

Lecture Notes:
Convergence of Series and an Introduction to Taylor Polynomials

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May 8, 2025

*These notes aim to provide a clear and intuitive understanding of the topics covered,
supplemented with rigorous definitions, theorems, and illustrative examples.*

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Course Announcements

- **Office Hours Update:** Please note that I will be unable to hold office hours immediately after today's lecture as I need to return to my office promptly. However, I will be available for consultation tomorrow. You are welcome to meet with me then, roughly after 12:00 PM, perhaps around 12:15 PM. Please feel free to reach out if you'd like to arrange a specific time.

1 Infinite Series: The Leibniz Test for Alternating Series

1.1 A Brief Recap: Understanding Convergence

We've been exploring the fascinating world of infinite series. Recall that an infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

To understand whether such a sum "makes sense" or converges to a finite value, we look at its sequence of **partial sums**, S_k :

$$S_k = \sum_{n=1}^k a_n = a_1 + a_2 + \cdots + a_k$$

If the sequence of partial sums $\{S_k\}_{k=1}^{\infty}$ converges to a limit L , i.e., $\lim_{k \rightarrow \infty} S_k = L$, then we say the series $\sum a_n$ **converges** to L . Otherwise, the series **diverges**.

Calculating the limit of partial sums can often be very challenging. This is why we've developed various **convergence tests**. These tests don't always tell us *what* the sum is, but they can tell us *whether* the series converges or diverges. We've encountered several, such as the comparison tests and the necessary condition for convergence (if $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$).

Today, we add another powerful tool to our arsenal, specifically designed for a common type of series.

1.2 The Leibniz Test for Alternating Series

Many series involve terms that alternate in sign. For example, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. Such series are called **alternating series**.

Definition 1.1 (Alternating Series). An **alternating series** is a series whose terms alternate in sign. It can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

or

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 + \cdots$$

where $a_n > 0$ for all n . (Sometimes $a_n \geq 0$ and the condition $a_n > 0$ holds for n large enough). The common form we will use is $\sum (-1)^{n-1} a_n$ or $\sum (-1)^{n+1} a_n$, ensuring the first term is positive if a_1 is part of the sum.

The Leibniz Test (also known as the Alternating Series Test) provides simple conditions for the convergence of such series.

Theorem 1.2 (The Leibniz Test for Alternating Series). *An alternating series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ (or $\sum_{n=1}^{\infty} (-1)^n a_n$) converges if all three of the following conditions are met:*

- (i) $a_n > 0$ for all n (or for n sufficiently large).
- (ii) The sequence $\{a_n\}$ is eventually monotonically decreasing, i.e., $a_{n+1} \leq a_n$ for all n greater than some integer N .
- (iii) $\lim_{n \rightarrow \infty} a_n = 0$.

Remark 1.3. The Leibniz Test is a powerful tool, but it only tells us about convergence. It does *not* tell us the sum of the series. Also, if the conditions are not met (e.g., if $\lim_{n \rightarrow \infty} a_n \neq 0$), the series might diverge. In particular, if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the general term $(-1)^{n-1} a_n$ does not tend to zero, so the series diverges by the Term Test (necessary condition for convergence).

1.3 Examples using the Leibniz Test

Let's see the Leibniz Test in action with a few examples.

Example 1.4 (The Alternating Harmonic Series). Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. Here, $a_n = \frac{1}{n}$. Let's check the conditions for the Leibniz Test:

- (i) $a_n = \frac{1}{n} > 0$ for all $n \geq 1$. (Condition met)
- (ii) Is $\{a_n\}$ decreasing? We need to check if $a_{n+1} \leq a_n$. $a_{n+1} = \frac{1}{n+1}$ and $a_n = \frac{1}{n}$. Since $n+1 > n$, we have $\frac{1}{n+1} < \frac{1}{n}$. So, $a_{n+1} \leq a_n$ for all $n \geq 1$. (Condition met)
- (iii) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$. (Condition met)

Since all three conditions are satisfied, the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges by the Leibniz Test. (It is known to converge to $\ln 2$).

Example 1.5 (A p -Series with Alternating Signs). For which values of p does the series $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ converge? Here, $a_n = \frac{1}{n^p}$. We need $a_n > 0$, which is true if n^p is positive. For $n \geq 1$, this means we are interested in real p .

Let's analyze based on p :

Case 1: $p \leq 0$. Let $p = -q$ where $q \geq 0$. Then $a_n = \frac{1}{n^{-q}} = n^q$.

- If $q > 0$ (i.e., $p < 0$), then $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^q = \infty$.
- If $q = 0$ (i.e., $p = 0$), then $a_n = n^0 = 1$, so $\lim_{n \rightarrow \infty} a_n = 1$.

In both sub-cases ($p < 0$ and $p = 0$), $\lim_{n \rightarrow \infty} a_n \neq 0$. Thus, the third condition of the Leibniz Test is not met. More fundamentally, the general term of the series, $(-1)^{n-1} a_n$, does not tend to zero. Therefore, by the Term Test (necessary condition for convergence), the series diverges if $p \leq 0$.

Case 2: $p > 0$.

- (i) $a_n = \frac{1}{n^p} > 0$ for all $n \geq 1$ since $p > 0$. (Condition met)
- (ii) Is $\{a_n\}$ decreasing? Consider $f(x) = \frac{1}{x^p} = x^{-p}$. Then $f'(x) = -px^{-p-1} = -\frac{p}{x^{p+1}}$. Since $p > 0$ and $x \geq 1$, $f'(x) < 0$. Thus, $f(x)$ is decreasing for $x \geq 1$. This implies that $a_n = f(n)$ is a decreasing sequence. So, $a_{n+1} \leq a_n$. (Condition met)
- (iii) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ since $p > 0$. (Condition met)

Since all three conditions are met for $p > 0$, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ converges by the Leibniz Test if $p > 0$.

Conclusion for this example: The series converges if and only if $p > 0$.

Example 1.6 (An Alternating Series with Trigonometric Terms). Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{2\pi}{n}\right)$. Let $a_n = \sin\left(\frac{2\pi}{n}\right)$. We check the Leibniz conditions:

- (i) **Positivity of a_n :** For $n = 1$, $\frac{2\pi}{n} = 2\pi$, so $\sin(2\pi) = 0$. For $n = 2$, $\frac{2\pi}{n} = \pi$, so $\sin(\pi) = 0$. For $n = 3$, $\frac{2\pi}{n} = \frac{2\pi}{3}$. Since $0 < \frac{2\pi}{3} < \pi$, $\sin(\frac{2\pi}{3}) > 0$. For $n = 4$, $\frac{2\pi}{n} = \frac{\pi}{2}$, so $\sin(\frac{\pi}{2}) = 1 > 0$. For $n > 4$, we have $0 < \frac{2\pi}{n} < \frac{\pi}{2}$. In the interval $(0, \pi/2)$, the sine function is positive. So, $a_n = \sin\left(\frac{2\pi}{n}\right) \geq 0$ for $n \geq 1$, and $a_n > 0$ for $n \geq 3$. The conditions of the Leibniz test are often stated for n sufficiently large, so this is acceptable. Let's consider the tail from $n = 3$ or $n = 4$.

- (ii) **Monotonically decreasing a_n (for n large enough):** We need to show that $a_{n+1} \leq a_n$ for n large enough. That is, $\sin\left(\frac{2\pi}{n+1}\right) \leq \sin\left(\frac{2\pi}{n}\right)$. For $n \geq 4$, we have $0 < \frac{2\pi}{n+1} < \frac{2\pi}{n} \leq \frac{\pi}{2}$. The function $\sin(x)$ is monotonically increasing on the interval $[0, \pi/2]$. Since $\frac{2\pi}{n+1} < \frac{2\pi}{n}$ and both arguments are in the interval $(0, \pi/2]$ where $\sin(x)$ is increasing, we can apply \sin to both sides preserving the inequality (after considering the order): Because $\frac{2\pi}{n} > \frac{2\pi}{n+1}$, and $\sin(x)$ is increasing on $(0, \pi/2]$, it follows that $\sin\left(\frac{2\pi}{n}\right) \geq \sin\left(\frac{2\pi}{n+1}\right)$ for $n \geq 4$. So, the sequence $\{a_n\}$ is decreasing for $n \geq 4$. (Condition met for $n \geq 4$)
- (iii) **Limit of a_n :** As $n \rightarrow \infty$, $\frac{2\pi}{n} \rightarrow 0$. Since $\sin(x)$ is continuous at $x = 0$, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sin\left(\frac{2\pi}{n}\right) = \sin\left(\lim_{n \rightarrow \infty} \frac{2\pi}{n}\right) = \sin(0) = 0$. (Condition met)

Since all three conditions are satisfied (for $n \geq 4$, which is sufficient for the tail of the series), the series $\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{2\pi}{n}\right)$ converges by the Leibniz Test.

Follow-up: Conditional vs. Absolute Convergence Does the series converge absolutely? That is, does $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \sin\left(\frac{2\pi}{n}\right)\right| = \sum_{n=1}^{\infty} \sin\left(\frac{2\pi}{n}\right)$ converge? (Assuming $n \geq 3$ so $\sin(2\pi/n) > 0$).

We can use the Limit Comparison Test. Let $b_n = \sin\left(\frac{2\pi}{n}\right)$ (for $n \geq 3$) and compare with $c_n = \frac{2\pi}{n}$. We know that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. As $n \rightarrow \infty$, $\frac{2\pi}{n} \rightarrow 0$. So,

$$\lim_{n \rightarrow \infty} \frac{b_n}{c_n} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} = 1.$$

Since this limit is a finite positive number ($1 \neq 0, \infty$), the series $\sum b_n$ and $\sum c_n$ either both converge or both diverge. The series $\sum c_n = \sum \frac{2\pi}{n} = 2\pi \sum \frac{1}{n}$ is a multiple of the harmonic series, which diverges. Therefore, $\sum_{n=1}^{\infty} \sin\left(\frac{2\pi}{n}\right)$ also diverges.

Since the original series $\sum (-1)^{n+1} \sin\left(\frac{2\pi}{n}\right)$ converges, but the series of absolute values $\sum \sin\left(\frac{2\pi}{n}\right)$ diverges, the original series converges **conditionally**.

Remark 1.7. The convergence of a series depends on the behavior of its "tail." If the conditions of a test are met for $n \geq N$ for some integer N , that is sufficient to determine the convergence or divergence of the entire series, as the finite sum of the first $N - 1$ terms does not affect convergence.

2 An Introduction to Taylor Polynomials

2.1 Motivation: Approximating Functions

Many functions in mathematics and science are complex. Calculating their values (e.g., $\sin(0.1)$, $e^{1.7}$) or performing operations like differentiation and integration can be difficult or even impossible in terms of elementary functions. For instance, the integral $\int e^{-x^2} dx$, crucial in probability (related to the Gaussian or normal distribution), does not have an elementary antiderivative. We often rely on tables or numerical methods, which are themselves based on approximations.

The central idea we'll explore is: **Can we approximate complicated functions with simpler ones, at least locally (i.e., near a specific point)?**

What are the "simplest" functions we know? Polynomials! A polynomial $P(x) = c_0 + c_1x + c_2x^2 + \cdots + c_kx^k$ is wonderful because:

- Its value at any point x can be found using only addition and multiplication.
- It's easy to differentiate: the derivative of a polynomial is another polynomial.

- It's easy to integrate: the integral of a polynomial is another polynomial.

Our goal is to develop a systematic way to find a polynomial that "looks like" a given function $f(x)$ near a particular point $x = a$. This is the essence of Taylor polynomials.

2.2 A Brief Review of Higher-Order Derivatives

Before we construct these approximating polynomials, let's refresh our notation for derivatives. Let f be a function.

- $f^{(0)}(x) := f(x)$ (the zeroth derivative is the function itself).
- $f^{(1)}(x) := f'(x)$ (the first derivative).
- $f^{(2)}(x) := f''(x)$ (the second derivative).
- In general, $f^{(n)}(x)$ denotes the n -th derivative of f .

The derivatives can be defined recursively: $f^{(n+1)}(x) = \frac{d}{dx} (f^{(n)}(x))$.

It's important to remember that for $f^{(n)}(a)$ to exist, the function f must be $(n-1)$ -times differentiable in a neighborhood of a , and $f^{(n-1)}(x)$ must be differentiable at a . This also implies that $f, f', \dots, f^{(n-1)}$ are continuous in a neighborhood of a (if $f^{(n)}$ exists throughout that neighborhood).

2.2.1 Examples of Higher-Order Derivatives

Let's practice finding some higher-order derivatives.

Example 2.1 ($f(x) = x^3$).

$$\begin{aligned} f^{(0)}(x) &= x^3 \\ f^{(1)}(x) &= 3x^2 \\ f^{(2)}(x) &= 6x \\ f^{(3)}(x) &= 6 \\ f^{(4)}(x) &= 0 \\ f^{(n)}(x) &= 0 \quad \text{for } n \geq 4. \end{aligned}$$

Example 2.2 ($f(x) = x^N$ for a positive integer N). Let k be the order of the derivative.

$$\begin{aligned} f^{(0)}(x) &= x^N \\ f^{(1)}(x) &= Nx^{N-1} \\ f^{(2)}(x) &= N(N-1)x^{N-2} \\ &\vdots \\ f^{(k)}(x) &= N(N-1) \cdots (N-k+1)x^{N-k} \quad \text{for } k \leq N \\ &= \frac{N!}{(N-k)!} x^{N-k} \quad \text{for } k \leq N. \end{aligned}$$

In particular,

- $f^{(N)}(x) = N(N-1) \cdots (1)x^{N-N} = N!x^0 = N!$.
- $f^{(k)}(x) = 0$ for $k > N$.

Example 2.3 ($f(x) = e^x$). This is a particularly simple one!

$$f^{(n)}(x) = e^x \quad \text{for all } n \geq 0.$$

Example 2.4 ($f(x) = \sin x$). Let's list the first few derivatives:

$$\begin{aligned} f^{(0)}(x) &= \sin x \\ f^{(1)}(x) &= \cos x \\ f^{(2)}(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \quad (\text{The pattern repeats every 4 derivatives!}) \end{aligned}$$

We can write general formulas:

- For even derivatives, $n = 2m$: $f^{(2m)}(x) = (-1)^m \sin x$. (e.g., $m = 0 \Rightarrow n = 0 : (-1)^0 \sin x = \sin x$; $m = 1 \Rightarrow n = 2 : (-1)^1 \sin x = -\sin x$)
- For odd derivatives, $n = 2m + 1$: $f^{(2m+1)}(x) = (-1)^m \cos x$. (e.g., $m = 0 \Rightarrow n = 1 : (-1)^0 \cos x = \cos x$; $m = 1 \Rightarrow n = 3 : (-1)^1 \cos x = -\cos x$)

Alternatively, one can use $f^{(n)}(x) = \sin(x + n\pi/2)$.

Example 2.5 ($f(x) = \cos x$ (For you to verify)). You can similarly find a pattern for the derivatives of $\cos x$:

$$\begin{aligned} f^{(0)}(x) &= \cos x \\ f^{(1)}(x) &= -\sin x \\ f^{(2)}(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

The general formulas are:

- For even derivatives, $n = 2m$: $f^{(2m)}(x) = (-1)^m \cos x$.
- For odd derivatives, $n = 2m + 1$: $f^{(2m+1)}(x) = (-1)^{m+1} \sin x$.

Alternatively, $f^{(n)}(x) = \cos(x + n\pi/2)$.

2.3 Developing the Idea of Polynomial Approximation

Suppose we want to approximate a function $f(x)$ near a point $x = a$. We know the value $f(a)$. What's the simplest approximation for $f(x)$ when x is close to a (say $x = a + h$ for small h)?

Zeroth-Order Approximation (Constant Approximation): The most basic approximation is simply $f(x) \approx f(a)$. Let $P_0(x) = f(a)$. This is a constant polynomial. It matches $f(x)$ at $x = a$, i.e., $P_0(a) = f(a)$.

First-Order Approximation (Linear Approximation): Can we do better? We can try to match not only the value of f at a , but also its slope (i.e., its first derivative). Recall the geometric interpretation of the derivative: $f'(a)$ is the slope of the tangent line to the graph of $y = f(x)$ at $x = a$. The equation of this tangent line is $L(x) = f(a) + f'(a)(x - a)$. This linear function $L(x)$ is our **first-order approximation**, or linear approximation, to $f(x)$ near $x = a$. Let's call it $P_1(x)$:

$$P_1(x) = f(a) + f'(a)(x - a)$$

Notice two key properties of $P_1(x)$:

1. $P_1(a) = f(a) + f'(a)(a - a) = f(a)$. (Matches the function's value at a)
2. $P_1'(x) = f'(a)$. So, $P_1'(a) = f'(a)$. (Matches the function's first derivative at a)

If we let $x = a + h$, then $x - a = h$, and the approximation becomes $f(a + h) \approx f(a) + f'(a)h$. This is often useful when h is small.

Second-Order Approximation (Quadratic Approximation): To get an even better approximation, we might try to match the function's value, its first derivative, AND its second derivative at $x = a$. The second derivative, $f''(a)$, tells us about the concavity of f at a . Let's look for a quadratic polynomial $P_2(x)$ of the form:

$$P_2(x) = A_0 + A_1(x - a) + A_2(x - a)^2$$

We want to choose A_0, A_1, A_2 such that:

1. $P_2(a) = f(a)$
2. $P_2'(a) = f'(a)$
3. $P_2''(a) = f''(a)$

Let's find the coefficients:

1. $P_2(a) = A_0 + A_1(0) + A_2(0)^2 = A_0$. So, $A_0 = f(a)$.
2. $P_2'(x) = A_1 + 2A_2(x - a)$. $P_2'(a) = A_1 + 2A_2(0) = A_1$. So, $A_1 = f'(a)$.
3. $P_2''(x) = 2A_2$. $P_2''(a) = 2A_2$. So, $2A_2 = f''(a) \implies A_2 = \frac{f''(a)}{2}$.

Substituting these coefficients back, we get the **second-order approximation**:

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Notice the pattern: the coefficient of $(x - a)^k$ involves $f^{(k)}(a)$ and a factorial $k!$ (since $0! = 1, 1! = 1, 2! = 2$).

2.4 The Taylor Polynomial: General Definition

This pattern leads us to the general definition of the Taylor polynomial. To get an N -th order approximation to $f(x)$ near $x = a$, we construct a polynomial $P_N(x)$ of degree at most N such that $P_N(x)$ and its first N derivatives match $f(x)$ and its first N derivatives at $x = a$. That is: $P_N^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \dots, N$.

Definition 2.6 (Taylor Polynomial). Let f be a function that is N times differentiable at $x = a$. The **Taylor polynomial of order N** for f centered at (or around) a is given by:

$$\begin{aligned} P_N(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(N)}(a)}{N!}(x - a)^N \\ &= \sum_{k=0}^N \frac{f^{(k)}(a)}{k!}(x - a)^k \end{aligned}$$

(Recall that $0! = 1$ and $f^{(0)}(a) = f(a)$.)

Definition 2.7 (Maclaurin Polynomial). A special case of the Taylor polynomial occurs when $a = 0$. This is called the **Maclaurin polynomial of order N** for f :

$$P_N(x) = \sum_{k=0}^N \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(N)}(0)}{N!}x^N$$

The idea is that for x close to a , $f(x) \approx P_N(x)$. Generally, higher order N gives a better approximation over a wider interval around a .

2.5 Example: Approximating $f(x) = \sqrt{x}$

Let's find Taylor polynomial approximations for $f(x) = \sqrt{x}$ centered at $a = 1$. We want to use these to approximate $f(1.21) = \sqrt{1.21}$. We need the derivatives of $f(x) = x^{1/2}$ evaluated at $a = 1$:

- $f(x) = x^{1/2} \implies f(1) = 1^{1/2} = 1.$
- $f'(x) = \frac{1}{2}x^{-1/2} \implies f'(1) = \frac{1}{2}(1)^{-1/2} = \frac{1}{2}.$
- $f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{4}x^{-3/2} \implies f''(1) = -\frac{1}{4}(1)^{-3/2} = -\frac{1}{4}.$
- $f'''(x) = -\frac{1}{4}\left(-\frac{3}{2}\right)x^{-5/2} = \frac{3}{8}x^{-5/2} \implies f'''(1) = \frac{3}{8}(1)^{-5/2} = \frac{3}{8}.$

Now, let's construct the polynomials and approximate $\sqrt{1.21}$ (where $x = 1.21$, so $x - a = 1.21 - 1 = 0.21$). The actual value is $\sqrt{1.21} = 1.1$.

Zeroth-Order Taylor Polynomial ($P_0(x)$):

$$P_0(x) = f(1) = 1$$

Approximation: $P_0(1.21) = 1$. (Error = $1.1 - 1 = 0.1$)

First-Order Taylor Polynomial ($P_1(x)$):

$$\begin{aligned} P_1(x) &= f(1) + f'(1)(x - 1) \\ &= 1 + \frac{1}{2}(x - 1) \end{aligned}$$

Approximation: $P_1(1.21) = 1 + \frac{1}{2}(1.21 - 1) = 1 + \frac{1}{2}(0.21) = 1 + 0.105 = 1.105$. (Error = $1.1 - 1.105 = -0.005$)

Second-Order Taylor Polynomial ($P_2(x)$):

$$\begin{aligned} P_2(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 \\ &= 1 + \frac{1}{2}(x - 1) + \frac{-1/4}{2}(x - 1)^2 \\ &= 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 \end{aligned}$$

Approximation:

$$\begin{aligned} P_2(1.21) &= 1 + \frac{1}{2}(0.21) - \frac{1}{8}(0.21)^2 \\ &= 1.105 - \frac{1}{8}(0.0441) \\ &= 1.105 - 0.0055125 \\ &= 1.0994875 \end{aligned}$$

(Error = $1.1 - 1.0994875 \approx 0.0005125$)

Third-Order Taylor Polynomial ($P_3(x)$):

$$\begin{aligned} P_3(x) &= P_2(x) + \frac{f'''(1)}{3!}(x - 1)^3 \\ &= 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{3/8}{6}(x - 1)^3 \\ &= 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3 \end{aligned}$$

Approximation:

$$\begin{aligned}P_3(1.21) &= 1.0994875 + \frac{1}{16}(0.21)^3 \\&= 1.0994875 + \frac{1}{16}(0.009261) \\&= 1.0994875 + 0.0005788125 \\&= 1.1000663125\end{aligned}$$

(Error = $1.1 - 1.1000663125 \approx -0.0000663$)

As you can see, as we increase the order N of the Taylor polynomial, the approximation $P_N(1.21)$ gets closer to the actual value of $\sqrt{1.21} = 1.1$.

2.6 A Note on Visualizing Taylor Approximations

It is highly recommended to use a graphing tool (like Desmos, GeoGebra, or Mathematica/Matlab) to plot $f(x) = \sqrt{x}$ along with $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$ centered at $a = 1$. You will visually see how these polynomial approximations "hug" the curve of \sqrt{x} more closely near $x = 1$ as the order N increases. This provides excellent intuition for why Taylor polynomials are so effective.

For example, in Desmos, you could plot:

- $y = \sqrt{x}$
- $y = 1$
- $y = 1 + 0.5(x - 1)$
- $y = 1 + 0.5(x - 1) - (1/8)(x - 1)^2$
- $y = 1 + 0.5(x - 1) - (1/8)(x - 1)^2 + (1/16)(x - 1)^3$

Focus on the behavior near $x = 1$.

This concludes our introduction to Taylor polynomials. We will explore them further, including how to estimate the error in these approximations.