Analysis

Gaotang Li

August 3, 2022

Contents

I Foundations 3

1 Groups and Homomorphism 4
- 1.1 Basics . . . 4
- 1.2 Subgroup . . . 5
- 1.3 Cosets . . . 5
- 1.4 Homomorphisms . . . 6

2 Rings, Fields and Polynomials 8
- 2.1 Rings . . . 8
- 2.2 Consequence of Ring Definitions . . . 9
- 2.3 Field . . . 9
- 2.4 Ordered Field . . . 10
- 2.5 Formal Power Series . . . 10
- 2.6 Polynomials . . . 11

3 The Real Numbers 12
- 3.1 Order Completeness . . . 12
- 3.2 The Consequence of Order Completeness . . . 13

4 Vector Spaces, Affine Spaces and Algebras 14
- 4.1 Vector Spaces . . . 14

II Convergence 15

5 Convergence of Sequences 16
- 5.1 Sequences . . . 16
- 5.2 Metric Space . . . 16
- 5.3 Convergence . . . 17

6 Normed Vector Space 19
- 6.1 Norms . . . 19
- 6.2 The Space of Bounded Functions . . . 20
- 6.3 Inner Product Spaces . . . 21
- 6.4 Euclidean Spaces . . . 22

7 Infinite Limits 24
- 7.1 Convergence to $\pm\infty$ . . . 24
- 7.2 The Limit Superior and Limit Inferior . . . 24

8 Completeness 26
- 8.1 Cauchy Sequences . . . 26
- 8.2 Banach Spaces . . . 26

III Continuous Functions **27**

9 Continuity **28**
9.1 Elementary Properties and Examples 28
9.2 Sequential Continuity 29
9.3 Addition and Multiplication of Continuous Functions 29
9.4 One-Sided Continuity 30

Part I

Foundations

Chapter 1

Groups and Homomorphism

1.1 Basics

Definition 1.1.1 (Group). A pair $(G, \odot)$ consisting of a nonempty set G and an operation $\odot$ is called a **group** if the following holds:

- G is closed under the operation $\odot$
- $\odot$ is associative
- $\odot$ has an identity element e
- Each $g \in G$ has an **inverse** $h \in G$ such that $g \odot h = h \odot g = e$

Definition 1.1.2 (Abelian group). A group $G, \odot$ is called **commutative** or **Abelian** if $\odot$ is a commutative operation on G.

Remark. Let $G = (G, \odot)$

(a) the identity element e is unique

(b) Each $g \in G$ has a unique inverse which we denote by g^b. In particular $e^b = e$.

(c) For each $g \in G$, we have $\left(g^b\right)^b = g$.

(d) For arbitrary group elements g and h, $(g \odot h)^b = h^b \odot g^b$

Example. (a) Let $G := \{e\}$ be a one element set. Then $\{G, \odot\}$ is an Abelian group, the **trivial group**, with the (only possible) operation $e \odot e = e$.

(b) Let X be a nonempty set, and S_X be the set of all bijections from X to itself. Then $S_X := (S_X, \circ)$ is a group with identity element id_X when $\circ$ denotes the composition of functions. Further, the inverse function f^{-1} is the inverse of $f \in S_X$ in the group. When X is finite, the element of S_X are called permutations and S_X is called the **permutation group** of X.

(c) Let X be a nonempty set and $G, \odot$ a group. With the induced operation $\odot$, $\left(G^X, \odot\right)$ is a group. The inverse of $f \in G^X$ is the function

$$f^b \colon X \to G, \quad x \mapsto (f(x))^b$$

(d) Let $G_1, \ldots, G_m$ be groups. Then $G_1 \times \cdots \times G_m$ with the operation defined analogously to (d) is a group called the **direct product** of $G_1, \ldots, G_m$.

1.2 Subgroup

Definition 1.2.1 (Subgroup). Let $G = (G, \odot)$ be a group and H a nonempty subset of G, if

- $H \odot H \subseteq H$
- $h^b \in H$ for all $h \in H$

then $H := (H, \odot)$ is itself a group and is called a **subgroup** of G.

Remark. Here we use the same symbol $\odot$ for the restriction of the operation to H. Since H is nonempty, there is some $h \in H$ and so, from the two axioms above, $e = h^b \odot h$ is also in H.

Example. Let $G = (G, \odot)$ be a group.

(a) The trivial subgroup $\{e\}$ and G itself are subgroups of G, the smallest and largest subgroups with respect to inclusion

(b) If H_α, $\alpha \in A$ are subgroups of G, then $\bigcap_\alpha H_\alpha$ is also a subgroup of G.

1.3 Cosets

Definition 1.3.1 (Coset). Let N be a subgroup of G and $g \in G$. Then $g \odot N$ is the **left coset** and $N \odot g$ is the **right coset** of $g \in G$ with respect to N.

Remark. The definition of coset is related to the particular element.

Note. If we define

$$g \sim h \Leftrightarrow g \in h \odot N \tag{1.1}$$

Then $\sim$ is an equivalence on G.

Proof. $\sim$ is reflexive because $e \in N$
Let $g \in h \odot N$ and $h \in k \odot N$, then

$$g \in (k \odot N) \odot N = k \odot (N \odot N) = k \odot N$$

Let $g \in h \odot N$, then there is some $n \in N$ with $g = h \odot n$. Then it follows that $h = g \odot n^b \in N$. ■

Here 1.1 defines an equivalence relation on G. For the equivalence classes $[\cdot]$ with respect to $\sim$, we have

$$[g] = g \odot N, \quad g \in G. \tag{1.2}$$

For this reason, we denote $G/\sim$ by G/N, and call G/N the **set of left cosets** of G **modulo** N. Particularly, we have subgroups N such that

$$g \odot N = N \odot g, \quad g \in G. \tag{1.3}$$

Such a subgroup 1.3 is called a **normal subgroup** of G. We call $g \odot N$ the **coset of** g **modulo** N since each left coset is a right coset and vice versa. We have a well-defined operation on G/N where N is the normal subgroup of G, induced from $\odot$, such that

$$(G/N) \times (G/N) \to G/N, \quad (g \odot N, h \odot N) \mapsto (g \odot h) \odot N \tag{1.4}$$

Proposition 1.3.1. Let G be a group and N a normal subgroup of G. Then G/N with the induced operation is a group, the **quotient group of G modulo** N.

Proof. It is easy to check that the operation is associative. Since $(e \odot N) \odot (g \odot N) = (e \odot g) \odot N = g \odot N$, the identity element of G/N is $N = e \odot N$. Also

$$\left(g^b \odot N\right) \odot (g \odot N) = \left(g^b \odot g\right) \odot N = N$$

∎

Remark. (a) In notion of 1.1, $[e] = N$ is the identity element of G/N and $[g]^b = [g^b]$ is the inverse of $[g] \in G/N$. We also have $[g] \odot h = [g \odot h]$, $g, h \in G$.

(b) Any subgroup N of an Abelian group G is normal and so G/N is a group. Meanwhile, G/N is Abelian.

1.4 Homomorphisms

Definition 1.4.1 (Homomorphism)**.** Let $G = (G, \odot)$ and $G' = (G', \circledast)$ be groups... A function $\varphi \colon G \to G'$ is called a **(group) homomorphism** if

$$\varphi(g \odot h) = \varphi(g) \circledast \varphi(h), \quad g, h \in G$$

Definition 1.4.2 (Endomorphism)**.** A homomorphism from G to itself

Remark. (a) Let e and e' be the identity elements of G and G' respectively, and let $\varphi \colon G \to G'$ be a homomorphism. Then

$$\varphi(e) = e' \quad \text{and} \quad (\varphi(g))^b = \varphi(g^b), \quad g \in G$$

Proof. $e' \circledast \varphi(e) = \varphi(e) = \varphi(e \odot e) = \varphi(e) \circledast \varphi(e)$
$e' = \varphi(e) = \varphi\left(g^b \odot g\right) = \varphi\left(g^b\right) \circledast \varphi(g)$ ∎

(b) Let $\varphi \colon G \to G'$ be a homomorphism. The **kernel** of φ, $\ker(\varphi)$, defined by

$$\ker(\varphi) \coloneqq \varphi^{-1}(e') = \{g \in G\,;\, \varphi(g) = e'\}$$

is a normal subgroup of G.

Proof. First, try to prove $\ker(\varphi)$ is a subgroup of G. For all $g, h \in G$,

- $\varphi(g \odot h) = \varphi(g) \circledast \varphi(h) = e' \circledast e' = e'$
- $\varphi\left(g^b\right) = (\varphi(g))^b = (e')^b = e'$

Second, try to prove it is a normal subgroup. Let $h \in g \odot \ker(\varphi)$. Then we there is some $n \in G$ such that $\varphi(n) = e'$ and $h = g \odot n$. For $m \coloneqq g \odot n \odot g^b$, we have

$$\varphi(m) = \varphi(g) \circledast \varphi(n) \circledast \varphi(g^b) = \varphi(g) \circledast \varphi(g^b) = e'$$

and hence $m \in \ker(\varphi)$. Since $m \odot g = g \odot m = h$, this implies that $h \in \ker(\varphi) \odot g$. So $\ker(\varphi) \odot g \subseteq g \odot \ker(\varphi)$. Similarly one can show $g \odot \ker(\varphi) \subset \ker(\varphi) \odot g$. ∎

(c) Let $\varphi \colon G \to G'$ be a homomorphism and $N \coloneqq \ker(\varphi)$. Then

$$g \odot N = \varphi^{-1}(\varphi(g)), \quad g \in G,$$

and so

$$g \sim h \Leftrightarrow \varphi(g) = \varphi(h), \quad g, h \in G,$$

where $\sim$ denotes the equivalence relation 1.1.

(d) A homomorphism is injective if and only if its kernel is trivial, that is, $\ker(\varphi) = \{e\}$

(e) The image $\mathrm{im}(\varphi)$ of a homomorphism $\varphi\colon G \to G'$ is a subgroup of G'.

Example. (a) The constant function $G \to G'$, $g \mapsto e'$ is a homorphism, the **trivial** homomorphism.

(b) The identity function $\mathrm{id}_G\colon G \to G$ is an endomorphism.

(c) Compositions of homomorphisms (endomorphisms) are homomorphisms (endomorphisms).

(d) If $\varphi\colon G \to G'$ is a bijective homomorphism, then so is $\varphi^{-1}\colon G \to G'$

Definition 1.4.3 (Isomorphism)**.** A homomorphism $\varphi\colon G \to G'$ is called a (**group**) **isomorphism** from G to G' if φ is bijective.
In this circumstance, we say that the groups G and G' are **isomorphic** and write $G \cong G'$.

Definition 1.4.4 (Automorphism)**.** An isomorphism from G to itself.

Chapter 2

Rings, Fields and Polynomials

2.1 Rings

Definition 2.1.1 (Ring). A triple $(R, +, \cdot)$ consisting of a nonempty set R and operations, **addition** $+$ and **multiplication** $\cdot$, is called a **ring** if

- $(R, +)$ is an Abelian group
- Multiplication is associative
- The **distributive law** holds:

$$(a+b) \cdot c = a \cdot c + b \cdot, \quad c \cdot (a+b) = c \cdot a + c \cdot b, \quad a, b, c \in R$$

Note. A ring is called **commutative** if multiplication is commutative.
If there is an identity element with respect to multiplication, then it is written as 1_R or simply 1, and is called the **unity** (or **multiplicative identity**) of R, and we say $(R, +, \cdot)$ is a **ring with unity**.
When the addition and multiplication operations are clear from context, we write simply R instead of $(R, +, \cdot)$.

Example. (a) The **trivial ring** has exactly one element 0 and is itself denoted by 0. A ring with more than one element is **nontrivial**. If R is a ring with unity, then it follows from $1_R \cdot a = a$ for each $a \in R$, that R is trivial if and only if $1_R == 0_R$.

(b) Suppose R is a ring and S is a nonempty subset of R that satisfies the following:

- S is a subgroup of $(R, +)$.
- $S \cdot S \subseteq S$

Then S itself is a ring, a **subring** of R, and R is called an **overring** of S. If R is commutative then so is S, but the converse is not true in general.

(c) Intersections of subrings are subrings.

Definition 2.1.2 (Ring Homomorphism). Let R and R' be rings. A (**ring**) **homomorphism** is a function $\varphi\colon R \to R'$ which is compatible with the ring operations, that is,

$$\varphi(a+b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in R \tag{2.1}$$

Note. If, in addition, φ is bijective, then φ is called a (**ring**) **isomorphism** and R and R' are **isomorphic**.

A homomorphism φ from R to itself is a (**ring**) **endomorphism**. If φ is an isomorphism, then it is a (**ring**) **automorphism**.

Example. (a) A ring homomorphism $\varphi \colon R \to R'$ is, in particular, a group homomorphism from $(R,+)$ to $(R',+)$. The **kernel**, $\ker(\varphi)$, of φ is defined to be the kernel of this group homomorphism, that is,

$$\ker(\varphi) = \{a \in R; \varphi(a) = 0\} = \varphi^{-1}(0)$$

(b) The **zero function** $R \to R'$, $a \mapsto 0_{R'}$ is a homomorphism with $\ker(\varphi) = R$.

(c) Let R and R' be rings with unity and $\varphi \colon R \to R'$ a homomorphism. As (b) shows, it does not necessarily follow that $\varphi(1_R) = 1_{R'}$. This can be seen as a consequence of the fact that, with respect to multiplication, a ring is not a group.

2.2 Consequence of Ring Definitions

Definition 2.2.1 (The Binomial Theorem). Let a and b be two commuting elements ($ab = ba$) of a ring R with unity. Then, for all $n \in \mathbb{N}$,

$$(a+b)^b = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Lemma 2.2.1. For $m \in \mathbb{N}$ with $m \geq 2$, an element $\alpha = (\alpha_1 \dots \alpha_m) \in \mathbb{N}^m$ is called a **multi-index**. The **length** $|\alpha|$ of a multi-index $\alpha \in \mathbb{N}^m$ is defined by

$$|\alpha| \coloneqq \sum_{j=1}^{m} \alpha_j$$

Set also

$$\alpha! \coloneqq \prod_{j=1}^{m} (\alpha_j)!,$$

and define the **natural** (**partial**) **order** on $\mathbb{N}^m$ by

$$\alpha \leq \beta \rightleftarrows (\alpha_j \leq \beta_j,\ \ 1 \leq j \leq m).$$

for $a = (a_1, \dots, a_m) \in R^m$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$

Definition 2.2.2 (The Multinomial Theorem). Let R be a commutative ring with unity. Then for all $m \geq 2$,

$$(\sum_{j=1}^{m} \alpha)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^\alpha, \quad a = (a_1, \dots, a_m) \in R^m, \ \ k \in \mathbb{N}.$$

2.3 Field

Definition 2.3.1 (Field). K is a **field** when the following are satisfied:

- K is a commutative ring with unity.
- $0 \neq 1$
- $K^\times \coloneqq K \backslash \{0\}$ is an Abelian group with respect to multiplication.

Note. The Abelian group $K^\times := (K^\times, \cdot)$ is called the **multiplicative group** of K.

Remark. Let K be a field.

(a) For all $a \in K^\times$, $\left(a^{-1}\right)-1 = a$

(b) A field has no zero divisors

(c) Let $a \in K^\times$ and $a \in K^\times$ and $b \in K$. Then there is an unique $x \in K$ with $ax = b$, namely the **quotient** $\frac{b}{a} := b/a := ba^{-1}$

(d) Let K' be a field and $\varphi\colon K \to K'$ a homomorphism with $\varphi \neq 0$. Then

$$\varphi(1_K) = 1_{K'} \quad \text{and} \quad \varphi(a^{-1}) = \varphi(a)^{-1},\ a \in K^\times$$

2.4 Ordered Field

Definition 2.4.1 (Ordered Ring)**.** A ring R with an ordered $\leq$ is called an **ordered ring** if the following holds:

- $(R, \leq)$ is totally ordered.
- $x < y \Rightarrow x + z < y + z, z \in R$
- $x, y > 0 \Rightarrow xy > 0$

Note. This leads to a series of basic arithmetic rules.
We may define absolute value function from $K \mapsto K$.

Proposition 2.4.1. Let K be an ordered field and $x, y, a, \epsilon \in K$ with $\epsilon > 0$.

(i) $x = |x|\text{sign}\,(x)$, $|x| = x\text{sign}\,(x)$

(ii) $|x| = |-x|, x \leq |x|$

(iii) $|xy| = |x||y|$

(iv) $|x| \geq 0$ and $(|x| = 0 \Leftrightarrow x = 0)$

(v) $|x - a| < \epsilon \leftrightarrow a - \epsilon < x < a + \epsilon$

(vi) $|x + y| \leq |x| + |y|$ (**triangular inequality**)

Corollary 2.4.1 (reversed triangular inequality)**.** In any ordered field K we have

$$|x - y| \geq \left||x| - |y|\right|, \quad x, y \in K.$$

2.5 Formal Power Series

Definition 2.5.1 (formal power series)**.** Let R be a nontrivial ring with unity. On the set $R^{\mathbb{N}} = \text{Funct}\,(\mathbb{N}, R)$ define addition by

$$(p + q)_n := p_n + q_n, \qquad n \in \mathbb{N},$$

and multiplication by **convolution**,

$$(pq)_n := (p \cdot q)_n := \sum_{j=0}^{n} p_j q_{n-1} = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0$$

for $n \in \mathbb{N}$. Here p_n denotes the value of $p \in R^{\mathbb{N}}$ at $n \in \mathbb{N}$ and is called the n^{th} **coefficient** of p. In this situation an element $p \in R^{\mathbb{N}}$ is called a **formal power series over** R, and we set $R[X] := \left(R^{\mathbb{N}}, +, \cdot\right)$

Proposition 2.5.1. $R[X]$ is a ring with unity, the **formal power series ring over** R. If R is commutative, then so is $R[X]$

2.6 Polynomials

Definition 2.6.1 (Polynomial)**.** A **polynomial over** R is a formal power seres $p \in R[X]$ such that $\{n; p_n \neq 0\}$ is finite, in other words, $p_n = 0$ "almost everywhere".

Chapter 3

The Real Numbers

Starting words: we seek an ordered **extension field** of $\mathbb{Q}$ in which the equation $x^2 = a$ is solvable for each $a > 0$.

3.1 Order Completeness

We say a totally ordered set X is **order complete** (or X satisfies the **completeness axiom**) if every nonempty subset of X which is bounded above has a supremum.

Proposition 3.1.1. Let X be a totally ordered set. Then the followings are equivalent:

(i) X is order complete.

(ii) Every nonempty subset of X which is bounded below has an infimum.

(iii) For all nonempty subsets A, B of X such that $a \leq b$ for all $(a, b) \in A \times B$, there is some $c \in X$ such that $a \leq c \leq b$ for all $(a, b) \in A \times B$ (**Dedekind cut property**)

Note. A relation $\leq$ on X is a **partial order** on X if it is reflexive, transitive and **anti-symmetric**, that is,

$$(x \leq y)(y \leq x) \Rightarrow x = y$$

If $\leq$ is a partial order on X, then the pair $(X, \leq)$ is called a **partially ordered set**. If, in addition,

$$\forall x, y \in X : (x \leq y) \vee (y \leq x)$$

then $\leq$ is called a **total order** on X and $(X, \leq)$ is a **totally ordered set**.

Corollary 3.1.1. A totally ordered set is order complete if and only if every nonempty bounded subset has a supremum and an infimum.

Theorem 3.1.1 (Dedekind's Construction of the Real Numbers)**.** There is, up to isomorphism, a unique order complete extension field $\mathbb{R}$ of $\mathbb{Q}$. This extension is called **the field of real numbers**.

Proposition 3.1.2 (A Characterization of Supremum and Infimum)**.** Followed from natural order defined by $\mathbb{R}$.

(i) If $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then

  (a) $x < \sup(A) \Leftrightarrow \exists a \in A$ such that $x < a$.

  (b) $x < \inf(A) \Leftrightarrow \exists a \in A$ such that $x > a$.

(ii) Every subset A of $\mathbb{R}$ has a supremum and an infimum in $\mathbb{R}$

3.2 The Consequence of Order Completeness

The Archimedean Property

Proposition 3.2.1 (Archimedes). $\mathbb{N}$ is not bounded above in $\mathbb{R}$, that is, for each $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ such that $n > x$.

Corollary 3.2.1. Equivalent statements as the above proposition

(a) Let $a \in \mathbb{R}$. If $0 \leq a \leq 1/n$ for all $n \in \mathbb{N}^\times$.

(b) For each $a \in \mathbb{R}$ with $a > 0$ there is some $n \in \mathbb{N}^\times$ such that $1/n < a$.

The Density of the Rational/Irrational Numbers in $\mathbb{R}$

Proposition 3.2.2. For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $r \in \mathbb{Q}$ such that $a < r < b$.

Proposition 3.2.3 (n^{th} Roots). For all $a \in \mathbb{R}^+$ and $n \in \mathbb{N}^\times$, there is a unique $x \in \mathbb{R}^+$ such that $x^n = a$

Proposition 3.2.4. For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $\xi \in \mathbb{R} \backslash \mathbb{Q}$ such that $a < \xi < b$.

Intervals

An **interval** is a subset J of $\mathbb{R}$ such that

$$(x, y \in J, x < y) \Rightarrow (z \in J for x < z < y)$$

If J is a nonempty interval, then $\inf(J) \in \bar{\mathbb{R}}$ is the **left endpoint** and $\sup(J) \in \bar{\mathbb{R}}$ is the **right endpoint** of J. J is **closed on the left** if $a := \inf(J)$ is in J, and otherwise it is **open on the left**. The same applies to the other side.

An interval is **perfect** if it contains at least two points. It is **bounded** if both endpoints are in $\mathbf{R}$ and is **unbounded** otherwise. If J is a bounded interval, then the nonnegative number $|J| := \sup(J)$ - $\inf(J)$ is called the **length** of J.

Chapter 4

Vector Spaces, Affine Spaces and Algebras

4.1 Vector Spaces

Definition 4.1.1 (Vector Space). A **vector space over the field** K (or simply, a K**-vector space**) is a triple $(V, +, \cdot)$ consisting of a nonempty set V, an 'inner' operation $+$ on V called **addition**, and an 'outer' operation

$$K \times V \to V, \quad (\lambda, v) \mapsto \lambda \cdot v,$$

called **scalar multiplication** which satisfy the following axioms:

- $(V, +)$ is an Abelian group
- The distributive law holds:

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w, (\lambda + u) \cdot v = \lambda \cdot v + \mu \cdot v, \quad \lambda, \mu \in K, \quad v, w \in V$$

- $\lambda \cdot (\mu v) = (\lambda \mu) \cdot v$, $1 \cdot v = v$ for $\lambda, \mu \in K, v \in V$

A vector space is called **real** if $K = \mathbb{R}$ and **complex** if $K = \mathbb{C}$.

Note (Linear Functions). Let V and W be vector spaces over K. THen a function $T\colon V \mapsto W$ is $(K-)$**linear** if

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w), \quad \lambda, \mu \in K, \ v, w \in V$$

In this regard, a linear function is simply a function which is compatible with the vector space operations, in other words, it is a (**vector space**) **homomorphism**. The set of all linear functions from V to W is denoted by $\mathrm{Hom}(V, W)$ or $\mathrm{Hom}_K(V, W)$, and $\mathrm{End}(V) \coloneqq \mathrm{Hom}(V, V)$ is the set of all (vector space) **endomorphisms**. A bijective homomorphism $T \in \mathrm{Hom}(V, W)$ is a (vector space) **isomorphism**.

Remark. (a) A vector space homomorphism $T\colon V \mapsto W$ is, in particular, a group homomorphism $T\colon (V, +) \mapsto (W, +)$.

(b)

Part II

Convergence

Chapter 5

Convergence of Sequences

5.1 Sequences

Definition 5.1.1 (Sequence)**.** Let X be a set. A **sequence** (in X) is simply a function from $\mathbb{N}$ to X. If $\varphi\colon \mathbb{N} \mapsto X$ is a sequence, we write also

$$(x_n), (x_n)_{n\in\mathbb{N}} \text{ or } (x_0, x_1, x_2, \ldots)$$

for φ, where $x_n := \varphi(n)$ is the n^{th} term of the sequence $\varphi = (x_0, x_1, x_2, \ldots)$.

Remark. (a) A sequence (x_n) is different from its image $\{x_n; n \in \mathbb{N}\}$.

(b) Let (x_n) be a sequence in X and E a property. Then we say E holds for **almost all** terms of (x_n) if there is some $m \in \mathbb{N}$ such that $E(x_n)$ is true for all $n \geq m$, that is, if E holds for all but finitely many of the x_n. If there is a subset $N \subseteq \mathbb{N}$ with $\mathrm{Num}(N) = \infty$ and $E(x_n)$ is true for each $n \in N$ then E is true for **infinitely many** terms.

(c) For $m \in \mathbb{N}^\times$, a function $\Phi\colon m + \mathbb{N} \mapsto X$ is also called a sequence in X.

5.2 Metric Space

Definition 5.2.1 (Metric Space)**.** Let X be a set. A function $d\colon X \times X \mapsto \mathbb{R}^+$ is called a **metric** on X if the following hold:

- $d(x, y) = 0 \leftrightarrow x = y$.
- $d(x, y) = d(y, x)$, $x, y \in X$ (symmetry).
- $d(x, y) \leq d(x, z) + d(y, z)$, $x, y, z \in X$ (triangle inequality).

Note. If d is a metric on X, then (X, d) is called a **metric space**. We call $d(x, y)$ the **distance** between the **points** x and y in the metric space X.

In the metric space (X, d), for $a \in X$ and $r > 0$, the set

$$\mathbb{B}(a, r) := \mathbb{B}_X(a, r) := \{x \in X; d(a, x) < r\}$$

is called the **open ball** with center at a and radius r, while

$$\bar{\mathbb{B}}(a, r) := \bar{\mathbb{B}}_X(a, r) := \{x \in X; d(a, x) \leq r\}$$

is called the **closed ball** with center at a and radius r.

Example. (a) $\mathbb{K}$ is a metric space with the **natural metric**

$$\mathbb{K} \times \mathbb{K} \mapsto \mathbb{R}^+, \quad (x, y) \mapsto |x - y|$$

(b) Let (X, d) be a metric space and Y a nonempty subset of X. Then the restriction of d to $Y \times Y$, $d_Y := d|Y \times Y$, is a metric on Y, the **induced metric**, and (Y, d_Y) is a metric space, a **metric subspace** of X.

(c) Let X be a nonempty set. Then the function $d\,(x, y) := 1$ for $x \neq y$ and $d\,(x, x) := 0$ is a metric, called the **discrete metric** on X.

(d) Let (X_j, d_j), $1 \leq j \leq m$, be metric spaces and $X := X_1 \times \cdots \times X_m$. Then the function

$$d(x, y) := \underset{1 \leq j \leq m}{max}\, d_j(x_j, y_j)$$

for $x := (x_1, \ldots, x_m) \in X$ and $y := (y_1, \ldots, y_m) \in X$ is a metric on X called the **product metric**. The metric space $X := (X, d)$ is called the **product of the metric spaces** (X_j, d_j)

Proposition 5.2.1. Let (X, d) be a metric space. Then for all $x, y, z \in X$ we have

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

Note. A subset U of a metric space X is called a **neighborhood** of $a \in X$ if there is some $r > 0$ such that $\mathbb{B}\,(a, r) \subseteq U$. The **set of all neighborhoods of the point** a is denoted by $\mu\,(a)$, that is,

$$\mu(a) := \mu_X(a) := \{U \subseteq X; U \text{ is a neighborhood of } a\} \subseteq P(X)$$

Cluster Point

Definition 5.2.2 (Cluster Point)**.** We call $a \in X$ a **cluster point** of (x_n) if every neighborhood of a contains infinitely many terms of the sequence.

Proposition 5.2.2. The following are equivalent:

(i) a is a cluster point of (x_n).

(ii) For each $U \in \mu\,(a)$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in U$.

(iii) For each $\epsilon > 0$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in \mathbb{B}\,(a, \epsilon)$

5.3 Convergence

Definition 5.3.1 (Convergence)**.** A sequence (x_n) **converges** (or is **convergent**) with **limit** a if each neighborhood of a contains almost all terms of the sequence. In this case we write

$$\lim_{n \to \infty} x_n = a \text{ or } x_n \to a(n \to \infty)$$

and we say that (x_n) **converges to** a **as** n **goes to** ∞. A sequence (x_n) that is not convergent is called **divergent** and we say (x_n) **diverges**.

Proposition 5.3.1. The following statements are equivalent:

(i) $lim_{x_n} = a$.

(ii) For each $U \in \mu(a)$, there is some $N := N(U)$ such that $x_n \in U$ for all $n \geq N$.

(iii) For each $\epsilon > 0$, there is some $N := N(\epsilon)$ such that $x_n \in \mathbb{B}(a, \epsilon)$ for all $n \geq N$.

Bounded Sets

Definition 5.3.2. A subset $Y \subseteq X$ is called d-**bounded** or **bounded in** X (with respect to the metric d) if there is some $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in Y$. In this circumstance the **diameter** of Y, defined by

$$\operatorname{diam}(Y) := \sup_{x,y \in Y} d(x, y)$$

is finite. A sequence (x_n) is **bounded** if its image $\{x_n; n \in \mathbb{N}\}$ is bounded.

Proposition 5.3.2. Any convergent sequence is bounded.

Proof. Suppose that $x_n \to a$. Then there is some N such that $x_n \in \mathbb{B}(a, 1)$ for all $n \geq N$. It follows from the triangle inequality that

$$d(x_m, x_n) \leq d(x_m, a) + d(a, x_n) \leq 2, \quad m, n \in \mathbb{N}$$

Since there is also some $M \geq 0$ such that $d(x_j, x_k) \leq M$ for all $j, k \leq N$, we have $d(x_n, x_m) \leq M+2$ for all $m, n \in \mathbb{N}$. ■

Uniqueness of the Limit

Proposition 5.3.3. Let (x_n) be convergent with limit a. Then a is the unique cluster point of (x_n).

Corollary 5.3.1. The limit of a convergent sequence is unique.

Subsequence

Let $\varphi = (x_n)$ be a sequence in X and $\Phi \colon \mathbb{N} \to \mathbb{N}$ a strictly increasing function, then $\varphi \circ \Phi \in X^{\mathbb{N}}$ is called a **subsequence** of φ. Extending the notation $(x_n)_{n \in \mathbb{N}}$ introduced above for the sequence φ, we write $(x_{n_k})_{k \in \mathbb{N}}$ for the subsequence $\varphi \circ \Phi$ where $n_k := \Phi(k)$.

Proposition 5.3.4. If (x_n) is a convergent sequence with limit a, then each subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) is convergent with $\lim_{k \to \infty} x_{n_k} = a$.

Proposition 5.3.5. A point a is a cluster point of a sequence (x_n) if and only if there is some subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) which converges to a.

Chapter 6

Normed Vector Space

6.1 Norms

Definition 6.1.1 (Norm). Let E be a vector space over $\mathbb{K}$. A function $||\cdot||\colon E \to \mathbb{R}^+$ is called a **norm** if the following hold:

- $||x|| = 0 \Leftrightarrow x = 0$.
- $||\lambda x|| = |\lambda| ||x||$, $x \in E$, $\lambda \in \mathbb{K}$ (positive homogeneity)
- $||x + y|| \leq ||x|| + ||y||$, $x, y \in E$ (triangle inequality). A pair $(E, ||\cdot||)$ consisting of a vector space E and a norm $||\cdot||$ is called a **normed vector space**. If the norm is clear from context, we write E instead of $(E, ||\cdot||)$.

Remark. Let $E := (E, ||\cdot||)$ be a normed vector space.

(a) The function
$$d\colon E \times E \to \mathbb{R}^+, \quad (x, y) \mapsto ||x - y||$$
is a metric on E, the **metric induced from the norm**. Hence any normed vector space is also a metric space.

(b) The **reversed triangle inequality** holds for the norm:
$$||x - y|| \geq |||x|| - ||y|||, \quad x, y \in E$$

(c) All statements from previous chapter also hold in normed vector space.

Balls

For $a \in E$ and $r > 0$, we define the **open** and **closed balls** with center at a and radius r by
$$\mathbb{B}_E(a, r) := \mathbb{B}(a, r) := x \in E; ||x - a|| < r$$
and
$$\bar{\mathbb{B}}_E(a, r) := \bar{\mathbb{B}}_E := x \in E; ||x - a|| \leq r.$$

These definitions agree with those for the metric space (E, d) when d is induced from norm. We also write
$$\mathbb{B} := \mathbb{B}(0, 1) = x \in E; ||x|| < 1 \quad \text{and} \quad \bar{\mathbb{B}} := \bar{\mathbb{B}}(0, 1) = x \in E; ||x|| \leq 1$$
for the **open** and **closed unit balls** in E. We have
$$r\mathbb{B} = \mathbb{B}(0, r), \quad r\bar{\mathbb{B}} = \bar{\mathbb{B}}(0, r)$$

Bounded Sets

A subset X of E is called **bounded in** E (or **norm bounded**) if it is bounded in the induced metric space.

Remark. Let $E := (E, ||\cdot||)$ be a normed vector space

(a) $X \subseteq E$ is bounded if and only if there is some $r > 0$ such that $X \subseteq r\mathbb{B}$, that is, $||x|| < r$ for all $x \in X$.

(b) If X and Y are nonempty bounded subsets of E, then so are $X \cup Y$, $X + Y$ and λX with $\lambda \in \mathbb{K}$.

Example. (a) The absolute value $|\cdot|$ is a norm on the vector space $\mathbb{K}$.

(b) Let F be a subspace of a normed vector space $E := (E, \|\cdot\|)$. Then the restriction $\|\cdot\|_F := \|\cdot\||F$ of $\|\cdot\|$ to F is a norm on F. Thus $F := (F, \|\cdot\|_F)$ is a normed vector space with this **induced norm**.

(c) Let $E_j, \|\cdot\|_j$, $1 \leq j \leq m$, be normed vector space over $\mathbb{K}$. Then

$$\|x\|_\infty := \max_{1\leq j\leq m} \|x_j\|_j, \qquad x = (x_1, \ldots, x_m) \in E := E_1 \times \cdots \times E_m$$

defines a norm, called the **product norm**, on the product vector space E. The metric on E induced from this norm coincides with the product metric from 5.2 example (d), where d_j is the metric induced on E_j from $\|\cdot\|_j$.

(d) For $m \in \mathbb{N}^\times, \mathbb{K}^m$ is a normed vector space with the **maximum norm**

$$|x|_\infty := \max_{1\leq j\leq m} |x_j|, \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m.$$

In this case m = 1.

6.2 The Space of Bounded Functions

Let X be a nonempty set and $(E, \|\cdot\|)$ a normed vector space. A function $u \in E^X$ is called **bounded** if the image of u in E is bounded. For $u \in E^X$, define

$$\|u\|_\infty := \|u\|_{\infty,X} := \sup_{x\in X} \|u(x)\| \in \mathbb{R}^+ \cup \infty$$

Remark. (a) For $u \in E^X$, the followings are equivalent:

(i) u is bounded.

(ii) $u\,(X)$ is bounded in E.

(iii) There is some $r > 0$ such that $\|u\,(x)\| \leq r$ for all $x \in X$.

(iv) $\|u\|_\infty < \infty$.

(b) Clearly id $\in \mathbb{K}^\mathbb{K}$ is not bounded, that is, $\|\mathrm{id}\|_\infty = \infty$.
(b) shows that $\|\cdot\|_\infty$ may not be a norm on the vector space E^X when E is not trivial. We therefore set

$$B(X, E) := u \in E^X \ ; \ u \ \text{ is bounded}$$

and call $B\,(X, E)$ the **space of bounded functions** from X to E.

Proposition 6.2.1. $B\,(X, E)$ is a subspace of E^X and $\|\cdot\|_\infty$ is a norm, called the **supremum norm**, on $B\,(X, E)$.

Remark. (a) If $X := \mathbb{N}$, then $B(X, E)$ is the normed vector space of bounded sequences in E. In the special case $E := \mathbb{K}$, $B(\mathbb{N}, \mathbb{K})$ is denoted by ℓ_∞, that is,

$$\ell_\infty := \ell\infty(\mathbb{K}) := B(\mathbb{N}, \mathbb{K})$$

is the normed **vector space of bounded sequences** with the supremum norm

$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}}|x_n|, \quad (x_n) \in \ell_\infty$$

(b) If $X = \{1, \ldots, m\}$ for some $m \in \mathbb{N}^\times$, then

$$B(X, E) = (E^m, \|\cdot\|_\infty)$$

6.3 Inner Product Spaces

Definition 6.3.1 (Inner Product Space). Let E be a vector space over the filed $\mathbb{K}$. A function

$$(\cdot|\cdot)\colon E \times E \to \mathbb{K}, \quad (x, y) \mapsto (x|y)$$

is called a **scalar product** or **inner product** on E if the following hold:

- $(x|y) = \overline{(y|x)}$, $x, y \in E$
- $(\lambda x + \mu y|z) = \lambda\,(x|z) + \mu\,(y|z)$, $x, y, z \in E$, $\lambda, \mu \in \mathbb{K}$.
- $(x|x) \geq 0$, $x \in E$, and $(x|x) = 0 \Leftrightarrow 0$.

A vector space E with a scalar product $(\cdot|\cdot)$ is called an **inner product space** and is written in $(E, (\cdot|\cdot))$.

Remark. (a) In the real case $\mathbb{K} = \mathbb{R}$, the first point can be written as

$$(x|y) = (y|x), \quad x, y \in E$$

In other words, the function is **symmetric** when E is a real vector space. In the case $\mathbb{K} = \mathbb{C}$, the function is said to be **Hermitian** when the first point holds.

(b) From the first two points it follows that

$$(x|\lambda y + \mu z) = \bar{\lambda}(x|y) + \bar{\mu}(x|z), \quad x, y, z \in E, \quad \lambda, \mu \in \mathbb{K},$$

that is, for each fixed $x \in E$, the function $(x|\cdot) : E \to \mathbb{K}$ is **conjugate linear**. If $\mathbb{K} = \mathbb{R}$, it is **bilinear**.

(c) $(x|0) = 0$ for all $x \in E$.

Let $m \in \mathbb{N}^\times$. For $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ in $\mathbb{K}^m$, define

$$(x|y) := \sum_{j=1}^{m} x_j \bar{y}_j$$

to be the **Euclidean Inner product** on $\mathbb{K}^m$.

The Cauchy-Schwarz Inequality

Theorem 6.3.1 (Cauchy-Schwarz Inequality). Let $(E, (\cdot|\cdot))$ be an inner product space. Then

$$|(x|y)|^2 \leq (x|x)(y|y) \quad x, y \in E$$

and the equality occurs if and only if x and y are linearly dependent.

Theorem 6.3.2. Let $(E, (\cdot|\cdot))$ be an inner product space and

$$\|x\| := \sqrt{(x|x)}, \qquad x \in E$$

Then $\|\cdot\|$ is a norm on E, the **norm induced from the scalar product** $(\cdot|\cdot)$. A norm which is induced from a scalar product is also called a **Hilbert norm**.

Corollary 6.3.1. Let $(E, (\cdot|\cdot))$ be an inner product space. Then

$$|(x|y)| \leq \|x\|\|y\| \qquad x, y \in E$$

6.4 Euclidean Spaces

Convention: Unless otherwise stated, we consider $\mathbb{K}^m$ to be endowed with the Euclidean inner product $(\cdot|\cdot)$ and the induced norm

$$|x| := \sqrt{(x|x)} = \sqrt{\sum_{j=1}^{m} |x_j|^2} \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m$$

the **Euclidean norm**. In the real case, we write also $x \cdot y$ for $(x|y)$.

We further define the norm

$$|x|_1 := \sum_{j=1}^{m} |x_j|, \qquad x = (x_1, \ldots, x_m) \in \mathbb{K}^m$$

Proposition 6.4.1. Let $m \in \mathbb{N}^\times$. Then

$$|x|_\infty \leq |x| \leq \sqrt{m}|x|_\infty, \qquad \frac{1}{\sqrt{m}}|x|_1 \leq |x| \leq |x|_1, \qquad x \in \mathbb{K}^m$$

Equivalent Norm

Let E be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E are **equivalent** if there is some $K \geq 1$ such that

$$\frac{1}{K}\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1, \quad x \in E$$

In this case we write $\|\cdot\|_1 \ \|\cdot\|_2$.

Remark. (a) is an equivalence relation on the set of all norms of a fixed vector space.

(b) $\|\cdot\|_1 \ \|\cdot\| \ \|\cdot\|_\infty$ on $\mathbb{K}^m$.

(c) We write $\mathbb{B}^m$ for the **real open Euclidean unit ball**, that is, $\mathbb{B}^m := \mathbb{B}_{\mathbb{R}^m}$ and $\mathbb{B}_1^m$ and $\mathbb{B}_\infty^m$ for the unit balls in $(\mathbb{R}^m, |\cdot|_1)$ and in $(\mathbb{R}^m, |\cdot|_\infty)$ respectively. We have

$$\mathbb{B}^m \subseteq \mathbb{B}_\infty^m \subseteq \sqrt{m}\mathbb{B}^m \qquad \mathbb{B}_1^m \subseteq \mathbb{B}^m \subseteq \sqrt{m}\mathbb{B}_1^m$$

(d) Let $E = (E, \|\cdot\|)$ be a normed vector space and $\|\cdot\|_1$ a norm on E which is equivalent to $\|\cdot\|$. Set $E_1 := (E, \|\cdot\|_1)$. Then

$$\mathcal{U}_E(a) = \mathcal{U}_{E_1}(a), \qquad a \in E$$

that is, the set of neighborhoods of a depends only on the equivalence class of the norm. Equivalent norms produce the same set of neighborhoods.

Convergence in Product Spaces

Proposition 6.4.2. Let $m \in \mathbb{N}^\times$ and $x_n = \left(x_n^1, \ldots, x_n^m\right) \in \mathbb{K}^m$ for $n \in \mathbb{N}$. Then the followings are equivalent:

1. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x = \left(x^1, \ldots, x^m\right)$ in $\mathbb{K}^m$.
2. For each $k \in \{1, \ldots, m\}$, the sequence $\left(x_n^k\right)_{n\in\mathbb{N}}$ converges to x^k in $\mathbb{K}$.

Chapter 7

Infinite Limits

7.1 Convergence to $\pm\infty$

Sequences in $\mathbb{R}$ can usually be considered to converge to $+\infty$ or $-\infty$ in the extended number line $\bar{\mathbb{R}}$. A subset $U \subseteq \bar{\mathbb{R}}$ is called a **neighborhood of** ∞ (or of $-\infty$) if there is some $K > 0$ such that $(K, \infty) \subseteq U$ (or such that $(-\infty, -K) \subsetneqq U$). The set of neighborhoods of $\pm\infty$ is denoted by $\mathcal{U}(\pm\infty)$, that is,

$$\mathcal{U}(\pm\infty) := U \subseteq \bar{\mathbb{R}};\, U \text{ is neighborhood of } \pm\infty$$

Now let (x_n) be a sequence in $\mathbb{R}$. Then $\pm\infty$ is called a **cluster point** (or **limit**) of (x_n), if each neighborhood U of $\pm\infty$ contains infinitely many (or almost all) terms of (x_n). If $\pm\infty$ is the limit of (x_n), we usually write

$$\lim_{n\to\infty} x_n = \pm\infty \qquad \text{or} \qquad x_n \to \pm\infty\,(n \to \infty)$$

The sequence (x_n) **converges** in $\bar{\mathbb{R}}$ if there is some $x \in \bar{\mathbb{R}}$ such that $\lim_{n\to\infty} x_n = x$. The sequence (x_n) **diverges** in $bar\mathbb{R}$, if it does not converge in $\bar{\mathbb{R}}$. With this definition, any sequence which converge in $\mathbb{R}$, also converges in $\bar{\mathbb{R}}$, and any sequence which diverges in $\bar{\mathbb{R}}$, also diverges in $\mathbb{R}$. On the other hand there are divergent sequences in $\mathbb{R}$ which converge in $\bar{\mathbb{R}}$ (to $\pm\infty$). In this case the sequence is said to converge **improperly**.

Proposition 7.1.1. Every monotone sequence (x_n) in $\mathbb{R}$ converges in $\bar{\mathbb{R}}$, and

$$\lim x_n = \begin{cases} \sup x_n; n \in \mathbb{N}, & \text{if } (x_n) \text{ is increasing,} \\ \inf x_n; n \in \mathbb{N}, & \text{if } (x_n) \text{ is decreasing.} \end{cases}$$

7.2 The Limit Superior and Limit Inferior

Definition 7.2.1. Let (x_n) be a sequence in $\mathbb{R}$. We can define two new sequences (y_n) and (z_n) by

$$y_n := \sup_{k\geq n} x_k := \sup x_k; k \geq n,$$

$$z_n := \inf_{k\geq n} x_k := \inf x_k; k \geq n.$$

Clearly (y_n) is increasing and (z_n) is decreasing. By the above proposition, these sequences converge in $\bar{\mathbb{R}}$:

$$\limsup_{n\to\infty} x_n := \overline{\lim}_{n\to\infty} x_n := \lim_{n\to\infty}(\sup_{k\geq n} x_k).$$

the **limit superior**, and

$$\liminf_{n\to\infty} x_n := \underline{\lim}_{n\to\infty} x_n := \lim_{n\to\infty}(\inf_{k\geq n} x_k).$$

the **limit inferior** of the sequences (x_n). We also have

$$\mathrm{limsup} x_n = \inf_{n\in\mathbb{N}}(\sup_{k\geq n} x_k) \quad \text{and} \quad \mathrm{liminf} x_n = \sup_{n\in\mathbb{N}}(\inf_{k\geq n} x_k).$$

Theorem 7.2.1. Any sequence (x_n) in $\mathbb{R}$ has a smallest cluster point x_* and a greatest cluster point x^* in $\bar{\mathbb{R}}$ and these satisfy

$$\mathrm{liminf} x_n = x_* \quad \text{and} \quad \mathrm{limsup} x_n = x^*$$

Theorem 7.2.2. Let (x_n) be a sequence in $\mathbb{R}$. Then

$$(x_n) \text{ converges in } \bar{\mathbb{R}} \Leftrightarrow \overline{\lim} x_n \leq \underline{\lim} x_n$$

When the sequence converges, the limit x satisfies

$$x = \lim x_n = \underline{\lim} x_n = \overline{\lim} x_n.$$

Theorem 7.2.3 (Bolzano-Weierstrass). Every bounded sequence in $\mathbb{K}^m$ has a convergent subsequence, that is, a cluster point.

Chapter 8

Completeness

8.1 Cauchy Sequences

In the following $X = (X, d)$ is a metric space.

A sequence (x_n) in X is called a **Cauchy sequence** if, for each $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

Similarly, if (x_n) is a sequence in a normed vector space $E = (E, \|\cdot\|)$, then (x_n) is a Cauchy sequence if and only if for each $\epsilon > 0$ there is some N such that $\|x_n - x_m\| < \epsilon$ for all $m, n \geq N$. In particular, Cauchy sequences in E are "translation invariant", that is, if (x_n) is a Cauchy sequence and a is an arbitrary vector in E, then the 'translated' sequence $(x_n + a)$ is also a Cauchy sequence. This implies that Cauchy sequences cannot be defined using neighborhoods.

Proposition 8.1.1. Every convergent sequence is a Cauchy sequence.

Proposition 8.1.2. Every Cauchy sequence is bounded.

Proposition 8.1.3. If a Cauchy sequence has a convergent subsequence, then it is itself convergent.

8.2 Banach Spaces

A metric space X is called **complete** if every Cauchy sequence in X converges. A complete normed vector space is called a **Banach space**.

Theorem 8.2.1. $\mathbb{K}^m$ is a Banach space.

Theorem 8.2.2. Let X be a nonempty set and $E = (E, \|\cdot\|)$ a Banach space. Then $B(X, E)$ is also a Banach space.

Remark.
(a) A direct consequence of the previous two theorems is that For every nonempty set X, $B(X, \mathbb{R})$, $B(X, \mathbb{C})$, and $B(X, \mathbb{K}^m)$ are Banach spaces.

(b) The completeness of a normed vector space E is invariant under changes to equivalent norms.

(c) A complete inner product space is called a **Hilbert space**.

Part III

Continuous Functions

Chapter 9

Continuity

9.1 Elementary Properties and Examples

Let $f\colon X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y). Then f is **continuous** at $x_0 \in X$ if, for each neighborhood V of $f(x_0)$ in Y, there is a neighborhood U of x_0 in X such that $f(U) \subseteq V$.

Hence to prove the continuity of f at x_0, one supposes that an arbitrary neighborhood V of $f(x_0)$ is given and then shows that there is a neighborhood U of x_0 such that $f(U) \subseteq V$, that is, $f(x) \in V$ for all $x \in U$.

The function $f\colon X \to Y$ is **continuous** if it is continuous at each point of X. We say f is **discontinuous at** x_0 if f is not continuous at x_0. f is **discontinuous** if it is discontinuous at (at least) one point of X. The set of all continuous functions from X to Y is denoted $C(X, Y)$, a subset of Y^X.

Proposition 9.1.1. A function $f\colon X \to Y$ is continuous at $x_0 \in X$ if and only if, for each $\epsilon > 0$, there is some $\delta := \delta(x_0, \epsilon) > 0$ with the property that

$$d(f(x_0), f(x)) < \epsilon \text{ for all } x \in X \text{ such that } d(x_0, x) < \delta$$

Corollary 9.1.1. Let E and F be normed vector spaces and $X \subseteq E$. Then $f\colon X \to F$ is continuous at $x_0 \in X$ if and only if, for each $\epsilon > 0$, there is some $\delta := \delta(x_0, \epsilon) > 0$ satisfying that

$$\|f(x) - f(x_0)\|_F < \epsilon \quad \text{for all } x \in X \quad \text{such that } \|x - x_0\|_E < \delta$$

Example. In the following examples, X and Y are metric spaces.

(a) The square root function $\mathbb{R}^+ \to \mathbb{R}^+$, $x \to \sqrt{x}$ is continuous.

(b) The floor function $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{R}$, $x \to \lfloor x\rfloor := \max\{k \in \mathbb{Z}; k \le x\}$ is continuous at $x_0 \in \mathbb{R}\backslash\mathbb{Z}$ and discontinuous at $x_0 \in \mathbb{Z}$.

(c) The **Dirichlet function** $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases}$$

is nowhere continuous, that is, it is discontinuous at every $x_0 \in \mathbb{R}$.

(d) Suppose that $f\colon X \to \mathbb{R}$ is continuous at $x_0 \in X$ and $f(x_0) > 0$. Then there is a neighborhood U of x_0 such that $f(x) > 0$ for all $x \in U$.

(e) A function $f\colon X \to Y$ is **Lipschitz continuous** with **Lipschitz constant** $\alpha > 0$ if

$$d(f(x), f(y)) \le \alpha d(x, y), \qquad x, y \in X$$

Every Lipschitz continuous function is continuous.

(f) Any constant function $X \to Y$, $x \mapsto y_0$ is Lipschitz continuous.

(g) The identity function $\mathrm{id}\colon X \to X$, $x \mapsto x$ is Lipschitz continuous.

(h) If $E_1, \ldots, E_m$ are normed vector spaces, then $E := E_1 \times \cdots \times E_m$ is a normed vector space with respect to the product norm $\|\cdot\|_\infty$. The canonical projections

$$\mathrm{pr}_k\colon E \to E_k, \quad x = (x_1, \ldots, x_m) \mapsto x_k, \quad 1 \le k \le m,$$

are Lipschitz continuous. In particular, the projections $\mathrm{pr}_k\colon \mathbb{K}^m \to \mathbb{K}$ are Lipschitz continuous.

(i) Let E be a normed vector space. Then the norm function

$$\|\cdot\|\colon E \to \mathbb{R}, \quad x \mapsto \|x\|$$

is Lipschitz continuous.

(j) If $A \subseteq X$ and $f\colon X \to Y$ is continuous at $x_0 \in A$, then $f|A\colon A \to Y$ is continuous at x_0. Here A has the metric induced from X.

(k) Let $M \subseteq X$ be a nonempty subset of X. For each $x \in X$,

$$d(x, M) := \inf_{m \in M} d(x, m)$$

is called the **distance** from x to M. The **distance function**

$$d(\cdot, M)\colon X \to \mathbb{R}, \quad x \mapsto d(x, M)$$

is Lipschitz continuous.

(l) For any inner product space $(E, (\cdot|\cdot))$, the scalar product $(\cdot|\cdot) : E \times E \to \mathbb{K}$ is continuous.

(m) Let E and F be normed vector spaces and $X \subseteq E$. Then the continuity of $f\colon X \to F$ at $x_0 \in X$ is independent of the choice of equivalent norms on E and on F.

(n) A function f between metric spaces X and Y is **isometric** (or an **isometry**) if $d\left(f(x), f(x')\right) = d(x, x')$ for all $x, x' \in X$, that is, if f 'preserves distances'. Such a function is Lipschitz continuous and is a bijection from X to its image $f(X)$. If E and F are normed vector spaces and $T\colon E \to F$ is linear, then T is isometric if and only if $\|Tx\| = \|x\|$ for all $x \in E$. If, in addition, T is surjective then T is an **isometric isomorphism** from E to F, and T^{-1} is also isometric.

9.2 Sequential Continuity

Definition 9.2.1 (Sequential Continuity). A function $f\colon X \to Y$ between metric spaces X and Y is called **sequentially continuous** at $x \in X$, if, for every sequences (x_k) in X such that $\lim_{x_k} = x$, we have $\lim_{f(x_k)} = f(x)$.

Theorem 9.2.1 (sequence criterion). Let X, Y be metric spaces. Then a function $f\colon X \to Y$ is continuous at x if and only if it is sequentially continuous at x.

Let $f\colon X \to Y$ be a continuous function between metric spaces. Then for any convergent sequence (x_k) in X we have

$$\lim f(x_k) = f(\lim x_k)$$

That's why we say 'continuous functions respect the taking of limits'.

9.3 Addition and Multiplication of Continuous Functions

Proposition 9.3.1. Suppose that X is a metric space, F is a normed vector space, and

$$f\colon \operatorname{dom}(f) \subseteq X \to F, \quad g\colon \operatorname{dom}(g) \subseteq X \to F$$

are continuous at $x_0 \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$.

- $f + g$ and λf are continuous at x_0.
- If $F = \mathbb{K}$, then $f \cdot g$ is continuous at x_0.
- If $F = \mathbb{K}$ and $g(x_0) \neq 0$, then f/g is continuous at x_0.

Corollary 9.3.1. (i) Rational functions are continuous.

(ii) Polynomials in n variables are continuous (on $\mathbb{K}^m$).

(iii) $C(X, F)$ is a subspace of F^X, the **vector space of continuous functions** from X to F.

Theorem 9.3.1 (continuity of compositions)**.** Let X, Y and Z be metric spaces. Suppose that $f\colon X \to Y$ is continuous at $x \in X$, and $g\colon Y \to Z$ is continuous at $f(x) \in Y$. Then the composition $g \circ f\colon X \to Z$ is continuous at x.

Example. Let X be a metric space and E be a normed vector space.

Let $f\colon X \to E$ be continuous at x_0. Then the **norm of** f,

$$\|f\|\colon X \to \mathbb{R}, \quad x \mapsto \|f(x)\|,$$

is continuous at x_0.

9.4 One-Sided Continuity

Let X be a subset of $\mathbb{R}$ and $x_0 \in X$. For $\delta > 0$, the set $X \cap (x_0 - \delta, x_0]$ (or $X \cap [x_0, x_0 + \delta)$) is called a **left** (or **right**) **δ-neighborhood** of x_0.

Let Y be a metric space. Then $f\colon X \to Y$ is **left** (or **right**) **continuous** at x_0, if, for each neighborhood V of $f(x_0)$ in Y, there is some $\delta > 0$ such that $f(X \cap (x_0 - \delta, x_0]) \subseteq V$ (or $f(X \cap [x_0, x_0 + \delta)) \subseteq V$).