Exercise 1

(1) $F_X(x) = \begin{cases} 0 & x<-1 \\ \frac{x}{4}+\frac{1}{2} & -1\le x<1 \\ 1 & x\ge 1 \end{cases}$

(i) $P(X=0) = F_X(0) - \lim_{x\to 0^-} F_X(x) = \frac{1}{2}-\frac{1}{2} = 0$

$P(X=1) = F_X(1) - \lim_{x\to 1^-} F_X(x) = 1-\frac{3}{4} = \frac{1}{4}$

$P(X=-1) = F_X(-1) - \lim_{x\to -1^-} F_X(x) = \frac{1}{4} - 0 = \frac{1}{4}$

(ii) $P(X<1) = \lim_{x\to 1^-} F_X(x) = \frac{3}{4}$

$P(X>-1) = 1-F_X(-1) = 1-\frac{1}{4} = \frac{3}{4}$

(iii) $P(|X|>\frac{1}{2}) = P(X<-\frac{1}{2}) + P(X>\frac{1}{2}) = \lim_{x\to -\frac{1}{2}^-} F_X(x) + 1 - F_X(\frac{1}{2}) = \frac{3}{8} + 1 - \frac{5}{8} = \frac{3}{4}$

$P(X\le 0) = F_X(0) = \frac{1}{2}$

$P(X<0) = \lim_{x\to 0^-} F_X(x) = \frac{1}{2}$

(2) Suppose $X\sim Exp(\lambda)$ then $f_X(x) = \lambda e^{-\lambda x}$ and $F_X(x) = \int_0^x \lambda e^{-\lambda t}dt = -[e^{-\lambda t}]_0^x = 1-e^{-\lambda x}$.

(a) $Y = \sum_{j=0}^{\infty} j\,\mathbb{I}_{X\in[j,j+1]}$ hence Y is discrete, getting the values $0,1,2,\dots$.

$P_Y(k) = P_X(k\le X\le k+1) = F_X(k+1) - F_X(k)$

$F_Y(k) = F_X(k+1)$

(b) $Z = X\,\mathbb{I}_{X\in[1,2]}$ hence $Z = \begin{cases} 0 & X<1 \\ X & 1\le X\le 2 \\ 0 & 2<X \end{cases}$ $\quad P_Z(z) = \begin{cases} 1-P_X(1\le X\le 2) & 0 \\ f(z) & 1\le z\le 2 \end{cases}$

$$F_Z(z) = \begin{cases} 0 & z<0 \\ 1-P_X(1\le X\le 2) & 0\le z<1 \\ 1-P_X(1\le X\le 2) + F_X(z) - F_X(1) & 1\le z\le 2 \\ 1 & 2<z \end{cases} = \begin{cases} 0 & z<0 \\ 1-e^{-\lambda}(1-e^{-\lambda}) & 0\le z<1 \\ 1+e^{-2\lambda}-e^{-\lambda z} & 1\le z\le 2 \\ 1 & 2<z \end{cases}$$

$P_X(1\le X\le 2) = F_X(2) - F_X(1) = 1-e^{-2\lambda}-1+e^{-\lambda} = e^{-\lambda}-e^{-2\lambda} = e^{-\lambda}(1-e^{-\lambda})$

$1-P_X(1\le X\le 2) + F_X(z) - F_X(1) = 1-e^{-\lambda}+e^{-2\lambda}+1-e^{-\lambda z}-1+e^{-\lambda} = 1+e^{-2\lambda}-e^{-\lambda z}$

(c) $V = (X-c)\,\mathbb{I}_{X>c}$ hence V gets the values $[0,\infty)$.

$P_V(v) = \begin{cases} 0 & v<0 \\ F_X(c) & v=0 \\ f_X(v+c) & v>0 \end{cases}$ $\quad F_V(v) = \begin{cases} 0 & v<0 \\ F_X(c) & v=0 \\ F_X(v+c) & 0<v \end{cases}$

(3)

(a) Suppose F is the CDF of a purely continuous random variable, meaning that it is strictly monotonically increasing on the support, with $Im\,F = (0,1)$. Let $G = F^{-1}$ then G maps from $(0,1)$ to the support of the distribution. Let $X\sim U(0,1)$, $Y = G(X)$, and $y\in Im\,G$, then $\exists x'\in(0,1)$ s.t. $G(x') = y$. Since $X\sim U(0,1)$, $F_Y(y) = P_{Uni}(X<x') = x'$. But $G(x') = y \leftrightarrow F^{-1}(x') = y \leftrightarrow x' = F(y)$, meaning that $F_Y(y) = F(y)$ and $F_Y = F$.

(b) Assume now that X has the CDF of F, that is $F_X = F$ and let $Y = F(X)$, meaning that Y gets the values $(0,1)$. Let $y \in (0,1)$ then $\exists x' \in \mathbb{R}$ s.t. $F_X(x') = y$. $F_Y(y) = P_X(X < x') = y$, implying that $Y \sim U(0,1)$.