1 Introduction

This a very raough draft, rather a skeleton of a technical manual of FRT as implemented in python. I contains theoretical background and selected technical specifics on the implementation. All relevant modifications will be documented here. Currently, it also includes references to the fortran code – which serves as the base of the python distribution, too.

2 Radiation field components

2.1 Radiative transfer equation

2.1.1 General background

FRT is based on the radiative transfer equation in plant canopies, which can be written for the radiance I following the notation of Knyazikhin and Marshak (1991):

$$(\mathbf{\Omega}\cdot\nabla)I(\mathbf{r},\mathbf{\Omega})+\sigma(\mathbf{r},\mathbf{\Omega})I(\mathbf{r},\mathbf{\Omega})=\int\limits_{4\pi}d\Omega' I(\mathbf{r},\mathbf{\Omega}')\sigma_S(\mathbf{r},\mathbf{\Omega}'\rightarrow\mathbf{\Omega})+Q_0(\mathbf{r},\mathbf{\Omega}), \tag{1}$$

where $I(\mathbf{r},\mathbf{\Omega})$ is the radiance of the scattered radiation field in the direction $\mathbf{\Omega}$ at the point $\mathbf{r}$ inside the canopy, $\sigma=u_L(\mathbf{r})G(\mathbf{\Omega})$ is the volume extinction coefficient, $\sigma_S(\mathbf{r},\mathbf{\Omega}'\rightarrow\mathbf{\Omega})=\frac{1}{\pi}u_L(\mathbf{r})\Gamma(\mathbf{\Omega}'\rightarrow\mathbf{\Omega})$ is the volume scattering coefficient, $u_L(\mathbf{r})$ is the leaf area density (in $\mathrm{m}^2\mathrm{m}^{-3}$), and Q_0 is the source function due to incident radiation:

$$Q_0(\mathbf{r},\mathbf{\Omega})=\int\limits_{4\pi}d\Omega' I_0(\mathbf{r},\mathbf{\Omega}')\sigma_S(\mathbf{r},\mathbf{\Omega}'\rightarrow\mathbf{\Omega}), \tag{2}$$

where I_0 is the radiance of incident direct solar radiation. The volume extinction coefficient $\sigma(\mathbf{r},\mathbf{\Omega})$ is defined as the fraction of radiant energy traveling in the direction $\mathbf{\Omega}$ intercepted by a unit volume of the vegetation canopy at the point $\mathbf{r}$. Similarly, the volume scattering coefficient $\sigma_S(\mathbf{r},\mathbf{\Omega}'\rightarrow\mathbf{\Omega})$ gives the fraction of radiant energy traveling in the direction $\mathbf{\Omega}'$ that is scattered by unit canopy volume into unit solid angle around $\mathbf{\Omega}$.

Simultaneously with Eq. (1) we specify the boundary conditions

$$I(\mathbf{r},\mathbf{\Omega})=B(\mathbf{r},\mathbf{\Omega}),\ \mathbf{r}\in\delta V,\ \mathbf{n}(\mathbf{r})\cdot\mathbf{\Omega}<0, \tag{3}$$

where δV is the canopy boundary, $\mathbf{n}(\mathbf{r})$ is the outward normal at the point $\mathbf{r}\in\delta V$, and $B(\mathbf{r},\mathbf{\Omega})$ is a wavelength-independent function defined on δV. The formulation of RTE as given by Eq. (1) and implemented in FRT assumes that $B(\mathbf{r},\mathbf{\Omega})$ at the canopy upper boundary δV_{top} equals the diffuse sky radiance $I_{dif}(\mathbf{\Omega})$, the source term Q_0 includes only the incident direct radiation, $I_0(\mathbf{r},\mathbf{\Omega}')=I_S\delta(\mathbf{\Omega}'-\mathbf{\Omega}_S)$;$I_S(\mathbf{r})=F_S(\mathbf{r})/\cos\theta_S$ is the radiance of direct solar radiation, $F_S(L)$ is direct solar irradiance (flux density on a horizontal surface) and θ_S is solar zenith angle. At the bottom canopy surface δV_{bottom}, $B(\mathbf{r},\mathbf{\Omega})$ equals the sum of 1) ground-reflected canopy-scattered diffuse sky and direct radiation; 2) canopy-transmitted direct flux:

$$\begin{cases} B(\mathbf{r},\mathbf{\Omega})=I_{dif}(\mathbf{r},\mathbf{\Omega}) & \mathbf{r}\in\delta V_{top},\ \mathbf{n}(\mathbf{r})\cdot\mathbf{\Omega}<0 \\ B(\mathbf{r},\mathbf{\Omega})=\int\limits_{\mathbf{n}(\mathbf{r})\cdot\mathbf{\Omega}>0} d\Omega' I(\mathbf{r},\mathbf{\Omega}')\rho_{gnd}(\mathbf{r},\mathbf{\Omega}'\rightarrow\mathbf{\Omega})+F_S(\mathbf{r}) & \mathbf{r}\in\delta V_{bottom},\ \mathbf{n}(\mathbf{r})\cdot\mathbf{\Omega}<0\cdot \end{cases} \tag{4}$$

For computing multiple scattering, FRT uses a model assuming a horizontally homogeneous canopy. In this case, the RTE can be simplified to

$$\mu(\mathbf{\Omega})\frac{dI(L,\mathbf{\Omega})}{dL}=G(\mathbf{\Omega})I(L,\mathbf{\Omega})-\frac{1}{\pi}\int\limits_{4\pi}I(L,\mathbf{\Omega}')\Gamma(\mathbf{\Omega}'\rightarrow\mathbf{\Omega})d\mathbf{\Omega}'-\frac{1}{\pi}I_S(L)\Gamma(\mathbf{\Omega}_S\rightarrow\mathbf{\Omega}), \tag{5}$$

where $\mu(\mathbf{\Omega})$ is the cosine of the polar angle of the direction $\mathbf{\Omega}$ (Knyazikhin & Marshak, 1991, Eq. 1.15), $dL=-u_L(z)dz$ u_L is the leaf area density, Γ is the area scattering phase function, and G is the Ross-Nilson G-function,

$$G(L,\mathbf{\Omega}) = \frac{1}{2\pi}\int\limits_{2\pi^+} d\mathbf{\Omega}_L g_L(L,\mathbf{\Omega}_L)\left|\mathbf{\Omega}_L\cdot\mathbf{\Omega}\right|, \tag{6}$$

$\mathbf{\Omega}_L$ is the leaf normal, g_L is the leaf normal probability density normalized as

$$\int\limits_{2\pi^+} d\mathbf{\Omega}_L g_L(L,\mathbf{\Omega}_L) = 2\pi. \tag{7}$$

Here, μ contains the information about the hemisphere[1]: for downward directions, $\mu < 0$. Together with the source term, boundary conditions need to be given. The boundary conditions are as before – the crown layer of a forest canopy is

- bounded from above a non-reflective sky with a given downwelling spectral diffuse irradiance;
- bounded from below by *forest floor*, a reflective surface with either a given reflectance factor, or assumed to consist of a soil covered by a two-layer horizontally homogeneous understory.

2.1.2 Twostream equations

Equation (5) can be multiplied by dL integrated over the upward and downward-facing hemispheres to obtain two directional fluxes. For the upward direction we get

$$\begin{aligned}\int\limits_{2\pi^+} d\Omega\left[\mu(\mathbf{\Omega})dI(L,\mathbf{\Omega})\right] =& dL\int\limits_{2\pi^+} d\mathbf{\Omega}G(\mathbf{\Omega})I(L,\mathbf{\Omega}) - \frac{1}{\pi}dL\int\limits_{2\pi^+} d\mathbf{\Omega}\int\limits_{4\pi} I(L,\mathbf{\Omega}')\Gamma(\mathbf{\Omega}'\to\Omega)d\mathbf{\Omega}'\\ &- \frac{1}{\pi}dL\int\limits_{2\pi^+} d\mathbf{\Omega}I_S(L)\Gamma(\mathbf{\Omega}_S\to\mathbf{\Omega}),\end{aligned} \tag{8}$$

where $2\pi^+$ is the upper hemisphere and we have assumed G and Γ to be independent from L. Next, we note that the upward flux

$$dF^+(L) = \int\limits_{2\pi^+} d\mathbf{\Omega}\left[\mu(\mathbf{\Omega})dI(L,\mathbf{\Omega})\right]$$

and make the assumption that the radiance within the hemisphere isotropic, so that we can calculate the integral

$$\begin{aligned}\int\limits_{2\pi^+} d\mathbf{\Omega}G(\mathbf{\Omega})I(L,\mathbf{\Omega}) &= \frac{\int\limits_{2\pi^+} d\mathbf{\Omega}G(\mathbf{\Omega})I(L,\mathbf{\Omega})}{\int\limits_{2\pi^+} I(L,\mathbf{\Omega})\mu d\mathbf{\Omega}}\times\int\limits_{2\pi^+} I(L,\mathbf{\Omega})\mu d\mathbf{\Omega}\\ &= F^+(L)\end{aligned}$$

as, according to Miller's theorem (1967)

$$\int\limits_{2\pi} d\mathbf{\Omega}G(\mathbf{\Omega}) = \pi, \tag{9}$$

and

$$\int\limits_{2\pi}\mu d\mathbf{\Omega} = \int\limits_0^{2\pi} d\phi\int\limits_0^{\pi/2}\cos\vartheta\sin\vartheta d\vartheta = 2\pi\int\limits_0^1\sin\vartheta\, d(\sin\vartheta) = \pi.$$

[1] θ_S is in the upper hemisphere, as opposed to the direction of solar rays, ϑ_S, which is directed downwards.

An identical result is obtained for horizontal leaves for which $G(\Omega) \equiv \mu$ and the radiance is indeed isotropic. Next, We denote

$$\Gamma^{\uparrow\uparrow} = \int_{2\pi^+} d\mathbf{\Omega}' \int_{2\pi^+} d\mathbf{\Omega}\Gamma(\mathbf{\Omega}' \to \mathbf{\Omega})$$
$$\Gamma^{\downarrow\uparrow} = \int_{2\pi^-} d\mathbf{\Omega}' \int_{2\pi^+} d\mathbf{\Omega}\Gamma(\mathbf{\Omega}' \to \mathbf{\Omega})$$
$$\Gamma^{S\uparrow} = \int_{2\pi^+} d\mathbf{\Omega}\Gamma(\mathbf{\Omega}_S \to \mathbf{\Omega})$$

and assume that the direct beam is a delta function,

$$I_S(L, \mathbf{\Omega}') = I_S(L)\delta(\mathbf{\Omega}' - \mathbf{\Omega}_S),$$

so that

$$\int_{2\pi^+} d\mathbf{\Omega}' I_S(L)\delta(\mathbf{\Omega}' - \mathbf{\Omega}_S) = \frac{F_S(L)}{\cos\vartheta_S},$$

where F_S is the direct solar flux (i.e., irradiance on a horizontal surface). Finally, we obtain for the upward-traveling flux

$$\frac{dF^+(L)}{dL} = F^+ - \frac{1}{\pi}F^+(L)\Gamma^{\uparrow\uparrow} - \frac{1}{\pi}F^-(L)\Gamma^{\downarrow\uparrow} - \frac{1}{\pi}\frac{F_S(L)}{\cos\vartheta_S}\Gamma^{S\uparrow}.$$

Similarly, we get for the downward flux $F^-(L)$

$$\frac{dF^-(L)}{dL} = F^- - \frac{1}{\pi}F^-(L)\Gamma^{\downarrow\downarrow} - \frac{1}{\pi}F^+(L)\Gamma^{\uparrow\downarrow} - \frac{1}{\pi}\frac{F_S(L)}{\cos\vartheta_S}\Gamma^{S\downarrow},$$

$$\Gamma^{\downarrow\downarrow} = \int_{2\pi^-} d\mathbf{\Omega}' \int_{2\pi^-} d\mathbf{\Omega}\Gamma(\mathbf{\Omega}' \to \mathbf{\Omega})$$
$$\Gamma^{\uparrow\downarrow} = \int_{2\pi^+} d\mathbf{\Omega}' \int_{2\pi^-} d\mathbf{\Omega}\Gamma(\mathbf{\Omega}' \to \mathbf{\Omega})$$
$$\Gamma^{S\downarrow} = \int_{2\pi^-} d\mathbf{\Omega}\Gamma(\mathbf{\Omega}_S \to \mathbf{\Omega}).$$

XXX

The equation for radiance in the direction of observation can simply be written as

$$\mu(\Omega_V)\frac{dI(L,\Omega_V)}{dL} = G(L,\Omega_V)I(L,\Omega_V) - \frac{1}{\pi}\int_{4\pi} d\Omega' I(L,\Omega')\Gamma(L,\Omega' \to \Omega_V) - \frac{1}{\pi}\int_{4\pi} I_S(L,\Omega')\Gamma(L,\Omega' \to \Omega_V)d\Omega'.$$

The associated flux is defined as (in accordance with E_o of Verhoef, 1984) DEF: E_V

$$E_V = \pi I_V = \pi I(\Omega_V).$$

Another flux contributed by radiance in the direction Ω_V on a horizontal surface is defined as DEF: F_V

$$F_V = I(L,\Omega_V)\,|\cos\vartheta_V|$$

Denoting $\mu_V = |\cos\vartheta_V|$, for an upward direction (i.e., looking downwards on the canopy), $\mu_V = \cos\vartheta_V$ (direct radiation is treated as a delta function),

$$\mu_V\frac{dI_V(L)}{dL} = G(L,\Omega_V)I_V(L) - \frac{1}{\pi}\int_{4\pi} d\Omega' I(L,\Omega')\Gamma(L,\Omega' \to \Omega_V) - \frac{1}{\pi}I_S(L)\Gamma(L,\Omega_S \to \Omega),$$

for a downward direction $\mu_V = -\cos\vartheta_V$

$$-\mu_V \frac{dI_V(L)}{dL} = G(L,\Omega_V)I_V(L) - \frac{1}{\pi}\int\limits_{4\pi} d\Omega' I(L,\Omega')\Gamma(L,\Omega' \to \Omega_V) - \frac{1}{\pi} I_S(L)\Gamma(L,\Omega_S \to \Omega_V).$$

Denoting DEF: B_V

$$B_V(L) = \frac{1}{\pi\mu_V}\int\limits_{4\pi} d\Omega' I(L,\Omega')\Gamma(L,\Omega' \to \Omega_V)$$

and DEF: S_V

$$S_V = \frac{1}{\pi\mu_V} I_S(L)\Gamma(L,\Omega_S \to \Omega_V) = \frac{1}{\pi\mu_V \cos\theta_S} F_S(L)\Gamma(L,\Omega_S \to \Omega_V),$$

the equations for radiance in the view direction become

$$\pm\mu_V \frac{dI_V(L)}{dL} = G(L,\Omega_V)I_V(L) - \mu_V\left(B_V(L) + S_V(L)\right),$$

for the fluxes

$$\begin{aligned}
\pm\mu_V \frac{dE_V(L)}{dL} &= G(L,\Omega_V)E_V(L) - \pi\mu_V\left(B_V(L) + S_V(L)\right) \\
\pm\mu_V \frac{dF_V(L)}{dL} &= G(L,\Omega_V)F_V(L) - \mu_V^2\left(B_V(L) + S_V(L)\right)
\end{aligned}$$

XXX

$$\begin{aligned}
\frac{dE_s}{dx} &= kE_s \\
\frac{dE_-}{dx} &= -sE_s + aE_- - \sigma E_+ \\
\frac{dE_+}{dx} &= s'E_s + \sigma E_- - aE_+ \\
\frac{dE_V}{dx} &= wE_s + vE_- + uE_+ - KE_V.
\end{aligned}$$

2.1.3 Reflectance and transmittance factors and submodels

FRT outputs reflectance factors defined as

$$\rho_X = \frac{L_X}{\frac{1}{\pi}F},$$

where L_X is a radiance produced by some canopy element or process, and $\frac{1}{\pi}F$ is the radiance produced by a non-absorbing horizontal Lambertian scatterer at top-of-canopy (TOC) level. Evidently, this makes F the total (diffuse+direct) irradiance at TOC.

FRT computes separately first-order and diffuse reflectance factors using two different submodels, geometric-optical (GO) and two-stream (2S), and for two different incidence types, the direct beam and diffuse (isotropic) TOC irradiance. The irradiation conditions are quantified using the ratio of direct to total irradiance, q_S. FRT uses computes the radiance in two opposing view directions, one above and one below the canopy (reflectance and transmittace coefficients, respectively). The following components are added to obtain forest reflectance and transmittance:

1. first-order contribution to reflectance of sunlit ground using direct transmittances T_0, T_V from GO,
2. the crown-contributed first-order reflectance ρ^1_{CR} (GO),
3. the crown-contributed first-order transmittance τ^1_{CR} (GO),
4. the higher-order reflectance (order higher than 1) contribution from canopy elements, including all orders of diffuse-sky reflectance component, $R_{hd,hi,can}$ (2S),

5. the higher-order reflectance (order higher than 1) contribution from ground, including all orders of diffuse-sky reflectance component, $R_{hd,hi,gnd}$ (2S),

6. the higher-order transmittance (order higher than 1) contribution from canopy elements, including all orders of diffuse-sky reflectance component, $T_{hd,hi}$ (2S).

The components are computed from layer scattering factors, computed for a black soil. Latere, for reflectance components 3–5, a reflecting ground is added (section 2.5.). The following layer scattering functions are used in frt:

1. ρ^1_{CR} , crown-contributed first-order reflectance (GO),

2. τ^1_{CR}, crown-contributed first-order transmittance (GO),

3. t_{di}, flux transmittance for direct incidence (currently [2022] computed from 2S),

4. r_{id}, directional reflectance for diffuse incidence (2S),

5. t_{id}, directional transmittance for diffuse incidence (2S),

6. r_{ii}, bi-hemispherical transmittance (2S),

7. t_{ii}, bi–hemispherical transmittance (2S),

8. $r_{dd,high}$, the higher-order reflectance (order higher than 1) contribution from canopy elements, direct beam incidence (2S),

9. $t_{dd,high}$, the higher-order transmittance (order higher than 1) contribution from canopy elements, direct beam incidence (2S).

These layer scattering factors follow the general formalism of Kuusk (2001) & Verhoef (1984) with differences in index letters. The scattering factors are distinct from reflectance and transmittance factors in the sense that the formalism contains two independent incident fluxes, hence it is possible to have a scattering factor from direct (index "s" by Kuusk, 2001) to isotropic incidence (index "d" by Kuusk, 2001) traveling in the same direction. The index letters denote four types of fluxes or radiances:

1. downward isotropic (denoted by letter "i", "d" by Kuusk, 2001) flux,

2. downard directional (denoted by "d", "s" by Kuusk, 2001) flux,

3. upward isotropic (denoted by letter "i", "d" by Kuusk, 2001) flux,

4. radiance towards the sensor (denoted by "d" , "o" by Kuusk, 2001).

The scattering factors are denoted by r_{xy} or t_{xy} with x being the source flux and y the destination flux or radiance ($x \rightarrow y$). As component #4 is a radiance, it cannot serve as x. Hence, "d" as the first index always denotes the solar beam (#2) and in the second position, radiance towards the sensor (#4). The other two scattering factors are considered symmetrical (in the current implementation), r_{ii} denotes backscatter from #1 to #3, but also from #3 to #1 (note: this may change some time in the furture?). The directional solar flux (#2) cannot be in the place of y as it can only extinguished in the canopy. All scattering factors are denoted by lowercase letters here and are computed for a black ground (i.e., a layer in vacuum). The nomenclature here is set up to be in line with the canopy reflectance and transmittance factors, but instead of "hemispherical", "isotropic" is used to explicate the exclusion of the direct solar beam (#2) from the downward isotropic flux (#1). For traditional reflectace factors, h indicates hemispherical and d directional.

The layer scattering factors "dy" (i.e., from the direct beam #2 into any other direction) can be split into first-order and higher-order components. This is not done for the isotropic fluxes. The factors $t_{dd,high}$and $r_{dd,high}$ are needed for this harmonization of the GO and 2S models (section (2.1.3)),

2.1.4 Introduction of shoot-level clumping

Canopy transmittance can be obtained from (5) by setting $\Gamma = 0$ and integrating from TOC to bottom-of-canopy (BOC, $L(z) = \mathrm{LAI}$)

$$\begin{aligned}\int_{I(\mathrm{TOC})}^{I(\mathrm{BOC})} \frac{dI(L,\Omega)}{I(L,\Omega)} &= \int_0^{\mathrm{LAI}} \frac{G(\Omega)}{\mu(\Omega)} dL \\ -\ln \frac{I(\mathrm{BOC})}{I(\mathrm{TOC})} &= \frac{G(\Omega)}{\mu(\Omega)} \mathrm{LAI} \\ t_{CAN} &= \exp\left[-\frac{G(\Omega)}{\mu(\Omega)} \mathrm{LAI}\right], \end{aligned} \tag{10}$$

where we have assumed that G is independent of the canopy depth $L(z)$ and the canopy transmittance $t_{CAN} = \frac{I(\mathrm{BOC})}{I(\mathrm{TOC})}$. Shoot-level clumping (grouping) has traditionally be introduced to canopy transmittance (Oker-Blom and Kellomäki, 1983; Oker-Blom and Smolander, 1988; Chen et al., 1994; Stenberg et al., 2014; Nilson, 1999) by modifying the extinction coefficient, i.e., introducing a clumping index κ_{sh} (`ShootClumping`, fortran: `clmpsh`):

$$t_{CAN} = \exp\left[-\kappa_{sh} \frac{G(\Omega)}{\mu(\Omega)} \mathrm{LAI}\right].$$

This effectively introduces the effective LAI $\kappa_{sh}\mathrm{LAI}$ and the corresponding downward cumulative effective leaf area $\kappa_{sh}L(z)$. Note that $\kappa_{sh}\mathrm{LAI}$ only accounts for shoot-level clumping, other "effecive LAI" definitions will be introduced later. The radiative transfer equation (5) for $\kappa_{sh}L$ still holds under the original assumptions,

$$\mu(\Omega)\frac{dI(L,\Omega)}{\kappa_{sh}dL} = G(\Omega)I(\kappa_{sh}L,\Omega) - \frac{1}{\pi}\int_{4\pi} I(\kappa_{sh}L,\Omega')\Gamma(\Omega' \to \Omega)d\Omega' - \frac{1}{\pi}I_S(\kappa_{sh}L)\Gamma(\Omega_S \to \Omega).$$

In the case of a 3D canopy, we can introduce clumping in a similar way, by defining the effective leaf area density, $\kappa_{sh}u(\mathbf{r})$ and using it in Eq. (1) instead of $u(\mathbf{r})$.

2.1.5 Scaling needle reflectance

If shoot is treated as the basic scattering unit, instead of the leaf (needle) reflectance, shoot reflectance should be used instead. The most straightforward way to do this would be using the photon recollision probability p, which can be calculated from the clumping index $\kappa_{sh} = 4\overline{\mathrm{STAR}}$ as $p = 1 - \kappa_{sh}$ (Smolander and Stenberg, 2003). We can easily scale the leaf albedo $\omega_L = r_L + t_L$ to the shoot abedo as

$$\omega_{sh} = \frac{(1-p)\,\omega_L}{1 - p\omega_L}$$

(Rautiainen et al., 2012); however, little information is avaiable on the relationship between needle reflectance and shoot reflectance (scattering in the backward hemisphere relative to incident light travel), and needle and shoot transmittance (scattering into forward hemisphere). Only very few studies are available on shoot scattering phase function, e.g. by (Nilson and Ross, 1997; Mõttus et al., 2012) and these are somewhat contorversial; an eqution relating shoot diffuse (non-direct) transmittance to its structure is given by Rautiainen et al. (2018), although based only on 10 Scots pine shoots. Without reliable knowledge, the frt does not change scattering directionality, i.e., it uses a flat leaf phase function with scaled reflectance and transmittance, $r_{sh} = (1-p)\,r_L/\,(1-p\omega_L)$, $t_{sh} = (1-p)\,t_L/\,(1-p\omega_L)$.

2.2 First-order scattering

The theory here should be based on the work by Nilson and Kuusk (1989); Nilson and Peterson (1991).

2.2.1 First-order radiance

TBW from RTE (5):

$$\mu(\Omega)\frac{dI(L,\Omega)}{dL} = G(L,\Omega)I(L,\Omega) - \frac{1}{\pi}\int_{4\pi} I(L,\Omega')\Gamma(L,\Omega' \to \Omega)d\Omega' - \frac{1}{\pi}I_S(L)\Gamma(L,\Omega_S \to \Omega),$$

2.2.2 Calculating ρ^1_{CR}

First-order scattering (reflectance for upward scattering, transmittance for downward) coefficient is calculated as

$$\rho^1_{CR}(\mathbf{\Omega}_1, \mathbf{\Omega}_2) = \sum_{j=1}^{m} \rho^1_{CR,j}(\mathbf{\Omega}_1, \mathbf{\Omega}_2),$$

$$\rho^1_{CR,j}(\mathbf{\Omega}_1, \mathbf{\Omega}_2) = \lambda_j c_j \int_{V_j} u_j \Gamma(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \frac{p_{00,j}(x, y, z, \mathbf{\Omega}_1, \mathbf{\Omega}_2)}{\cos\vartheta_1} dV,$$

where m is the number of tree classes, λ_j is stand density, u_j is foliage area density within a tree crown, $\Gamma(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$ is the area scattering phase function, $p_{00,j}$ is bidirectional gap probability at the point (x, y, z) within a tree crown, c_j is a parameter to account for tree distribution, V_j is the crown envelope.

$\Gamma(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$ is a weighted average of all canopy elements, it can be calculated analytically for bi-Lambertian leaves & random orientation.

$p_{00,j}$ is a product of two independent probabilities, $p_{00,j} = p_1 p_2$, $p_1$is within-tree gap probability, p_2 is between-crown gap probability

p_1 = ? [now idea yet how it's calculated]

p_2 according to Nilson and Peterson (1991),

$$p_2 = a_S(z_1, \vartheta_1) a_S(z_1, \vartheta_2) C_{HS2}(z_1, z_2, l_{12}, \mathbf{\Omega}_1, \mathbf{\Omega}_2),$$

$a_S(z, \vartheta)$ is the average proprotion of gaps in forest canopy at height z in the direction ϑ, $C_{HS2}(z_1, z_2, l_{12}, \mathbf{\Omega}_1, \mathbf{\Omega}_2)$ is a hot-spot correction factor (crown-level).

By Kuusk and Nilson (2000):

$$a_S(z, \vartheta) = \exp\left\{ -\sum_j N_j b_{1,j}(z, \vartheta) \left[S_{crown,j}(z, \vartheta) + S_{trunk,j}(z, \vartheta) \right] \right\}, \tag{11}$$

$$b_{1,j} = \frac{\ln\left[1 - (1 - a_{1,j}(z, \vartheta))(1 - c_j)\right]}{1 - c_j},$$

$S_{crown,j}$ is the crown projection area on a horizontal surface at height z, $S_{trunk,j}$ is the trunk projection area on a horizontal surface at height z, c_j is Fischer's grouping index, $a_{1,j}$ is the gap probability inside a crown at height z,

$$a_{1,j}(z, \vartheta) = \exp\left[-u_j \frac{V_j(z)}{S_{crown,j}(z, \vartheta) \cos\vartheta} \right],$$

$V_j(z)$ is the volume of part of the tree crown that is above z.

C_{HS2}: maybe later sth on that, too

2.3 Canopy grouping index

Fisher's grouping index c is the relative varianceof number of trees in the projection area of a crown. It depends on the projection area of a tree crown and thus on viewing angle; its estimation is complicated. For a positive binomial distribution, $c = 1 - N\sigma$, where N is the number of trees per unit ground area and σ is the 'living area' of a single tree [same gound units] (Nilson, 1999); for a Poisson distribution, $c = 1$.

The model requires as an input a correction factor (tree distribution) parameter, `TreeClumping`)

$$c_B = \frac{\ln c}{1 - c},$$

used for calculating canopy cover

$$C_{CAN} = 1 - \exp(-c_B C_{CR}), \tag{12}$$

where C_{CR} is crown cover (the sum of crown projections per unit ground area). The equation can be readily derived from Eq. (11). Fisher's grouping index c used in the model is calculated by numerically inverting the

equation for c_B. Eq. (12) is the same as (7a) for the vertical direction by Nilson (1999). It holds for transparent crowns if the crown area in computing of $C_{CR}$is also reduced by the area of gaps in crown projections. For opaque crowns – or, equivalently, C_{CAN} including only between-crown gaps – we get the identical Eq. (10) by Nilson (1999). In the model, `CanopyCover` and `CrownCover` (`cano` and `vliit`, respectively, in fortran; Nilson (1999) uses "crown closure" for C_{CAN} – however, *closure* is commonly used in forestry for transmittance integrated over the hemiphere while *cover* is measured in the vertical direction) are computed assuming opaque corwns, within-crown gaps are ignored. Similarly, from here onward, we assume opaque crown in computing these two quantities.

The equations abové yield the following relationship:

$$c_B = -\ln \frac{1 - C_{CAN}}{C_{CR}}$$

and when no crown overlapping occurs, $C_{CAN} = C_{CR} = 1 - c$ (Eq. above (2) by Nilson and Kuusk, 2004). The limit of no overlapping crowns is the physical limit for c. Mathematically, c can further decrease (more regular tree distribution), but this leads to the physically impossible situation where $C_{CAN} > C_{CR}$.

For a forest stand consisting of a single tree class, the situation can be readily solved by limiting $c \geq 1 - C_{CR}$.C_{CR} can be simply determined from tree density and crown geometry and hence this restriction can be easily implemented – as is done if `correctFGI` is `True` in the python code. This flag forces the minimum c individually for all tree classes: in the case of an inital $c_B$value in the configuration file causing an unphysical layer, this leads to setting

$$c = 1 - C_{CR}$$

in the `frt_model.configure_frt()` function the tree class violating physics.

The correction has its shortcomings. While it implements the natural limit that no single tree class can have a canopy cover exceeding its crown cover, it ignores interactions between tree classes. In the current theoretical framework, classes are independent and forest transmittance is computed by a simple multiplication of the transmittances of the classes. A decrease of c beyond its physical limit computed from its own crown cover would decrease simulated canopy transmittance (increase C_{CAN}) as if the class interacted with trees from other classes. In othe words, the decreased transmittance could counterbalance the (probably) incorrect assumption that trees of different classes do not interact. The descritption here is speculative and, furthermore, Nilson (1999) has stated that "for the within-tree gaps it tends to provide systematically underestimated values of the gap proportion when the tree distribution pattern is of a regular type" – there may be no need to increase the simulated transmittance.

An alternative approach – not implemeted in the model could be using the multi-class version of (12),

$$1 - C_{CAN} = \exp\left\{-\sum_j c_{B,j} C_{CR,j}\right\}.$$

To force physicality, we can correct the $c_{B,j}$ at the limit of non-overlapping crowns ($C_{CR} = C_{CAN}$) by introducing a correction factor k_{cB}, so that

$$1 - C_{CR} = \exp\left\{-\sum_j k_{cB} c_{B,j} C_{CR,j}\right\}$$

$$k_{cB} = -\frac{\ln(1 - C_{CR})}{\sum_j c_{B,j} C_{CR,j}}$$

and multiply all $c_{B,j}$ with this value. Documentation will be changed if and when this will be implemented.

2.4 Diffuse reflectance (2-stream component)

XXX explain the different components for the black soil case – there is some text already in this document

2.5 The effect of reflecting soil surface (2-stream)

As described in section (2.1.3), reflectance factors are computed for black soil separately for direct and diffuse illumination and a reflective surface is added later. Computation of total reflectance starts with the fractions of fluxes that are reflected by the canopy (no soil effect), to which then we ad the component that has been reflected by the surface (soil) once + the component that has been reflected by the surface twice + ...

We split the reflectance signal for this case into three more subcomponents:

- contributed by photons which were intercepted by the canopy and arrive to the sensor after one or more interactions, the last being with a a canopy element, R_{can},
- the contribution by the photons which have been directly transmitted by the canopy, reflected by the ground and reach the sensor after one or more interactions, the last being with a a canopy element, R_{trans},
- and the contribution of photons which reach the sensor from the ground, either sunlit or shaded, without hitting any leaves after the last interaction with the ground, R_{ground} .

The total reflectance of the canopy–soil system is

$$R = R_{can} + R_{trans} + R_{ground}.$$

Later, the three components will be split into scattering orders.

Considering that a fraction of q_S photons arrive at TOC as the direct beam and using the notations in section (2.1.3), we get for the sum of scattering orders

$$\begin{aligned}
R_{can}(2\pi \to \mathbf{\Omega}_{view}) &= q_S\left[\rho^1_{CR} + r_{dd,high}\right] + (1-q_S)r_{id} + \left[q_S t_{di} + (1-q_S)t_{ii}\right] R_{soil,hh} t_{id} \\
&\quad + \left(q_S t_{di} + (1-q_S)t_{ii}\right) R_{soil,hh} r_{ii} R_{soil,hh} t_{id} \\
&\quad + \left(q_S t_{di} + (1-q_S)t_{ii}\right) R_{soil,hh} r_{ii} R_{soil,hh} t_{ii} R_{soil,hh} t_{id} + \ldots \\
&= q_S\left[\rho^1_{CR} + r_{dd,high}\right] + (1-q_S)r_{id} + \left(q_S t_{di} + (1-q_S)t_{ii}\right) t_{id} R_{soil,hh} \\
&\quad + \left(q_S t_{di} + (1-q_S)t_{ii}\right) t_{ii} r_{ii} R^2_{soil,hh} + \left(q_S t_{di} + (1-q_S)t_{ii}\right) t_{id} r^2_{ii} R^3_{soil,hh} + \ldots \\
&= q_S\left[\rho^1_{CR} + r_{dd,high}\right] + (1-q_S)r_{id} + \left(q_S t_{di} + (1-q_S)t_{ii}\right) t_{id} R_{soil,hh} \sum_{i=0}^{\infty} \left(r_{ii} R_{soil}\right)^i \\
&= q_S\left[\rho^1_{CR} + r_{dd,high}\right] + (1-q_S)r_{id} + \frac{\left[q_S t_{di} + (1-q_S)t_{ii}\right] t_{id} R_{soil,hh}}{1 - r_{ii} R_{soil,hh}},
\end{aligned} \tag{13}$$

where q_S is the ration of direct to diffuse flux, r_{id} is the hemispherical-directional reflectance of the canopy layer (black ground), r_{ii} is the bihemispherical reflectance of the canopy layer (black ground and true diffuse sky, but as the model is symmetrical to up/down directions, it is assumed to be the same if illuminated and viewed from below), t_{di} is the directional hemispherical transmittance (black ground), t_{ii} is the bihemispherical transmittance (black ground, as viewed from below, but as the model is symmetrical to up/down directions, it is assumed to be the same for a canopy illuminated diffusely from above).

The second component $R_{trans}(2\pi \to \mathbf{\Omega}_{view})$ describes photons which are first transmitted by the canopy, reflected by the ground (so they create a source there), and then hit a canopy element at least once with their last interaction being with a canopy element. This R_{trans} does not contain a diffuse component weighted by $(1 - q_S)$ as for diffuse radiation, diffuse and direct transmitttance are not separabe and this component is included in R_{can} via t_{ii}.

$$
\begin{aligned}
R_{trans} =& q_S T_0 R_{soil,dh} t_{id} + q_S T_0 R_{soil,dh} r_{ii} R_{soil,hh} t_{id} + q_S T_0 R_{soil,dh} r_{ii} R_{soil,hh} r_{ii} R_{soil,hh} t_{id} + \ldots \\
=& q_S T_0 R_{soil,dh} t_{id} + q_S T_0 R_{soil,dh} r_{ii} R_{soil,hh} t_{id} + q_S T_0 R_{soil,dh} r_{ii}^2 R_{soil,hh}^2 t_{id} + \ldots \\
=& q_S T_0 R_{soil,dh} t_{id} \left\{1 + r_{ii} R_{soil,hh} + r_{ii}^2 R_{soil,hh}^2 + \ldots\right\} \\
=& q_S T_0 R_{soil,dh} t_{id} \sum_{i=0}^{\infty} \left(r_{ii} R_{soil,hh}\right)^i \\
=& \frac{q_S T_0 R_{soil,dh} t_{id}}{1 - r_{ii} R_{soil,hh}}.
\end{aligned}
$$

The third component is contributed by ground-hitting direct photons not interacting with the canopy on their exit,

$$
\begin{aligned}
R_{ground} =& q_S R_{soil,dd} p_{gap}(\mathbf{\Omega}_S, \mathbf{\Omega}_V) \\
&+ q_S t_{di} R_{soil,hd} T_V + (1 - q_S) t_{ii} R_{soil,hd} T_V \\
&+ q_S T_0 R_{soil,dh} r_{ii} R_{soil,hd} T_V + \ldots \\
&+ q_S t_{di} R_{soil,hh} r_{ii} R_{soil,hd} T_V + (1 - q_S) t_{ii} R_{soil,hh} r_{ii} R_{soil,hd} T_V + \ldots \\
=& q_S R_{soil,dd} p_{gap}(\mathbf{\Omega}_S, \mathbf{\Omega}_V) \\
&+ q_S t_{di} R_{soil,hd} T_V \left\{1 + r_{ii} R_{soil,hh} + r_{ii}^2 R_{soil,hh}^2 + \ldots\right\} \\
&+ \left(1 - q_S\right) t_{ii} R_{soil,hd} T_V \left\{1 + r_{ii} R_{soil,hh} + r_{ii}^2 R_{soil,hh}^2 + \ldots\right\} \\
&+ q_S T_0 R_{soil,dh} r_{ii} R_{soil,hd} T_V \left\{1 + r_{ii} R_{soil,hh} + r_{ii}^2 R_{soil,hh}^2 + \ldots\right\} \\
=& q_S R_{soil,dd} p_{gap}(\mathbf{\Omega}_S, \mathbf{\Omega}_V) \\
&+ R_{soil,hd} T_V \left\{q_S t_{di} + \left(1 - q_S\right) t_{ii} + q_S T_0 R_{soil,dh} r_{ii}\right\} \\
&\times \left\{1 + r_{ii} R_{soil,hh} + r_{ii}^2 R_{soil,hh}^2 + \ldots\right\} \\
=& q_S R_{soil,dd} p_{gap}(\mathbf{\Omega}_S, \mathbf{\Omega}_V) \\
&+ \frac{\left\{q_S t_{di} + \left(1 - q_S\right) t_{ii} + q_S T_0 R_{soil,dh} r_{ii}\right\} R_{soil,hd} T_V}{1 - r_{ii} R_{soil,hh}}
\end{aligned}
$$

where $R_{soil,dd}$ is the bidirectional reflectance of the forest floor and $p_{gap}(\mathbf{\Omega}_S, \mathbf{\Omega}_V)$ is the bidirectional gap probability.

Total flux reflectace is thus

$$
\begin{aligned}
R(2\pi \rightarrow \mathbf{\Omega}_{view}) &= R_{can} + R_{ground} + R_{trans} \\
&= q_S \left[\rho_{CR}^1 + r_{dd,high}\right] + (1 - q_S) r_{id} + q_S R_{soil,dd} p_{gap}(\mathbf{\Omega}_S, \mathbf{\Omega}_V) \\
&\quad + \frac{\left[q_S t_{di} + (1 - q_S) t_{ii}\right] t_{id} R_{soil,hh} + q_S T_0 R_{soil,dh} t_{id}}{1 - r_{ii} R_{soil,hh}} \\
&\quad + \frac{\left\{q_S t_{di} + \left(1 - q_S\right) t_{ii} + q_S T_0 R_{soil,dh} r_{ii}\right\} R_{soil,hd} T_V}{1 - r_{ii} R_{soil,hh}}.
\end{aligned} \tag{14}
$$

The transmittance $T(2\pi \rightarrow \mathbf{\Omega}_{view})$ of a similar layer can be calculated as follows:

$$
\begin{aligned}
T &= q_S \left(\tau_{CR}^1 + t_{dd,high}\right) + (1 - q_S) t_{id} \\
&\quad + \left\{q_S \left[T_0 R_{soil,dh} + t_{di} R_{soil,hh}\right] + (1 - q_S) t_{ii} R_{soil,hh}\right\} r_{id} \\
&\quad + \left\{q_S \left[T_0 R_{soil,dh} + t_{di} R_{soil,hh}\right] + (1 - q_S) t_{ii} R_{soil,hh}\right\} r_{ii} R_{soil,hh} r_{id}(\mathbf{\Omega}_V) + \ldots \\
&= q_S \left(\tau_{CR}^1 + t_{dd,high}\right) + (1 - q_S) t_{id} \\
&\quad + \left\{q_S \left[T_0 R_{soil,dh} + t_{di} R_{soil,hh}\right] + (1 - q_S) t_{ii} R_{soil,hh}\right\} \\
&\quad \times \left(1 + R_{soil,hh} r_{ii} + R_{soil,hh}^2 r_{ii}^2 + \ldots\right) r_{id} \\
&= q_S \left(\tau_{CR}^1 + t_{dd,high}\right) + (1 - q_S) t_{id} \\
&\quad + \frac{\left\{q_S \left[T_0 R_{soil,dh} + t_{di} R_{soil,hh}\right] + (1 - q_S) t_{ii} R_{soil,hh}\right\} r_{id}}{1 - r_{ii} R_{soil,hh}}.
\end{aligned} \tag{15}
$$

2.6 Matching 1st and higher order reflectance (correcting model,...)

The model consists of two parts: the geometric-optical submodel for calculating direct transmittance and first-order scattering of the incident direct solar beam, and the two-stream submodel, used for calculating higher-order scattering and first-order scattering of the direct beam. Without restricting generality, let us now look at the case of direct illumination only, where the ratio of direct to total irradiance $q_S = 1$.

As the first-order reflectances of the two submodels are different, the model is not energy-conservative. To compensate for that, the different components are calculated for a conservative environment ($\omega_L = r_L + t_L = 1$), no diffuse sky radiation and a black background, and a correction factor is found. For more details, see (Mõttus et al., 2007).

For a conservative environment and black background, an energy-conservative model yields

$$\left\{\rho^1_{CR} + \tau^1_{CR} + r_{dd,high} + t_{dd,high}\right\}_{\omega_L=1} + T_0 = 1, \tag{16}$$

where ρ^1_{CR} is canopy first-order reflectance, τ^1_{CR} is first order diffuse transmittance, $r_{dd,high}$ is higher-order reflectance, $t_{dd,high}$ is higher-order transmittance and T_0 is direct transmittance, integrated over 2π. For (16) to hold, the model must be bound from below by black surface (or, equivalently, no surface at all). For a geometric-optical model consisting of two submodels, (16) need not hold automatically. Assuming that first-order components are calculated correctly, we can introduce a correction factor C which modifies the higher-order components are corrected as

$$\left\{\rho^1_{CR} + \tau^1_{CR} + C_1\left[r_{dd,high} + t_{dd,high}\right]\right\}_{\omega_L=1} + T_0 = 1,$$

where the components now denote reflectances and transmittances predicted by FRT. The correction factor can be calculated as

$$C_1 = \left\{\frac{1 - \left(\rho^1_{CR} + \tau^1_{CR} + T_0\right)}{r_{dd,high} + t_{dd,high}}\right\}_{\omega_L=1}. \tag{17}$$

If $C_1 > 1$, the two-stream submodel overestimates first-order scattering reflectance component. A source must be added to compensate for this lost radiation. Multiplying by C_1 increases the output of this submodel, and as the equation of RT is linear, it is equivalent to increasing its input. Increasing input can be viewed as adding an extra source at the places where first-order scattering takes place. As the difference of first-order scattering reflectance component of the two submodels depends linearly on leaf single-scattering albedo (and not on multiple-scattering effects) and so does the source created by C_1, this correction has to be constant with leaf albedo (or wavelength). Thus, C_1 calculated under these conditions can be applied at all wavelengths. The only uncertainty in this algorithm is the vertical distribution of this energy loss at first scattering, but a more detailed source cannot be inserted into the two-stream submodel (or it is most likely not worth the trouble).

Also, the predicion of first-order scattering may depend on the ratio of leaf reflectance to transmittance–if it is not constant with wavelength, the correction is not accurate. The model can to calculate the correction factor for each wavelength; alternatively the dependence is assumed to be weak and in the input file, a wavelength may be specified for which C_1 is calculated. If given, the wavelength determines only the calculation of the ratio of leaf reflectance to leaf transmittance.

The final task is identifying the terms to which the correction factor C_1 in (17) should be applied. As described in in sections (2.5) and quantified in (2.5), the reflectance and transmittance of the forest canopy–forest floor system is computed first for black forest floor, and multiple scattering between the floor and canopy is accounted for later using the two-stream submodel. C_1 should only be applied to the terms in (14) and (15) to the terms describing the first interaction, and not the ones including the effect of the forest floor. Specifically, in the first line of (13), these are the quantities which depend on the incident direction and are computed using the 2S submodel, $r_{dd,high}$ and t_{di}. t_{id} included in that line should not be corrected as it describes the transmittance of the canopy layer after the interaction with the ground and is a part of the inclusion of a reflecting forest floor. Similarly, in (15) the corrected quantities are $t_{dd,high}$ and t_{di}. The corrected equation for T becomes

$$\begin{aligned} T =& q_S\left(\tau^1_{CR} + C_1 t_{dd,high}\right) + (1 - q_S)t_{id} \\ &+ \frac{\left\{q_S\left[T_0 R_{soil,dh} + C_1 t_{di} R_{soil,hh}\right] + (1 - q_S)t_{ii} R_{soil,hh}\right\} r_{id}}{1 - r_{ii} R_{soil,hh}}. \end{aligned} \tag{18}$$

In the fortran code, the correction is implemented in twostr.f and diffor.f. The code does not compute the layer scattering factor $t_{dd,high}$ required the equations above. Instead, a black box (based on Verhoef???) is implemented in bgrdd.f, which apparently outputs higher-order refelctance, which, based on 18 is (after deleting the first-order component, note that C_1 is obviously not used in the code)

$$\begin{aligned} T_{can,high} =& q_S t_{dd,high} + (1-q_S) t_{id} \\ &+ \frac{\left\{q_S \left[T_0 R_{soil,dh} + t_{di} R_{soil,hh}\right] + (1-q_S) t_{ii} R_{soil,hh}\right\} r_{id}}{1 - r_{ii} R_{soil,hh}}. \end{aligned}$$

Setting $q_S = 1$ and $R_{soil,xx} = 0$, we get

$$T_{can,high}|_{BS,direct} = t_{dd,high}.$$

Other layer scattering coefficients are computed in layer.f. All scattering factors are corrected and converted to the canopy scattering and transmittance factors (with the inclusion of the effect of the underlying surface) in twostream.f.

2.7 Canopy absorption

Total canopy absorption for reflecting ground surface (required by e.g, RAMI calculations) can be calculated by considering the conservation of energy: the part of phhotons not reflected is either absorbed by the ground or by the canopy. Absorption by ground can be calculated as

$$A_{gr} \quad = \quad t_0 \left(1 - R_{dh,gr}\right) + t_{dif} \left(1 - R_{hh,gr}\right),$$

where $R_{dh,gr}$ is ground directional-hemispherical reflectance, t_0 is the direct component of forest transmittance ($t_0 = q_S T_0$) calculated by the model, t_{dif} is the diffuse component of forest tranmisttance calculated by the model, $R_{hh,gr}$ is ground hemispherical-hemispherical reflectance. All components must be calculated using the actual ground reflectance.

Canopy absorption is now

$$A = 1 - r_{flux} - A_{gr},$$

where r_{flux} is forest flux reflectance for specified illumination conditions (integrated over the upper hemisphere).

3 Leaf reflectance and transmittance

3.1 General

Leaf reflectance and transmittance are read from a file or fixed in input file. Leaf albedo can be scaled (if the appropriate flag is set in the input file) to shoot level using the p-theory and κ_{sh} as follows: the leaf albedo $\omega_L = r_L + t_L$ is transformed into shoot albedo as $\omega_{sh} = \omega_L(1-p)/\left[(1-p)\omega\right]$, where $p = 1 - \kappa_{sh}$; leaf scattering directionality is preserved. Specifically, for shoot reflectance this yields $r_{sh} = r_L(1-p)/\left[(1-p)\omega\right]$.

The mean optical properties of scattering elements are calculated

- For each tree class separately by calculating the weighted average of leaf and branch reflectance and transmittance factors (excl. trunks). The transmittance of branches is taken zero. The weights are κ_{sh}LAI and BAI, respectively. For broadleaves, $\kappa_{sh} \equiv 1$. These values are used in calculating the first-order crown reflectance component ρ^1_{CR}.
- For all canopy, a single element spectral reflectance and an element spectral transmittance by weighing the treeclass-mean factors with the plant area (LAI+BAI) of the corresponding class. These factors are used for calculating the diffuse scattering and the flux reflectance and transmittance of the canopy.

Contribution of trunks is computed separately.

3.2 Leaf scattering coefficients

Let us start with the definition of the Ross-Nilson G-function:

$$G(L,\Omega) = \frac{1}{2\pi} \int\limits_{2\pi+} d\Omega_L g_L(L,\Omega_L) \left|\Omega_L \cdot \Omega\right|,$$

where $g_L(\Omega_L)$ is the leaf normal distribution function normalized so that

$$\int\limits_{2\pi+} d\Omega_L g_L(\Omega_L) = 2\pi$$

and $\gamma_L(\Omega_L, \Omega' \to \Omega)$ is the leaf scattering phase function. From (Knyazikhin and Marshak, 1991) Eq. 1.16[2]

$$\Gamma(\Omega' \to \Omega) = \frac{1}{2} \int\limits_{2\pi+} d\Omega_L g_L(\Omega_L) \left|\Omega' \cdot \Omega_L\right| \gamma_L(\Omega_L, \Omega' \to \Omega), \tag{19}$$

For a bi-Lamberian leaf (ibid, unnumbered eq. between 1.17 & 1.18), but for a bi-facial leaf,

$$\gamma_L(\Omega_L, \Omega' \to \Omega) \quad = \quad \begin{cases} \frac{1}{\pi} r_{L,1} \left|\Omega \cdot \Omega_L\right|, & (\Omega \cdot \Omega_L) > 0 \,\&\, (\Omega' \cdot \Omega_L) < 0 \\ \frac{1}{\pi} r_{L,2} \left|\Omega \cdot \Omega_L\right|, & (\Omega \cdot \Omega_L) < 0 \,\&\, (\Omega' \cdot \Omega_L) > 0 \\ \frac{1}{\pi} t_{L,1} \left|\Omega \cdot \Omega_L\right|, & (\Omega \cdot \Omega_L) < 0 \,\&\, (\Omega' \cdot \Omega_L) < 0 \\ \frac{1}{\pi} t_{L,2} \left|\Omega \cdot \Omega_L\right|, & (\Omega \cdot \Omega_L) > 0 \,\&\, (\Omega' \cdot \Omega_L) > 0 \end{cases}, \tag{20}$$

where r_{L1} is the adaxial (upper) leaf surface reflectance factor, r_{L2} abaxial (lower) reflectance, $t_{L,1}$ leaf transmittance when the adaxial (upper) surface is illuminated, $t_{L,2}$ transmittance when abaxial (lower) surface is illuminated. The directions are as follows: Ω_L the normal of adaxial (upper) leaf surface, Ω' direction of the travel of illuminating (e.g., solar) photons (i.e., direction to the leaf), Ω the direction from the leaf to the sensor (viewer).Similarly, the area scattering phase function becomes

$$\Gamma(\Omega' \to \Omega) = \frac{1}{2\pi} X_{LY} \int\limits_{2\pi+} d\Omega_L g_L(\Omega_L) \left|\Omega' \cdot \Omega_L\right| \left|\Omega \cdot \Omega_L\right|,$$

where X is either r or t and $Y$is 1 or 2, depending on the conditions in (20).

3.3 Specular reflectance

Specular reflectance calculated as

$$r_S = \left(\frac{1 - n_{wax}}{1 + n_{wax}}\right)^2,$$

where $n_{wax}$is wax refractive index, is subtracted from total reflectance and diffuse reflectance is stored in `rlfcl`. This specular reflectance value corresponds to normal incidence of radiation,

$$r_S = \left(\frac{n_{wax} - n_{air}}{n_{wax} + n_{air}}\right)^2,$$

where $n_{air} = 1$. For other angles, specular reflection depends on polarization (Fresnel's equations):

$$\begin{aligned} r_{S,\parallel} &= \frac{\tan^2(\vartheta - \vartheta_t)}{\tan^2(\vartheta + \vartheta_t)} \\ r_{S,\perp} &= \frac{\sin^2(\vartheta - \vartheta_t)}{\sin^2(\vartheta + \vartheta_t)} \\ \sin\vartheta_t &= \frac{n_{air}}{n_{wax}} \sin\vartheta, \end{aligned}$$

[2]This is valid if leaf normal is always in the upper hemisphere as is assumed here, i.e., abaxial side is facing downwards. If not, integrate over 4π and multiply the integral by $\frac{1}{4}$ instead of $\frac{1}{2}$.

ϑ_t is the angle at which light is transmitted through the surface.

Wax refractive index is read from a file (originating from the PROSPECT model) and is for tree crown layer corrected by a correction factor.- The correction factor may – or may not – be related to the K_0mentioned by Nilson and Kuusk (1989)

Different subroutines can add the specular component (`optmean`, diffuse fluxes in `smrcm` and `diffor`) using the sam.e equation; specular reflection is used in first order scattering in `smrcm` and `hetk8o`.

Transmittance is taken as calculated by a submodel or given in the input file.

To ignore specular reflectance, wax refractive index and wax refractive index correction factor can be set to 1 in the input file, or the correction factor can be set to a value so that refractive index $\times$ refractive index correction factor < 1 – then the program corrects the refractive index to 1.

4 Mathematical implementation

4.1 Integration quadratures

FRT contains 3-D integration over hemisphere or full sphere to

1. compute crown gaps and scattering, and
2. compute fluxes and flux reflectance factors.

Hence, two different quadratures are applied,

1. in cubell9 & cubcirc11 for integrating over ellipsoids consisting of `ncub` (defined in `frtpar.h`) directions. The quadrature is defined in the parameters in the `/volint/` common block;
2. in `comprt`, `grndref` and elsewhre to integrate directional reflectance and transmittance factors to obtain flux reflectance and transmittance, respectively.

Case #1 is described by a reference in the code. Case #2 uses a Gaussian quadrature computed by `gauleg()`.

The reflected (or transmitted) irradiance F_{out} is obtained by integrating the radiance I_{out} over the hemisphere, $F_{out} = \int_{2\pi} \cos\vartheta\, I_{out}(\vec{\Omega})\, d\vec{\Omega}$. Using the flux reflectance $R_F = F_{out}/F_{in}$ and the directional reflectance factor $R(\vec{\Omega})$=$I_{out}(\vec{\Omega})/I_{out}^{Lamb}$, were I_{out}^{Lamb} is the radiance produced by a non-absorbing Lambertian surface under similar conditions, $I_{out}^{Lamb} = F_{in}/\pi$, we get

$$R_F = \frac{\int_{2\pi} \cos\vartheta\, I_{out}(\vec{\Omega})\, d\vec{\Omega}}{F_{in}} = \frac{\int_{2\pi} \cos\vartheta\, R(\vec{\Omega}) F_{in}\, d\vec{\Omega}}{F_{in}\pi} = \frac{1}{\pi} \int_{2\pi} \cos\vartheta\, R(\vec{\Omega})\, d\vec{\Omega}.$$

Switching now to polar coordinates (ϑ, ϕ), $d\vec{\Omega} = \sin\vartheta\, d\vartheta\, d\phi$, $R_F = \frac{1}{\pi} \int_0^{\pi/2} d\vartheta\, \cos\vartheta\, \sin\vartheta\, \int_0^{2\pi} d\phi\, R(\vartheta, \phi)$. As $R(\vartheta, \phi)$ is a periodic function wrt. ϕ, the best quadrature for ϕ is regular sampling. To integrate over ϑ,we can

1. choose the best quadrature over $[0, \pi/2]$ and calculate $R(\vartheta, \phi)\, \sin\vartheta\, \cos\vartheta$ on cubature nodes, or
2. choose the best quadrature over $[0, 1]$ and calculate $\mu\, R(\arccos\mu)$ on cubature nodes (as $\int_0^{\pi/2} d\vartheta\, f(\vartheta)\, \cos\vartheta\, \sin\vartheta = \int_0^1 f(\arccos\mu)\, \mu\, d\mu$).

For historical reasons, option #1 is used in FRT. In classical RT approaches, the (slightly more accurate?) approach #2 is more common. In FRT, a Gauss-Legendre quadrature of order `nquad_t` is created over $[0, \pi/2]$ and the nodes are stored in `xquad_t`. The nodes for the quadrature over ϕ is are stored in `xquad_p`. The

functions are later evaluated on the quadrature as

$$\begin{aligned}\frac{1}{\pi}\int_{2\pi}\cos\vartheta\, f(\vec{\Omega})\, d\vec{\Omega} &= \frac{1}{\pi}\sum_{i=1}^{N(\vartheta)}\sum_{j=1}^{N(\phi)} w_{\vartheta,i}\, w_{\phi,j}\, f(\vartheta_i,\phi_j)\, \cos\vartheta_i\, \sin\vartheta_i \\ &= \sum_{i=1}^{N(\vartheta)}\frac{1}{\pi}w_{\vartheta,i}\, w_{\phi,j}\, \cos\vartheta_i\, \sin\vartheta_i \sum_{j=1}^{N(\phi)} f(\vartheta_i,\phi_j) \\ &= \sum_{i=1}^{N(\vartheta)}\sum_{j=1}^{N(\phi)} w_{i,j} f(\vartheta_i,\phi_j).\end{aligned}$$

The combined weights $w_{i,j}$ hence depend on the choice of the distribution of nodes along the zenith and aziputh angles (ϑ_i and ϕ_j, respectively) and are stored in `quad_q`. In FRT, Gauss-Legendre quadrature is used for ϑ_i and regular sampling over ϕ_j. As forest reflectance is symmetric wrt. the principal plane, ϕ_j is sampled between $[0,\pi]$ and the weights are constant, $w_{\phi,j} = 2\pi/N_\phi$, where N_ϕ is the number of nodes over ϕ. As a note, Gauss-Legendre quadrature over $[0,\pi]$ may also be considered at some day as the function is not periodic in this interval. The quadrature can therefore be used to calculate easily integrals like

$$\frac{1}{\pi}\int_{2\pi}\cos\vartheta\, f(\vec{\Omega})\, d\vec{\Omega} = 2\int_0^{\pi/2} d\vartheta\, \cos\vartheta\, \sin\vartheta\, f(\vartheta)$$

with weights

$$w_{i,j} \equiv w_i = w_{\vartheta,i}\, \cos\vartheta_i\, \sin\vartheta_i/\pi \times 2\pi/N_\phi = \frac{2w_{\vartheta,i}\, \cos\vartheta_i\, \sin\vartheta_i}{N_\phi},$$

where $w_{\vartheta,j}$ are the weigths of Gauss-Legendre quadrature over $[0,\pi/2]$. The cubature allows also to easily calculate e.g. effective (optical) LAI from the Miller's theorem ,

$$L = -2\int_0^{\pi/2} d\vartheta\, \cos\vartheta\, \sin\vartheta\, \ln\left(i_0(\vartheta)\right).$$

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