Probability for Statisticians - Course 52534
Exercise 1

Instructions: Answer the following questions. Explain and justify your answers. If a proof is required, present all steps of the proof.

1. Consider the random variable X with the following cumulative distribution function (CDF):
$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{x}{4} + \frac{1}{2} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$
Calculate the following probabilities:
   (i) $\mathrm{P}(X = 0)$, $\mathrm{P}(X = 1)$, $\mathrm{P}(X = -1)$
   (ii) $\mathrm{P}(X < 1)$, $\mathrm{P}(X > -1)$
   (iii) $\mathrm{P}(|X| \geq 1/2)$, $\mathrm{P}(X \leq 0)$, $\mathrm{P}(X < 0)$

2. Let $X \sim \mathrm{Exp}(\lambda)$. Find the cumulative distribution function (CDF) of the following random variables:
   (a) $Y = \sum_{j=0}^{\infty} j\mathbb{I}_{X\in[j,j+1]}$
   (b) $Z = X\mathbb{I}_{X\in[1,2]}$
   (c) $V = (X - c)\mathbb{I}_{X\geq c}$

3. Let F be the CDF of a purely continuous random variable. Assume F is strictly monotonically increasing on the support of the distribution and continuous, with its image being the interval $(0, 1)$. Since F is strictly monotonically increasing, its inverse function $G = F^{-1}$ exists, is also strictly monotonically increasing, and maps from $(0, 1)$ to the support of the distribution.
   (a) Let $X \sim U(0, 1)$. Show that the distribution of $Y = G(X)$ is F, i.e., the CDF of Y, F_Y, satisfies $F_Y = F$.

(b) Assume that X has the distribution F, i.e., $F_X = F$. Let $Y = F(X)$. Show that $Y \sim U(0, 1)$.