Determinant

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1 General Concept

- The most intuitive definition of **determinant**:

  If A is an n x n matrix, the *determinant* of A is defined to be the number

  $$det(A) = \sum_P sgn(P) prod_A(P),$$

  where the sum is taken over all possible patterns P in the matrix A.

- **Definition - Pattern**
  - A *patternP* in A is a list of n entries of A that contains exactly one entry from each row of A and one entry from each column of A
  - an *inversion* in P is a pair $(a_{i_1 j_1}, a_{i_2 j_2})$ in P such that $i_1 < i_2$ but $j_1 > j_2$ (pairwise speaking, one entry is at the right top of the other)
  - *sgn* is defined to be the indicator function of P, where if there are even number of inversions then sgn(P) = 1; otherwise (odd number of inversions) sgn(P) = -1

- Laplace expansions

  we can fix one row/column and split a big determinant calculations into smaller ones. More specific definitions are in worksheet/textbook

2 Understandings

- The determinant of a triangular matrix is the product of the diagonal entries
- $\det(A^T) = \det(A)$
- Determinant is **multilinear** on both the columns and the rows of square matrices (proved by the distributive property of $prod_A(P)$ inside the summation)
- For any n x n matrices A and B, $\det(AB) = (\det A)(\det B)$

- First, prove $\det(EA) = (\det E)(\det A)$, where E is the elementary matrix obtained from I_n by scaling, interchanging rows, or adding n times a row to another row
- then discuss two conditions. If A/B not invertible, we proved the claim; if they both invertible then make them being the product of elementary matrices and then we prove the claim

- If a square matrix A is invertible, then $\det(A) \neq 0$ and $\det(A^{-1}) = (\det A)^{-1}$
  - $det((E_1....E_k)A) = det(I_n) = 1$
  - Using the property of spliting elementary matrices determinant, we could get
  - $\det(E_k....E_1)\det(A) = 1$
  - which is $\det(A^{-1})\det(A) = 1$
- If A and B are similar n x n matrices, then $\det B = \det A$
- $\det(A^n) = (det A)^n$
- The determinant of an orthogonal matrix is $\pm$ 1
- In higher dimensions, the "n-dimensional volume" of the n-dimensional parallelepiped $(\overrightarrow{v_1},...,\overrightarrow{v_n})$ determined by the linearly independent vectors $\overrightarrow{v_1},...,\overrightarrow{v_n}$ in $\mathbb{R}^n$ is given by

  $$\prod_{k=1}^{n} \|\overrightarrow{v_k} - proj_{V_k}(\overrightarrow{v_k})\|, \text{ where } V_k = \text{Span}(\overrightarrow{v_1},...,\overrightarrow{v_{k-1}})$$

- The above can be used to show the the volume of the parallelepiped is $\|det[\overrightarrow{v_1}...\overrightarrow{v_n}]\|$
- The determinant of an isomorphic linear transformation is defined to be

  $$\det T = \det[T]_B$$

  This makes sense of the determinant of similar matrix always equal to each other
- We can think the determinant of a linear isomorphic transformation as an expansion factor. Namely we have the following formula:

  $$\text{Vol}(T[A]) = \|det T\| \text{ Vol}(A)$$