Understanding the MGF's Foundation:

Why the Existence Condition is Key

A Note for Probability Students

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The Role of the Existence Condition

We've defined the Moment Generating Function (MGF) for a random variable X as $M_X(t) = \mathbb{E}[e^{tX}]$. A crucial caveat accompanies this definition: it's only valid if this expectation actually yields a finite number. The standard safeguard is the *existence condition*: we require that there exists some positive number $\delta > 0$ for which $\mathbb{E}[e^{\delta|X|}]$ is finite.

Why is this specific condition sufficient? Let's walk through the argument to see how it guarantees that $M_X(t)$ is well-defined (i.e., finite) for all values of t within the important interval $(-\delta, \delta)$ centered around zero. This interval is precisely where the MGF needs to be well-behaved for us to extract moments via differentiation.

Demonstration of Sufficiency. Our objective is to rigorously show that if we assume $\mathbb{E}[e^{\delta|X|}] < \infty$ for some $\delta > 0$, then it logically follows that $\mathbb{E}[e^{tX}]$ must be finite for any t satisfying $|t| < \delta$.

Let's select an arbitrary real number t such that it falls within the specified interval, meaning $|t| < \delta$. Our task is to confirm that the value $\mathbb{E}[e^{tX}]$ is finite.

The core idea is to relate the quantity e^{tX} (whose expectation we're interested in) to $e^{\delta|X|}$ (whose expectation we know is finite). We can construct a chain of simple inequalities to bridge this gap:

1. **Relating tX to $|tX|$:** For any real number a, it's always true that $a \leq |a|$. Applying this fundamental property to the exponent tX, we have:

$$tX \leq |tX|$$

2. **Using Properties of Absolute Value:** The absolute value of a product is the product of the absolute values: $|tX| = |t||X|$. Substituting this gives:

$$tX \leq |t||X|$$

3. **Incorporating the Condition $|t| < \delta$:** Here's the crucial step where our assumption about t enters. Since $|X|$ is non-negative, and we know $0 \leq |t| < \delta$, multiplying $|X|$ by $|t|$ results in a value less than or equal to multiplying it by δ:

$$|t||X| \leq \delta|X|$$

(We use $\leq$ rather than $<$ to correctly handle the case where $X = 0$).

4. **Combining the Inequalities:** Linking these steps together provides the key relationship:

$$tX \leq |t||X| \leq \delta|X|$$

Therefore, we have established that $tX \leq \delta|X|$.

Now, we leverage the fact that the exponential function $f(u) = e^u$ is *monotonically increasing*. This means that if $a \leq b$, then $e^a \leq e^b$. Applying this to our inequality $tX \leq \delta|X|$, we obtain:

$$e^{tX} \leq e^{\delta|X|}$$

This inequality holds true for every possible outcome of the random variable X.

The final step involves the expectation operator, $\mathbb{E}[\cdot]$. A fundamental property of expectation is that it preserves inequalities for non-negative random variables. Since both e^{tX} and $e^{\delta|X|}$ are always positive, we can take the expectation of both sides of the inequality:

$$\mathbb{E}[e^{tX}] \leq \mathbb{E}[e^{\delta|X|}]$$

By our initial assumption (the existence condition), we know that $\mathbb{E}[e^{\delta|X|}]$ is a finite number. Let's denote this finite value by K. Substituting this into our inequality yields:

$$\mathbb{E}[e^{tX}] \leq K < \infty$$

Since the expectation $\mathbb{E}[e^{tX}]$ is bounded above by the finite number K, it must necessarily be finite itself.

Conclusion: We have successfully demonstrated that the condition $\mathbb{E}[e^{\delta|X|}] < \infty$ for some $\delta > 0$ is indeed sufficient to guarantee that the MGF $M_X(t) = \mathbb{E}[e^{tX}]$ exists as a finite value for all t in the open interval $(-\delta, \delta)$. This foundational result ensures the MGF is well-defined in the neighborhood of $t = 0$, allowing us to confidently proceed with using its derivatives to generate moments. $\square$