Eigenstuff

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1 General concept

- **Definition**

  Let V be a vector space of dimension n, and let $T : V \to V$ be a linear transformation

  The scalar $\lambda \in \mathbb{R}$ is an *eigenvalue* of T if there is a non-zero vector $v \in V$ such that

  $$T(v) = \lambda v;$$

  such a vector is called an *eigenvector* of T, with corresponding eigenvalue λ.

- The geometric intuition for eigenvector is that some special vectors in the domain of T such that would only stretch itself when doing matrix multiplication with a certain matrix. Such vectors are called eigen vectors and the degree of "stretching" is called eigenvalues.

- The set

  $$E_\lambda = \{v \in V : T(v) = \lambda v\}$$

  is called the *eigenspace* corresponding to λ. Eigenspace is a subspace of V because it contains the zero vector, and closed under vector addition and scalar multiplication

- **Definition**

  The *characteristic polynomial* of the linear transformation $T : V \to V$ is the polynomial f_T (which we write here in the variable λ) given by

  $$f_T(\lambda) = det(\lambda I - T),$$

  where $I : V \to V$ is the identity transformation. Here we are thinking of $\lambda I - T$ as a new linear transformation from V to V, defined by

$$(\lambda I - T)(v) = \lambda v - T(v) \text{ for all } v \in V$$

- The above definition leads us a systematic way of finding eigenvalues. If v is an eigenvector of T with corresponding eigenvalue λ, then,

  $$(\lambda I - T)(v) = 0$$

  since here we assume the nonzero eigenvector exists, then the linear transformation $\lambda I - T$ has a nontrivial kernel, hence it is not invertible, and we thus have

  $$det(\lambda I - T) = 0$$

  Conversely, reversing the arugment shows that if the determinant is 0, then λ is an eigenvalue of T. This shows that *the eigenvalues of T are just the roots of the characteristic polynomial of T*

- **Definition**

  Let λ be an eigenvalue of T.

  The *algebraic multiplicity* "almu(λ)" of λ is the number of times that λ occurs as a root of the characteristic polynomial f_T of T; that is, the largest power of the root.

  The *geometric multiplicity* "gemu(λ)" of λ is the dimension of the corresponding eigenspace E_λ

- **Definition - Diagonalization**

  A linear transformation $T : V \to V$ of the finite-dimensional vector space V is said to be *diagonalizable* if there there is a basis B of V such that $[T]_B$ is diagnoal. A square matrix A is said to be *diagonalizable* if the linear transformation T_A is diagonalizable, or equivalently if A is similar to a diagonal matrix

2 Understanding of the concept

- Eigenvectors that are of distinct eigenvalues are linearly independent
- The intersection of two eigenvectors that have distinct eigenvalues is only the zero vector
- we can think about diagonalizable matrix means that the matrix is the $\mathcal{B}$-matrix, where the basis is just the eigenbasis. In other words, the direct sum of all eigenspaces equals the vector space.
  - Two important theorems to bridge the gap:
  - if $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis of V and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & \\ & ... & \\ & & \lambda_n \end{bmatrix} \text{ if and only if } T(v_i) = \lambda_i v_i \text{ for all } 1 \leq i \leq n$$

  - T is diagonalizable if and only if there is a basis $\mathcal{B}$ of V consisting of eigenvectors of T. (Such a basis of V is called an *eigenbasis* of T)
  - Similarly, we have theorems for just a n x n matrix A.
    * if the n x n matrix $P = [\vec{v}_1 \cdots \vec{v}_n]$ is invertible and if $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then

      $$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & ... & \\ & & \lambda_n \end{bmatrix} \text{ if and only if } A\vec{v}_i = \lambda_i \vec{v}_i \text{ for all i}$$

    * A is diagonalizable if and only is there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of A. (Again, such basis of $\mathbb{R}^n$ is called an *eigenbasis* for A)

- In Summary of above's theorems, we finally bridge the gap between diagonizalition and eigenvectors/eigenvalues through the idea of similarity and (eigen)basis. Hence, we have the following understanding:

  To *diagonalize* a linear transformation $T : V \to V$ means to find a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal. Similarly, to *diagonalize* a square matrix A means to factor A as $A = PDP^{-1}$ where D is diagonal

- Another interpretation of the lemma of eigenspace spans the vector space is :

  The n x n matrix A is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues of A is n. Equivalently, the n x n matrix A is diagonalizable if and only if the sum of the algebraic multiplicities of the eigenvalues of A is n and additionally for each eigenvalue λ of A, the algebraic and geometric multiplicities of λ are equal.

  Note: The set of eigenvalues are also called the *spectrum*

- Similar to abstract vector space we have studied before, we can apply eigenbasis/diagonalization on linear spaces.

- Suppose $\lambda_1, \ldots, \lambda_r$ are *distinct* eigenvalues of the n x n matrix A, and for each $1 \leq k \leq r$ let $\mathcal{B}_k$ be a basis of the subspace

  $$E_{\lambda_k} = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda_k \vec{v}\}$$

  Then, the set $\mathcal{B} = \mathcal{B}_1 \bigcup \cdots \bigcup \mathcal{B}_r$ is linearly independent

- Two useful theorems:

  - an n x n matrix with n distinct eigenvalues must be diagonalizable

- If A is a diagonizable n x n matrix with only one eigenvalue, then A is already a diagonal matrix

- For all similar n x n matrices A and B, they have the same
  - characteristic polynomial
  - eigenvalues
  - determinant
  - trace

- Some interesting examples given a square matrix A that is:
  - invertible but not diagonalizable $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
  - diagonalizable but not invertible $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  - neither diagonalizable nor invertible $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
  - both diagonalizable and invertible $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- every eigenvector of A belongs to $\ker(A)$ or $\text{im}(A)$

- Spectral Theorem implications
  - if λ and μ are distinct eigenvalues of the symmetric matrix A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then $\vec{v} \cdot \vec{w} = 0$.
  - If A is not invertible (columns are linearly independent), then 0 is always the eigenvalue. This follows from the fact that $E_\lambda = \ker(A - \lambda I) = \ker(A - 0I)$ if A is non-invertible. The geometric multiplicity is the dimension of the kernal, equivalently, the number of linearly dependent columns
  - If the matrix A is symmetric, we know
    - $A = Q^T DQ = Q^{-1}DQ$ for orthogonal matrix Q and diagonal matrix D consisting of eigenvalues
    - A has orthonormal eigenbasis

- Special Matrices:
  - reflections and projections are diagonalizable over $\mathbb{R}$ and rotations are diagonalizable over $\mathbb{C}$ but not necessarily $\mathbb{R}$.
  - Projection matrix have eigenvalue 0 or 1. If you project $v \in V$ onto V, then $\text{proj}_v = v$, so it has eigenvalue 1; if $v \in V^\perp$, then $\text{proj}_v = 0$, so it has eigenvalue of 0. Dimension of E_1 is the dimension of image V and the dimension of E_0 is the dimension of the Kernel. This from another perspective explaning the geometric multiplicities summing to n and always diagonalizable.

- Only non-invertible matrices have eigenvectors inside its Kernel. Generally, we may asuume all eigenvectors lie in the image.

- For any 2 x 2 matrix A the characteristic polynomial of A is

$$x^2 - tr(A)x + det(A)$$