# Eigenvalues and Eigenvectors

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## 1 General concept

- **Definition**

  Let $V$ be a vector space of dimension $n$, and let $T : V \to V$ be a linear transformation

  The scalar $\lambda \in \mathbb{R}$ is an *eigenvalue* of $T$ if there is a non-zero vector $v \in V$ such that

  $$T(v) = \lambda v;$$

  such a vector is called an *eigenvector* of $T$, with corresponding eigenvalue $\lambda$.

- The geometric intuition for eigenvector is that some special vectors in the domain of $T$ such that would only stretch itself when doing matrix multiplication with a certain matrix. Such vectors are called eigen vectors and the degree of "stretching" is called eigenvalues.

- The set

  $$E_\lambda = \{v \in V : T(v) = \lambda v\}$$

  is called the *eigenspace* corresponding to $\lambda$. Eigenspace is a subspace of $V$ because it contains the zero vector, and closed under vector addition and scalar multiplication

- **Definition**

  The *characteristic polynomial* of the linear transformation $T : V \to V$ is the polynomial $f_T$ (which we write here in the variable $\lambda$) given by

  $$f_T(\lambda) = det(\lambda I - T),$$

  where $I : V \to V$ is the identity transformation. Here we are thinking of $\lambda I - T$ as a new linear transformation from $V$ to $V$, defined by

$$(\lambda I - T)(v) = \lambda v - T(v) \text{ for all } v \in V$$

- The above definition leads us a systematic way of finding eigenvalues. If $v$ is an eigenvector of $T$ with corresponding eigenvalue $\lambda$, then,

  $$(\lambda I - T)(v) = 0$$

  since here we assume the nonzero eigenvector exists, then the linear transformation $\lambda I - T$ has a nontrivial kernel, hence it is not invertible, and we thus have

  $$det(\lambda I - T) = 0$$

  Conversely, reversing the arugment shows that if the determinant is 0, then $\lambda$ is an eigenvalue of $T$. This shows that *the eigenvalues of $T$ are just the roots of the characteristic polynomial of $T$*

- **Definition**

  Let $\lambda$ be an eigenvalue of $T$.

  The *algebraic multiplicity* "almu($\lambda$)" of $\lambda$ is the number of times that $\lambda$ occurs as a root of the characteristic polynomial $f_T$ of $T$; that is, the largest power of the root.

  The *geometric multiplicity* "gemu($\lambda$)" of $\lambda$ is the dimension of the corresponding eigenspace $E_\lambda$

- **Definition - Diagonalization**

  A linear transformation $T : V \to V$ of the finite-dimensional vector space $V$ is said to be *diagonalizable* if there there is a basis $B$ of $V$ such that $[T]_B$ is diagnoal. A square matrix $A$ is said to be *diagonalizable* if the linear transformation $T_A$ is diagonalizable, or equivalently if A is similar to a diagonal matrix

# 2 Understanding of the concept

- Eigenvectors that are of distinct eigenvalues are linearly independent
- The intersection of two eigenvectors that have distinct eigenvalues is only the zero vector
- we can think about diagonalizable matrix means that the matrix is the $\mathcal{B}$-matrix, where the basis is just the eigenbasis. In other words, the direct sum of all eigenspaces equals the vector space.
  - Two important theorems to bridge the gap:
  - if $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis of $V$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & \\ & ... & \\ & & \lambda_n \end{bmatrix} \text{ if and only if } T(v_i) = \lambda_i v_i \text{ for all } 1 \leq i \leq n$$

  - $T$ is diagonalizable if and only if there is a basis $\mathcal{B}$ of $V$ consisting of eigenvectors of $T$. (Such a basis of $V$ is called an *eigenbasis* of $T$)
  - Similarly, we have theorems for just a $n$ x $n$ matrix A.
    * if the $n$ x $n$ matrix $P = [\vec{v}_1 \cdots \vec{v}_n]$ is invertible and if $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, then

      $$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & ... & \\ & & \lambda_n \end{bmatrix} \text{ if and only if } A\vec{v}_i = \lambda_i \vec{v}_i \text{ for all i}$$

    * A is diagonalizable if and only is there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of $A$. (Again, such basis of $\mathbb{R}^n$ is called an *eigenbasis* for $A$)

- In Summary of above's theorems, we finally bridge the gap between diagonizalition and eigenvectors/eigenvalues through the idea of similarity and (eigen)basis. Hence, we have the following understanding:

  To *diagonalize* a linear transformation $T : V \to V$ means to find a basis $\mathcal{B}$ of $V$ such that $[T]_{\mathcal{B}}$ is diagonal. Similarly, to *diagonalize* a square matrix $A$ means to factor $A$ as $A = PDP^{-1}$ where $D$ is diagonal

- Another interpretation of the lemma of eigenspace spans the vector space is :

  The $n$ x $n$ matrix $A$ is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues of $A$ is $n$. Equivalently, the $n$ x $n$ matrix $A$ is diagonalizable if and only if the sum of the algebraic multiplicities of the eigenvalues of $A$ is $n$ and additionally for each eigenvalue $\lambda$ of $A$, the algebraic and geometric multiplicities of $\lambda$ are equal.

  Note: The set of eigenvalues are also called the *spectrum*

- Similar to abstract vector space we have studied before, we can apply eigenbasis/diagonalization on linear spaces.

- Suppose $\lambda_1, \dots, \lambda_r$ are *distinct* eigenvalues of the $n$ x $n$ matrix $A$, and for each $1 \leq k \leq r$ let $\mathcal{B}_k$ be a basis of the subspace

  $$E_{\lambda_k} = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda_k \vec{v}\}$$

  Then, the set $\mathcal{B} = \mathcal{B}_1 \bigcup \cdots \bigcup \mathcal{B}_r$ is linearly independent

- Two useful theorems:

  - an $n$ x $n$ matrix with $n$ distinct eigenvalues must be diagonalizable

- If $A$ is a diagonizable $n$ x $n$ matrix with only one eigenvalue, then $A$ is already a diagonal matrix

- For all similar $n$ x $n$ matrices $A$ and $B$, they have the same
  - characteristic polynomial
  - eigenvalues
  - determinant
  - trace
- Some interesting examples given a square matrix A that is:
  - invertible but not diagonalizable $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
  - diagonalizable but not invertible $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  - neither diagonalizable nor invertible $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
  - both diagonalizable and invertible $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$