The Art of Computer Programming

Explicit Solutions

Keaun Moughari

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§ 1.2.4

42. re: Lower-bound on internal path length of extended binary trees (p. 400)

(a) Prove that $\sum_{k=1}^{n} a_k = na_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k), \quad \text{if } n > 0.$

Solution:

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= na_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k) \\
&= na_n + \sum_{k=1}^{n-1} ka_k - \sum_{k=1}^{n-1} ka_{k+1} \\
&= na_n + (a_1 + 2a_2 + \cdots + (n-1)a_{n-1}) \\
&\quad - (a_2 + 2a_3 + \cdots + (n-2)a_{n-1} + (n-1)a_n) \\
&= na_n - (n-1)a_n + (a_1 + 2a_2 + \cdots + (n-1)a_{n-1}) \\
&\quad - (a_2 + 2a_3 + \cdots + (n-2)a_{n-1}) \\
&= a_n + \sum_{k=1}^{n-1} ka_k - \sum_{k=1}^{n-2} ka_{k+1} \\
&= a_n + \sum_{k=0}^{n-2} (k+1)a_{k+1} - \sum_{k=1}^{n-2} ka_{k+1} \\
&= a_n + a_1 + \sum_{k=1}^{n-2} (k+1)a_{k+1} - \sum_{k=1}^{n-2} ka_{k+1} \\
&= a_n + a_1 + \sum_{k=1}^{n-2} (k+1)a_{k+1} - ka_{k+1} \\
&= a_n + a_1 + \sum_{k=1}^{n-2} a_{k+1} \\
&= a_n + a_1 + \sum_{k=2}^{n-1} a_k \\
&= \sum_{k=1}^{n} a_k
\end{aligned}$$

(b) The preceding formula is useful for evaluating certain sums involving the floor function. Prove that, if b is an integer ≥ 2,

$$\sum_{k=1}^{n} \lfloor \log_b k \rfloor = (n+1)\lfloor \log_b n \rfloor - (b^{\lfloor \log_b n \rfloor + 1} - b)/(b-1).$$

Aside:

(1) $f = \lfloor \log_b n \rfloor ; b \geq 2$

$$\begin{aligned}
S_{n-1} &= \sum_{k=1}^{f-1} b^k \\
&= b^{\lfloor \log_b 1 \rfloor} - b^{\lfloor \log_b n \rfloor} + \sum_{k=1}^{f-1} b^{k+1} \\
&= 1 - b^{\lfloor \log_b n \rfloor} + b \sum_{k=1}^{f-1} b^k \\
&= 1 - b^{\lfloor \log_b n \rfloor} + bS_{n-1} \\
(1-b)S_{n-1} &= 1 - b^{\lfloor \log_b n \rfloor} \\
\implies S_{n-1} &= \frac{b^{\lfloor \log_b n \rfloor} - 1}{b-1}
\end{aligned}$$

(2) $\sum_{1}^{f-1} = \sum_{j}^{f-1} + \sum_{1}^{j-1}$

$$\implies \sum_{j}^{f-1} = \sum_{1}^{f-1} - \sum_{1}^{j-1}$$

(3) $f = \lfloor \log_b n \rfloor ; b \geq 2$

$$\begin{aligned}
S_n &= \sum_{k=1}^{n-1} k \ [k+1 \text{ is a power of } b] \\
&= \sum_{k=1}^{f} (b^k - 1) \\
&= \sum_{k=1}^{f} b^k - \sum_{k=1}^{f} 1 \\
&= \sum_{k=1}^{f} b^k - f \\
&= \left[fb^f - \sum_{k=1}^{f-1} k(b^{k+1} - b^k) \right] - f \qquad (a) \\
&= f(b^f - 1) - \sum_{k=1}^{f-1} k(b^{k+1} - b^k) \\
&= f(b^f - 1) - (b-1) \sum_{k=1}^{f-1} kb^k \\
&= f(b^f - 1) - (b-1) \sum_{j=1}^{f-1} \sum_{k=j}^{f-1} b^k
\end{aligned}$$

$$
\begin{aligned}
&= f(b^f - 1) - (b-1)\sum_{j=1}^{f-1}\left(\sum_{k=1}^{f-1} b^k - \sum_{k=1}^{j-1} b^k\right) && \textit{(2)}\\
&= f(b^f - 1) - (b-1)\sum_{j=1}^{f-1}\left(\frac{b^f - 1}{b-1} - \frac{b^j - 1}{b-1}\right)\\
&= f(b^f - 1) - (b-1)\sum_{j=1}^{f-1}\left(\frac{b^f}{b-1} - \frac{b^j}{b-1}\right)\\
&= f(b^f - 1) - (b-1)\left[\frac{1}{b-1}\left(\sum_{j=1}^{f-1} b^f - \sum_{j=1}^{f-1} b^j\right)\right]\\
&= f(b^f - 1) - \left(\sum_{j=1}^{f-1} b^f - \sum_{j=1}^{f-1} b^j\right)\\
&= f(b^f - 1) - \left(b^f(f-1) - \sum_{j=1}^{f-1} b^j\right)\\
&= f(b^f - 1) - \left(b^f(f-1) - \left(\frac{b^f - 1}{b-1}\right)\right)\\
&= f(b^f - 1) - b^f(f-1) + \left(\frac{b^f - 1}{b-1}\right)\\
&= fb^f - f - fb^f + b^f + \left(\frac{b^f - 1}{b-1}\right)\\
&= b^f - f + \left(\frac{b^f - 1}{b-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
(b-1)S_n &= (b-1)\left(b^f - f\right) + \left(b^f - 1\right)\\
&= b^{f+1} - fb + f - b^f + b^f - 1\\
&= b^{f+1} - 1 - fb + f\\
&= b^{f+1} - 1 - f(b-1)\\
S_n &= \left(\frac{b^{f+1} - 1}{b-1}\right) - f\\
\Longrightarrow S_n &= \frac{b^{\lfloor \log_b n \rfloor + 1} - 1}{b-1} - \lfloor \log_b n \rfloor && **
\end{aligned}
$$

e.o.a.

Solution:

$$
\begin{aligned}
\sum_{k=1}^{n} \lfloor \log_b k \rfloor &= (n+1)\lfloor \log_b n \rfloor - \frac{b^{\lfloor \log_b n \rfloor + 1} - b}{b-1}\\
&= n\lfloor \log_b n \rfloor - \frac{b^{\lfloor \log_b n \rfloor + 1} - b}{b-1} + \lfloor \log_b n \rfloor\\
&= \cdots
\end{aligned}
$$

§ 6.2.1

25. re: ... (p. 425)

Suppose that a binary tree has a_k internal nodes and b_k external nodes on level k, for $k = 0, 1, \ldots$ (The root is at level zero.) Thus in Fig. 8 we have $(a_0, a_1, \ldots, a_5) = (1, 2, 4, 4, 1, 0)$ and $(b_0, b_1, \ldots, b_5) = (0, 0, 0, 4, 7, 2)$. *needs figure*