MAT221
Calculus & Applied Math

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Chapter 1: Linear Equations

Chapter 1: Linear Equations

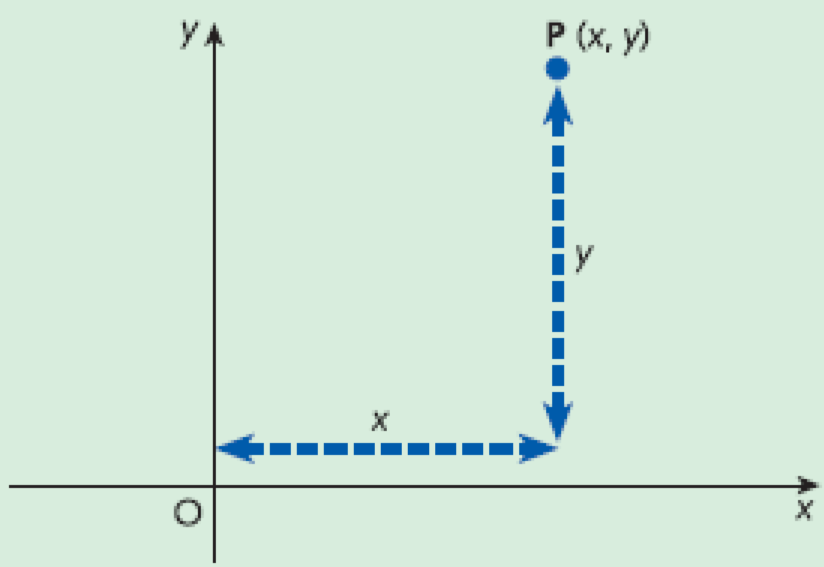
- **Graphs of linear equations**
- **Algebraic solution of simultaneous linear equations**
- **Supply and demand analysis**
- **National income determination**

- The main aim of this chapter is to introduce the mathematics of linear equations
- You are also shown how to solve simultaneous linear equations
- It introduces the fundamental concept of an economic function and describes how to calculate equilibrium prices and quantities in supply and demand theory

1.1 Graphs of linear equations

- Plot points on graph paper given their coordinates

Figure 1.1

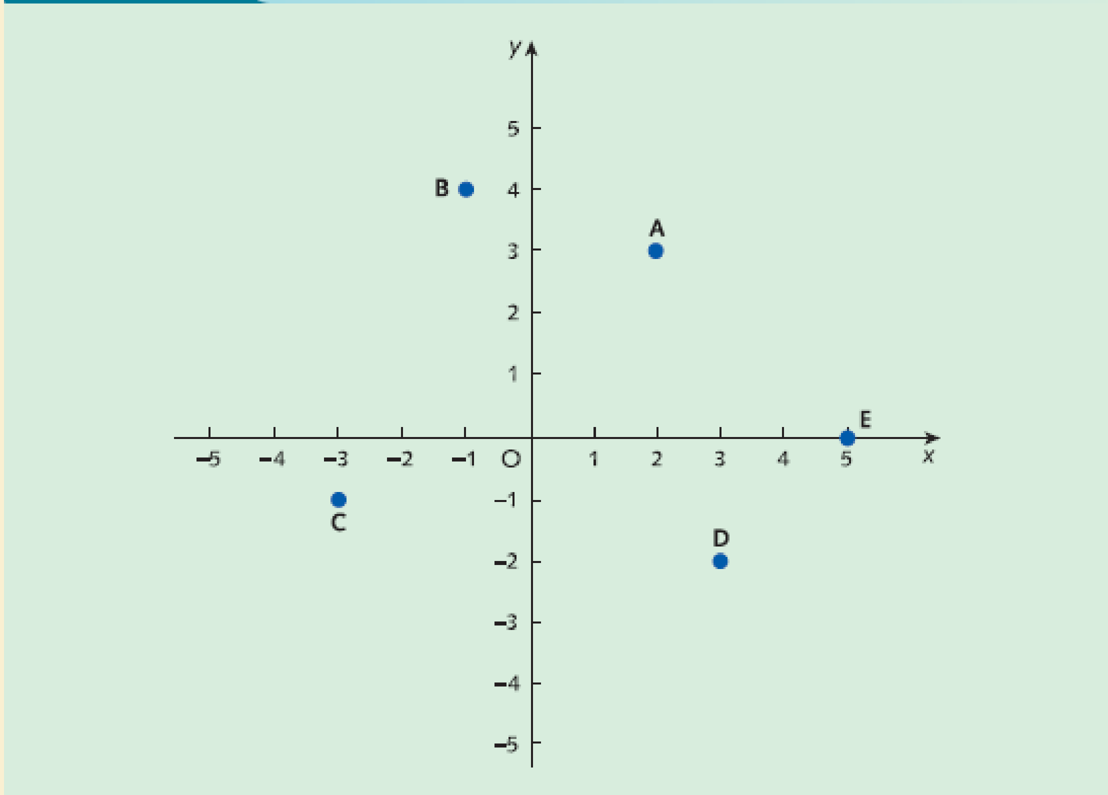


X denotes the horizontal distance along the x axis and y denotes the vertical distance along the y axis

1.1 Graphs of linear equations

- Plot points on graph paper given their coordinates
  - Ex: A(2, 3), B(−1, 4), C(−3, −1), D(3, −2) and E(5, 0)

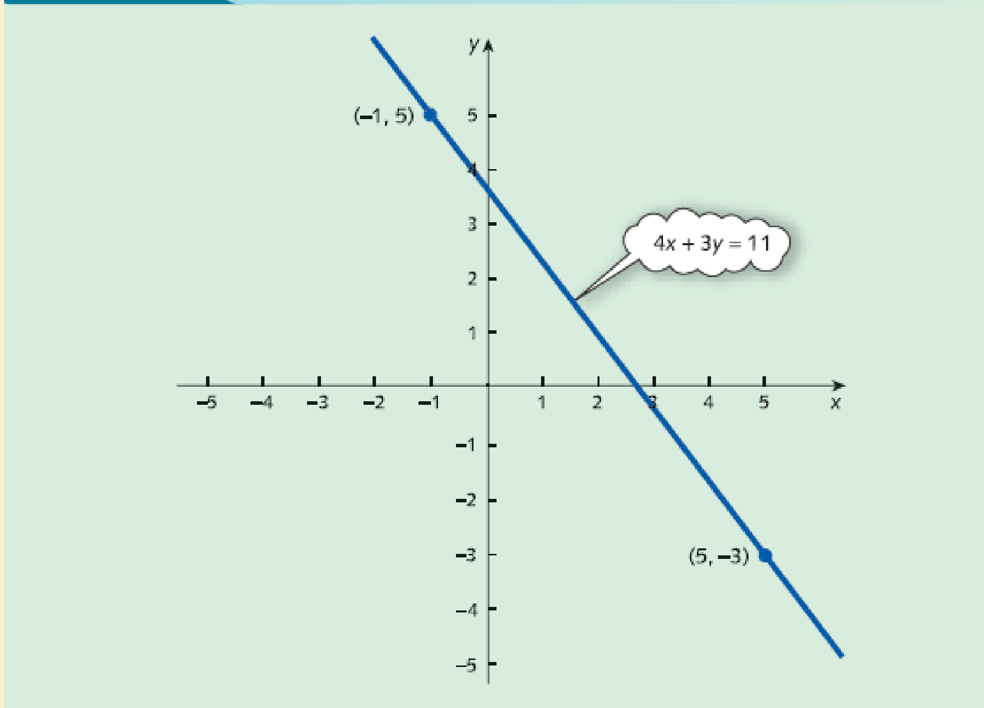
Figure 1.2



1.1 Graphs of linear equations

- Sketch a line by finding the coordinates of two points on the line
  - Sketch curves represented by equations

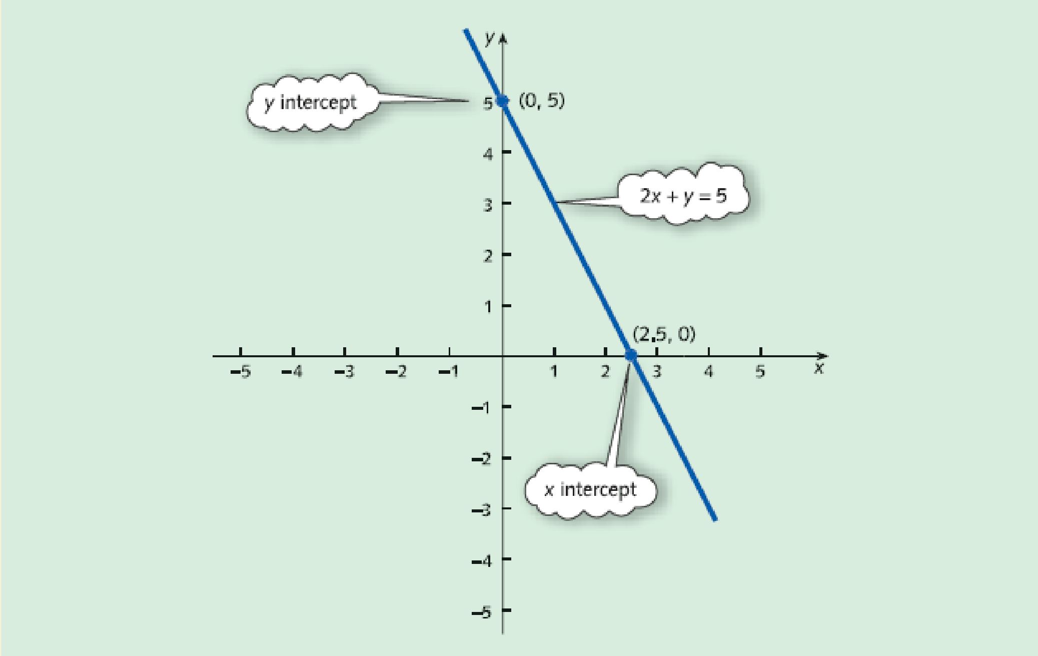
Figure 1.3



1.1 Graphs of linear equations

- Sketch a line by finding the coordinates of two points on the line
  - The easiest thing to do is to put x = 0 and find y and then to put y = 0 and find x

Figure 1.4



1.1 Graphs of linear equations

- Solve simultaneous linear equations graphically
  - In economics
    - The supply equation and the demand equation
    - Both involve the same variables quantity Q and price P
  - Equilibrium quantity and price to be determined by finding the point of intersection of the two lines

Find the point of intersection of the two lines

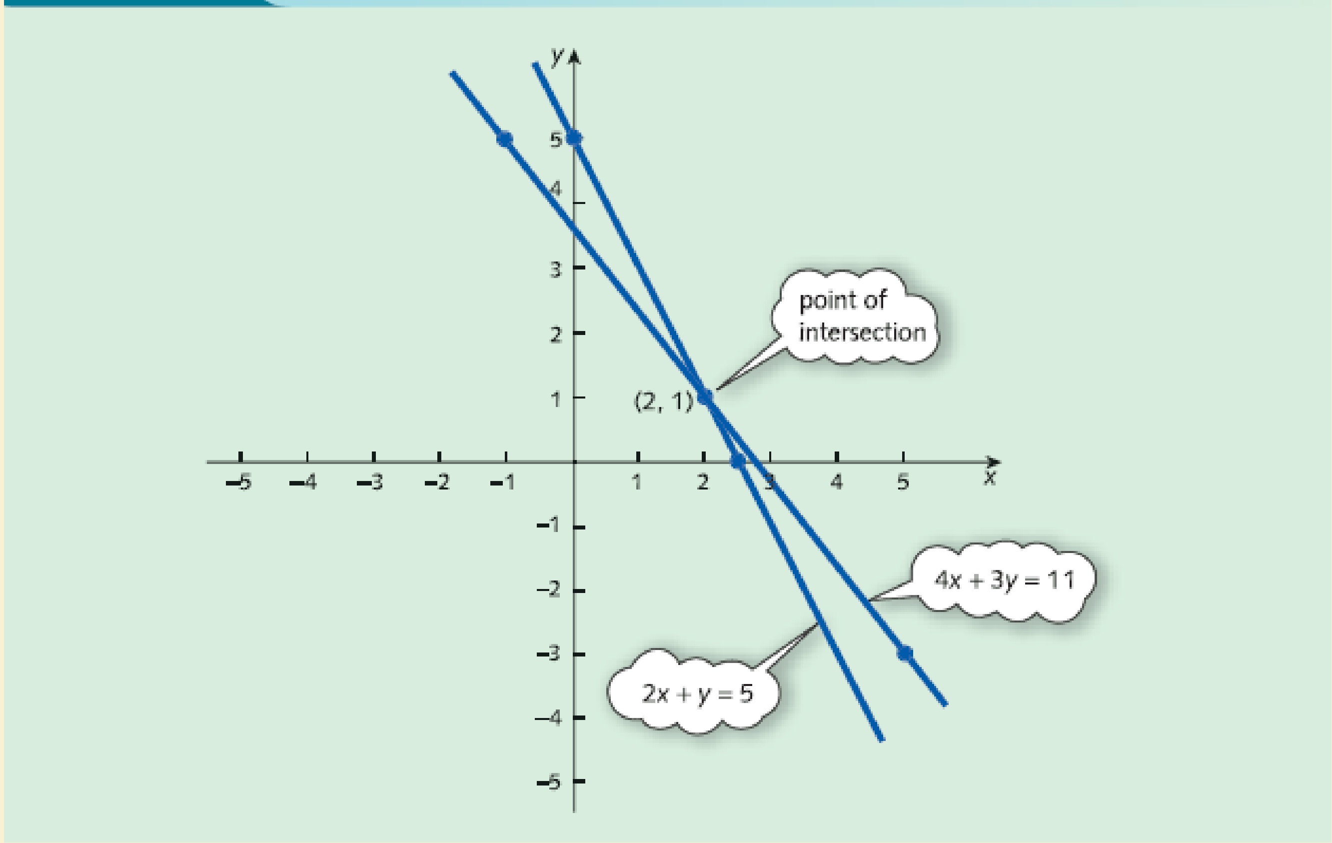
$$4x + 3y = 11$$

$$2x + y = 5$$

1.1 Graphs of linear equations

- Solve simultaneous linear equations graphically

Figure 1.5



1.1 Graphs of linear equations

- Sketch a line by using its slope and intercept

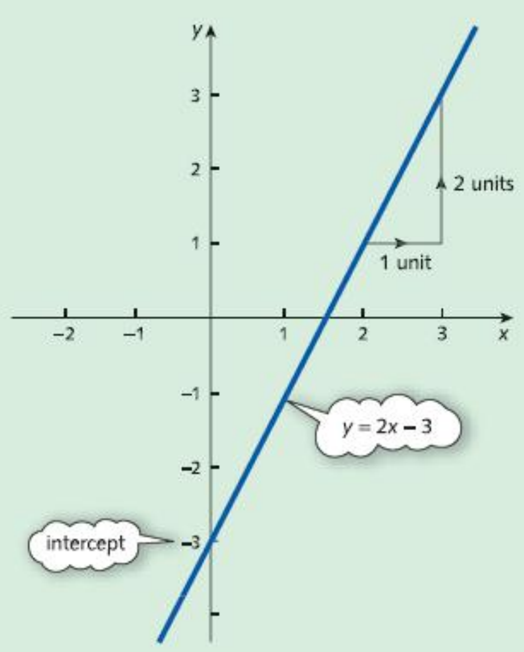
$$dx + ey = f$$

can be rearranged into the special form

$$y = ax + b$$

b represents the ***intercept*** on the y axis
a, the coefficient of x, determines the ***slope***

Figure 1.6



1.1 Graphs of linear equations

Key Terms

Coefficient A numerical multiplier of the variables in an algebraic term, such as the numbers 4 and 7 in the expression, $4x + 7yz^2$.

Coordinates A set of numbers that determine the position of a point relative to a set of axes.

Intercept The point(s) where a graph crosses one of the coordinate axes.

Linear equation An equation of the form $y = ax + b$.

Origin The point where the coordinate axes intersect.

Simultaneous linear equations A set of linear equations in which there are (usually) the same number of equations and unknowns. The solution consists of values of the unknowns which satisfy all of the equations at the same time.

Slope of a line Also known as the gradient, it is the change in the value of y when x increases by 1 unit.

x *axis* The horizontal coordinate axis pointing from left to right.

y *axis* The vertical coordinate axis pointing upwards.

1.1 Graphs of linear equations

Practice Problems

14 Solve the following pairs of simultaneous linear equations graphically:

(a) $-2x + y = 2$
$2x + y = -6$

(b) $3x + 4y = 12$
$x + 4y = 8$

(c) $2x + y = 4$
$4x - 3y = 3$

(d) $x + y = 1$
$6x + 5y = 15$

15 Use the slope–intercept approach to sketch the lines

(a) $y = -x$

(b) $x - 2y = 6$

1.2 Algebraic solution of simultaneous linear equations

- Solve a system of two simultaneous linear equations in two unknowns using elimination
  - Graphical method has several drawbacks
    - Not always easy to decide on a suitable scale for the axes
    - Accuracy of the graphical solution (decimals e.g. 231/571)
    - Quite frequently in economics we need to solve three equations in three unknowns or maybe four equations in four unknowns
  - Elimination method
    - Exact solution
    - Can be applied to systems equations larger than just two equations in two unknowns

1.2 Algebraic solution of simultaneous linear equations

- Elimination method
  - Step 1: Add/subtract a multiple of one equation to/from a multiple of the other to eliminate x
  - Step 2: Solve the resulting equation for y

$$\begin{array}{r} 4x + 3y = 11 \\ \underline{4x + 2y = 10} - \\ y = 1 \end{array}$$

the x's cancel when you subtract

  - Step 3: Substitute the value of y into one of the original equations to deduce x

$$\begin{aligned} 4x + 3(1) &= 11 && \text{(substitute } y = 1\text{)} \\ 4x + 3 &= 11 \\ 4x &= 8 && \text{(subtract 3 from both sides)} \\ x &= 2 && \text{(divide both sides by 4)} \end{aligned}$$

  - Step 4: Check that no mistakes have been made by substituting both x and y into the other original equation

1.2 Algebraic solution of simultaneous linear equations

- Elimination method

Practice Problem

1 **(a)** Solve the equations

$$3x - 2y = 4$$
$$x - 2y = 2$$

by eliminating one of the variables.

(b) Solve the equations

$$3x + 5y = 19$$
$$-5x + 2y = -11$$

by eliminating one of the variables.

1.2 Algebraic solution of simultaneous linear equations

- System of equations does not have a solution

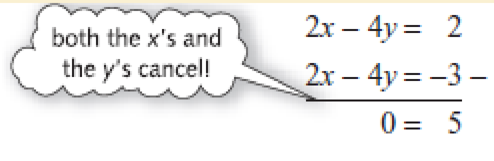
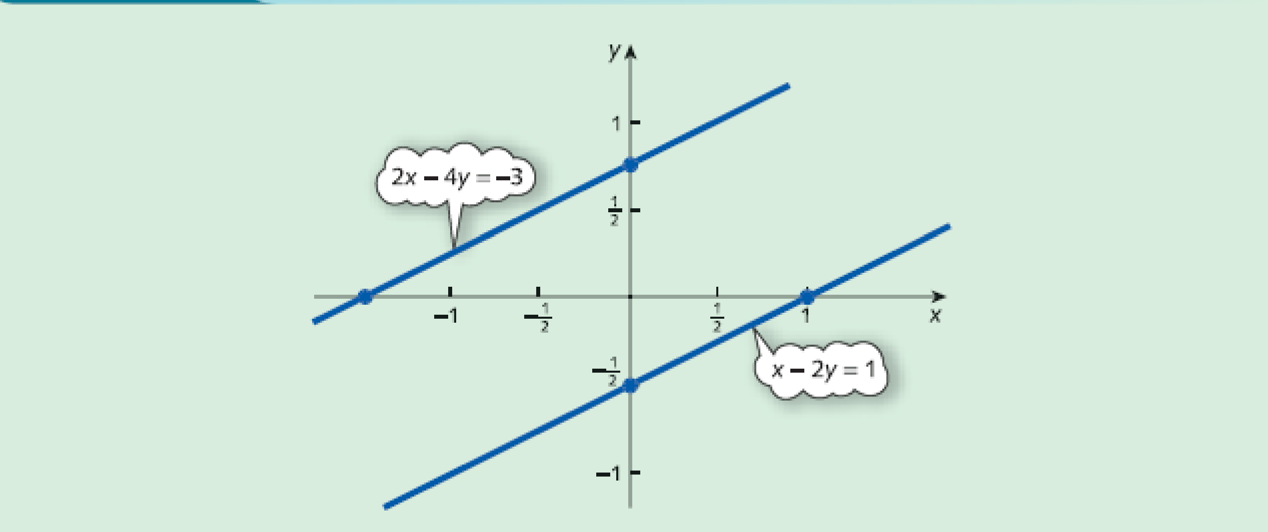


Figure 1.11



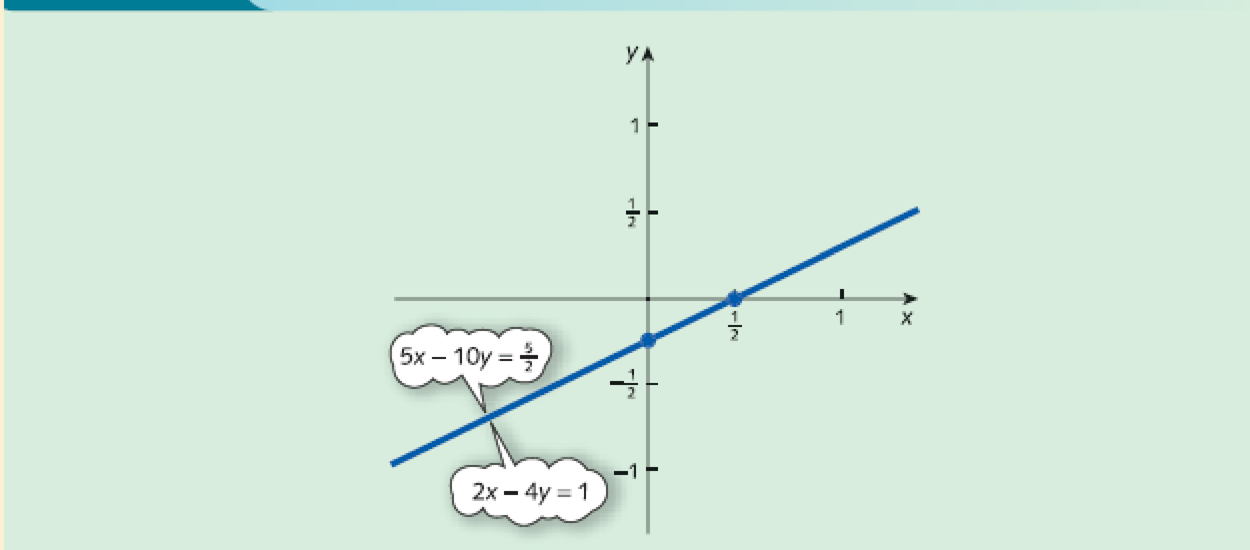
1.2 Algebraic solution of simultaneous linear equations

- System of equations has infinitely many solutions

$$\begin{array}{r} 10x - 20y = 5 \\ 10x - 20y = 5 \; - \\ \hline 0 = 0 \end{array}$$

everything cancels including the right-hand side!

Figure 1.12



1.2 Algebraic solution of simultaneous linear equations

- Solve a system of three simultaneous linear equations in three unknowns using elimination
  - Step 1: Add/subtract multiples of the first equation to/from multiples of the second and third equations to eliminate x

$$\begin{aligned} ?x + ?y + ?z &= ? \\ ?x + ?y + ?z &= ? \\ ?x + ?y + ?z &= ? \end{aligned} \qquad \begin{aligned} ?x + ?y + ?z &= ? \\ ?y + ?z &= ? \\ ?y + ?z &= ? \end{aligned}$$

  - Step 2: Add/subtract a multiple of the second equation to/from a multiple of the third to eliminate y

$$\begin{aligned} ?x + ?y + ?z &= ? \\ ?y + ?z &= ? \\ ?z &= ? \end{aligned}$$

  - Step 3: Solve the last equation for z. Substitute the value of z into the second equation to deduce y. Finally, substitute the values of both y and z into the first equation to deduce x
  - Step 4: Check that no mistakes

1.2 Algebraic solution of simultaneous linear equations

- Solve a system of three simultaneous linear equations in three unknowns using elimination

Practice Problem

3 Solve the following system of equations:

$$2x + 2y - 5z = -5 \qquad (1)$$

$$x - y + z = 3 \qquad (2)$$

$$-3x + y + 2z = -2 \qquad (3)$$

1.2 Algebraic solution of simultaneous linear equations

Key Terms

Elimination method The method in which variables are removed from a system of simultaneous equations by adding (or subtracting) a multiple of one equation to (or from) a multiple of another.

1.2 Algebraic solution of simultaneous linear equations

Practice Problems

6 Use the elimination method to attempt to solve the following systems of equations. Comment on the nature of the solution in each case.

(a) $-3x + 5y = 4$

$9x - 15y = -12$

(b) $6x - 2y = 3$

$15x - 5y = 4$

7 Solve the following systems of equations:

(a)
$$x - 3y + 4z = 5 \quad (1)$$
$$2x + y + z = 3 \quad (2)$$
$$4x + 3y + 5z = 1 \quad (3)$$

(b)
$$3x + 2y - 2z = -5 \quad (1)$$
$$4x + 3y + 3z = 17 \quad (2)$$
$$2x - y + z = -1 \quad (3)$$

8 Attempt to solve the following systems of equations. Comment on the nature of the solution in each case.

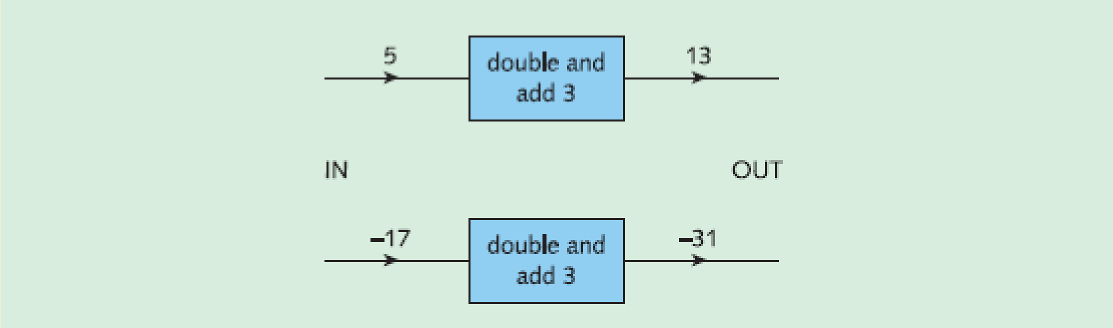
(a)
$$x - 2y + z = -2 \quad (1)$$
$$x + y - 2z = 4 \quad (2)$$
$$-2x + y + z = 12 \quad (3)$$

(b)
$$2x + 3y - z = 13 \quad (1)$$
$$x - 2y + 2z = -3 \quad (2)$$
$$3x + y + z = 10 \quad (3)$$

1.3 Supply and demand analysis

- Use the function notation, y = f (x)
  - Microeconomics: economic theory and policy of individual firms and markets
  - Equilibrium price and quantity ??
  - Function, f, is a rule which assigns to each incoming number, x, a uniquely defined outgoing number, y

Figure 1.13



$$y = 2x + 3 \quad \text{or} \quad f(x) = 2x + 3$$

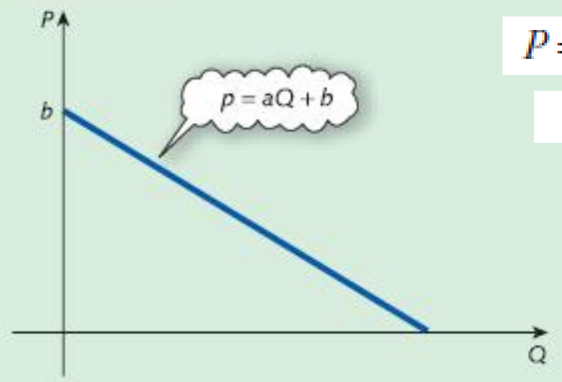
1.3 Supply and demand analysis

- Identify and sketch a linear demand function
  - In microeconomics the quantity demanded, Q, of a good depends on the market price, P; then, conversely, P must be related to Q

$Q = f(P)$ $P = g(Q)$ f and g, are said to be *inverse* functions:

  - P is a decreasing function of Q

Figure 1.14



$P = aQ + b$

$a < 0$

$b > 0$

1.3 Supply and demand analysis

- Identify and sketch a linear demand function

Example

Sketch a graph of the demand function

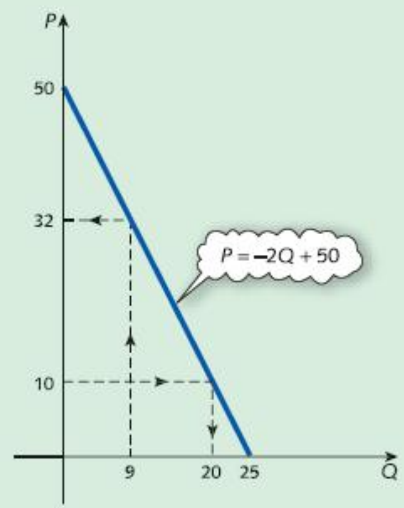
$$P = -2Q + 50$$

Hence, or otherwise, determine the value of

(a) P when $Q = 9$

(b) Q when $P = 10$

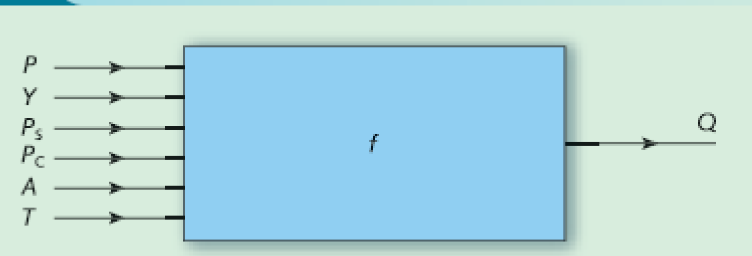
Figure 1.15



1.3 Supply and demand analysis

- Endogenous and exogenous variables in an economic model
  - In practice, Q depends on other factors than P as well:
    - incomes of consumers, Y,
    - price of substitutable goods, P_S, (e.g. buses-taxis)
    - price of complementary goods, P_C, (e.g. CDs and hi-fi)
    - advertising expenditure, A,
    - consumers' tastes,T,

Figure 1.16



1.3 Supply and demand analysis

- Endogenous and exogenous variables in an economic model

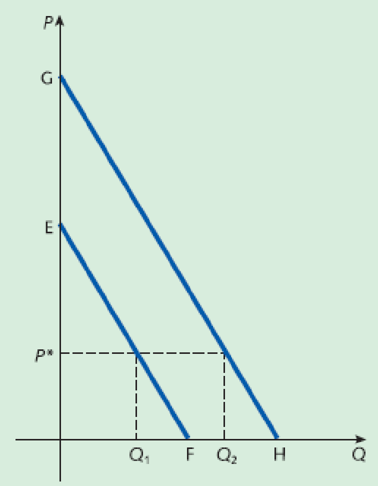
$$Q = f(P, Y, P_S, P_C, A, T)$$

- Q and P endogenous variables are allowed to vary and are determined within the model
- The remaining variables are called exogenous, are constant and are determined outside the model

1.3 Supply and demand analysis

- Endogenous and exogenous variables
  - Assumption Y, P_S, P_C, A and T are all constant
  - Suppose that income, Y, increases
    - » We conclude that if one of the exogenous variables changes then the whole demand curve moves, whereas if one of the endogenous variables changes, we simply move along the fixed curve.

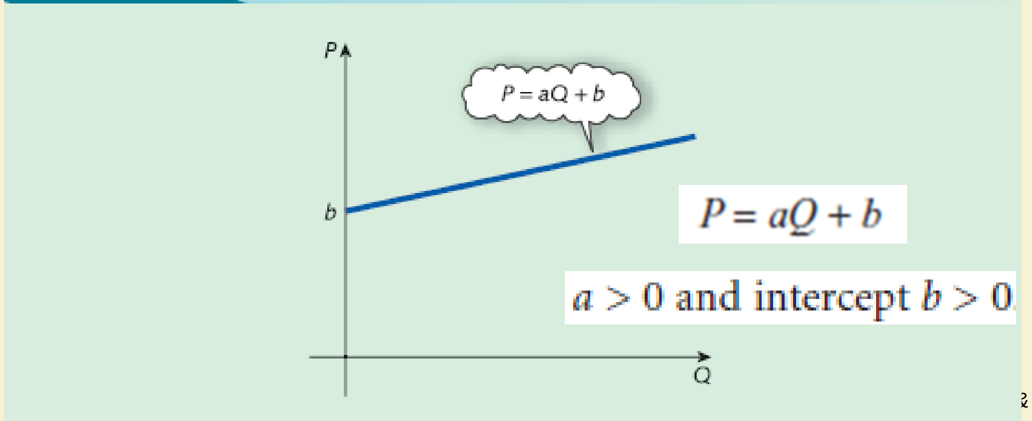
Figure 1.17



1.3 Supply and demand analysis

- Identify and sketch a linear supply function
  - P is an increasing function of Q
  - Q, is influenced by things other than price
    - Exogenous variables
      - » the prices of factors of production (land, capital, labour and enterprise)
      - » the profits obtainable on alternative goods, and technology

Figure 1.18



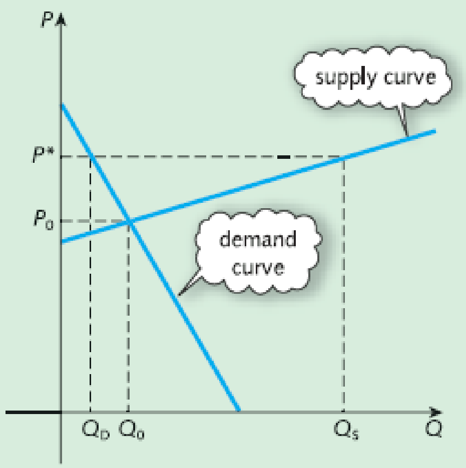
$P = aQ + b$

$a > 0$ and intercept $b > 0$

1.3 Supply and demand analysis

- Determine the equilibrium price and quantity for a single-commodity market
  - » Market price, P*,
  - » Equilibrium price, P_0,
  - » Quantity supplied, Q_S,
  - » Quantity demanded, Q_D,

Figure 1.19



1.3 Supply and demand analysis

- Equilibrium price and quantity

Example

The demand and supply functions of a good are given by

$$P = -2Q_D + 50$$

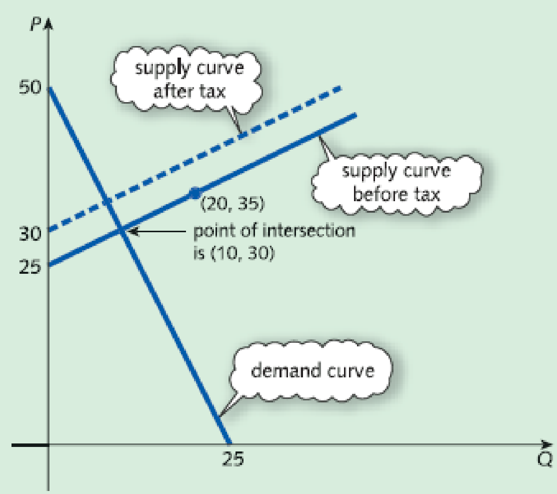
$$P = \tfrac{1}{2}Q_S + 25$$

where P, Q_D and Q_S denote the price, quantity demanded and quantity supplied respectively.

(a) Determine the equilibrium price and quantity.

(b) Determine the effect on the market equilibrium if the government decides to impose a fixed tax of $5 on each good.

Figure 1.20



Or Algebraic solution; simultaneous equations for two unknowns P and Q, using the elimination method

1.3 Supply and demand analysis

- Determine the equilibrium price and quantity for a multi-commodity market by solving simultaneous linear equations
  - More realistic model of supply and demand
  - Taking into account substitutable and complementary goods

$$Q_{D_1} = 10 - 2P_1 + P_2$$

$$Q_{D_2} = 5 + 2P_1 - 2P_2$$

$$Q_{S_1} = -3 + 2P_1$$

$$Q_{S_2} = -2 + 3P_2$$

$Q_{D_1} = Q_{S_1}$ and $Q_{D_2} = Q_{S_2}$

which can be solved by elimination.

1.3 Supply and demand analysis

Key Terms

Complementary goods A pair of goods consumed together. As the price of either goes up, the demand for both goods goes down.

Decreasing function A function, $y = f(x)$, in which y decreases as x increases.

Demand function A relationship between the quantity demanded and various factors that affect demand, including price.

Dependent variable A variable whose value is determined by that taken by the independent variables; in $y = f(x)$, the dependent variable is y.

Endogenous variable A variable whose value is determined within a model.

Equilibrium This state occurs when quantity supplied and quantity demanded are equal.

Exogenous variable A variable whose value is determined outside a model.

Function A rule that assigns to each incoming number, x, a uniquely defined outgoing number, y.

Increasing function A function, $y = f(x)$, in which y increases as x increases.

Independent variable A variable whose value determines that of the dependent variable; in $y = f(x)$, the independent variable is x.

1.3 Supply and demand analysis

Inferior good A good whose demand decreases as income increases.

Inverse function A function, written f^{-1}, which reverses the effect of a given function, f, so that $x = f^{-1}(y)$ when $y = f(x)$.

Modelling The creation of piece of mathematical theory which represents (a simplification of) some aspect of practical economics.

Parameter A constant whose value affects the specific values but not the general form of a mathematical expression such as the constants a, b and c in $ax^2 + bx + c$.

Substitutable goods A pair of goods that are alternatives to each other. As the price of one of them goes up, the demand for the other rises.

Superior good A good whose demand increases as income increases.

Supply function A relationship between the quantity supplied and various factors that affect supply, including price.

1.3 Supply and demand analysis

13 The demand and supply functions for three interdependent commodities are

$$Q_{D_1} = 15 - P_1 + 2P_2 + P_3$$

$$Q_{D_2} = 9 + P_1 - P_2 - P_3$$

$$Q_{D_3} = 8 + 2P_1 - P_2 - 4P_3$$

$$Q_{S_1} = -7 + P_1$$

$$Q_{S_2} = -4 + 4P_2$$

$$Q_{S_3} = -5 + 2P_3$$

where Q_{D_i}, Q_{S_i} and P_i denote the quantity demanded, quantity supplied and price of good i respectively. Determine the equilibrium price and quantity for this three-commodity model.

Quiz questions

Practice Problem 1.2

3 Solve the following system of equations:

$$\begin{aligned} 2x + 2y - 5z &= -5 \qquad (1) \\ x - y + z &= 3 \qquad (2) \\ -3x + y + 2z &= -2 \qquad (3) \end{aligned}$$

8 Attempt to solve the following systems of equations. Comment on the nature of the solution in each case.

(a)
$$\begin{aligned} x - 2y + z &= -2 \qquad (1) \\ x + y - 2z &= 4 \qquad (2) \\ -2x + y + z &= 12 \qquad (3) \end{aligned}$$

(b)
$$\begin{aligned} 2x + 3y - z &= 13 \qquad (1) \\ x - 2y + 2z &= -3 \qquad (2) \\ 3x + y + z &= 10 \qquad (3) \end{aligned}$$

Example

71

The demand and supply functions for two interdependent commodities are given by

$$Q_{D_1} = 10 - 2P_1 + P_2$$

$$Q_{D_2} = 5 + 2P_1 - 2P_2$$

$$Q_{S_1} = -3 + 2P_1$$

$$Q_{S_2} = -2 + 3P_2$$

where Q_{D_i}, Q_{S_i} and P_i denote the quantity demanded, quantity supplied and price of good i respectively. Determine the equilibrium price and quantity for this two-commodity model.

10 The demand and supply functions of a good are given by

$$P = -5Q_D + 80$$

$$P = 2Q_S + 10$$

where P, Q_D and Q_S denote price, quantity demanded and quantity supplied respectively.

(1) Find the equilibrium price and quantity

(a) graphically **(b)** algebraically

(2) If the government deducts, as tax, 15% of the market price of each good, determine the new equilibrium price and quantity.

13 The demand and supply functions for three interdependent commodities are

$$Q_{D_1} = 15 - P_1 + 2P_2 + P_3$$

$$Q_{D_2} = 9 + P_1 - P_2 - P_3$$

$$Q_{D_3} = 8 + 2P_1 - P_2 - 4P_3$$

$$Q_{S_1} = -7 + P_1$$

$$Q_{S_2} = -4 + 4P_2$$

$$Q_{S_3} = -5 + 2P_3$$

where Q_{D_i}, Q_{S_i} and P_i denote the quantity demanded, quantity supplied and price of good i respectively. Determine the equilibrium price and quantity for this three-commodity model.

Chapter 7: Matrices

Chapter 7: Matrices

- **Basic matrix operations**
- **Matrix inversion**
- **Cramer's rule**

- It introduces the concept of a matrix, which is a convenient mathematical way of representing information displayed in a table
- Inverses matrix provide an alternative way of solving systems of simultaneous linear equations
- Cramer's rule method is a useful way of solving economic models where only a selection of endogenous variables need to be determined

7.1 Basic matrix operations

- Understand the notation and terminology of matrix algebra

Table 7.1

		Monthly sales for goods		
		G1	G2	G3
Sold to customer	*C1*	7	3	4
	C2	1	5	6

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

- any rectangular array of numbers surrounded by a pair of brackets is called a matrix
- the individual numbers constituting the array are called entries or elements

7.1 Basic matrix operations

- Understand the notation and terminology of matrix algebra
  - In general, a matrix of order m × n has m rows and n columns
  - Matrix A has two rows and three columns and is said to have order 2 × 3
  - a_{ij} stands for the element of A which occurs in row i and column j
  - $a_{12} = 3$ (row 1 and column 2 of A)

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

- D of order 3 × 2 would be written $\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{bmatrix}$

7.1 Basic matrix operations

- Transposition

Table 7.2

		Sold to customer	
		C1	C2
Monthly sales for goods	*G1*	7	1
	G2	3	5
	G3	4	6

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}} = \mathbf{B}$$

read 'A transpose equals B'

$$\mathbf{B} = \begin{bmatrix} 7 & 1 \\ 3 & 5 \\ 4 & 6 \end{bmatrix}$$

- The transpose of a matrix is found by replacing rows by columns

7.1 Basic matrix operations

Example

Write down the transpose of the matrices

$$\mathbf{D} = \begin{bmatrix} 1 & 7 & 0 & 3 \\ 2 & 4 & 6 & 0 \\ 5 & 1 & 9 & 2 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

column vector.

Solution

The transpose of the 3×4 matrix $\mathbf{D}$ is the 4×3 matrix

$$\mathbf{D}^T = \begin{bmatrix} 1 & 2 & 5 \\ 7 & 4 & 1 \\ 0 & 6 & 9 \\ 3 & 0 & 2 \end{bmatrix}$$

The transpose of the 2×1 matrix $\mathbf{E}$ is the 1×2 matrix

$$\mathbf{E}^T = [-6 \quad 3]$$

row vector,

7.1 Basic matrix operations

- Addition and subtraction for any two m×n matrices

» two-customer three-product example

sales for the month of January.

$$A = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

for February

$$B = \begin{bmatrix} 6 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}$$

This means, for example, that customer C1 buys 7 items of G1 in January and 6 items of G1 in February. Customer C1 therefore buys a total of

$$7 + 6 = 13$$

$$C = \begin{bmatrix} 7+6 & 3+2 & 4+1 \\ 1+0 & 5+4 & 6+4 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \\ 1 & 9 & 10 \end{bmatrix}$$

$$C = A + B$$

7.1 Basic matrix operations

- Scalar multiplication (e.g. for each month)

Example

If

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

find

(a) $2\mathbf{A}$ (b) $-\mathbf{A}$ (c) $0\mathbf{A}$

Solution

(a) $$2\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

(b) $$-\mathbf{A} = (-1)\mathbf{A} = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix}$$

(c) $$0\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$$

$$k(l\mathbf{A}) = (kl)\mathbf{A}$$

7.1 Basic matrix operations

Example

Let

$$A = \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 2 \\ -1 & 6 \\ 3 & 4 \end{bmatrix}$$

Find

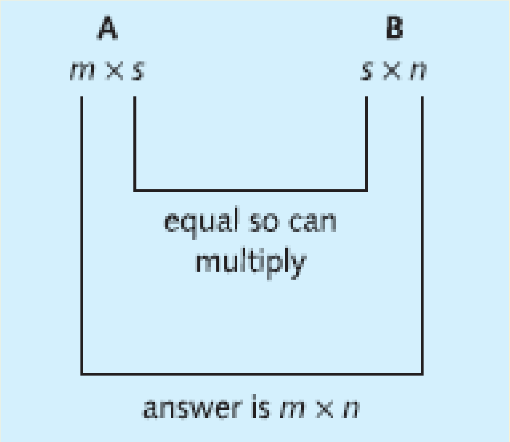
(a) $A + B$ (b) $A - B$ (c) $A - A$

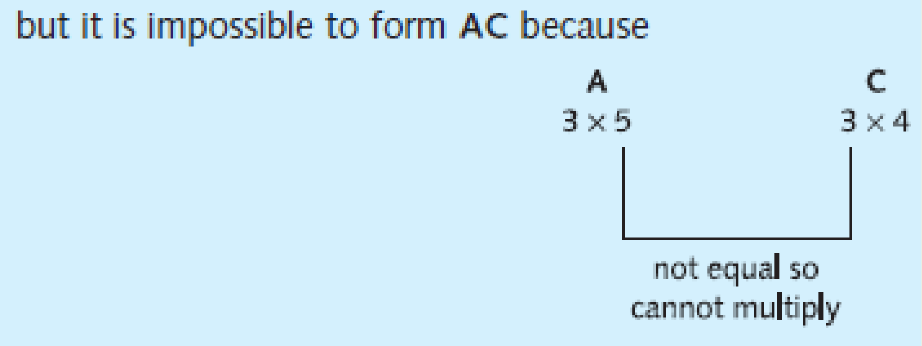
zero matrix

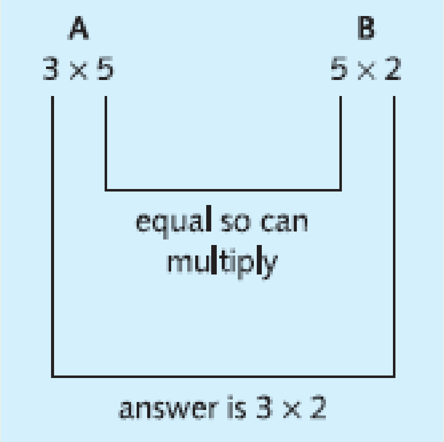
$$[0] \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.1 Basic matrix operations

- Matrix multiplication







7.1 Basic matrix operations

- Matrix multiplication

We begin by showing you how to multiply a row vector by a column vector. To illustrate this let us suppose that goods G1, G2 and G3 sell at \$50, \$30 and \$20, respectively, and let us introduce the row vector

$$\mathbf{p} = [50 \quad 30 \quad 20]$$

If the firm sells a total of 100, 200 and 175 goods of type G1, G2 and G3, respectively, then we can write this information as the column vector

$$\mathbf{q} = \begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix}$$

The total revenue received from the sale of G1 is found by multiplying the price, \$50, by the quantity, 100, to get

$$\$50 \times 100 = \$5000$$

Similarly, the revenue from G2 and G3 is

$$\$30 \times 200 = \$6000$$

and

$$20 \times 175 = \$3500$$

respectively. The total revenue of the firm is therefore

$$TR = \$5000 + \$6000 + \$3500 = \$14\,500$$

The value of TR is a single number and can be regarded as a 1×1 matrix: that is,

$$[14\,500]$$

This 1×1 matrix is obtained by multiplying together the price vector, **p**, and the quantity vector, **q**, to get

$$[50 \quad 30 \quad 20]\begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix} = [14\,500]$$

7.1 Basic matrix operations

- Matrix multiplication

$$\begin{bmatrix} 50 & 30 & 20 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix} = [5000 + 6000 + 3500] = [14\,500]$$

Example

If

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 6 \\ 9 \\ 2 \end{bmatrix}$$

find **ab** and **ac**.

7.1 Basic matrix operations

- Matrix multiplication

Practice Problem

5 Let

$$\mathbf{a} = [1 \quad -1 \quad 0 \quad 3 \quad 2], \quad \mathbf{b} = [1 \quad 2 \quad 9], \quad \mathbf{c} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Find (where possible)

(a) $\mathbf{ac}$ (b) $\mathbf{bd}$ (c) $\mathbf{ad}$

7.1 Basic matrix operations

- Matrix multiplication

Example

Find AB in the case when

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 5 & 4 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 5 & 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ \boxed{1 \quad 0 \quad 4} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 1 \\ 5 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 0 \\ 4 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 1 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 2 \\ 1 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 7 & 2 & 5 & 4 \\ \boxed{23} & \boxed{17} & \boxed{6} & \boxed{5} \end{bmatrix}$$

7.1 Basic matrix operations

- Matrix multiplication

Practice Problem

6 Write down the order of the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Hence verify that it is possible to form the matrix product

$$\mathbf{C} = \mathbf{AB}$$

and write down the order of **C**. Calculate all of the elements of **C**.

7.1 Basic matrix operations

- Matrix multiplication

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

$$A(BC) = (AB)C$$

Example

If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

evaluate **AB** and **BA**.

7.1 Basic matrix operations

- Matrix multiplication

Practice Problems

7 Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 1 & 0 \\ -1 & 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Find (where possible)

(a) AB (b) BA (c) CD (d) DC
(e) AE (f) EA (g) DE (h) ED

7.1 Basic matrix operations

- Matrix multiplication

8 Evaluate the matrix product **Ax**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 6 & 5 \\ 8 & 9 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence show that the system of linear equations

$$\begin{aligned} x + 4y + 7z &= -3 \\ 2x + 6y + 5z &= 10 \\ 8x + 9y + 5z &= 1 \end{aligned}$$

can be written as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{b} = \begin{bmatrix} -3 \\ 10 \\ 1 \end{bmatrix}$$

7.1 Basic matrix operations

- Matrix multiplication

15 Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (ad - bc \neq 0)$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Show that

(a) $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IA} = \mathbf{A}$ **(b)** $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{AA}^{-1} = \mathbf{I}$ **(c)** $\mathbf{Ix} = \mathbf{x}$

7.1 Basic matrix operations

- Matrix multiplication

Key Terms

Column vector A matrix with one column.

Elements The individual numbers inside a matrix. (Also called entries.)

Matrix A rectangular array of numbers, set out in rows and columns, surrounded by a pair of brackets. (Plural matrices.)

Order The dimensions of a matrix. A matrix with m rows and n columns has order $m \times n$.

Row vector A matrix with one row.

Transpose of a matrix The matrix obtained from a given matrix by interchanging rows and columns. The transpose of a matrix A is written A^T.

Zero matrix A matrix in which every element is zero.

7.2 Matrix inversion

- Square matrices: number of rows and columns are equal, we concentrate on 2×2 and 3×3 matrices
- The matrix I is called the identity matrix and is analogous to the number 1 in ordinary arithmetic
- The matrix A^{-1} is said to be the inverse of A and is analogous to the reciprocal of a number
- The number $ad - bc$ is called the determinant of A and is written as det(A) or |A|
- If the matrix has a non-zero determinant, it is said to be non-singular; otherwise it is said to be singular.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7.2 Matrix inversion

- 2 × 2 Matrices| Construction of A^{-1}

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

swap
a and *d*

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

change signs
of *b* and *c*

Finally, we multiply the matrix by the scalar

$$\frac{1}{ad - bc}$$

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

divide each element
by $ad - bc$

We deduce that the inverse of a matrix exists only if the matrix has a non-zero determinant. This is comparable to the situation in arithmetic where a reciprocal of a number exists provided the number is non-zero.

Example

Find the inverse of the following matrices. Are these matrices singular or non-singular?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}$$

Solution

We begin by calculating the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

to see whether or not the inverse exists.

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = 4 - 6 = -2$$

We see that $\det(\mathbf{A}) \neq 0$, so the matrix is non-singular and the inverse exists. To find $\mathbf{A}^{-1}$ we swap the diagonal elements, 1 and 4, change the sign of the off-diagonal elements, 2 and 3, and divide by the determinant, -2. Hence

$$\mathbf{A}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Of course, if $\mathbf{A}^{-1}$ really is the inverse of $\mathbf{A}$, then $\mathbf{A}^{-1}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{-1}$ should multiply out to give $\mathbf{I}$. As a check:

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

To discover whether or not the matrix

$$\mathbf{B} = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}$$

has an inverse we need to find its determinant.

$$\det(\mathbf{B}) = \begin{vmatrix} 2 & 5 \\ 4 & 10 \end{vmatrix} = 2(10) - 5(4) = 20 - 20 = 0$$

We see that $\det(\mathbf{B}) = 0$, so this matrix is singular and the inverse does not exist.

7.2 Matrix inversion

- 2 × 2 Matrices|

Practice Problem

1 Find (where possible) the inverse of the following matrices. Are these matrices singular or non-singular?

$$\mathbf{A} = \begin{bmatrix} 6 & 4 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$$

7.2 Matrix inversion

- 2×2 Matrices| Solve matrix equations
  - One reason for calculating the inverse of a matrix is that it helps us to solve matrix equations in the same way that the reciprocal of a number is used to solve algebraic equations.
  - We have already seen in Section 7.1 how to express a system of linear equations in matrix form.

$$ax + by = e$$

$$cx + dy = f$$

can be written as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} e \\ f \end{bmatrix}$$

7.2 Matrix inversion

- 2 × 2 Matrices| Solve matrix equations

The coefficient matrix, **A**, and right-hand-side vector, **b**, are assumed to be given and the problem is to determine the vector of unknowns, **x**. Multiplying both sides of

$$\mathbf{Ax} = \mathbf{b}$$

by $\mathbf{A}^{-1}$ gives

$$\mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b}$$

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad \text{(associative property)}$$

$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \quad \text{(definition of an inverse)}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad \text{(Practice Problem 15(c) in Section 7.1)}$$

The solution vector **x** can therefore be found simply by multiplying $\mathbf{A}^{-1}$ by **b**. We are assuming here that $\mathbf{A}^{-1}$ exists. If the coefficient matrix is singular then the inverse cannot be found and the system of linear equations does not possess a unique solution; there are either infinitely many solutions or no solution.

Example

The equilibrium prices P_1 and P_2 for two goods satisfy the equations

$$-4P_1 + P_2 = -13$$

$$2P_1 - 5P_2 = -7$$

Express this system in matrix form and hence find the values of P_1 and P_2.

Solution

Using the notation of matrices, the simultaneous equations

$$-4P_1 + \quad P_2 = -13$$

$$2P_1 - 5P_2 = -7$$

can be written as

$$\begin{bmatrix} -4 & 1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

that is, as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 \\ 2 & -5 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

The matrix $\mathbf{A}$ has determinant

$$\begin{vmatrix} -4 & 1 \\ 2 & -5 \end{vmatrix} = (-4)(-5) - (1)(2) = 20 - 2 = 18$$

To find $\mathbf{A}^{-1}$ we swap the diagonal elements, -4 and -5, change the sign of the off-diagonal elements, 1 and 2, and divide by the determinant, 18, to get

$$\mathbf{A}^{-1} = \frac{1}{18}\begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}$$

Finally, to calculate $\mathbf{x}$ we multiply $\mathbf{A}^{-1}$ by $\mathbf{b}$ to get

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ &= \frac{1}{18}\begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -13 \\ -7 \end{bmatrix} \\ &= \frac{1}{18}\begin{bmatrix} 72 \\ 54 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \end{aligned}$$

Hence $P_1 = 4$ and $P_2 = 3$.

7.2 Matrix inversion

- 2 × 2 Matrices| Solve matrix equations

Practice Problem

2 The equilibrium prices P_1 and P_2 for two goods satisfy the equations

$$9P_1 + P_2 = 43$$

$$2P_1 + 7P_2 = 57$$

Express this system in matrix form and hence find the values of P_1 and P_2.

7.2 Matrix inversion

- 2 × 2 Matrices

Practice Problem

3 The general linear supply and demand equations for a one-commodity market model are given by

$$P = aQ_S + b \qquad (a > 0,\ b > 0)$$

$$P = -cQ_D + d \qquad (c > 0,\ d > 0)$$

Show that in matrix notation the equilibrium price, P, and quantity, Q, satisfy

$$\begin{bmatrix} 1 & -a \\ 1 & c \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Solve this system to express P and Q in terms of a, b, c and d. Write down the multiplier for Q due to changes in b and deduce that an increase in b leads to a decrease in Q.

7.2 Matrix inversion

- 3×3 Matrices| Cofactors

Before we can discuss the determinant and inverse of a 3×3 matrix we need to introduce an additional concept known as a *cofactor*. Corresponding to each element a_{ij} of a matrix $\mathbf{A}$, there is a cofactor, A_{ij}. A 3×3 matrix has nine elements, so there are nine cofactors to be computed. The cofactor, A_{ij}, is defined to be the determinant of the 2×2 matrix obtained by deleting row i and column j of $\mathbf{A}$, prefixed by a '+' or '−' sign according to the following pattern

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example, suppose we wish to calculate A_{23}, which is the cofactor associated with a_{23} in the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

7.2 Matrix inversion

The element a_{23} lies in the second row and third column. Consequently, we delete the second row and third column to produce the 2 × 2 matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{bmatrix}$$

The cofactor, A_{23}, is the determinant of this 2 × 2 matrix prefixed by a '–' sign because from the pattern

$$\begin{bmatrix} + & - & + \\ - & + & \boxed{-} \\ + & - & + \end{bmatrix}$$

we see that a_{23} is in a minus position. In other words,

$$\begin{aligned} \mathbf{A}_{23} &= -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= -(a_{11}a_{32} - a_{12}a_{31}) \\ &= -a_{11}a_{32} + a_{12}a_{31} \end{aligned}$$

Example

Find all the cofactors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

Solution

Let us start in the top left-hand corner and work row by row. For cofactor A_{11}, the element $a_{11} = 2$ lies in the first row and first column, so we delete this row and column to produce the 2×2 matrix

$$\begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{1} \\ \cancel{4} & 3 & 7 \\ \cancel{2} & 1 & 3 \end{bmatrix}$$

Cofactor A_{11} is the determinant of this 2×2 matrix, prefixed by a '+' sign because from the pattern

$$\begin{bmatrix} \boxed{+} & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

we see that a_{11} is in a plus position. Hence

$$\begin{aligned} A_{11} &= +\begin{vmatrix} 3 & 7 \\ 1 & 3 \end{vmatrix} \\ &= +(3(3) - 7(1)) \\ &= 9 - 7 \\ &= 2 \end{aligned}$$

For cofactor A_{12}, the element $a_{12} = 4$ lies in the first row and second column, so we delete this row and column to produce the 2×2 matrix

$$\begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{1} \\ 4 & \cancel{3} & 7 \\ 2 & \cancel{1} & 3 \end{bmatrix}$$

Cofactor A_{12} is the determinant of this 2×2 matrix, prefixed by a '–' sign because from the pattern

$$\begin{bmatrix} + & \boxed{-} & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

we see that a_{12} is in a minus position. Hence

$$\begin{aligned} A_{12} &= +\begin{vmatrix} 4 & 7 \\ 2 & 3 \end{vmatrix} \\ &= -(4(3) - 7(2)) \\ &= -(12 - 14) \\ &= 2 \end{aligned}$$

We can continue in this way to find the remaining cofactors

$$A_{13} = +\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = -2$$

$$A_{21} = -\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = -11$$

$$A_{22} = +\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4$$

$$A_{23} = -\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = 6$$

$$A_{31} = +\begin{vmatrix} 4 & 1 \\ 3 & 7 \end{vmatrix} = 25$$

$$A_{32} = -\begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} = -10$$

$$A_{33} = +\begin{vmatrix} 2 & 4 \\ 4 & 3 \end{vmatrix} = -10$$

7.2 Matrix inversion

- 3 × 3 Matrices| Cofactors

Practice Problem

4 Find all the cofactors of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

7.2 Matrix inversion

- 3 × 3 Matrices| Determinant

We are now in a position to describe how to calculate the determinant and inverse of a 3 × 3 matrix. The determinant is found by multiplying the elements in any one row or column by their corresponding cofactors and adding together. It does not matter which row or column is chosen; exactly the same answer is obtained in each case. If we expand along the first row of the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we get

$$\det(\mathbf{A}) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Similarly, if we expand down the second column, we get

$$\det(\mathbf{A}) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

7.2 Matrix inversion

- 3 × 3 Matrices| Determinant

Practice Problem

5 Find the determinants of

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 270 & -372 & 0 \\ 552 & 201 & 0 \\ 999 & 413 & 0 \end{bmatrix}$$

[Hint: you might find your answer to Practice Problem 4 useful when calculating the determinant of **A**.]

7.2 Matrix inversion

- 3 × 3 Matrices| Construction of A^{-1}

The inverse of the 3 × 3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

7.2 Matrix inversion

- 3×3 Matrices| Construction of A^{-1}

Once the cofactors of **A** have been found, it is easy to construct $\mathbf{A}^{-1}$. We first stack the cofactors in their natural positions

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

called the adjugate matrix

Secondly, we take the transpose to get

$$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

called the adjoint matrix

7.2 Matrix inversion

- 3 × 3 Matrices| Construction of A^{-1}

Finally, we multiply by the scalar

$$\frac{1}{|\mathbf{A}|}$$

to get

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

divide each element by the determinant

The last step is impossible if

$$|\mathbf{A}| = 0$$

because we cannot divide by zero. Under these circumstances the inverse does not exist and the matrix is singular.

7.2 Matrix inversion

- 3 × 3 Matrices| Construction of A^{-1}

Practice Problem

6 Find (where possible) the inverses of

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 270 & -372 & 0 \\ 552 & 201 & 0 \\ 999 & 413 & 0 \end{bmatrix}$$

[Hint: you might find your answers to Practice Problems 4 and 5 useful.]

7.2 Matrix inversion

- 3 × 3 Matrices| Solve matrix equations

Inverses of 3 × 3 matrices can be used to solve systems of three linear equations in three unknowns. The general system

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

can be written as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

7.2 Matrix inversion

- 3 × 3 Matrices| Solve matrix equations

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The vector of unknowns, $\mathbf{x}$, can be found by inverting the coefficient matrix, $\mathbf{A}$, and multiplying by the right-hand-side vector, $\mathbf{b}$, to get

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

7.2 Matrix inversion

- 3 × 3 Matrices| Solve matrix equations

Practice Problem

7 Determine the equilibrium prices of three interdependent commodities that satisfy

$$P_1 + 3P_2 + 3P_3 = 32$$

$$P_1 + 4P_2 + 3P_3 = 37$$

$$P_1 + 3P_2 + 4P_3 = 35$$

[Hint: you might find your answer to Practice Problem 6 useful.]

7.2 Matrix inversion

- n × n Matrices| Construction of A^{-1}

Key Terms

Cofactor (of an element) The cofactor of the element, a_{ij}, is the determinant of the matrix left when row i and column j are deleted, multiplied by +1 or –1, depending on whether $i + j$ is even or odd, respectively.

Determinant A determinant can be expanded as the sum of the products of the elements in any one row or column and their respective cofactors.

Identity matrix An $n \times n$ matrix, **I**, in which every element on the main diagonal is 1 and the other elements are all 0. If **A** is any $n \times n$ matrix then $\mathbf{AI} = \mathbf{I} = \mathbf{IA}$.

Inverse matrix A matrix, $\mathbf{A}^{-1}$ with the property that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$.

Non-singular matrix A square matrix with a non-zero determinant.

Singular matrix A square matrix with a zero determinant. A singular matrix fails to possess an inverse.

Square matrix A matrix with the same number of rows as columns.

7.3 Cramer's rule

- Limitations of using inverses
  - The method described in previous section can be extended to larger matrices of order n × n.
  - However, the cofactor approach is very inefficient.
  - The amount of working rises dramatically as n increases, making this method inappropriate for large matrices.
  - 16 cofactors to be calculated…

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 5 & 4 & 1 \\ 0 & 7 & -3 & 6 \\ 2 & 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -24 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

7.3 Cramer's rule

- It frequently happens in economics that only a few of the variables x_i are actually needed.
- For instance, it could be that the variable x_3 is the only one of interest.
- In this section we describe an alternative method that finds the value of one variable at a time.
- This new method requires less effort if only a selection of the variables is required.
- It is known as Cramer's rule and makes use of matrix determinants.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 5 & 4 & 1 \\ 0 & 7 & -3 & 6 \\ 2 & 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -24 \\ 15 \end{bmatrix}$$

7.3 Cramer's rule

$n \times n$ system, $\mathbf{Ax} = \mathbf{b}$, states that the ith variable, x_i, can be found from

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

where $\mathbf{A}_i$ is the $n \times n$ matrix found by replacing the ith column of $\mathbf{A}$ by the right-hand-side vector $\mathbf{b}$. To understand this, consider the simple 2×2 system

$$\begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

and suppose that we need to find the value of the second variable, x_2, say. According to Cramer's rule, this is given by

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

where

$$\mathbf{A} = \begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 7 & -6 \\ 4 & 12 \end{bmatrix}$$

7.3 Cramer's rule

$$\mathbf{A} = \begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 7 & -6 \\ 4 & 12 \end{bmatrix}$$

Notice that x_2 is given by the quotient of two determinants. The one on the bottom is that of the original coefficient matrix **A**. The one on the top is that of the matrix found from **A** by replacing the second column (since we are trying to find the second variable) by the right-hand-side vector

$$\begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

In this case the determinants are easily worked out to get

$$\det(\mathbf{A}_2) = \begin{vmatrix} 7 & -6 \\ 4 & 12 \end{vmatrix} = 7(12) - (-6)(4) = 108$$

$$\det(\mathbf{A}) = \begin{vmatrix} 7 & 2 \\ 4 & 5 \end{vmatrix} = 7(5) - 2(4) = 27$$

Hence

$$x_2 = \frac{108}{27} = 4$$

7.3 Cramer's rule

Example

Solve the system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 1 & 6 \\ 2 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ 13 \end{bmatrix}$$

using Cramer's rule to find x_1.

Solution

Cramer's rule gives

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

where **A** is the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 1 & 6 \\ 2 & 7 & 5 \end{bmatrix}$$

and $\mathbf{A}_1$ is constructed by replacing the first column of **A** by the right-hand-side vector

$$\begin{bmatrix} 9 \\ -9 \\ 13 \end{bmatrix}$$

which gives

$$\mathbf{A}_1 = \begin{bmatrix} 9 & 2 & 3 \\ -9 & 1 & 6 \\ 13 & 7 & 5 \end{bmatrix}$$

If we expand each of these determinants along the top row, we get

$$\det(\mathbf{A}_1) = \begin{vmatrix} 9 & 2 & 3 \\ -9 & 1 & 6 \\ 13 & 7 & 5 \end{vmatrix}$$

$$= 9\begin{vmatrix} 1 & 6 \\ 7 & 5 \end{vmatrix} - 2\begin{vmatrix} -9 & 6 \\ 13 & 5 \end{vmatrix} + 3\begin{vmatrix} -9 & 1 \\ 13 & 7 \end{vmatrix}$$

$$= 9(-37) - 2(-123) + 3(-76)$$

$$= -315$$

and

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 1 & 6 \\ 2 & 7 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 6 \\ 7 & 5 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 5 \end{vmatrix} + 3\begin{vmatrix} -4 & 1 \\ 2 & 7 \end{vmatrix}$$

$$= 1(-37) - 2(-32) + 3(-30)$$

$$= -63$$

Hence

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{-315}{-63} = 5$$

7.3 Cramer's rule

Practice Problem

1 **(a)** Solve the system of equations

$$2x_1 + 4x_2 = 16$$
$$3x_1 - 5x_2 = -9$$

using Cramer's rule to find x_2.

(b) Solve the system of equations

$$\begin{aligned} 4x_1 + x_2 + 3x_3 &= 8 \\ -2x_1 + 5x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 4x_3 &= 9 \end{aligned}$$

using Cramer's rule to find x_3.

7.3 Cramer's rule

Key Terms

Cramer's rule A method of solving simultaneous equations, $\mathbf{Ax} = \mathbf{b}$, by the use of determinants. The ith variable x_i can be computed using $\det(\mathbf{A}_i)/\det(\mathbf{A})$ where $\mathbf{A}_i$ is the determinant of the matrix obtained from $\mathbf{A}$ by replacing the ith column by $\mathbf{b}$.

Chapter 2: Non-linear Equations

Chapter 2: Non-linear Equations

» **Quadratic functions**
» **Revenue, cost and profit**
» **Indices and logarithms**
» **The exponential and natural logarithm functions**

» Sketching the graphs of quadratic functions
» It are illustrated by finding the equilibrium price and quantity for quadratic supply and demand functions
» You are also shown how to Sketch the graphs of quadratic revenue and profit functions and to find their maximum values

2.1 Quadratic functions

- Simplest non-linear function: Quadratic

$$f(x) = ax^2 + bx + c$$

Consider the elementary equation

$$x^2 - 9 = 0$$

It is easy to see that the expression on the left-hand side is a special case of the above with $a = 1$, $b = 0$ and $c = -9$. To solve this equation we add 9 to both sides to get

$$x^2 = 9$$

x^2 is an abbreviation for $x \times x$

These two solutions are called the *square roots* of 9. The symbol √ is reserved for the positive square root, so in this notation the solutions are $\sqrt{9}$ and $-\sqrt{9}$. These are usually combined and written $\pm\sqrt{9}$. The equation

Example

Solve the following quadratic equations:

(a) $5x^2 - 80 = 0$ (b) $x^2 + 64 = 0$ (c) $(x + 4)^2 = 81$

Solution

(a) $5x^2 - 80 = 0$

$5x^2 = 80$ (add 80 to both sides)

$x^2 = 16$ (divide both sides by 5)

$x = \pm 4$ (square root both sides)

(b) $x^2 + 64 = 0$

$x^2 = -64$ (subtract 64 from both sides)

This equation does not have a solution because you cannot square a real number and get a negative answer.

(c) $(x + 4)^2 = 81$

$x + 4 = \pm 9$ (square root both sides)

2.1 Quadratic functions

- Simplest non-linear function: Quadratic

Problem

1 Solve the following quadratic equations. (Round your solutions to 2 decimal places if necessary.)

(a) $x^2 - 100 = 0$ **(b)** $2x^2 - 8 = 0$ **(c)** $x^2 - 3 = 0$ **(d)** $x^2 - 5.72 = 0$

(e) $x^2 + 1 = 0$ **(f)** $3x^2 + 6.21 = 0$ **(g)** $x^2 = 0$

All of the equations considered in Problem 1 are of the special form

$ax^2 + c = 0$

2.1 Quadratic functions

- Simplest non-linear function: Quadratic

$$ax^2 + bx + c = 0$$

has solutions

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

The following example describes how to use this formula. It also illustrates the fact (which you have already discovered in Practice Problem 1) that a quadratic equation can have two solutions, one solution or no solutions.

2.1 Quadratic functions

- Simplest non-linear function: Quadratic

Example

Solve the quadratic equations

(a) $2x^2 + 9x + 5 = 0$

(b) $x^2 - 4x + 4 = 0$

(c) $3x^2 - 5x + 6 = 0$

2.1 Quadratic functions

- Quadratic| Discriminant

This example demonstrates the three cases that can occur when solving quadratic equations. The precise number of solutions that an equation can have depends on whether the number under the square root sign is positive, zero or negative. The number $b^2 - 4ac$ is called the *discriminant* because the sign of this number discriminates between the three cases that can occur.

- If $b^2 - 4ac > 0$ then there are two solutions

$$x = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

- If $b^2 - 4ac = 0$ then there is one solution

$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}$$

- If $b^2 - 4ac < 0$ then there are no solutions because $\sqrt{(b^2 - 4ac)}$ does not exist.

2.1 Quadratic functions

- Quadratic| Discriminant

Practice Problem

2 Solve the following quadratic equations (where possible):

(a) $2x^2 - 19x - 10 = 0$ (b) $4x^2 + 12x + 9 = 0$

(c) $x^2 + x + 1 = 0$ (d) $x^2 - 3x + 10 = 2x + 4$

2.1 Quadratic functions

- Quadratic| Factorization
  - You may be familiar with another method for solving quadratic equations. This is based on the factorization of a quadratic into the product of two linear factors.

$$(x+1)(x+2) = x^2 + 3x + 2$$

Consequently, the solutions of the equation

$$x^2 + 3x + 2 = 0$$

are the same as those of

$$(x+1)(x+2) = 0$$

The quadratic equation

$$x^2 + 3x + 2 = 0$$

therefore has two solutions, $x = -1$ and $x = -2$.

2.1 Quadratic functions

- Quadratic| Factorization

Example

Write down the solutions to the following quadratic equations:

(a) $x(3x - 4) = 0$ (b) $(x - 7)^2 = 0$

Solution

(a) If $x(3x - 4) = 0$ then either $x = 0$ or $3x - 4 = 0$

The first gives the solution $x = 0$ and the second gives $x = 4/3$.

(b) If $(x - 7)(x - 7) = 0$ then either $x - 7 = 0$ or $x - 7 = 0$

Both options lead to the same solution, $x = 7$.

2.1 Quadratic functions

- Quadratic| Factorization

Practice Problem

3 Write down the solutions to the following quadratic equations. (There is no need to multiply out the brackets.)

(a) $(x - 4)(x + 3) = 0$

(b) $x(10 - 2x) = 0$

(c) $(2x - 6)^2 = 0$

The difficulty with this approach is that it is impossible, except in very simple cases, to work out the factorization from any given quadratic, so the preferred method is to use the formula.

2.1 Quadratic functions

- Quadratic| Sketching the graphs
  - One important feature of linear functions is that their graphs are always straight lines.
  - Now, whenever you are asked to produce a graph of an unfamiliar function, it is often a good idea to tabulate the function,
  - to plot these points on graph paper and to join them up with a smooth curve.
  - The precise number of points to be taken depends on the function but, as a general rule, between 5 and 10 points usually produce a good picture.

2.1 Quadratic functions

- Quadratic| Sketching the graphs

Example

Sketch a graph of the square function, $f(x) = x^2$.

Solution

A table of values for the simple square function

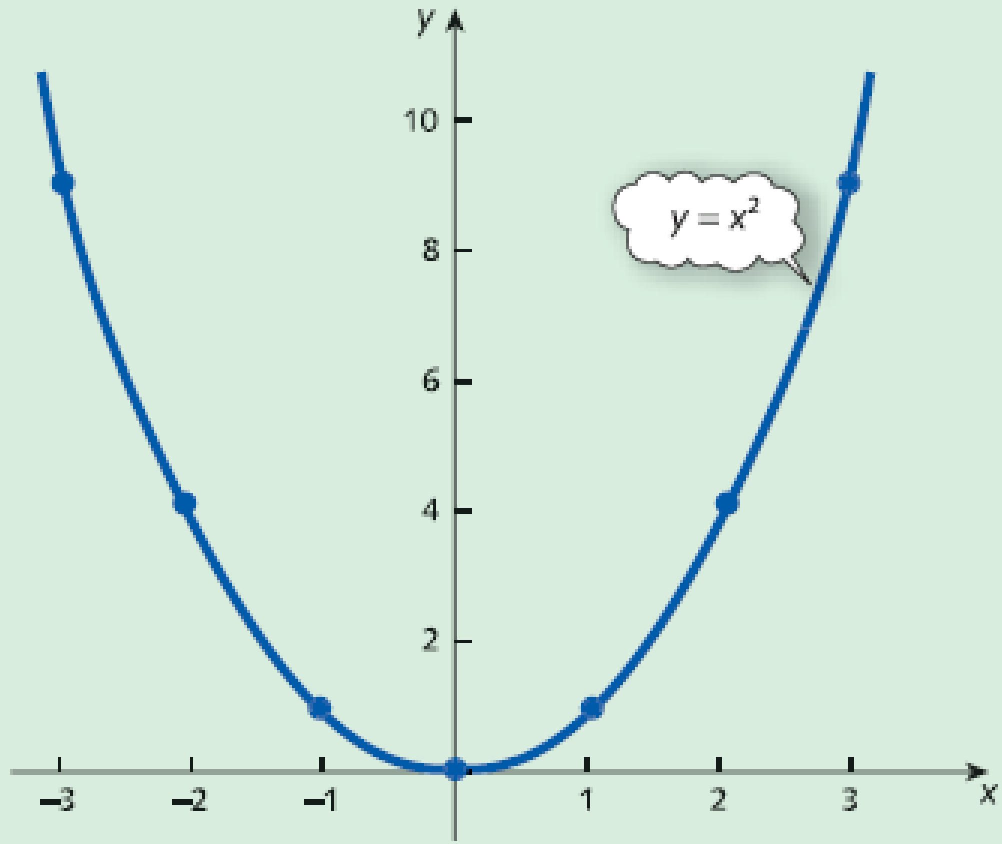
$$f(x) = x^2$$

is given by

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

The first row of the table gives a selection of 'incoming' numbers, x, while the second row shows the corresponding 'outgoing' numbers, y. Points with coordinates (x, y) are then plotted on graph paper to produce the curve shown in Figure 2.1. For convenience, different scales are used on the x and y axes.

Figure 2.1



Mathematicians call this curve a parabola, whereas economists refer to it as U-shaped. Notice that the graph is symmetric about the y axis with a minimum point at the origin; if a mirror is placed along the y axis then the left-hand part is the image of the right-hand part.

2.1 Quadratic functions

- Quadratic| Sketching the graphs

Practice Problem

4 Complete the following tables of function values and hence sketch a graph of each quadratic function.

(a) $f(x) = 4x^2 - 12x + 5$

x	-1	0	1	2	3	4
$f(x)$						

(b) $f(x) = -x^2 + 6x - 9$

x	0	1	2	3	4	5	6
$f(x)$							

(c) $f(x) = -2x^2 + 4x - 6$

x	-2	-1	0	1	2	3	4
$f(x)$							

2.1 Quadratic functions

- Quadratic| Graphs| Strategy for sketching

$$f(x) = ax^2 + bx + c$$

Step 1

Determine the basic shape. The graph has a U shape if $a > 0$, and an inverted U shape if $a < 0$.

Step 2

Determine the y intercept. This is obtained by substituting $x = 0$ into the function, which gives $y = c$.

Step 3

Determine the x intercepts (if any). These are obtained by solving the quadratic equation

$$ax^2 + bx + c = 0$$

This three-step strategy is illustrated in the following example.

Example

Give a rough sketch of the graph of the following quadratic function:

$$f(x) = -x^2 + 8x - 12$$

Solution

For the function

$$f(x) = -x^2 + 8x - 12$$

the strategy is as follows.

Step 1

The coefficient of x^2 is -1, which is negative, so the graph is a 'sad' parabola with an inverted U shape.

Step 2

The constant term is -12, so the graph crosses the vertical axis at $y = -12$.

Whenever the coefficient of x^2 is positive, the graph bends upwards and is a 'happy' parabola (U shape). Similarly, when the coefficient of x^2 is negative, the graph bends downwards and is a 'sad' parabola (inverted U shape).

Step 3

For the quadratic equation

$$-x^2 + 8x - 12 = 0$$

the formula gives

$$x = \frac{-8 \pm \sqrt{(8^2 - 4(-1)(-12))}}{2(-1)} = \frac{-8 \pm \sqrt{(64 - 48)}}{-2}$$

$$= \frac{-8 \pm \sqrt{16}}{-2} = \frac{-8 \pm 4}{-2}.$$

so the graph crosses the horizontal axis at

$$x = \frac{-8 + 4}{-2} = 2$$

and

$$x = \frac{-8 - 4}{-2} = 6$$

The information obtained in steps 1–3 is sufficient to produce the sketch shown in Figure 2.4.

In fact, we can go even further in this case and locate the coordinates of the turning point – that is, the maximum point – on the curve. By symmetry, the x coordinate of this point occurs exactly halfway between $x = 2$ and $x = 6$: that is, at

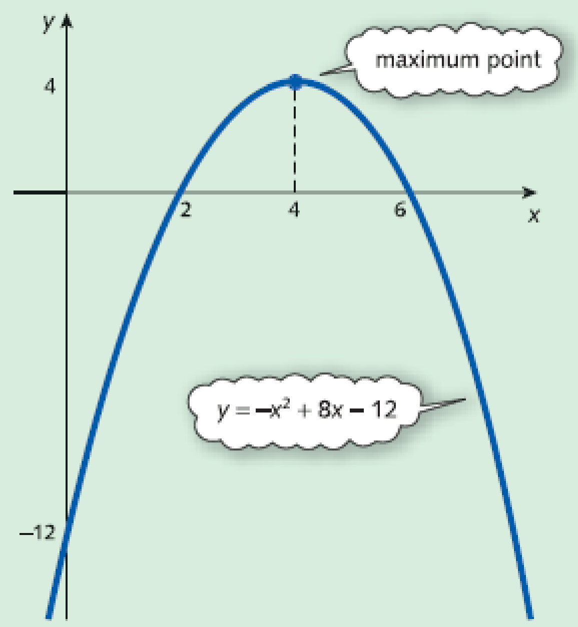
$$x = \tfrac{1}{2}(2 + 6) = 4$$

The corresponding y coordinate is found by substituting $x = 4$ into the function to get

$$f(4) = -(4)^2 + 8(4) - 12 = 4$$

The maximum point on the curve therefore has coordinates (4, 4).

Figure 5.2



2.1 Quadratic functions

- Quadratic| Graphs| Strategy for sketching

Practice Problem

5 Use the three-step strategy to produce rough graphs of the following quadratic functions:

(a) $f(x) = 2x^2 - 11x - 6$ (b) $f(x) = x^2 - 6x + 9$

2.1 Quadratic functions

- Microeconomics
  - We conclude this section by seeing how to solve a particular problem in microeconomics.
  - In Chapter 1 the concept of market equilibrium was introduced and in each of the problems the supply and demand functions were always given to be linear.
  - The following example shows this to be an unnecessary restriction and indicates that it is almost as easy to manipulate quadratic supply and demand functions.

Example

Given the supply and demand functions

$$P = Q_S^2 + 14Q_S + 22$$
$$P = -Q_D^2 - 10Q_D + 150$$

calculate the equilibrium price and quantity.

Solution

In equilibrium, $Q_S = Q_D$, so if we denote this equilibrium quantity by Q, the supply and demand functions become

$$P = Q^2 + 14Q + 22$$
$$P = -Q^2 - 10Q + 150$$

Hence

$$Q^2 + 14Q + 22 = -Q^2 - 10Q + 150$$

since both sides are equal to P. Collecting like terms gives

$$2Q^2 + 24Q - 128 = 0$$

which is just a quadratic equation in the variable Q. Before using the formula to solve this it is a good idea to divide both sides by 2 to avoid large numbers. This gives

$$Q^2 + 12Q - 64 = 0$$

and so

$$Q = \frac{-12 \pm \sqrt{((12^2) - 4(1)(-64))}}{2(1)}$$

$$= \frac{-12 \pm \sqrt{(400)}}{2}$$

$$= \frac{-12 \pm 20}{2}$$

The quadratic equation has solutions $Q = -16$ and $Q = 4$. Now the solution $Q = -16$ can obviously be ignored because a negative quantity does not make sense. The equilibrium quantity is therefore 4. The equilibrium price can be calculated by substituting this value into either the original supply or demand equation.

From the supply equation,

$$P = 4^2 + 14(4) + 22 = 94$$

As a check, the demand equation gives

$$P = -(4)^2 - 10(4) + 150 = 94 \qquad \checkmark$$

2.1 Quadratic functions

- Microeconomics

Practice Problem

6 Given the supply and demand functions

$$P = 2Q_S^2 + 10Q_S + 10$$

$$P = -Q_D^2 - 5Q_D + 52$$

calculate the equilibrium price and quantity.

2.1 Quadratic functions

Key Terms

Discriminant The number $b^2 - 4ac$ which is used to indicate the number of solutions of the quadratic equation $ax^2 + bx + c = 0$.

Parabola The shape of the graph of a quadratic function.

Quadratic function A function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$.

Square root A number that when multiplied by itself equals a given number; the solutions of the equation $x^2 = c$ which are written $\pm\sqrt{c}$.

U-shaped curve A term used by economists to describe a curve, such as a parabola, which bends upwards, like the letter U.

2.3 Indices and logarithms

- Index notation

$$\boxed{b^n = b \times b \times b \times \ldots b}$$

a total of n b's multiplied together

$$\boxed{b^0 = 1}$$

and

$$\boxed{b^{-n} = \frac{1}{b^n}}$$

where n is any positive whole number.

1 **(1)** Without using a calculator evaluate

(a) 10^2 **(b)** 10^1 **(c)** 10^0 **(d)** 10^{-1} **(e)** 10^{-2} **(f)** $(-1)^{100}$

(g) $(-1)^{99}$ **(h)** 7^{-3} **(i)** $(-9)^2$ **(j)** $(72\ 101)^1$ **(k)** $(2.718)^0$

2.3 Indices and logarithms

- Index notation

$$\boxed{b^{1/n} = n\text{th root of } b}$$

By this we mean that $b^{1/n}$ is a number which, when raised to the power n, produces b. In symbols, if $c = b^{1/n}$ then $c^n = b$. Using this definition,

$9^{1/2}$ = square root of 9 $= 3$ (because $3^2 = 9$)

$8^{1/3}$ = cube root of 8 $= 2$ (because $2^3 = 8$)

$625^{1/4}$ = fourth root of 625 $= 5$ (because $5^4 = 625$)

2.3 Indices and logarithms

- Index notation

$$\boxed{b^{p/q} = (b^p)^{1/q} = (b^{1/q})^p}$$

$$16^3 = 16 \times 16 \times 16 = 4096$$

and taking the fourth root of this gives

$$16^{3/4} = (4096)^{1/4} = 8 \quad (\text{because } 8^4 = 4096)$$

On the other hand, taking the fourth root first gives

$$16^{1/4} = 2 \quad (\text{because } 2^4 = 16)$$

and cubing this gives

$$16^{3/4} = 2^3 = 8$$

2.3 Indices and logarithms

- Index notation

Example

Evaluate

(a) $8^{4/3}$ (b) $25^{-3/2}$

Solution

(a) To evaluate $8^{4/3}$ we need both to raise the number to the power of 4 and to find a cube root. Choosing to find the cube root first,

$$8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$

(b) Again it is easy to find the square root of 25 first before raising the number to the power of –3, so

$$25^{-3/2} = (25^{1/2})^{-3} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

For this particular exponential form we have actually carried out three distinct operations. The minus sign tells us to reciprocate, the fraction ½ tells us to take the square root and the 3 tells us to cube. You might like to check for yourself that you get the same answer irrespective of the order in which these three operations are performed.

2.3 Indices and logarithms

- Index notation

Practice Problem

2 (1) Without using your calculator, evaluate

(a) $16^{1/2}$ (b) $27^{1/3}$ (c) $4^{5/2}$ (d) $8^{-2/3}$ (e) $1^{-17/25}$

(2) Confirm your answer to part (1) using a calculator.

2.3 Indices and logarithms

- Rules of indices

Rule 1	$b^m \times b^n = b^{m+n}$
Rule 2	$b^m \div b^n = b^{m-n}$
Rule 3	$(b^m)^n = b^{mn}$
Rule 4	$(ab)^n = a^n b^n$

Practice Problem

3 Simplify

(a) $(x^{3/4})^8$ **(b)** $\dfrac{x^2}{x^{3/2}}$ **(c)** $(x^2y^4)^3$ **(d)** $\sqrt{x}(x^{5/2} + y^3)$

[Hint: in part (d) note that $\sqrt{x} = x^{1/2}$ and multiply out the brackets.]

2.3 Indices and logarithms

- To be homogeneous..

In general, a function

$$Q = f(K, L)$$

is said to be *homogeneous* if

$$f(\lambda K, \lambda L) = \lambda^n f(K, L)$$

for some number, n. This means that when both variables K and L are multiplied by λ we can pull out all of the λs as a common factor, λ^n. The power, n, is called the **degree of homogeneity**. In the previous example we showed that

$$f(\lambda K, \lambda L) = \lambda^{5/6} f(K, L)$$

and so it is homogeneous of degree 5/6. In general, if the degree of homogeneity, n, satisfies:

- $n < 1$, the function is said to display *decreasing returns to scale*
- $n = 1$, the function is said to display *constant returns to scale*
- $n > 1$, the function is said to display *increasing returns to scale*.

2.3 Indices and logarithms

- To be homogeneous..

Example

Show that the following production function is homogeneous and find its degree of homogeneity:

$$Q = 2K^{1/2}L^{3/2}$$

Does this function exhibit decreasing returns to scale, constant returns to scale or increasing returns to scale?

Solution

We are given that

$$f(K, L) = 2K^{1/2}L^{3/2}$$

so replacing K by λK and L by λL gives

$$f(\lambda K, \lambda L) = 2(\lambda K)^{1/2}(\lambda L)^{3/2}$$

2.3 Indices and logarithms

- To be homogeneous..

We can pull out all of the λs by using rule 4 to get

$$2\lambda^{1/2}K^{1/2}\lambda^{3/2}L^{3/2}$$

and then using rule 1 to get

$$\lambda^2(2K^{1/2}L^{3/2})$$

$$\lambda^{1/2}\lambda^{3/2} = \lambda^{1/2+3/2} = \lambda^2$$

We have therefore shown that

$$f(\lambda K, \lambda L) = \lambda^2 f(K, L)$$

and so the function is homogeneous of degree 2. Moreover, since $2 > 1$ we deduce that it has increasing returns to scale.

2.3 Indices and logarithms

- To be homogeneous..

Practice Problem

4 Show that the following production functions are homogeneous and comment on their returns to scale:

(a) $Q = 7KL^2$ **(b)** $Q = 50K^{1/4}L^{3/4}$

2.3 Indices and logarithms

- Logarithms

$$\text{if} \quad M = b^n \quad \text{then} \quad \log_b M = n$$

when n is called the *logarithm of* M *to base* b.

Example

Evaluate

(a) $\log_3 9$ (b) $\log_4 2$ (c) $\log_7 {}^1\!/_7$

Solution

(a) To find the value of $\log_3 9$ we convert the problem into one involving powers. From the definition of a logarithm to base 3 we see that the statement

2.3 Indices and logarithms

- Logarithms

$$\text{if} \quad M = b^n \quad \text{then} \quad \log_b M = n$$

when n is called the *logarithm of* M *to base* b.

$$\log_3 9 = n$$

is equivalent to

$$9 = 3^n$$

The problem of finding the logarithm of 9 to base 3 is exactly the same as that of writing 9 as a power of 3. The solution of this equation is clearly $n = 2$ since

$$9 = 3^2$$

Hence $\log_3 9 = 2$.

(b) Again to evaluate $\log_4 2$ we merely rewrite

if $M = b^n$ then $\log_b M = n$

$$\log_4 2 = n$$

in exponential form as

$$2 = 4^n$$

The problem of finding the logarithm of 2 to base 4 is exactly the same as that of writing 2 as a power of 4. The value of 2 is obtained from 4 by taking the square root, which involves raising 4 to the power of $^1/_2$, so

$$2 = 4^{1/2}$$

Hence $\log_2 4 = {}^1/_2$.

(c) If

$$\log_7 {}^1/_7 = n$$

then

$$^1/_7 = 7^n$$

The value of $^1/_7$ is found by taking the reciprocal of 7, which involves raising 7 to the power of –1: that is,

$$^1/_7 = 7^{-1}$$

Hence $\log_7 {}^1/_7 = -1$.

2.3 Indices and logarithms

- Logarithms

Practice Problem

5 (1) Write down the values of n which satisfy

(a) $1000 = 10^n$ (b) $100 = 10^n$ (c) $10 = 10^n$

(d) $1 = 10^n$ (e) $\frac{1}{10} = 10^n$ (f) $\frac{1}{100} = 10^n$

(2) Use your answer to part (1) to write down the values of

(a) $\log_{10}1000$ (b) $\log_{10}100$ (c) $\log_{10}10$

(d) $\log_{10}1$ (e) $\log_{10}{}^1/_{10}$ (f) $\log_{10}{}^1/_{100}$

(3) Confirm your answer to part (2) using a calculator.

2.3 Indices and logarithms

- Logarithms

Rule 1 $\log_b(x \times y) = \log_b x + \log_b y$
Rule 2 $\log_b(x \div y) = \log_b x - \log_b y$
Rule 3 $\log_b x^m = m\log_b x$

Practice Problem

6 Use the rules of logs to express each of the following as a single logarithm:

(a) $\log_b x - \log_b y + \log_b z$ (b) $4\log_b x + 2\log_b y$

Slide

Example

(b) To solve

$$5^x = 2(3)^x$$

we take logarithms of both sides to get

$$\log(5^x) = \log(2 \times 3^x)$$

The right-hand side is the logarithm of a product and, according to rule 1, can be written as the sum of the logarithms, so the equation becomes

$$\log(5^x) = \log(2) + \log(3^x)$$

As in part (a) the key step is to use rule 3 to 'bring down the powers'. If rule 3 is applied to both $\log(5^x)$ and $\log(3^x)$ then the equation becomes

$$x\log(5) = \log(2) + x\log(3)$$

This is now the type of equation that we know how to solve. We collect x's on the left-hand side to get

$$x\log(5) - x\log(3) = \log(2)$$

and then pull out a common factor of x to get

$$x[\log(5) - \log(3)] = \log(2)$$

Now, by rule 2, the difference of two logarithms is the same as the logarithm of their quotient, so

$$\log(5) - \log(3) = \log(5 \div 3)$$

Hence the equation becomes

$$x\log\left(\frac{5}{3}\right) = \log(2)$$

so

$$x = \frac{\log(2)}{\log(5/3)}$$

Finally, taking logarithms to base 10 using a calculator gives

Finally, taking logarithms to base 10 using a calculator gives

$$x = \frac{0.301\,029\,996}{0.221\,848\,750} = 1.36$$

As a check, the original equation

$$5^x = 2(3)^x$$

becomes

$$5^{1.36} = 2(3)^{1.36}$$

that is,

$$8.92 = 8.91 \qquad \checkmark$$

Again the slight discrepancy is due to rounding errors in the value of x.

2.3 Indices and logarithms

- Logarithms

Practice Problem

7 Solve the following equations for x:

(a) $3^x = 7$ **(b)** $5(2)^x = 10^x$

2.3 Indices and logarithms

Key Terms

Constant returns to scale Exhibited by a production function when a given percentage increase in input leads to the same percentage increase in output: $f(\lambda K, \lambda L) = \lambda f(K, L)$.

Decreasing returns to scale Exhibited by a production function when a given percentage increase in input leads to a smaller percentage increase in output: $f(\lambda K, \lambda L) = \lambda^n f(K, L)$ where $0 < n < 1$.

Exponent A superscript attached to a variable; the number 5 is the exponent in the expression, $2x^5$.

Exponential form A representation of a number which is written using powers. For example, 2^5 is the exponential form of the number 32.

Factors of production The inputs into the production of goods and services: labour, land, capital and raw materials.

Homogeneous functions A function with the property that when all of the inputs are multiplied by a constant, λ, the output is multiplied by λ^n where n is the degree of homogeneity.

Increasing returns to scale Exhibited by a production function when a given percentage increase in input leads to a larger percentage increase in output: $f(\lambda K, \lambda L) = \lambda^n f(K, L)$ where $n > 1$.

Index Another word for exponent.

Labour All forms of human input to the production process.

Logarithm The power to which a base must be raised to yield a particular number.

Power Another word for exponent. If this is a positive integer then it gives the number of

Production function The relationship between the output of a good and the inputs used to produce it.

2.4 The exponential and natural logarithm functions

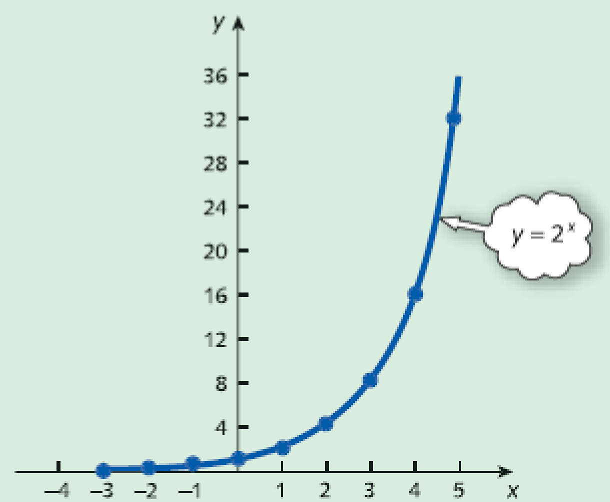
- Sketch graphs of exponential functions

Example

Sketch the graphs of the functions

(a) $f(x) = 2^x$ (b) $g(x) = 2^{-x}$

Figure 2.14



x	−3	−2	−1	0	1	2	3	4	5
2^x	0.125	0.25	0.5	1	2	4	8	16	32

2.4 The exponential and natural logarithm functions

- The number e

$$\left(1+\frac{1}{m}\right)^m$$

approaches a limiting value of 2.718 281 828 . . . , which we choose to denote by the letter e. In symbols we write

$$e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$$

The number e has a similar status in mathematics as the number π and is just as useful. It arises in the mathematics of finance

2.4 The exponential and natural logarithm functions

- The number e

Example

The percentage, y, of households possessing refrigerators, t years after they have been introduced in a developed country, is modelled by

$$y = 100 - 95e^{-0.15t}$$

(1) Find the percentage of households that have refrigerators

(a) at their launch

(b) after 1 year

(c) after 10 years

(d) after 20 years

(2) What is the market saturation level?

(3) Sketch a graph of y against t and hence give a qualitative description of the growth of refrigerator ownership over time.

Solution

(1) To calculate the percentage of households possessing refrigerators now and in 1, 10 and 20 years' time, we substitute $t = 0, 1, 10$ and 20 into the formula

$$y = 100 - 95e^{-0.15t}$$

to get

(a) $y(0) = 100 - 95e^{0} = 5\%$

(b) $y(1) = 100 - 95e^{-0.15} = 18\%$

(c) $y(10) = 100 - 95e^{-1.5} = 79\%$

(d) $y(20) = 100 - 95e^{-3.0} = 95\%$

check these numbers on your own calculator

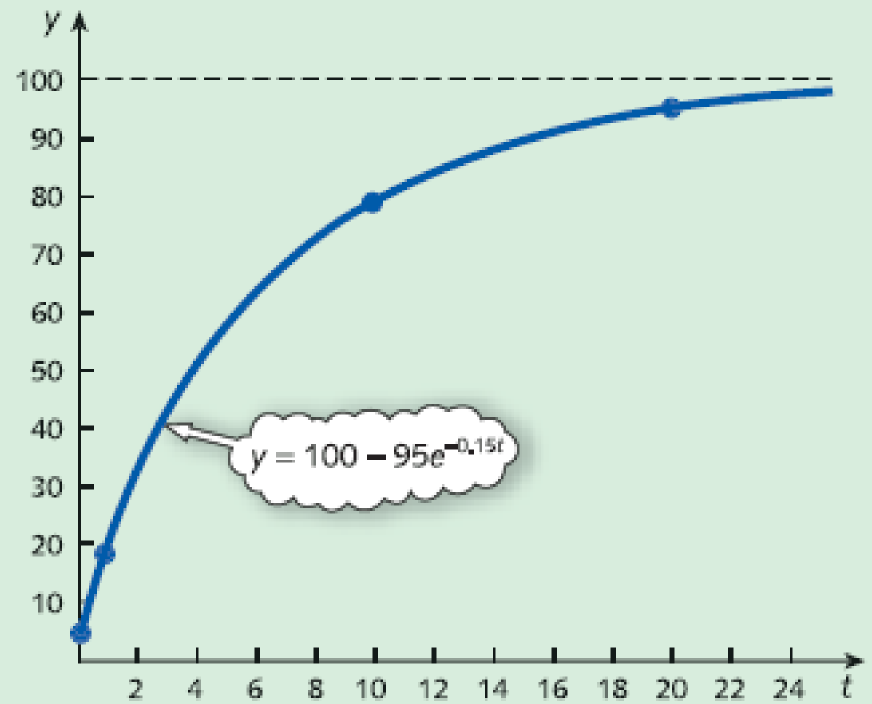
(2) To find the saturation level we need to investigate what happens to y as t gets ever larger. We know that the graph of a negative exponential function has the basic shape shown in Figure 2.15. Consequently, the value of $e^{-0.15t}$ will eventually approach zero as t increases. The market saturation level is therefore given by

$$y = 100 - 95(0) = 100\%$$

(3) A graph of y against t, based on the information obtained in parts (1) and (2), is sketched in Figure 2.16.

This shows that y grows rapidly to begin with, but slows down as the market approaches saturation level. A saturation level of 100% indicates that eventually all households are expected to possess refrigerators, which is not surprising given the nature of the product.

Figure 2.16



2.4 The exponential and natural loga functions

- The number e

Practice Problem

3 The percentage, y, of households possessing camcorders t years after they have been launched is modelled by

$$y = \frac{55}{1 + 800e^{-0.3t}}$$

(1) Find the percentage of households that have camcorders

(a) at their launch

(b) after 10 years

(c) after 20 years

(d) after 30 years

(2) What is the market saturation level?

(3) Sketch a graph of y against t and hence give a qualitative description of the growth of camcorder ownership over time.

2.4 The exponential and natural logarithm functions

- Natural logarithms

In Section 2.3 we noted that if a number M can be expressed as b^n then n is called the logarithm of M to base b. In particular, for base e,

if $\quad M = e^n \quad$ then $\quad n = \log_e M$

We call logarithms to base e *natural logarithms*. These occur sufficiently frequently to warrant their own notation. Rather than writing $\log_e M$ we simply put $\ln M$ instead. The three rules of logs can then be stated as

Rule 1 $\quad \ln(x \times y) = \ln x + \ln y$
Rule 2 $\quad \ln(x \div y) = \ln x - \ln y$
Rule 3 $\quad \ln x^m = m \ln x$

Example

Use the rules of logs to express

(a) $\ln\left(\frac{x}{\sqrt{y}}\right)$ in terms of $\ln x$ and $\ln y$

(b) $3 \ln p + \ln q - 2 \ln r$ as a single logarithm

Solution

(a) In this part we need to 'expand', so we read the rules of logs from left to right:

$$\begin{aligned} \ln\left(\frac{x}{\sqrt{y}}\right) &= \ln x - \ln \sqrt{y} && \text{(rule 2)} \\ &= \ln x - \ln y^{1/2} && \text{(fractional powers denote roots)} \\ &= \ln x - \frac{1}{2}\ln y && \text{(rule 3)} \end{aligned}$$

(b) In this part we need to reverse this process and so read the rules from right to left:

$$\begin{aligned} 3 \ln p + \ln q - 2 \ln r &= \ln p^3 + \ln q - \ln r^2 && \text{(rule 3)} \\ &= \ln(p^3 q) - \ln r^2 && \text{(rule 1)} \\ &= \ln\left(\frac{p^3 q}{r^2}\right) && \text{(rule 2)} \end{aligned}$$

2.4 The exponential and natural loga functions

- Natural logarithms

Practice Problem

4 Use the rules of logs to express

(a) $\ln(a^2b^3)$ in terms of $\ln a$ and $\ln b$

(b) $\frac{1}{2}\ln x - 3\ln y$ as a single logarithm

Example

An economy is forecast to grow continuously so that the gross national product (GNP), measured in billions of dollars, after t years is given by

$$GNP = 80e^{0.02t}$$

After how many years is GNP forecast to be \$88 billion?

Solution

We need to solve

$$88 = 80e^{0.02t}$$

for t. Dividing through by 80 gives

$$1.1 = e^{0.02t}$$

Using the definition of natural logarithms we know that

$$\text{if} \quad M = e^n \quad \text{then} \quad n = \ln M$$

If we apply this definition to the equation

$$1.1 = e^{0.02t}$$

we deduce that

$$0.02t = \ln 1.1 = 0.095\,31\ldots \quad \text{(check this using your own calculator)}$$

so

$$t = \frac{0.095\,31}{0.02} = 4.77$$

We therefore deduce that GNP reaches a level of \$88 billion after 4.77 years.

2.4 The exponential and natural loga functions

- Natural logarithms

Practice Problem

5 During a recession a firm's revenue declines continuously so that the revenue, TR (measured in millions of dollars), in t years' time is modelled by

$$TR = 5e^{-0.15t}$$

(a) Calculate the current revenue and also the revenue in 2 years' time.

(b) After how many years will the revenue decline to \$2.7 million?

2.4 The exponential and natural logarithm functions

Key Terms

The exponential function The function, $f(x) = e^x$; an exponential function in which the base is number, e = 2.718 28 . . .

Natural logarithm A logarithm to base, e; if $M = e^n$ then n is the natural logarithm of M and we write, $n = \ln M$

Chapter 3: Mathematics of Finance

Chapter 3: Mathematics of Finance

- Percentages
- Compound interest
- Investment appraisal (evaluation)

» Percentages are used to calculate and interpret index numbers, and to adjust value data for inflation

» The exponential function is used to solve problems in which interest is compounded continuously

» Discounting (calculating the present value given a future value) can be used to appraise different investment projects

3.1 Percentages

- Idea of a percentage
  - In order to be able to handle financial calculations, it is necessary to use percentages proficiently.
  - The word ‘percentage’ literally means ‘per cent’, i.e. per hundredth, so that whenever we speak of r% of something, we simply mean the fraction (r/100)ths of it.

For example,

25% is the same as $\frac{25}{100} = \frac{1}{4}$

30% is the same as $\frac{30}{100} = \frac{3}{10}$

50% is the same as $\frac{50}{100} = \frac{1}{2}$

3.1 Percentages

- Idea of a percentage

Example

Calculate

(a) 15% of 12 (b) 98% of 17 (c) 150% of 290

Solution

(a) 15% of 12 is the same as

$$\frac{15}{100} \times 12 = 0.15 \times 12 = 1.8$$

(b) 98% of 17 is the same as

$$\frac{98}{100} \times 17 = 0.98 \times 17 = 16.66$$

(c) 150% of 290 is the same as

$$\frac{150}{100} \times 290 = 1.5 \times 290 = 435$$

3.1 Percentages

- Idea of a percentage
  - Whenever any numerical quantity increases or decreases, it is customary to refer to this change in percentage terms. The following example serves to remind you how to perform calculations involving percentage changes.

Example

(a) An investment rises from \$2500 to \$3375. Express the increase as a percentage of the original.

(b) At the beginning of a year, the population of a small village is 8400. If the annual rise in population is 12%, find the population at the end of the year.

(c) In a sale, all prices are reduced by 20%. Find the sale price of a good originally costing \$580.

Solution

(a) The rise in the value of the investment is

$$3375 - 2500 = 875$$

As a fraction of the original this is

$$\frac{875}{2500} = 0.35$$

This is the same as 35 hundredths, so the percentage rise is 35%.

(b) As a fraction

$$12\% \text{ is the same as } \frac{12}{100} = 0.12$$

so the rise in population is

$$0.12 \times 8400 = 1008$$

Hence the final population is

$$8400 + 1008 = 9408$$

(c) As a fraction

$$20\% \text{ is the same as } \frac{20}{100} = 0.2$$

so the fall in price is

$$0.2 \times 580 = 116$$

Hence the final price is

$$580 - 116 = \$464$$

3.1 Percentages

- Idea of a percentage

Practice Problem

2 **(a)** A firm's annual sales rise from 50 000 to 55 000 from one year to the next. Express the rise as a percentage of the original.

(b) The government imposes a 15% tax on the price of a good. How much does the consumer pay for a good priced by a firm at \$1360?

(c) Investments fall during the course of a year by 7%. Find the value of an investment at the end of the year if it was worth \$9500 at the beginning of the year.

3.1 Percentages

- Scale factor
  - let us suppose that the price of good is set to rise by 9%, and that its current price is \$78. The new price consists of the original (which can be thought of as 100% of the \$78) plus the increase (which is 9% of \$78).

The final price is therefore

$$100\% + 9\% = 109\% \text{ (of the \$78)}$$

which is the same as

$$\frac{109}{100} = 1.09$$

  - In other words, in order to calculate the final price all we have to do is to multiply by the ***scale factor,*** 1.09.
  - Hence the new price is $1.09 \times 78 = \$85.02$

3.1 Percentages

- Scale factor

One advantage of this approach is that it is then just as easy to go backwards and work out the original price from the new price. To go backwards in time we simply *divide* by the scale factor. For example, if the final price of a good is \$1068.20 then before a 9% increase the price would have been

$$1068.20 \div 1.09 = \$980$$

In general, if the percentage rise is r% then the final value consists of the original (100%) together with the increase (r%), giving a total of

$$\frac{100}{100} + \frac{r}{100} = 1 + \frac{r}{100}$$

To go forwards in time we multiply by this scale factor, whereas to go backwards we divide.

Example

(a) If the annual rate of inflation is 4%, find the price of a good at the end of a year if its price at the beginning of the year is \$25.

(b) The cost of a good is \$799 including 17.5% VAT (value added tax). What is the cost excluding VAT?

(c) Express the rise from 950 to 1007 as a percentage.

Solution

(a) The scale factor is

$$1 + \frac{4}{100} = 1.04$$

We are trying to find the price *after* the increase, so we *multiply* to get

$$25 \times 1.04 = \$26$$

(b) The scale factor is

$$1 + \frac{17.5}{100} = 1.175$$

This time we are trying to find the price *before* the increase, so we *divide* by the scale factor to get

$$799 \div 1.175 = \$680$$

(c) The scale factor is

$$\frac{\text{new value}}{\text{old value}} = \frac{1007}{950} = 1.06$$

which can be thought of as

$$1 + \frac{6}{100}$$

so the rise is 6%.

3.1 Percentages

- Scale factor

Practice Problem

3 (a) The value of a good rises by 13% in a year. If it was worth $6.5 million at the beginning of the year, find its value at the end of the year.

(b) The GNP of a country has increased by 63% over the past 5 years and is now $124 billion. What was the GNP 5 years ago?

(c) Sales rise from 115 000 to 123 050 in a year. Find the annual percentage rise.

3.1 Percentages

- Scale factor

It is possible to use scale factors to solve problems involving percentage decreases. To be specific, suppose that an investment of $76 falls by 20%. The new value is the original (100%) less the decrease (20%), so is 80% of the original. The scale factor is therefore 0.8, giving a new value of

$$0.8 \times 76 = \$60.80$$

In general, the scale factor for an r% decrease is

$$\frac{100}{100} - \frac{r}{100} = 1 - \frac{r}{100}$$

Once again, you multiply by this scale factor when going forwards in time and divide when going backwards.

3.1 Percentages

- Index numbers
  - Economic data often take the form of a time series; values of economic indicators are available on an annual, quarterly or monthly basis, and we are interested in analysing the rise and fall of these numbers over time.
  - Index numbers enable us to identify trends and relationships in the data.
  - The following example shows you how to calculate index numbers and how to interpret them.

3.1 Percentages

- Index numbers

Example

Table 3.1 shows the values of household spending (in billions of dollars) during a 5-year period. Calculate the index numbers when 2000 is taken as the base year and give a brief interpretation.

Table 3.1

	Year				
	1999	2000	2001	2002	2003
Household spending	686.9	697.2	723.7	716.6	734.5

When finding index numbers, a base year is chosen and the value of 100 is allocated to that year. In this example, we are told to take 2000 as the base year, so the index number of 2000 is 100. To find the index number of the year 2001 we work out the scale factor associated with the change in household spending from the base year, 2000 to 2001, and then multiply the answer by 100.

$$\boxed{\text{index number} = \text{scale factor from base year} \times 100}$$

In this case, we get

$$\frac{723.7}{697.2} \times 100 = 103.8$$

This shows that the value of household spending in 2001 was 103.8% of its value in 2000. In other words, household spending increased by 3.8% during 2001.

For the year 2002, the value of household spending was 716.6, giving an index number

$$\frac{716.6}{697.2} \times 100 = 102.8$$

This shows that the value of household spending in 2002 was 102.8% of its value in 2000. In other words, household spending increased by 2.8% between 2000 and 2002. Notice that this is less than that calculated for 2001, reflecting the fact that spending actually fell slightly during 2002. The remaining two index numbers are calculated in a similar way and are shown in Table 3.2.

Table 3.2

			Year		
	1999	**2000**	**2001**	**2002**	**2003**
Household spending	686.9	697.2	723.7	716.6	734.5
Index number	98.5	100	103.8	102.8	105.3

3.1 Percentages

- Index numbers

Practice Problem

6 Calculate the index numbers for the data shown in Table 3.1, this time taking 1999 as the base year.

Example

Find the index numbers of each share price shown in Table 3.3, taking April as the base month. Hence compare the performances of these two share prices during this period.

Table 3.3

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Share A	0.31	0.28	0.31	0.34	0.40	0.39	0.45	0.52
Share B	6.34	6.40	6.45	6.52	6.57	6.43	6.65	7.00

3.1 Percentages

Key Terms

Annual rate of inflation The percentage increase in the level of prices over a 12-month period.

Index numbers The scale factor of a variable measured from the base year multiplied by 100.

Laspeyre index An index number for groups of data which are weighted by the quantities used in the base year.

Nominal data Monetary values prevailing at the time that they were measured.

Paasche index An index number for groups of data which are weighted by the quantities used in the current year.

Real data Monetary values adjusted to take inflation into account.

Scale factor The multiplier that gives the final value in percentage problems.

Time series A sequence of numbers indicating the variation of data over time.

3.2 Compound interest

- Compound interest

Example

Find the value, in 4 years' time, of \$10 000 invested at 5% interest compounded annually.

Solution

- At the end of year 1 the interest is $0.05 \times 10\,000 = 500$, so the investment is 10 500.
- At the end of year 2 the interest is $0.05 \times 10\,500 = 525$, so the investment is 11 025.
- At the end of year 3 the interest is $0.05 \times 11\,025 = 551.25$, so the investment is 11 576.25.
- At the end of year 4 the interest is $0.05 \times 11\,576.25 = 578.81$ rounded to 2 decimal places, so the final investment is \$12 155.06 to the nearest cent.

3.2 Compound interest

- Compound interest

Practice Problem

1 Find the value, in 10 years' time, of $1000 invested at 8% interest compounded annually.

3.2 Compound interest

- Another method of calculating
  - The original sum of money is called the ***principal*** and is denoted by ***P***, and the final sum is called the ***future value*** and is denoted by ***S***.
  - In general, if the ***interest rate*** is ***r%*** compounded annually then the scale factor is

$$1 + \frac{r}{100}$$

so, after n years,

$$\boxed{S = P\left(1 + \frac{r}{100}\right)^n}$$

3.2 Compound interest

- Another method of calculating

Example

Find the value, in 4 years' time, of \$10 000 invested at 5% interest compounded annually.

Solution

In this problem, $P = 10\,000$, $r = 5$ and $n = 4$, so the formula $S = P\left(1 + \frac{r}{100}\right)^n$ gives

$$S = 10\,000\left(1 + \frac{5}{100}\right)^4 = 10\,000(1.05)^4 = 12\,155.06$$

which is, of course, the same answer as before.

3.2 Compound interest

- Another method of calculating

Practice Problem

2 Use the formula

$$S = P\left(1 + \frac{r}{100}\right)^n$$

to find the value, in 10 years' time, of $1000 invested at 8% interest compounded annually. [You might like to compare your answer with that obtained in Practice Problem 1.]

Example

A principal of \$25 000 is invested at 12% interest compounded annually. After how many years will the investment first exceed \$250 000?

Solution

We want to save a total of \$250 000 starting with an initial investment of \$25 000. The problem is to determine the number of years required for this on the assumption that the interest is fixed at 12% throughout this time. The formula for compound interest is

$$S = P\left(1 + \frac{r}{100}\right)^n$$

We are given that

$$P = 25\,000,\ S = 250\,000,\ r = 12$$

so we need to solve the equation

$$250\,000 = 25\,000\left(1 + \frac{12}{100}\right)^n$$

for n.

Provided that we know any three of these, we can use the formula to determine the remaining variable.

One way of doing this would just be to keep on guessing values of n until we find the one that works. However, a more mathematical approach is to use logarithms, because we are being asked to solve an equation in which the unknown occurs as a power. Following the method described in Section 2.3, we first divide both sides by 25 000 to get

$$10 = (1.12)^n$$

Taking logarithms of both sides gives

$$\log(10) = \log(1.12)^n$$

and if you apply rule 3 of logarithms you get

Example

A principal of \$10 is invested at 12% interest for 1 year. Determine the future value if the interest is compounded

(a) annually (b) semi-annually (c) quarterly (d) monthly (e) weekly

Solution

The formula for compound interest gives

$$S = 10\left(1 + \frac{r}{100}\right)^n$$

(a) If the interest is compounded annually then $r = 12$, $n = 1$, so

$$S = \$10(1.12)^1 = \$11.20$$

(b) If the interest is compounded semi-annually then an interest of 12/2 = 6% is added on every 6 months and, since there are two 6-month periods in a year,

$$S = \$10(1.06)^2 = \$11.24$$

(c) If the interest is compounded quarterly then an interest of 12/4 = 3% is added on every 3 months and, since there are four 3-month periods in a year,

$$S = \$10(1.03)^4 = \$11.26$$

(d) If the interest is compounded monthly then an interest of 12/12 = 1% is added on every month and, since there are 12 months in a year,

$$S = \$10(1.01)^{12} = \$11.27$$

(e) If the interest is compounded weekly then an interest of 12/52 = 0.23% is added on every week and, since there are 52 weeks in a year,

$$S = \$10(1.0023)^{52} = \$11.27$$

3.2 Compound interest

- Advanced method

$$\boxed{S = Pe^{rt/100}}$$

where e is the number

2.718 281 828 459 045 235 36 (to 20 decimal places)

If $r = 12$, $t = 1$ and $P = 10$ then this formula gives

$$S = \$10e^{12\times1/100} = \$10e^{0.12} = \$11.27$$

check this using your own calculator

which is in agreement with the limiting value obtained in the previous example.

Example

A principal of \$2000 is invested at 10% interest compounded continuously. After how many days will the investment first exceed \$2100?

Solution

We want to save a total of \$2100 starting with an initial investment of \$2000. The problem is to determine the number of days required for this on the assumption that the interest rate is 10% compounded continuously. The formula for continuous compounding is

$$S = Pe^{rt/100}$$

We are given that

$$S = 2100,\ P = 2000,\ r = 10$$

so we need to solve the equation

$$2100 = 2000e^{10t/100}$$

for t. Dividing through by 2000 gives

$$1.05 = e^{0.1t}$$

Provided that we know any three of these, we can use the formula to determine the remaining variable.

Example

A firm decides to increase output at a constant rate from its current level of 50 000 to 60 000 during the next 5 years. Calculate the annual rate of increase required to achieve this growth.

Solution

If the rate of increase is r% then the scale factor is $1 + \frac{r}{100}$ so after 5 years, output will be

$$50\,000\left(1 + \frac{r}{100}\right)^5$$

To achieve a final output of 60 000, the value of r is chosen to satisfy the equation

$$50\,000\left(1 + \frac{r}{100}\right)^5 = 60\,000$$

Dividing both sides by 50 000 gives

$$\left(1 + \frac{r}{100}\right)^5 = 1.2$$

to get

$$1 + \frac{r}{100} = (1.2)^{1/5} = 1.037$$

Hence $r = 3.7$%.

3.2 Compound interest

- Advanced method

Practice Problems

4 (1) A principal, \$30, is invested at 6% interest for 2 years. Determine the future value if the interest is compounded

(a) annually (b) semi-annually (c) quarterly

(d) monthly (e) weekly (f) daily

(2) Use the formula

$$S = Pe^{rt/100}$$

to determine the future value of \$30 invested at 6% interest compounded continuously for 2 years. Confirm that it is in agreement with the results of part (1).

5 Determine the rate of interest required for a principal of \$1000 to produce a future value of \$4000 after 10 years compounded continuously.

3.2 Compound interest

- Advanced method

Practice Problem

3 A firm estimates that its sales will rise by 3% each year and that it needs to sell at least 10 000 goods each year in order to make a profit. Given that its current annual sales are only 9000, how many years will it take before the firm breaks even?

Practice Problem

6 Determine the annual percentage rate of interest if the nominal rate is 12% compounded quarterly.

Practice Problem

7 The turnover of a leading supermarket chain, A, is currently $560 million and is expected to increase at a constant rate of 1.5% a year. Its nearest rival, supermarket B, has a current turnover of $480 million and plans to increase this at a constant rate of 3.4% a year. After how many years will supermarket B overtake supermarket A?

3.2 Compound interest

Key Terms

Annual percentage rate The equivalent annual interest paid for a loan, taking into account the compounding over a variety of time periods.

Compound interest The interest which is added on to the initial investment, so that this will itself gain interest in subsequent time periods.

Continuous compounding The limiting value when interest is compounded with ever-increasing frequency.

Future value The final value of an investment after one or more time periods.

Principal The value of the original sum invested.

Simple interest The interest which is paid direct to the investor instead of being added to the original amount.