The Art of Computer Programming

Explicit Solutions

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October 14, 2022

§ 1.2.4

42. re: Lower-bound of internal path length of extended binary trees (p. 400)

(a) Prove that $\sum_{k=1}^{n} a_k = na_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k), \quad \text{if } n > 0.$

Solution:

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= na_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k) \\
&= na_n + \sum_{k=1}^{n-1} ka_k - \sum_{k=1}^{n-1} ka_{k+1} \\
&= na_n + (a_1 + 2a_2 + \cdots + (n-1)a_{n-1}) \\
&\quad - (a_2 + 2a_3 + \cdots + (n-2)a_{n-1} + (n-1)a_n) \\
&= na_n - (n-1)a_n + (a_1 + 2a_2 + \cdots + (n-1)a_{n-1}) \\
&\quad - (a_2 + 2a_3 + \cdots + (n-2)a_{n-1}) \\
&= a_n + \sum_{k=1}^{n-1} ka_k - \sum_{k=1}^{n-2} ka_{k+1} \\
&= a_n + \sum_{k=0}^{n-2} (k+1)a_{k+1} - \sum_{k=1}^{n-2} ka_{k+1} \\
&= a_n + a_1 + \sum_{k=1}^{n-2} (k+1)a_{k+1} - \sum_{k=1}^{n-2} ka_{k+1} \\
&= a_n + a_1 + \sum_{k=1}^{n-2} (k+1)a_{k+1} - ka_{k+1} \\
&= a_n + a_1 + \sum_{k=1}^{n-2} a_{k+1} \\
&= a_n + a_1 + \sum_{k=2}^{n-1} a_k \\
&= \sum_{k=1}^{n} a_k
\end{aligned}$$

(b) The preceding formula is useful for evaluating certain sums involving the floor function. Prove that, if b is an integer ≥ 2,

$$\sum_{k=1}^{n} \lfloor \log_b k \rfloor = (n+1)\lfloor \log_b n \rfloor - (b^{\lfloor \log_b n \rfloor + 1} - b)/(b-1).$$

Aside:

$$(1) \quad f = \lfloor \log_b n \rfloor; b \geq 2 \qquad\qquad (2) \quad \sum_{1}^{f-1} = \sum_{j}^{f-1} + \sum_{1}^{j-1}$$

$$\Longrightarrow \sum_{j}^{f-1} = \sum_{1}^{f-1} - \sum_{1}^{j-1}$$

$$\begin{aligned}
S_{n-1} &= \sum_{k=1}^{f-1} b^k \\
&= b - b^{\lfloor \log_b n \rfloor} + \sum_{k=1}^{f-1} b^{k+1} \\
&= b - b^{\lfloor \log_b n \rfloor} + b \sum_{k=1}^{f-1} b^k \\
&= b - b^{\lfloor \log_b n \rfloor} + bS_{n-1} \\
(1-b)S_{n-1} &= b - b^{\lfloor \log_b n \rfloor} \\
\Longrightarrow S_{n-1} &= \frac{b^{\lfloor \log_b n \rfloor} - b}{b-1}
\end{aligned}$$

$$(3) \quad f = \lfloor \log_b n \rfloor; b \geq 2$$

$$\begin{aligned}
S_n &= \sum_{k=1}^{n-1} k \; [k+1 \text{ is a power of } b] \\
&= \sum_{k=1}^{f} (b^k - 1) \\
&= \sum_{k=1}^{f} b^k - \sum_{k=1}^{f} 1 \\
&= \sum_{k=1}^{f} b^k - f \\
&= \left[fb^f - \sum_{k=1}^{f-1} k(b^{k+1} - b^k) \right] - f \qquad (a) \\
&= f(b^f - 1) - \sum_{k=1}^{f-1} k(b^{k+1} - b^k) \\
&= f(b^f - 1) - (b-1) \sum_{k=1}^{f-1} kb^k \\
&= f(b^f - 1) - (b-1) \sum_{j=1}^{f-1} \sum_{k=j}^{f-1} b^k
\end{aligned}$$

$$
\begin{aligned}
&= f(b^f-1)-(b-1)\sum_{j=1}^{f-1}\left(\sum_{k=1}^{f-1}b^k-\sum_{k=1}^{j-1}b^k\right) && \text{(2)}\\
&= f(b^f-1)-(b-1)\sum_{j=1}^{f-1}\left(\frac{b^f-b}{b-1}-\frac{b^j-b}{b-1}\right)\\
&= f(b^f-1)-(b-1)\sum_{j=1}^{f-1}\left(\frac{b^f}{b-1}-\frac{b^j}{b-1}\right)\\
&= f(b^f-1)-(b-1)\left[\frac{1}{b-1}\left(\sum_{j=1}^{f-1}b^f-\sum_{j=1}^{f-1}b^j\right)\right]\\
&= f(b^f-1)-\left(\sum_{j=1}^{f-1}b^f-\sum_{j=1}^{f-1}b^j\right)\\
&= f(b^f-1)-\left(b^f(f-1)-\sum_{j=1}^{f-1}b^j\right)\\
&= f(b^f-1)-\left(b^f(f-1)-\left(\frac{b^f-b}{b-1}\right)\right)\\
&= f(b^f-1)-b^f(f-1)+\left(\frac{b^f-b}{b-1}\right)\\
&= fb^f-f-fb^f+b^f+\left(\frac{b^f-b}{b-1}\right)\\
&= b^f-f+\left(\frac{b^f-b}{b-1}\right)\\
(b-1)S_n &= (b-1)\left(b^f-f\right)+\left(b^f-b\right)\\
&= b^{f+1}-fb+f-b^f+b^f-b\\
&= b^{f+1}-b-fb+f\\
&= b^{f+1}-b-f(b-1)\\
S_n &= \left(\frac{b^{f+1}-b}{b-1}\right)-f\\
\Longrightarrow S_n &= \frac{b^{\lfloor \log_b n\rfloor+1}-b}{b-1}-\lfloor \log_b n\rfloor
\end{aligned}
$$

e.o.a.

Solution:

$$
\begin{aligned}
\sum_{k=1}^{n}\lfloor \log_b k\rfloor &= n\lfloor \log_b n\rfloor - S_n\\
&= n\lfloor \log_b n\rfloor-\left[\frac{b^{\lfloor \log_b n\rfloor+1}-b}{b-1}-\lfloor \log_b n\rfloor\right] && (3)\\
&= n\lfloor \log_b n\rfloor-\frac{b^{\lfloor \log_b n\rfloor+1}+b}{b-1}+\lfloor \log_b n\rfloor
\end{aligned}
$$

$$= (n+1)\lfloor \log_b n \rfloor - \frac{b^{\lfloor \log_b n \rfloor + 1} + b}{b-1}$$

§ 2.3.4.5

5. re: Generating function of BT with n nodes and internal path length p (p. 404)

Let

$$B(w,z) = \sum_{n,p\geq 0} b_{np} w^p z^n,$$

where b_{np} is the number of binary trees with n nodes and internal path length p.

(a) Find a functional relation that characterizes $B(w,z)$, generalizing $zB(z)^2 = B(z) - 1$.

(b) Use the result of (a) to determine the average internal path length of a binary tree with n nodes, assuming that each of the $\frac{1}{n+1}\binom{2n}{n}$ trees is equally probable.

(c) Find the asymptotic value of this quantity.

§ 6.2.1

25. re: Relation between internal and external nodes in extended BT (p. 425)

Suppose that a binary tree has a_k internal nodes and b_k external nodes on level k, for $k = 0, 1, \ldots$ (The root is at level zero.) Thus in Fig. 8 we have $(a_0, a_1, \ldots, a_5) = (1, 2, 4, 4, 1, 0)$ and $(b_0, b_1, \ldots, b_5) = (0, 0, 0, 4, 7, 2)$.

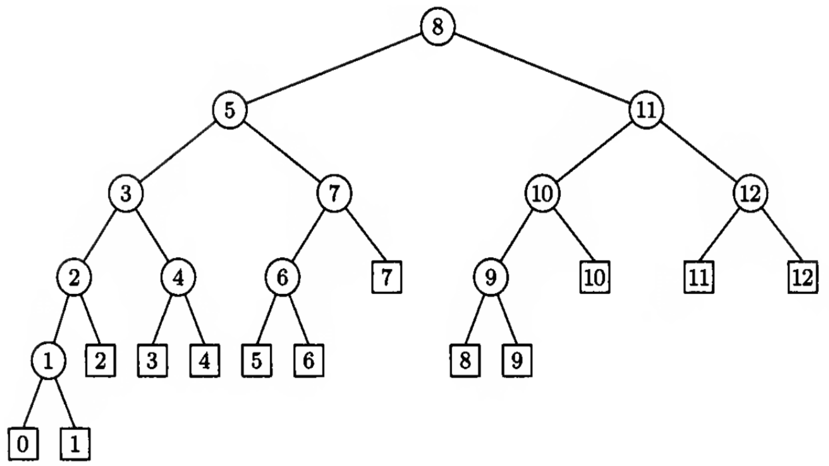

Fig. 8. The Fibonacci tree of order 6.

(a) Show that a simple algebraic relationship holds between the generating functions $A(z) = \sum_k a_k z^k$ and $B(z) = \sum_k b_k z^k$.

(b) The probability distribution for a successful search in a binary tree has the generating function $g(z) = zA(z)/N$, and for an unsuccessful search the generating function is $h(z) = B(z)/(N+1)$. (Thus in the text's notation we have $C_N = \text{mean}(g), C_N' = \text{mean}(h)$, and Eq. 2 ($C_N = (1+\frac{1}{N})C_N' - 1$) gives a relation between these quantities.) Find a relation between $\text{var}(g)$ and $\text{var}(h)$.