

Production Networks

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Background

- ▶ Goods and services are produced through supply chains.
 - ▶ Coal is used in steel production; steel is used in truck production; trucks are used in postal services (sector-to-sector).
 - ▶ Hyundai sells battery packs to Volkswagen; Volkswagen sells busses to a bus company (firm-to-firm).
- ▶ How are such production networks formed?
- ▶ Related questions:¹
 - ▶ How does a shock in a particular sector/firm transmit through production networks?
 - ▶ Anticipating geopolitical risks, how should a country form supply chain networks with foreign countries?

¹These are not addressed in the model we will focus on, though.

Literature (1)

- ▶ There is a massive and still growing literature on production networks.
- ▶ Classic
 - ▶ Hulten (1978)
- ▶ Sector-to-sector
 - ▶ Baqaee and Farhi (2019a), Baqaee and Farhi (2024), Baqaee (2018), Baqaee and Farhi (2019b)
 - ▶ Acemoglu and Azar (2020), Acemoglu et al. (2012)
 - ▶ Kopytov et al. (2024): the one we will study here
 - ▶ Liu (2019), Liu and Tsyvinski (2024)

Literature (2)

- ▶ Firm-to-firm
 - ▶ Discrete²
 - ▶ Dhyne et al. (2023), Dhyne et al. (2022), Carvalho et al. (2020)
 - ▶ Continuous³
 - ▶ Lim (2018), Miyauchi (2024), Huneus (2020), Eaton et al. (2023)
 - ▶ Empirics
 - ▶ Dhyne et al. (2020), Bernard et al. (2019)⁴, Baqaee et al. (2023)
 - ▶ Pure theory (or mainly theory)
 - ▶ Oberfield (2018)⁵, Acemoglu and Tahbaz-Salehi (2024), Grossman et al. (2023), Grossman et al. (2024a),⁶ Grossman et al. (2024b)

²Only quantitative models listed

³Only quantitative models listed

⁴Theory and its test

⁵Can be relabeled as sector-to-sector, though.

⁶They calibrated their model, but did not use data on firm-to-firm trade.

Trade-off

- ▶ Which would you source from?
 - ▶ an expensive, but stable supplier,
 - ▶ a cheap, but unstable supplier.

Overview of the model

- ▶ Here we study the model of Kopytov, Mishra, Nimark, and Taschereau-Dumouchel (2024).
- ▶ There is one representative firm in each sector.
- ▶ The representative firm chooses the exponents in its production function.
 - ▶ To what extent does the firm rely on each sector?
- ▶ The firm is owned by households. As such, it takes risks into account.
 - ▶ The intermediate good from this sector enhances my production, but the productivity of this sector is volatile...
- ▶ The unique equilibrium is (ex-ante) efficient.
- ▶ Therefore, we can characterize equilibrium networks and allocations as a solution to the planner's problem.
- ▶ Eventually, the firm's problem reduces to choosing Domar weights.

Setup

- ▶ There are n sectors, indexed by $i \in \{1, \dots, n\}$.
- ▶ Each sector produces a differentiated good.
- ▶ In each sector, there is a representative firm.
 - ▶ We use sector i , firm i , and product i interchangeably.
- ▶ Firms face perfect competition. Equilibrium profits are zero.

Production functions and techniques

- ▶ Representative firm i has access to a set of production techniques \mathcal{A}_i and chooses only one technique $\alpha_i \in \mathcal{A}_i$.
- ▶ The production function of i is

$$F(\alpha_i, L_i, X_i) = e^{\varepsilon_i} A_i(\alpha_i) \zeta(\alpha_i) L_i^{1 - \sum_{j=1}^n \alpha_{ij}} \prod_{j=1}^n X_{ij}^{\alpha_{ij}}, \quad (1)$$

- ▶ L_i is labor inputs,
- ▶ $X_i = (X_{i1}, \dots, X_{in})^\top$ is a vector of intermediate inputs,
- ▶ ε_i is the stochastic component of firm i 's total factor productivity,
- ▶ $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in})^\top \in \mathcal{A}_i$ is a production technique that determines intermediate input shares and affects total factor productivity through $A_i(\alpha_i)$,
- ▶ $A_i(\alpha_i)$ is a productivity shifter,
- ▶ $\zeta(\alpha_i)$ is just a normalization to simplify the cost function.⁷

⁷ $[\zeta(\alpha_i)]^{-1} = (1 - \sum_{j=1}^n \alpha_{ij})^{1 - \sum_{j=1}^n \alpha_{ij}} \prod_{j=1}^n \alpha_{ij}^{\alpha_{ij}}.$

Production techniques and intermediate input shares

- ▶ We define i 's set of feasible production techniques as

$$\mathcal{A}_i = \{\alpha_i \in [0, 1]^n : \sum_{j=1}^n \alpha_{ij} \leq \bar{\alpha}_i\},$$

where $0 < 1 - \bar{\alpha}_i < 1$ is the lower bound of the share of labor (and the upper bound of the sum of the share of intermediate inputs).

- ▶ Define $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ (a Cartesian product).
 - ▶ $\alpha \in \mathcal{A}$ is a production network in this economy.
 - ▶ α represents to what extent each sector relies on intermediate inputs from other sectors.
 - ▶ α can be expressed as a matrix.

Productivity shifter $A_i(\alpha_i)$ (1)

- ▶ The production technique α_i influences i 's total factor productivity through $A_i(\alpha_i)$.
- ▶ The authors' example: "beach towels and flowers are not very useful when making a car, and a technique that relies only on these inputs would have a low A_i ."

Assumption 1

$A_i(\alpha_i)$ is smooth and strictly log-concave.

- ▶ I interpret "smooth" as A_i is differentiable as many times as we wish.
- ▶ Let M be a convex subset of R^n . Function $f : M \rightarrow R_+$ is strictly log-concave if

$$f(\theta x + (1 - \theta)y) > f(x)^\theta f(y)^{1-\theta},$$

for any $x, y \in M$ and $0 < \theta < 1$.

- ▶ Note that then $\log f$ is strictly concave if f is strictly positive.

Productivity shifter $A_i(\alpha_i)$ (2)

Why do we need Assumption 1?

1. There exists a unique technique that solves the optimization problem of the firm.
2. For each sector i , there is a unique vector of *ideal* input shares α_i° that maximizes A_i .
 - ▶ This represents the most productive way to combine intermediate goods to produce good i .
 - ▶ But, this is not necessarily i 's technique choice in equilibrium.
 - ▶ Why? Because maximizing A_i is not the same as maximizing i 's risk-adjusted expected profits.⁸

Without loss of generality, normalize $A_i(\alpha_i^\circ)$ for all i .

⁸This is at the core of this paper. We will see it later.

Productivity shifter $A_i(\alpha_i)$ (3)

Example

One example of a function $A_i(\alpha_i)$ that satisfies Assumption 1 is the quadratic form

$$\log A_i(\alpha_i) = \frac{1}{2}(\alpha_i - \alpha_i^\circ)^\top \bar{H}_i(\alpha_i - \alpha_i^\circ), \quad (2)$$

where \bar{H}_i is a negative-definite matrix that is also the Hessian of $\log A_i$.

Sectoral productivity shocks ε_i

- ▶ Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^\top$ be a vector of sectoral productivity shocks.
- ▶ We assume that the vector is normally distributed $\varepsilon \sim \mathcal{N}(\mu, \Sigma)$.
- ▶ μ determines the expected levels of sectoral productivities.
- ▶ Σ determines uncertainty of individual elements of ε and their correlations.
- ▶ ε is the only source of uncertainty in this economy.
- ▶ Each firm i chooses α_i before ε is realized.
- ▶ A high μ_i leads to a low unit cost and a low price of good i .
- ▶ A high Σ_{ii} leads to a volatile price of good i .
- ▶ A high Σ_{ij} leads to more correlated prices of good i and j .
- ▶ These affect the sourcing decisions of the firms.

Households (1)

- ▶ There is one risk-averse representative household in this economy.
- ▶ She chooses $C = (C_1, \dots, C_n)$ to maximize

$$u \left(\left(\frac{C_1}{\beta_1} \right)^{\beta_1} \times \dots \times \left(\frac{C_n}{\beta_n} \right)^{\beta_n} \right), \quad (3)$$

where $\beta_i > 0$ for all i and $\sum_{i=1}^n \beta_i = 1$.

- ▶ We refer to $Y = \prod_{i=1}^n (\beta_i^{-1} C_i)^{\beta_i}$ as aggregate consumption or GDP.
- ▶ The utility function $u(\cdot)$ is CRRA with coefficient of relative risk aversion ρ . That is, (3) is rewritten as

$$\frac{Y^{1-\rho}}{1-\rho}.$$

Households (2)

- ▶ The household makes consumption decisions after uncertainty is resolved.
- ▶ In each state of the world, the household faces the budget constraint

$$\sum_{i=1}^n P_i C_i \leq 1,$$

where P_i is the price of good i , and the wage is used as a numeraire.

- ▶ Firms are owned by the representative household.
- ▶ Firms maximize expected profits discounted by the household's stochastic discount factor

$$\Lambda = u'(Y)/\bar{P}, \tag{4}$$

where $\bar{P} = \prod_{i=1}^n P_i^{\beta_i}$.

Households (3)

- ▶ By solving the household's optimization, we can show that

$$y = -\beta^\top p$$

where $y = \log Y$, $\beta = (\beta_1, \dots, \beta_n)^\top$ and $p = (p_1, \dots, p_n)^\top$.

- ▶ For any i , $p_i = \log P_i$.

Representative firms' optimization: two steps

1. The firm decides which production technique to use.
 - ▶ This choice is made before the random productivity vector ε is realized.
2. The firm chooses labor and intermediate inputs after the realization of ε .
 - ▶ And the household chooses consumption after the realization of ε .
 - ▶ That is, the final demand for each good is also determined after the realization of ε .

We solve these problems backwardly.

The firm's second-stage problem

- ▶ Under a given technique α_i , the cost minimization problem of firm i is

$$K_i(\alpha_i, P) = \min_{L_i, X_i} \left(L_i + \sum_{j=1}^n P_j X_j \right), \quad (5)$$

subject to

$$F(\alpha_i, L_i, X_i) \geq 1.$$

- ▶ The solution to this problem implicitly defines the unit cost of production $K_i(\alpha_i, P)$.
- ▶ Using the production function (1), the unit cost function is

$$K_i(\alpha_i, P) = \frac{1}{e^{\varepsilon_i} A_i(\alpha_i)} \prod_{j=1}^n P_j^{\alpha_{ij}}. \quad (6)$$

The firm's first-stage problem

- ▶ Given an expression for K_i , the first stage of the representative firm's problem is to pick a technique $\alpha_i \in \mathcal{A}_i$ to maximize expected discounted profits

$$\alpha_i^* \in \arg \max_{\alpha_i \in \mathcal{A}_i} E[\Lambda Q_i(P_i - K_i(\alpha_i, P))]. \quad (7)$$

- ▶ Q_i is the equilibrium demand for good i ,
 - ▶ the profits in different states of the world are weighted by the household's stochastic discount factor Λ .
- ▶ The representative firm takes P , Q_i , and Λ as given.
- ▶ Therefore, this problem reduces to $\min_{\alpha_i \in \mathcal{A}_i} E[\Lambda Q_i K_i(\alpha_i, P)]$.
- ▶ The firm minimizes the weighted expectation of the total cost $Q_i K_i(\alpha_i, P)$ with the weights being Λ .
- ▶ The firm inherits the risk attitude of the representative household.

Equilibrium prices

- In equilibrium, competitive pressure pushes prices to be equal to unit costs

$$P_i = K_i(\alpha_i, P) \quad (8)$$

for all $i \in \{1, \dots, n\}$.

Definition 1 (equilibrium)

An equilibrium is a choice of technique $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$ and a stochastic tuple $(P^*, C^*, L^*, X^*, Q^*)$ such that:

1. (optimal technique choice) For each $i \in \{1, \dots, n\}$, the technique choice $\alpha_i^* \in \mathcal{A}_i$ solves (7) given prices P^* , demand Q_i^* , and the stochastic discount factor Λ^* given by (4).
2. (Optimal input choice) For each $i \in \{1, \dots, n\}$, factor demands per unit of output L_i^*/Q_i^* and X_i^*/Q_i^* are a solution to (5) given prices P^* and the chosen technique α_i^* .
3. (Consumer maximization) The consumption vector C^* maximizes (3).

Definition 1 (equilibrium): continued

4. (Unit cost pricing) For each $i \in \{1, \dots, n\}$, P_i^* solves (8) where $K_i(\alpha_i^*, P^*)$ is given by (6).
5. (Market clearing) For each $i \in \{1, \dots, n\}$,

$$C_i^* + \sum_{j=1}^n X_{ji}^* = Q_i^* = F_i(\alpha_i^*, L_i^*, X_i^*), \text{ and } \sum_{i=1}^n L_i^* = 1.$$

Comments on Definition 1

- ▶ Conditions 2-5 correspond to the standard competitive equilibrium conditions for an economy with a fixed production network.
 - ▶ Firms and the household optimize in a competitive environment.
 - ▶ All markets clear given equilibrium prices.
- ▶ Condition 1 emphasizes that production techniques, and hence the production network represented by the matrix α^* , are equilibrium objects.

How we proceed

- ▶ Fixed production networks,
- ▶ Endogenous production networks.

Two objects: the Leontief inverse and the Domar weight

- ▶ The Leontief inverse is

$$\mathcal{L}(\alpha) = (I - \alpha)^{-1} = I + \alpha + \alpha^2 + \cdots,$$

where

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix},$$

and I is the $n \times n$ identity matrix.

- ▶ Define the Domar weight ω_i of sector i as the ratio of its sales to nominal GDP

$$\omega_i = \frac{P_i Q_i}{P^\top C}.$$

- ▶ The vector of Domar weights $\omega = (\omega_1, \cdots, \omega_n)^\top$ satisfies $\omega^\top = \beta^\top \mathcal{L}(\alpha) > 0$.

Lemma 1

Under a given network α , the vector of log prices is given by

$$p(\alpha) = -\mathcal{L}(\alpha)(\varepsilon + a(\alpha)),$$

and log GDP is given by

$$y(\alpha) = \omega(\alpha)^\top (\varepsilon + a(\alpha)),$$

where $a(\alpha) = (\log A_1(\alpha_1), \dots, \log A_n(\alpha_n))^\top$.

- ▶ Lemma 1 describes how prices and GDP depend on the productivity vector $\varepsilon + a(\alpha)$ and the production network α .
- ▶ An increase in productivity pushes down prices through the Leontief matrix $\mathcal{L}(\alpha)$.
- ▶ An increase in productivity has a linear and positive effect on GDP with the coefficient being the Domar weight.

Mean and variance of GDP

Under a fixed network α ,

$$E[y(\alpha)] = \omega(\alpha)^\top (\mu + a(\alpha)) \quad (9)$$

and

$$V[y(\alpha)] = \omega(\alpha)^\top \Sigma \omega(\alpha). \quad (10)$$

Corollary 1

For a fixed production network α , the following hold

1. The impact of a change in expected TFP μ_i on the moments of log GDP is given by

$$\frac{\partial E[y]}{\partial \mu_i} = \omega_i, \text{ and } \frac{\partial V[y]}{\partial \mu_i} = 0.$$

2. The impact of a change in volatility Σ_{ij} on the moments of log GDP is given by

$$\frac{\partial E[y]}{\partial \Sigma_{ij}} = 0, \text{ and } \frac{\partial V[y]}{\partial \Sigma_{ij}} = \omega_i \omega_j.$$

- ▶ The first part is the Hulten theorem.
- ▶ For the second part, first think about the case $j = i$.

Firm decisions (1)

- ▶ We have discussed the case of fixed production networks.
- ▶ Now we move on to endogenous production networks.
- ▶ Let α^* be the equilibrium network.
- ▶ Let $\lambda(\alpha^*) = \log \Lambda(\alpha^*)$ (the log of the stochastic discount factor).
- ▶ Let $k_i(\alpha_i, \alpha^*) = \log K_i(\alpha_i, P^*(\alpha^*))$ (the log of the unit cost).
- ▶ Using these notations, we can reorganize firm i 's maximization problem (7) as

$$\alpha_i^* \in \arg \min_{\alpha_i \in \mathcal{A}_i} E[k_i(\alpha_i, \alpha^*)] + \text{Cov}[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)]. \quad (11)$$

- ▶ We can rewrite like this because $\lambda(\alpha^*)$, $p_i(\alpha^*)$, and $k_i(\alpha_i, \alpha^*)$ are normally distributed.
- ▶ See the supplementary material for details.

Firm decisions (2)

- ▶ Taking the expected value of the log of (6), we have

$$E[k_i(\alpha_i, \alpha^*)] = -\mu_i - a_i(\alpha_i) + \sum_{j=1}^n \alpha_{ij} E[p_j].$$

- ▶ That is, the firm prefers techniques that have high productivity a_i and that rely on inputs that are expected to be cheap.
- ▶ The second term in (11) captures the importance of aggregate risk for the firm's decision.
 - ▶ The firm prefers to have a low unit cost in states of the world in which the marginal utility of consumption is high.
 - ▶ The economy is in a bad situation.
 - ⇒ Aggregate consumption (GDP) is low.
 - ⇒ The marginal utility is low.
 - ⇒ The firm really wants to have low costs in such a situation.

Lemma 2

In equilibrium, the technique choice problem of the representative firm in sector i is

$$\alpha_i^* \in \arg \max_{\alpha_i \in \mathcal{A}_i} a_i(\alpha_i) - \sum_{j=1}^n \alpha_{ij} \mathcal{R}_j(\alpha^*), \quad (12)$$

where

$$\mathcal{R}(\alpha^*) = E[p(\alpha^*)] + \text{Cov}[p(\alpha^*), \lambda(\alpha^*)]$$

is the vector of equilibrium risk-adjusted prices, and where

$$E[p(\alpha^*)] = -\mathcal{L}(\alpha^*)(\mu + a(\alpha^*))$$

and

$$\text{Cov}[p(\alpha^*), \lambda(\alpha^*)] = (\rho - 1)\mathcal{L}(\alpha^*)\Sigma[\mathcal{L}(\alpha^*)]^\top \beta.$$

Comments on Lemma 2

- ▶ All the equilibrium information needed for the firm's problem is contained in the vector of risk-adjusted prices \mathcal{R} .
- ▶ \mathcal{R} quantity provides an overall measure of the desirability of an input that depends on its expected price and on how its price covaries with the stochastic discount factor.
- ▶ Goods that are cheap when aggregate consumption is low are particularly attractive as inputs, controlling for expected prices.

The Hessian matrix of a_i

- Define H_i by the Hessian matrix of a_i

$$H_i = \begin{bmatrix} \frac{\partial^2 a_i}{\partial \alpha_{i1}^2} & \frac{\partial^2 a_i}{\partial \alpha_{i1} \partial \alpha_{i2}} & \cdots & \frac{\partial^2 a_i}{\partial \alpha_{i1} \partial \alpha_{in}} \\ \frac{\partial^2 a_i}{\partial \alpha_{i2} \partial \alpha_{i1}} & \frac{\partial^2 a_i}{\partial \alpha_{i2}^2} & \cdots & \frac{\partial^2 a_i}{\partial \alpha_{i2} \partial \alpha_{in}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 a_i}{\partial \alpha_{in} \partial \alpha_{i1}} & \frac{\partial^2 a_i}{\partial \alpha_{in} \partial \alpha_{i2}} & \cdots & \frac{\partial^2 a_i}{\partial \alpha_{in}^2} \end{bmatrix}.$$

- Taking the first order condition of (12) and applying the implicit function theorem, we can show that

$$\frac{\partial \alpha_{ij}}{\partial \mathcal{R}_k} = [H_i^{-1}(\alpha_i)]_{jk},$$

where $[\cdot]_{jk}$ denotes the (j, k) element of a matrix.

Complements vs substitutes: the Hessian matters

- ▶ Does an increase in k 's risk-adjusted price \mathcal{R}_k lead to a decrease or an increase in the share of another input $j \neq k$?
- ▶ If $[H_i^{-1}]_{jk} > 0$, we say j and k are substitutes (for i).
 - ▶ $\mathcal{R}_k \uparrow \Rightarrow \alpha_{ij} \uparrow$
 - ▶ Substituting away from k to j
- ▶ If $[H_i^{-1}]_{jk} < 0$, we say j and k are complements (for i).
 - ▶ $\mathcal{R}_k \uparrow \Rightarrow \alpha_{ij} \downarrow$
 - ▶ If k is too expensive to buy, i doesn't need j either
- ▶ One sufficient condition for a Hessian matrix H_i to exhibit complementarity for all sectors is $[H_i]_{jk} \geq 0$ for all $j \neq k$.

Example: 4 sectors in partial equilibrium

Input 1 Steel

Input 2 Equipment

- ▶ Milling machines and lathes to transform raw steel into usable components

Input 3 Carbon fiber

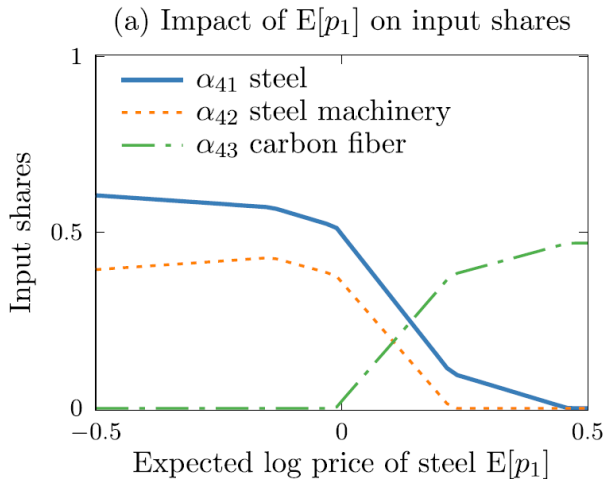
- ▶ Carbon fiber can replace steel

Sector 4 Car manufacturing

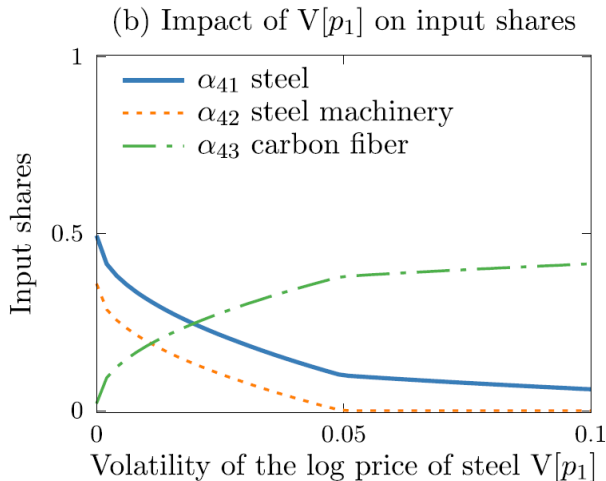
- ▶ This car manufacturer has a TFP shifter function

$$a_4(\alpha_4) = - \sum_{j=1}^4 \kappa_j (\alpha_{4j} - \alpha_{4j}^{\circ})^2 - \psi_1 (\alpha_{41} - \alpha_{42})^2 \\ - \psi_2 [(\alpha_{41} + \alpha_{43}) - (\alpha_{41}^{\circ} + \alpha_{43}^{\circ})^2].$$

Input shares and $E[p_1]$



Input shares and $V[p_1]$



Lemma 3

An efficient production network α^* solves

$$\mathcal{W} := \max_{\alpha \in \mathcal{A}} W(\alpha, \mu, \sigma),$$

where W is a measure of the welfare of the household, and where

$$W(\alpha, \mu, \sigma) := E[y(\alpha)] - \frac{1}{2}(\rho - 1)V[y(\alpha)] \quad (13)$$

is welfare under a given network α .

- ▶ The relative risk aversion ρ determines the relative importance of the expected log GDP $E[y(\alpha)]$ and the variance of the log GDP $V[y(\alpha)]$.
- ▶ The welfare depends on only first and second moments of log GDP. This is because preferences are CRRA and log GDP (aggregate consumption) is normally distributed.

Toward Domar weights (1)

- ▶ Using (9) and (10), we can rewrite the objective function (13) as

$$\omega^\top \mu + \omega^\top a(\alpha) - \frac{1}{2}(\rho - 1)\omega^\top \Sigma \omega.$$

- ▶ Remember that μ and Σ are exogenous parameters.
 - ▶ So the social planner cannot choose them.
- ▶ Only endogenous variables are ω and $a(\alpha)$.
- ▶ Moreover, the only term that does not depend exclusively on ω is $\omega^\top a(\alpha)$.
- ▶ We want to rewrite this in terms of ω alone.

Toward Domar weights (2)

- ▶ Consider the optimization problem

$$\bar{a}(\omega) := \max_{\alpha \in \mathcal{A}} \omega^\top a(\omega), \quad (14)$$

subject to the definition of the Domar weights given by $\omega^\top = \beta^\top \mathcal{L}(\alpha)$.

- ▶ We refer to the value function \bar{a} as the aggregate TFP shifter function.
- ▶ We denote by $\alpha(\omega)$ the solution to (14).
- ▶ For given ω , \bar{a} and $\alpha(\omega)$ do not depend on μ or Σ .

Example

- ▶ We can explicitly solve for \bar{a} and $\alpha(\omega)$ under the quadratic TFP shifter function (2).
- ▶ At an interior solution $\alpha \in \text{int}\mathcal{A}$, the optimal production network $\alpha(\omega)$ for a given vector of Domar weights ω satisfies

$$\alpha_i(\omega) - \alpha_i^\circ = H_i^{-1} \left(\sum_{j=1}^n \omega_j H_j^{-1} \right)^{-1} \left(\omega - \beta - \sum_{j=1}^n \omega_j \alpha_j^\circ \right)$$

for all i , and the associated value function \bar{a} is

$$\bar{a}(\omega) = \frac{1}{2} \sum_{i=1}^n \omega_i (\alpha_i(\omega) - \alpha_i^\circ)^\top H_i (\alpha_i(\omega) - \alpha_i^\circ).$$

- ▶ The gradients ∇a_i of the TFP shifter functions are all equal to each other such that

$$\nabla a_i = \nabla a_j$$

for all i, j .

Corollary 2

The efficient Domar weight vector ω^* solves

$$\mathcal{W} = \max_{\omega \in \mathcal{O}} \underbrace{\omega^\top \mu + \bar{a}(\omega)}_{E[y]} - \frac{1}{2}(\rho - 1) \underbrace{\omega^\top \Sigma \omega}_{V[y]}, \quad (15)$$

where $\mathcal{O} = \{\omega \in R_+^n : \omega \geq \beta \text{ and } \mathbf{1} \geq \omega^\top (\mathbf{1} - \bar{\alpha})\}$ and $\bar{\alpha}(\omega)$ is given by (14).

- ▶ The set \mathcal{O} contains the vectors ω such that the corresponding production network $\alpha(\omega) \in \mathcal{A}$.
- ▶ The first inequality follows from $\alpha_{ij} \geq 0$ for all i, j .
- ▶ The second inequality, where $\mathbf{1}$ denotes the $n \times 1$ all-one column vector, follows from $\sum_j \alpha_{ij} \leq \bar{\alpha}_i$ for all i .

Lemma 4

The objective function of the planner's problem (15) is strictly concave. Furthermore, there is a unique vector of Domar weights ω^* that solves that problem, and there is a unique production network $\alpha(\omega^*)$ associated with that solution.

- Therefore, the first-order conditions will characterize the unique efficient network.

Proposition 1

There exists a unique equilibrium, and it is efficient.

Taking stock

- ▶ There is a unique equilibrium, and it is efficient.
- ▶ Finding the efficient network reduces to finding the Domar weights associated with the network.
- ▶ Therefore, finding the equilibrium network reduces to finding the efficient Domar weights.

"Beliefs"

- ▶ Somehow, in the paper, the authors call μ and Σ "beliefs."
- ▶ They are just the mean vector and the covariance matrix of the stochastic part of (log) TFP.
- ▶ Maybe they refer to how households and producers "perceive" the level and uncertainty of productivity.
- ▶ We will look at some of the results in the paper about how beliefs affect equilibrium outcomes.

Impacts of beliefs

1. Impacts on Domar weights,
2. Impacts on welfare.

Proposition 2

The Domar weight ω_i of sector i is (weakly) increasing in μ_i and (weakly) decreasing in Σ_{ij} .

Risk-adjusted productivity shocks

- ▶ We define a risk-adjusted version of the productivity vector ε

$$\mathcal{E} = \underbrace{\mu}_{E[\varepsilon]} - \underbrace{(\rho - 1)\Sigma\omega}_{Cov[\varepsilon, \lambda]}.$$

- ▶ This measures how higher exposure to ε affects the household's utility.
- ▶ Remember that λ denotes the log of the stochastic discount factor Λ .
- ▶ Let $\mathbf{1}_i$ be the column vector with a 1 only in the i -th element and zeros otherwise. Then

$$\frac{\partial \mathcal{E}}{\partial \mu_i} = \mu_i,$$

and

$$\frac{\partial \mathcal{E}}{\partial \Sigma_{ij}} = -\frac{1}{2}(\rho - 1)(\omega_j \mathbf{1}_i + \omega_i \mathbf{1}_j).$$

Proposition 3

Let γ denote either μ_i or Σ_{ij} . If $\omega \in \text{int}\mathcal{O}$, then

$$\frac{d\omega}{d\gamma} = \underbrace{-\mathcal{H}^{-1}}_{\text{propagation}} \times \underbrace{\frac{\partial \mathcal{E}}{\partial \gamma}}_{\text{impulse}},$$

where the $n \times n$ negative definite matrix \mathcal{H} is given by

$$\mathcal{H} = \nabla^2 \bar{a} + \frac{d\mathcal{E}}{d\omega},$$

and where the matrix $\nabla^2 \bar{a}$ is the Hessian of the aggregate TFP shifter function \bar{a} , and $\frac{d\mathcal{E}}{d\omega} = -\frac{d\text{Cov}[\varepsilon, \lambda]}{d\omega} = -(\rho - 1)\Sigma$ is the Jacobian matrix of the risk adjusted TFP vector \mathcal{E} .

Comments on Proposition 3

- ▶ The impulse (the 2nd part on the RHS) captures the direct effect on the risk-adjusted TFP.
- ▶ The propagation (the 1st part on the RHS) captures the global, economy-wide substitution patterns between sectors.
 - ▶ Contrast it with H_i^{-1} (local, firm-level substitution).
- ▶ If $\mathcal{H}_{ij}^{-1} < 0$, i and j are global complements.
 - ▶ $\mathcal{E}_i \uparrow \Rightarrow \omega_j \uparrow$
- ▶ If $\mathcal{H}_{ij}^{-1} > 0$, i and j are global substitutes.
 - ▶ $\mathcal{E}_i \uparrow \Rightarrow \omega_j \downarrow$

What's \mathcal{H} ?

$$\mathcal{H} = \nabla^2 \bar{a} - \underbrace{(\rho - 1)\Sigma}_{\frac{d\text{Cov}[\varepsilon, \lambda]}{d\omega}}$$

Two forces:

1. Aggregate TFP shifter function \bar{a}

- ▶ Local substitution patterns in (a_1, \dots, a_n) contribute to global substitution patterns

2. Covariance matrix Σ

- ▶ Suppose that ω_i increases because of a positive shock in i .
- ▶ In response to an increase in ω_i , the planner puts a lower ω_j as Σ_{ij} increases.

$$\frac{\partial \mathcal{H}_{ij}^{-1}}{\partial \Sigma_{ij}} > 0$$

Proposition 5

Let denote γ either μ_i or Σ_{ij} . Under an endogenous network, welfare responds to a marginal change in γ as if the network were fixed at its equilibrium value α^* , that is

$$\frac{d\mathcal{W}(\mu, \Sigma)}{d\gamma} = \frac{\partial W(\alpha^*, \mu, \Sigma)}{\partial \gamma}.$$

How about non-infinitesimal change?

- ▶ Let $\alpha^*(\mu, \Sigma)$ be the equilibrium production network under (μ, Σ) .

$$\begin{aligned} & \underbrace{\mathcal{W}(\mu', \Sigma') - \mathcal{W}(\mu, \Sigma)}_{\text{Change in welfare under a flexible network}} \\ & \geq \underbrace{\mathcal{W}(\alpha^*(\mu, \Sigma), \mu', \Sigma') - \mathcal{W}(\alpha^*(\mu, \Sigma), \mu, \Sigma)}_{\text{Change in welfare under a fixed network}}. \end{aligned}$$

Corollary 4

The impact of an increase in μ_i on welfare is given by

$$\frac{d\mathcal{W}}{d\mu_i} = \omega_i,$$

and the impact of an increase in Σ_{ij} on welfare is given by

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = -\frac{1}{2}(\rho - 1)\omega_i\omega_j.$$

- This is a direct result from Corollary 1, Proposition 5, and (13).

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