Analysis

Gaotang Li

July 22, 2022

Abstract

Notice that since in this course, the cross-referencing between theorems, lemmas, and propositions are quite complex and hard to keep track of, hence in this note, whenever you see a **!** over =, like $\overset{!}{=}$, then that **!** is *clickable*! It will direct you to the corresponding theorem, lemma, or proposition we're using to deduce that particular equality.

Notice that there are some proofs is **intended** left as assignments, and for completeness, I put them in **??**, use it in your **own risks**! You'll lose the chance to practice and really understand the materials.

Additionally, we'll use Folland[**folland1999real**] as our main text, while using Tao[**tao2013introduction**] and Axler[**axler2019measure**] as supplementary references.

This course is taken in Winter 2022, and the date on the covering page is the last updated time.

Contents

Chapter 1

Foundations

Lecture 1: Groups and Homomorphism

22 July. 01:17

1.1 Basics

Definition 1.1.1 (Group)**.** A pair $(G, \odot)$ consisting of a nonempty set G and an operation $\odot$ is called a **group** if the following holds:

- G is closed under the operation $\odot$
- $\odot$ is associative
- $\odot$ has an identity element e
- Each $g \in G$ has an **inverse** $h \in G$ such that $g \odot h = h \odot g = e$

Definition 1.1.2 (Abelian group)**.** A group $G, \odot$ is called **commutative** or **Abelian** if $\odot$ is a commutative operation on G.

Remark. Let $G = (G, \odot)$

(a) the identity element e is unique

(b) Each $g \in G$ has a unique inverse which we denote by g^b. In particular $e^b = e$.

(c) For each $g \in G$, we have $\left(g^b\right)^b = g$.

(d) For arbitrary group elements g and h, $(g \odot h)^b = h^b \odot g^b$

Example. (a) Let $G := \{e\}$ be a one element set. Then $\{G, \odot\}$ is an Abelian group, the **trivial group**, with the (only possible) operation $e \odot e = e$.

(b) Let X be a nonempty set, and S_X be the set of all bijections from X to itself. Then $S_X := (S_X, \circ)$ is a group with identity element id_X when $\circ$ denotes the composition of functions. Further, the inverse function f^{-1} is the inverse of $f \in S_X$ in the group. When X is finite, the element of S_X are called permutations and S_X is called the **permutation group** of X.

(c) Let X be a nonempty set and $G, \odot$ a group. With the induced operation $\odot$

1.2 σ-algebras

We start from the definition of the most fundamental element in measure theory.

Definition 1.2.1 (σ-algebra)**.** Let X be a set. A collection $\mathcal{A}$ of subsets of X, i.e., $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra on X* if

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under complements. i.e., if $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$.
- $\mathcal{A}$ is closed under countable unions. i.e., if $A_i \in \mathcal{A}$, then $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{A}$.
- wd

Remark. There are some easy properties we can immediately derive.

- $X \in \mathcal{A}$ from $X = X \setminus \underbrace{\varnothing}_{\in \mathcal{A}}$ and $\mathcal{A}$ is closed under complement.
- $\bigcap\limits_{i=1}^{\infty} A_i = \left(\bigcup\limits_{i=1}^{\infty} A_i^c \right)^c$, namely $\mathcal{A}$ is closed under countable intersections.
- $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \varnothing \cup \varnothing \cup \ldots$, hence $\mathcal{A}$ is closed under finite unions and intersections.

Note. The definition of σ-algebra should remind us the definition of topological basis, and this is indeed the case. We can consider a topological space and put some structure on the σ-algebra $\mathcal{A}$, which gives us the following.