Calculus II Lecture Notes
Techniques of Integration: Part I

Your Friendly Math Teacher

April 2025

Welcome Back, Integrators!

It's wonderful to see you all diving into the depths of Calculus II! This semester is indeed a mathematical feast, juggling Linear Algebra 2, Probability, and our beloved Calculus II. Let's sharpen our tools and continue our exploration of the fascinating world of integrals.

First, a couple of quick housekeeping items:

- **Exam Appeals:** While I appreciate diligence, please only submit exam appeals if you suspect a genuine, significant error in grading. Minor point-chasing rarely succeeds and consumes valuable time for everyone. Focus your energy on mastering the material moving forward!
- **Homework Reminder:** Don't forget, the integration by parts homework assignment we discussed is due this coming **Sunday**. Please make sure to submit it on time.

Now, let's embark on today's mathematical adventure!

1 Setting the Stage: Revisiting the Fundamentals

Before we learn new tricks, let's ensure our foundations are solid. Remember, integration is fundamentally about reversing differentiation.

1.1 The Antiderivative (Our Hero!)

A function $F(x)$ earns the title of an **antiderivative** (or *primitive function*) of $f(x)$ on an interval I if its derivative brings us back to $f(x)$. Formally:

$$F'(x) = f(x) \quad \text{for all } x \in I$$

1.2 The Indefinite Integral (The Family of Heroes)

The **indefinite integral**, symbolized by $\int f(x)\,\mathrm{d}x$, represents the *entire family* of antiderivatives for $f(x)$. Since the derivative of a constant is zero, any two antiderivatives of the same function can only differ by a constant. We capture this idea with the constant of integration, C:

$$\int f(x)\,\mathrm{d}x = F(x) + C, \quad \text{where } F'(x) = f(x)$$

1.3 Integration by Parts (The Product Rule's Inverse)

This powerful technique comes directly from the product rule for derivatives. It allows us to tackle integrals of products. The magic formula is:

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

Or, more explicitly using function notation:

$$\int u(x)v'(x) \, \mathrm{d}x = u(x)v(x) - \int u'(x)v(x) \, \mathrm{d}x$$

The art here lies in choosing u (the part you'll differentiate) and v' (the part you'll integrate) wisely to make the *new* integral, $\int v \, \mathrm{d}u$, simpler than the original.

2 Mastering a Classic: The Homework Problem $\int e^x \sin(x) \, \mathrm{d}x$

This integral is a beautiful illustration of integration by parts, requiring a clever "boomerang" approach. Both e^x and $\sin(x)$ (or $\cos(x)$) seem reluctant to disappear under differentiation or integration.

The Strategy: Apply integration by parts twice, consistently choosing the types of functions for u and v', aiming to recover the original integral on the right-hand side, allowing us to solve algebraically.

1. **First Pass:** Let $u = e^x \implies \mathrm{d}u = e^x \, \mathrm{d}x$. Let $\mathrm{d}v = \sin(x) \, \mathrm{d}x \implies v = -\cos(x)$.

$$\int e^x \sin(x) \, \mathrm{d}x = e^x(-\cos(x)) - \int (-\cos(x))(e^x \, \mathrm{d}x) = -e^x \cos(x) + \int e^x \cos(x) \, \mathrm{d}x$$

2. **Second Pass (on the new integral):** Let $u = e^x \implies \mathrm{d}u = e^x \, \mathrm{d}x$. (Consistency is key!) Let $\mathrm{d}v = \cos(x) \, \mathrm{d}x \implies v = \sin(x)$.

$$\int e^x \cos(x) \, \mathrm{d}x = e^x \sin(x) - \int \sin(x)(e^x \, \mathrm{d}x) = e^x \sin(x) - \int e^x \sin(x) \, \mathrm{d}x$$

3. **Putting it Together and Solving:** Substitute the result from Pass 2 back into the equation from Pass 1:

$$\int e^x \sin(x) \, \mathrm{d}x = -e^x \cos(x) + \left(e^x \sin(x) - \int e^x \sin(x) \, \mathrm{d}x \right)$$

Let $I = \int e^x \sin(x) \, \mathrm{d}x$. Our equation becomes:

$$I = -e^x \cos(x) + e^x \sin(x) - I$$

Now, solve for I:

$$2I = e^x(\sin(x) - \cos(x))$$
$$I = \frac{e^x(\sin(x) - \cos(x))}{2}$$

Finally, don't forget our constant companion:

$$\boxed{\int e^x \sin(x) \, \mathrm{d}x = \frac{e^x(\sin(x) - \cos(x))}{2} + C}$$

Teacher's Note: Why add $+C$ only at the end? It's pure convenience! Any constants generated during intermediate steps would just combine into one final arbitrary constant.

3 The Power of Substitution: Undoing the Chain Rule

Integration by parts helps with products; substitution helps with compositions! It's the integration analogue of the chain rule.

3.1 The Substitution Rule (First Kind: $t = \phi(x)$)

The Core Idea: If you recognize an integrand as a composite function $g(\phi(x))$ multiplied by the derivative of the inner function $\phi'(x)$, integration becomes much simpler.

The Formal Rule: If G is an antiderivative of g (i.e., $G' = g$), then:

$$\int g(\phi(x))\phi'(x)\,\mathrm{d}x = G(\phi(x)) + C$$

Intuition: The $\phi'(x)\,\mathrm{d}x$ part effectively "cancels out" the effect of the inner function $\phi(x)$ during integration, allowing us to integrate the outer function g directly (resulting in G) and then evaluate it at the inner function $\phi(x)$.

3.2 The Practical Workflow (Making it Easier!)

While the formal rule is the foundation, we usually use a more hands-on approach with a temporary variable, typically t (or u).

1. **Choose Substitution:** Identify an "inner function" $\phi(x)$ whose derivative $\phi'(x)$ (or a constant multiple of it) is also present as a factor in the integrand. Let $t = \phi(x)$.
2. **Find Differential** dt**:** Differentiate t with respect to x: $\frac{\mathrm{d}t}{\mathrm{d}x} = \phi'(x)$. Now, think of this notationally (it's mathematically justified!) as $dt = \phi'(x)\,\mathrm{d}x$.
3. **Substitute:** Replace $\phi(x)$ with t and $\phi'(x)\,\mathrm{d}x$ with dt. The entire integral should now be in terms of t.
4. **Integrate:** Compute the (hopefully simpler) integral with respect to t. Let's say $\int g(t)\,\mathrm{d}t = G(t) + C$.
5. **Substitute Back:** Replace t with its original expression in x, i.e., $\phi(x)$. The final answer is $G(\phi(x)) + C$.

A Note on Notation: The Leibniz notation $\frac{\mathrm{d}f}{\mathrm{d}x}$ is incredibly useful here, allowing the seemingly magical step $dt = \phi'(x)\,\mathrm{d}x$. Though it looks like algebraic manipulation of symbols, the notation is designed precisely so this process yields mathematically correct results for substitution.

3.3 Examples in Action (First Kind)

Let's see this practical workflow shine!

- **Example 1:** $\int \sin^2(x)\cos(x)\,\mathrm{d}x$
  - Let $t = \sin(x)$. (Inner function of the square, derivative $\cos(x)$ is present).

- $dt = \cos(x)\,\mathrm{d}x$.
- Substitute: $\int t^2\,\mathrm{d}t$.
- Integrate: $\frac{t^3}{3} + C$.
- Substitute back: $\boxed{\dfrac{\sin^3(x)}{3} + C}$.

- **Example 2:** $\int e^x(e^{2x}+1)\,\mathrm{d}x$
  - Let $t = e^x$. Note $e^{2x} = (e^x)^2 = t^2$.
  - $dt = e^x\,\mathrm{d}x$.
  - Substitute: $\int (t^2+1)\,\mathrm{d}t$.
  - Integrate: $\frac{t^3}{3} + t + C$.
  - Substitute back: $\boxed{\dfrac{e^{3x}}{3} + e^x + C}$.
- **Example 3:** $\int \frac{\mathrm{d}x}{x\sqrt{1-\ln^2(x)}} = \int \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{1}{x}\,\mathrm{d}x$
  - Let $t = \ln(x)$. (Inner function, derivative $1/x$ is present).
  - $dt = \frac{1}{x}\,\mathrm{d}x$.
  - Substitute: $\int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t$.
  - Integrate (Recognize Arcsine!): $\arcsin(t) + C$.
  - Substitute back: $\boxed{\arcsin(\ln x) + C}$.
- **Example 4:** $\int \frac{e^{\tan(x)}}{\cos^2(x)}\,\mathrm{d}x = \int e^{\tan(x)} \cdot \sec^2(x)\,\mathrm{d}x$
  - Let $t = \tan(x)$. (Inner function, derivative $\sec^2(x)$ is present).
  - $dt = \sec^2(x)\,\mathrm{d}x = \frac{1}{\cos^2(x)}\,\mathrm{d}x$.
  - Substitute: $\int e^t\,\mathrm{d}t$.
  - Integrate: $e^t + C$.
  - Substitute back: $\boxed{e^{\tan(x)} + C}$.

3.4 Recognizing Special Patterns

Sometimes, substitution reveals common, useful patterns.

- **Example 5:** $\int 2x\sin(x^2+1)\,\mathrm{d}x$
  - Let $t = x^2 + 1$. (Including the $+1$ is convenient, its derivative is 0).
  - $dt = 2x\,\mathrm{d}x$. (Perfect match!)
  - Substitute: $\int \sin(t)\,\mathrm{d}t$.

- Integrate: $-\cos(t) + C$.
- Substitute back: $\boxed{-\cos(x^2 + 1) + C}$.

- **Example 6:** $\int xe^{x^2}\,\mathrm{d}x$
  - Let $t = x^2$.
  - $dt = 2x\,\mathrm{d}x$. We only have $x\,\mathrm{d}x$. We need a 2!
  - *Method 1: Manipulate dt.* $x\,\mathrm{d}x = \frac{1}{2}dt$. Substitute: $\int e^t \left(\frac{1}{2}\,\mathrm{d}t\right) = \frac{1}{2}\int e^t\,\mathrm{d}t = \frac{1}{2}e^t + C$.
  - *Method 2: Adjust the Integral.* $\int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}\int 2xe^{x^2}\,\mathrm{d}x$. Now substitute: $\frac{1}{2}\int e^t\,\mathrm{d}t = \frac{1}{2}e^t + C$.
  - Substitute back (either method): $\boxed{\dfrac{1}{2}e^{x^2} + C}$.

- **Pattern 1: Linear Argument** $\int f(\alpha x + \beta)\,\mathrm{d}x$ Using $t = \alpha x + \beta$, we find $dt = \alpha\,\mathrm{d}x \implies \mathrm{d}x = \frac{1}{\alpha}dt$. The integral becomes $\int f(t)\frac{1}{\alpha}dt = \frac{1}{\alpha}\int f(t)dt = \frac{1}{\alpha}F(t) + C$.

$$\boxed{\int f(\alpha x + \beta)\,\mathrm{d}x = \frac{1}{\alpha}F(\alpha x + \beta) + C} \quad (\alpha \neq 0)$$

  Where F is the antiderivative of f.

- **Pattern 2: Derivative over Function** $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$ Let $t = f(x)$. Then $dt = f'(x)\,\mathrm{d}x$. The integral becomes $\int \frac{dt}{t}$.

$$\boxed{\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C}$$

  The absolute value is crucial for the logarithm's domain! If $f(x) > 0$, it can be omitted.
  - $\int \frac{e^x}{e^x+5}\,\mathrm{d}x = \ln|e^x + 5| + C = \ln(e^x + 5) + C$ (since $e^x + 5 > 0$)
  - $\int \frac{x^4}{x^5-9}\,\mathrm{d}x = \frac{1}{5}\int \frac{5x^4}{x^5-9}\,\mathrm{d}x = \frac{1}{5}\ln|x^5 - 9| + C$
  - $\int \tan(x)\,\mathrm{d}x = \int \frac{\sin(x)}{\cos(x)}\,\mathrm{d}x = -\int \frac{-\sin(x)}{\cos(x)}\,\mathrm{d}x = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$

4 Changing Perspective: Substitution (Second Kind: $x = \phi(t)$)

Sometimes, it's more effective to define the *original* variable x in terms of a *new* variable t. This is useful when the integrand has awkward expressions in x (like complicated roots) that simplify if x is replaced by a suitable function $\phi(t)$.

The Workflow:

1. **Choose Substitution:** Select $x = \phi(t)$ to simplify the integrand.
2. **Find Differential** dx**:** Differentiate x with respect to t: $\frac{\mathrm{d}x}{\mathrm{d}t} = \phi'(t)$. Then $dx = \phi'(t)\,\mathrm{d}t$.
3. **Substitute:** Replace every x with $\phi(t)$ and dx with $\phi'(t)\,\mathrm{d}t$.
4. **Integrate:** Compute the integral with respect to t.

5. **Substitute Back:** Express t back in terms of x using the inverse function $t = \phi^{-1}(x)$ and substitute into the result.

Example 7: $\int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt[3]{x})}$

- The terms $x^{1/2}$ and $x^{1/3}$ are the issue. To clear both fractional powers, we need an exponent divisible by both 2 and 3. Let $x = t^6$.
- Find dx: $\frac{\mathrm{d}x}{\mathrm{d}t} = 6t^5 \implies dx = 6t^5\,\mathrm{d}t$.
- Substitute:
$$\int \frac{6t^5\,\mathrm{d}t}{\sqrt{t^6}(1+\sqrt[3]{t^6})} = \int \frac{6t^5\,\mathrm{d}t}{t^3(1+t^2)} = \int \frac{6t^2}{1+t^2}\,\mathrm{d}t$$
- Integrate: Use the add/subtract trick:
$$6\int \frac{t^2+1-1}{1+t^2}\,\mathrm{d}t = 6\int \left(\frac{t^2+1}{1+t^2} - \frac{1}{1+t^2}\right)\mathrm{d}t = 6\int \left(1 - \frac{1}{1+t^2}\right)\mathrm{d}t$$
$$= 6(t - \arctan(t)) + C$$
- Substitute back: Since $x = t^6$, we have $t = x^{1/6} = \sqrt[6]{x}$.
$$\boxed{6(\sqrt[6]{x} - \arctan(\sqrt[6]{x})) + C}$$

5 A Touch of Elegance: Trigonometric Substitution

This is a special, powerful case of the second kind of substitution, designed for integrands containing expressions involving sums or differences of squares, especially under square roots. The key is using Pythagorean identities.

Common Scenarios:

- For $\sqrt{a^2 - x^2}$: Try $x = a\sin(t)$ (with $t \in [-\pi/2, \pi/2]$)
- For $\sqrt{a^2 + x^2}$: Try $x = a\tan(t)$ (with $t \in (-\pi/2, \pi/2)$)
- For $\sqrt{x^2 - a^2}$: Try $x = a\sec(t)$ (with appropriate t range)

Example 8: $\int \sqrt{1-x^2}\,\mathrm{d}x$ (This integral represents the area of a semicircle!)

- Matches $\sqrt{a^2 - x^2}$ with $a = 1$. Let $x = \sin(t)$.
- **Restrict Domain:** Choose $t \in [-\pi/2, \pi/2]$. This ensures $\sin(t)$ is one-to-one and, importantly, $\cos(t) \geq 0$.
- Find dx: $\frac{\mathrm{d}x}{\mathrm{d}t} = \cos(t) \implies dx = \cos(t)\,\mathrm{d}t$.
- Substitute: $\int \sqrt{1 - \sin^2(t)} \cdot (\cos(t)\,\mathrm{d}t)$.
- Simplify: $\int \sqrt{\cos^2(t)} \cdot \cos(t)\,\mathrm{d}t = \int |\cos(t)|\cos(t)\,\mathrm{d}t$. Since $t \in [-\pi/2, \pi/2]$, $|\cos(t)| = \cos(t)$. So we have $\int \cos^2(t)\,\mathrm{d}t$.

- Integrate: Use the identity $\cos^2(t) = \frac{1+\cos(2t)}{2}$.

$$\int \frac{1+\cos(2t)}{2}\,\mathrm{d}t = \frac{1}{2}\int (1+\cos(2t))\,\mathrm{d}t = \frac{1}{2}\left(t + \frac{\sin(2t)}{2}\right) + C = \frac{t}{2} + \frac{\sin(2t)}{4} + C$$

- Substitute back to x:
  - $x = \sin(t) \implies t = \arcsin(x)$.
  - Use $\sin(2t) = 2\sin(t)\cos(t)$. We know $\sin(t) = x$. We also know $\cos(t) = \sqrt{1-\sin^2(t)} = \sqrt{1-x^2}$ (positive because of t's range). So, $\sin(2t) = 2x\sqrt{1-x^2}$.

  Substitute these into the result:

$$\frac{\arcsin(x)}{2} + \frac{2x\sqrt{1-x^2}}{4} + C = \frac{\arcsin(x)}{2} + \frac{x\sqrt{1-x^2}}{2} + C$$

$$\boxed{\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left(\arcsin(x) + x\sqrt{1-x^2}\right) + C}$$

  (Doesn't this look geometrically satisfying? It relates the area to an angle (arcsin) and a triangle's area!)

6 Looking Ahead: Practice Makes Perfect!

We've explored several powerful techniques today:

- Integration by Parts (especially the "boomerang" trick)
- Substitution (Type 1: $t = \phi(x)$ and Type 2: $x = \phi(t)$)
- Trigonometric Substitution

Mastery comes from practice! Recognizing which technique to apply, and executing it correctly, is an art form developed by working through many examples.

7 Action Items & Reminders

✓ **Submit Homework:** Due Sunday (covering Integration by Parts).

✓ **Practice Integrals:** Work through problems applying all the techniques discussed today. Focus on identifying the best approach.

✓ **Review Trig Identities:** Especially Pythagorean and double-angle formulas – they are essential!

✓ **Review Class Examples:** Make sure you can reproduce the solutions and understand the reasoning behind each step.

Keep up the fantastic work, everyone! Happy integrating!