Analysis

Gaotang Li

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Part I

Foundations

Chapter 1

Groups and Homomorphism

1.1 Basics

Definition 1.1.1 (Group). A pair $(G, \odot)$ consisting of a nonempty set G and an operation $\odot$ is called a **group** if the following holds:

- G is closed under the operation $\odot$
- $\odot$ is associative
- $\odot$ has an identity element e
- Each $g \in G$ has an **inverse** $h \in G$ such that $g \odot h = h \odot g = e$

Definition 1.1.2 (Abelian group). A group $G, \odot$ is called **commutative** or **Abelian** if $\odot$ is a commutative operation on G.

Remark. Let $G = (G, \odot)$

(a) the identity element e is unique

(b) Each $g \in G$ has a unique inverse which we denote by g^b. In particular $e^b = e$.

(c) For each $g \in G$, we have $\left(g^b\right)^b = g$.

(d) For arbitrary group elements g and h, $(g \odot h)^b = h^b \odot g^b$

Example. (a) Let $G := \{e\}$ be a one element set. Then $\{G, \odot\}$ is an Abelian group, the **trivial group**, with the (only possible) operation $e \odot e = e$.

(b) Let X be a nonempty set, and S_X be the set of all bijections from X to itself. Then $S_X := (S_X, \circ)$ is a group with identity element id_X when $\circ$ denotes the composition of functions. Further, the inverse function f^{-1} is the inverse of $f \in S_X$ in the group. When X is finite, the element of S_X are called permutations and S_X is called the **permutation group** of X.

(c) Let X be a nonempty set and $G, \odot$ a group. With the induced operation $\odot$, $\left(G^X, \odot\right)$ is a group. The inverse of $f \in G^X$ is the function

$$f^b \colon X \to G, \quad x \mapsto (f(x))^b$$

(d) Let $G_1, \ldots, G_m$ be groups. Then $G_1 \times \cdots \times G_m$ with the operation defined analogously to (d) is a group called the **direct product** of $G_1, \ldots, G_m$.

1.2 Subgroup

Definition 1.2.1 (Subgroup). Let $G = (G, \odot)$ be a group and H a nonempty subset of G, if

- $H \odot H \subseteq H$
- $h^b \in H$ for all $h \in H$

then $H := (H, \odot)$ is itself a group and is called a **subgroup** of G.

Remark. Here we use the same symbol $\odot$ for the restriction of the operation to H. Since H is nonempty, there is some $h \in H$ and so, from the two axioms above, $e = h^b \odot h$ is also in H.

Example. Let $G = (G, \odot)$ be a group.

(a) The trivial subgroup $\{e\}$ and G itself are subgroups of G, the smallest and largest subgroups with respect to inclusion

(b) If H_α, $\alpha \in A$ are subgroups of G, then $\bigcap_\alpha H_\alpha$ is also a subgroup of G.

1.3 Cosets

Definition 1.3.1 (Coset). Let N be a subgroup of G and $g \in G$. Then $g \odot N$ is the **left coset** and $N \odot g$ is the **right coset** of $g \in G$ with respect to N.

Remark. The definition of coset is related to the particular element.

Note. If we define

$$g \sim h \Leftrightarrow g \in h \odot N \tag{1.1}$$

Then $\sim$ is an equivalence on G.

Proof. $\sim$ is reflexive because $e \in N$
Let $g \in h \odot N$ and $h \in k \odot N$, then

$$g \in (k \odot N) \odot N = k \odot (N \odot N) = k \odot N$$

Let $g \in h \odot N$, then there is some $n \in N$ with $g = h \odot n$. Then it follows that $h = g \odot n^b \in N$. ■

Here 1.1 defines an equivalence relation on G. For the equivalence classes $[\cdot]$ with respect to $\sim$, we have

$$[g] = g \odot N, \quad g \in G. \tag{1.2}$$

For this reason, we denote $G/\sim$ by G/N, and call G/N the **set of left cosets** of G **modulo** N. Particularly, we have subgroups N such that

$$g \odot N = N \odot g, \quad g \in G. \tag{1.3}$$

Such a subgroup 1.3 is called a **normal subgroup** of G. We call $g \odot N$ the **coset of** g **modulo** N since each left coset is a right coset and vice versa. We have a well-defined operation on G/N where N is the normal subgroup of G, induced from $\odot$, such that

$$(G/N) \times (G/N) \to G/N, \quad (g \odot N, h \odot N) \mapsto (g \odot h) \odot N \tag{1.4}$$

Proposition 1.3.1. Let G be a group and N a normal subgroup of G. Then G/N with the induced operation is a group, the **quotient group of G modulo** N.

Proof. It is easy to check that the operation is associative. Since $(e \odot N) \odot (g \odot N) = (e \odot g) \odot N = g \odot N$, the identity element of G/N is $N = e \odot N$. Also

$$\left(g^b \odot N\right) \odot (g \odot N) = \left(g^b \odot g\right) \odot N = N$$

∎

Remark. (a) In notion of 1.1, $[e] = N$ is the identity element of G/N and $[g]^b = [g^b]$ is the inverse of $[g] \in G/N$. We also have $[g] \odot h = [g \odot h]$, $g, h \in G$.

(b) Any subgroup N of an Abelian group G is normal and so G/N is a group. Meanwhile, G/N is Abelian.

1.4 Homomorphisms

Definition 1.4.1 (Homomorphism). Let $G = (G, \odot)$ and $G' = (G', \circledast)$ be groups... A function $\varphi \colon G \to G'$ is called a **(group) homomorphism** if

$$\varphi(g \odot h) = \varphi(g) \circledast \varphi(h), \quad g, h \in G$$

Definition 1.4.2 (Endomorphism). A homomorphism from G to itself

Remark. (a) Let e and e' be the identity elements of G and G' respectively, and let $\varphi \colon G \to G'$ be a homomorphism. Then

$$\varphi(e) = e' \quad \text{and} \quad (\varphi(g))^b = \varphi(g^b), \quad g \in G$$

Proof. $e' \circledast \varphi(e) = \varphi(e) = \varphi(e \odot e) = \varphi(e) \circledast \varphi(e)$
$e' = \varphi(e) = \varphi\left(g^b \odot g\right) = \varphi\left(g^b\right) \circledast \varphi(g)$ ∎

(b) Let $\varphi \colon G \to G'$ be a homomorphism. The **kernel** of φ, $\ker(\varphi)$, defined by

$$\ker(\varphi) := \varphi^{-1}(e') = \{g \in G\,;\, \varphi(g) = e'\}$$

is a normal subgroup of G.

Proof. First, try to prove $\ker(\varphi)$ is a subgroup of G. For all $g, h \in G$,

- $\varphi(g \odot h) = \varphi(g) \circledast \varphi(h) = e' \circledast e' = e'$
- $\varphi\left(g^b\right) = (\varphi(g))^b = (e')^b = e'$

Second, try to prove it is a normal subgroup. Let $h \in g \odot \ker(\varphi)$. Then we there is some $n \in G$ such that $\varphi(n) = e'$ and $h = g \odot n$. For $m := g \odot n \odot g^b$, we have

$$\varphi(m) = \varphi(g) \circledast \varphi(n) \circledast \varphi(g^b) = \varphi(g) \circledast \varphi(g^b) = e'$$

and hence $m \in \ker(\varphi)$. Since $m \odot g = g \odot m = h$, this implies that $h \in \ker(\varphi) \odot g$. So $\ker(\varphi) \odot g \subseteq g \odot \ker(\varphi)$. Similarly one can show $g \odot \ker(\varphi) \subset \ker(\varphi) \odot g$. ∎

(c) Let $\varphi \colon G \to G'$ be a homomorphism and $N := \ker(\varphi)$. Then

$$g \odot N = \varphi^{-1}(\varphi(g)), \quad g \in G,$$

and so

$$g \sim h \Leftrightarrow \varphi(g) = \varphi(h), \quad g, h \in G,$$

where $\sim$ denotes the equivalence relation 1.1.

(d) A homomorphism is injective if and only if its kernel is trivial, that is, $\ker(\varphi) = \{e\}$

(e) The image $\operatorname{im}(\varphi)$ of a homomorphism $\varphi\colon G \to G'$ is a subgroup of G'.

Example. (a) The constant function $G \to G'$, $g \mapsto e'$ is a homorphism, the **trivial** homomorphism.

(b) The identity function $\mathrm{id}_G\colon G \to G$ is an endomorphism.

(c) Compositions of homomorphisms (endomorphisms) are homomorphisms (endomorphisms).

(d) If $\varphi\colon G \to G'$ is a bijective homomorphism, then so is $\varphi^{-1}\colon G \to G'$

Definition 1.4.3 (Isomorphism)**.** A homomorphism $\varphi\colon G \to G'$ is called a (**group**) **isomorphism** from G to G' if φ is bijective.
In this circumstance, we say that the groups G and G' are **isomorphic** and write $G \cong G'$.

Definition 1.4.4 (Automorphism)**.** An isomorphism from G to itself.

Chapter 2

Rings, Fields and Polynomials

2.1 Rings

Definition 2.1.1 (Ring). A triple $(R, +, \cdot)$ consisting of a nonempty set R and operations, **addition** $+$ and **multiplication** $\cdot$, is called a **ring** if

- $(R, +)$ is an Abelian group
- Multiplication is associative
- The **distributive law** holds:

$$(a + b) \cdot c = a \cdot c + b \cdot, \quad c \cdot (a + b) = c \cdot a + c \cdot b, \quad a, b, c \in R$$

Note. A ring is called **commutative** if multiplication is commutative.
If there is an identity element with respect to multiplication, then it is written as 1_R or simply 1, and is called the **unity** (or **multiplicative identity**) of R, and we say $(R, +, \cdot)$ is a **ring with unity**.
When the addition and multiplication operations are clear from context, we write simply R instead of $(R, +, \cdot)$.

Example. (a) The **trivial ring** has exactly one element 0 and is itself denoted by 0. A ring with more than one element is **nontrivial**. If R is a ring with unity, then it follows from $1_R \cdot a = a$ for each $a \in R$, that R is trivial if and only if $1_R == 0_R$.

(b) Suppose R is a ring and S is a nonempty subset of R that satisfies the following:

- S is a subgroup of $(R, +)$.
- $S \cdot S \subseteq S$

Then S itself is a ring, a **subring** of R, and R is called an **overring** of S. If R is commutative then so is S, but the converse is not true in general.

(c) Intersections of subrings are subrings.

Definition 2.1.2 (Ring Homomorphism). Let R and R' be rings. A (**ring**) **homomorphism** is a function $\varphi\colon R \to R'$ which is compatible with the ring operations, that is,

$$\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in R \tag{2.1}$$

Note. If, in addition, φ is bijective, then φ is called a (**ring**) **isomorphism** and R and R' are **isomorphic**.

A homomorphism φ from R to itself is a **(ring) endomorphism**. If φ is an isomorphism, then it is a **(ring) automorphism**.

Example. (a) A ring homomorphism $\varphi\colon R \to R'$ is, in particular, a group homomorphism from $(R, +)$ to $(R', +)$. The **kernel**, $\ker(\varphi)$, of φ is defined to be the kernel of this group homomorphism, that is,

$$\ker(\varphi) = \{a \in R; \varphi(a) = 0\} = \varphi^{-1}(0)$$

(b) The **zero function** $R \to R'$, $a \mapsto 0_{R'}$ is a homomorphism with $\ker(\varphi) = R$.

(c) Let R and R' be rings with unity and $\varphi\colon R \to R'$ a homomorphism. As (b) shows, it does not necessarily follow that $\varphi(1_R) = 1_{R'}$. This can be seen as a consequence of the fact that, with respect to multiplication, a ring is not a group.

2.2 Consequence of Ring Definitions

Definition 2.2.1 (The Binomial Theorem). Let a and b be two commuting elements ($ab = ba$) of a ring R with unity. Then, for all $n \in \mathbb{N}$,

$$(a+b)^b = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Lemma 2.2.1. For $m \in \mathbb{N}$ with $m \geq 2$, an element $\alpha = (\alpha_1 \dots \alpha_m) \in \mathbb{N}^m$ is called a **multi-index**. The **length** $|\alpha|$ of a multi-index $\alpha \in \mathbb{N}^m$ is defined by

$$|\alpha| := \sum_{j=1}^{m} \alpha_j$$

Set also

$$\alpha! := \prod_{j=1}^{m} (\alpha_j)!,$$

and define the **natural (partial) order** on $\mathbb{N}^m$ by

$$\alpha \leq \beta \rightleftarrows (\alpha_j \leq \beta_j,\ \ 1 \leq j \leq m).$$

for $a = (a_1, \dots, a_m) \in R^m$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$

Definition 2.2.2 (The Multinomial Theorem). Let R be a commutative ring with unity. Then for all $m \geq 2$,

$$(\sum_{j=1}^{m} \alpha)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^\alpha, \quad a = (a_1, \dots, a_m) \in R^m, \ \ k \in \mathbb{N}.$$

2.3 Field

Definition 2.3.1 (Field). K is a **field** when the following are satisfied:

- K is a commutative ring with unity.
- $0 \neq 1$
- $K^\times := K \backslash \{0\}$ is an Abelian group with respect to multiplication.

Note. The Abelian group $K^\times := (K^\times, \cdot)$ is called the **multiplicative group** of K.

Remark. Let K be a field.

(a) For all $a \in K^\times$, $\left(a^{-1}\right)-1 = a$

(b) A field has no zero divisors

(c) Let $a \in K^\times$ and $a \in K^\times$ and $b \in K$. Then there is an unique $x \in K$ with $ax = b$, namely the **quotient** $\frac{b}{a} := b/a := ba^{-1}$

(d) Let K' be a field and $\varphi\colon K \to K'$ a homomorphism with $\varphi \neq 0$. Then

$$\varphi(1_K) = 1_{K'} \quad \text{and} \quad \varphi(a^{-1}) = \varphi(a)^{-1},\ a \in K^\times$$

2.4 Ordered Field

Definition 2.4.1 (Ordered Ring)**.** A ring R with an ordered $\leq$ is called an **ordered ring** if the following holds:

- $(R, \leq)$ is totally ordered.
- $x < y \Rightarrow x + z < y + z, z \in R$
- $x, y > 0 \Rightarrow xy > 0$

Note. This leads to a series of basic arithmetic rules.
We may define absolute value function from $K \mapsto K$.

Proposition 2.4.1. Let K be an ordered field and $x, y, a, \epsilon \in K$ with $\epsilon > 0$.

(i) $x = |x|\operatorname{sign}(x)$, $|x| = x\operatorname{sign}(x)$

(ii) $|x| = |-x|, x \leq |x|$

(iii) $|xy| = |x||y|$

(iv) $|x| \geq 0$ and $(|x| = 0 \Leftrightarrow x = 0)$

(v) $|x - a| < \epsilon \leftrightarrow a - \epsilon < x < a + \epsilon$

(vi) $|x + y| \leq |x| + |y|$ (**triangular inequality**)

Corollary 2.4.1 (reversed triangular inequality)**.** In any ordered field K we have

$$|x - y| \geq ||x| - |y||\,, \quad x, y \in K.$$

2.5 Formal Power Series

Definition 2.5.1 (formal power series)**.** Let R be a nontrivial ring with unity. On the set $R^{\mathbb{N}} = \operatorname{Funct}(\mathbb{N}, R)$ define addition by

$$(p + q)_n := p_n + q_n, \qquad n \in \mathbb{N},$$

and multiplication by **convolution**,

$$(pq)_n \coloneqq (p \cdot q)_n \coloneqq \sum_{j=0}^{n} p_j q_{n-1} = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0$$

for $n \in \mathbb{N}$. Here p_n denotes the value of $p \in R^{\mathbb{N}}$ at $n \in \mathbb{N}$ and is called the n^{th} **coefficient** of p. In this situation an element $p \in R^{\mathbb{N}}$ is called a **formal power series over** R, and we set $R[X] \coloneqq \left(R^{\mathbb{N}}, +, \cdot\right)$

Proposition 2.5.1. $R[X]$ is a ring with unity, the **formal power series ring over** R. If R is commutative, then so is $R[X]$

2.6 Polynomials

Definition 2.6.1 (Polynomial)**.** A **polynomial over** R is a formal power seres $p \in R[X]$ such that $\{n; p_n \neq 0\}$ is finite, in other words, $p_n = 0$ "almost everywhere".

Chapter 3

The Real Numbers

Starting words: we seek an ordered **extension field** of $\mathbb{Q}$ in which the equation $x^2 = a$ is solvable for each $a > 0$.

3.1 Order Completeness

We say a totally ordered set X is **order complete** (or X satisfies the **completeness axiom**) if every nonempty subset of X which is bounded above has a supremum.

Proposition 3.1.1. Let X be a totally ordered set. Then the followings are equivalent:

(i) X is order complete.

(ii) Every nonempty subset of X which is bounded below has an infimum.

(iii) For all nonempty subsets A, B of X such that $a \leq b$ for all $(a, b) \in A \times B$, there is some $c \in X$ such that $a \leq c \leq b$ for all $(a, b) \in A \times B$ (**Dedekind cut property**)

Note. A relation $\leq$ on X is a **partial order** on X if it is reflexive, transitive and **anti-symmetric**, that is,

$$(x \leq y)(y \leq x) \Rightarrow x = y$$

If $\leq$ is a partial order on X, then the pair $(X, \leq)$ is called a **partially ordered set**. If, in addition,

$$\forall x, y \in X : (x \leq y) \vee (y \leq x)$$

then $\leq$ is called a **total order** on X and $(X, \leq)$ is a **totally ordered set**.

Corollary 3.1.1. A totally ordered set is order complete if and only if every nonempty bounded subset has a supremum and an infimum.

Theorem 3.1.1 (Dedekind's Construction of the Real Numbers)**.** There is, up to isomorphism, a unique order complete extension field $\mathbb{R}$ of $\mathbb{Q}$. This extension is called **the field of real numbers**.

Proposition 3.1.2 (A Characterization of Supremum and Infimum)**.** Followed from natural order defined by $\mathbb{R}$.

(i) If $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then

(a) $x < \sup(A) \Leftrightarrow \exists a \in A$ such that $x < a$.

(b) $x < \inf(A) \Leftrightarrow \exists a \in A$ such that $x > a$.

(ii) Every subset A of $\mathbb{R}$ has a supremum and an infimum in $\mathbb{R}$

3.2 The Consequence of Order Completeness

The Archimedean Property

Proposition 3.2.1 (Archimedes). $\mathbb{N}$ is not bounded above in $\mathbb{R}$, that is, for each $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ such that $n > x$.

Corollary 3.2.1. Equivalent statements as the above proposition

(a) Let $a \in \mathbb{R}$. If $0 \leq a \leq 1/n$ for all $n \in \mathbb{N}^{\times}$.

(b) For each $a \in \mathbb{R}$ with $a > 0$ there is some $n \in \mathbb{N}^{\times}$ such that $1/n < a$.

The Density of the Rational/Irrational Numbers in $\mathbb{R}$

Proposition 3.2.2. For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $r \in \mathbb{Q}$ such that $a < r < b$.

Proposition 3.2.3 (n^{th} Roots). For all $a \in \mathbb{R}^{+}$ and $n \in \mathbb{N}^{\times}$, there is a unique $x \in \mathbb{R}^{+}$ such that $x^n = a$

Proposition 3.2.4. For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $\xi \in \mathbb{R} \backslash \mathbb{Q}$ such that $a < \xi < b$.

Intervals

An **interval** is a subset J of $\mathbb{R}$ such that

$$(x, y \in J, x < y) \Rightarrow (z \in J \text{for} x < z < y)$$

If J is a nonempty interval, then $\inf(J) \in \bar{\mathbb{R}}$ is the **left endpoint** and $\sup(J) \in \bar{\mathbb{R}}$ is the **right endpoint** of J. J is **closed on the left** if $a := \inf(J)$ is in J, and otherwise it is **open on the left**. The same applies to the other side.

An interval is **perfect** if it contains at least two points. It is **bounded** if both endpoints are in $\mathbf{R}$ and is **unbounded** otherwise. If J is a bounded interval, then the nonnegative number $|J| := \sup(J)$ - $\inf(J)$ is called the **length** of J.

Chapter 4

Vector Spaces, Affine Spaces and Algebras

4.1 Vector Spaces

Definition 4.1.1 (Vector Space). A **vector space over the field** K (or simply, a K**-vector space**) is a triple $(V, +, \cdot)$ consisting of a nonempty set V, an 'inner' operation $+$ on V called **addition**, and an 'outer' operation

$$K \times V \to V, \quad (\lambda, v) \mapsto \lambda \cdot v,$$

called **scalar multiplication** which satisfy the following axioms:

- $(V, +)$ is an Abelian group
- The distributive law holds:

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w, (\lambda + u) \cdot v = \lambda \cdot v + \mu \cdot v, \quad \lambda, \mu \in K, \quad v, w \in V$$

- $\lambda \cdot (\mu v) = (\lambda\mu) \cdot v$, $1 \cdot v = v$ for $\lambda, \mu \in K, v \in V$

A vector space is called **real** if $K = \mathbb{R}$ and **complex** if $K = \mathbb{C}$.

Note (Linear Functions). Let V and W be vector spaces over K. THen a function $T\colon V \mapsto W$ is $(K-)$**linear** if

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w), \quad \lambda, \mu \in K, \ v, w \in V$$

In this regard, a linear function is simply a function which is compatible with the vector space operations, in other words, it is a (**vector space**) **homomorphism**. The set of all linear functions from V to W is denoted by $\mathrm{Hom}(V, W)$ or $\mathrm{Hom}_K(V, W)$, and $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ is the set of all (vector space) **endomorphisms**. A bijective homomorphism $T \in \mathrm{Hom}(V, W)$ is a (vector space) **isomorphism**.

Remark. (a) A vector space homomorphism $T\colon V \mapsto W$ is, in particular, a group homomorphism $T\colon (V, +) \mapsto (W, +)$.

(b)

Part II

Convergence

Chapter 5

Convergence of Sequences

5.1 Sequences

Definition 5.1.1 (Sequence). Let X be a set. A **sequence** (in X) is simply a function from $\mathbb{N}$ to X. If $\varphi\colon \mathbb{N} \mapsto X$ is a sequence, we write also

$$(x_n), (x_n)_{n\in\mathbb{N}} \text{ or } (x_0, x_1, x_2, \ldots)$$

for φ, where $x_n \coloneqq \varphi(n)$ is the n^{th} term of the sequence $\varphi = (x_0, x_1, x_2, \ldots)$.

Remark. (a) A sequence (x_n) is different from its image $\{x_n; n \in \mathbb{N}\}$.

(b) Let (x_n) be a sequence in X and E a property. Then we say E holds for **almost all** terms of (x_n) if there is some $m \in \mathbb{N}$ such that $E(x_n)$ is true for all $n \geq m$, that is, if E holds for all but finitely many of the x_n. If there is a subset $N \subseteq \mathbb{N}$ with $\mathrm{Num}(N) = \infty$ and $E(x_n)$ is true for each $n \in N$ then E is true for **infinitely many** terms.

(c) For $m \in \mathbb{N}^\times$, a function $\Phi\colon m + \mathbb{N} \mapsto X$ is also called a sequence in X.

5.2 Metric Space

Definition 5.2.1 (Metric Space). Let X be a set. A function $d\colon X \times X \mapsto \mathbb{R}^+$ is called a **metric** on X if the following hold:

- $d(x, y) = 0 \leftrightarrow x = y$.
- $d(x, y) = d(y, x)$, $x, y \in X$ (symmetry).
- $d(x, y) \leq d(x, z) + d(y, z)$, $x, y, z \in X$ (triangle inequality).

Note. If d is a metric on X, then (X, d) is called a **metric space**. We call $d(x, y)$ the **distance** between the **points** x and y in the metric space X.

In the metric space (X, d), for $a \in X$ and $r > 0$, the set

$$\mathbb{B}(a, r) \coloneqq \mathbb{B}_X(a, r) \coloneqq \{x \in X; d(a, x) < r\}$$

is called the **open ball** with center at a and radius r, while

$$\bar{\mathbb{B}}(a, r) \coloneqq \bar{\mathbb{B}}_X(a, r) \coloneqq \{x \in X; d(a, x) \leq r\}$$

is called the **closed ball** with center at a and radius r.

Example. (a) $\mathbb{K}$ is a metric space with the **natural metric**

$$\mathbb{K} \times \mathbb{K} \mapsto \mathbb{R}^{+}, \quad (x, y) \mapsto |x - y|$$

(b) Let (X, d) be a metric space and Y a nonempty subset of X. Then the restriction of d to $Y \times Y$, $d_Y := d|Y \times Y$, is a metric on Y, the **induced metric**, and (Y, d_Y) is a metric space, a **metric subspace** of X.

(c) Let X be a nonempty set. Then the function $d\,(x, y) := 1$ for $x \neq y$ and $d\,(x, x) := 0$ is a metric, called the **discrete metric** on X.

(d) Let (X_j, d_j), $1 \leq j \leq m$, be metric spaces and $X := X_1 \times \cdots \times X_m$. Then the function

$$d(x, y) := \underset{1 \leq j \leq m}{max}\, d_j(x_j, y_j)$$

for $x := (x_1, \ldots, x_m) \in X$ and $y := (y_1, \ldots, y_m) \in X$ is a metric on X called the **product metric**. The metric space $X := (X, d)$ is called the **product of the metric spaces** (X_j, d_j)

Proposition 5.2.1. Let (X, d) be a metric space. Then for all $x, y, z \in X$ we have

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

Note. A subset U of a metric space X is called a **neighborhood** of $a \in X$ if there is some $r > 0$ such that $\mathbb{B}\,(a, r) \subseteq U$. The **set of all neighborhoods of the point** a is denoted by $\mu\,(a)$, that is,

$$\mu(a) := \mu_X(a) := \{U \subseteq X; U \text{ is a neighborhood of } a\} \subseteq P(X)$$

Cluster Point

Definition 5.2.2 (Cluster Point)**.** We call $a \in X$ a **cluster point** of (x_n) if every neighborhood of a contains infinitely many terms of the sequence.

Proposition 5.2.2. The following are equivalent:

(i) a is a cluster point of (x_n).

(ii) For each $U \in \mu\,(a)$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in U$.

(iii) For each $\epsilon > 0$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in \mathbb{B}\,(a, \epsilon)$

5.3 Convergence

Definition 5.3.1 (Convergence)**.** A sequence (x_n) **converges** (or is **convergent**) with **limit** a if each neighborhood of a contains almost all terms of the sequence. In this case we write

$$\lim_{n \to \infty} x_n = a \text{ or } x_n \to a(n \to \infty)$$

and we say that (x_n) **converges to** a **as** n **goes to** ∞. A sequence (x_n) that is not convergent is called **divergent** and we say (x_n) **diverges**.

Proposition 5.3.1. The following statements are equivalent:

(i) $lim_{x_n} = a$.

(ii) For each $U \in \mu(a)$, there is some $N := N(U)$ such that $x_n \in U$ for all $n \geq N$.

(iii) For each $\epsilon > 0$, there is some $N := N(\epsilon)$ such that $x_n \in \mathbb{B}(a, \epsilon)$ for all $n \geq N$.

Bounded Sets

Definition 5.3.2. A subset $Y \subseteq X$ is called d-**bounded** or **bounded in** X (with respect to the metric d) if there is some $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in Y$. In this circumstance the **diameter** of Y, defined by

$$\operatorname{diam}(Y) := \sup_{x,y \in Y} d(x, y)$$

is finite. A sequence (x_n) is **bounded** if its image $\{x_n; n \in \mathbb{N}\}$ is bounded.

Proposition 5.3.2. Any convergent sequence is bounded.

Proof. Suppose that $x_n \to a$. Then there is some N such that $x_n \in \mathbb{B}(a, 1)$ for all $n \geq N$. It follows from the triangle inequality that

$$d(x_m, x_n) \leq d(x_m, a) + d(a, x_n) \leq 2, \quad m, n \in \mathbb{N}$$

Since there is also some $M \geq 0$ such that $d(x_j, x_k) \leq M$ for all $j, k \leq N$, we have $d(x_n, x_m) \leq M + 2$ for all $m, n \in \mathbb{N}$. ■

Uniqueness of the Limit

Proposition 5.3.3. Let (x_n) be convergent with limit a. Then a is the unique cluster point of (x_n).

Corollary 5.3.1. The limit of a convergent sequence is unique.

Subsequence

Let $\varphi = (x_n)$ be a sequence in X and $\Phi\colon \mathbb{N} \to \mathbb{N}$ a strictly increasing function, then $\varphi \circ \Phi \in X^{\mathbb{N}}$ is called a **subsequence** of φ. Extending the notation $(x_n)_{n \in \mathbb{N}}$ introduced above for the sequence φ, we write $(x_{n_k})_{k \in \mathbb{N}}$ for the subsequence $\varphi \circ \Phi$ where $n_k := \Phi(k)$.

Proposition 5.3.4. If (x_n) is a convergent sequence with limit a, then each subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) is convergent with $\lim_{k \to \infty} x_{n_k} = a$.

Proposition 5.3.5. A point a is a cluster point of a sequence (x_n) if and only if there is some subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) which converges to a.

Chapter 6

Normed Vector Space

6.1 Norms

Definition 6.1.1 (Norm). Let E be a vector space over $\mathbb{K}$. A function $||\cdot||\colon E \to \mathbb{R}^+$ is called a **norm** if the following hold:

- $||x|| = 0 \Leftrightarrow x = 0$.
- $||\lambda x|| = |\lambda|||x||$, $x \in E$, $\lambda \in \mathbb{K}$ (positive homogeneity)
- $||x+y|| \leq ||x|| + ||y||$, $x, y \in E$ (triangle inequality). A pair $(E, ||\cdot||)$ consisting of a vector space E and a norm $||\cdot||$ is called a **normed vector space**. If the norm is clear from context, we write E instead of $(E, ||\cdot||)$.

Remark. Let $E := (E, ||\cdot||)$ be a normed vector space.

(a) The function
$$d\colon E \times E \to \mathbb{R}^+, \quad (x, y) \mapsto ||x - y||$$
is a metric on E, the **metric induced from the norm**. Hence any normed vector space is also a metric space.

(b) The **reversed triangle inequality** holds for the norm:
$$||x - y|| \geq |||x|| - ||y|||, \quad x, y \in E$$

(c) All statements from previous chapter also hold in normed vector space.

Balls

For $a \in E$ and $r > 0$, we define the **open** and **closed balls** with center at a and radius r by
$$\mathbb{B}_E(a, r) := \mathbb{B}(a, r) := x \in E; ||x - a|| < r$$
and
$$\bar{\mathbb{B}}_E(a, r) := \bar{\mathbb{B}}_E := x \in E; ||x - a|| \leq r.$$

These definitions agree with those for the metric space (E, d) when d is induced from norm. We also write
$$\mathbb{B} := \mathbb{B}(0, 1) = x \in E; ||x|| < 1 \quad \text{and} \quad \bar{\mathbb{B}} := \bar{\mathbb{B}}(0, 1) = x \in E; ||x|| \leq 1$$
for the **open** and **closed unit balls** in E. We have
$$r\mathbb{B} = \mathbb{B}(0, r), \quad r\bar{\mathbb{B}} = \bar{\mathbb{B}}(0, r)$$

Bounded Sets

A subset X of E is called **bounded in** E (or **norm bounded**) if it is bounded in the induced metric space.

Remark. Let $E := (E, ||\cdot||)$ be a normed vector space

(a) $X \subseteq E$ is bounded if and only if there is some $r > 0$ such that $X \subseteq r\mathbb{B}$, that is, $||x|| < r$ for all $x \in X$.

(b) If X and Y are nonempty bounded subsets of E, then so are $X \cup Y$, $X + Y$ and λX with $\lambda \in \mathbb{K}$.

Example. (a) The absolute value $|\cdot|$ is a norm on the vector space $\mathbb{K}$.

(b) Let F be a subspace of a normed vector space $E := (E, \|\cdot\|)$. Then the restriction $\|\cdot\|_F := \|\cdot\||F$ of $\|\cdot\|$ to F is a norm on F. Thus $F := (F, \|\cdot\|_F)$ is a normed vector space with this **induced norm**.

(c) Let $E_j, \|\cdot\|_j$, $1 \leq j \leq m$, be normed vector space over $\mathbb{K}$. Then

$$\|x\|_\infty := \max_{1\leq j\leq m}\|x_j\|_j, \qquad x = (x_1, \dots, x_m) \in E := E_1 \times \cdots \times E_m$$

defines a norm, called the **product norm**, on the product vector space E. The metric on E induced from this norm coincides with the product metric from 5.2 example (d), where d_j is the metric induced on E_j from $\|\cdot\|_j$.

(d) For $m \in \mathbb{N}^\times, \mathbb{K}^m$ is a normed vector space with the **maximum norm**

$$|x|_\infty := \max_{1\leq j\leq m}|x_j|, \qquad x = (x_1, \dots, x_m) \in \mathbb{K}^m.$$

In this case m = 1.

6.2 The Space of Bounded Functions

Let X be a nonempty set and $(E, \|\cdot\|)$ a normed vector space. A function $u \in E^X$ is called **bounded** if the image of u in E is bounded. For $u \in E^X$, define

$$\|u\|_\infty := \|u\|_{\infty,X} := \sup_{x\in X}\|u(x)\| \in \mathbb{R}^+ \cup \infty$$

Remark. (a) For $u \in E^X$, the followings are equivalent:

(i) u is bounded.

(ii) $u(X)$ is bounded in E.

(iii) There is some $r > 0$ such that $\|u(x)\| \leq r$ for all $x \in X$.

(iv) $\|u\|_\infty < \infty$.

(b) Clearly id $\in \mathbb{K}^\mathbb{K}$ is not bounded, that is, $\|\mathrm{id}\|_\infty = \infty$.
(b) shows that $\|\cdot\|_\infty$ may not be a norm on the vector space E^X when E is not trivial. We therefore set

$$B(X, E) := u \in E^X \ ; \ u \text{ is bounded}$$

and call $B(X, E)$ the **space of bounded functions** from X to E.

Proposition 6.2.1. $B(X, E)$ is a subspace of E^X and $\|\cdot\|_\infty$ is a norm, called the **supremum norm**, on $B(X, E)$.

Remark. (a) If $X := \mathbb{N}$, then $B(X, E)$ is the normed vector space of bounded sequences in E. In the special case $E := \mathbb{K}$, $B(\mathbb{N}, \mathbb{K})$ is denoted by ℓ_∞, that is,

$$\ell_\infty := \ell\infty(\mathbb{K}) := B(\mathbb{N}, \mathbb{K})$$

is the normed **vector space of bounded sequences** with the supremum norm

$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}}|x_n|, \quad (x_n) \in \ell_\infty$$

(b) If $X = \{1, \ldots, m\}$ for some $m \in \mathbb{N}^\times$, then

$$B(X, E) = (E^m, \|\cdot\|_\infty)$$

6.3 Inner Product Spaces

Definition 6.3.1 (Inner Product Space). Let E be a vector space over the filed $\mathbb{K}$. A function

$$(\cdot|\cdot)\colon E \times E \to \mathbb{K}, \quad (x, y) \mapsto (x|y)$$

is called a **scalar product** or **inner product** on E if the following hold:

- $(x|y) = \overline{(y|x)}$, $x, y \in E$
- $(\lambda x + \mu y|z) = \lambda\,(x|z) + \mu\,(y|z)$, $x, y, z \in E$, $\lambda, \mu \in \mathbb{K}$.
- $(x|x) \geq 0$, $x \in E$, and $(x|x) = 0 \Leftrightarrow 0$.

A vector space E with a scalar product $(\cdot|\cdot)$ is called an **inner product space** and is written in $(E, (\cdot|\cdot))$.

Remark. (a) In the real case $\mathbb{K} = \mathbb{R}$, the first point can be written as

$$(x|y) = (y|x), \quad x, y \in E$$

In other words, the function is **symmetric** when E is a real vector space. In the case $\mathbb{K} = \mathbb{C}$, the function is said to be **Hermitian** when the first point holds.

(b) From the first two points it follows that

$$(x|\lambda y + \mu z) = \bar{\lambda}(x|y) + \bar{\mu}(x|z), \quad x, y, z \in E, \quad \lambda, \mu \in \mathbb{K},$$

that is, for each fixed $x \in E$, the function $(x|\cdot) : E \to \mathbb{K}$ is **conjugate linear**. If $\mathbb{K} = \mathbb{R}$, it is **bilinear**.

(c) $(x|0) = 0$ for all $x \in E$.

Let $m \in \mathbb{N}^\times$. For $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ in $\mathbb{K}^m$, define

$$(x|y) := \sum_{j=1}^{m} x_j \bar{y}_j$$

to be the **Euclidean Inner product** on $\mathbb{K}^m$.

The Cauchy-Schwarz Inequality

Theorem 6.3.1 (Cauchy-Schwarz Inequality). Let $(E, (\cdot|\cdot))$ be an inner product space. Then

$$|(x|y)|^2 \leq (x|x)(y|y) \quad x, y \in E$$

and the equality occurs if and only if x and y are linearly dependent.

Theorem 6.3.2. Let $(E, (\cdot|\cdot))$ be an inner product space and

$$\|x\| := \sqrt{(x|x)}, \qquad x \in E$$

Then $\|\cdot\|$ is a norm on E, the **norm induced from the scalar product** $(\cdot|\cdot)$. A norm which is induced from a scalar product is also called a **Hilbert norm**.

Corollary 6.3.1. Let $(E, (\cdot|\cdot))$ be an inner product space. Then

$$|(x|y)| \leq \|x\|\|y\| \qquad x, y \in E$$

6.4 Euclidean Spaces

Convention: Unless otherwise stated, we consider $\mathbb{K}^m$ to be endowed with the Euclidean inner product $(\cdot|\cdot)$ and the induced norm

$$|x| := \sqrt{(x|x)} = \sqrt{\sum_{j=1}^{m} |x_j|^2} \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m$$

the **Euclidean norm**. In the real case, we write also $x \cdot y$ for $(x|y)$.

We further define the norm

$$|x|_1 := \sum_{j=1}^{m} |x_j|, \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m$$

Proposition 6.4.1. Let $m \in \mathbb{N}^{\times}$. Then

$$|x|_\infty \leq |x| \leq \sqrt{m}|x|_\infty, \qquad \frac{1}{\sqrt{m}}|x|_1 \leq |x| \leq |x|_1, \qquad x \in \mathbb{K}^m$$

Equivalent Norm

Let E be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E are **equivalent** if there is some $K \geq 1$ such that

$$\frac{1}{K}\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1, \quad x \in E$$

In this case we write $\|\cdot\|_1 \ \|\cdot\|_2$.

Remark. (a) is an equivalence relation on the set of all norms of a fixed vector space.

(b) $\|\cdot\|_1 \ \|\cdot\| \ \|\cdot\|_\infty$ on $\mathbb{K}^m$.

(c) We write $\mathbb{B}^m$ for the **real open Euclidean unit ball**, that is, $\mathbb{B}^m := \mathbb{B}_{\mathbb{R}^m}$ and $\mathbb{B}_1^m$ and $\mathbb{B}_\infty^m$ for the unit balls in $(\mathbb{R}^m, |\cdot|_1)$ and in $(\mathbb{R}^m, |\cdot|_\infty)$ respectively. We have

$$\mathbb{B}^m \subseteq \mathbb{B}_\infty^m \subseteq \sqrt{m}\mathbb{B}^m \qquad \mathbb{B}_1^m \subseteq \mathbb{B}^m \subseteq \sqrt{m}\mathbb{B}_1^m$$

(d) Let $E = (E, \|\cdot\|)$ be a normed vector space and $\|\cdot\|_1$ a norm on E which is equivalent to $\|\cdot\|$. Set $E_1 := (E, \|\cdot\|_1)$. Then

$$\mathcal{U}_E(a) = \mathcal{U}_{E_1}(a), \qquad a \in E$$

that is, the set of neighborhoods of a depends only on the equivalence class of the norm. Equivalent norms produce the same set of neighborhoods.

Convergence in Product Spaces

Proposition 6.4.2. Let $m \in \mathbb{N}^\times$ and $x_n = \left(x_n^1, \ldots, x_n^m\right) \in \mathbb{K}^m$ for $n \in \mathbb{N}$. Then the followings are equivalent:

1. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x = \left(x^1, \ldots, x^m\right)$ in $\mathbb{K}^m$.
2. For each $k \in \{1, \ldots, m\}$, the sequence $\left(x_n^k\right)_{n\in\mathbb{N}}$ converges to x^k in $\mathbb{K}$.

Chapter 7

Infinite Limits

7.1 Convergence to $\pm\infty$

Sequences in $\mathbb{R}$ can usually be considered to converge to $+\infty$ or $-\infty$ in the extended number line $\bar{\mathbb{R}}$. A subset $U \subseteq \bar{\mathbb{R}}$ is called a **neighborhood of** ∞ (or of $-\infty$) if there is some $K > 0$ such that $(K, \infty) \subseteq U$ (or such that $(-\infty, -K) \subsetneqq U$). The set of neighborhoods of $\pm\infty$ is denoted by $\mathcal{U}(\pm\infty)$, that is,

$$\mathcal{U}(\pm\infty) := U \subseteq \bar{\mathbb{R}};\ U \ \text{ is neighborhood of } \pm\infty$$

Now let (x_n) be a sequence in $\mathbb{R}$. Then $\pm\infty$ is called a **cluster point** (or **limit**) of (x_n), if each neighborhood U of $\pm\infty$ contains infinitely many (or almost all) terms of (x_n). If $\pm\infty$ is the limit of (x_n), we usually write

$$\lim_{n\to\infty} x_n = \pm\infty \qquad \text{or} \qquad x_n \to \pm\infty\ (n \to \infty)$$

The sequence (x_n) **converges** in $\bar{\mathbb{R}}$ if there is some $x \in \bar{\mathbb{R}}$ such that $\lim\limits_{n\to\infty} x_n = x$. The sequence (x_n) **diverges** in $bar\mathbb{R}$, if it does not converge in $\bar{\mathbb{R}}$. With this definition, any sequence which converge in $\mathbb{R}$, also converges in $\bar{\mathbb{R}}$, and any sequence which diverges in $\bar{\mathbb{R}}$, also diverges in $\mathbb{R}$. On the other hand there are divergent sequences in $\mathbb{R}$ which converge in $\bar{\mathbb{R}}$ (to $\pm\infty$). In this case the sequence is said to converge **improperly**.

Proposition 7.1.1. Every monotone sequence (x_n) in $\mathbb{R}$ converges in $\bar{\mathbb{R}}$, and

$$\lim x_n = \begin{cases} \sup x_n; n \in \mathbb{N}, & \text{if } (x_n) \text{ is increasing,} \\ \inf x_n; n \in \mathbb{N}, & \text{if } (x_n) \text{ is decreasing.} \end{cases}$$

7.2 The Limit Superior and Limit Inferior

Definition 7.2.1. Let (x_n) be a sequence in $\mathbb{R}$. We can define two new sequences (y_n) and (z_n) by

$$y_n := \sup_{k\geq n} x_k := \sup x_k; k \geq n,$$

$$z_n := \inf_{k\geq n} x_k := \inf x_k; k \geq n.$$

Clearly (y_n) is increasing and (z_n) is decreasing. By the above proposition, these sequences converge in $\bar{\mathbb{R}}$:

$$\limsup_{n\to\infty} x_n := \overline{\lim}_{n\to\infty} x_n := \lim_{n\to\infty}(\sup_{k\geq n} x_k).$$

the **limit superior**, and

$$\liminf_{n\to\infty} x_n := \underline{\lim}_{n\to\infty} x_n := \lim_{n\to\infty}(\inf_{k\geq n} x_k).$$

the **limit inferior** of the sequences (x_n). We also have

$$\text{limsup} x_n = \inf_{n\in\mathbb{N}}(\sup_{k\geq n} x_k) \quad \text{and} \quad \text{liminf} x_n = \sup_{n\in\mathbb{N}}(\inf_{k\geq n} x_k).$$

Theorem 7.2.1. Any sequence (x_n) in $\mathbb{R}$ has a smallest cluster point x_* and a greatest cluster point x^* in $\bar{\mathbb{R}}$ and these satisfy

$$\text{liminf} x_n = x_* \quad \text{and} \quad \text{limsup} x_n = x^*$$

Theorem 7.2.2. Let (x_n) be a sequence in $\mathbb{R}$. Then

$$(x_n) \text{ converges in } \bar{\mathbb{R}} \Leftrightarrow \overline{\lim} x_n \leq \underline{\lim} x_n$$

When the sequence converges, the limit x satisfies

$$x = \lim x_n = \underline{\lim} x_n = \overline{\lim} x_n.$$

Theorem 7.2.3 (Bolzano-Weierstrass)**.** Every bounded sequence in $\mathbb{K}^m$ has a convergent subsequence, that is, a cluster point.

Chapter 8

Completeness

8.1 Cauchy Sequences

In the following $X = (X, d)$ is a metric space.

A sequence (x_n) in X is called a **Cauchy sequence** if, for each $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

Similarly, if (x_n) is a sequence in a normed vector space $E = (E, \|\cdot\|)$, then (x_n) is a Cauchy sequence if and only if for each $\epsilon > 0$ there is some N such that $\|x_n - x_m\| < \epsilon$ for all $m, n \geq N$. In particular, Cauchy sequences in E are "translation invariant", that is, if (x_n) is a Cauchy sequence and a is an arbitrary vector in E, then the 'translated' sequence $(x_n + a)$ is also a Cauchy sequence. This implies that Cauchy sequences cannot be defined using neighborhoods.

Proposition 8.1.1. Every convergent sequence is a Cauchy sequence.

Proposition 8.1.2. Every Cauchy sequence is bounded.

Proposition 8.1.3. If a Cauchy sequence has a convergent subsequence, then it is itself convergent.

8.2 Banach Spaces

A metric space X is called **complete** if every Cauchy sequence in X converges. A complete normed vector space is called a **Banach space**.

Theorem 8.2.1. $\mathbb{K}^m$ is a Banach space.

Theorem 8.2.2. Let X be a nonempty set and $E = (E, \|\cdot\|)$ a Banach space. Then $B(X, E)$ is also a Banach space.

Remark.

(a) A direct consequence of the previous two theorems is that For every nonempty set X, $B(X, \mathbb{R})$, $B(X, \mathbb{C})$, and $B(X, \mathbb{K}^m)$ are Banach spaces.

(b) The completeness of a normed vector space E is invariant under changes to equivalent norms.

(c) A complete inner product space is called a **Hilbert space**.

Part III

Continuous Functions

Chapter 9

Continuity

9.1 Elementary Properties and Examples

Let $f\colon X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y). Then f is **continuous** at $x_0 \in X$ if, for each neighborhood V of $f(x_0)$ in Y, there is a neighborhood U of x_0 in X such that $f(U) \subseteq V$.

Hence to prove the continuity of f at x_0, one supposes that an arbitrary neighborhood V of $f(x_0)$ is given and then shows that there is a neighborhood U of x_0 such that $f(U) \subseteq V$, that is, $f(x) \in V$ for all $x \in U$.

The function $f\colon X \to Y$ is **continuous** if it is continuous at each point of X. We say f is **discontinuous at** x_0 if f is not continuous at x_0. f is **discontinuous** if it is discontinuous at (at least) one point of X. The set of all continuous functions from X to Y is denoted $C(X, Y)$, a subset of Y^X.

Proposition 9.1.1. A function $f\colon X \to Y$ is continuous at $x_0 \in X$ if and only if, for each $\epsilon > 0$, there is some $\delta := \delta(x_0, \epsilon) > 0$ with the property that

$$d(f(x_0), f(x)) < \epsilon \text{ for all } x \in X \text{ such that } d(x_0, x) < \delta$$

Corollary 9.1.1. Let E and F be normed vector spaces and $X \subseteq E$. Then $f\colon X \to F$ is continuous at $x_0 \in X$ if and only if, for each $\epsilon > 0$, there is some $\delta := \delta(x_0, \epsilon) > 0$ satisfying that

$$\|f(x) - f(x_0)\|_F < \epsilon \quad \text{for all } x \in X \quad \text{such that } \|x - x_0\|_E < \delta$$

Example. In the following examples, X and Y are metric spaces.

(a) The square root function $\mathbb{R}^+ \to \mathbb{R}^+$, $x \to \sqrt{x}$ is continuous.

(b) The floor function $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{R}$, $x \to \lfloor x \rfloor := \max\{k \in \mathbb{Z}; k \leq x\}$ is continuous at $x_0 \in \mathbb{R}\backslash\mathbb{Z}$ and discontinuous at $x_0 \in \mathbb{Z}$.

(c) The **Dirichlet function** $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases}$$

is nowhere continuous, that is, it is discontinuous at every $x_0 \in \mathbb{R}$.

(d) Suppose that $f\colon X \to \mathbb{R}$ is continuous at $x_0 \in X$ and $f(x_0) > 0$. Then there is a neighborhood U of x_0 such that $f(x) > 0$ for all $x \in U$.

(e) A function $f\colon X \to Y$ is **Lipschitz continuous** with **Lipschitz constant** $\alpha > 0$ if

$$d(f(x), f(y)) \leq \alpha d(x, y), \qquad x, y \in X$$

Every Lipschitz continuous function is continuous.

(f) Any constant function $X \to Y$, $x \mapsto y_0$ is Lipschitz continuous.

(g) The identity function $\mathrm{id}\colon X \to X$, $x \mapsto x$ is Lipschitz continuous.

(h) If $E_1, \ldots, E_m$ are normed vector spaces, then $E := E_1 \times \cdots \times E_m$ is a normed vector space with respect to the product norm $\|\cdot\|_\infty$. The canonical projections

$$\mathrm{pr}_k\colon E \to E_k, \quad x = (x_1, \ldots, x_m) \mapsto x_k, \quad 1 \le k \le m,$$

are Lipschitz continuous. In particular, the projections $\mathrm{pr}_k\colon \mathbb{K}^m \to \mathbb{K}$ are Lipschitz continuous.

(i) Let E be a normed vector space. Then the norm function

$$\|\cdot\|\colon E \to \mathbb{R}, \quad x \mapsto \|x\|$$

is Lipschitz continuous.

(j) If $A \subseteq X$ and $f\colon X \to Y$ is continuous at $x_0 \in A$, then $f|A\colon A \to Y$ is continuous at x_0. Here A has the metric induced from X.

(k) Let $M \subseteq X$ be a nonempty subset of X. For each $x \in X$,

$$d(x, M) := \inf_{m \in M} d(x, m)$$

is called the **distance** from x to M. The **distance function**

$$d(\cdot, M)\colon X \to \mathbb{R}, \quad x \mapsto d(x, M)$$

is Lipschitz continuous.

(l) For any inner product space $(E, (\cdot|\cdot))$, the scalar product $(\cdot|\cdot) : E \times E \to \mathbb{K}$ is continuous.

(m) Let E and F be normed vector spaces and $X \subseteq E$. Then the continuity of $f\colon X \to F$ at $x_0 \in X$ is independent of the choice of equivalent norms on E and on F.

(n) A function f between metric spaces X and Y is **isometric** (or an **isometry**) if $d\left(f\left(x\right), f\left(x'\right)\right) = d\left(x, x'\right)$ for all $x, x' \in X$, that is, if f 'preserves distances'. Such a function is Lipschitz continuous and is a bijection from X to its image $f\left(X\right)$. If E and F are normed vector spaces and $T\colon E \to F$ is linear, then T is isometric if and only if $\|Tx\| = \|x\|$ for all $x \in E$. If, in addition, T is surjective then T is an **isometric isomorphism** from E to F, and T^{-1} is also isometric.

9.2 Sequential Continuity

Definition 9.2.1 (Sequential Continuity)**.** A function $f\colon X \to Y$ between metric spaces X and Y is called **sequentially continuous** at $x \in X$, if, for every sequences (x_k) in X such that $\lim_{x_k} = x$, we have $\lim_{f(x_k)} = f\left(x\right)$.

Theorem 9.2.1 (sequence criterion)**.** Let X, Y be metric spaces. Then a function $f\colon X \to Y$ is continuous at x if and only if it is sequentially continuous at x.

Let $f\colon X \to Y$ be a continuous function between metric spaces. Then for any convergent sequence (x_k) in X we have

$$\lim f(x_k) = f(\lim x_k)$$

That's why we say 'continuous functions respect the taking of limits'.

9.3 Addition and Multiplication of Continuous Functions

Proposition 9.3.1. Suppose that X is a metric space, F is a normed vector space, and

$$f\colon \operatorname{dom}(f) \subseteq X \to F, \quad g\colon \operatorname{dom}(g) \subseteq X \to F$$

are continuous at $x_0 \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$.

- $f + g$ and λf are continuous at x_0.
- If $F = \mathbb{K}$, then $f \cdot g$ is continuous at x_0.
- If $F = \mathbb{K}$ and $g(x_0) \neq 0$, then f/g is continuous at x_0.

Corollary 9.3.1. (i) Rational functions are continuous.

(ii) Polynomials in n variables are continuous (on $\mathbb{K}^m$).

(iii) $C(X, F)$ is a subspace of F^X, the **vector space of continuous functions** from X to F.

Theorem 9.3.1 (continuity of compositions)**.** Let X, Y and Z be metric spaces. Suppose that $f\colon X \to Y$ is continuous at $x \in X$, and $g\colon Y \to Z$ is continuous at $f(x) \in Y$. Then the composition $g \circ f\colon X \to Z$ is continuous at x.

Example. Let X be a metric space and E be a normed vector space.

Let $f\colon X \to E$ be continuous at x_0. Then the **norm of** f,

$$\|f\|\colon X \to \mathbb{R}, \quad x \mapsto \|f(x)\|,$$

is continuous at x_0.

9.4 One-Sided Continuity

Let X be a subset of $\mathbb{R}$ and $x_0 \in X$. For $\delta > 0$, the set $X \cap (\, x_0 - \delta, x_0]$ (or $X \cap [\, x_0, x_0 + \delta)$) is called a **left** (or **right**) **δ-neighborhood** of x_0.

Let Y be a metric space. Then $f\colon X \to Y$ is **left** (or **right**) **continuous** at x_0, if, for each neighborhood V of $f(x_0)$ in Y, there is some $\delta > 0$ such that $f(X \cap (\, x_0 - \delta, x_0]\,) \subseteq V$ (or $f(X \cap [\, x_0, x_0 + \delta)\,) \subseteq V$).

Chapter 10

The Fundamentals of Topology

10.1 Open Sets

Definition 10.1.1. Let $X := (X, d)$ be a metric space. An element a of a subset A of X is called an **interior point** of A if there is a neighborhood U of a such that $U \subseteq A$. The set A is called **open** if every point of A is an interior point.

Example. The open ball $\mathbb{B}(a, r)$ is open.

Remark. (a) The concepts "interior point" and "open set" depend on the surrounding metric space X. It is sometimes useful to make this explicit by saying that 'a is an interior point of A with respect to X', or 'A is open in X'.

(b) If A is open with respect to a particular norm, it is open with respect to all equivalent norms.

(c) Every point in a metric space has an open neighborhood

Proposition 10.1.1. Let $\tau := \{O \subseteq X;\ O \text{ is open}\}$ be a family of open sets.

(i) $\emptyset, X \in \tau$

(ii) If $O_\alpha \in \tau$ for all $\alpha \in \mathcal{A}$, then $\bigcup_\alpha O_\alpha \in \tau$. That is, arbitrary unions of open sets are open.

(iii) If $O_0, \dots, O_n \in \tau$, then $\bigcap_{k=0}^n O_k \in \tau$. That is, finite intersections of open sets are open.

Let M be a set and $\tau \subseteq \mathcal{P}(M)$, a set of subset satisfying (i)-(iii). Then τ is called a **topology** on M, and the elements of τ are called the **open sets** with respect to τ. The pair (M, τ) is called a **topological space**.

Remark. (a) Let $\tau \in \mathcal{P}(X)$ be the family of open sets. Then τ is called the **topology on X induced from the metric** d. If X is a normed vector space with metric induced from the norm, then τ is called the **norm topology**.

(b) Let $(X, \|\cdot\|)$ be a normed vector space, and $\|\cdot\|_1$ a norm on X which is equivalent to $\|\cdot\|$. Let $\tau_{\|\cdot\|}$ and $\tau_{\|\cdot\|_1}$ be the norm topologies induced from $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$. Then they coincide, that is, equivalent norms induce the same topology on X.

10.2 Closed Sets

A subset A of the metric space X is called **closed** in X if A^c is open in X.

Proposition 10.2.1. (i) $\emptyset$ and X are closed.

(ii) Arbitrary intersections of closed sets are closed.

(iii) Finite unions of closed sets are closed.

Remark.
- Infinite intersections of open sets need not be open.
- Infinite unions of closed sets need not be closed.

Definition 10.2.1. Let $A \subseteq X$ and $x \in X$. We call x an **accumulation point** of A if every neighborhood of x in X has a nonempty intersection with A. The element $x \in X$ is called a **limit point** of A if every neighborhood of x in X contains a point of A other than x. We set

$$\overline{A} := \{x \in X;\ x \ \text{ is an accumulation point of } A\}.$$

Clearly any element of A and any limit point of A is an accumulation point of A.

Proposition 10.2.2. Let A be a subset of a metric space X.

(i) $A \subseteq \overline{A}$

(ii) $A = \overline{A} \Leftrightarrow A$ is closed.

Proposition 10.2.3. An element x of X is a limit point of A if and only if there is a sequence (x_k) in $A \backslash \{x\}$ which converges to x.

Corollary 10.2.1. An element x of X is an accumulation point of A if and only if there is a sequence (x_k) in A such that $x_k \to x$.

Proposition 10.2.4. For $A \subseteq X$, the followings are equivalent:

- A is closed.
- A contains all its limit points.
- Every sequence in A which converges in X, has its limits in A.

10.3 The Closure and Interior of a Set

Definition 10.3.1. Let A be a subset of a metric space X. Define the **closure of** A by

$$\mathrm{cl}(A) := \mathrm{cl}_X(A) := \bigcap_{B \in M} B$$

with

$$M := \{B \subseteq X; B \supseteq A \ \text{ and } B \ \text{ is closed in } X\}.$$

.

Remark. The closure of A is the smallest closed set which contains A. Any closed set which contains A, also contains $\mathrm{cl}(A)$.

Proposition 10.3.1. Let A be a subset of a metric space X. Then $\overline{A} = \mathrm{cl}\,(A)$.

Corollary 10.3.1. Let A and B be subsets of X.

(i) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.

(ii) $\overline{(\overline{A})} = \overline{A}$.

(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Definition 10.3.2 (Interior of a Set).

$$\operatorname{int}(A) := \operatorname{int}_X(A) := \bigcup \{O \subseteq A; O \text{ is open in X }\}$$

$\operatorname{int}(A)$ is the largest open subset of A.

Definition 10.3.3.

$$\mathring{A} := \{a \in A; a \text{ is an interior point of } A\}.$$

Proposition 10.3.2. Let A be a subset of a metric space X. Then $\mathring{A} = \operatorname{int}(A)$.

Corollary 10.3.2. Let A and B be subsets of X.

(i) $A \subseteq B \Rightarrow \mathring{A} \subseteq \mathring{B}$.

(ii) $\left(\mathring{A}\right)^{\circ} = \mathring{A}$.

(iii) A is open $\Leftrightarrow A = \mathring{A}$.

10.4 The Hausdorff Condition

Definition 10.4.1. For a subset A of a metric space X, the (topological) **boundary of** A is defined by $\partial A := \overline{A} \backslash \mathring{A}$.

Proposition 10.4.1. Let A be a subset of X.

- ∂A is closed.
- x is in ∂A if and only if every neighborhood of x has nonempty intersection with both A and A^c.

The following proposition shows that, in metric spaces, any two distinct points have disjoint neighborhoods.

Proposition 10.4.2 (Hausdorff Condition). Let $x, y \in X$ be such that $x \neq y$. Then there are a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.

Corollary 10.4.1. Any one element subset of a metric space is closed.

10.5 A Characterization of Continuous Functions

Theorem 10.5.1. Let $f: X \to Y$ be a function between metric spaces X and Y. Then the followings are equivalent:

(i) f is continuous.

(ii) $f^{-1}(O)$ is open in X for each open set O in Y.

(iii) $f^{-1}(A)$ is closed in X for each closed set A in Y.

Remark. According to this theorem, a function is continuous if and only if the preimage of any open set is open, if and only if the preimage of any closed set is closed. We denote the topology of a metric space X by τ_X, that is,

$$\tau_X \coloneqq \{O \subseteq X; O \text{ is open in } X\}$$

Then,

$$f\colon X \to Y \text{ is continuous } \iff f^{-1}\colon \tau_Y \to \tau_X$$

10.6 Continuous Extensions

Let X and Y be metric spaces. Suppose that $D \subseteq X$, $f\colon D \to Y$ is continuous and $a \in X$ is a limit point of D. If D is not closed, then a may not be in D and so f is not defined at a. In this section we consider whether $f(a)$ can be defined so that f is continuous on $D \cup \{a\}$. If such extensions exist, then, for any sequence (x_n) in D which converges to a, $(f(x_n))$ converges to $f(a)$.

Remark. (a) The followings are equivalent:

- $\lim_{x\to a} f(x) = y$.
- For each neighborhood V of y in Y, there is a neighborhood U of a in X such that $f(U \cap D) \subseteq V$.

(b) If $a \in D$ is a limit point of D, then

$$\lim_{x\to a} f(x) = f(a) \iff f \text{ is continuous at } a.$$

10.7 Relative Topology

Let X be a metric space and Y a subset of X. Then Y is itself a metric space with respect to the metric $d_Y \coloneqq d|Y \times Y$ induced from X, and so 'open in (Y, d_Y)' and 'closed in (Y, d_Y)' are well-defined concepts.

Proposition 10.7.1. Let X be a metric space and $M \subseteq Y \subseteq X$. Then M is open (or closed) in Y if and only if M is open (or closed) in (Y, d_Y).

Corollary 10.7.1. If $M \subseteq Y \subseteq X$, then M is open in Y if and only if $Y \setminus M$ is closed in Y.

Chapter 11

Compactness

We see that continuous images of open sets may not be open, and continuous images of closed sets may not be closed. This chapter investigates certain properties that are preserved by continuity.

11.1 Covers

In the following, $X := (X, d)$ is a metric space.

A family of sets $\{A_\alpha \subseteq X; \alpha \in \mathcal{A}\}$ is called a **cover** of the subset $K \subseteq X$ if $K \subseteq \bigcup_\alpha A_\alpha$. A cover is called **open** if each A_α is open in X. A subset $K \subseteq X$ is called **compact** if every open cover of K has a finite subfamily which is also a cover of K. In other words, $K \subseteq X$ is compact if every open cover of K has a **finite subcover**.

Example. (a) Let (x_k) be a convergent sequence in X with limit a. Then the set $K := \{a\} \cup \{x_k; k \in \mathbb{N}\}$ is compact.

(b) The state of (a) is false, in general, if the limit a is not included in K.

(c) The set of natural numbers $\mathbb{N}$ is not compact in $\mathbb{R}$.

Proposition 11.1.1. Any compact set $K \subseteq X$ is closed and bounded in X.

11.2 A Characterization of Compact Set

Definition 11.2.1. A subset K of X is **totally bounded** if, for each $r > 0$, there are $m \in \mathbb{N}$ and $x_0, \ldots, x_m \in K$ such that $K \subseteq \bigcup_{k=0}^{m} \mathbb{B}(x_k, r)$.

Theorem 11.2.1. A subset $K \subseteq X$ is compact if and only if every sequence in K has a cluster point K.

11.3 Sequential Compactness

Definition 11.3.1. A subset $K \subseteq X$ is **sequentially compact** if every sequence in K has a subsequence which converges to an element of K.

Theorem 11.3.1. A subset of a metric space is compact if and only if it is sequentially compact.

Theorem 11.3.2 (Heine-Borel)**.** A subset of $\mathbb{K}^n$ is compact if and only if it is closed and bounded. In particular, an interval is compact if and only if it is closed and bounded.

11.4 Continuous Functions on Compact Spaces

Theorem 11.4.1. Let X and Y be metric spaces and $f\colon X \to Y$ continuous. If X is compact, then $f(X)$ is compact. That is, continuous images of compact sets are compact.

Corollary 11.4.1. Let X and Y be metric spaces and $f\colon X \to Y$. If X is compact, then $f(X)$ is bounded.

Theorem 11.4.2 (extreme value theorem)**.** Let X be a compact metric space and $f\colon X \to \mathbb{R}$ continuous. Then there are $x_0, x_1 \in X$ such that

$$f(x_0) = \min_{x\in X} f(x) \quad \text{and} \quad f(x_1) = \max_{x\in X} f(x)$$

11.5 Total Boundedness

Theorem 11.5.1. A subset of a metric space is compact if and only if it is complete and totally bounded.

11.6 Uniform continuity

Definition 11.6.1. Let X and Y be metric spaces and $f\colon X \to Y$. Then f is called **uniformly continuous** if, for each $\epsilon > 0$, there is some $\delta(\epsilon) > 0$ such that

$$d(f(x), f(y)) < \epsilon \quad \text{for all } x, y \in X \text{ such that } d(x, y) < \delta(\epsilon).$$

Example. Lipschitz continuous functions are uniformly continuous.

Theorem 11.6.1. Suppose that X and Y are metric spaces with X compact. If $f\colon X \to Y$ is continuous, then f is uniformly continuous. That is, continuous functions on compact sets are uniformly continuous.

11.7 Compactness in General Topological Spaces

Remark.

(a) Let $X = (X, \tau)$ be a topological space. Then X is **compact** if X is a Hausdorff space and every open cover of X has a finite subcover. The space X is **sequentially compact** if it is a Hausdorff space and every sequence has a convergent subsequence. A subset $Y \subseteq X$ is **compact** (or **sequentially compact**) if the topological subspace (Y, τ_Y) is compact (or sequentially compact).

(b) Any compact subset K of a Hausdorff space X is closed. For each $x_0 \in K^c$ there are disjoint open sets U and V in X such that $K \subseteq U$ and $x_0 \in V$. In other words, a compact subset of a Hausdorff space and a point, not in that subset, can be separated by open neighborhoods.

(c) Any closed subset of a compact space is compact.

(d) Let X be compact and Y Hausdorff. Then the image of any continuous function $f\colon X \to Y$ is compact.

(e) In general topological spaces, compactness and sequential compactness are distinct concepts. That is, a compact space need not to be sequentially compact, and a sequentially compact space need not be compact.

(f) Uniform continuity is undefined in general topological spaces since the definition above makes essential use of the metric.

Chapter 12

Connectivity

Definition 12.0.1 (connectedness)**.** A metric space X is called **connected** if X cannot be represented as the union of two disjoint nonempty open subsets. Thus X is connected if and only if

$$\nexists\, O_1, O_2 \subseteq X, \text{ open, nonempty, with } O_1 \cap O_2 = \emptyset \text{ and } O_1 \cup O_2 = X$$

A subset M of X is called **connected** in X if M is connected with respect to the metric induced from X.

Proposition 12.0.1. For any metric space X, the followings are equivalent:

- X is connected.
- X is the only nonempty subset of X which is both open and closed.

Theorem 12.0.1 (Connectivity in $\mathbb{R}$)**.** A subset of $\mathbb{R}$ is connected if and only if it is an interval.

12.1 The Generalized Intermediate Value Theorem

Theorem 12.1.1. Let X and Y be metric spaces and $f\colon X \to Y$ continuous. If X is connected, then so is $f(X)$. That is, continuous images of connected sets are connected.

Corollary 12.1.1. Continuous images of intervals are connected.

Theorem 12.1.2 (Generalized Intermediate Value Theorem)**.** Let X be a connected metric space and $f\colon X \to \mathbb{R}$ continuous. Then $f(X)$ is an interval. In particular, f takes on every value between any two given function values.

12.2 Path Connectivity

Definition 12.2.1. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. A continuous function $w\colon [\alpha, \beta] \to X$ is called a **continuous path** connecting **end points** $w(\alpha)$ and $w(\beta)$.

Definition 12.2.2. A metric space X is called **path connected** if, for each pair $(x, y) \in X \times X$, there is a continuous path in X connecting x and y. A subset of a metric space is called **path connected** if it is a path connected metric space with respect to the induced metric.

Proposition 12.2.1. Any path connected space is connected.

Let E be a normed vector space and $a, b \in E$. The linear structure of E allows us to consider 'straight' paths in E:

$$v\colon [0,1] \to E, \quad t \mapsto (1-t)a + tb$$

we denote the image of the path v by $[a, b]$

A subset X of E is called **convex** if, for each pair $(a, b) \in X \times X$, $[a, b]$ is contained in X.

Remark. Let E be a normed vector space.

(a) Every convex subset of E is path connected and connected.

(b) For all $a \in E$ and $r > 0$, the balls $\mathbb{B}_E(a, r)$ are convex.

(c) A subset of $\mathbb{R}$ is convex if and only if it is an interval.

Theorem 12.2.1. Let X be a nonempty, open and connected subset of a normed vector space. Then any pair of points of X can be connected by a polygonal path in X.

Corollary 12.2.1. An open subset of a normed vector space is connected if and only if it is path connected.

Part IV

Differentiation in One Variable

Chapter 13

Differentiability