A Weakly Undominated Nash Equilibrium Always Exists in a Finite Game

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We want to show that every game with finitely many players and finitely many actions has at least one Nash equilibrium in which none of the players use weakly dominated strategies. First we show an auxiliary statement (Statement 1), and then we show the main statement (Statement 2).

Throughout this note we consider a strategic game G:

- $N = \{1, \cdots, n\}$: the finite set of players,
- A_i: the finite strategy space for $i \in N$,
- $u_i^0 : \times_{i=1}^n A_i \to \mathbb{R}$: the payoff function for $i \in N$.

For any player i, let z^i be the number of pure strategies, and let x^i be the number of weakly dominated (pure) strategies. Denote the set of weakly dominated strategies by $C_i = \{a_i^1, \cdots, a_i^{x^i}\}$ for any player i. We can write $A_i = \{a_i^1, \cdots, a_i^{x^i}, a_i^{x^i+1}, \cdots, a_i^{z^i}\}$. Let $B_i = A_i \setminus C_i = \{a_i^{x^i+1}, \cdots, a_i^{z^i}\}$. B_i is the set of "weakly undominated" strategies.

Since a_i^1 is weakly dominated, there exists a mixed strategy $\sigma_i^1 \in \Delta(A_i)$ such that

- for any $a_{-i} \in \times_{j \neq i} A_j$,
$$u_i^0(\sigma_i^1, a_{-i}) \geq u_i^0(a_i^1, a_{-i}), \tag{1}$$
- for some $a_{-i} \in \times_{j \neq i} A_j$,
$$u_i^0(\sigma_i^1, a_{-i}) > u_i^0(a_i^1, a_{-i}). \tag{2}$$

Rewriting (1),
$$\sum_{k=1}^{z^i} \sigma_i^1(a_i^k) u_i(a_i^k, a_{-i}) \geq u_i(a_i^1, a_{-i}). \tag{3}$$

Consider a new mixed strategy σ_i^{1*} defined by

$$\sigma_i^{1*}(a_i^k) = \begin{cases} \frac{\sigma_i^1(a_i^k)}{1-\sigma_i^1(a_i^1)} & \text{for } k \neq 1, \\ 0 & \text{for } k = 1. \end{cases} \tag{4}$$

We shifted the probability mass from a_i^1 to the other pure strategies in making σ_i^{1*} from σ_i^1. Then, mixed strategy σ_i^{1*} also weakly dominates a_i^1.

Since a_i^2 is weakly dominated, there exists a mixed strategy $\sigma_i^2 \in \Delta(A_i)$ which satisfies conditions similar to (1) and (2). Then, first, we consider a mixed strategy σ_i^{2*} defined by

$$\sigma_i^{2*}(a_i^k) = \begin{cases} \sigma_i^{2*}(a_i^k) + \sigma_i^{1*}(a_i^k) & \text{for } k \neq 1, \\ 0 & \text{for } k = 1. \end{cases} \tag{5}$$

We shifted the probability mass from a_i^1 to σ_i^{1*} in making σ_i^{2*} from σ_i^2. Mixed strategy σ_i^{2*} weakly dominates a_i^2, and does not assign a positive probability weight on a_i^1. We consider another mixed strategy σ_i^{2**} defined by

$$\sigma_i^{2**}(a_i^k) = \begin{cases} \frac{\sigma_i^{2*}(a_i^k)}{1-\sigma_i^{2*}(a_i^2)} & \text{for } k \neq 2, \\ 0 & \text{for } k = 2. \end{cases} \tag{6}$$

In making σ_i^{2**} from σ_i^{2*}, we shifted the probability mass from a_i^2 to the other pure strategies. Mixed strategy σ_i^{2**} weakly dominates a_i^k and does not assign positive probability weights on a_i^1 and a_i^2.

We iterate this process for $k = 3, \cdots, x^i$. Then, we can construct a mixed strategy such that

1. it weakly dominates $a_i^{x^i}$,
2. it assigns positive probability weights only on elements in B_i.

By reshuffling the order (superscripts) of $a_i^1, \cdots, a_i^{x^i}$ and iterating the same process, we can say the following.

Statement 1. *For any weakly dominated strategy a_i, there exists a mixed strategy $\sigma_i \in \Delta(A_i)$ such that*

1. *σ_i weakly dominates a_i,*
2. *σ_i assigns positive probability weights only on elements in the set of "weakly undominated" strategies B_i.*

Using Statement 1, we show the main statement:

Statement 2. *Every game with finitely many players and finitely many actions has at least one Nash equilibrium in which none of the players use weakly dominated strategies.*

Consider a reduced game G' defined by

- $N = \{1, \cdots, n\}$: the set of players,
- B_i: the strategy space for $i \in N$,
- $u_i' : \times_{i=1}^n B_i \to \mathbb{R}$: the payoff function for $i \in N$,

where $u_i'(b_i, b_{-i}) \equiv u_i(b_i, b_{-i})$ for any $b_i \in B_i$ and $b_{-i} \in \times_{j \neq i} B_j$. In words, G' is the game in which all weakly dominated strategies are removed from G. By Nash's theorem, there exists a Nash equilibrium $(\sigma_i')_{i=1}^n$ for G'.

We define a profile of mixed strategies $(\sigma_i)_{i=1}^n$ in the original game G by

$$\sigma_i(a_i) = \begin{cases} \sigma_i'(a_i) & \text{for } a_i \in B_i, \\ 0 & \text{for } a_i \in C_i. \end{cases} \tag{7}$$

Observe that σ_i does not assign a positive probability weight on any weakly dominated strategy.

We argue that $(\sigma_i)_{i=1}^n$ is a Nash equilibrium in the original game G. We show this by the way of contradiction. Suppose, to the contrary, that player j has a profitable deviation from σ_j shifting a probability weight $\delta \in (0, 1]$ from b_j to c_j for some $b_j \in B_j$ and $c_j \in C_j$. Since c_j is weakly dominated, by Statement 1, there exists a mixed strategy $\hat{\sigma}_j$ such that $\hat{\sigma}_j$ puts positive probability weights only on elements in B_j, and $\hat{\sigma}_j$ weakly dominates c_j. Therefore, shifting probability weight δ from b_j to $\hat{\sigma}_j$ is also a profitable deviation from σ_j in G. Define a mixed strategy $\hat{\sigma}_j'$ in the reduced game G' by

$$\hat{\sigma}_j'(b_j) = \hat{\sigma}_j(b_j) \tag{8}$$

for any $b_j \in B_j$. $\hat{\sigma}_j'$ is a profitable deviation from σ_j' against σ_{-j}'. But, then σ_j would not be a best response against σ_{-j} in the reduced game G', which contradicts the hypothesis that $(\sigma_i')_{i=1}^n$ is a Nash equilibrium in G'.