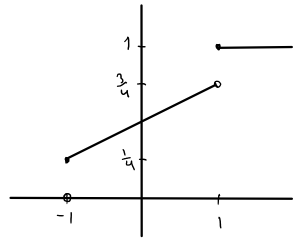
Exercise 2

1. Suppose X is a random variable s.t. $F_X(x) = \begin{cases} 0 & x<-1 \\ \frac{x}{4}+\frac{1}{2} & -1\le x<1 \\ 1 & x\ge 1 \end{cases}$ $\quad P_X(x) = \begin{cases} 0 & x<-1 \\ \frac{1}{4} & -1<x<1 \\ 0 & x>1 \end{cases}$



$$E(X) = \int_{-\infty}^{-1} x\cdot 0\,dx + (-1)\cdot\frac{1}{4} + \int_{-1}^{1} x\cdot\frac{1}{4}\,dx + 1\cdot\frac{1}{4} + \int_{1}^{\infty} x\cdot 0\,dx = \frac{1}{8}\left[x^2\right]_{-1}^{1} = 0$$

$$E(X^2) = \int_{-\infty}^{-1} x^2\cdot 0\,dx + (-1)^2\cdot\frac{1}{4} + \int_{-1}^{1} x^2\frac{1}{4}\,dx + 1^2\cdot\frac{1}{4} + \int_{1}^{\infty} x^2\cdot 0\,dx = \frac{1}{2} + \frac{1}{12}\left[x^3\right]_{-1}^{1} = \frac{2}{3}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{3} - 0^2 = \frac{2}{3}$$

2. Suppose $X \sim Exp(\lambda)$, then $P_X(x) = \lambda e^{-\lambda x}$.

(a) Let $Y = \sum_{j=0}^{\infty} j\, I_{X\in[j,j+1]}$ then $Im\,Y = \{0\}\cup\mathbb{N}$ and $P_Y(y) = F_X(y+1) - F_X(y) = \lambda\int_y^{y+1} e^{-\lambda t}\,dt = -\left[e^{-\lambda t}\right]_y^{y+1} = e^{-\lambda y} - e^{-\lambda y-\lambda} = e^{-\lambda y}(1-e^{-\lambda})$. This is the PMF of a geometric distribution with failure probability $q = e^{-\lambda}$.

$E(Y) = \sum_{y=0}^{\infty} y q^y p = p\sum_{y=0}^{\infty} y q^y$. $\sum_{y=0}^{\infty} q^y = \frac{1}{1-q} \xrightarrow{d/dq} \sum_{y=1}^{\infty} y q^{y-1} = \frac{1}{(1-q)^2} \to \sum_{y=1}^{\infty} y q^y = \frac{q}{(1-q)^2} \to E(Y) = \frac{pq}{(1-q)^2} = \frac{q}{p} = \frac{e^{-\lambda}}{1-e^{-\lambda}}$.

$E(Y^2) = \sum_{y=0}^{\infty} y^2 q^y p$ $\quad \sum_{y=0}^{\infty} q^y = \frac{1}{1-q} \xrightarrow{d^2/dq^2} \sum_{y=2}^{\infty} y(y-1)q^{y-2} = \frac{2}{(1-q)^3} \to \sum_{y=2}^{\infty} y^2 q^y - y q^y = \frac{2q^2}{(1-q)^3} \to \sum_{y=0}^{\infty} y^2 q^y = \frac{2q^2}{p^3} + \frac{q}{p^2} = \frac{q(q+1)}{p^3}$ and

$E(Y^2) = \frac{q(q+1)}{p^2}$. $\quad Var(Y) = E(Y^2) - E(Y)^2 = \frac{q(q+1)}{p^2} - \frac{q^2}{p^2} = \frac{q}{p^2} = \frac{e^{-\lambda}}{(1-e^{-\lambda})^2}$.

(b) Let $Z = X\, I_{X\in[1,2]}$, then $Im\,Z = \{0\}\cup[1,2]$ and $P_Z(0) = 1 - \int_1^2 \lambda e^{-\lambda t}\,dt = 1 + \left[e^{-\lambda t}\right]_1^2 = 1 + e^{-2\lambda} - e^{-\lambda}$.

$\forall_{1\le z\le 2}\ f_Z(z) = \lambda e^{-\lambda z}$. $\quad E(Z) = 0\cdot P_Z(0) + \int_1^2 z\lambda e^{-\lambda z}\,dz = -e^{-\lambda z}\left(z+\frac{1}{\lambda}\right)\Big]_1^2 = e^{-\lambda}\frac{\lambda+1}{\lambda} - e^{-2\lambda}\frac{2\lambda+1}{\lambda}$.

$$E(Z^2) = 0^2\cdot P_Z(0) + \int_1^2 z^2\lambda e^{-\lambda z}\,dz = -e^{-\lambda z}\left(z^2 + \frac{2z}{\lambda} + \frac{2}{\lambda^2}\right)\Big]_1^2 = e^{-\lambda}\left(1+\frac{2}{\lambda}+\frac{2}{\lambda^2}\right) - e^{-2\lambda}\left(4+\frac{4}{\lambda}+\frac{2}{\lambda^2}\right)$$

$$Var(Z) = E(Z^2) - E(Z)^2 = e^{-\lambda}\left(1+\frac{2}{\lambda}+\frac{2}{\lambda^2}\right) - e^{-2\lambda}\left(4+\frac{4}{\lambda}+\frac{2}{\lambda^2}\right) - \left(e^{-\lambda}\frac{\lambda+1}{\lambda} - e^{-2\lambda}\frac{2\lambda+1}{\lambda}\right)^2$$

(c) Let $V = (X-c)\, I_{X\ge c}$ then $Im\,V = [0,\infty)$, $f_V(0) = F_X(c) = \int_0^c \lambda e^{-\lambda t}\,dt = -\left[e^{-\lambda t}\right]_0^c = 1 - e^{-\lambda c}$. $\forall v\ge 0$

$f_V(v) = \lambda e^{-\lambda(v+c)}$. $\quad E(V) = 0\,(1-e^{-\lambda c}) + \int_0^{\infty} v\lambda e^{-\lambda(v+c)}\,dv = -e^{-\lambda(v+c)}\left(v+\frac{1}{\lambda}\right)\Big]_0^{\infty} = \frac{1}{\lambda}e^{-\lambda c} - \lim_{v\to\infty}\frac{\lambda v+1}{e^{\lambda(v+c)}} \overset{\text{l'Hospital}}{=} \frac{1}{\lambda}e^{-\lambda c}$

$E(V^2) = 0^2(1-e^{-\lambda c}) + \int_0^{\infty} v^2\lambda e^{-\lambda(v+c)}\,dv = -e^{-\lambda(v+c)}\left(v^2+\frac{2v}{\lambda}+\frac{2}{\lambda^2}\right)\Big]_0^{\infty} = \frac{2}{\lambda^2}e^{-\lambda c}$ $\quad Var(V) = E(V^2) - E(V)^2 =$

$\frac{2}{\lambda^2}e^{-\lambda c} - \frac{1}{\lambda^2}e^{-2\lambda c} = \frac{1}{\lambda^2}e^{-\lambda c}\left(2-e^{-\lambda c}\right)$.

3. The support of the Beta distribution $B(\alpha,\beta)$ with $\alpha,\beta\in\mathbb{R}_+$ is $(0,1)$. Its PDF is $f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$

where the Beta function is $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and the Gamma function is $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}\,dt$.

$\frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = 1 \to \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = B(\alpha,\beta)$. Also $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}\,dt = -\left[e^{-t}t^{\alpha-1}\right]_0^{\infty} + (\alpha-1)\int_0^{\infty} t^{\alpha-2}e^{-t}$,

implying that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.

(a) Assume $\alpha=\beta=1$, then $\Gamma(1) = \int_0^{\infty} e^{-t}\,dt = -e^{-t}\Big]_0^{\infty} = 1$, $\Gamma(2) = \int_0^{\infty} te^{-t}\,dt = -e^{-t}(t+1)\Big]_0^{\infty} = 1 \to B(\alpha,\beta) = 1 \to$

$f(x|1,1) = 1$, meaning that $B(1,1)$ is the uniform distribution on the interval $[0,1]$.

(b) Suppose $X\sim B(\alpha,\beta)$, then $E(X) = \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\,\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)} =$

$\frac{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}$. $\quad E(X^2) = \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}\,\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} =$

$$\frac{\alpha(\alpha+1)\Gamma(\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta)(\alpha+\beta+1)\Gamma(\alpha+\beta)\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}. \quad Var(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha(\alpha+1)(\alpha+\beta)-\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(c) Let $Y = 1 - X$, then $\mathrm{Im}\,Y = (0,1)$. $f_Y(y) = \begin{cases} 0 & y \notin (0,1) \\ f_X(1-y|\alpha,\beta) & y \in (0,1) \end{cases}$, hence the support of Y is $(0,1)$.

$f_Y(y) = f_X(1-y|\alpha,\beta) = \frac{1}{B(\alpha,\beta)}(1-y)^{\alpha-1} y^{\beta-1} = f_X(y|\beta,\alpha)$. Therefore, $Y \sim B(\beta,\alpha)$.

4. Suppose X, Y are random variables with finite second moments. $\forall a,b \in \mathbb{R}$ $(a-b)^2 \geq 0 \to a^2+b^2 \geq 2ab \to 2a^2+2b^2 \geq (a+b)^2$.

$E((X+Y)^2) = \int_x\int_y (x+y)^2 f(x,y)\,dy\,dx \leq \int_x\int_y (2x^2+2y^2) f(x,y)\,dy\,dx = 2\int_x\int_y x^2 f(x,y)\,dy\,dx + 2\int_x\int_y y^2 f(x,y)\,dy\,dx =$

$2E(X^2) + 2E(Y^2) < \infty$. Therefore, $X+Y$ also has a finite second moment.

5. Suppose $X \sim N(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$.

(a) $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. $M_X(t) = E(e^{tX}) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(x^2-2(\mu+\sigma^2 t)x+\mu^2)}{2\sigma^2}} dx =$

$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}+(\mu t+\frac{1}{2}\sigma^2 t^2)} dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\mu t+\frac{1}{2}\sigma^2 t^2}\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx$, but $\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$,

hence $M_X(t) = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$

(b) Let $Y = aX+b$ for some $a,b \in \mathbb{R}$, and let $y \in \mathbb{R}$ then $f_Y(y) = f_X(\frac{y-b}{a}) = \frac{1}{|a|\sqrt{2\pi\sigma^2}} e^{-\frac{(\frac{y-b}{a}-\mu)^2}{2\sigma^2}}$.

$M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = E(e^{atx}e^{bt}) = e^{bt}E(e^{(at)X}) = e^{bt}M_X(at) = e^{bt}e^{\mu at+\frac{1}{2}\sigma^2(at)^2} =$

$e^{(a\mu+b)\mu+\frac{1}{2}(a\sigma)^2 t^2}$ which is the generating function of a normal distribution with mean $a\mu+b$ and variace $a^2\sigma^2$.

$(\sigma^2 at-b-\mu a)(\sigma^2 at-b-\mu a) = \sigma^4 a^2 t^2 - ab\sigma^2 t - a^2\mu\sigma^2 t - ab\sigma^2 t + b^2 + ab\mu - a^2\mu\sigma^2 t + ab\mu + a^2\mu^2 =$

$\sigma^4 a^2 t^2 - 2ab\sigma^2 t - 2a^2\mu\sigma^2 t + b^2 + 2ab\mu + a^2\mu^2$