Question 5

Motoaki Takahashi

October 21, 2021

We want to show the following statement.

Statement 1. *Every game with finitely many players and finitely many actions has at least one Nash equilibrium in which none of the players use weakly dominated strategies.*

Proof. Consider any such game G_0:

- $N = \{1, \cdots, n\}$: the set of players,
- A_i: the set of actions for $i \in N$,
- $u_i^0 : \times_{i=1}^n A_i \to R$: i's payoff function.

Let C_i be the set of weakly dominated strategies for any $i \in N$. Let $B_i \equiv A_i \setminus C_i$ for any $i \in N$.

Then consider a "reduced" game G_1 such that

- $N = \{1, \cdots, n\}$: the set of players,
- B_i: the set of actions for $i \in N$,
- $u_i^1 : \times_{i=1}^n B_i \to R$: i's payoff function,

where $u_i^1(b_i, b_{-i}) \equiv u_i^0(b_i, b_{-i})$ for any $b_i \in B_i$ and $b_{-i} \in \times_{j \neq i} B_j$. By Nash's theorem, there exists a Nash equilibrium $(\sigma_i)_{i=1}^n$ for G_1, where $\sigma_i \in \Delta(B_i)$ and $\Delta(B_i)$ denotes the set of probability distributions on B_i.

We reinterpret (or abuse) $(\sigma_i)_{i=1}^n$ such that σ_i is now a probability distribution on A_i, but still assigns positive probability masses only on elements in B_i, for any $i \in N$.

We argue that $(\sigma_i)_{i=1}^n$ is a Nash equilibrium in the original game G_0.

I show this by the way of contradiction. Suppose, to the contrary, that player j has a profitable deviation from σ_j shifting a probability weight $\delta \in (0, 1]$ from b_j to c_j for some $b_j \in B_j$ and $c_j \in C_j$. **Since c_j is weakly dominated, by Statement 2, there exists a mixed**

strategy $\hat{\sigma}_j$ such that $\hat{\sigma}_j$ puts positive weights only on elements in B_j, and $\hat{\sigma}_j$ weakly dominates c_j. Therefore, shifting δ from b_j to $\hat{\sigma}_j$ is also a profitable deviation, and $\hat{\sigma}_j$ can be played in G_1. But, then σ_j would not be a best response against σ_{-j} in the reduced game G_1, which contradicts the hypothesis that $(\sigma_i)_{i=1}^n$ is a Nash equilibrium in G_1.

$\square$

Notations of the set of actions A_j, the set of weakly dominated strategies C_j, and $B_j \equiv A_j \setminus C_j$ carry over to the next statement.

Statement 2. *Let $c_j \in C_j$. Then there exists a mixed strategy $\hat{\sigma}_j$ such that*

1. *$\hat{\sigma}_j$ assigns positive probability masses only on elements in B_j,*
2. *$\hat{\sigma}_j$ weakly dominates c_j.*