The Eaton-Kortum Model

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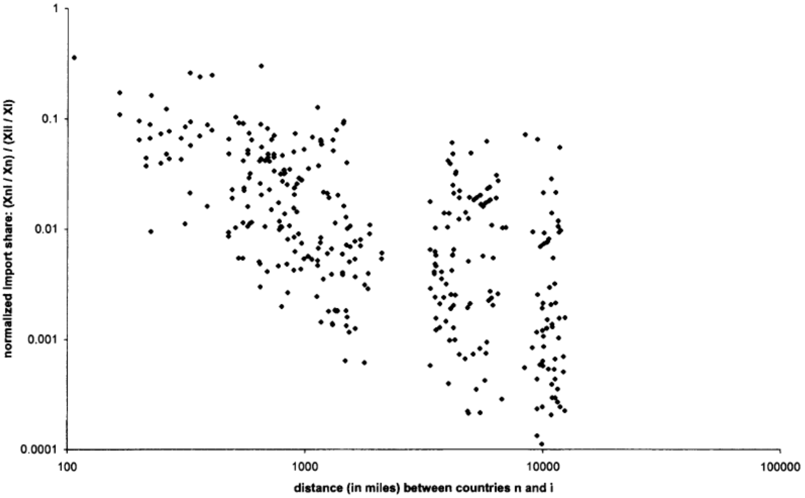
Motivation

- So far, we have mainly studied models of two-country settings.
  - Home vs Foreign.
- But, the actual world economy consists of many countries.
  - Germany, France, Switzerland, USA, China, $\cdots$.
- Can we solve an equilibrium for a model of many countries?
  - With only paper and a pencil, no.
  - With a computer, yes.
- The Eaton-Kortum model is a static quantitaive many-country model of international trade.
  - "Quantitaive" means you can compute numerical solutions of equilibria.
  - Therefore, you can get a number of a welfare change induced by a change in productivity/trade costs/populations.
- An advantage is that gains from trade are expressed as widely available trade values and one key parameter: trade elasticity.

Observation: Bilateral trade

X_{ni}: the (manufacturing) trade value from country i to country n (as of 1986)



Model features we want

- The further two countries are, the less they trade.
- Controlling for population sizes and geographic locations, large exporters tend to be rich.
  - In 1986, they were USA, Japan, and West Germany.
  - This "competitiveness" will be represented as parameters of productivity.
- In the Ricardian tradition of one-factor models, being rich means earning high real wages.

Setup

- There are N countries: $i, n = 1, \cdots, N$.
- There is a unit continuum of varieties $j \in [0, 1]$.
- Consumers in country i have the following utility function

$$U_i = \left[\int_0^1 Q_i(j)^{(\sigma-1)/\sigma} dj \right]^{\sigma/(\sigma-1)}.$$

  - σ: the parameter of the elasticity of substitution, $\sigma > 0$.

CES price index (1)

- Let $p_i(j)$ be the price of variety j in country i.
- We solve the following expenditure minimization problem

$$\min \int_0^1 p_i(j) Q_i(j) dj \tag{1}$$

subject to

$$\left[\int_0^1 Q_i(j)^{(\sigma-1)/\sigma} dj \right]^{\sigma/(\sigma-1)} \geq 1.$$

- That is, you want to minimize total expenditure given that you must enjoy one unit of utility.

CES price index (2)

- We solve this problem with the Lagrangian multiplier method.
- Let L be the Lagrangian and λ be its multiplier. Then,

$$L = \int_0^1 p_i(j)Q_i(j)dj + \lambda\left(1 - \left[\int_0^1 Q_i(j)^{(\sigma-1)/\sigma}dj\right]^{\sigma/(\sigma-1)}\right).$$

- The first-order conditions are

$$\frac{\partial L}{\partial Q_i(j)} = p_i(j) - \lambda\frac{\sigma}{\sigma-1}\left[\int_0^1 Q_i(j)^{\frac{\sigma-1}{\sigma}}dj\right]^{\frac{\sigma}{\sigma-1}-1} \cdot \frac{\sigma-1}{\sigma}Q_i(j)^{\frac{\sigma-1}{\sigma}-1} = 0 \qquad (2)$$

for any $j \in [0,1]$ and

$$\frac{\partial L}{\partial \lambda} = 1 - \left[\int_0^1 Q_i(j)^{(\sigma-1)/\sigma}dj\right]^{\sigma/(\sigma-1)} = 0. \qquad (3)$$

CES price index (3)

- Rewriting (2),

$$p_i(j) = \lambda \left[\int_0^1 Q_i(j)^{(\sigma-1)/\sigma} dj \right]^{1/(\sigma-1)} Q_i(j)^{-\frac{1}{\sigma}}.$$

- This holds for different varieties $j \neq j'$

$$\frac{p_i(j')}{p_i(j)} = \frac{Q_i(j')^{-\frac{1}{\sigma}}}{Q_i(j)^{-\frac{1}{\sigma}}} = \frac{Q_i(j)^{\frac{1}{\sigma}}}{Q_i(j')^{\frac{1}{\sigma}}}.$$

  Rewriting this, we have

$$Q_i(j)^{\frac{1}{\sigma}} p_i(j) p_i(j')^{-1} = Q_i(j')^{\frac{1}{\sigma}}.$$

  Raise both sides to the $\sigma - 1$ power,

$$Q_i(j)^{\frac{\sigma-1}{\sigma}} p_i(j)^{\sigma-1} p_i(j')^{1-\sigma} = Q_i(j')^{\frac{\sigma-1}{\sigma}}.$$

CES price index (4)

- Integrate both sides with respect to j' (not j)

$$Q_i(j)^{\frac{\sigma-1}{\sigma}} p_i(j)^{\sigma-1} \int_0^1 p_i(j')^{1-\sigma} dj' = \int_0^1 Q_i(j')^{\frac{\sigma-1}{\sigma}} dj'.$$

  Raise both sides to the $\frac{\sigma}{\sigma-1}$ power

$$Q_i(j) p_i(j)^{\sigma} \left[\int_0^1 p_i(j')^{1-\sigma} dj' \right]^{\frac{\sigma}{\sigma-1}} = \underbrace{\left[\int_0^1 Q_i(j')^{\frac{\sigma-1}{\sigma}} dj' \right]^{\frac{\sigma}{\sigma-1}}}_{=1 \text{ because of } (3)}.$$

- Therefore the optimal (expenditure minimizing) demand for variety j is

$$Q_i(j) = p_i(j)^{-\sigma} \left[\int_0^1 p_i(j')^{1-\sigma} dj' \right]^{-\frac{\sigma}{\sigma-1}}. \tag{4}$$

CES price index (5)

- Inserting the optimal demand (4) into the objective function (1) yields

$$
\begin{aligned}
&\int_0^1 p_i(j) Q_i(j) dj \\
&= \int_0^1 p_i(j)^{1-\sigma} \left[\int_0^1 p_i(j')^{1-\sigma} dj' \right]^{-\frac{\sigma}{\sigma-1}} dj \\
&= \left[\int_0^1 p_i(j')^{1-\sigma} dj' \right]^{-\frac{\sigma}{\sigma-1}} \int_0^1 p_i(j)^{1-\sigma} dj \\
&= \left[\int_0^1 p_i(j')^{1-\sigma} dj' \right]^{-\frac{\sigma}{\sigma-1}+1} \\
&= \left[\int_0^1 p_i(j')^{1-\sigma} dj' \right]^{\frac{-\sigma+(\sigma-1)}{\sigma-1}} \\
&= \left[\int_0^1 p_i(j')^{1-\sigma} dj' \right]^{\frac{1}{1-\sigma}} .
\end{aligned}
$$

Costs given productivity

- We first discuss prices of varieties consumers face *given producers' productivity*.
- Within a country, many infinitisimal[1] and identical producers produce a variety $j \in [0, 1]$.
- These producers' production exhibits constant returns to scale.
- Therefore we can treat their behavior as behavior of a representative firm.
- The cost of a bundle of inputs in country i is c_i.
- Productivity of variety j in country i is $z_i(j)$.
- The cost of producing a unit of variety j is, then, $c_i/z_i(j)$.

[1] "Infinitesimal" means very small.

Iceberg trade costs and prices given productivity

- You want to ship one unit of variety from country i to country n.
- During shipment, a part of your goods shipped is lost.
  - You send salt. A part of the salt is melted to the sea.
  - Pirates can steal your computers once in ten times.
- Since Paul Samuelson, this situation is expressed as iceberg trade costs.
- Delivering one unit from country i to country n requires producing d_{ni} in i.
  - For example, if $d_{ni} = 1.05$, to deliver one unit of a variety to country n, you need to ship 1.05 units from country i.
  - In this case, 5 percent of the iceberg is melted down.
- For any three countries i, k, and n, $d_{ni} \leq d_{nk} d_{ki}$.
  - This is called the triangle inequality.
  - Trade through a third country costs more than direct trade.

Prices given productivity

- Then, the price of a variety j produced in i and sold in n is

$$p_{ni}(j) = \frac{c_i d_{ni}}{z_i(j)}.$$

- Country n buys variety $j \in [0, 1]$ from the country that sells it at the lowest price.
- Therefore, the unit price country n actually pays for variety j is

$$p_n(j) = \min\{p_{ni}(j); i = 1, \cdots, N\}. \tag{5}$$

Technology (1)

- The productivity of variety j in country i is drawn from the country-specific (cumulative) probability distribution

$$F_i(z) = e^{-T_i z^{-\theta}}. \tag{6}$$

  - $T_i > 0$ and $\theta > 0$.
- Different varieties in country i draw productivity from the independent and identical distributions (6).
- Quick recap: For a real-valued random variable Z, the (cumulative) distribution function is $F(z) = Pr[Z \leq z]$.
  - If $F(\cdot)$ is differentiable, $f(z) = F'(z)$ is the probability density function.
- The probability distribution (6) is the Fréchet distribution[2] with the location parameter T_i and the shape parameter θ.

[2]Or the type-II extreme value distribution.

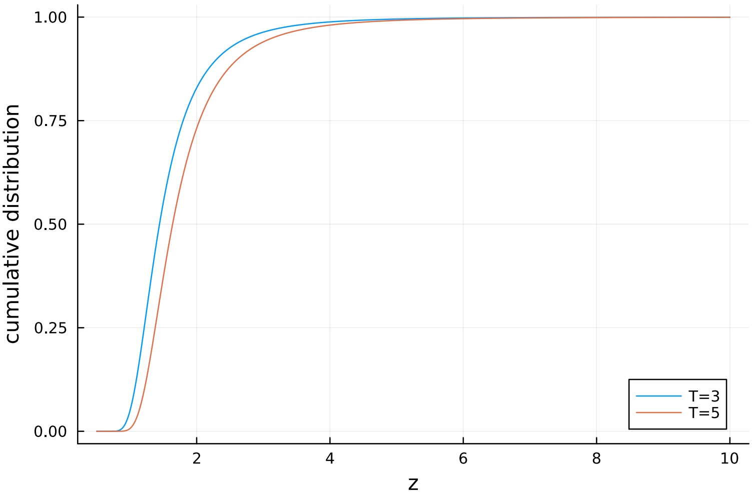
Technology (2)

$$F_i(z) = e^{-T_i z^{-\theta}}$$

- A bigger T_i implies that a high productivity draw for variety j is more likely.
  - In this sense, T_i is often called country i's (average) productivity level.[3]
- A bigger θ implies less variability.

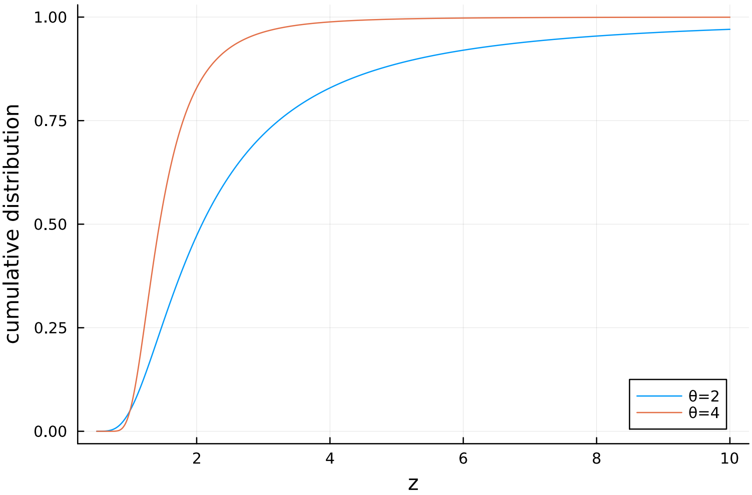
[3] This governs the average, but is not the average itself.

Fréchet distribution: Example (1)



$\theta = 4$.

Fréchet distribution: Example (2)



$T = 3$.

From Technology to Prices

- We assumed a probability distribution for productivity.
- Let P_{ni} be the random variable that represents the price of a variety produced in i and sold in n.
- Then the distribution function for P_{ni} is

$$\begin{aligned} G_{ni}(p) &= Pr[P_{ni} \leq p] \\ &= 1 - F_i(c_i d_{ni}/p) \\ &= 1 - e^{-[T_i(c_i d_{ni})^{-\theta}]p^{\theta}}. \end{aligned}$$

- But, according to (5), what really matters for consumers in n is the distribution of

$$P_n = \min\{P_{ni}; i = 1, \cdots, N\}.$$

- Let $G_n(\cdot)$ denotes the distribution function of P_n.
  - That is, $G_n(p) = Pr\left[P_n \leq p\right]$

Price distribution

$$
\begin{aligned}
G_n(p) &= Pr[Pn \leq p] \\
&= Pr\left[\min_{i=1,\cdots,N} P_{ni} \leq p\right] \\
&= 1 - Pr\left[p \leq P_{n1} \text{ and } p \leq P_{n2} \text{ and } \cdots \text{ and } p \leq P_{nN}\right] \\
&= 1 - Pr[p \leq P_{n1}] \cdot Pr[p \leq P_{n2}] \cdot \cdots \cdot Pr[p \leq P_{nN}] \\
&= 1 - (1 - Pr[P_{n1} \leq p]) \cdot (1 - Pr[P_{n2} \leq p]) \cdot \cdots \cdot (1 - Pr[P_{nN} \leq p]) \\
&= 1 - (1 - G_{n1}(p)) \cdot (1 - G_{n2}(p)) \cdot \cdots \cdot (1 - G_{nN}(p)) \\
&= 1 - e^{-[T_1(c_1 d_{n1})^{-\theta}]p^\theta} \cdot e^{-[T_2(c_2 d_{n2})^{-\theta}]p^\theta} \cdot \cdots \cdot e^{-[T_N(c_N d_{nN})^{-\theta}]p^\theta} \\
&= 1 - e^{-\Phi_n p^\theta},
\end{aligned}
$$

where

$$
\Phi_n = \sum_{i=1}^{N} T_i (c_i d_{ni})^{-\theta}.
$$

Trade shares (1)

- Let the set of all countries be $\mathcal{N} = \{1, 2, \cdots, N\}$.
- The probability that country i serves an infinitesimal variety to country n at the lowest price is[4]

$$\begin{aligned}
\pi_{ni} &= Pr\left[P_{ni} \leq \min_{k \in \mathcal{N} \setminus \{i\}} P_{nk}\right] \\
&= \int_0^\infty Pr\left[\min_{k \in \mathcal{N} \setminus \{i\}} P_{nk} \geq p\right] dG_{ni}(p) \\
&= \int_0^\infty Pr[P_{nk} \geq p \text{ for all } k \in \mathcal{N} \setminus \{i\}] dG_{ni}(p) \\
&= \int_0^\infty \prod_{k \in \mathcal{N} \setminus \{i\}} Pr[P_{nk} \geq p] dG_{ni}(p) \\
&= \int_0^\infty \prod_{k \in \mathcal{N} \setminus \{i\}} (1 - G_{nk}(p)) dG_{ni}(p).
\end{aligned}$$

[4] The following calculation follows Allen and Arkolakis' notes.

Trade shares (2)

- The probability density function of prices produced in country i and sold in n is

$$g_{ni}(p) = \frac{dG_{ni}(p)}{dp} = e^{-T_i(c_i d_{ni})^{-\theta} p^{\theta}} T_i(c_i d_{ni})^{-\theta} \theta p^{\theta-1}.$$

- Then, we have

$$\begin{aligned}
\pi_{ni} &= \int_0^{\infty} \prod_{k \in \mathcal{N} \setminus \{i\}} (1 - G_{nk}(p)) dG_{ni}(p) \\
&= \int_0^{\infty} \prod_{k \in \mathcal{N} \setminus \{i\}} (1 - G_{nk}(p)) g_{ni}(p) dp \\
&= \int_0^{\infty} \left(\prod_{k \in \mathcal{N} \setminus \{i\}} e^{-[T_k(c_k d_{nk})^{-\theta}] p^{\theta}} \right) e^{-T_i(c_i d_{ni})^{-\theta} p^{\theta}} T_i(c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp
\end{aligned}$$

Trade shares (3)

[continued]

$$
\begin{aligned}
\pi_{ni} &= \int_0^\infty e^{-\Phi_n p^\theta} T_i (c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp \\
&= \left(-\frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n} \right) \int_0^\infty \left(-\Phi_n \theta p^{\theta-1} e^{-\Phi_n p^\theta} \right) dp \\
&= \left(-\frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n} \right) \int_0^\infty (e^{-\Phi_n p^\theta})' dp \\
&= \left(-\frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n} \right) \left[e^{-\Phi_n p^\theta} \right]_0^\infty \\
&= \left(-\frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n} \right) (0 - 1) \\
&= \frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n} = \frac{T_i (c_i d_{ni})^{-\theta}}{\sum_{k=1}^N T_k (c_k d_{nk})^{-\theta}}.
\end{aligned}
$$

Price index (1)

- Remember that the price index in country i is

$$p_i = \left[\int_0^1 p_i(j)^{1-\sigma} dj \right]^{\frac{1}{1-\sigma}}$$

- Using the distribution function of prices in i, G_i, we rewrite this

$$\begin{aligned} p_i^{1-\sigma} &= \int_0^1 p_i(j)^{1-\sigma} dj \\ &= \int_0^\infty p^{1-\sigma} dG_i(p) \\ &= \int_0^\infty p^{1-\sigma} g_i(p) dp. \end{aligned}$$

Price index (2)

- The probability density function of prices in country i is

$$g_i(p) = \frac{dG_i(p)}{dp} = e^{-\Phi_i p^\theta} \theta \Phi_i p^{\theta-1}.$$

- Using this, we further compute the price index

$$\begin{aligned} p_i^{1-\sigma} &= \int_0^\infty p^{1-\sigma} e^{-\Phi_i p^\theta} \theta \Phi_i p^{\theta-1} dp \\ &= \int_0^\infty \theta \Phi_i p^{\theta-\sigma} e^{-\Phi_i p^\theta} dp. \end{aligned}$$

Price index (3)

- We change the variable of integration from p to $x = \Phi_i p^\theta$.

p	0	$\to$	∞
x	0	$\to$	∞

- Other relevant information about this change of the integration variable:

$$\frac{dx}{dp} = \theta \Phi_i p^{\theta - 1}.$$

Therefore,

$$\begin{aligned} dp &= \frac{dx}{\theta \Phi_i p^{\theta-1}} \\ &= \frac{dx}{\theta x p^{-1}} \\ &= \frac{dx}{\theta x \left(\frac{x}{\Phi_i}\right)^{-\frac{1}{\theta}}}. \end{aligned}$$

Price index (4)

- Then we continue the calculation of $p_i^{1-\sigma}$

$$
\begin{aligned}
p_i^{1-\sigma} &= \int_0^\infty \theta x \left(\frac{x}{\Phi_i}\right)^{-\frac{\sigma}{\theta}} e^{-x} \frac{dx}{\theta x \left(\frac{x}{\Phi_i}\right)^{-\frac{1}{\theta}}} \\
&= \int_0^\infty \left(\frac{x}{\Phi_i}\right)^{\frac{1-\sigma}{\theta}} e^{-x} dx \\
&= \Phi_i^{-\frac{1-\sigma}{\theta}} \underbrace{\int_0^\infty x^{\frac{1-\sigma}{\theta}} e^{-x} dx}_{=\Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)},
\end{aligned}
$$

  where $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$ is the Gamma function.

- Therefore, the price index is

$$
p_i = \gamma \Phi_i^{-\frac{1}{\theta}},
$$

  where γ is just a constant $\gamma = \Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)^{1/(1-\sigma)}$.

Closing the model (1)

- Assume that there is only one sector (manufacturing).
- Assume trade balances.
  - No trade surplus/deficit.
- Let X_i and Y_i be i's total spending and gross production, respectively.
- Let X_{ni} be the trade value from i to n.
- Then, we have

$$Y_i = \sum_{n=1}^{N} X_{ni}. \tag{7}$$

and

$$X_i = \sum_{n=1}^{N} X_{in}. \tag{8}$$

Closing the model (2)

- Trade balances mean
$$\underbrace{\sum_{n \neq i} X_{ni}}_{i\text{'s export value}} = \underbrace{\sum_{n \neq i} X_{in}}_{i\text{'s import value}} .$$
- Adding X_{ii} (the home purchase in i) to both sides,
$$\sum_{n=1}^{N} X_{ni} = \sum_{n=1}^{N} X_{in}.$$
- This, (7), and (8) yield
$$Y_i = X_i.$$

Closing the model (3)

- Assume the Cobb-Douglas production function so that the cost function takes the form of
$$c_i = w_i^{\beta} p_i^{1-\beta}.$$
- This implies
$$w_i L_i = \beta Y_i = \beta X_i. \tag{9}$$
- Using (7) and (9), we have
$$\begin{aligned} w_i L_i &= \sum_{n=1}^{N} w_n L_n \pi_{ni} \\ &= \sum_{n=1}^{N} w_n L_n \frac{T_i (c_i d_{ni})^{-\theta}}{\sum_{k=1}^{N} T_k (c_k d_{nk})^{-\theta}} \\ &= \sum_{n=1}^{N} w_n L_n \frac{T_i (w_i^{\beta} p_i^{1-\beta} d_{ni})^{-\theta}}{\sum_{k=1}^{N} T_k (w_k^{\beta} p_k^{1-\beta} d_{nk})^{-\theta}}. \end{aligned}$$

Closing the model (4)

- We can rewrite the price index

$$
\begin{aligned}
p_i &= \gamma \left(\sum_{n=1}^{N} T_n (c_n d_{in})^{-\theta} \right)^{-\frac{1}{\theta}} \\
&= \gamma \left(\sum_{n=1}^{N} T_n (w_n^{\beta} p_n^{1-\beta} d_{in})^{-\theta} \right)^{-\frac{1}{\theta}} .
\end{aligned}
$$

Equilibrium conditions

- An equilibrium is characterized by a tuple of $\{w_i\}_{i=1}^N$ and $\{p_i\}_{i=1}^N$ such that

$$w_i = \frac{1}{L_i}\sum_{n=1}^{N} w_n L_n \frac{T_i(w_i^{\beta} p_i^{1-\beta} d_{ni})^{-\theta}}{\sum_{k=1}^{N} T_k(w_k^{\beta} p_k^{1-\beta} d_{nk})^{-\theta}} \tag{10}$$

  and

$$p_i = \gamma \left(\sum_{n=1}^{N} T_n(w_n^{\beta} p_n^{1-\beta} d_{in})^{-\theta}\right)^{-\frac{1}{\theta}} \tag{11}$$

  for $i = 1, \cdots, N$.
- This is a system of $2N$ equations for $2N$ unknowns.
- This does not guarantee the existence and uniqueness of an equilibrium.
- But, Alvarez and Lucas (2007) established the existence and uniqueness. No worry about them.

Let's compute it

- We'll compute an equilibrium with Julia.
- That is, we'll find a solution $\{w_i\}_{i=1}^N$ and $\{p_i\}_{i=1}^N$ for equations (10) and (11).