

Analysis

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Part I

Foundations

Chapter 1

Groups and Homomorphism

1.1 Basics

Definition 1.1.1 (Group). A pair (G, \odot) consisting of a nonempty set G and an operation \odot is called a **group** if the following holds:

- G is closed under the operation \odot
- \odot is associative
- \odot has an identity element e
- Each $g \in G$ has an **inverse** $h \in G$ such that $g \odot h = h \odot g = e$

Definition 1.1.2 (Abelian group). A group G, \odot is called **commutative** or **Abelian** if \odot is a commutative operation on G .

Remark. Let $G = (G, \odot)$

- (a) the identity element e is unique
- (b) Each $g \in G$ has a unique inverse which we denote by g^b . In particular $e^b = e$.
- (c) For each $g \in G$, we have $(g^b)^b = g$.
- (d) For arbitrary group elements g and h , $(g \odot h)^b = h^b \odot g^b$

Example. (a) Let $G := \{e\}$ be a one element set. Then $\{G, \odot\}$ is an Abelian group, the **trivial group**, with the (only possible) operation $e \odot e = e$.

(b) Let X be a nonempty set, and S_X be the set of all bijections from X to itself. Then $S_X := (S_X, \circ)$ is a group with identity element id_X when \circ denotes the composition of functions. Further, the inverse function f^{-1} is the inverse of $f \in S_X$ in the group. When X is finite, the element of S_X are called permutations and S_X is called the **permutation group** of X .

(c) Let X be a nonempty set and G, \odot a group. With the induced operation \odot , (G^X, \odot) is a group. The inverse of $f \in G^X$ is the function

$$f^b: X \rightarrow G, \quad x \mapsto (f(x))^b$$

(d) Let G_1, \dots, G_m be groups. Then $G_1 \times \dots \times G_m$ with the operation defined analogously to (d) is a group called the **direct product** of G_1, \dots, G_m .

1.2 Subgroup

Definition 1.2.1 (Subgroup). Let $G = (G, \odot)$ be a group and H a nonempty subset of G , if

- $H \odot H \subseteq H$
- $h^b \in H$ for all $h \in H$

then $H := (H, \odot)$ is itself a group and is called a **subgroup** of G .

Remark. Here we use the same symbol \odot for the restriction of the operation to H . Since H is nonempty, there is some $h \in H$ and so, from the two axioms above, $e = h^b \odot h$ is also in H .

Example. Let $G = (G, \odot)$ be a group.

- The trivial subgroup $\{e\}$ and G itself are subgroups of G , the smallest and largest subgroups with respect to inclusion
- If H_α , $\alpha \in A$ are subgroups of G , then $\bigcap_\alpha H_\alpha$ is also a subgroup of G .

1.3 Cosets

Definition 1.3.1 (Coset). Let N be a subgroup of G and $g \in G$. Then $g \odot N$ is the **left coset** and $N \odot g$ is the **right coset** of $g \in G$ with respect to N .

Remark. The definition of coset is related to the particular element.

Note. If we define

$$g \sim h \Leftrightarrow g \in h \odot N \quad (1.1)$$

Then \sim is an equivalence on G .

Proof. \sim is reflexive because $e \in N$

Let $g \in h \odot N$ and $h \in k \odot N$, then

$$g \in (k \odot N) \odot N = k \odot (N \odot N) = k \odot N$$

Let $g \in h \odot N$, then there is some $n \in N$ with $g = h \odot n$. Then it follows that $h = g \odot n^b \in N$. ■

Here 1.1 defines an equivalence relation on G . For the equivalence classes $[\cdot]$ with respect to \sim , we have

$$[g] = g \odot N, \quad g \in G. \quad (1.2)$$

For this reason, we denote G/\sim by G/N , and call G/N the **set of left cosets** of G **modulo** N . Particularly, we have subgroups N such that

$$g \odot N = N \odot g, \quad g \in G. \quad (1.3)$$

Such a subgroup 1.3 is called a **normal subgroup** of G . We call $g \odot N$ the **coset of g modulo N** since each left coset is a right coset and vice versa. We have a well-defined operation on G/N where N is the normal subgroup of G , induced from \odot , such that

$$(G/N) \times (G/N) \rightarrow G/N, \quad (g \odot N, h \odot N) \mapsto (g \odot h) \odot N \quad (1.4)$$

Proposition 1.3.1. Let G be a group and N a normal subgroup of G . Then G/N with the induced

operation is a group, the **quotient group of G modulo N** .

Proof. It is easy to check that the operation is associative. Since $(e \odot N) \odot (g \odot N) = (e \odot g) \odot N = g \odot N$, the identity element of G/N is $N = e \odot N$. Also

$$(g^b \odot N) \odot (g \odot N) = (g^b \odot g) \odot N = N$$

■

Remark. (a) In notion of 1.1, $[e] = N$ is the identity element of G/N and $[g]^b = [g^b]$ is the inverse of $[g] \in G/N$. We also have $[g] \odot h = [g \odot h]$, $g, h \in G$.

(b) Any subgroup N of an Abelian group G is normal and so G/N is a group. Meanwhile, G/N is Abelian.

1.4 Homomorphisms

Definition 1.4.1 (Homomorphism). Let $G = (G, \odot)$ and $G' = (G', \otimes)$ be groups... A function $\varphi: G \rightarrow G'$ is called a **(group) homomorphism** if

$$\varphi(g \odot h) = \varphi(g) \otimes \varphi(h), \quad g, h \in G$$

Definition 1.4.2 (Endomorphism). A homomorphism from G to itself

Remark. (a) Let e and e' be the identity elements of G and G' respectively, and let $\varphi: G \rightarrow G'$ be a homomorphism. Then

$$\varphi(e) = e' \quad \text{and} \quad (\varphi(g))^b = \varphi(g^b), \quad g \in G$$

Proof. $e' \otimes \varphi(e) = \varphi(e) = \varphi(e \odot e) = \varphi(e) \otimes \varphi(e)$
 $e' = \varphi(e) = \varphi(g^b \odot g) = \varphi(g^b) \otimes \varphi(g)$

■

(b) Let $\varphi: G \rightarrow G'$ be a homomorphism. The **kernel** of φ , $\ker(\varphi)$, defined by

$$\ker(\varphi) := \varphi^{-1}(e') = \{g \in G; \varphi(g) = e'\}$$

is a normal subgroup of G .

Proof. First, try to prove $\ker(\varphi)$ is a subgroup of G . For all $g, h \in G$,

- $\varphi(g \odot h) = \varphi(g) \otimes \varphi(h) = e' \otimes e' = e'$
- $\varphi(g^b) = (\varphi(g))^b = (e')^b = e'$

Second, try to prove it is a normal subgroup. Let $h \in g \odot \ker(\varphi)$. Then we there is some $n \in G$ such that $\varphi(n) = e'$ and $h = g \odot n$. For $m := g \odot n \odot g^b$, we have

$$\varphi(m) = \varphi(g) \otimes \varphi(n) \otimes \varphi(g^b) = \varphi(g) \otimes \varphi(g^b) = e'$$

and hence $m \in \ker(\varphi)$. Since $m \odot g = g \odot m = h$, this implies that $h \in \ker(\varphi) \odot g$. So $\ker(\varphi) \odot g \subseteq g \odot \ker(\varphi)$. Similarly one can show $g \odot \ker(\varphi) \subseteq \ker(\varphi) \odot g$. ■

(c) Let $\varphi: G \rightarrow G'$ be a homomorphism and $N := \ker(\varphi)$. Then

$$g \odot N = \varphi^{-1}(\varphi(g)), \quad g \in G,$$

and so

$$g \sim h \Leftrightarrow \varphi(g) = \varphi(h), \quad g, h \in G,$$

where \sim denotes the equivalence relation 1.1.

- (d) A homomorphism is injective if and only if its kernel is trivial, that is, $\ker(\varphi) = \{e\}$
- (e) The image $\text{im}(\varphi)$ of a homomorphism $\varphi: G \rightarrow G'$ is a subgroup of G' .

Example. (a) The constant function $G \rightarrow G', g \mapsto e'$ is a homomorphism, the **trivial** homomorphism.

(b) The identity function $\text{id}_G: G \rightarrow G$ is an endomorphism.

(c) Compositions of homomorphisms (endomorphisms) are homomorphisms (endomorphisms).

(d) If $\varphi: G \rightarrow G'$ is a bijective homomorphism, then so is $\varphi^{-1}: G' \rightarrow G$.

Definition 1.4.3 (Isomorphism). A homomorphism $\varphi: G \rightarrow G'$ is called a **(group) isomorphism** from G to G' if φ is bijective.

In this circumstance, we say that the groups G and G' are **isomorphic** and write $G \cong G'$.

Definition 1.4.4 (Automorphism). An isomorphism from G to itself.

Chapter 2

Rings, Fields and Polynomials

2.1 Rings

Definition 2.1.1 (Ring). A triple $(R, +, \cdot)$ consisting of a nonempty set R and operations, **addition** $+$ and **multiplication** \cdot , is called a **ring** if

- $(R, +)$ is an Abelian group
- Multiplication is associative
- The **distributive law** holds:

$$(a + b) \cdot c = a \cdot c + b \cdot c, \quad c \cdot (a + b) = c \cdot a + c \cdot b, \quad a, b, c \in R$$

Note. A ring is called **commutative** if multiplication is commutative.

If there is an identity element with respect to multiplication, then it is written as 1_R or simply 1 , and is called the **unity** (or **multiplicative identity**) of R , and we say $(R, +, \cdot)$ is a **ring with unity**.

When the addition and multiplication operations are clear from context, we write simply R instead of $(R, +, \cdot)$.

Example. (a) The **trivial ring** has exactly one element 0 and is itself denoted by 0 . A ring with more than one element is **nontrivial**. If R is a ring with unity, then it follows from $1_R \cdot a = a$ for each $a \in R$, that R is trivial if and only if $1_R = 0_R$.

(b) Suppose R is a ring and S is a nonempty subset of R that satisfies the following:

- S is a subgroup of $(R, +)$.
- $S \cdot S \subseteq S$

Then S itself is a ring, a **subring** of R , and R is called an **overring** of S . If R is commutative then so is S , but the converse is not true in general.

(c) Intersections of subrings are subrings.

Definition 2.1.2 (Ring Homomorphism). Let R and R' be rings. A **(ring) homomorphism** is a function $\varphi: R \rightarrow R'$ which is compatible with the ring operations, that is,

$$\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in R \quad (2.1)$$

Note. If, in addition, φ is bijective, then φ is called a **(ring) isomorphism** and R and R' are **isomorphic**.

A homomorphism φ from R to itself is a **(ring) endomorphism**. If φ is an isomorphism, then it is a **(ring) automorphism**.

Example. (a) A ring homomorphism $\varphi: R \rightarrow R'$ is, in particular, a group homomorphism from $(R, +)$ to $(R', +)$. The **kernel**, $\ker(\varphi)$, of φ is defined to be the kernel of this group homomorphism, that is,

$$\ker(\varphi) = \{a \in R; \varphi(a) = 0\} = \varphi^{-1}(0)$$

(b) The **zero function** $R \rightarrow R'$, $a \mapsto 0_{R'}$ is a homomorphism with $\ker(\varphi) = R$.

(c) Let R and R' be rings with unity and $\varphi: R \rightarrow R'$ a homomorphism. As (b) shows, it does not necessarily follow that $\varphi(1_R) = 1_{R'}$. This can be seen as a consequence of the fact that, with respect to multiplication, a ring is not a group.

2.2 Consequence of Ring Definitions

Definition 2.2.1 (The Binomial Theorem). Let a and b be two commuting elements ($ab = ba$) of a ring R with unity. Then, for all $n \in \mathbb{N}$,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Lemma 2.2.1. For $m \in \mathbb{N}$ with $m \geq 2$, an element $\alpha = (\alpha_1 \dots \alpha_m) \in \mathbb{N}^m$ is called a **multi-index**. The **length** $|\alpha|$ of a multi-index $\alpha \in \mathbb{N}^m$ is defined by

$$|\alpha| := \sum_{j=1}^m \alpha_j$$

Set also

$$\alpha! := \prod_{j=1}^m (\alpha_j)!,$$

and define the **natural (partial) order** on \mathbb{N}^m by

$$\alpha \leq \beta \Leftrightarrow (\alpha_j \leq \beta_j, \ 1 \leq j \leq m).$$

for $a = (a_1, \dots, a_m) \in R^m$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$

Definition 2.2.2 (The Multinomial Theorem). Let R be a commutative ring with unity. Then for all $m \geq 2$,

$$\left(\sum_{j=1}^m a_j\right)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^\alpha, \quad a = (a_1, \dots, a_m) \in R^m, \ k \in \mathbb{N}.$$

2.3 Field

Definition 2.3.1 (Field). K is a **field** when the following are satisfied:

- K is a commutative ring with unity.
- $0 \neq 1$
- $K^\times := K \setminus \{0\}$ is an Abelian group with respect to multiplication.

Note. The Abelian group $K^\times := (K^\times, \cdot)$ is called the **multiplicative group** of K .

Remark. Let K be a field.

- (a) For all $a \in K^\times$, $(a^{-1})^{-1} = a$
- (b) A field has no zero divisors
- (c) Let $a \in K^\times$ and $b \in K$. Then there is a unique $x \in K$ with $ax = b$, namely the **quotient** $\frac{b}{a} := b/a := ba^{-1}$
- (d) Let K' be a field and $\varphi: K \rightarrow K'$ a homomorphism with $\varphi \neq 0$. Then

$$\varphi(1_K) = 1_{K'} \quad \text{and} \quad \varphi(a^{-1}) = \varphi(a)^{-1}, \quad a \in K^\times$$

2.4 Ordered Field

Definition 2.4.1 (Ordered Ring). A ring R with an ordered \leq is called an **ordered ring** if the following holds:

- (R, \leq) is totally ordered.
- $x < y \Rightarrow x + z < y + z, z \in R$
- $x, y > 0 \Rightarrow xy > 0$

Note. This leads to a series of basic arithmetic rules.

We may define absolute value function from $K \mapsto K$.

Proposition 2.4.1. Let K be an ordered field and $x, y, a, \epsilon \in K$ with $\epsilon > 0$.

- (i) $x = |x|\text{sign}(x)$, $|x| = x\text{sign}(x)$
- (ii) $|x| = |-x|, x \leq |x|$
- (iii) $|xy| = |x||y|$
- (iv) $|x| \geq 0$ and $(|x| = 0 \Leftrightarrow x = 0)$
- (v) $|x - a| < \epsilon \Leftrightarrow a - \epsilon < x < a + \epsilon$
- (vi) $|x + y| \leq |x| + |y|$ (**triangular inequality**)

Corollary 2.4.1 (reversed triangular inequality). In any ordered field K we have

$$|x - y| \geq ||x| - |y||, \quad x, y \in K.$$

2.5 Formal Power Series

Definition 2.5.1 (formal power series). Let R be a nontrivial ring with unity. On the set $R^\mathbb{N} = \text{Func}(\mathbb{N}, R)$ define addition by

$$(p + q)_n := p_n + q_n, \quad n \in \mathbb{N},$$

and multiplication by **convolution**,

$$(pq)_n := (p \cdot q)_n := \sum_{j=0}^n p_j q_{n-j} = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0$$

for $n \in \mathbb{N}$. Here p_n denotes the value of $p \in R^{\mathbb{N}}$ at $n \in \mathbb{N}$ and is called the n^{th} **coefficient** of p . In this situation an element $p \in R^{\mathbb{N}}$ is called a **formal power series over R** , and we set $R[X] := (R^{\mathbb{N}}, +, \cdot)$

Proposition 2.5.1. $R[X]$ is a ring with unity, the **formal power series ring over R** . If R is commutative, then so is $R[X]$

2.6 Polynomials

Definition 2.6.1 (Polynomial). A **polynomial over R** is a formal power series $p \in R[X]$ such that $\{n; p_n \neq 0\}$ is finite, in other words, $p_n = 0$ "almost everywhere".

Chapter 3

The Real Numbers

Starting words: we seek an ordered **extension field** of \mathbb{Q} in which the equation $x^2 = a$ is solvable for each $a > 0$.

3.1 Order Completeness

We say a totally ordered set X is **order complete** (or X satisfies the **completeness axiom**) if every nonempty subset of X which is bounded above has a supremum.

Proposition 3.1.1. Let X be a totally ordered set. Then the followings are equivalent:

- (i) X is order complete.
- (ii) Every nonempty subset of X which is bounded below has an infimum.
- (iii) For all nonempty subsets A, B of X such that $a \leq b$ for all $(a, b) \in A \times B$, there is some $c \in X$ such that $a \leq c \leq b$ for all $(a, b) \in A \times B$ (**Dedekind cut property**)

Note. A relation \leq on X is a **partial order** on X if it is reflexive, transitive and **anti-symmetric**, that is,

$$(x \leq y)(y \leq x) \Rightarrow x = y$$

If \leq is a partial order on X , then the pair (X, \leq) is called a **partially ordered set**. If, in addition,

$$\forall x, y \in X: (x \leq y) \vee (y \leq x)$$

then \leq is called a **total order** on X and (X, \leq) is a **totally ordered set**.

Corollary 3.1.1. A totally ordered set is order complete if and only if every nonempty bounded subset has a supremum and an infimum.

Theorem 3.1.1 (Dedekind's Construction of the Real Numbers). There is, up to isomorphism, a unique order complete extension field \mathbb{R} of \mathbb{Q} . This extension is called **the field of real numbers**.

Proposition 3.1.2 (A Characterization of Supremum and Infimum). Followed from natural order defined by \mathbb{R} .

- (i) If $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then
 - (a) $x < \sup(A) \Leftrightarrow \exists a \in A$ such that $x < a$.
 - (b) $x < \inf(A) \Leftrightarrow \exists a \in A$ such that $x > a$.

(ii) Every subset A of \mathbb{R} has a supremum and an infimum in \mathbb{R}

3.2 The Consequence of Order Completeness

The Archimedean Property

Proposition 3.2.1 (Archimedes). \mathbb{N} is not bounded above in \mathbb{R} , that is, for each $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ such that $n > x$.

Corollary 3.2.1. Equivalent statements as the above proposition

- (a) Let $a \in \mathbb{R}$. If $0 \leq a \leq 1/n$ for all $n \in \mathbb{N}^\times$.
- (b) For each $a \in \mathbb{R}$ with $a > 0$ there is some $n \in \mathbb{N}^\times$ such that $1/n < a$.

The Density of the Rational/Irrational Numbers in \mathbb{R}

Proposition 3.2.2. For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $r \in \mathbb{Q}$ such that $a < r < b$.

Proposition 3.2.3 (n^{th} Roots). For all $a \in \mathbb{R}^+$ and $n \in \mathbb{N}^\times$, there is a unique $x \in \mathbb{R}^+$ such that $x^n = a$

Proposition 3.2.4. For all $a, b \in \mathbb{R}$ such that $a < b$, there is some $\xi \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < \xi < b$.

Intervals

An **interval** is a subset J of \mathbb{R} such that

$$(x, y \in J, x < y) \Rightarrow (z \in J \text{ for } x < z < y)$$

If J is a nonempty interval, then $\inf(J) \in \bar{\mathbb{R}}$ is the **left endpoint** and $\sup(J) \in \bar{\mathbb{R}}$ is the **right endpoint** of J . J is **closed on the left** if $a := \inf(J)$ is in J , and otherwise it is **open on the left**. The same applies to the other side.

An interval is **perfect** if it contains at least two points. It is **bounded** if both endpoints are in \mathbb{R} and is **unbounded** otherwise. If J is a bounded interval, then the nonnegative number $|J| := \sup(J) - \inf(J)$ is called the **length** of J .

Chapter 4

Vector Spaces, Affine Spaces and Algebras

4.1 Vector Spaces

Definition 4.1.1 (Vector Space). A **vector space over the field K** (or simply, a **K -vector space**) is a triple $(V, +, \cdot)$ consisting of a nonempty set V , an 'inner' operation $+$ on V called **addition**, and an 'outer' operation

$$K \times V \rightarrow V, \quad (\lambda, v) \mapsto \lambda \cdot v,$$

called **scalar multiplication** which satisfy the following axioms:

- $(V, +)$ is an Abelian group
- The distributive law holds:

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w, (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v, \quad \lambda, \mu \in K, \quad v, w \in V$$

- $\lambda \cdot (\mu v) = (\lambda\mu) \cdot v, 1 \cdot v = v$ for $\lambda, \mu \in K, v \in V$

A vector space is called **real** if $K = \mathbb{R}$ and **complex** if $K = \mathbb{C}$.

Note (Linear Functions). Let V and W be vector spaces over K . Then a function $T: V \mapsto W$ is **(K -)linear** if

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w), \quad \lambda, \mu \in K, \quad v, w \in V$$

In this regard, a linear function is simply a function which is compatible with the vector space operations, in other words, it is a **(vector space) homomorphism**. The set of all linear functions from V to W is denoted by $\text{Hom}(V, W)$ or $\text{Hom}_K(V, W)$, and $\text{End}(V) := \text{Hom}(V, V)$ is the set of all (vector space) **endomorphisms**. A bijective homomorphism $T \in \text{Hom}(V, W)$ is a (vector space) **isomorphism**.

Remark. (a) A vector space homomorphism $T: V \mapsto W$ is, in particular, a group homomorphism $T: (V, +) \mapsto (W, +)$.

(b)

Part II

Convergence

Chapter 5

Convergence of Sequences

5.1 Sequences

Definition 5.1.1 (Sequence). Let X be a set. A **sequence** (in X) is simply a function from \mathbb{N} to X . If $\varphi: \mathbb{N} \mapsto X$ is a sequence, we write also

$$(x_n), (x_n)_{n \in \mathbb{N}} \text{ or } (x_0, x_1, x_2, \dots)$$

for φ , where $x_n := \varphi(n)$ is the n^{th} term of the sequence $\varphi = (x_0, x_1, x_2, \dots)$.

Remark. (a) A sequence (x_n) is different from its image $\{x_n; n \in \mathbb{N}\}$.

(b) Let (x_n) be a sequence in X and E a property. Then we say E holds for **almost all** terms of (x_n) if there is some $m \in \mathbb{N}$ such that $E(x_n)$ is true for all $n \geq m$, that is, if E holds for all but finitely many of the x_n . If there is a subset $N \subseteq \mathbb{N}$ with $\text{Num}(N) = \infty$ and $E(x_n)$ is true for each $n \in N$ then E is true for **infinitely many** terms.

(c) For $m \in \mathbb{N}^\times$, a function $\Phi: m + \mathbb{N} \mapsto X$ is also called a sequence in X .

5.2 Metric Space

Definition 5.2.1 (Metric Space). Let X be a set. A function $d: X \times X \mapsto \mathbb{R}^+$ is called a **metric** on X if the following hold:

- $d(x, y) = 0 \leftrightarrow x = y$.
- $d(x, y) = d(y, x)$, $x, y \in X$ (symmetry).
- $d(x, y) \leq d(x, z) + d(y, z)$, $x, y, z \in X$ (triangle inequality).

Note. If d is a metric on X , then (X, d) is called a **metric space**. We call $d(x, y)$ the **distance** between the **points** x and y in the metric space X .

In the metric space (X, d) , for $a \in X$ and $r > 0$, the set

$$\mathbb{B}(a, r) := \mathbb{B}_X(a, r) := \{x \in X; d(a, x) < r\}$$

is called the **open ball** with center at a and radius r , while

$$\bar{\mathbb{B}}(a, r) := \bar{\mathbb{B}}_X(a, r) := \{x \in X; d(a, x) \leq r\}$$

is called the **closed ball** with center at a and radius r .

Example. (a) \mathbb{K} is a metric space with the **natural metric**

$$\mathbb{K} \times \mathbb{K} \mapsto \mathbb{R}^+, \quad (x, y) \mapsto |x - y|$$

- (b) Let (X, d) be a metric space and Y a nonempty subset of X . Then the restriction of d to $Y \times Y$, $d_Y := d|_{Y \times Y}$, is a metric on Y , the **induced metric**, and (Y, d_Y) is a metric space, a **metric subspace** of X .
- (c) Let X be a nonempty set. Then the function $d(x, y) := 1$ for $x \neq y$ and $d(x, x) := 0$ is a metric, called the **discrete metric** on X .
- (d) Let (X_j, d_j) , $1 \leq j \leq m$, be metric spaces and $X := X_1 \times \cdots \times X_m$. Then the function

$$d(x, y) := \max_{1 \leq j \leq m} d_j(x_j, y_j)$$

for $x := (x_1, \dots, x_m) \in X$ and $y := (y_1, \dots, y_m) \in X$ is a metric on X called the **product metric**. The metric space $X := (X, d)$ is called the **product of the metric spaces** (X_j, d_j)

Proposition 5.2.1. Let (X, d) be a metric space. Then for all $x, y, z \in X$ we have

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

Note. A subset U of a metric space X is called a **neighborhood** of $a \in X$ if there is some $r > 0$ such that $\mathbb{B}(a, r) \subseteq U$. The **set of all neighborhoods of the point** a is denoted by $\mu(a)$, that is,

$$\mu(a) := \mu_X(a) := \{U \subseteq X; U \text{ is a neighborhood of } a\} \subseteq P(X)$$

Cluster Point

Definition 5.2.2 (Cluster Point). We call $a \in X$ a **cluster point** of (x_n) if every neighborhood of a contains infinitely many terms of the sequence.

Proposition 5.2.2. The following are equivalent:

- (i) a is a cluster point of (x_n) .
- (ii) For each $U \in \mu(a)$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in U$.
- (iii) For each $\epsilon > 0$ and $m \in \mathbb{N}$, there is some $n \geq m$ such that $x_n \in \mathbb{B}(a, \epsilon)$

5.3 Convergence

Definition 5.3.1 (Convergence). A sequence (x_n) **converges** (or is **convergent**) with **limit** a if each neighborhood of a contains almost all terms of the sequence. In this case we write

$$\lim_{n \rightarrow \infty} x_n = a \text{ or } x_n \rightarrow a (n \rightarrow \infty)$$

and we say that (x_n) **converges to a as n goes to ∞** . A sequence (x_n) that is not convergent is called **divergent** and we say (x_n) **diverges**.

Proposition 5.3.1. The following statements are equivalent:

- (i) $\lim_{n \rightarrow \infty} x_n = a$.

- (ii) For each $U \in \mu(a)$, there is some $N := N(U)$ such that $x_n \in U$ for all $n \geq N$.
- (iii) For each $\epsilon > 0$, there is some $N := N(\epsilon)$ such that $x_n \in \mathbb{B}(a, \epsilon)$ for all $n \geq N$.

Bounded Sets

Definition 5.3.2. A subset $Y \subseteq X$ is called **d-bounded** or **bounded in** X (with respect to the metric d) if there is some $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in Y$. In this circumstance the **diameter** of Y , defined by

$$\text{diam}(Y) := \sup_{x, y \in Y} d(x, y)$$

is finite. A sequence (x_n) is **bounded** if its image $\{x_n; n \in \mathbb{N}\}$ is bounded.

Proposition 5.3.2. Any convergent sequence is bounded.

Proof. Suppose that $x_n \rightarrow a$. Then there is some N such that $x_n \in \mathbb{B}(a, 1)$ for all $n \geq N$. It follows from the triangle inequality that

$$d(x_m, x_n) \leq d(x_m, a) + d(a, x_n) \leq 2, \quad m, n \in \mathbb{N}$$

Since there is also some $M \geq 0$ such that $d(x_j, x_k) \leq M$ for all $j, k \leq N$, we have $d(x_n, x_m) \leq M + 2$ for all $m, n \in \mathbb{N}$. ■

Uniqueness of the Limit

Proposition 5.3.3. Let (x_n) be convergent with limit a . Then a is the unique cluster point of (x_n) .

Corollary 5.3.1. The limit of a convergent sequence is unique.

Subsequence

Let $\varphi = (x_n)$ be a sequence in X and $\Phi: \mathbb{N} \rightarrow \mathbb{N}$ a strictly increasing function, then $\varphi \circ \Phi \in X^{\mathbb{N}}$ is called a **subsequence** of φ . Extending the notation $(x_n)_{n \in \mathbb{N}}$ introduced above for the sequence φ , we write $(x_{n_k})_{k \in \mathbb{N}}$ for the subsequence $\varphi \circ \Phi$ where $n_k := \Phi(k)$.

Proposition 5.3.4. If (x_n) is a convergent sequence with limit a , then each subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) is convergent with $\lim_{k \rightarrow \infty} x_{n_k} = a$.

Proposition 5.3.5. A point a is a cluster point of a sequence (x_n) if and only if there is some subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) which converges to a .

Chapter 6

Normed Vector Space

6.1 Norms

Definition 6.1.1 (Norm). Let E be a vector space over \mathbb{K} . A function $\|\cdot\|: E \rightarrow \mathbb{R}^+$ is called a **norm** if the following hold:

- $\|x\| = 0 \Leftrightarrow x = 0$.
- $\|\lambda x\| = |\lambda| \|x\|$, $x \in E$, $\lambda \in \mathbb{K}$ (positive homogeneity)
- $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in E$ (triangle inequality). A pair $(E, \|\cdot\|)$ consisting of a vector space E and a norm $\|\cdot\|$ is called a **normed vector space**. If the norm is clear from context, we write E instead of $(E, \|\cdot\|)$.

Remark. Let $E := (E, \|\cdot\|)$ be a normed vector space.

- (a) The function

$$d: E \times E \rightarrow \mathbb{R}^+, \quad (x, y) \mapsto \|x - y\|$$

is a metric on E , the **metric induced from the norm**. Hence any normed vector space is also a metric space.

- (b) The **reversed triangle inequality** holds for the norm:

$$\|x - y\| \geq \left| \|x\| - \|y\| \right|, \quad x, y \in E$$

- (c) All statements from previous chapter also hold in normed vector space.