Notes on Eigenvalues and Eigenvectors

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Fall 2021

1 General concept

- **Definition**

  Let V be a vector space of dimension n, and let $T : V \to V$ be a linear transformation

  The scalar $\lambda \in \mathbb{R}$ is an *eigenvalue* of T if there is a non-zero vector $v \in V$ such that

  $$T(v) = \lambda v;$$

  such a vector is called an *eigenvecto* of T, with corresponding eigenvalue λ.

- The geometric intuition for eigenvector is that some special vectors in the domain of T such that would only stretch itself when doing matrix multiplication with a certain matrix. Such vectors are called eigen vectors and the degree of "stretching" is called eigenvalues.

- The set

  $$E_\lambda = \{v \in V : T(v) = \lambda v\}$$

  is called the *eigenspace* corresponding to λ. Eigenspace is a subspace of V because it contains the zero vector, and closed under vector addition and scalar multiplication

- **Definition**

  The *characteristic polynomial* of the linear transformation $T : V \to V$ is the polynomial f_T (which we write here in the variable λ) given by

  $$f_T(\lambda) = det(\lambda I - T),$$

  where $I : V \to V$ is the identity transformation. Here we are thinking of $\lambda I - T$ as a new linear transformation from V to V, defined by

$$(\lambda I - T)(v) = \lambda v - T(v) \text{ for all } v \in V$$

- The above definition leads us a systematic way of finding eigenvalues. If v is an eigenvector of T with corresponding eigenvalue λ, then,

  $$(\lambda I - T)(v) = 0$$

  since here we assume the nonzero eigenvector exists, then the linear transformation $\lambda I - T$ has a nontrivial kernel, hence it is not invertible, and we thus have

  $$det(\lambda I - T) = 0$$

  Conversely, reversing the arugment shows that if the determinant is 0, then λ is an eigenvalue of T. This shows that *the eigenvalues of T are just the roots of the characteristic polynomial of T*

- **Definition**

  Let λ be an eigenvalue of T.

  The *algebraic multiplicity* "almu(λ)" of λ is the number of times that λ occurs as a root of the characteristic polynomial f_T of T; that is, the largest power of the root.

  The *geometric multiplicity* "gemu(λ)" of λ is the dimension of the corresponding eigenspace E_λ

- **Definition - Diagonalization**

  A linear transformation $T : V \to V$ of the finite-dimensional vector space V is said to be *diagonalizable* if there there is a basis B of V such that $[T]_B$ is diagnoal. A square matrix A is said to be *diagonalizable* if the linear transformation T_A is diagonalizable, or equivalently if A is similar to a diagonal matrix

2 Understanding of the concept

- Eigenvectors that are of distinct eigenvalues are linearly independent
- The intersection of two eigenvectors that have distinct eigenvalues is only the zero vector
- A linear transformation $T : V \to V$ of the finite-dimensional vector space V is diagonalizable if and only if there is a basis B of V consisting of eigenvectors of T (Such a basis of V is called an *eigenbasis* of T)
- An n x n sqaure matrix A is diagonalizable if and only if there is a basis B of $\mathbb{R}^n$ consisting of eigenvectors of A (Such a basis of V is called an *eigenbasis* of T)

- In either of the above cases, the matrix is similar to the diagonal matrix that all of the entries are eigenvalues
- To *diagonalize* a square matrix A means factor A as $A = PDP^{-1}$ where D is diagonal