# Analysis

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# Contents

# Part I

# Foundations

# Chapter 1

# Groups and Homomorphism

## 1.1 Basics

**Definition 1.1.1** (Group). A pair $(G, \odot)$ consisting of a nonempty set $G$ and an operation $\odot$ is called a **group** if the following holds:

- $G$ is closed under the operation $\odot$
- $\odot$ is associative
- $\odot$ has an identity element $e$
- Each $g \in G$ has an **inverse** $h \in G$ such that $g \odot h = h \odot g = e$

**Definition 1.1.2** (Abelian group). A group $G, \odot$ is called **commutative** or **Abelian** if $\odot$ is a commutative operation on $G$.

**Remark.** Let $G = (G, \odot)$

(a) the identity element $e$ is unique

(b) Each $g \in G$ has a unique inverse which we denote by $g^b$. In particular $e^b = e$.

(c) For each $g \in G$, we have $\left(g^b\right)^b = g$.

(d) For arbitrary group elements $g$ and $h$, $(g \odot h)^b = h^b \odot g^b$

**Example.** (a) Let $G \coloneqq \{e\}$ be a one element set. Then $\{G, \odot\}$ is an Abelian group, the **trivial group**, with the (only possible) operation $e \odot e = e$.

(b) Let $X$ be a nonempty set, and $S_X$ be the set of all bijections from $X$ to itself. Then $S_X \coloneqq (S_X, \circ)$ is a group with identity element $id_X$ when $\circ$ denotes the composition of functions. Further, the inverse function $f^{-1}$ is the inverse of $f \in S_X$ in the group. When $X$ is finite, the element of $S_X$ are called permutations and $S_X$ is called the **permutation group** of $X$.

(c) Let $X$ be a nonempty set and $G, \odot$ a group. With the induced operation $\odot$, $\left(G^X, \odot\right)$ is a group. The inverse of $f \in G^X$ is the function

$$f^b \colon X \to G, \quad x \mapsto (f(x))^b$$

(d) Let $G_1, \ldots, G_m$ be groups. Then $G_1 \times \cdots \times G_m$ with the operation defined analogously to (d) is a group called the **direct product** of $G_1, \ldots, G_m$.

## 1.2 Subgroup

**Definition 1.2.1** (Subgroup). Let $G = (G, \odot)$ be a group and $H$ a nonempty subset of $G$, if

- $H \odot H \subseteq H$
- $h^b \in H$ for all $h \in H$

then $H := (H, \odot)$ is itself a group and is called a **subgroup** of $G$.

**Remark.** Here we use the same symbol $\odot$ for the restriction of the operation to $H$. Since $H$ is nonempty, there is some $h \in H$ and so, from the two axioms above, $e = h^b \odot h$ is also in $H$.

**Example.** Let $G = (G, \odot)$ be a group.

(a) The trivial subgroup $\{e\}$ and $G$ itself are subgroups of $G$, the smallest and largest subgroups with respect to inclusion

(b) If $H_\alpha$, $\alpha \in A$ are subgroups of $G$, then $\bigcap_\alpha H_\alpha$ is also a subgroup of $G$.

## 1.3 Cosets

**Definition 1.3.1** (Coset). Let $N$ be a subgroup of $G$ and $g \in G$. Then $g \odot N$ is the **left coset** and $N \odot g$ is the **right coset** of $g \in G$ with respect to $N$.

**Remark.** The definition of coset is related to the particular element.

**Note.** If we define

$$g \sim h \Leftrightarrow g \in h \odot N \tag{1.1}$$

Then $\sim$ is an equivalence on $G$.

**Proof.** $\sim$ is reflexive because $e \in N$
Let $g \in h \odot N$ and $h \in k \odot N$, then

$$g \in (k \odot N) \odot N = k \odot (N \odot N) = k \odot N$$

Let $g \in h \odot N$, then there is some $n \in N$ with $g = h \odot n$. Then it follows that $h = g \odot n^b \in N$. ■

Here 1.1 defines an equivalence relation on $G$. For the equivalence classes $[\cdot]$ with respect to $\sim$, we have

$$[g] = g \odot N, \quad g \in G. \tag{1.2}$$

For this reason, we denote $G/\sim$ by $G/N$, and call $G/N$ the **set of left cosets** of $G$ **modulo** $N$. Particularly, we have subgroups $N$ such that

$$g \odot N = N \odot g, \quad g \in G. \tag{1.3}$$

Such a subgroup 1.3 is called a **normal subgroup** of $G$. We call $g \odot N$ the **coset of** $g$ **modulo** $N$ since each left coset is a right coset and vice versa. We have a well-defined operation on $G/N$ where $N$ is the normal subgroup of $G$, induced from $\odot$, such that

$$(G/N) \times (G/N) \to G/N, \quad (g \odot N, h \odot N) \mapsto (g \odot h) \odot N \tag{1.4}$$

**Proposition 1.3.1.** Let $G$ be a group and $N$ a normal subgroup of $G$. Then $G/N$ with the induced operation is a group, the **quotient group of $G$ modulo** N.

**Proof.** It is easy to check that the operation is associative. Since $(e \odot N) \odot (g \odot N) = (e \odot g) \odot N = g \odot N$, the identity element of $G/N$ is $N = e \odot N$. Also

$$\left(g^b \odot N\right) \odot (g \odot N) = \left(g^b \odot g\right) \odot N = N$$

■

**Remark.** (a) In notion of 1.1, $[e] = N$ is the identity element of $G/N$ and $[g]^b = [g^b]$ is the inverse of $[g] \in G/N$. We also have $[g] \odot h = [g \odot h]$, $g, h \in G$.

(b) Any subgroup $N$ of an Abelian group $G$ is normal and so $G/N$ is a group. Meanwhile, $G/N$ is Abelian.

## 1.4 Homomorphisms

**Definition 1.4.1** (Homomorphism). Let $G = (G, \odot)$ and $G' = (G', \circledast)$ be groups...A function $\varphi \colon G \to G'$ is called a (**group**) **homomorphism** if

$$\varphi(g \odot h) = \varphi(g) \circledast \varphi(h), \quad g, h \in G$$

**Definition 1.4.2** (Endomorphism). A homomorphism from $G$ to itself

**Remark.** (a) Let $e$ and $e'$ be the identity elements of $G$ and $G'$ respectively, and let $\varphi \colon G \to G'$ be a homomorphism. Then

$$\varphi(e) = e' \quad \text{and} \quad (\varphi(g))^b = \varphi(g^b), \quad g \in G$$

**Proof.** $e' \circledast \varphi(e) = \varphi(e) = \varphi(e \odot e) = \varphi(e) \circledast \varphi(e)$
$e' = \varphi(e) = \varphi\left(g^b \odot g\right) = \varphi\left(g^b\right) \circledast \varphi(g)$ ■

(b) Let $\varphi \colon G \to G'$ be a homomorphism. The **kernel** of $\varphi$, $\ker(\varphi)$, defined by

$$\ker(\varphi) \coloneqq \varphi^{-1}(e') = \{g \in G\,;\, \varphi(g) = e'\}$$

is a normal subgroup of $G$.

**Proof.** First, try to prove $\ker(\varphi)$ is a subgroup of $G$. For all $g, h \in G$,

- $\varphi(g \odot h) = \varphi(g) \circledast \varphi(h) = e' \circledast e' = e'$
- $\varphi\left(g^b\right) = (\varphi(g))^b = (e')^b = e'$

Second, try to prove it is a normal subgroup. Let $h \in g \odot \ker(\varphi)$. Then we there is some $n \in G$ such that $\varphi(n) = e'$ and $h = g \odot n$. For $m \coloneqq g \odot n \odot g^b$, we have

$$\varphi(m) = \varphi(g) \circledast \varphi(n) \circledast \varphi(g^b) = \varphi(g) \circledast \varphi(g^b) = e'$$

and hence $m \in \ker(\varphi)$. Since $m \odot g = g \odot m = h$, this implies that $h \in \ker(\varphi) \odot g$. So $\ker(\varphi) \odot g \subseteq g \odot \ker(\varphi)$. Similarly one can show $g \odot \ker(\varphi) \subset \ker(\varphi) \odot g$. ■

(c) Let $\varphi \colon G \to G'$ be a homomorphism and $N \coloneqq \ker(\varphi)$. Then

$$g \odot N = \varphi^{-1}(\varphi(g)), \quad g \in G,$$

and so

$$g \sim h \Leftrightarrow \varphi(g) = \varphi(h), \quad g, h \in G,$$

where $\sim$ denotes the equivalence relation 1.1.

(d) A homomorphism is injective if and only if its kernel is trivial, that is, $\ker(\varphi) = \{e\}$

(e) The image $\mathrm{im}(\varphi)$ of a homomorphism $\varphi\colon G \to G'$ is a subgroup of $G'$.

**Example.** (a) The constant function $G \to G'$, $g \mapsto e'$ is a homorphism, the **trivial** homomorphism.

(b) The identity function $\mathrm{id}_G\colon G \to G$ is an endomorphism.

(c) Compositions of homomorphisms (endomorphisms) are homomorphisms (endomorphisms).

(d) If $\varphi\colon G \to G'$ is a bijective homomorphism, then so is $\varphi^{-1}\colon G \to G'$

**Definition 1.4.3** (Isomorphism)**.** A homomorphism $\varphi\colon G \to G'$ is called a (**group**) **isomorphism** from $G$ to $G'$ if $\varphi$ is bijective.
In this circumstance, we say that the groups $G$ and $G'$ are **isomorphic** and write $G \cong G'$.

**Definition 1.4.4** (Automorphism)**.** An isomorphism from $G$ to itself.

# Chapter 2

# Rings, Fields and Polynomials

## 2.1 Rings

**Definition 2.1.1** (Ring). A triple $(R, +, \cdot)$ consisting of a nonempty set $R$ and operations, **addition** $+$ and **multiplication** $\cdot$, is called a **ring** if

- $(R, +)$ is an Abelian group
- Multiplication is associative
- The **distributive law** holds:

$$(a+b) \cdot c = a \cdot c + b \cdot, \quad c \cdot (a+b) = c \cdot a + c \cdot b, \quad a, b, c \in R$$

**Note.** A ring is called **commutative** if multiplication is commutative.
If there is an identity element with respect to multiplication, then it is written as $1_R$ or simply 1, and is called the **unity** (or **multiplicative identity**) of $R$, and we say $(R, +, \cdot)$ is a **ring with unity**.
When the addition and multiplication operations are clear from context, we write simply $R$ instead of $(R, +, \cdot)$.

**Example.** (a) The **trivial ring** has exactly one element 0 and is itself denoted by 0. A ring with more than one element is **nontrivial**. If $R$ is a ring with unity, then it follows from $1_R \cdot a = a$ for each $a \in R$, that $R$ is trivial if and only if $1_R == 0_R$.

(b) Suppose $R$ is a ring and $S$ is a nonempty subset of $R$ that satisfies the following:

- $S$ is a subgroup of $(R, +)$.
- $S \cdot S \subseteq S$

Then $S$ itself is a ring, a **subring** of $R$, and $R$ is called an **overring** of $S$. If $R$ is commutative then so is $S$, but the converse is not true in general.

(c) Intersections of subrings are subrings.

**Definition 2.1.2** (Ring Homomorphism). Let $R$ and $R'$ be rings. A (**ring**) **homomorphism** is a function $\varphi \colon R \to R'$ which is compatible with the ring operations, that is,

$$\varphi(a+b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \ a, b \in R \tag{2.1}$$

**Note.** If, in addition, $\varphi$ is bijective, then $\varphi$ is called a (**ring**) **isomorphism** and $R$ and $R'$ are **isomorphic**.

A homomorphism $\varphi$ from $R$ to itself is a (**ring**) **endomorphism**. If $\varphi$ is an isomorphism, then it is a (**ring**) **automorphism**.

**Example.** (a) A ring homomorphism $\varphi \colon R \to R'$ is, in particular, a group homomorphism from $(R, +)$ to $(R', +)$. The **kernel**, $\ker(\varphi)$, of $\varphi$ is defined to be the kernel of this group homomorphism, that is,

$$\ker(\varphi) = \{a \in R; \varphi(a) = 0\} = \varphi^{-1}(0)$$

(b) The **zero function** $R \to R'$, $a \mapsto 0_{R'}$ is a homomorphism with $\ker(\varphi) = R$.

(c) Let $R$ and $R'$ be rings with unity and $\varphi \colon R \to R'$ a homomorphism. As (b) shows, it does not necessarily follow that $\varphi(1_R) = 1_{R'}$. This can be seen as a consequence of the fact that, with respect to multiplication, a ring is not a group.

## 2.2 Consequence of Ring Definitions

**Definition 2.2.1** (The Binomial Theorem). Let $a$ and $b$ be two commuting elements ($ab = ba$) of a ring $R$ with unity. Then, for all $n \in \mathbb{N}$,

$$(a+b)^b = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

**Lemma 2.2.1.** For $m \in \mathbb{N}$ with $m \geq 2$, an element $\alpha = (\alpha_1 \ldots \alpha_m) \in \mathbb{N}^m$ is called a **multi-index**. The **length** $|\alpha|$ of a multi-index $\alpha \in \mathbb{N}^m$ is defined by

$$|\alpha| := \sum_{j=1}^{m} \alpha_j$$

Set also

$$\alpha! := \prod_{j=1}^{m} (\alpha_j)!,$$

and define the **natural** (**partial**) **order** on $\mathbb{N}^m$ by

$$\alpha \leq \beta \rightleftarrows (\alpha_j \leq \beta_j,\ \ 1 \leq j \leq m).$$

for $a = (a_1, \ldots, a_m) \in R^m$ and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$

**Definition 2.2.2** (The Multinomial Theorem). Let $R$ be a commutative ring with unity. Then for all $m \geq 2$,

$$(\sum_{j=1}^{m} \alpha)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^\alpha, \quad a = (a_1, \ldots, a_m) \in R^m, \ \ k \in \mathbb{N}.$$

## 2.3 Field

**Definition 2.3.1** (Field). $K$ is a **field** when the following are satisfied:

- $K$ is a commutative ring with unity.
- $0 \neq 1$
- $K^\times := K \backslash \{0\}$ is an Abelian group with respect to multiplication.

**Note.** The Abelian group $K^\times := (K^\times, \cdot)$ is called the **multiplicative group** of $K$.

**Remark.** Let $K$ be a field.

(a) For all $a \in K^\times$, $\left(a^{-1}\right)-1 = a$

(b) A field has no zero divisors

(c) Let $a \in K^\times$ and $a \in K^\times$ and $b \in K$. Then there is an unique $x \in K$ with $ax = b$, namely the **quotient** $\frac{b}{a} := b/a := ba^{-1}$

(d) Let $K'$ be a field and $\varphi \colon K \to K'$ a homomorphism with $\varphi \neq 0$. Then

$$\varphi(1_K) = 1_{K'} \quad \text{and} \quad \varphi(a^{-1}) = \varphi(a)^{-1}, \; a \in K^\times$$

## 2.4 Ordered Field

**Definition 2.4.1** (Ordered Ring)**.** A ring $R$ with an ordered $\leq$ is called an **ordered ring** if the following holds:

- $(R, \leq)$ is totally ordered.
- $x < y \Rightarrow x + z < y + z, z \in R$
- $x, y > 0 \Rightarrow xy > 0$

**Note.** This leads to a series of basic arithmetic rules.
We may define absolute value function from $K \mapsto K$.

**Proposition 2.4.1.** Let $K$ be an ordered field and $x, y, a, \epsilon \in K$ with $\epsilon > 0$.

(i) $x = |x| \operatorname{sign}(x)$, $|x| = x \operatorname{sign}(x)$

(ii) $|x| = |-x|, x \leq |x|$

(iii) $|xy| = |x||y|$

(iv) $|x| \geq 0$ and $(|x| = 0 \Leftrightarrow x = 0)$

(v) $|x - a| < \epsilon \leftrightarrow a - \epsilon < x < a + \epsilon$

(vi) $|x + y| \leq |x| + |y|$ (**triangular inequality**)

**Corollary 2.4.1** (reversed triangular inequality)**.** In any ordered field $K$ we have

$$|x - y| \geq \left||x| - |y|\right|, \quad x, y \in K.$$

## 2.5 Formal Power Series

**Definition 2.5.1** (formal power series)**.** Let $R$ be a nontrivial ring with unity. On the set $R^{\mathbb{N}} = \operatorname{Funct}(\mathbb{N}, R)$ define addition by

$$(p + q)_n := p_n + q_n, \qquad n \in \mathbb{N},$$

and multiplication by **convolution**,

$$(pq)_n := (p \cdot q)_n := \sum_{j=0}^{n} p_j q_{n-1} = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0$$

for $n \in \mathbb{N}$. Here $p_n$ denotes the value of $p \in R^{\mathbb{N}}$ at $n \in \mathbb{N}$ and is called the $n^{\text{th}}$ **coefficient** of $p$. In this situation an element $p \in R^{\mathbb{N}}$ is called a **formal power series over** $R$, and we set $R[X] := \left(R^{\mathbb{N}}, +, \cdot\right)$

**Proposition 2.5.1.** $R[X]$ is a ring with unity, the **formal power series ring over** $R$. If $R$ is commutative, then so is $R[X]$

## 2.6 Polynomials

**Definition 2.6.1** (Polynomial). A **polynomial over** $R$ is a formal power seres $p \in R[X]$ such that $\{n; p_n \neq 0\}$ is finite, in other words, $p_n = 0$ "almost everywhere".

# Chapter 3

# The Real Numbers

Starting words: we seek an ordered **extension field** of $\mathbb{Q}$ in which the equation $x^2 = a$ is solvable for each $a > 0$.

## 3.1 Order Completeness

We say a totally ordered set $X$ is **order complete** (or $X$ satisfies the **completeness axiom**) if every nonempty subset of $X$ which is bounded above has a supremum.

**Proposition 3.1.1.** Let $X$ be a totally ordered set. Then the followings are equivalent:

(i) $X$ is order complete.

(ii) Every nonempty subset of $X$ which is bounded below has an infimum.

(iii) For all nonempty subsets $A, B$ of $X$ such that $a \leq b$ for all $(a, b) \in A \times B$, there is some $c \in X$ such that $a \leq c \leq b$ for all $(a, b) \in A \times B$ (**Dedekind cut property**)

**Note.** A relation $\leq$ on $X$ is a **partial order** on $X$ if it is reflexive, transitive and **anti-symmetric**, that is,

$$(x \leq y)(y \leq x) \Rightarrow x = y$$

If $\leq$ is a partial order on $X$, then the pair $(X, \leq)$ is called a **partially ordered set**. If, in addition,

$$\forall x, y \in X : (x \leq y) \vee (y \leq x)$$

then $\leq$ is called a **total order** on $X$ and $(X, \leq)$ is a **totally ordered set**.

**Corollary 3.1.1.** A totally ordered set is order complete if and only if every nonempty bounded subset has a supremum and an infimum.

**Theorem 3.1.1 (Dedekind's Construction of the Real Numbers).** There is, up to isomorphism, a unique order complete extension field $\mathbb{R}$ of $\mathbb{Q}$. This extension is called **the field of real numbers**.

**Proposition 3.1.2 (A Characterization of Supremum and Infimum).** Followed from natural order defined by $\mathbb{R}$.

(i) If $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then

(a) $x < \sup(A) \Leftrightarrow \exists a \in A$ such that $x < a$.

(b) $x < \inf(A) \Leftrightarrow \exists a \in A$ such that $x > a$.

(ii) Every subset $A$ of $\mathbb{R}$ has a supremum and an infimum in $\mathbb{R}$

## 3.2 The Consequence of Order Completeness

### The Archimedean Property

**Proposition 3.2.1** (Archimedes). $\mathbb{N}$ is not bounded above in $\mathbb{R}$, that is, for each $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ such that $n > x$.

**Corollary 3.2.1.** Equivalent statements as the above proposition

(a) Let $a \in \mathbb{R}$. If $0 \leq a \leq 1/n$ for all $n \in \mathbb{N}^{\times}$.

(b) For each $a \in \mathbb{R}$ with $a > 0$ there is some $n \in \mathbb{N}^{\times}$ such that $1/n < a$.

### The Density of the Rational/Irrational Numbers in $\mathbb{R}$

**Proposition 3.2.2.** For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $r \in \mathbb{Q}$ such that $a < r < b$.

**Proposition 3.2.3** ($n^{\text{th}}$ Roots). For all $a \in \mathbb{R}^{+}$ and $n \in \mathbb{N}^{\times}$, there is a unique $x \in \mathbb{R}^{+}$ such that $x^n = a$

**Proposition 3.2.4.** For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $\xi \in \mathbb{R} \backslash \mathbb{Q}$ such that $a < \xi < b$.

### Intervals

An **interval** is a subset $J$ of $\mathbb{R}$ such that

$$(x, y \in J, x < y) \Rightarrow (z \in J \text{for} x < z < y)$$

If $J$ is a nonempty interval, then $\inf(J) \in \bar{\mathbb{R}}$ is the **left endpoint** and $\sup(J) \in \bar{\mathbb{R}}$ is the **right endpoint** of $J$. $J$ is **closed on the left** if $a := \inf(J)$ is in $J$, and otherwise it is **open on the left**. The same applies to the other side.

An interval is **perfect** if it contains at least two points. It is **bounded** if both endpoints are in $\mathbf{R}$ and is **unbounded** otherwise. If $J$ is a bounded interval, then the nonnegative number $|J| := \sup(J)$ - $\inf(J)$ is called the **length** of $J$.

# Chapter 4

# Vector Spaces, Affine Spaces and Algebras

## 4.1 Vector Spaces

**Definition 4.1.1** (Vector Space). A **vector space over the field** $K$ (or simply, a $K$**-vector space**) is a triple $(V, +, \cdot)$ consisting of a nonempty set $V$, an 'inner' operation $+$ on $V$ called **addition**, and an 'outer' operation

$$K \times V \to V, \quad (\lambda, v) \mapsto \lambda \cdot v,$$

called **scalar multiplication** which satisfy the following axioms:

- $(V, +)$ is an Abelian group
- The distributive law holds:

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w, (\lambda + u) \cdot v = \lambda \cdot v + \mu \cdot v, \quad \lambda, \mu \in K, \quad v, w \in V$$

- $\lambda \cdot (\mu v) = (\lambda\mu) \cdot v$, $1 \cdot v = v$ for $\lambda, \mu \in K, v \in V$

A vector space is called **real** if $K = \mathbb{R}$ and **complex** if $K = \mathbb{C}$.

**Note** (Linear Functions). Let $V$ and $W$ be vector spaces over $K$. THen a function $T \colon V \mapsto W$ is $(K-)$**linear** if

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w), \quad \lambda, \mu \in K, \ v, w \in V$$

In this regard, a linear function is simply a function which is compatible with the vector space operations, in other words, it is a (**vector space**) **homomorphism**. The set of all linear functions from $V$ to $W$ is denoted by $\mathrm{Hom}(V, W)$ or $\mathrm{Hom}_K(V, W)$, and $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ is the set of all (vector space) **endomorphisms**. A bijective homomorphism $T \in \mathrm{Hom}(V, W)$ is a (vector space) **isomorphism**.

**Remark.** (a) A vector space homomorphism $T \colon V \mapsto W$ is, in particular, a group homomorphism $T \colon (V, +) \mapsto (W, +)$.

(b)

# Part II

# Convergence

# Chapter 5

# Convergence of Sequences

## 5.1 Sequences

**Definition 5.1.1** (Sequence). Let $X$ be a set. A **sequence** (in $X$) is simply a function from $\mathbb{N}$ to $X$. If $\varphi\colon \mathbb{N} \mapsto X$ is a sequence, we write also

$$(x_n), (x_n)_{n\in\mathbb{N}} \text{ or } (x_0, x_1, x_2, \ldots)$$

for $\varphi$, where $x_n := \varphi(n)$ is the $n^{\text{th}}$ term of the sequence $\varphi = (x_0, x_1, x_2, \ldots)$.

**Remark.**

(a) A sequence $(x_n)$ is different from its image $\{x_n; n \in \mathbb{N}\}$.

(b) Let $(x_n)$ be a sequence in $X$ and $E$ a property. Then we say $E$ holds for **almost all** terms of $(x_n)$ if there is some $m \in \mathbb{N}$ such that $E(x_n)$ is true for all $n \geq m$, that is, if $E$ holds for all but finitely many of the $x_n$. If there is a subset $N \subseteq \mathbb{N}$ with $\mathrm{Num}(N) = \infty$ and $E(x_n)$ is true for each $n \in N$ then $E$ is true for **infinitely many** terms.

(c) For $m \in \mathbb{N}^\times$, a function $\Phi\colon m + \mathbb{N} \mapsto X$ is also called a sequence in $X$.

## 5.2 Metric Space

**Definition 5.2.1** (Metric Space). Let $X$ be a set. A function $d\colon X \times X \mapsto \mathbb{R}^+$ is called a **metric** on $X$ if the following hold:

- $d(x, y) = 0 \leftrightarrow x = y$.
- $d(x, y) = d(y, x)$, $x, y \in X$ (symmetry).
- $d(x, y) \leq d(x, z) + d(y, z)$, $x, y, z \in X$ (triangle inequality).

**Note.** If $d$ is a metric on $X$, then $(X, d)$ is called a **metric space**. We call $d(x, y)$ the **distance** between the **points** $x$ and $y$ in the metric space $X$.

In the metric space $(X, d)$, for $a \in X$ and $r > 0$, the set

$$\mathbb{B}(a, r) := \mathbb{B}_X(a, r) := \{x \in X; d(a, x) < r\}$$

is called the **open ball** with center at $a$ and radius $r$, while

$$\bar{\mathbb{B}}(a, r) := \bar{\mathbb{B}}_X(a, r) := \{x \in X; d(a, x) \leq r\}$$

is called the **closed ball** with center at $a$ and radius $r$.

**Example.** (a) $\mathbb{K}$ is a metric space with the **natural metric**

$$\mathbb{K} \times \mathbb{K} \mapsto \mathbb{R}^+, \quad (x, y) \mapsto |x - y|$$

(b) Let $(X, d)$ be a metric space and $Y$ a nonempty subset of $X$. Then the restriction of $d$ to $Y \times Y$, $d_Y := d|Y \times Y$, is a metric on $Y$, the **induced metric**, and $(Y, d_Y)$ is a metric space, a **metric subspace** of $X$.

(c) Let $X$ be a nonempty set. Then the function $d(x, y) := 1$ for $x \neq y$ and $d(x, x) := 0$ is a metric, called the **discrete metric** on $X$.

(d) Let $(X_j, d_j)$, $1 \leq j \leq m$, be metric spaces and $X := X_1 \times \cdots \times X_m$. Then the function

$$d(x, y) := \underset{1 \leq j \leq m}{max}\, d_j(x_j, y_j)$$

for $x := (x_1, \ldots, x_m) \in X$ and $y := (y_1, \ldots, y_m) \in X$ is a metric on $X$ called the **product metric**. The metric space $X := (X, d)$ is called the **product of the metric spaces** $(X_j, d_j)$

**Proposition 5.2.1.** Let $(X, d)$ be a metric space. Then for all $x, y, z \in X$ we have

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

**Note.** A subset $U$ of a metric space $X$ is called a **neighborhood** of $a \in X$ if there is some $r > 0$ such that $\mathbb{B}(a, r) \subseteq U$. The **set of all neighborhoods of the point** $a$ is denoted by $\mu(a)$, that is,

$$\mu(a) := \mu_X(a) := \{U \subseteq X; U \text{ is a neighborhood of } a\} \subseteq P(X)$$

### Cluster Point

**Definition 5.2.2** (Cluster Point)**.** We call $a \in X$ a **cluster point** of $(x_n)$ if every neighborhood of $a$ contains infinitely many terms of the sequence.

**Proposition 5.2.2.** The following are equivalent:

(i) $a$ is a cluster point of $(x_n)$.

(ii) For each $U \in \mu(a)$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in U$.

(iii) For each $\epsilon > 0$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in \mathbb{B}(a, \epsilon)$

## 5.3 Convergence

**Definition 5.3.1** (Convergence)**.** A sequence $(x_n)$ **converges** (or is **convergent**) with **limit** $a$ if each neighborhood of $a$ contains almost all terms of the sequence. In this case we write

$$\lim_{n \to \infty} x_n = a \text{ or } x_n \to a(n \to \infty)$$

and we say that $(x_n)$ **converges to** $a$ **as** $n$ **goes to** $\infty$. A sequence $(x_n)$ that is not convergent is called **divergent** and we say $(x_n)$ **diverges**.

**Proposition 5.3.1.** The following statements are equivalent:

(i) $lim_{x_n} = a$.

(ii) For each $U \in \mu(a)$, there is some $N := N(U)$ such that $x_n \in U$ for all $n \geq N$.

(iii) For each $\epsilon > 0$, there is some $N := N(\epsilon)$ such that $x_n \in \mathbb{B}(a, \epsilon)$ for all $n \geq N$.

### Bounded Sets

**Definition 5.3.2.** A subset $Y \subseteq X$ is called d-**bounded** or **bounded in** $X$ (with respect to the metric $d$) if there is some $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in Y$. In this circumstance the **diameter** of $Y$, defined by

$$\operatorname{diam}(Y) := \sup_{x,y \in Y} d(x, y)$$

is finite. A sequence $(x_n)$ is **bounded** if its image $\{x_n; n \in \mathbb{N}\}$ is bounded.

**Proposition 5.3.2.** Any convergent sequence is bounded.

**Proof.** Suppose that $x_n \to a$. Then there is some $N$ such that $x_n \in \mathbb{B}(a, 1)$ for all $n \geq N$. It follows from the triangle inequality that

$$d(x_m, x_n) \leq d(x_m, a) + d(a, x_n) \leq 2, \quad m, n \in \mathbb{N}$$

Since there is also some $M \geq 0$ such that $d(x_j, x_k) \leq M$ for all $j, k \leq N$, we have $d(x_n, x_m) \leq M+2$ for all $m, n \in \mathbb{N}$. ■

### Uniqueness of the Limit

**Proposition 5.3.3.** Let $(x_n)$ be convergent with limit $a$. Then $a$ is the unique cluster point of $(x_n)$.

**Corollary 5.3.1.** The limit of a convergent sequence is unique.

### Subsequence

Let $\varphi = (x_n)$ be a sequence in $X$ and $\Phi \colon \mathbb{N} \to \mathbb{N}$ a strictly increasing function, then $\varphi \circ \Phi \in X^{\mathbb{N}}$ is called a **subsequence** of $\varphi$. Extending the notation $(x_n)_{n \in \mathbb{N}}$ introduced above for the sequence $\varphi$, we write $(x_{n_k})_{k \in \mathbb{N}}$ for the subsequence $\varphi \circ \Phi$ where $n_k := \Phi(k)$.

**Proposition 5.3.4.** If $(x_n)$ is a convergent sequence with limit $a$, then each subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)$ is convergent with $\lim_{k \to \infty} x_{n_k} = a$.

**Proposition 5.3.5.** A point $a$ is a cluster point of a sequence $(x_n)$ if and only if there is some subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)$ which converges to $a$.

Chapter 6

# Normed Vector Space

## 6.1 Norms

**Definition 6.1.1** (Norm). Let $E$ be a vector space over $\mathbb{K}$. A function $||\cdot||\colon E \to \mathbb{R}^+$ is called a **norm** if the following hold:

- $||x|| = 0 \Leftrightarrow x = 0$.
- $||\lambda x|| = |\lambda|||x||$, $x \in E$, $\lambda \in \mathbb{K}$ (positive homogeneity)
- $||x+y|| \leq ||x|| + ||y||$, $x, y \in E$ (triangle inequality). A pair $(E, ||\cdot||)$ consisting of a vector space $E$ and a norm $||\cdot||$ is called a **normed vector space**. If the norm is clear from context, we write $E$ instead of $(E, ||\cdot||)$.

**Remark.** Let $E := (E, ||\cdot||)$ be a normed vector space.

(a) The function

$$d\colon E \times E \to \mathbb{R}^+, \quad (x, y) \mapsto ||x-y||$$

is a metric on $E$, the **metric induced from the norm**. Hence any normed vector space is also a metric space.

(b) The **reversed triangle inequality** holds for the norm:

$$||x-y|| \geq \left| ||x|| - ||y|| \right|, \quad x, y \in E$$

(c) All statements from previous chapter also hold in normed vector space.

### Balls

For $a \in E$ and $r > 0$, we define the **open** and **closed balls** with center at $a$ and radius $r$ by

$$\mathbb{B}_E(a, r) := \mathbb{B}(a, r) := x \in E; ||x-a|| < r$$

and

$$\bar{\mathbb{B}}_E(a, r) := \bar{\mathbb{B}}_E := x \in E; ||x-a|| \leq r.$$

These definitions agree with those for the metric space $(E, d)$ when $d$ is induced from norm. We also write

$$\mathbb{B} := \mathbb{B}(0, 1) = x \in E; ||x|| < 1 \quad \text{and} \quad \bar{\mathbb{B}} := \bar{\mathbb{B}}(0, 1) = x \in E; ||x|| \leq 1$$

for the **open** and **closed unit balls** in $E$. We have

$$r\mathbb{B} = \mathbb{B}(0, r), \quad r\bar{\mathbb{B}} = \bar{\mathbb{B}}(0, r)$$

### Bounded Sets

A subset $X$ of $E$ is called **bounded in** $E$ (or **norm bounded**) if it is bounded in the induced metric space.

**Remark.** Let $E := (E, ||\cdot||)$ be a normed vector space

(a) $X \subseteq E$ is bounded if and only if there is some $r > 0$ such that $X \subseteq r\mathbb{B}$, that is, $||x|| < r$ for all $x \in X$.

(b) If $X$ and $Y$ are nonempty bounded subsets of $E$, then so are $X \cup Y$, $X + Y$ and $\lambda X$ with $\lambda \in \mathbb{K}$.

**Example.** (a) The absolute value $|\cdot|$ is a norm on the vector space $\mathbb{K}$.

(b) Let $F$ be a subspace of a normed vector space $E := (E, \|\cdot\|)$. Then the restriction $\|\cdot\|_F := \|\cdot\||F$ of $\|\cdot\|$ to $F$ is a norm on $F$. Thus $F := (F, \|\cdot\|_F)$ is a normed vector space with this **induced norm**.

(c) Let $E_j, \|\cdot\|_j$, $1 \leq j \leq m$, be normed vector space over $\mathbb{K}$. Then

$$\|x\|_\infty := \max_{1\leq j\leq m} \|x_j\|_j, \qquad x = (x_1, \ldots, x_m) \in E := E_1 \times \cdots \times E_m$$

defines a norm, called the **product norm**, on the product vector space $E$. The metric on $E$ induced from this norm coincides with the product metric from 5.2 example (d), where $d_j$ is the metric induced on $E_j$ from $\|\cdot\|_j$.

(d) For $m \in \mathbb{N}^\times, \mathbb{K}^m$ is a normed vector space with the **maximum norm**

$$|x|_\infty := \max_{1\leq j\leq m} |x_j|, \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m.$$

In this case m = 1.

## 6.2 The Space of Bounded Functions

Let $X$ be a nonempty set and $(E, \|\cdot\|)$ a normed vector space. A function $u \in E^X$ is called **bounded** if the image of $u$ in $E$ is bounded. For $u \in E^X$, define

$$\|u\|_\infty := \|u\|_{\infty,X} := \sup_{x\in X} \|u(x)\| \in \mathbb{R}^+ \cup \infty$$

**Remark.** (a) For $u \in E^X$, the followings are equivalent:

(i) $u$ is bounded.

(ii) $u(X)$ is bounded in $E$.

(iii) There is some $r > 0$ such that $\|u(x)\| \leq r$ for all $x \in X$.

(iv) $\|u\|_\infty < \infty$.

(b) Clearly $\mathrm{id} \in \mathbb{K}^\mathbb{K}$ is not bounded, that is, $\|\mathrm{id}\|_\infty = \infty$.

(b) shows that $\|\cdot\|_\infty$ may not be a norm on the vector space $E^X$ when $E$ is not trivial. We therefore set

$$B(X, E) := u \in E^X \ ; \ u \ \text{ is bounded}$$

and call $B(X, E)$ the **space of bounded functions** from $X$ to $E$.

**Proposition 6.2.1.** $B(X, E)$ is a subspace of $E^X$ and $\|\cdot\|_\infty$ is a norm, called the **supremum norm**, on $B(X, E)$.

**Remark.** (a) If $X := \mathbb{N}$, then $B(X, E)$ is the normed vector space of bounded sequences in $E$. In the special case $E := \mathbb{K}$, $B(\mathbb{N}, \mathbb{K})$ is denoted by $\ell_\infty$, that is,

$$\ell_\infty := \ell\infty(\mathbb{K}) := B(\mathbb{N}, \mathbb{K})$$

is the normed **vector space of bounded sequences** with the supremum norm

$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}}|x_n|, \quad (x_n) \in \ell_\infty$$

(b) If $X = \{1, \dots, m\}$ for some $m \in \mathbb{N}^\times$, then

$$B(X, E) = (E^m, \|\cdot\|_\infty)$$

## 6.3 Inner Product Spaces

**Definition 6.3.1** (Inner Product Space). Let $E$ be a vector space over the filed $\mathbb{K}$. A function

$$(\cdot|\cdot)\colon E \times E \to \mathbb{K}, \quad (x, y) \mapsto (x|y)$$

is called a **scalar product** or **inner product** on $E$ if the following hold:

- $(x|y) = \overline{(y|x)}$, $x, y \in E$
- $(\lambda x + \mu y|z) = \lambda\,(x|z) + \mu\,(y|z)$, $x, y, z \in E$, $\lambda, \mu \in \mathbb{K}$.
- $(x|x) \geq 0$, $x \in E$, and $(x|x) = 0 \Leftrightarrow 0$.

A vector space $E$ with a scalar product $(\cdot|\cdot)$ is called an **inner product space** and is written in $(E, (\cdot|\cdot))$.

**Remark.** (a) In the real case $\mathbb{K} = \mathbb{R}$, the first point can be written as

$$(x|y) = (y|x), \quad x, y \in E$$

In other words, the function is **symmetric** when $E$ is a real vector space. In the case $\mathbb{K} = \mathbb{C}$, the function is said to be **Hermitian** when the first point holds.

(b) From the first two points it follows that

$$(x|\lambda y + \mu z) = \bar{\lambda}(x|y) + \bar{\mu}(x|z), \quad x, y, z \in E, \quad \lambda, \mu \in \mathbb{K},$$

that is, for each fixed $x \in E$, the function $(x|\cdot) : E \to \mathbb{K}$ is **conjugate linear**. If $\mathbb{K} = \mathbb{R}$, it is **bilinear**.

(c) $(x|0) = 0$ for all $x \in E$.

Let $m \in \mathbb{N}^\times$. For $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ in $\mathbb{K}^m$, define

$$(x|y) := \sum_{j=1}^{m} x_j \bar{y}_j$$

to be the **Euclidean Inner product** on $\mathbb{K}^m$.

### The Cauchy-Schwarz Inequality

**Theorem 6.3.1** (Cauchy-Schwarz Inequality). Let $(E, (\cdot|\cdot))$ be an inner product space. Then

$$|(x|y)|^2 \leq (x|x)(y|y) \quad x, y \in E$$

and the equality occurs if and only if $x$ and $y$ are linearly dependent.

**Theorem 6.3.2.** Let $(E, (\cdot|\cdot))$ be an inner product space and

$$\|x\| := \sqrt{(x|x)}, \qquad x \in E$$

Then $\|\cdot\|$ is a norm on $E$, the **norm induced from the scalar product** $(\cdot|\cdot)$. A norm which is induced from a scalar product is also called a **Hilbert norm**.

**Corollary 6.3.1.** Let $(E, (\cdot|\cdot))$ be an inner product space. Then

$$|(x|y)| \leq \|x\|\|y\| \qquad x, y \in E$$

## 6.4 Euclidean Spaces

Convention: Unless otherwise stated, we consider $\mathbb{K}^m$ to be endowed with the Euclidean inner product $(\cdot|\cdot)$ and the induced norm

$$|x| := \sqrt{(x|x)} = \sqrt{\sum_{j=1}^{m} |x_j|^2} \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m$$

the **Euclidean norm**. In the real case, we write also $x \cdot y$ for $(x|y)$.

We further define the norm

$$|x|_1 := \sum_{j=1}^{m} |x_j|, \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m$$

**Proposition 6.4.1.** Let $m \in \mathbb{N}^\times$. Then

$$|x|_\infty \leq |x| \leq \sqrt{m}|x|_\infty, \qquad \frac{1}{\sqrt{m}}|x|_1 \leq |x| \leq |x|_1, \qquad x \in \mathbb{K}^m$$

### Equivalent Norm

Let $E$ be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $E$ are **equivalent** if there is some $K \geq 1$ such that

$$\frac{1}{K}\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1, \quad x \in E$$

In this case we write $\|\cdot\|_1 \ \|\cdot\|_2$.

**Remark.** (a) is an equivalence relation on the set of all norms of a fixed vector space.

(b) $\|\cdot\|_1 \ \|\cdot\| \ \|\cdot\|_\infty$ on $\mathbb{K}^m$.

(c) We write $\mathbb{B}^m$ for the **real open Euclidean unit ball**, that is, $\mathbb{B}^m := \mathbb{B}_{\mathbb{R}^m}$ and $\mathbb{B}_1^m$ and $\mathbb{B}_\infty^m$ for the unit balls in $(\mathbb{R}^m, |\cdot|_1)$ and in $(\mathbb{R}^m, |\cdot|_\infty)$ respectively. We have

$$\mathbb{B}^m \subseteq \mathbb{B}_\infty^m \subseteq \sqrt{m}\mathbb{B}^m \qquad \mathbb{B}_1^m \subseteq \mathbb{B}^m \subseteq \sqrt{m}\mathbb{B}_1^m$$

(d) Let $E = (E, \|\cdot\|)$ be a normed vector space and $\|\cdot\|_1$ a norm on $E$ which is equivalent to $\|\cdot\|$. Set $E_1 := (E, \|\cdot\|_1)$. Then

$$\mathcal{U}_E(a) = \mathcal{U}_{E_1}(a), \qquad a \in E$$

that is, the set of neighborhoods of $a$ depends only on the equivalence class of the norm. Equivalent norms produce the same set of neighborhoods.

## Convergence in Product Spaces

**Proposition 6.4.2.** Let $m \in \mathbb{N}^\times$ and $x_n = \left(x_n^1, \ldots, x_n^m\right) \in \mathbb{K}^m$ for $n \in \mathbb{N}$. Then the followings are equivalent:

1. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x = \left(x^1, \ldots, x^m\right)$ in $\mathbb{K}^m$.
2. For each $k \in \{1, \ldots, m\}$, the sequence $\left(x_n^k\right)_{n\in\mathbb{N}}$ converges to $x^k$ in $\mathbb{K}$.

# Chapter 7

# Infinite Limits

## 7.1 Convergence to $\pm\infty$

Sequences in $\mathbb{R}$ can usually be considered to converge to $+\infty$ or $-\infty$ in the extended number line $\bar{\mathbb{R}}$. A subset $U \subseteq \bar{\mathbb{R}}$ is called a **neighborhood of** $\infty$ (or of $-\infty$) if there is some $K > 0$ such that $(K, \infty) \subseteq U$ (or such that $(-\infty, -K) \subsetneqq U$). The set of neighborhoods of $\pm\infty$ is denoted by $\mathcal{U}(\pm\infty)$, that is,

$$\mathcal{U}(\pm\infty) := U \subseteq \bar{\mathbb{R}};\, U \ \text{ is neighborhood of } \pm\infty$$

Now let $(x_n)$ be a sequence in $\mathbb{R}$. Then $\pm\infty$ is called a **cluster point** (or **limit**) of $(x_n)$, if each neighborhood $U$ of $\pm\infty$ contains infinitely many (or almost all) terms of $(x_n)$. If $\pm\infty$ is the limit of $(x_n)$, we usually write

$$\lim_{n\to\infty} x_n = \pm\infty \qquad \text{or} \qquad x_n \to \pm\infty \; (n \to \infty)$$

The sequence $(x_n)$ **converges** in $\bar{\mathbb{R}}$ if there is some $x \in \bar{\mathbb{R}}$ such that $\lim\limits_{n\to\infty} x_n = x$. The sequence $(x_n)$ **diverges** in $bar\mathbb{R}$, if it does not converge in $\bar{\mathbb{R}}$. With this definition, any sequence which converge in $\mathbb{R}$, also converges in $\bar{\mathbb{R}}$, and any sequence which diverges in $\bar{\mathbb{R}}$, also diverges in $\mathbb{R}$. On the other hand there are divergent sequences in $\mathbb{R}$ which converge in $\bar{\mathbb{R}}$ (to $\pm\infty$). In this case the sequence is said to converge **improperly**.

**Proposition 7.1.1.** Every monotone sequence $(x_n)$ in $\mathbb{R}$ converges in $\bar{\mathbb{R}}$, and

$$\lim x_n = \begin{cases} \sup x_n; n \in \mathbb{N}, & \text{if } (x_n) \text{ is increasing,} \\ \inf x_n; n \in \mathbb{N}, & \text{if } (x_n) \text{ is decreasing.} \end{cases}$$

## 7.2 The Limit Superior and Limit Inferior

**Definition 7.2.1.** Let $(x_n)$ be a sequence in $\mathbb{R}$. We can define two new sequences $(y_n)$ and $(z_n)$ by

$$y_n := \sup_{k\geq n} x_k := \sup x_k; k \geq n,$$

$$z_n := \inf_{k\geq n} x_k := \inf x_k; k \geq n.$$

Clearly $(y_n)$ is increasing and $(z_n)$ is decreasing. By the above proposition, these sequences converge in $\bar{\mathbb{R}}$:

$$\limsup_{n\to\infty} x_n := \overline{\lim}_{n\to\infty} x_n := \lim_{n\to\infty}(\sup_{k\geq n} x_k).$$

the **limit superior**, and

$$\liminf_{n\to\infty} x_n := \underline{\lim}_{n\to\infty} x_n := \lim_{n\to\infty}(\inf_{k\geq n} x_k).$$

the **limit inferior** of the sequences $(x_n)$. We also have

$$\text{limsup} x_n = \inf_{n\in\mathbb{N}}(\sup_{k\geq n} x_k) \quad \text{and} \quad \text{liminf} x_n = \sup_{n\in\mathbb{N}}(\inf_{k\geq n} x_k).$$

**Theorem 7.2.1.** Any sequence $(x_n)$ in $\mathbb{R}$ has a smallest cluster point $x_*$ and a greatest cluster point $x^*$ in $\bar{\mathbb{R}}$ and these satisfy

$$\text{liminf} x_n = x_* \quad \text{and} \quad \text{limsup} x_n = x^*$$

**Theorem 7.2.2.** Let $(x_n)$ be a sequence in $\mathbb{R}$. Then

$$(x_n) \text{ converges in } \bar{\mathbb{R}} \Leftrightarrow \overline{\lim} x_n \leq \underline{\lim} x_n$$

When the sequence converges, the limit $x$ satisfies

$$x = \lim x_n = \underline{\lim} x_n = \overline{\lim} x_n.$$

**Theorem 7.2.3** (Bolzano-Weierstrass)**.** Every bounded sequence in $\mathbb{K}^m$ has a convergent subsequence, that is, a cluster point.

# Chapter 8

# Completeness

## 8.1 Cauchy Sequences

In the following $X = (X, d)$ is a metric space.

A sequence $(x_n)$ in $X$ is called a **Cauchy sequence** if, for each $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

Similarly, if $(x_n)$ is a sequence in a normed vector space $E = (E, \|\cdot\|)$, then $(x_n)$ is a Cauchy sequence if and only if for each $\epsilon > 0$ there is some $N$ such that $\|x_n - x_m\| < \epsilon$ for all $m, n \geq N$. In particular, Cauchy sequences in $E$ are "translation invariant", that is, if $(x_n)$ is a Cauchy sequence and $a$ is an arbitrary vector in $E$, then the 'translated' sequence $(x_n + a)$ is also a Cauchy sequence. This implies that Cauchy sequences cannot be defined using neighborhoods.

**Proposition 8.1.1.** Every convergent sequence is a Cauchy sequence.

**Proposition 8.1.2.** Every Cauchy sequence is bounded.

**Proposition 8.1.3.** If a Cauchy sequence has a convergent subsequence, then it is itself convergent.

## 8.2 Banach Spaces

A metric space $X$ is called **complete** if every Cauchy sequence in $X$ converges. A complete normed vector space is called a **Banach space**.

**Theorem 8.2.1.** $\mathbb{K}^m$ is a Banach space.

**Theorem 8.2.2.** Let $X$ be a nonempty set and $E = (E, \|\cdot\|)$ a Banach space. Then $B(X, E)$ is also a Banach space.

**Remark.** (a) A direct consequence of the previous two theorems is that For every nonempty set $X$, $B(X, \mathbb{R})$, $B(X, \mathbb{C})$, and $B(X, \mathbb{K}^m)$ are Banach spaces.

(b) The completeness of a normed vector space $E$ is invariant under changes to equivalent norms.

(c) A complete inner product space is called a **Hilbert space**.