רגרסיה ומודלים סטטיסטיים- תרגיל 0- **רשות**

מרחבים ותתי מרחבים:

4.13. Show that the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (1, 5, 8)$ span $\mathbf{R}^3$.

We need to show that an arbitrary vector $v = (a, b, c)$ in $\mathbf{R}^3$ is a linear combination of u_1, u_2, u_3. Set $v = xu_1 + yu_2 + zu_3$; that is, set

$$(a, b, c) = x(1, 1, 1) + y(1, 2, 3) + z(1, 5, 8) = (x + y + z, \quad x + 2y + 5z, \quad x + 3y + 8z)$$

Form the equivalent system and reduce it to echelon form:

$$\begin{aligned} x + \ \ y + \ \ z &= a \\ x + 2y + 5z &= b \\ x + 3y + 8z &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + \ \ z &= a \\ y + 4z &= b - a \\ 2y + 7c &= c - a \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + \ \ z &= a \\ y + 4z &= b - a \\ -z &= c - 2b + a \end{aligned}$$

The above system is in echelon form and is consistent; in fact,

$$x = -a + 5b - 3c, \quad y = 3a - 7b + 4c, \quad z = a + 2b - c$$

is a solution. Thus, u_1, u_2, u_3 span $\mathbf{R}^3$.

4.12. Prove Theorem 4.3: The intersection of any number of subspaces of V is a subspace of V.

Let $\{W_i : i \in I\}$ be a collection of subspaces of V and let $W = \cap(W_i : i \in I)$. Because each W_i is a subspace of V, we have $0 \in W_i$, for every $i \in I$. Hence, $0 \in W$. Suppose $u, v \in W$. Then $u, v \in W_i$, for every $i \in I$. Because each W_i is a subspace, $au + bv \in W_i$, for every $i \in I$. Hence, $au + bv \in W$. Thus, W is a subspace of V.

המרחב המשלים האורתוגונלי:

7.11. Let W be the subspace of $\mathbf{R}^5$ spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$. Find a basis of the orthogonal complement $W^\perp$ of W.

We seek all vectors $w = (x, y, z, s, t)$ such that

$$\begin{aligned} \langle w, u \rangle &= \ \ x + 2y + 3z - \ \ s + 2t = 0 \\ \langle w, v \rangle &= 2x + 4y + 7z + 2s - \ \ t = 0 \end{aligned}$$

Eliminating x from the second equation, we find the equivalent system

$$\begin{aligned} x + 2y + 3z - s + 2t &= 0 \\ z + 4s - 5t &= 0 \end{aligned}$$

The free variables are y, s, and t. Therefore,

(1) Set $y = -1$, $s = 0$, $t = 0$ to obtain the solution $w_1 = (2, -1, 0, 0, 0)$.

(2) Set $y = 0$, $s = 1$, $t = 0$ to find the solution $w_2 = (13, 0, -4, 1, 0)$.

(3) Set $y = 0$, $s = 0$, $t = 1$ to obtain the solution $w_3 = (-17, 0, 5, 0, 1)$.

The set $\{w_1, w_2, w_3\}$ is a basis of $W^\perp$.

PROBLEM 4. Let U and W be subspaces of a finite dimensional inner product space V. Show that

(1) $(U + W)^\perp = U^\perp \cap W^\perp$

Suppose $x \in (U + W)^\perp$, i.e., x is orthogonal to any vector in $U + V$. Since $u \in U$ also belongs to $U + V$, x must be orthogonal to any $u \in U$. Similarly, it must be orthogonal to any $v \in V$. Hence $x \in U^\perp$ and $x \in V^\perp$, or, equivalently, $x \in U^\perp \cap V^\perp$.

Conversely, suppose $x \in U^\perp \cap V^\perp$. Then x is orthogonal to any $u \in U$ and any $v \in V$. But then x is orthogonal to $u + v$, that is,

$$\langle x, u + v \rangle = \langle x, u \rangle + \langle x, v \rangle = 0 + 0 = 0.$$

Thus $x \in (U + W)^\perp$.

(2) $(U \cap W)^\perp = U^\perp + W^\perp$

Denote $A := (U \cap W)^\perp$ and $B = U^\perp + W^\perp$. Then by the previous question

$$B^\perp = (U^\perp + W^\perp)^\perp = U \cap W = A^\perp$$

and hence $B = A$.

העתקות לינאריות:

5.10. Suppose $F : \mathbf{R}^3 \to \mathbf{R}^2$ is defined by $F(x, y, z) = (x + y + z,\ 2x - 3y + 4z)$. Show that F is linear.

We argue via matrices. Writing vectors as columns, the mapping F may be written in the form $F(v) = Av$, where $v = [x, y, z]^T$ and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$$

Then, using properties of matrices, we have

$$F(v + w) = A(v + w) = Av + Aw = F(v) + F(w)$$

and
$$F(kv) = A(kv) = k(Av) = kF(v)$$

Thus, F is linear.

5.11. Show that the following mappings are not linear:

(a) $F : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (xy, x)$

(b) $F : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $F(x, y) = (x + 3,\ 2y,\ x + y)$

(c) $F : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (|x|,\ y + z)$

(a) Let $v = (1, 2)$ and $w = (3, 4)$; then $v + w = (4, 6)$. Also,

$$F(v) = (1(2), 1) = (2, 1) \qquad \text{and} \qquad F(w) = (3(4), 3) = (12, 3)$$

Hence,

$$F(v + w) = (4(6), 4) = (24, 6) \neq F(v) + F(w)$$

(b) Because $F(0, 0) = (3, 0, 0) \neq (0, 0, 0)$, F cannot be linear.

(c) Let $v = (1, 2, 3)$ and $k = -3$. Then $kv = (-3, -6, -9)$. We have

$$F(v) = (1, 5) \quad \text{and} \quad kF(v) = -3(1, 5) = (-3, -15).$$

Thus,

$$F(kv) = F(-3, -6, -9) = (3, -15) \neq kF(v)$$

Accordingly, F is not linear.

5.18. Consider the matrix mapping $A : \mathbf{R}^4 \to \mathbf{R}^3$, where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$. Find a basis and the dimension of (a) the image of A, (b) the kernel of A.

(a) The column space of A is equal to $\operatorname{Im} A$. Now reduce A^T to echelon form:

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{(1, 1, 3), (0, 1, 2)\}$ is a basis of $\operatorname{Im} A$, and $\dim(\operatorname{Im} A) = 2$.

(b) Here $\operatorname{Ker} A$ is the solution space of the homogeneous system $AX = 0$, where $X = \{x, y, z, t)^T$. Thus, reduce the matrix A of coefficients to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + 2y + 3z + t &= 0 \\ y + 2z - 3t &= 0 \end{aligned}$$

The free variables are z and t. Thus, $\dim(\operatorname{Ker} A) = 2$.

(i) Set $z = 1$, $t = 0$ to get the solution $(1, -2, 1, 0)$.

(ii) Set $z = 0$, $t = 1$ to get the solution $(-7, 3, 0, 1)$.

Thus, $(1, -2, 1, 0)$ and $(-7, 3, 0, 1)$ form a basis for $\operatorname{Ker} A$.

5.22. Prove Theorem 5.3: Let $F : V \to U$ be linear. Then,

(a) Im F is a subspace of U, (b) Ker F is a subspace of V.

(a) Because $F(0) = 0$, we have $0 \in \text{Im } F$. Now suppose $u, u' \in \text{Im } F$ and $a, b \in K$. Because u and u' belong to the image of F, there exist vectors $v, v' \in V$ such that $F(v) = u$ and $F(v') = u'$. Then

$$F(av + bv') = aF(v) + bF(v') = au + bu' \in \text{Im } F$$

Thus, the image of F is a subspace of U.

(b) Because $F(0) = 0$, we have $0 \in \text{Ker } F$. Now suppose $v, w \in \text{Ker } F$ and $a, b \in K$. Because v and w belong to the kernel of F, $F(v) = 0$ and $F(w) = 0$. Thus,

$$F(av + bw) = aF(v) + bF(w) = a0 + b0 = 0 + 0 = 0, \qquad \text{and so} \qquad av + bw \in \text{Ker } F$$

Thus, the kernel of F is a subspace of V.

וקטור קואורדינאטות:

4.62. Find the coordinate vector of $v = (a, b, c)$ in $\mathbf{R}^3$ relative to

(a) the usual basis $E = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$,

(b) the basis $S = \{u_1, u_2, u_3\} = \{(1, 1, 1),\ (1, 1, 0),\ (1, 0, 0)\}$.

(a) Relative to the usual basis E, the coordinates of $[v]_E$ are the same as v. That is, $[v]_E = [a, b, c]$.

(b) Set v as a linear combination of u_1, u_2, u_3 using unknown scalars x, y, z. This yields

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x + y + z &= a \\ x + y \quad\; &= b \\ x \qquad\quad &= c \end{aligned}$$

Solving the system yields $x = c$, $y = b - c$, $z = a - b$. Thus, $[v]_S = [c,\ b - c,\ a - b]$.

מטריצה מייצגת העתקה:

6.2. Consider the following linear operator G on $\mathbf{R}^2$ and basis S:

$$G(x, y) = (2x - 7y,\ 4x + 3y) \qquad \text{and} \qquad S = \{u_1, u_2\} = \{(1, 3),\ (2, 5)\}$$

(a) Find the matrix representation $[G]_S$ of G relative to S.

(b) Verify $[G]_S[v]_S = [G(v)]_S$ for the vector $v = (4, -3)$ in $\mathbf{R}^2$.

First find the coordinates of an arbitrary vector $v = (a, b)$ in $\mathbf{R}^2$ relative to the basis S. We have

$$\begin{bmatrix} a \\ b \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \qquad \text{and so} \qquad \begin{aligned} x + 2y &= a \\ 3x + 5y &= b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -5a + 2b$, $y = 3a - b$. Thus,

$$(a, b) = (-5a + 2b)u_1 + (3a - b)u_2, \qquad \text{and so} \qquad [v] = [-5a + 2b,\quad 3a - b]^T$$

(a) Using the formula for (a, b) and $G(x, y) = (2x - 7y,\ 4x + 3y)$, we have

$$\begin{aligned} G(u_1) &= G(1, 3) = (-19, 13) = 121u_1 - 70u_2 \\ G(u_2) &= G(2, 5) = (-31, 23) = 201u_1 - 116u_2 \end{aligned} \qquad \text{and so} \qquad [G]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix}$$

(We emphasize that the coefficients of u_1 and u_2 are written as columns, not rows, in the matrix representation.)

(b) Use the formula $(a, b) = (-5a + 2b)u_1 + (3a - b)u_2$ to get

$$\begin{aligned} v &= (4, -3) = -26u_1 + 15u_2 \\ G(v) &= G(4, -3) = (20, 7) = -131u_1 + 80u_2 \end{aligned}$$

Then $\qquad [v]_S = [-26, 15]^T \qquad \text{and} \qquad [G(v)]_S = [-131, 80]^T$

Accordingly,

$$[G]_S[v]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix} \begin{bmatrix} -26 \\ 15 \end{bmatrix} = \begin{bmatrix} -131 \\ 80 \end{bmatrix} = [G(v)]_S$$

ליכסון:

1. Let

$$\mathbf{A} = \begin{bmatrix} 6.8 & 2.4 \\ 2.4 & 8.2 \end{bmatrix}$$

Compute the spectral decomposition of $\mathbf{A}$.

<u>Solution:</u>

First we will calculate the eigenvalues by solving the equation

$$|\mathbf{A} - \lambda I| = 0$$

$$\begin{vmatrix} 6.8-\lambda & 2.4 \\ 2.4 & 8.2-\lambda \end{vmatrix} = (6.8-\lambda)(8.2-\lambda) - 5.76 = 0$$

$$\Rightarrow \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10) = 0$$

$$\Rightarrow \lambda_1 = 5, \lambda_2 = 10.$$

In order to find the corresponding eigenvectors we will solve the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$
For $\lambda = 5$ we have

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1.8 & 2.4 \\ 2.4 & 3.2 \end{pmatrix}$$

The equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ leads to the equation $3x_1 + 4x_2 = 0$, which is solved by

$$v_1 = \begin{pmatrix} 1 \\ -0.75 \end{pmatrix}$$

For $\lambda = 10$ we have

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -3.2 & 2.4 \\ 2.4 & -1.8 \end{pmatrix}$$

The equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ leads to the equation $4x_1 - 3x_2 = 0$, which is solved by

$$v_2 = \begin{pmatrix} 0.75 \\ 1 \end{pmatrix},$$

By normalizing to length 1 the vectors v_1, v_2, we get

$$e_1 = \begin{pmatrix} 0.8 \\ -0.6 \end{pmatrix}, e_2 = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}.$$

The spectral decomposition of $\mathbf{A}$ is thus

$$\mathbf{A} = U\Lambda U^T$$

where

$$U = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$$