Abstract Vector Spaces and Coordinates

Oliver Li

Fall 2021

1 General Concept

- a *vector space* is a set of elements, called *vectors*, on which two operations are defined: (1) vector addition (2) scalar multiplication
- **Definition - isomorphism**

  Given vector spaces V and W, an *isomorphism* from V to W is an invertible linear transformation from V to W.

  Note(1): The composition of isomorphisms is an isomorphism

  Note(2): Isomorphism is an equivalence relation
- **Definition - Coordinates**

  Suppose $\mathcal{B} = (\overrightarrow{v_1}, ..., \overrightarrow{v_n})$ is an ordered basis of the finite dimensional space V, and let $\overrightarrow{v_i} \in V$. The $\mathcal{B}$ **-coordinate vector** of $\overrightarrow{v}$, written $[\overrightarrow{v}]_{\mathcal{B}}$ is the unique vector

  $$[\overrightarrow{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ ... \\ a_n \end{bmatrix} \in \mathbb{R}$$

  such that $\overrightarrow{v} = a_1\overrightarrow{v_1} + \cdots + a_n\overrightarrow{v_n}$

  The map $L_{\mathcal{B}} : V \to \mathbb{R}^n$ defined by $L_{\mathcal{B}}(\overrightarrow{v}) = [\overrightarrow{v}]_{\mathcal{B}}$ is called the $\mathcal{B}$ **-coordinate isomorphism**
- **Definition - Change-of-coordinates matrix**

  If $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ and $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_n)$ are two ordered bases of the vector space V, the **change-of-coordiantes matrix** from the $\mathcal{B}$ to $\mathcal{C}$ is the standard matrix $L_{\mathcal{C}} \circ L_{\mathcal{B}}^{-1}$, where $L_{\mathcal{B}}$ and $L_{\mathcal{C}}$ are the $\mathcal{B}$- and $\mathcal{C}$-coordinate isomorphisms, respectively
- **Definition - $\mathcal{B}$-matrix**

  If $T : V \to V$ is a linear transformation of the vector space V and if $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is an ordered basis of V, then the $\mathcal{B}$**-matrix** of T is the standard matrix of $L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1}$, where $L_{\mathcal{B}} : V \to \mathbb{R}^n$ is the $\mathcal{B}$-coordinate isomorphism

- **Definition - similar**

  Let A and B be two n x n matrices. Then A is similar to B if there exists an invertible n x n matrix S such that $A = S^{-1}BS$

2 Understandings

- Let V and W be vector spaces, suppose $T : V \to W$ is an isomorphism, and let $\mathcal{B} = (v_1, ..., v_n)$ be a list of vectors in V. Then $\mathcal{B}$ is a basis of V if and only if $(T(v_1), ..., T(v_n))$ is a basis of W.

  - Forward Direction:
    - Let $\overrightarrow{w} \in W$; Since $\mathcal{B}$ spans V, there exists scalars $c_1, ..., c_n$ such that $T^{-1}(\overrightarrow{w}) = c_1\overrightarrow{v_1} + \cdots + c_n\overrightarrow{v_n}$
    - Applying T on both sides of the equation, we could get
      $\overrightarrow{w} = T(T^{-1}(\overrightarrow{w})) = T(\sum_{i=1}^{n} c_i\overrightarrow{v_i}) = \sum_{i=1}^{n} c_iT(\overrightarrow{v_i})$
    - The above proves that $T(\overrightarrow{v_i})$ spans W.
    - To see they are linearly independent. Let $c_1, ..., cn \in \mathbb{R}$ and suppose $\sum_{i=1}^{n} c_iT(\overrightarrow{v_i} = \overrightarrow{0})$
    - Again, applying T^{-1} to both sides we could get

$$\overrightarrow{0} = T^{-1}(\overrightarrow{0}) = T^{-1}(\sum_{i=1}^{n} c_iT(\overrightarrow{v_i})) = (T^{-1}\circ T)(\sum_{i=1}^{n} c_i\overrightarrow{v_i}) = \sum_{i=1}^{n} c_i\overrightarrow{v_i} \quad (1)$$

    - since we know $\overrightarrow{v_i}$ is basis so the coefficient c_i must be all 0 and thus they are linearly independent and we have completed the proof
  - Backward Direction follows by symmetry
  - The key is to think about isomorphism in the first place so we could use inverse of linear transformation and the following process is done naturally

- If a vector space V is finite-dimensional, then V has a finite basis

- If V and W are finite-dimensional vector spaces, then $V \cong W$ if and only if dim V = dim W (proved by using the help of L-mapping)

- We can interpret any L-mapping as **the mapping of isomorphism** between the vector space V and "flattened" constant space $\mathbb{R}^n$

  - For instance, the vector space $\mathbb{R}^{nxn}$ has isomorphism with the vector space $\mathbb{R}^{n^n}$

- Proved using the first bullet point in Understanding part. Given an isomorphism $T : V \to \mathbb{R}^n$, the list $(T^{-1}(\overrightarrow{e_1}), ..., T^{-1}\overrightarrow{e_n})$ is an ordered basis of V. The converse is proved before.

- The associated $\mathcal{B}$ or any basis based coordinates are the unique linear combinations of this particular set of basis for a certain vector

- We could interpret our intuitive recognition of any linear transformation to with respect to coordinate perspective, that is $A\overrightarrow{x} = T(\overrightarrow{x})$ could be seen as

$$A[\overrightarrow{x}]_\varepsilon = [T(\overrightarrow{x})]_\varepsilon$$

With this recognition, A is no more than the matrix of T *relative to the standard basis*, ε.

- **IMPORTANT**

$[T]_\epsilon = [[T(\overrightarrow{e_1})]_\epsilon \cdots [T(\overrightarrow{e_n})]_\epsilon]$

  - Elegant proof for $[T]_\epsilon$:
    If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for all $\overrightarrow{x} \in \mathbb{R}^n$,
    $$T(\overrightarrow{x}) = T(\textstyle\sum_{i=1}^n x_i \overrightarrow{e_i}) = \sum_{i=1}^n x_i T(\overrightarrow{e_i}) = [T(\overrightarrow{e_1}) \cdots T(\overrightarrow{e_n})]$$

Similarly, for any basis $\mathcal{B}$, we have

$[T]_\mathcal{B} = [[T(b_1)]_\mathcal{B} \cdots [T(b_n)]_\mathcal{B}]$

- For any column vectors that are consisted of all the basis vector B, we have the following relationship:

$$B[\overrightarrow{x}]_\mathcal{B}, \text{ and } B^{-1}\overrightarrow{x} = [\overrightarrow{x}]_\mathcal{B}$$

This implies that

$L_\mathcal{B} = B^{-1}$, and $L_\mathcal{B}^{-1} = B$

- **Change-of-Coordinates**

Let V be a vector space, and $\mathcal{B}$ and $\mathcal{C}$ be two bases of V. Then we have

  - $L_\mathcal{C} \circ L_\mathcal{B}^{-1}$ is an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$
  - Its standard matrix S will be an invertible n x n matrix which is called the *change-of-coordinates* matrix from $\mathcal{B}$ to $\mathcal{C}$, denoted $S = S_{\mathcal{B}\to\mathcal{C}}$
  - The i-th column of $S_{\mathcal{B}\to\mathcal{C}}$ is $S_{\mathcal{B}\to\mathcal{C}}\vec{e}_i = L_\mathcal{C}(L_\mathcal{B}^{-1}(\vec{e}_i)) = L_\mathcal{C}(\vec{b}_i) = [\vec{b}_i]_\mathcal{C}$
  - $S_{\mathcal{B}\to\mathcal{C}}[\vec{v}]_\mathcal{B} = [\vec{v}]_\mathcal{C}$
  - $S_{\mathcal{B}\to\mathcal{C}}$ and $S_{\mathcal{C}\to\mathcal{B}}$ are inverses of each other
  - $S_{\mathcal{B}\to\mathcal{C}} = C^{-1}B$ and $S_{\mathcal{C}\to\mathcal{B}} = B^{-1}C$ (True only when $V = \mathbb{R}^n$)

- All of the above to our definition of **$\mathcal{B}$-matrix of T**

  Let $\mathcal{B} = (\overrightarrow{b_1}, ..., \overrightarrow{b_n})$ be an ordered basis of the vector space V. Then for any linear transformation $T : V \to V$, there is an unique n x n matrix $[T]_{\mathcal{B}}$, called the $\mathcal{B}$-matrix of T, such that $[T(\overrightarrow{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\overrightarrow{x}]_{\mathcal{B}}$ for all $\overrightarrow{x} \in V$.

- The above definition naturally leads to the definition of **similarity**. Since the $\mathcal{B}$-matrix is just another mapping to a different basis-based $\mathbb{R}^n$ space, there exists isomorphism between the constant vector space and the original V. Since there are numerous basis, there are numerous isomorphisms. The composition of isomorphism is isomorphism, so that is how similarity arises.

- We hence have two systematic ways to calculate the $\mathcal{B}$-matrix of T
  - $[T]_{\mathcal{B}} = [[T(b_1)]_{\mathcal{B}} \cdots [T(b_n)]_{\mathcal{B}}]$
  - $[T]_{\mathcal{B}} = L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1}$

- In case forgetting the first fomula, here is the simple proof:

  In order to find the first column of the B-matrix of T, we should do

  $$[T]_{\mathcal{B}}\overrightarrow{e_1} = [T]_{\mathcal{B}}[\overrightarrow{b_1}]_{\mathcal{B}} = [T(b_1)]_{\mathcal{B}}$$

- The $\mathcal{B}$-matrix of T is unique
  - Proved by considering the linear transformation $U = L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. We know $[U]_\epsilon$ is unique and we can show that $[T]_{\mathcal{B}} = [U]_\epsilon$

- **similarity and coordinates**

  Two n x n matrices A and B are similar to each other if and only if there is a linear transformation T of an n-dimensional vector space V and a pair of ordered bases $\mathcal{B}$ and $\mathcal{C}$ of V such that $A = [T]_{\mathcal{B}}$ and $B = [T]_{\mathcal{C}}$