \cdot \operatorname{adj}(A) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$. What is $\det(A)$? Is A invertible? [cite: 29, 38]*

Hint 3.1.2.1. Compare the given matrix product with the general form $A \cdot \operatorname{adj}(A) = \det(A)I_n$[cite: 29]. What must $\det(A)$ be? What is n?

Hint 3.1.2.2. A matrix is invertible if and only if its determinant is non-zero[cite: 38]. Based on the value you found for $\det(A)$, can you conclude whether A is invertible?

3.2 Invertibility and the Adjoint

Problem 3.2. *Adjoint and Inverse*

Question 3.2.1. *Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find A^{-1} using the formula involving the adjugate[cite: 38].*

Hint 3.2.1.1. The formula is $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$, provided $\det(A) \neq 0$[cite: 38]. First, calculate $\det(A)$. Is it non-zero?

Hint 3.2.1.2. Next, calculate $\operatorname{adj}(A)$ using the shortcut for 2×2 matrices[cite: 16].

Hint 3.2.1.3. Substitute $\det(A)$ and $\operatorname{adj}(A)$ into the inverse formula.

Question 3.2.2. *Let A be an invertible $n \times n$ matrix. Express $\operatorname{adj}(A)$ in terms of A^{-1} and $\det(A)$[cite: 38, 41]. Use this to argue why $\operatorname{adj}(A)$ must also be invertible.*

Hint 3.2.2.1. Start with the formula for the inverse: $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$[cite: 38].

Hint 3.2.2.2. Rearrange the formula algebraically to solve for $\operatorname{adj}(A)$. Remember $\det(A)$ is just a non-zero scalar since A is invertible.

Hint 3.2.2.3. You should arrive at $\operatorname{adj}(A) = \det(A)A^{-1}$[cite: 41]. Since A is invertible, A^{-1} exists. Is the product of a non-zero scalar and an invertible matrix also invertible? Why?

Hint 3.2.2.4. Alternatively, consider the determinant of $\operatorname{adj}(A)$. We know $\det(\operatorname{adj}(A)) = (\det A)^{n-1}$[cite: 57]. If A is invertible, what does this tell you about $\det(\operatorname{adj}(A))$ (assuming $n > 1$)? [cite: 40]

Question 3.2.3. *If A is a singular 3×3 matrix, can $\operatorname{adj}(A)$ be invertible? Justify your answer using the relationship between the rank of A and the rank of $\operatorname{adj}(A)$[cite: 39, 40, 42, 80, 81, 82]. Consider the special case $n = 1$ separately[cite: 44].*

Hint 3.2.3.1. What does it mean for A to be singular? [cite: 39] What are the possible ranks for a singular 3×3 matrix A?

Hint 3.2.3.2. Recall the rules relating $\operatorname{rank}(A)$ and $\operatorname{rank}(\operatorname{adj}(A))$ for an $n \times n$ matrix[cite: 80]: If $\operatorname{rank}(A) = n$, then $\operatorname{rank}(\operatorname{adj}(A)) = n$. If $\operatorname{rank}(A) = n - 1$, then $\operatorname{rank}(\operatorname{adj}(A)) = 1$. If $\operatorname{rank}(A) \leq n - 2$, then $\operatorname{rank}(\operatorname{adj}(A)) = 0$.

Hint 3.2.3.3. Apply these rules for $n = 3$ and the possible ranks of the singular matrix A. What are the possible ranks for $\operatorname{adj}(A)$?

Hint 3.2.3.4. For a matrix to be invertible, what must its rank be? Can $\operatorname{adj}(A)$ have this rank if A is singular and $n = 3$? [cite: 40]

Hint 3.2.3.5. What happens in the $n = 1$ case? If $A = (0)$, what is $\operatorname{adj}(A)$? Is it invertible? [cite: 44, 141]

4 Matrix Identities Involving the Adjoint

Problem 4.1. *Exploring Adjoint Identities*

Question 4.1.1. *Prove that* $\mathrm{adj}(A^T) = (\mathrm{adj}(A))^T$ *for any* $n \times n$ *matrix* A*[cite: 45].*

Hint 4.1.1.1. Let $B = A^T$. We want to show $\mathrm{adj}(B) = (\mathrm{adj}(A))^T$. Let's look at the (i, j)-th entry of each side. The (i, j)-th entry of $(\mathrm{adj}(A))^T$ is the (j, i)-th entry of $\mathrm{adj}(A)$. What is this entry by definition? [cite: 9]

Hint 4.1.1.2. The (j, i)-th entry of $\mathrm{adj}(A)$ is $C_{ij}(A) = (-1)^{i+j} M_{ij}(A)$, where the cofactor and minor are calculated from A[cite: 9].

Hint 4.1.1.3. Now consider the (i, j)-th entry of $\mathrm{adj}(B) = \mathrm{adj}(A^T)$. By definition, this is $C_{ji}(B) = (-1)^{j+i} M_{ji}(B)$, where the cofactor and minor are calculated from $B = A^T$[cite: 9].

Hint 4.1.1.4. How does the minor $M_{ji}(B)$ relate to a minor of A? Recall that $B = A^T$. $M_{ji}(B)$ is the determinant of the matrix obtained by removing row j and column i from A^T[cite: 46]. How does this submatrix relate to the submatrix used to calculate $M_{ij}(A)$ (obtained by removing row i and column j from A)? [cite: 46]

Hint 4.1.1.5. Remember that $\det(M) = \det(M^T)$[cite: 46]. Use this to show that $M_{ji}(B) = M_{ij}(A)$.

Hint 4.1.1.6. Substitute $M_{ji}(B) = M_{ij}(A)$ into the expression for the (i, j)-th entry of $\mathrm{adj}(B)$ and compare it to the expression for the (i, j)-th entry of $(\mathrm{adj}(A))^T$. Are they equal? [cite: 47]

Question 4.1.2. *Let* A *and* B *be* 3×3 *matrices with* $\det(A) = 2$ *and* $\det(B) = -3$. *Calculate* $\det(\mathrm{adj}(AB))$.

Hint 4.1.2.1. We need $\det(\mathrm{adj}(M))$ where $M = AB$. Recall the identity relating the determinant of the adjugate to the determinant of the original matrix: $\det(\mathrm{adj}(M)) = (\det M)^{n-1}$[cite: 57]. What is n here?

Hint 4.1.2.2. First, find $\det(M) = \det(AB)$. How does the determinant of a product relate to the determinants of the factors? [cite: 51]

Hint 4.1.2.3. Calculate $\det(AB)$ using the given determinants of A and B.

Hint 4.1.2.4. Now substitute this value for $\det(M)$ and $n = 3$ into the formula $\det(\mathrm{adj}(M)) = (\det M)^{n-1}$.

Hint 4.1.2.5. Alternatively, use $\mathrm{adj}(AB) = \mathrm{adj}(B)\,\mathrm{adj}(A)$[cite: 49]. Then $\det(\mathrm{adj}(AB)) = \det(\mathrm{adj}(B)\,\mathrm{adj}(A))$. How does the determinant behave with products?

Hint 4.1.2.6. $\det(\mathrm{adj}(B)\,\mathrm{adj}(A)) = \det(\mathrm{adj}(B))\det(\mathrm{adj}(A))$. Now use $\det(\mathrm{adj}(X)) = (\det X)^{n-1}$ for $X = A$ and $X = B$[cite: 57]. Does this give the same result?

Question 4.1.3. *Let* A *be a* 4×4 *matrix and* $c = 2$. *If* $\mathrm{adj}(A) = M$, *what is* $\mathrm{adj}(cA)$ *in terms of* M *and* c*? [cite: 54]*

Hint 4.1.3.1. Recall the identity for the adjugate of a scalar multiple: $\mathrm{adj}(cA) = c^{n-1}\,\mathrm{adj}(A)$[cite: 54].

Hint 4.1.3.2. Identify the values of c and n in this problem.

Hint 4.1.3.3. Substitute these values and the fact that $\operatorname{adj}(A) = M$ into the identity.

Question 4.1.4. *If A is a 3×3 matrix with $\det(A) = 4$, find $\det(\operatorname{adj}(\operatorname{adj}(A)))$.*

Hint 4.1.4.1. Let $B = \operatorname{adj}(A)$. We want to find $\det(\operatorname{adj}(B))$. Use the identity $\det(\operatorname{adj}(B)) = (\det B)^{n-1}$[cite: 57]. What is n?

Hint 4.1.4.2. We need $\det(B) = \det(\operatorname{adj}(A))$. Use the same identity again: $\det(\operatorname{adj}(A)) = (\det A)^{n-1}$[cite: 57]. Calculate this value using the given $\det(A)$ and n.

Hint 4.1.4.3. Substitute the value of $\det(B)$ you just found back into the expression from the first hint: $\det(\operatorname{adj}(B)) = (\det B)^{n-1}$.

Hint 4.1.4.4. Alternatively, use the identity $\operatorname{adj}(\operatorname{adj}(A)) = (\det A)^{n-2}A$ for $n \geq 2$[cite: 67].

Hint 4.1.4.5. Take the determinant of both sides of $\operatorname{adj}(\operatorname{adj}(A)) = (\det A)^{n-2}A$. Remember that $\det(kA) = k^n \det(A)$ for a scalar k. Let $k = (\det A)^{n-2}$.

Hint 4.1.4.6. So, $\det(\operatorname{adj}(\operatorname{adj}(A))) = \det((\det A)^{n-2}A) = ((\det A)^{n-2})^n \det(A)$. Simplify this expression using $n = 3$ and $\det(A) = 4$. Does it match the previous result?

Question 4.1.5. *Let A be an invertible $n \times n$ matrix. Show that $(\operatorname{adj}(A))^{-1} = \operatorname{adj}(A^{-1})$.*

Hint 4.1.5.1. We want to show that $\operatorname{adj}(A) \cdot \operatorname{adj}(A^{-1}) = I$. Recall the identity $\operatorname{adj}(XY) = \operatorname{adj}(Y) \operatorname{adj}(X)$[cite: 49]. Apply this with $X = A$ and $Y = A^{-1}$.

Hint 4.1.5.2. What is $XY = AA^{-1}$? What is $\operatorname{adj}(AA^{-1})$? [cite: 141]

Hint 4.1.5.3. So we have $\operatorname{adj}(A^{-1}) \operatorname{adj}(A) = \operatorname{adj}(I) = I$[cite: 49, 141]. Does this directly show $(\operatorname{adj}(A))^{-1} = \operatorname{adj}(A^{-1})$? Yes, by the definition of an inverse.

Hint 4.1.5.4. Alternatively, start with $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$[cite: 38]. And $\operatorname{adj}(A^{-1}) = \det(A^{-1})(A^{-1})^{-1}$[cite: 41].

Hint 4.1.5.5. We know $\det(A^{-1}) = 1/\det(A)$ and $(A^{-1})^{-1} = A$. Substitute these into the second equation: $\operatorname{adj}(A^{-1}) = \frac{1}{\det(A)}A$.

Hint 4.1.5.6. Now, let's find $(\operatorname{adj}(A))^{-1}$. We know $\operatorname{adj}(A) = \det(A)A^{-1}$[cite: 41]. So $(\operatorname{adj}(A))^{-1} = (\det(A)A^{-1})^{-1}$. How does the inverse distribute over a scalar multiple and the matrix?

Hint 4.1.5.7. $(\det(A)A^{-1})^{-1} = (\det A)^{-1}(A^{-1})^{-1} = \frac{1}{\det A}A$. Compare this with the expression for $\operatorname{adj}(A^{-1})$. Are they equal?

Hint 4.1.5.8. Another way: Use the identity $\operatorname{adj}(A^k) = (\operatorname{adj}(A))^k$ [cite: 76] with $k = -1$. Does this identity hold for negative integers if A is invertible? Assume it does for a moment.

5 Relationship with Rank

Problem 5.1. *Rank Relationships*

Question 5.1.1. *Let A be a 5×5 matrix. Determine* $\operatorname{rank}(\operatorname{adj}(A))$ *in each of the following cases: (a)* $\operatorname{rank}(A) = 5$ *(b)* $\operatorname{rank}(A) = 4$ *(c)* $\operatorname{rank}(A) = 3$ *(d)* $\operatorname{rank}(A) = 0$ *[cite: 80, 81, 82]*

Hint 5.1.1.1. Recall the rules governing the relationship between $\operatorname{rank}(A)$ and $\operatorname{rank}(\operatorname{adj}(A))$ for an $n \times n$ matrix[cite: 80]. What is n here?

Hint 5.1.1.2. Case (a): If $\operatorname{rank}(A) = n$, what is $\operatorname{rank}(\operatorname{adj}(A))$? [cite: 80]

Hint 5.1.1.3. Case (b): If $\operatorname{rank}(A) = n - 1$, what is $\operatorname{rank}(\operatorname{adj}(A))$? [cite: 81]

Hint 5.1.1.4. Case (c): If $\operatorname{rank}(A) \le n - 2$, what is $\operatorname{rank}(\operatorname{adj}(A))$? Does $\operatorname{rank}(A) = 3$ satisfy this condition for $n = 5$? [cite: 82]

Hint 5.1.1.5. Case (d): If A is the zero matrix (rank 0), what is $\operatorname{adj}(A)$ (assuming $n > 1$)? What is its rank? [cite: 82, 140] Does this fit the general rule for $\operatorname{rank}(A) \le n - 2$?

Question 5.1.2. *Let A be a 3×3 matrix with* $\operatorname{rank}(A) = 2$. *Let* $\mathbf{v}$ *be a non-zero vector such that* $A\mathbf{v} = \mathbf{0}$. *Explain why every column of* $\operatorname{adj}(A)$ *must be a scalar multiple of* $\mathbf{v}$*[cite: 83, 84, 85, 86].*

Hint 5.1.2.1. If $\operatorname{rank}(A) = 2$ for a 3×3 matrix, what is $\det(A)$? [cite: 81, 83]

Hint 5.1.2.2. What does the fundamental identity $A \cdot \operatorname{adj}(A) = \det(A)I$ become in this case? [cite: 39, 83]

Hint 5.1.2.3. Let $\mathbf{c}_j$ be the j-th column of $\operatorname{adj}(A)$. Consider the product $A \cdot \operatorname{adj}(A)$. The j-th column of this product is $A\mathbf{c}_j$. What does the result from the previous hint tell you about $A\mathbf{c}_j$? [cite: 84]

Hint 5.1.2.4. If $A\mathbf{c}_j = \mathbf{0}$, what space does the vector $\mathbf{c}_j$ belong to? [cite: 84]

Hint 5.1.2.5. What is the dimension of the nullspace (kernel) of A if A is 3×3 and $\operatorname{rank}(A) = 2$? (Rank-Nullity Theorem)[cite: 85].

Hint 5.1.2.6. We are given that $A\mathbf{v} = \mathbf{0}$ with $\mathbf{v} \neq \mathbf{0}$. What does this mean about $\mathbf{v}$ in relation to the nullspace? Since the nullspace has dimension 1, how must any other vector in the nullspace (like the columns $\mathbf{c}_j$) relate to $\mathbf{v}$? [cite: 85, 86]

Hint 5.1.2.7. Also, what is the rank of $\operatorname{adj}(A)$ when $\operatorname{rank}(A) = n - 1 = 2$?[cite: 81]. How does the rank relate to the columns spanning only a 1D space (the nullspace)? [cite: 86]

6 Applications

Problem 6.1. *Inverse and System Solving*

Question 6.1.1. *Find the inverse of* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ *using the adjugate method[cite: 87, 88]. (You may have computed* $\operatorname{adj}(A)$ *in Problem 2.2.1).*

Hint 6.1.1.1. First, compute $\det(A)$. A cofactor expansion along the second row might be efficient here.

Hint 6.1.1.2. Retrieve or re-compute $\operatorname{adj}(A)$ using the standard procedure (minors, cofactors, transpose)[cite: 20, 24].

Hint 6.1.1.3. Apply the formula $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$[cite: 38, 87].

Hint 6.1.1.4. Verify your result by computing AA^{-1}. Should it equal I_3?

Question 6.1.2. *Consider the system* $A\mathbf{x} = \mathbf{b}$ *where* A *is an* $n \times n$ *singular matrix* ($\det(A) = 0$)*. Show that if a solution* $\mathbf{x}$ *exists, it must satisfy* $\operatorname{adj}(A)\mathbf{b} = \mathbf{0}$*[cite: 97, 98, 99, 100]. Is the converse true (if* $\operatorname{adj}(A)\mathbf{b} = \mathbf{0}$*, does a solution necessarily exist)?*

Hint 6.1.2.1. Start with the assumption that a solution $\mathbf{x}$ exists, so $A\mathbf{x} = \mathbf{b}$ holds[cite: 97].

Hint 6.1.2.2. Multiply both sides of the equation $A\mathbf{x} = \mathbf{b}$ on the left by $\operatorname{adj}(A)$[cite: 98].

Hint 6.1.2.3. Use the fundamental identity $\operatorname{adj}(A)A = \det(A)I_n$[cite: 29]. What does this become since A is singular? [cite: 39, 99]

Hint 6.1.2.4. Substitute the result from the previous hint into the equation from Hint 2. What condition must $\operatorname{adj}(A)\mathbf{b}$ satisfy? [cite: 100]

Hint 6.1.2.5. For the converse: $\operatorname{adj}(A)\mathbf{b} = \mathbf{0}$ is a necessary condition for consistency[cite: 100]. Does it guarantee a solution exists? Think about the relationship between $\mathbf{b}$ and the column space of A. The condition $\operatorname{adj}(A)\mathbf{b} = \mathbf{0}$ implies that $\mathbf{b}$ is orthogonal to the rows of $\operatorname{adj}(A)$. How do the rows of $\operatorname{adj}(A)$ relate to the nullspace of A^T? What is the relationship between the nullspace of A^T and the column space of A (Fundamental Theorem of Linear Algebra)? [cite: 102]

Hint 6.1.2.6. Consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Calculate $\operatorname{adj}(A)$ and $\operatorname{adj}(A)\mathbf{b}$. Does a solution to $A\mathbf{x} = \mathbf{b}$ exist? (The column space of A is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$).

7 Properties Under Transformations and Specific Matrix Types

Problem 7.1. *Adjoints of Special Matrices*

Question 7.1.1. *Let* A *be a* 3×3 *orthogonal matrix with* $\det(A) = -1$*. Find a simple expression for* $\operatorname{adj}(A)$ *in terms of* A *or* A^T*[cite: 130, 131].*

Hint 7.1.1.1. What does it mean for A to be orthogonal? What is the relationship between A^{-1} and A^T? [cite: 130]

Hint 7.1.1.2. Recall the formula relating the adjugate and the inverse for an invertible matrix: $\operatorname{adj}(A) = \det(A)A^{-1}$[cite: 128].

Hint 7.1.1.3. Substitute the relationship between A^{-1} and A^T (from Hint 1) and the given value of $\det(A)$ into the formula from Hint 2[cite: 131].

Question 7.1.2. *Prove that if A is symmetric ($A^T = A$), then* $\operatorname{adj}(A)$ *is also symmetric[cite: 132].*

Hint 7.1.2.1. We want to show that $(\operatorname{adj}(A))^T = \operatorname{adj}(A)$.

Hint 7.1.2.2. Recall the identity relating the adjugate of a transpose and the transpose of the adjugate: $(\operatorname{adj}(A))^T = \operatorname{adj}(A^T)$[cite: 45, 132].

Hint 7.1.2.3. Since A is symmetric, what is A^T equal to? Substitute this into the right side of the identity from Hint 2. Does this prove the claim? [cite: 132]

Question 7.1.3. *Let A be an $n \times n$ skew-symmetric matrix ($A^T = -A$). Determine whether* $\operatorname{adj}(A)$ *is symmetric or skew-symmetric, considering whether n is odd or even[cite: 133, 134, 135, 136, 137].*

Hint 7.1.3.1. We need to examine $(\operatorname{adj}(A))^T$. Start with the identity $(\operatorname{adj}(A))^T = \operatorname{adj}(A^T)$[cite: 45, 133].

Hint 7.1.3.2. Since A is skew-symmetric, substitute $A^T = -A$ into the identity: $(\operatorname{adj}(A))^T = \operatorname{adj}(-A)$[cite: 133].

Hint 7.1.3.3. Recall the identity for the adjugate of a scalar multiple: $\operatorname{adj}(cA) = c^{n-1}\operatorname{adj}(A)$[cite: 54]. Apply this with $c = -1$: $\operatorname{adj}(-A) = (-1)^{n-1}\operatorname{adj}(A)$[cite: 133].

Hint 7.1.3.4. Combine the results: $(\operatorname{adj}(A))^T = (-1)^{n-1}\operatorname{adj}(A)$[cite: 133].

Hint 7.1.3.5. Case 1: n is odd. What is $n-1$? What is $(-1)^{n-1}$? What does the equation from Hint 4 become? Does this mean $\operatorname{adj}(A)$ is symmetric or skew-symmetric? [cite: 135, 137]

Hint 7.1.3.6. Case 2: n is even. What is $n-1$? What is $(-1)^{n-1}$? What does the equation from Hint 4 become? Does this mean $\operatorname{adj}(A)$ is symmetric or skew-symmetric? [cite: 136, 137]

Question 7.1.4. *Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$. Describe the matrix* $\operatorname{adj}(D)$*[cite: 138, 139, 140]. What is* $\operatorname{adj}(D)$ *if one of the d_i is zero? What if at least two d_i's are zero (assume $n > 1$)?*

Hint 7.1.4.1. Consider the (i, j)-th entry of $\operatorname{adj}(D)$, which is $C_{ji}(D) = (-1)^{j+i}M_{ji}(D)$[cite: 9].

Hint 7.1.4.2. What is the minor $M_{ji}(D)$ if $j \neq i$? The submatrix used to calculate $M_{ji}(D)$ is obtained by removing row j and column i from the diagonal matrix D. Does this submatrix still have zeros everywhere off the main diagonal? Does it have a zero on its main diagonal? What is its determinant? [cite: 138, 305]

Hint 7.1.4.3. This implies that $C_{ji}(D) = 0$ if $j \neq i$. So, $\operatorname{adj}(D)$ must be a diagonal matrix[cite: 138].

Hint 7.1.4.4. Now consider the diagonal entries (i, i) of $\operatorname{adj}(D)$. These are $C_{ii}(D) = (-1)^{i+i}M_{ii}(D) = M_{ii}(D)$. What is the minor $M_{ii}(D)$? It's the determinant of the diagonal matrix formed by removing row i and column i from D[cite: 139].

Hint 7.1.4.5. The matrix for $M_{ii}(D)$ is $\text{diag}(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n)$. What is its determinant? [cite: 139] This is the i-th diagonal entry of $\text{adj}(D)$[cite: 140].

Hint 7.1.4.6. If exactly one $d_k = 0$. Consider the k-th diagonal entry of $\text{adj}(D)$. It's the product $\prod_{j \neq k} d_j$. Is this zero or non-zero? Now consider any other diagonal entry $i \neq k$. Its value is $\prod_{j \neq i} d_j$. Does this product include d_k? What is its value? What does $\text{adj}(D)$ look like?

Hint 7.1.4.7. If at least two entries, say $d_k = 0$ and $d_l = 0$ $(k \neq l)$, consider any diagonal entry $\prod_{j \neq i} d_j$ of $\text{adj}(D)$. Can this product ever be non-zero? (Think about whether d_k or d_l must be included in the product). What is $\text{adj}(D)$ in this case? [cite: 82]

8 Conceptual, Geometric, and Advanced Topics

Problem 8.1. *Properties and Interpretations*

Question 8.1.1. *Explain why the mapping $F : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ defined by $F(A) = \text{adj}(A)$ is a continuous function[cite: 142, 143, 144, 145].*

Hint 8.1.1.1. What are the entries of the matrix $\text{adj}(A)$? They are cofactors $C_{ji}(A)$[cite: 9].

Hint 8.1.1.2. How is a cofactor $C_{ji}(A)$ defined? It involves a minor $M_{ji}(A)$ and a sign[cite: 6].

Hint 8.1.1.3. How is a minor $M_{ji}(A)$ defined? It's the determinant of a submatrix of A[cite: 5].

Hint 8.1.1.4. Is the determinant function a polynomial function of the entries of a matrix? [cite: 144]

Hint 8.1.1.5. Therefore, are the minors polynomials in the entries of A? Are the cofactors? Are the entries of $\text{adj}(A)$? [cite: 144]

Hint 8.1.1.6. Are polynomial functions continuous? If each entry of $F(A)$ is a continuous function of the entries of A, what does that imply about the continuity of the matrix function $F(A)$? [cite: 145]

Question 8.1.2. *Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix}$ be a 3×3 matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$. Verify the formula stating that the *rows* of $\text{adj}(A)$ are given by $(\mathbf{a}_2 \times \mathbf{a}_3)^T$, $(\mathbf{a}_3 \times \mathbf{a}_1)^T$, and $(\mathbf{a}_1 \times \mathbf{a}_2)^T$[cite: 174, 175]. (Note: The source document discussion is slightly confusing, this formulation aligns with $A\,\text{adj}(A) = \det(A)I$).*

Hint 8.1.2.1. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. The first row of $\text{adj}(A)$ consists of the cofactors C_{11}, C_{21}, C_{31}[cite: 9, 167].

Hint 8.1.2.2. Calculate the cofactor $C_{11} = (-1)^{1+1} M_{11}$. Write out the 2×2 determinant for M_{11}.

Hint 8.1.2.3. Now calculate the cross product $\mathbf{a}_2 \times \mathbf{a}_3$. Write out the components of this vector using the determinant formula for the cross product or component-wise definition.

Hint 8.1.2.4. Compare the first component of $\mathbf{a}_2 \times \mathbf{a}_3$ with the cofactor C_{11}. Are they equal?

Hint 8.1.2.5. Now calculate the cofactor $C_{21} = (-1)^{2+1}M_{21}$. Write out the 2×2 determinant for M_{21}. Compare it with the second component of $\mathbf{a}_2 \times \mathbf{a}_3$. Are they equal? (Pay attention to the sign from $(-1)^{2+1}$).

Hint 8.1.2.6. Calculate $C_{31} = (-1)^{3+1}M_{31}$. Compare it with the third component of $\mathbf{a}_2 \times \mathbf{a}_3$. Are they equal?

Hint 8.1.2.7. Since the components of $(\mathbf{a}_2 \times \mathbf{a}_3)^T$ match the first row of $\text{adj}(A)$ (C_{11}, C_{21}, C_{31}), the first part of the formula is verified. Repeat the process for the second and third rows of $\text{adj}(A)$ using $\mathbf{a}_3 \times \mathbf{a}_1$ and $\mathbf{a}_1 \times \mathbf{a}_2$ respectively.

Question 8.1.3. *Let λ be an eigenvalue of an invertible $n \times n$ matrix A. Show that $\det(A)/\lambda$ is an eigenvalue of $\text{adj}(A)$[cite: 111, 112, 113]. What happens if A is singular?*

Hint 8.1.3.1. Case 1: A is invertible. Start with the relationship $\text{adj}(A) = \det(A)A^{-1}$[cite: 111].

Hint 8.1.3.2. If λ is an eigenvalue of A, what are the eigenvalues of A^{-1}? [cite: 112]

Hint 8.1.3.3. If $M = cN$, how do the eigenvalues of M relate to the eigenvalues of N? Use this with $M = \text{adj}(A)$, $N = A^{-1}$, and $c = \det(A)$ to find the eigenvalues of $\text{adj}(A)$ in terms of λ and $\det(A)$[cite: 113].

Hint 8.1.3.4. Case 2: A is singular ($\det(A) = 0$). Assume $\text{rank}(A) = n - 1$ (the most interesting singular case)[cite: 114]. What is the rank of $\text{adj}(A)$? [cite: 81, 114]

Hint 8.1.3.5. A matrix of rank 1 has at most one non-zero eigenvalue[cite: 115]. What are most of the eigenvalues of $\text{adj}(A)$ in this case? [cite: 116]

Hint 8.1.3.6. Let $A\mathbf{v} = \lambda\mathbf{v}$ with $\lambda \neq 0$. Consider $A\,\text{adj}(A) = 0$. Multiply by $\mathbf{v}$ on the right. Alternatively use $\text{adj}(A)A = 0$. Multiply by $\mathbf{v}$ on the right: $\text{adj}(A)A\mathbf{v} = \mathbf{0}$. Substitute $A\mathbf{v} = \lambda\mathbf{v}$[cite: 117, 118].

Hint 8.1.3.7. You should get $\lambda\,\text{adj}(A)\mathbf{v} = \mathbf{0}$[cite: 118]. Since $\lambda \neq 0$, what does this say about $\text{adj}(A)\mathbf{v}$? What eigenvalue of $\text{adj}(A)$ does the eigenvector $\mathbf{v}$ (corresponding to $\lambda \neq 0$ of A) correspond to? [cite: 119, 120]

Hint 8.1.3.8. What about the eigenvalue $\lambda = 0$ of A? Let $A\mathbf{w} = \mathbf{0}$ where $\mathbf{w}$ is in the nullspace of A. Since $\text{rank}(\text{adj}(A)) = 1$ and its columns are in the nullspace of A, the image of $\text{adj}(A)$ is spanned by $\mathbf{w}$[cite: 124, 125]. Thus $\text{adj}(A)\mathbf{w}$ must be a multiple of $\mathbf{w}$. $\text{adj}(A)\mathbf{w} = \mu\mathbf{w}$. This μ is the potentially non-zero eigenvalue of $\text{adj}(A)$[cite: 126]. Can we determine μ? It's given as $\text{tr}(\text{adj}(A))$[cite: 126].

Hint 8.1.3.9. What if $\text{rank}(A) \leq n - 2$? What is $\text{adj}(A)$? What are its eigenvalues? [cite: 127]

9 Challenge / Contest-Style Problems

Problem 9.1. *From Properties to Solutions*

Question 9.1.1. *If A is a real $n \times n$ matrix such that $\operatorname{adj}(A) = A^T$, what are the possible values for $\det(A)$?*

Hint 9.1.1.1. Start by taking the determinant of both sides of the equation $\operatorname{adj}(A) = A^T$.

Hint 9.1.1.2. Use the identities $\det(\operatorname{adj}(A)) = (\det A)^{n-1}$ [cite: 57] and $\det(A^T) = \det(A)$.

Hint 9.1.1.3. You should arrive at an equation involving only $\det(A)$ and n. Let $d = \det(A)$. The equation is $d^{n-1} = d$.

Hint 9.1.1.4. Solve the equation $d^{n-1} - d = 0$ or $d(d^{n-2} - 1) = 0$ for d. What are the possible real solutions for d? Consider the cases $n = 1$, $n = 2$, and $n > 2$.

Hint 9.1.1.5. If $n = 1$, $A = (a)$, $\operatorname{adj}(A) = (1)$, $A^T = (a)$. So $a = 1$. $\det(A) = 1$. Does $d(d^{1-2} - 1) = 0$ make sense? Let's use $d^{n-1} = d$. If $n = 1$, $d^0 = d$, so $1 = d$.

Hint 9.1.1.6. If $n = 2$, $d(d^0 - 1) = 0$, so $d(1 - 1) = 0$, which is $0 = 0$. This doesn't restrict d. Let's re-evaluate. For $n = 2$, $d^2 = d$. This identity gives no information for $n = 2$. Let's try $A \operatorname{adj}(A) = \det(A)I$. Substitute $\operatorname{adj}(A) = A^T$. So $AA^T = \det(A)I$. Take determinants: $\det(A)\det(A^T) = (\det A)^n$. $(\det A)^2 = (\det A)^n$. $d^2 = d^n$. $d^2(1 - d^{n-2}) = 0$. This implies $d = 0$ or $d^{n-2} = 1$. If $n = 2$, $d^0 = 1$, which is $1 = 1$. Still no restriction. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. So we need $d = a, -b = c, -c = b, a = d$. This means $a = d$ and $c = -b$. Matrix is $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. $\det(A) = a^2 + b^2$. Can this be any non-negative real number? Yes. So for $n = 2$, $\det(A)$ can be any value ≥ 0.

Hint 9.1.1.7. If $n > 2$, $d = 0$ or $d^{n-2} = 1$. Since A is real, $d^{n-2} = 1$ implies $d = 1$ or (if n is even) $d = -1$.

Hint 9.1.1.8. Summarize the possible values for $\det(A)$ based on n.

Question 9.1.2. *Prove the result cited in the document: If $A^2 = A$ (and A is $n \times n$) and $\operatorname{rank}(A) = n - 1$, then $\operatorname{adj}(A) = I - A$[cite: 262].*

Hint 9.1.2.1. Since $A^2 = A$ and $A \neq I$ (because $\operatorname{rank}(A) = n - 1 < n$), A must be singular. Why? [cite: 240] So $\det(A) = 0$.

Hint 9.1.2.2. What is the rank of $\operatorname{adj}(A)$ if $\operatorname{rank}(A) = n - 1$? [cite: 81, 243]

Hint 9.1.2.3. What is the rank of $I - A$? Recall that A and $I - A$ are complementary projections. Use the property $\operatorname{rank}(P) + \operatorname{rank}(I - P) = n$ for a projection P. [cite: 244]

Hint 9.1.2.4. So, both $\operatorname{adj}(A)$ and $I - A$ have rank 1. We need to show they are equal, not just proportional.

Hint 9.1.2.5. Consider the product $A \operatorname{adj}(A)$. What is it equal to, since $\det(A) = 0$? [cite: 39]

Hint 9.1.2.6. Now consider the product $A(I - A)$. Expand it using $A^2 = A$. Is it equal to $A\,\mathrm{adj}(A)$? [cite: 246] This shows columns of $\mathrm{adj}(A)$ and $I - A$ are in the nullspace of A.

Hint 9.1.2.7. Since $\mathrm{rank}(A) = n - 1$, the nullspace of A is 1-dimensional[cite: 85]. Both $\mathrm{adj}(A)$ and $I - A$ are rank 1 matrices whose columns span this nullspace. This means $\mathrm{adj}(A) = k(I - A)$ for some scalar k[cite: 249].

Hint 9.1.2.8. How can we find k? Take the trace of both sides: $\mathrm{tr}(\mathrm{adj}(A)) = k\,\mathrm{tr}(I - A)$[cite: 250].

Hint 9.1.2.9. What are the eigenvalues of an idempotent matrix A with rank $n-1$?[cite: 184, 251]. How many eigenvalues are 1 and how many are 0?

Hint 9.1.2.10. Use the eigenvalues to calculate $\mathrm{tr}(A)$. Then calculate $\mathrm{tr}(I - A) = \mathrm{tr}(I) - \mathrm{tr}(A)$[cite: 251, 252].

Hint 9.1.2.11. Recall from Problem 8.1.3 (or re-derive using characteristic polynomial coefficients) that the non-zero eigenvalue of $\mathrm{adj}(A)$ (when $\mathrm{rank}(A) = n - 1$) is $\mathrm{tr}(\mathrm{adj}(A))$[cite: 126]. Also, the coefficient c_1 of λ in the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$ is related to $\mathrm{tr}(\mathrm{adj}(A))$[cite: 257].

Hint 9.1.2.12. The characteristic polynomial for A is $p(\lambda) = \lambda^1(\lambda - 1)^{n-1}$[cite: 255]. Find the coefficient of λ^1 in this polynomial by expanding $(\lambda - 1)^{n-1}$ using the binomial theorem or by differentiation[cite: 259].

Hint 9.1.2.13. Relate this coefficient c_1 to $\mathrm{tr}(\mathrm{adj}(A))$ using $c_1 = (-1)^{n-1}\,\mathrm{tr}(\mathrm{adj}(A))$[cite: 257, 260]. Show that $\mathrm{tr}(\mathrm{adj}(A)) = 1$.

Hint 9.1.2.14. Substitute $\mathrm{tr}(\mathrm{adj}(A)) = 1$ and $\mathrm{tr}(I - A) = 1$ into the equation from Hint 8: $\mathrm{tr}(\mathrm{adj}(A)) = k\,\mathrm{tr}(I - A)$. What must k be? [cite: 261]

Hint 9.1.2.15. Since $k = 1$, we have $\mathrm{adj}(A) = I - A$[cite: 262].

Question 9.1.3. *Find all 2×2 real matrices A such that $\mathrm{adj}(A) = A$[cite: 207].*

Hint 9.1.3.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathrm{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$[cite: 15]. Set $A = \mathrm{adj}(A)$. What equations must a, b, c, d satisfy?

Hint 9.1.3.2. The equations are $a = d$, $b = -b$, $c = -c$, $d = a$. What do $b = -b$ and $c = -c$ imply about b and c?

Hint 9.1.3.3. So, $b = 0$ and $c = 0$. The matrix must be diagonal, $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. The condition $a = d$ still holds. So $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI_2$.

Hint 9.1.3.4. Now, let's check the general analysis. If $\mathrm{adj}(A) = A$, then $\det(A)$ must satisfy $d(d^{n-2}-1) = 0$. For $n = 2$, this is $d(d^0-1) = 0 \implies d(1-1) = 0 \implies 0 = 0$. This gave no restriction on $\det(A)$. However, we also have $A\,\mathrm{adj}(A) = \det(A)I$. If $\mathrm{adj}(A) = A$, then $A^2 = \det(A)I$.

Hint 9.1.3.5. Apply $A^2 = \det(A)I$ to our candidate solution $A = aI_2$. $(aI_2)^2 = a^2 I_2$. $\det(aI_2)I_2 = a^2 I_2$. So $a^2 I_2 = a^2 I_2$. This holds for any scalar a.

Hint 9.1.3.6. Therefore, any matrix of the form $A = aI_2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is a solution for $n = 2$.

10 Edge Cases

Problem 10.1. *Handling Extremes*

Question 10.1.1. *Let* $A = (a)$ *be a* 1×1 *matrix. Verify explicitly that* $\operatorname{adj}(A) = (1)$ *and that* $A \operatorname{adj}(A) = \det(A) I_1$*[cite: 300, 301, 302]. Why is* $\operatorname{adj}((0)) = (1)$ *not zero?*

Hint 10.1.1.1. For $A = (a)$, the cofactor matrix C has entry $C_{11} = (-1)^{1+1} M_{11}$[cite: 300].

Hint 10.1.1.2. The minor M_{11} is the determinant of the matrix obtained by removing row 1 and column 1. This is the 0×0 matrix. By convention, what is its determinant? [cite: 300]

Hint 10.1.1.3. Calculate C_{11} using the determinant from Hint 2. This gives the cofactor matrix $C = (C_{11})$.

Hint 10.1.1.4. The adjugate is the transpose of the cofactor matrix: $\operatorname{adj}(A) = C^T$[cite: 8, 301]. What is the transpose of a 1×1 matrix?

Hint 10.1.1.5. Now compute the product $A \operatorname{adj}(A) = (a)(1)$[cite: 302].

Hint 10.1.1.6. Compute the right side $\det(A) I_1$. What is $\det((a))$? What is I_1? [cite: 302] Compare both sides.

Hint 10.1.1.7. The fact that $\operatorname{adj}((0)) = (1)$ follows directly from the definition involving the 0×0 determinant being 1[cite: 301, 304]. It seems counterintuitive but is required for consistency, especially for the identity $A \operatorname{adj}(A) = \det(A) I$.

Question 10.1.2. *Verify that* $\operatorname{adj}(I_n) = I_n$ *for any* $n \geq 1$*[cite: 141, 307].*

Hint 10.1.2.1. We need the (i, j)-th entry of $\operatorname{adj}(I_n)$, which is $C_{ji}(I_n) = (-1)^{j+i} M_{ji}(I_n)$[cite: 9].

Hint 10.1.2.2. Consider the minor $M_{ji}(I_n)$. This is the determinant of I_n with row j and column i removed.

Hint 10.1.2.3. Case 1: $j = i$. The minor $M_{ii}(I_n)$ is the determinant of I_n with row i and column i removed. What matrix is this? What is its determinant? [cite: 305]

Hint 10.1.2.4. Case 2: $j \neq i$. The minor $M_{ji}(I_n)$ is the determinant of I_n with row j and column i removed. Can you show this resulting matrix must have a row or column of all zeros? (Consider the j-th row if $j > i$, or the i-th column if $i > j$). What is the determinant of a matrix with a zero row or column? [cite: 305]

Hint 10.1.2.5. So, $M_{ji}(I_n) = 1$ if $j = i$ and 0 if $j \neq i$.

Hint 10.1.2.6. Now find the cofactor $C_{ji}(I_n) = (-1)^{j+i} M_{ji}(I_n)$. Does the sign matter when $M_{ji} = 0$? What is $C_{ii}(I_n)$?

Hint 10.1.2.7. This shows that the entry (i, j) of $\operatorname{adj}(I_n)$, which is $C_{ji}(I_n)$, is 1 if $i = j$ and 0 if $i \neq j$. What matrix has these entries? [cite: 307]