MAT 205
Linear Algebra

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What is linear algebra ?

- **LINEAR ALGEBRA** is The branch of algebra in which one studies vector (linear) spaces, linear operators (linear mappings), and linear, bilinear, and quadratic functions (functionals and forms) on vector spaces… *(Encyclopedia of Mathematics, Kluwer Academic Press, 1990)*

- Linear algebra has become a central course for mathematics majors as well as students of science, business, and engineering. Its balance of computation, theory, and applications to real life, geometry, and other areas makes linear algebra unique among mathematics courses…

- One of the early goals of this course is to develop an algorithm that helps solve larger systems in an orderly manner and is amenable to computer implementation…

- Solving systems of linear equations is one of the most important applications of linear algebra, as we will see in chapter 1…

Chapter 1
Systems of Linear Equations

Chapter 1: Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations
1.2 Gaussian Elimination and Gauss-Jordan Elimination
1.3 Applications of Systems of Linear Equations

CHAPTER OBJECTIVES

- Recognize, graph, and solve a system of linear equations in *n* variables.
- Use back-substitution to solve a system of linear equations.
- Determine whether a system of linear equations is consistent or inconsistent.
- Determine if a matrix is in row-echelon form or reduced row-echelon form.
- Use elementary row operations with back-substitution to solve a system in row-echelon form.
- Use elimination to rewrite a system in row-echelon form.
- Write an augmented or coefficient matrix from a system of linear equations, or translate a matrix into a system of linear equations.
- Solve a system of linear equations using Gaussian elimination and Gaussian elimination with back-substitution.
- Solve a homogeneous system of linear equations.
- Set up and solve a system of equations to fit a polynomial function to a set of data points, as well as to represent a network.

1.1 Introduction to Systems of Linear Equations

- Linear Equations in n Variables
  - Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power.

A **linear equation in n variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient,** and x_1 is the **leading variable.**

  - Example 1 lists some equations that are linear and some that are not linear.

1.1 Introduction to Systems of Linear Equations

- Linear Equations in n Variables

EXAMPLE 1 Examples of Linear Equations and Nonlinear Equations

Each equation is linear.

(a) $3x + 2y = 7$ (b) $\frac{1}{2}x + y - \pi z = \sqrt{2}$

(c) $x_1 - 2x_2 + 10x_3 + x_4 = 0$ (d) $\left(\sin \frac{\pi}{2}\right)x_1 - 4x_2 = e^2$

Each equation is not linear.

(a) $xy + z = 2$ (b) $e^x - 2y = 4$

(c) $\sin x_1 + 2x_2 - 3x_3 = 0$ (d) $\frac{1}{x} + \frac{1}{y} = 4$

1.1 Introduction to Systems of Linear Equations

- Linear Equations in n Variables

EXAMPLE 2 Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

SOLUTION To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. If you solve for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable x_2 is **free**, which means that it can take on any real value. The variable x_1 is not free because its value depends on the value assigned to x_2. To represent the infinite number of solutions of this equation, it is convenient to introduce a third variable t called a **parameter.** By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \qquad x_2 = t, \quad t \text{ is any real number.}$$

Particular solutions can be obtained by assigning values to the parameter t. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$.

1.1 Introduction to Systems of Linear Equations

- Systems of Linear Equations

A system of m linear equations in n variables is a set of m equations, each of which is linear in the same n variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m .
\end{aligned}
$$

REMARK: The double-subscript notation indicates a_{ij} is the coefficient of x_j in the ith equation.

EXAMPLE 4 Systems of Two Equations in Two Variables

Solve each system of linear equations, and graph each system as a pair of straight lines.

(a) $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$ (b) $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$ (c) $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

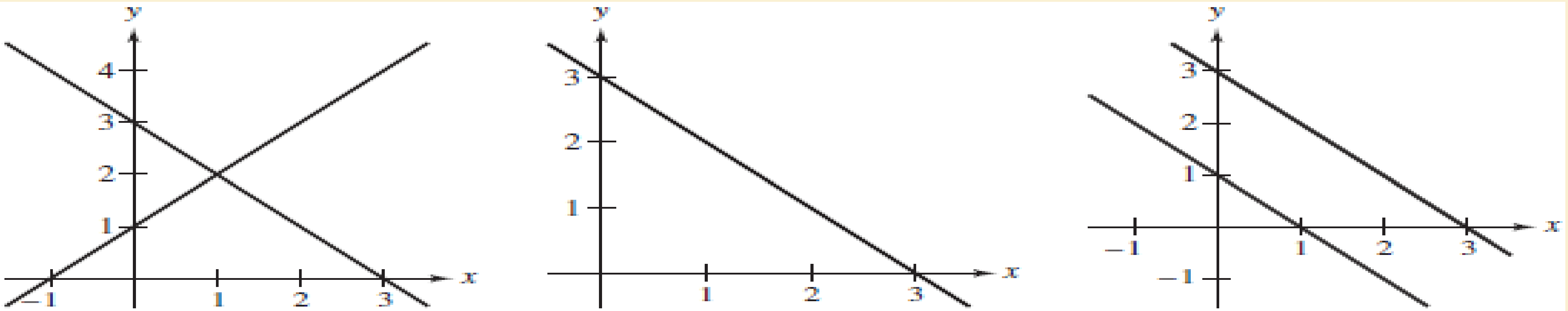
SOLUTION (a) This system has exactly one solution, $x = 1$ and $y = 2$. The solution can be obtained by adding the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is represented by two *intersecting* lines, as shown in Figure 1.1(a).

(b) This system has an infinite number of solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is shown as

$$x = 3 - t, \qquad y = t, \quad t \text{ is any real number.}$$

The graph of this system is represented by two *coincident* lines, as shown in Figure 1.1(b).

(c) This system has no solution because it is impossible for the sum of two numbers to be 3 and 1 simultaneously. The graph of this system is represented by two *parallel* lines, as shown in Figure 1.1(c).



(a) Two intersecting lines: (b) Two coincident lines: (c) Two parallel lines:

1.1 Introduction to Systems of Linear Equations

- Systems of Linear Equations

Discovery

Graph the two lines

$$3x - y = 1$$
$$2x - y = 0$$

in the xy-plane. Where do they intersect? How many solutions does this system of linear equations have?

Repeat this analysis for the pairs of lines

$$3x - y = 1 \qquad 3x - y = 1$$
$$3x - y = 0 \qquad 6x - 2y = 2.$$

In general, what basic types of solution sets are possible for a system of two equations in two unknowns?

1.1 Introduction to Systems of Linear Equations

- Systems of Linear Equations

For a system of linear equations in n variables, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has an infinite number of solutions (consistent system).
3. The system has no solution (inconsistent system).

1.1 Introduction to Systems of Linear Equations

- Solving a System of Linear Equations

Solving a System of Linear Equations

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad\qquad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it follows a stair-step pattern and has leading coefficients of 1. To solve such a system, use a procedure called **back-substitution.**

1.1 Introduction to Systems of Linear Equations

- Solving a System of Linear Equations

EXAMPLE 6 Using Back-Substitution to Solve a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 && \text{Equation 1} \\ y + 3z &= 5 && \text{Equation 2} \\ z &= 2 && \text{Equation 3} \end{aligned}$$

SOLUTION From Equation 3 you already know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$\begin{aligned} y + 3(2) &= 5 && \text{Substitute } z = 2. \\ y &= -1. && \text{Solve for } y. \end{aligned}$$

Finally, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 && \text{Substitute } y = -1, z = 2. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

1.1 Introduction to Systems of Linear Equations

- Solving a System of Linear Equations
  - To solve a system that is not in row-echelon form, first change it to an equivalent system that is in row-echelon form by using the operations listed below.

Each of the following operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

EXAMPLE 7 **Using Elimination to Rewrite a System in Row-Echelon Form**

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION Although there are several ways to begin, you want to use a systematic procedure that can be applied easily to large systems. Work from the upper left corner of the system, saving the x in the upper left position and eliminating the other x's from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that everything but the first x has been eliminated from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

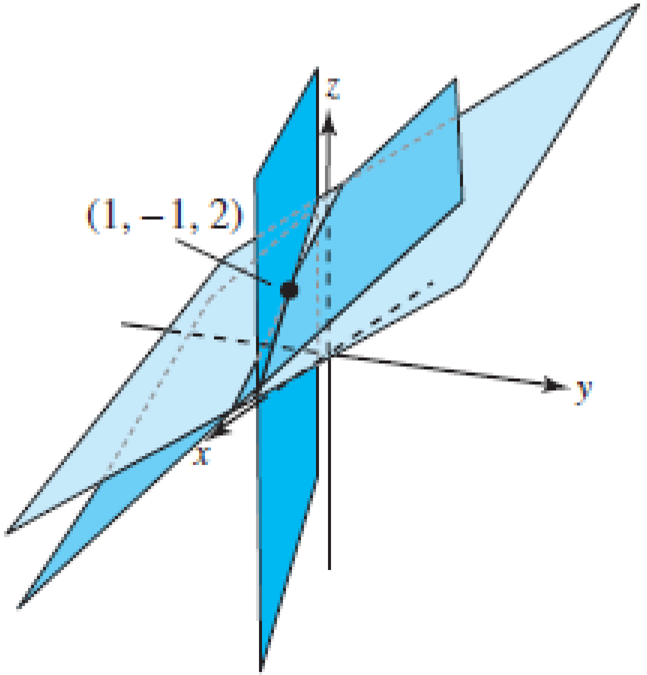
same system you solved in Example 6…

1.1 Introduction to Systems of Linear Equations

Each of the three equations in Example 7 is represented in a three-dimensional coordinate system by a plane. Because the unique solution of the system is the point

$$(x, y, z) = (1, -1, 2),$$

the three planes intersect at the point represented by these coordinates, as shown in Figure 1.2.



EXAMPLE 8 An Inconsistent System

Solve the system.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 2 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

SOLUTION

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 5x_2 - 4x_3 &= -2 \end{aligned}$$

Adding -1 times the first equation to the third equation produces a new third equation.

(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.) Now, continuing the elimination process, add -1 times the second equation to the third equation to produce a new third equation.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 0 &= -2 \end{aligned}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because the third "equation" is a false statement, this system has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

EXAMPLE 9 A System with an Infinite Number of Solutions

Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 \qquad - 3x_3 &= -1 \\ -x_1 + 3x_2 \qquad &= 1 \end{aligned}$$

SOLUTION Begin by rewriting the system in row-echelon form as follows.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 \qquad &= 1 \end{aligned}$$

The first two equations are interchanged.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Adding -3 times the second equation to the third equation eliminates the third equation.

Because the third equation is unnecessary, omit it to obtain the system shown below.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

To represent the solutions, choose x_3 to be the free variable and represent it by the parameter t. Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$$x_1 = 3t - 1, \qquad x_2 = t, \qquad x_3 = t, \quad t \text{ is any real number.}$$

1.1 Introduction to Systems of Linear Equations

- Exercises

In Exercises 65–68, state why each system of equations must have at least one solution. Then solve the system and determine if it has exactly one solution or an infinite number of solutions.

65.
$$\begin{aligned} 4x + 3y + 17z &= 0 \\ 5x + 4y + 22z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$$

66.
$$\begin{aligned} 2x + 3y \phantom{{}- z} &= 0 \\ 4x + 3y - z &= 0 \\ 8x + 3y + 3z &= 0 \end{aligned}$$

67.
$$\begin{aligned} 5x + 5y - z &= 0 \\ 10x + 5y + 2z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$$

68.
$$\begin{aligned} 12x + 5y + z &= 0 \\ 12x + 4y - z &= 0 \end{aligned}$$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Matrix
  - In Section 1.1, Gaussian elimination was introduced as a procedure for solving a system of linear equations. In this section you will study this procedure more thoroughly, beginning with some definitions. The first is the definition of a matrix.

If m and n are positive integers, then an $m \times n$ **matrix** is a rectangular array

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \\ \vdots \\ a \end{bmatrix}}\right\} m \text{ rows}$$

in which each **entry**, a_{ij}, of the matrix is a number. An $m \times n$ matrix (read "m by n") has m **rows** (horizontal lines) and n **columns** (vertical lines).

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Matrix

EXAMPLE 1 Examples of Matrices

Each matrix has the indicated size.

(a) Size: 1×1

$$\begin{bmatrix} 2 \end{bmatrix}$$

(b) Size: 2×2

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) Size: 1×4

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$

(d) Size: 3×2

$$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Matrix
  - One very common use of matrices is to represent systems of linear equations

System	*Augmented Matrix*	*Coefficient Matrix*
$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \quad - 4z &= 6 \end{aligned}$	$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by aligning the variables in the equations vertically.

Given System	*Align Variables*	*Augmented Matrix*
$\begin{aligned} x_1 + 3x_2 &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 - 5x_3 &= 0 \end{aligned}$	$\begin{aligned} x_1 + 3x_2 \qquad\quad &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 \qquad\quad - 5x_3 &= 0 \end{aligned}$	$\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{bmatrix}$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Elementary Row Operations

Elementary Row Operations

In the previous section you studied three operations that can be used on a system of linear equations to produce equivalent systems.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are said to be **row-equivalent** if one can be obtained from the other by a finite sequence of elementary row operations.

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

EXAMPLE 2 Elementary Row Operations

(a) Interchange the first and second rows.

Original Matrix	*New Row-Equivalent Matrix*	*Notation*
$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$

(b) Multiply the first row by $\frac{1}{2}$ to produce a new first row.

Original Matrix	*New Row-Equivalent Matrix*	*Notation*
$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\left(\frac{1}{2}\right)R_1 \rightarrow R_1$

(c) Add -2 times the first row to the third row to produce a new third row.

Original Matrix	*New Row-Equivalent Matrix*	*Notation*
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$

REMARK: Notice in Example 2(c) that adding -2 times row 1 to row 3 does not change row 1.

EXAMPLE 3 Using Elementary Row Operations to Solve a System

Linear System

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Associated Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Add the first row to the second row to produce a new second row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} \quad R_2 + R_1 \to R_2$$

Add -2 times the first equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Add -2 times the first row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \quad R_3 + (-2)R_1 \to R_3$$

Add the second equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Add the second row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad R_3 + R_2 \to R_3$$

to be followed in the next slide

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \left(\tfrac{1}{2}\right)R_3 \rightarrow R_3$$

Now you can use back-substitution to find the solution, as in Example 6 in Section 1.1. The solution is $x = 1$, $y = -1$, and $z = 2$.

- The last matrix in Example 3 is said to be in row-echelon form. The term echelon refers to the stair-step pattern formed by the nonzero elements of the matrix.
- To be in row-echelon form, a matrix must have the properties listed next.

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Row-Echelon Form of a Matrix

A matrix in **row-echelon form** has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

REMARK: A matrix in row-echelon form is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Row-Echelon Form of a Matrix

EXAMPLE 4 Row-Echelon Form

The matrices below are in row-echelon form.

(a) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The matrices shown in parts (b) and (d) are in *reduced* row-echelon form. The matrices listed below are not in row-echelon form.

(e) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Gaussian Elimination

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well as an algorithmic method for solving systems of linear equations. For this algorithm, the order in which the elementary row operations are performed is important. Move from *left to right by columns*, changing all entries directly below the leading 1's to zeros.

EXAMPLE 5 **Gaussian Elimination with Back-Substitution**

Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

SOLUTION The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

The first two rows are interchanged. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Adding -2 times the first row to the third row produces a new third row. $R_3 + (-2)R_1 \rightarrow R_3$

to be followed in the next slide

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$

Adding -1 times the first row to the fourth row produces a new fourth row. $R_4 + (-1)R_1 \rightarrow R_4$

Now that the first column is in the desired form, you should change the second column as shown below.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

Adding 6 times the second row to the fourth row produces a new fourth row. $R_4 + (6)R_2 \rightarrow R_4$

To write the third column in proper form, multiply the third row by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

Multiplying the third row by $\frac{1}{3}$ produces a new third row. $\left(\frac{1}{3}\right)R_3 \rightarrow R_3$

to be followed in the next slide

Similarly, to write the fourth column in proper form, you should multiply the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Multiplying the fourth row by $-\frac{1}{13}$ produces a new fourth row. $\left(-\frac{1}{13}\right)R_4 \rightarrow R_4$

The matrix is now in row-echelon form, and the corresponding system of linear equations is as shown below.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Using back-substitution, you can determine that the solution is

$$x_1 = -1, \quad x_2 = 2, \quad x_3 = 1, \quad x_4 = 3.$$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Gauss-Jordan Elimination

  - With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form.
  - A second method of elimination, called ***Gauss-Jordan elimination*** after Carl Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a ***reduced row-echelon*** form is obtained. This procedure is demonstrated in the next example.

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Gauss-Jordan Elimination

EXAMPLE 7 Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION In Example 3, Gaussian elimination was used to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now, rather than using back-substitution, apply elementary row operations until you obtain a matrix in reduced row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

to be followed in the next slide

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 + (2)R_2 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_2 + (-3)R_3 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 + (-9)R_3 \rightarrow R_1$$

Now, converting back to a system of linear equations, you have

$$\begin{aligned} x \quad\quad\quad &= 1 \\ y \quad\quad &= -1 \\ z &= 2. \end{aligned}$$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Homogeneous Systems of Linear Equations

Homogeneous Systems of Linear Equations

As the final topic of this section, you will look at systems of linear equations in which each of the constant terms is zero. We call such systems **homogeneous.** For example, a homogeneous system of m equations in n variables has the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0.
\end{aligned}$$

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have an infinite number of solutions.

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Exercises

In Exercises 23–36, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

23. $\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$

24. $\begin{aligned} 2x + 6y &= 16 \\ -2x - 6y &= -16 \end{aligned}$

25. $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

26. $\begin{aligned} 2x - y &= -0.1 \\ 3x + 2y &= 1.6 \end{aligned}$

27. $\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$

28. $\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$

29. $\begin{aligned} x_1 \qquad\quad - 3x_3 &= -2 \\ 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$

30. $\begin{aligned} 2x_1 - x_2 + 3x_3 &= 24 \\ 2x_2 - x_3 &= 14 \\ 7x_1 - 5x_2 \qquad &= 6 \end{aligned}$

31. $\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$

32. $\begin{aligned} 2x_1 + \qquad 3x_3 &= 3 \\ 4x_1 - 3x_2 + 7x_3 &= 5 \\ 8x_1 - 9x_2 + 15x_3 &= 10 \end{aligned}$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Exercises

48. Consider the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & k \\ 4 & -2 & 6 \end{bmatrix}$.

(a) If A is the *augmented* matrix of a system of linear equations, determine the number of equations and the number of variables.

(b) If A is the *augmented* matrix of a system of linear equations, find the value(s) of k such that the system is consistent.

(c) If A is the *coefficient* matrix of a *homogeneous* system of linear equations, determine the number of equations and the number of variables.

(d) If A is the *coefficient* matrix of a *homogeneous* system of linear equations, find the value(s) of k such that the system is consistent.

1.3 Applications of Systems of Linear Equations

– Systems of linear equations arise in a wide variety of applications and are one of the central themes in linear algebra. In this section you will look at two such applications, and you will see many more in subsequent chapters. The first application shows how to fit a polynomial function to a set of data points in the plane. The second application focuses on networks and Kirchhoff’s Laws for electricity.

EXAMPLE 1 Polynomial Curve Fitting

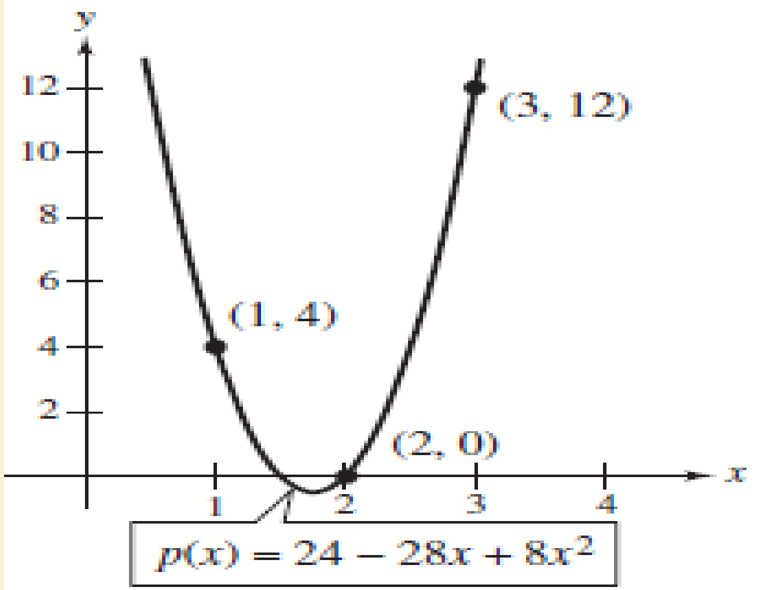
Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points (1, 4), (2, 0), and (3, 12).

SOLUTION Substituting $x = 1$, 2, and 3 into $p(x)$ and equating the results to the respective y-values produces the system of linear equations in the variables a_0, a_1, and a_2 shown below.

$$\begin{aligned} p(1) &= a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4 \\ p(2) &= a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0 \\ p(3) &= a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12 \end{aligned}$$

The solution of this system is $a_0 = 24$, $a_1 = -28$, and $a_2 = 8$, so the polynomial function is

$$p(x) = 24 - 28x + 8x^2.$$



EXAMPLE 6 **Analysis of an Electrical Network**

Determine the currents I_1, I_2, and I_3 for the electrical network shown in Figure 1.13.

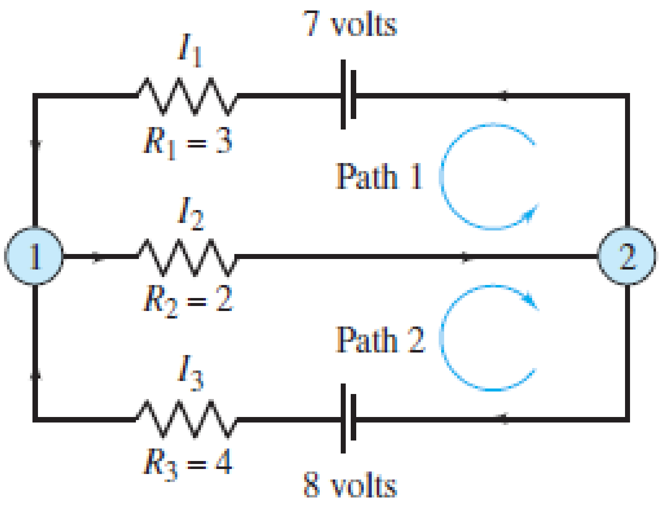


Figure 1.13

SOLUTION Applying Kirchhoff's first law to either junction produces

$$I_1 + I_3 = I_2 \qquad \text{Junction 1 or Junction 2}$$

and applying Kirchhoff's second law to the two paths produces

$$R_1I_1 + R_2I_2 = 3I_1 + 2I_2 = 7 \qquad \text{Path 1}$$
$$R_2I_2 + R_3I_3 = 2I_2 + 4I_3 = 8. \qquad \text{Path 2}$$

So, you have the following system of three linear equations in the variables I_1, I_2, and I_3.

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ 3I_1 + 2I_2 &= 7 \\ 2I_2 + 4I_3 &= 8 \end{aligned}$$

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{bmatrix}$$

produces the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which means $I_1 = 1$ amp, $I_2 = 2$ amps, and $I_3 = 1$ amp.

Chapter 2
Matrices

Chapter 2: Matrices

2.1 Operations with Matrices
2.2 Properties of Matrix Operations
2.3 The Inverse of a Matrix
2.4 Elementary Matrices
2.5 Applications of Matrix Operations

CHAPTER OBJECTIVES

- Write a system of linear equations represented by a matrix, as well as write the matrix form of a system of linear equations.
- Write and solve a system of linear equations in the form $A\mathbf{x} = \mathbf{b}$.
- Use properties of matrix operations to solve matrix equations.
- Find the transpose of a matrix, the inverse of a matrix, and the inverse of a matrix product (if they exist).
- Factor a matrix into a product of elementary matrices, and determine when they are invertible.
- Find and use the *LU*-factorization of a matrix to solve a system of linear equations.
- Use a stochastic matrix to measure consumer preference.
- Use matrix multiplication to encode and decode messages.
- Use matrix algebra to analyze economic systems (Leontief input-output models).
- Use the method of least squares to find the least squares regression line for a set of data.

2.1 Operations with Matrices

- Fundamentals of matrix theory
  - In Section 1.2 you used matrices to solve systems of linear equations. Matrices, however, can be used to do much more than that.
  - There is a rich mathematical theory of matrices, and its applications are numerous.
  - This section and the next introduce some fundamentals of matrix theory:
    - Equality of Matrices
    - Addition of Matrices
    - Scalar Multiplication and Matrix Subtraction
    - Finding the Product of Two Matrices
    - …
    - …

2.1 Operations with Matrices

- Fundamentals of matrix theory

Matrix Notation

1. A matrix can be denoted by an uppercase letter such as

$$A, B, C, \ldots .$$

2. A matrix can be denoted by a representative element enclosed in brackets, such as

$$[a_{ij}], [b_{ij}], [c_{ij}], \ldots .$$

3. A matrix can be denoted by a rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

2.1 Operations with Matrices

- Fundamentals of matrix theory

Definition of Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if they have the same size $(m \times n)$ and

$$a_{ij} = b_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

2.1 Operations with Matrices

- Fundamentals of matrix theory

Definition of Equality of Matrices

EXAMPLE 1 Equality of Matrices

Consider the four matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 3 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}.$$

Matrices A and B are not equal because they are of different sizes. Similarly, B and C are not equal. Matrices A and D are equal if and only if $x = 3$.

REMARK: The phrase "if and only if" means the statement is true in both directions. For example, "p if and only if q" means that p implies q and q implies p.

2.1 Operations with Matrices

- Fundamentals of matrix theory

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their sum is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different sizes is undefined.

2.1 Operations with Matrices

- Fundamentals of matrix theory

Definition of Matrix Addition

EXAMPLE 2 Addition of Matrices

(a) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined.

2.1 Operations with Matrices

- Fundamentals of matrix theory

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

You can use $-A$ to represent the scalar product $(-1)A$. If A and B are of the same size, $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B. \qquad \text{Subtraction of matrices}$$

EXAMPLE 3 Scalar Multiplication and Matrix Subtraction

For the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

find (a) $3A$, (b) $-B$, and (c) $3A - B$.

SOLUTION

(a)
$$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b)
$$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c)
$$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

2.1 Operations with Matrices

- Fundamentals of matrix theory

Matrix Multiplication

The third basic matrix operation is **matrix multiplication.** To see the usefulness of this operation, consider the following application in which matrices are helpful for organizing information.

A football stadium has three concession areas, located in the south, north, and west stands. The top-selling items are peanuts, hot dogs, and soda. Sales for a certain day are recorded in the first matrix below, and the prices (in dollars) of the three items are given in the second matrix.

$$\begin{array}{lccc|l}
 & \multicolumn{3}{c}{\textit{Number of Items Sold}} & \\
 & \textit{Peanuts} & \textit{Hot Dogs} & \textit{Soda} & \textit{Selling Price}
\end{array}$$

$$\begin{array}{l} \textit{South stand} \\ \textit{North stand} \\ \textit{West stand} \end{array}
\begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix}
\quad
\begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix}
\begin{array}{l} \textit{Peanuts} \\ \textit{Hot Dogs} \\ \textit{Soda} \end{array}$$

To calculate the total sales of the three top-selling items at the south stand, you can multiply each entry in the first row of the matrix on the left by the corresponding entry in the price column matrix on the right and add the results. The south stand sales are

$$(120)(2.00) + (250)(3.00) + (305)(2.75) = \$1828.75. \qquad \textbf{South stand sales}$$

Similarly, you can calculate the sales for the other two stands as follows.

$$(207)(2.00) + (140)(3.00) + (419)(2.75) = \$1986.25 \qquad \textbf{North stand sales}$$
$$(29)(2.00) + (120)(3.00) + (190)(2.75) = \$940.50 \qquad \textbf{West stand sales}$$

To calculate the total sales of the three top-selling items at the south stand, you can multiply each entry in the first row of the matrix on the left by the corresponding entry in the price column matrix on the right and add the results. The south stand sales are

$$(120)(2.00) + (250)(3.00) + (305)(2.75) = \$1828.75. \qquad \text{South stand sales}$$

Similarly, you can calculate the sales for the other two stands as follows.

$$(207)(2.00) + (140)(3.00) + (419)(2.75) = \$1986.25 \qquad \text{North stand sales}$$

$$(29)(2.00) + (120)(3.00) + (190)(2.75) = \$940.50 \qquad \text{West stand sales}$$

The preceding computations are examples of matrix multiplication. You can write the product of the 3×3 matrix indicating the number of items sold and the 3×1 matrix indicating the selling prices as follows.

$$\begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = \begin{bmatrix} 1828.75 \\ 1986.25 \\ 940.50 \end{bmatrix}$$

The product of these matrices is the 3×1 matrix giving the total sales for each of the three stands.

2.1 Operations with Matrices

- Fundamentals of matrix theory

Matrix Multiplication

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}.$$

EXAMPLE 4 Finding the Product of Two Matrices

Find the product AB, where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

– How to Multiply Two Matrices Together ?

Be sure you understand that for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is,

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} \quad = \quad \underset{m \times p}{AB}.$$

equal

size of AB

So, the product BA is not defined for matrices such as A and B in Example 4.

SOLUTION First note that the product AB is defined because A has size 3×2 and B has size 2×2. Moreover, the product AB has size 3×2 and will take the form

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

To find c_{11} (the entry in the first row and first column of the product), multiply corresponding entries in the first row of A and the first column of B. That is,

$$c_{11} = (-1)(-3) + (3)(-4) = -9$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Similarly, to find c_{12}, multiply corresponding entries in the first row of A and the second column of B to obtain

$$c_{12} = (-1)(2) + (3)(1) = 1$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Continuing this pattern produces the results shown below.

$$\begin{aligned} c_{21} &= (4)(-3) + (-2)(-4) = -4 \\ c_{22} &= (4)(2) + (-2)(1) = 6 \\ c_{31} &= (5)(-3) + (0)(-4) = -15 \\ c_{32} &= (5)(2) + (0)(1) = 10 \end{aligned}$$

The product is

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}.$$

EXAMPLE 5 Matrix Multiplication

(a) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 3 \qquad 2 \times 3$

(b) $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

(c) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

(d) $\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

$1 \times 3 \qquad 3 \times 1 \qquad 1 \times 1$

(e) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$

$3 \times 1 \qquad 1 \times 3 \qquad 3 \times 3$

2.1 Operations with Matrices

- Fundamentals of matrix theory

Discovery

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Calculate $A + B$ and $B + A$.

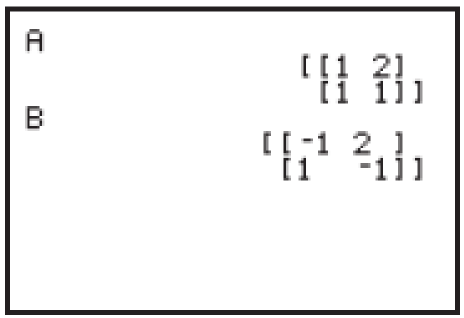
In general, is the operation of matrix addition commutative? Now calculate AB and BA. Is matrix multiplication commutative?

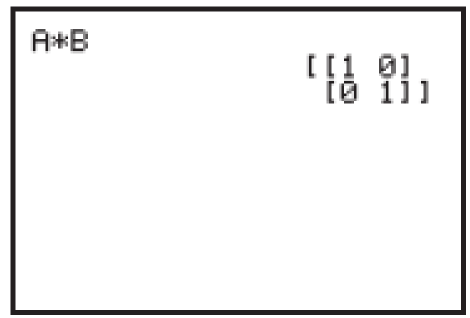
2.1 Operations with Matrices

- Fundamentals of matrix theory

Technology Note

Most graphing utilities and computer software programs can perform matrix addition, scalar multiplication, and matrix multiplication. If you are using a graphing utility, your screens for Example 5(c) may look like:





Keystrokes and programming syntax for these utilities/programs applicable to Example 5(c) are provided in the **Online Technology Guide,** available at *college.hmco.com/pic/larsonELA6e*.

2.1 Operations with Matrices

- Systems of Linear Equations

Systems of Linear Equations

One practical application of matrix multiplication is representing a system of linear equations. Note how the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

can be written as the matrix equation $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix of the system, and $\mathbf{x}$ and $\mathbf{b}$ are column matrices. You can write the system as

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} \underset{=}{=} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}.$$

2.1 Operations with Matrices

- Exercises

In Exercises 1–6, find (a) $A + B$, (b) $A - B$, (c) $2A$, (d) $2A - B$, and (e) $B + \frac{1}{2}A$.

1. $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$

2.1 Operations with Matrices

- Exercises

In Exercises 11–18, find (a) AB and (b) BA (if they are defined).

11. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

13. $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

15. $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

16. $A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

2.1 Operations with Matrices

- Exercises

In Exercises 29–36, write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve this matrix equation for $\mathbf{x}$.

29. $$\begin{aligned} -x_1 + x_2 &= 4 \\ -2x_1 + x_2 &= 0 \end{aligned}$$

30. $$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 &= 10 \end{aligned}$$

31. $$\begin{aligned} -2x_1 - 3x_2 &= -4 \\ 6x_1 + x_2 &= -36 \end{aligned}$$

32. $$\begin{aligned} -4x_1 + 9x_2 &= -13 \\ x_1 - 3x_2 &= 12 \end{aligned}$$

33. $$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 9 \\ -x_1 + 3x_2 - x_3 &= -6 \\ 2x_1 - 5x_2 + 5x_3 &= 17 \end{aligned}$$

34. $$\begin{aligned} x_1 + x_2 - 3x_3 &= -1 \\ -x_1 + 2x_2 \phantom{{}+x_3} &= 1 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

2.1 Operations with Matrices

- Exercises

True or False? In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

67. (a) For the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix.

(b) The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as a linear combination, where the coefficients of the linear combination are a solution of the system.

68. (a) If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the product AB is an $m \times r$ matrix.

(b) The matrix equation $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix and $\mathbf{x}$ and $\mathbf{b}$ are column matrices, can be used to represent a system of linear equations.

2.2 Properties of Matrix Operations

- Algebra of matrices
  - In Section 2.1 you concentrated on the mechanics of the three basic matrix operations: matrix addition, scalar multiplication, and matrix multiplication.
  - This section begins to develop the algebra of matrices.
  - You will see that this algebra shares many (but not all) of the properties of the algebra of real numbers.
  - Several properties of matrix addition and scalar multiplication are listed below.

2.2 Properties of Matrix Operations

- Algebra of matrices

THEOREM 2.1
Properties of Matrix Addition and Scalar Multiplication

If A, B, and C are $m \times n$ matrices and c and d are scalars, then the following properties are true.

1. $A + B = B + A$ — Commutative property of addition
2. $A + (B + C) = (A + B) + C$ — Associative property of addition
3. $(cd)A = c(dA)$ — Associative property of multiplication
4. $1A = A$ — Multiplicative identity
5. $c(A + B) = cA + cB$ — Distributive property
6. $(c + d)A = cA + dA$ — Distributive property

2.2 Properties of Matrix Operations

- Algebra of matrices

THEOREM 2.1
Properties of Matrix Addition and Scalar Multiplication

EXAMPLE 1 Addition of More than Two Matrices

By adding corresponding entries, you can obtain the sum of four matrices shown below.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

2.2 Properties of Matrix Operations

- Algebra of matrices

THEOREM 2.2 Properties of Zero Matrices

If A is an $m \times n$ matrix and c is a scalar, then the following properties are true.

1. $A + O_{mn} = A$
2. $A + (-A) = O_{mn}$
3. If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$.

2.2 Properties of Matrix Operations

- Algebra of matrices

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the solutions below.

Real Numbers (Solve for x.)	$m \times n$ Matrices (Solve for X.)
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

The process of solving a matrix equation is demonstrated in Example 2.

2.2 Properties of Matrix Operations

- Algebra of matrices

EXAMPLE 2 **Solving a Matrix Equation**

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

SOLUTION Begin by solving the equation for X to obtain

$$3X = B - A \Rightarrow X = \tfrac{1}{3}(B - A).$$

Now, using the given matrices A and B, you have

$$X = \tfrac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) = \tfrac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

2.2 Properties of Matrix Operations

- Properties of Matrix Multiplication

THEOREM 2.3
Properties of Matrix Multiplication

If A, B, and C are matrices (with sizes such that the given matrix products are defined) and c is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

EXAMPLE 3 **Matrix Multiplication Is Associative**

Find the matrix product ABC by grouping the factors first as $(AB)C$ and then as $A(BC)$. Show that the same result is obtained from both processes.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION Grouping the factors as $(AB)C$, you have

$$(AB)C = \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}.$$

Grouping the factors as $A(BC)$, you obtain the same result.

$$A(BC) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}$$

EXAMPLE 4 **Noncommutativity of Matrix Multiplication**

Show that AB and BA are not equal for the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

SOLUTION

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

$$AB \neq BA$$

Do not conclude from Example 4 that the matrix products AB and BA are *never* the same. Sometimes they are the same. For example, try multiplying the following matrices, first in the order AB and then in the order BA.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}$$

You will see that the two products are equal. The point is this: Although AB and BA are sometimes equal, AB and BA are usually not equal.

2.2 Properties of Matrix Operations

- Properties of Matrix Multiplication

EXAMPLE 5 An Example in Which Cancellation Is Not Valid

Show that $AC = BC$.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

SOLUTION

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$AC = BC$, even though $A \neq B$.

2.2 Properties of Matrix Operations

- Properties of Matrix Multiplication

THEOREM 2.4
Properties of the Identity Matrix

If A is a matrix of size $m \times n$, then the following properties are true.

1. $AI_n = A$
2. $I_mA = A$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\underset{2 \times 2}{I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \qquad \underset{3 \times 3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

2.2 Properties of Matrix Operations

- Properties of Matrix Multiplication

THEOREM 2.4
Properties of the Identity Matrix

EXAMPLE 6 Multiplication by an Identity Matrix

(a) $\begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

2.2 Properties of Matrix Operations

- The Transpose of a Matrix
  - The transpose of a matrix is formed by writing its columns as rows.

EXAMPLE 8 The Transpose of a Matrix

Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$

SOLUTION (a) $A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$ (b) $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ (c) $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $D^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$

2.2 Properties of Matrix Operations

- The Transpose of a Matrix

THEOREM 2.6
Properties of Transposes

If A and B are matrices (with sizes such that the given matrix operations are defined) and c is a scalar, then the following properties are true.

1. $(A^T)^T = A$ Transpose of a transpose
2. $(A + B)^T = A^T + B^T$ Transpose of a sum
3. $(cA)^T = c(A^T)$ Transpose of a scalar multiple
4. $(AB)^T = B^T A^T$ Transpose of a product

2.2 Properties of Matrix Operations

- The Transpose of a Matrix

Discovery

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$.

Calculate $(AB)^T$, $A^T B^T$, *and* $B^T A^T$. *Make a conjecture about the transpose of a product of two square matrices. Select two other square matrices to check your conjecture.*

2.2 Properties of Matrix Operations

- Exercises

In Exercises 9–14, perform the indicated operations, provided that $c = -2$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix},\quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

9. $B(CA)$ **10.** $C(BC)$ **11.** $(B + C)A$
12. $B(C + O)$ **13.** $(cB)(C + C)$ **14.** $B(cA)$

In Exercises 17 and 18, demonstrate that if $AB = O$, then it is *not* necessarily true that $A = O$ or $B = O$ for the following matrices.

17. $A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$

2.2 Properties of Matrix Operations

- Exercises

In Exercises 23–28, find (a) A^T, (b) A^TA, and (c) AA^T.

23. $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}$

25. $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 4 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

26. $A = \begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix}$

2.2 Properties of Matrix Operations

- Exercises

True or False? In Exercises 35 and 36, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

35. (a) Matrix addition is commutative.

(b) Matrix multiplication is associative.

(c) The transpose of the product of two matrices equals the product of their transposes; that is, $(AB)^T = A^T B^T$.

(d) For any matrix C, the matrix CC^T is symmetric.

36. (a) Matrix multiplication is commutative.

(b) Every matrix A has an additive inverse.

(c) If the matrices A, B, and C satisfy $AB = AC$, then $B = C$.

(d) The transpose of the sum of two matrices equals the sum of their transposes.

2.3 The Inverse of a Matrix

- The Inverse of a Matrix
  - Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices.
  - This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication.

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is called the *multiplicative inverse* of a because $a^{-1}a$ yields 1 (the identity element for multiplication). The definition of a multiplicative inverse of a matrix is similar.

2.3 The Inverse of a Matrix

- The Inverse of a Matrix

Definition of the Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) **inverse** of A. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

2.3 The Inverse of a Matrix

- The Inverse of a Matrix

EXAMPLE 1 The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

SOLUTION Using the definition of an inverse matrix, you can show that B is the inverse of A by showing that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

REMARK: Recall that it is not always true that $AB = BA$, even if both products are defined. If A and B are both square matrices and $AB = I_n$, however, then it can be shown that $BA = I_n$. Although the proof of this fact is omitted, it implies that in Example 1 you needed only to check that $AB = I_2$.

2.3 The Inverse of a Matrix

- The Inverse of a Matrix

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \vdots I]$. Note that you separate the matrices A and I by a dotted line. This process is called **adjoining** matrix I to matrix A.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \vdots I]$. The result will be the matrix $[I \vdots A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying AA^{-1} and $A^{-1}A$ to see that $AA^{-1} = I = A^{-1}A$.

$$\left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\underbrace{\hphantom{xx}}_{A}\ \underbrace{\hphantom{xx}}_{I} \qquad \underbrace{\hphantom{xx}}_{I}\ \underbrace{\hphantom{xx}}_{A^{-1}}$$

EXAMPLE 3 Finding the Inverse of a Matrix

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

SOLUTION Begin by adjoining the identity matrix to A to form the matrix

$$[A \vdots I] = \left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Now, using elementary row operations, rewrite this matrix in the form $[I \vdots A^{-1}]$, as follows.

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \qquad R_3 + (6)R_1 \rightarrow R_3$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \qquad R_3 + (4)R_2 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad (-1)R_3 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_2 + R_3 \rightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_1 + R_2 \rightarrow R_1$$

The matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Try confirming this by showing that $AA^{-1} = I = A^{-1}A$.

EXAMPLE 4 A Singular Matrix

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

SOLUTION Adjoin the identity matrix to A to form

$$[A \ \vdots \ I] = \left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

and apply Gauss-Jordan elimination as follows.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \quad R_2 + (-3)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right] \quad R_3 + (2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 0 & -7 & 2 & \vdots & -3 & 1 & 0 \\ 0 & 7 & -2 & \vdots & 2 & 0 & 1 \end{bmatrix} \qquad R_3 + (2)R_1 \to R_3$$

Now, notice that adding the second row to the third row produces a row of zeros on the left side of the matrix.

$$\begin{bmatrix} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 0 & -7 & 2 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 1 & 1 \end{bmatrix} \qquad R_3 + R_2 \to R_3$$

Because the "*A* portion" of the matrix has a row of zeros, you can conclude that it is not possible to rewrite the matrix $[A \vdots I]$ in the form $[I \vdots A^{-1}]$. This means that A has no inverse, or is noninvertible (or singular).

2.3 The Inverse of a Matrix

- For 2 x 2 Matrices
  - Using Gauss-Jordan elimination to find the inverse of a matrix works well for matrices of size 3 x 3 or greater.
  - For 2 x 2 matrices, however, you can use a formula to find the inverse instead of using Gauss-Jordan elimination. This simple formula is explained as follows.

If A is a 2×2 matrix represented by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is represented by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Try verifying this inverse by finding the product AA^{-1}.

2.3 The Inverse of a Matrix

- For 2 x 2 Matrices

EXAMPLE 5 Finding the Inverse of a 2 x 2 Matrix

If possible, find the inverse of each matrix.

(a) $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

SOLUTION (a) For the matrix A, apply the formula for the inverse of a 2×2 matrix to obtain $ad - bc = (3)(2) - (-1)(-2) = 4$. Because this quantity is not zero, the inverse is formed by interchanging the entries on the main diagonal and changing the signs of the other two entries, as follows.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

(b) For the matrix B, you have $ad - bc = (3)(2) - (-1)(-6) = 0$, which means that B is noninvertible.

REMARK: The denominator $ad - bc$ is called the **determinant** of A. You will study determinants in detail in Chapter 3.

2.3 The Inverse of a Matrix

- Properties of Inverses

THEOREM 2.8
Properties of Inverse Matrices

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdot \cdot \cdot A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \dfrac{1}{c}A^{-1}, c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$

EXAMPLE 6 The Inverse of the Square of a Matrix

Compute A^{-2} in two different ways and show that the results are equal.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION One way to find A^{-2} is to find $(A^2)^{-1}$ by squaring the matrix A to obtain

$$A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

and using the formula for the inverse of a 2×2 matrix to obtain

$$(A^2)^{-1} = \frac{1}{4}\begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Another way to find A^{-2} is to find $(A^{-1})^2$ by finding A^{-1}

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

and then squaring this matrix to obtain

$$(A^{-1})^2 = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Note that each method produces the same result.

2.3 The Inverse of a Matrix

- Properties of Inverses

THEOREM 2.9 The Inverse of a Product

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

2.3 The Inverse of a Matrix

- Properties of Inverses

Discovery

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$.

Calculate $(AB)^{-1}$, $A^{-1}B^{-1}$, *and* $B^{-1}A^{-1}$. *Make a conjecture about the inverse of a product of two nonsingular matrices. Select two other nonsingular matrices and see whether your conjecture holds.*

2.3 The Inverse of a Matrix

- Properties of Inverses

THEOREM 2.10
Cancellation Properties

If C is an invertible matrix, then the following properties hold.

1. If $AC = BC$, then $A = B$. **Right cancellation property**
2. If $CA = CB$, then $A = B$. **Left cancellation property**

2.3 The Inverse of a Matrix

- Systems of Equations

THEOREM 2.11
Systems of Equations with Unique Solutions

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

PROOF Because A is nonsingular, the steps shown below are valid.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

EXAMPLE 8 **Solving a System of Equations Using an Inverse Matrix**

Use an inverse matrix to solve each system.

(a) $\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$

(b) $\begin{aligned} 2x + 3y + z &= 4 \\ 3x + 3y + z &= 8 \\ 2x + 4y + z &= 5 \end{aligned}$

(c) $\begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$

SOLUTION First note that the coefficient matrix for each system is

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}.$$

Using Gauss-Jordan elimination, you can find A^{-1} to be

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

Using Gauss-Jordan elimination, you can find A^{-1} to be

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

To solve each system, use matrix multiplication, as follows.

(a) $$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

The solution is $x = 2$, $y = -1$, and $z = -2$.

(b) $$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$$

The solution is $x = 4$, $y = 1$, and $z = -7$.

(c) $$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is trivial: $x = 0$, $y = 0$, and $z = 0$.

2.3 The Inverse of a Matrix

- Exercises

In Exercises 1–4, show that B is the inverse of A.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

3. $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

In Exercises 5–24, find the inverse of the matrix (if it exists).

5. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

7. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

2.3 The Inverse of a Matrix

- Exercises

In Exercises 25–28, use an inverse matrix to solve each system of linear equations.

25. (a) $\begin{aligned} x + 2y &= -1 \\ x - 2y &= 3 \end{aligned}$

(b) $\begin{aligned} x + 2y &= 10 \\ x - 2y &= -6 \end{aligned}$

(c) $\begin{aligned} x + 2y &= -3 \\ x - 2y &= 0 \end{aligned}$

26. (a) $\begin{aligned} 2x - y &= -3 \\ 2x + y &= 7 \end{aligned}$

(b) $\begin{aligned} 2x - y &= -1 \\ 2x + y &= -3 \end{aligned}$

(c) $\begin{aligned} 2x - y &= 6 \\ 2x + y &= 10 \end{aligned}$

27. (a) $\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + x_3 &= -2 \end{aligned}$

(b) $\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= -3 \end{aligned}$

28. (a) $\begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= -1 \end{aligned}$

(b) $\begin{aligned} x_1 + x_2 - 2x_3 &= -1 \\ x_1 - 2x_2 + x_3 &= 2 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}$

2.3 The Inverse of a Matrix

- Exercises

In Exercises 33–36, use the inverse matrices to find (a) $(AB)^{-1}$, (b) $(A^T)^{-1}$, (c) A^{-2}, and (d) $(2A)^{-1}$.

33. $A^{-1} = \begin{bmatrix} 2 & 5 \\ -7 & 6 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 7 & -3 \\ 2 & 0 \end{bmatrix}$

34. $A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$

35. $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} & -2 \\ \frac{1}{4} & 1 & \frac{1}{2} \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 2 & 4 & \frac{5}{2} \\ -\frac{3}{4} & 2 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 2 \end{bmatrix}$

2.4 Elementary Matrices

- Elementary Matrix
  - In Section 1.2, the three elementary row operations for matrices listed below were introduced.
    - 1. Interchange two rows.
    - 2. Multiply a row by a nonzero constant.
    - 3. Add a multiple of a row to another row.
  - In this section, you will see how matrix multiplication can be used to perform these operations.

Definition of an Elementary Matrix

An $n \times n$ matrix is called an **elementary matrix** if it can be obtained from the identity matrix I_n by a single elementary row operation.

EXAMPLE 1 **Elementary Matrices and Nonelementary Matrices**

Which of the following matrices are elementary? For those that are, describe the corresponding elementary row operation.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

SOLUTION

(a) This matrix *is* elementary. It can be obtained by multiplying the second row of I_3 by 3.
(b) This matrix is *not* elementary because it is not square.
(c) This matrix is *not* elementary because it was obtained by multiplying the third row of I_3 by 0 (row multiplication must be by a *nonzero* constant).

(d) This matrix *is* elementary. It can be obtained by interchanging the second and third rows of I_3.
(e) This matrix *is* elementary. It can be obtained by multiplying the first row of I_2 by 2 and adding the result to the second row.
(f) This matrix is *not* elementary because two elementary row operations are required to obtain it from I_3.

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EXAMPLE 2 Elementary Matrices and Elementary Row Operations

(a) In the matrix product below, E is the elementary matrix in which the first two rows of I_3 have been interchanged.

$$\overset{E}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

Note that the first two rows of A have been interchanged by multiplying *on the left* by E.

(b) In the next matrix product, E is the elementary matrix in which the second row of I_3 has been multiplied by $\frac{1}{2}$.

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

Here the size of A is 3×4. A could, however, be any $3 \times n$ matrix and multiplication on the left by E would still result in multiplying the second row of A by $\frac{1}{2}$.

(c) In the product shown below, E is the elementary matrix in which 2 times the first row of I_3 has been added to the second row.

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & 5 \end{bmatrix}$$

Note that in the product EA, 2 times the first row of A has been added to the second row.

2.4 Elementary Matrices

- Elementary Matrix

THEOREM 2.12
Representing Elementary Row Operations

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

REMARK: Be sure to remember that in Theorem 2.12, A is multiplied *on the left* by the elementary matrix E. Right multiplication by elementary matrices, which involves column operations, will not be considered in this text.

2.4 Elementary Matrices

- Elementary Matrix

THEOREM 2.13
Elementary Matrices Are Invertible

If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

Elementary Matrix		*Inverse Matrix*	
$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$
$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$	$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$R_3 + (2)R_1 \rightarrow R_3$
$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$(\frac{1}{2})R_3 \rightarrow R_3$	$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$(2)R_3 \rightarrow R_3$

EXAMPLE 4 Writing a Matrix as the Product of Elementary Matrices

Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}.$$

SOLUTION Begin by finding a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

Matrix	*Elementary Row Operation*	*Elementary Matrix*
$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$	$(-1)R_1 \to R_1$	$E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	$R_2 + (-3)R_1 \to R_2$	$E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	$(\frac{1}{2})R_2 \to R_2$	$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$R_1 + (-2)R_2 \to R_1$	$E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Now, from the matrix product $E_4E_3E_2E_1A = I$, solve for A to obtain $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$. This implies that A is a product of elementary matrices.

$$A = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{E_1^{-1}} \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{E_2^{-1}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{E_3^{-1}} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{E_4^{-1}} = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

2.4 Elementary Matrices

- Elementary Matrix

THEOREM 2.15
Equivalent Conditions

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as the product of elementary matrices.

2.4 Elementary Matrices

- The LU-Factorization

Solving systems of linear equations is the most important application of linear algebra. At the heart of the most efficient and modern algorithms for solving linear systems, $A\mathbf{x} = \mathbf{b}$ is the so-called LU-factorization, in which the square matrix A is expressed as a product, $A = LU$. In this product, the square matrix L is **lower triangular,** which means all the entries above the main diagonal are zero. The square matrix U is **upper triangular,** which means all the entries below the main diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

3×3 lower triangular matrix $\qquad$ 3×3 upper triangular matrix

By writing $A\mathbf{x} = LU\mathbf{x}$ and letting $U\mathbf{x} = \mathbf{y}$, you can solve for $\mathbf{x}$ in two stages. First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$; then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Each system is easy to solve because the coefficient matrices are triangular. In particular, neither system requires any row operations.

2.4 Elementary Matrices

- The LU-Factorization

Definition of LU-Factorization

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an LU-factorization of A.

2.4 Elementary Matrices

- The LU-Factorization

EXAMPLE 5 ***LU*-Factorizations**

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = LU$

is an LU-factorization of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ as the product of the lower triangular matrix $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and the upper triangular matrix $U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$

is an LU-factorization of the matrix A.

2.5 Applications of Matrix Operations

- Cryptography
  - A cryptogram is a message written according to a secret code (the Greek word kryptos means “hidden”).
  - This section describes a method of using matrix multiplication to encode and decode messages.
    - Begin by assigning a number to each letter in the alphabet (with 0 assigned to a blank space), as above. Then the message is converted to numbers and partitioned into uncoded row matrices, each having entries, as demonstrated in Example 4.

$0 = \_$	$7 = G$	$14 = N$	$21 = U$
$1 = A$	$8 = H$	$15 = O$	$22 = V$
$2 = B$	$9 = I$	$16 = P$	$23 = W$
$3 = C$	$10 = J$	$17 = Q$	$24 = X$
$4 = D$	$11 = K$	$18 = R$	$25 = Y$
$5 = E$	$12 = L$	$19 = S$	$26 = Z$
$6 = F$	$13 = M$	$20 = T$	

2.5 Applications of Matrix Operations

- Cryptography

EXAMPLE 4 Forming Uncoded Row Matrices

Write the uncoded row matrices of size 1×3 for the message MEET ME MONDAY.

SOLUTION Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$\begin{matrix} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ M & E & E & T & \_ & M & E & \_ & M & O & N & D & A & Y & \_ \end{matrix}$$

Note that a blank space is used to fill out the last uncoded row matrix.

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EXAMPLE 5 **Encoding a Message**

Use the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

to encode the message MEET ME MONDAY.

SOLUTION The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 4 by the matrix A, as follows.

Uncoded Row Matrix	*Encoding Matrix A*	*Coded Row Matrix*

$$[13 \quad 5 \quad 5]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

$$[20 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [33 \quad -53 \quad -12]$$

$$[5 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [18 \quad -23 \quad -42]$$

$$[15 \quad 14 \quad 4]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [5 \quad -20 \quad 56]$$

$$[1 \quad 25 \quad 0]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [-24 \quad 23 \quad 77]$$

The sequence of coded row matrices is

$$[13 \quad -26 \quad 21][33 \quad -53 \quad -12][18 \quad -23 \quad -42][5 \quad -20 \quad 56][-24 \quad 23 \quad 77].$$

Finally, removing the brackets produces the cryptogram below.

$$13 \quad -26 \quad 21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77$$

For those who do not know the matrix A, decoding the cryptogram found in Example 5 is difficult. But for an authorized receiver who knows the matrix A, decoding is simple. The receiver need only multiply the coded row matrices by A^{-1} to retrieve the uncoded row matrices. In other words, if

$$X = [x_1 \ \ x_2 \ \cdot \ \cdot \ \cdot \ \ x_n]$$

is an uncoded $1 \times n$ matrix, then $Y = XA$ is the corresponding encoded matrix. The receiver of the encoded matrix can decode Y by multiplying on the right by A^{-1} to obtain

$$YA^{-1} = (XA)A^{-1} = X.$$

Chapter 3
Determinants

Chapter 3: Determinants

3.1 The Determinant of a Matrix

3.2 Evaluation of a Determinant Using Elementary Operations

3.3 Properties of Determinants

3.4 Introduction to Eigenvalues

3.5 Applications of Determinants

CHAPTER OBJECTIVES

- Find the determinants of a 2×2 matrix and a triangular matrix.
- Find the minors and cofactors of a matrix and use expansion by cofactors to find the determinant of a matrix.
- Use elementary row or column operations to evaluate the determinant of a matrix.
- Recognize conditions that yield zero determinants.
- Find the determinant of an elementary matrix.
- Use the determinant and properties of the determinant to decide whether a matrix is singular or nonsingular, and recognize equivalent conditions for a nonsingular matrix.
- Verify and find an eigenvalue and an eigenvector of a matrix.
- Find and use the adjoint of a matrix to find its inverse.
- Use Cramer's Rule to solve a system of linear equations.
- Use determinants to find the area of a triangle defined by three distinct points, to find an equation of a line passing through two distinct points, to find the volume of a tetrahedron defined by four distinct points, and to find an equation of a plane passing through three distinct points.

3.1 The Determinant of a Matrix

- Determinant of a Matrix; 2 x 2

Definition of the Determinant of a 2×2 Matrix

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

3.1 The Determinant of a Matrix

- Determinant of a Matrix; 2 x 2

EXAMPLE 1 The Determinant of a Matrix of Order 2

Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ (c) $C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$

SOLUTION (a) $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

(b) $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

(c) $|C| = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$

REMARK: The determinant of a matrix can be positive, zero, or negative.

3.1 The Determinant of a Matrix

- Minors and Cofactors

Definitions of Minors and Cofactors of a Matrix

If A is a square matrix, then the **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown in the diagram below.

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$\begin{aligned} C_{21} &= (-1)^{2+1} M_{21} \\ &= -M_{21} \end{aligned}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$\begin{aligned} C_{22} &= (-1)^{2+2} M_{22} \\ &= M_{22} \end{aligned}$$

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

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EXAMPLE 2 Find the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

SOLUTION To find the minor M_{11}, delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12}, delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = 4 \\ M_{21} = 2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = 5 & M_{32} = -3 & M_{33} = -6. \end{array}$$

Now, to find the cofactors, combine the checkerboard pattern of signs with these minors to obtain

$$\begin{array}{lll} C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\ C_{31} = 5 & C_{32} = 3 & C_{33} = -6. \end{array}$$

3.1 The Determinant of a Matrix

- Determinant of a Matrix

Definition of the Determinant of a Matrix

If A is a square matrix (of order 2 or greater), then the determinant of A is the sum of the entries in the first row of A multiplied by their cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

3.1 The Determinant of a Matrix

- Determinant of a Matrix

EXAMPLE 3 The Determinant of a Matrix of Order 3

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

SOLUTION This matrix is the same as the one in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \qquad C_{12} = 5, \qquad C_{13} = 4.$$

By the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First row expansion} \\ &= 0(-1) + 2(5) + 1(4) = 14. \end{aligned}$$

3.1 The Determinant of a Matrix

- Determinant of a Matrix

Although the determinant is defined as an expansion by the cofactors in the first row, it can be shown that the determinant can be evaluated by expanding by *any* row or column. For instance, you could expand the 3×3 matrix in Example 3 by the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= 3(-2) + (-1)(-4) + 2(8) = 14 \end{aligned}$$

Second row expansion

or by the first column to obtain

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= 0(-1) + 3(-2) + 4(5) = 14. \end{aligned}$$

First column expansion

Try other possibilities to confirm that the determinant of A can be evaluated by expanding by *any* row or column. This is stated in the theorem below, Laplace's Expansion of a Determinant, named after the French mathematician Pierre Simon de Laplace (1749–1827).

3.1 The Determinant of a Matrix

- Determinant of a Matrix

THEOREM 3.1
Expansion by Cofactors

Let A be a square matrix of order n. Then the determinant of A is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

EXAMPLE 4 The Determinant of a Matrix of Order 4

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}.$$

SOLUTION By inspecting this matrix, you can see that three of the entries in the third column are zeros. You can eliminate some of the work in the expansion by using the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because C_{23}, C_{33}, and C_{43} have zero coefficients, you need only find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

Expanding by cofactors in the second row yields

$$C_{13} = (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + 2(1)(-4) + 3(-1)(-7) = 13.$$

You obtain $|A| = 3(13) = 39$.

3.1 The Determinant of a Matrix

- Triangular Matrices

THEOREM 3.2
Determinant of a Triangular Matrix

If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{nn}.$$

Upper Triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

3.1 The Determinant of a Matrix

- Triangular Matrices

EXAMPLE 6 The Determinant of a Triangular Matrix

Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$

SOLUTION (a) The determinant of this lower triangular matrix is given by

$$|A| = (2)(-2)(1)(3) = -12.$$

(b) The determinant of this *diagonal* matrix is given by

$$|B| = (-1)(3)(2)(4)(-2) = 48.$$

3.1 The Determinant of a Matrix

- Exercises

In Exercises 1–12, find the determinant of the matrix.

1. $[1]$

2. $[-3]$

3. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$

8. $\begin{bmatrix} \frac{1}{3} & 5 \\ 4 & -9 \end{bmatrix}$

In Exercises 13–16, find (a) the minors and (b) the cofactors of the matrix.

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

14. $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$

16. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

3.1 The Determinant of a Matrix

- Exercises

In Exercises 19–34, use expansion by cofactors to find the determinant of the matrix.

19. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

22. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

24. $\begin{bmatrix} -0.4 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.2 \end{bmatrix}$

25. $\begin{bmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

26. $\begin{bmatrix} x & y & 1 \\ -2 & -2 & 1 \\ 1 & 5 & 1 \end{bmatrix}$

3.1 The Determinant of a Matrix

- Exercises

True or False? In Exercises 47 and 48, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

47. (a) The determinant of the 2×2 matrix A is $a_{21}a_{12} - a_{11}a_{22}$.

(b) The determinant of a matrix of order 1 is the entry of the matrix.

(c) The ij-cofactor of a square matrix A is the matrix defined by deleting the ith row and the jth column of A.

48. (a) To find the determinant of a triangular matrix, add the entries on the main diagonal.

(b) The determinant of a matrix can be evaluated using expansion by cofactors in any row or column.

(c) When expanding by cofactors, you need not evaluate the cofactors of zero entries.

3.1 The Determinant of a Matrix

- Exercises

In Exercises 49–54, solve for x.

49. $\begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0$

50. $\begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix} = 0$

51. $\begin{vmatrix} x+1 & -2 \\ 1 & x-2 \end{vmatrix} = 0$

52. $\begin{vmatrix} x+3 & 1 \\ -4 & x-1 \end{vmatrix} = 0$

53. $\begin{vmatrix} x-1 & 2 \\ 3 & x-2 \end{vmatrix} = 0$

54. $\begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix} = 0$

In Exercises 55–58, find the values of λ for which the determinant is zero.

55. $\begin{vmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{vmatrix}$

56. $\begin{vmatrix} \lambda-1 & 1 \\ 4 & \lambda-3 \end{vmatrix}$

57. $\begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{vmatrix}$

58. $\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda-2 \end{vmatrix}$

3.2 Evaluation of a Determinant Using Elementary Operations

- Determinant Using Elementary Operations

Which of the two determinants shown below is easier to evaluate?

$$|A| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{vmatrix} \quad \text{or} \quad |B| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$|B| = (1)(2)(-3)(-1) = 6.$$

$$|A| = 1\begin{vmatrix} -6 & 3 & 2 \\ 4 & -9 & -3 \\ -6 & 9 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 & 2 \\ -2 & -9 & -3 \\ 3 & 9 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & -6 & 2 \\ -2 & 4 & -3 \\ 3 & -6 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & -6 & 3 \\ -2 & 4 & -9 \\ 3 & -6 & 9 \end{vmatrix}.$$

Evaluating the determinants of these four 3×3 matrices produces

$$|A| = (1)(-60) + (2)(39) + (3)(-10) - (1)(-18) = 6.$$

EXAMPLE 2 **Evaluating a Determinant Using Elementary Row Operations**

Find the determinant of

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}.$$

SOLUTION Using elementary row operations, rewrite A in triangular form as follows.

$$\begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix}$$

Interchange the first two rows.

$$= -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

Add -2 times the first row to the second row to produce a new second row.

$$= 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

Factor -7 out of the second row.

$$= 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix}$$

Add -1 times the second row to the third row to produce a new third row.

Now, because the final matrix is triangular, you can conclude that the determinant is

$$|A| = 7(1)(1)(-1) = -7.$$

3.2 Evaluation of a Determinant Using Elementary Operations

- Exercises

In Exercises 25–38, use elementary row or column operations to evaluate the determinant.

25. $\begin{vmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{vmatrix}$

26. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$

27. $\begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix}$

28. $\begin{vmatrix} 3 & -1 & -3 \\ -1 & -4 & -2 \\ 3 & -1 & -1 \end{vmatrix}$

31. $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$

32. $\begin{vmatrix} 4 & -8 & 5 \\ 8 & -5 & 3 \\ 8 & 5 & 2 \end{vmatrix}$

33. $\begin{vmatrix} 4 & -7 & 9 & 1 \\ 6 & 2 & 7 & 0 \\ 3 & 6 & -3 & 3 \\ 0 & 7 & 4 & -1 \end{vmatrix}$

34. $\begin{vmatrix} 9 & -4 & 2 & 5 \\ 2 & 7 & 6 & -5 \\ 4 & 1 & -2 & 0 \\ 7 & 3 & 4 & 10 \end{vmatrix}$

3.3 Properties of Determinants

- Theorems

If A and B are square matrices of order n, then

$$\det(AB) = \det(A)\det(B).$$

If A is an $n \times n$ matrix and c is a scalar, then the determinant of cA is given by

$$\det(cA) = c^n \det(A).$$

A square matrix A is invertible (nonsingular) if and only if

$$\det(A) \neq 0.$$

3.3 Properties of Determinants

- Theorems

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

If A is a square matrix, then

$$\det(A) = \det(A^T).$$

3.3 Properties of Determinants

- Exercises

In Exercises 1–6, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$. Then verify that $|A||B| = |AB|$.

1. $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

In Exercises 45–48, find the value(s) of k such that A is singular.

45. $A = \begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix}$

46. $A = \begin{bmatrix} k-1 & 2 \\ 2 & k+2 \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \\ 4 & 2 & k \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{bmatrix}$

3.3 Properties of Determinants

- Exercises

In Exercises 11–14, find (a) $|A|$, (b) $|B|$, and (c) $|A + B|$. Then verify that $|A| + |B| \neq |A + B|$.

11. $A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$

13. $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

In Exercises 15–18, find (a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$.

15. $A = \begin{bmatrix} 6 & -11 \\ 4 & -5 \end{bmatrix}$

16. $A = \begin{bmatrix} -4 & 10 \\ 5 & 6 \end{bmatrix}$

17. $A = \begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & 6 \\ 3 & 2 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & -6 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

3.4 Introduction to Eigenvalues

- Eigenvalue problem

The central question of the **eigenvalue problem** can be stated as follows. If A is an $n \times n$ matrix, do there exist $n \times 1$ nonzero matrices $\mathbf{x}$ such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$? The scalar is usually denoted by λ (the Greek letter lambda) and is called an **eigenvalue** of A, and the nonzero column matrix $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ. The fundamental equation for the eigenvalue problem is

$$A\mathbf{x} = \lambda\mathbf{x}.$$

3.4 Introduction to Eigenvalues

- Eigenvalue problem

EXAMPLE 1 Verifying Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Verify that $\lambda_1 = 5$ is an eigenvalue of A corresponding to $\mathbf{x}_1$ and that $\lambda_2 = -1$ is an eigenvalue of A corresponding to $\mathbf{x}_2$.

SOLUTION To verify that $\lambda_1 = 5$ is an eigenvalue of A corresponding to $\mathbf{x}_1$, multiply the matrices A and $\mathbf{x}_1$, as follows.

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1\mathbf{x}_1$$

Similarly, to verify that $\lambda_2 = -1$ is an eigenvalue of A corresponding to $\mathbf{x}_2$, multiply A and $\mathbf{x}_2$.

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \lambda_2\mathbf{x}_2.$$

3.4 Introduction to Eigenvalues

- Eigenvalue problem

Provided with an $n \times n$ matrix A, how can you find the eigenvalues and corresponding eigenvectors? The key is to write the equation $A\mathbf{x} = \lambda\mathbf{x}$ in the equivalent form

$$(\lambda I - A)\mathbf{x} = \mathbf{0},$$

where I is the $n \times n$ identity matrix. This homogeneous system of equations has nonzero solutions if and only if the coefficient matrix $(\lambda I - A)$ is singular; that is, if and only if the determinant of $(\lambda I - A)$ is zero. The equation $\det(\lambda I - A) = 0$ is called the **characteristic equation** of A, and is a polynomial equation of degree n in the variable λ. Once you have found the eigenvalues of A, you can use Gaussian elimination to find the corresponding eigenvectors, as shown in the next two examples.

EXAMPLE 2 **Finding Eigenvalues and Eigenvectors**

Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

SOLUTION The characteristic equation of A is

$$\begin{aligned} |\lambda I - A| &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\ &= \lambda^2 - 4\lambda + 3 - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) = 0. \end{aligned}$$

This yields two eigenvalues, $\lambda_1 = 5$ and $\lambda_2 = -1$.

To find the corresponding eigenvectors, solve the homogeneous linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$. For $\lambda_1 = 5$, the coefficient matrix is

$$5I - A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 - 1 & -4 \\ -2 & 5 - 3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix},$$

3.4 Introduction to Eigenvalues

- Exercises

In Exercises 1–4, verify that λ_i is an eigenvalue of A and that $\mathbf{x}_i$ is a corresponding eigenvector.

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$; $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

$\lambda_2 = -3$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$; $\lambda_1 = 5$, $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$;

$\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; $\lambda_1 = 2$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$;

3.5 Applications of Determinants

– So far in this chapter, you have examined procedures for evaluating determinants, studied properties of determinants, and learned how determinants are used to find eigenvalues.

– In this section, you will study an explicit formula for the inverse of a nonsingular matrix and then use this formula to derive a theorem known as Cramer's Rule.

– You will then solve several applications of determinants using Cramer's Rule.

3.5 Applications of Determinants

- The Adjoint of a Matrix

a square matrix, then the **matrix of cofactors of** A has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

The transpose of this matrix is called the **adjoint** of A and is denoted by adj(A). That is,

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

3.5 Applications of Determinants

- The inverse of A

THEOREM 3.10
The Inverse of a Matrix Given by Its Adjoint

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

EXAMPLE 2 **Using the Adjoint of a Matrix to Find Its Inverse**

Use the adjoint of

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

to find A^{-1}.

SOLUTION The determinant of this matrix is 3. Using the adjoint of A (found in Example 1), you can find the inverse of A to be

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \tfrac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}.$$

You can check to see that this matrix is the inverse of A by multiplying to obtain

$$AA^{-1} = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.5 Applications of Determinants

- Cramer's Rule

THEOREM 3.11
Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix with a nonzero determinant $|A|$, then the solution of the system is given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}, \qquad \ldots, \qquad x_n = \frac{\det(A_n)}{\det(A)},$$

where the ith column of A_i is the column of constants in the system of equations.

3.5 Applications of Determinants

- Cramer's Rule

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

EXAMPLE 4 Using Cramer's Rule

Use Cramer's Rule to solve the system of linear equations for x.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \phantom{{}+ 2y} + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

SOLUTION The determinant of the coefficient matrix is

$$|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10.$$

Because $|A| \neq 0$, you know the solution is unique, and Cramer's Rule can be applied to solve for x, as follows.

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{(1)(-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10} = \frac{(1)(-1)(-8)}{10} = \frac{4}{5}$$

3.5 Applications of Determinants

- Area, Volume, and Equations of Lines and Planes

Area of a Triangle in the xy-Plane

The area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

$$\text{Area} = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix},$$

where the sign $(\pm)$ is chosen to yield a positive area.

3.5 Applications of Determinants

- Area, Volume, and Equations of Lines and Planes

EXAMPLE 5 **Finding the Area of a Triangle**

Find the area of the triangle whose vertices are $(1, 0)$, $(2, 2)$, and $(4, 3)$.

SOLUTION It is not necessary to know the relative positions of the three vertices. Simply evaluate the determinant

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{3}{2}$$

and conclude that the area of the triangle is $\frac{3}{2}$.

3.5 Applications of Determinants

- Area, Volume, and Equations of Lines and Planes

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

3.5 Applications of Determinants

- Area, Volume, and Equations of Lines and Planes

EXAMPLE 6 Finding an Equation of the Line Passing Through Two Points

Find an equation of the line passing through the points (2, 4) and (−1, 3).

SOLUTION Applying the determinant formula for the equation of a line passing through two points produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, expand by cofactors along the top row to obtain

$$x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 0$$

$$x - 3y + 10 = 0.$$

An equation of the line is $x - 3y = -10$.

3.5 Applications of Determinants

- Exercises

In Exercises 1–8, find the adjoint of the matrix A. Then use the adjoint to find the inverse of A, if possible.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

5. $A = \begin{bmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & -2 \end{bmatrix}$

3.5 Applications of Determinants

- Exercises

In Exercises 33–42, use a graphing utility or a computer software program with matrix capabilities and Cramer's Rule to solve for x_1, if possible.

37.
$$\begin{aligned} 4x_1 - x_2 + x_3 &= -5 \\ 2x_1 + 2x_2 + 3x_3 &= 10 \\ 5x_1 - 2x_2 + 6x_3 &= 1 \end{aligned}$$

38.
$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 2 \\ 2x_1 + 2x_2 - 3x_3 &= 3 \\ x_1 - 7x_2 + 8x_3 &= -4 \end{aligned}$$

39.
$$\begin{aligned} 3x_1 - 2x_2 + x_3 &= -29 \\ -4x_1 + x_2 - 3x_3 &= 37 \\ x_1 - 5x_2 + x_3 &= -24 \end{aligned}$$

3.5 Applications of Determinants

- Exercises

In Exercises 45–48, find the area of the triangle having the given vertices.

45. $(0, 0), (2, 0), (0, 3)$

46. $(1, 1), (2, 4), (4, 2)$

47. $(-1, 2), (2, 2), (-2, 4)$

48. $(1, 1), (-1, 1), (0, -2)$

In Exercises 49–52, determine whether the points are collinear.

49. $(1, 2), (3, 4), (5, 6)$

50. $(-1, 0), (1, 1), (3, 3)$

51. $(-2, 5), (0, -1), (3, -9)$

52. $(-1, -3), (-4, 7), (2, -13)$

Find the eigenvalues and corresponding eigenvectors of the matrix below.

$$\begin{bmatrix} 1 & 4 & 6 \\ 1 & 2 & 2 \\ -1 & -2 & -4 \end{bmatrix}$$

Chapter 4
Vector Spaces

Chapter 4: Vector Spaces

4.1 Vectors in R^n
4.2 Vector Spaces
4.3 Subspaces of Vector Spaces
4.4 Spanning Sets and Linear Independence
4.5 Basis and Dimension
4.6 Rank of a Matrix and Systems of Linear Equations
4.7 Coordinates and Change of Basis
4.8 Applications of Vector Spaces

CHAPTER OBJECTIVES

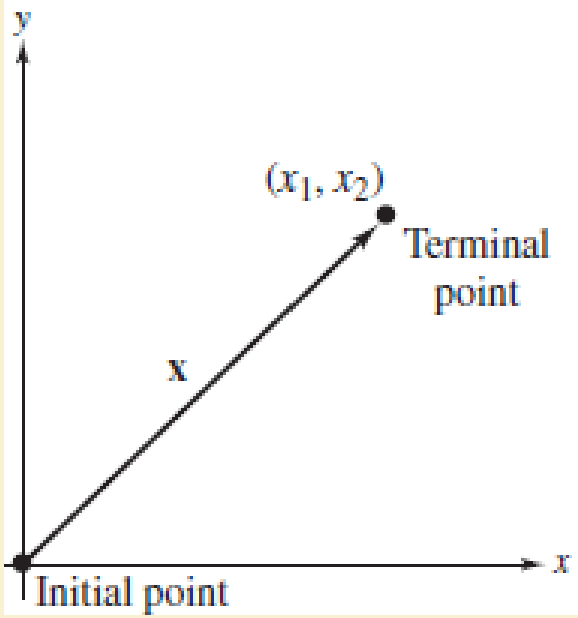
- Perform, recognize, and utilize vector operations on vectors in R^n.
- Determine whether a set of vectors with two operations is a vector space and recognize standard examples of vector spaces, such as: R^n, $M_{m,n}$, P_n, P, $C(-\infty, \infty)$, $C[a, b]$.
- Determine whether a subset W of a vector space V is a subspace.
- Write a linear combination of a finite set of vectors in V.
- Determine whether a set S of vectors in a vector space V is a spanning set of V.
- Determine whether a finite set of vectors in a vector space V is linearly independent.
- Recognize standard bases in the vector spaces R^n, $M_{m,n}$, and P_n.
- Determine if a vector space is finite dimensional or infinite dimensional.
- Find the dimension of a subspace of R^n, $M_{m,n}$, and P_n.
- Find a basis and dimension for the column or row space and a basis for the nullspace (nullity) of a matrix.
- Find a general solution of a consistent system $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{x}_p + \mathbf{x}_h$.
- Find $[\mathbf{x}]_B$ in R^n, $M_{m,n}$, and P_n.
- Find the transition matrix from the basis B to the basis B' in R^n.
- Find $[\mathbf{x}]_{B'}$ for a vector $\mathbf{x}$ in R^n.
- Determine whether a function is a solution of a differential equation and find the general solution of a given differential equation.

4.1 Vectors in R^n

– In physics and engineering, a vector is characterized by two quantities (length and direction) and is represented by a directed line segment.
– In this chapter you will see that these are only two special types of vectors.
– Their geometric representations can help you understand the more general definition of a vector.

4.1 Vectors in R^n

- Vectors in the Plane



$$\mathbf{x} = (x_1, x_2).$$

The coordinates x_1 and x_2 are called the **components** of the vector $\mathbf{x}$. Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

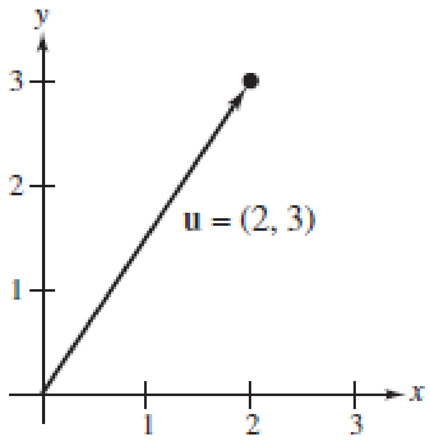
4.1 Vectors in R^n

- Vectors in the Plane

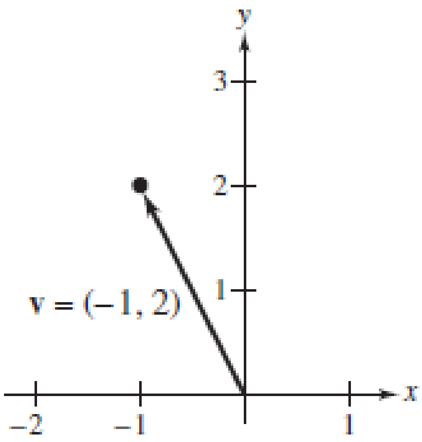
EXAMPLE 1 Vectors in the Plane

Use a directed line segment to represent each vector in the plane.
(a) $\mathbf{u} = (2, 3)$ (b) $\mathbf{v} = (-1, 2)$

SOLUTION To represent each vector, draw a directed line segment from the origin to the indicated terminal point, as shown in Figure 4.2.



(a)



(b)

Figure 4.2

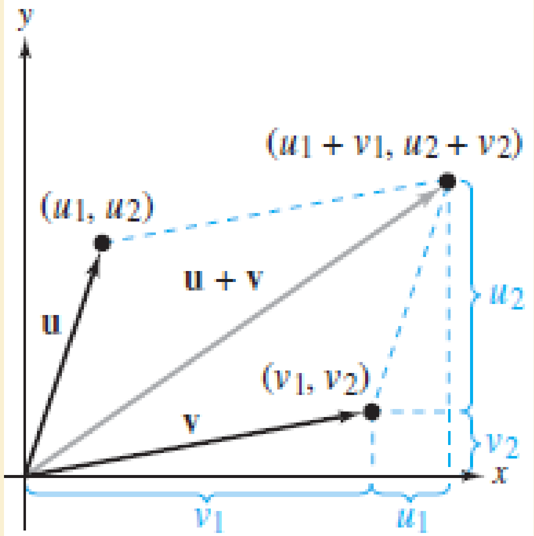
4.1 Vectors in R^n

- Vectors in the Plane

The first basic vector operation is **vector addition.** To add two vectors in the plane, add their corresponding components. That is, the sum of **u** and **v** is the vector

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

Geometrically, the sum of two vectors in the plane is represented as the diagonal of a parallelogram having **u** and **v** as its adjacent sides, as shown in Figure 4.3.



Vector Addition

EXAMPLE 2 Adding Two Vectors in the Plane

Find the sum of the vectors.

(a) $\mathbf{u} = (1, 4)$, $\mathbf{v} = (2, -2)$ (b) $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-3, 2)$ (c) $\mathbf{u} = (2, 1)$, $\mathbf{v} = (0, 0)$

SOLUTION

(a) $\mathbf{u} + \mathbf{v} = (1, 4) + (2, -2) = (3, 2)$

(b) $\mathbf{u} + \mathbf{v} = (3, -2) + (-3, 2) = (0, 0)$

(c) $\mathbf{u} + \mathbf{v} = (2, 1) + (0, 0) = (2, 1)$

Figure 4.4 gives the graphical representation of each sum.

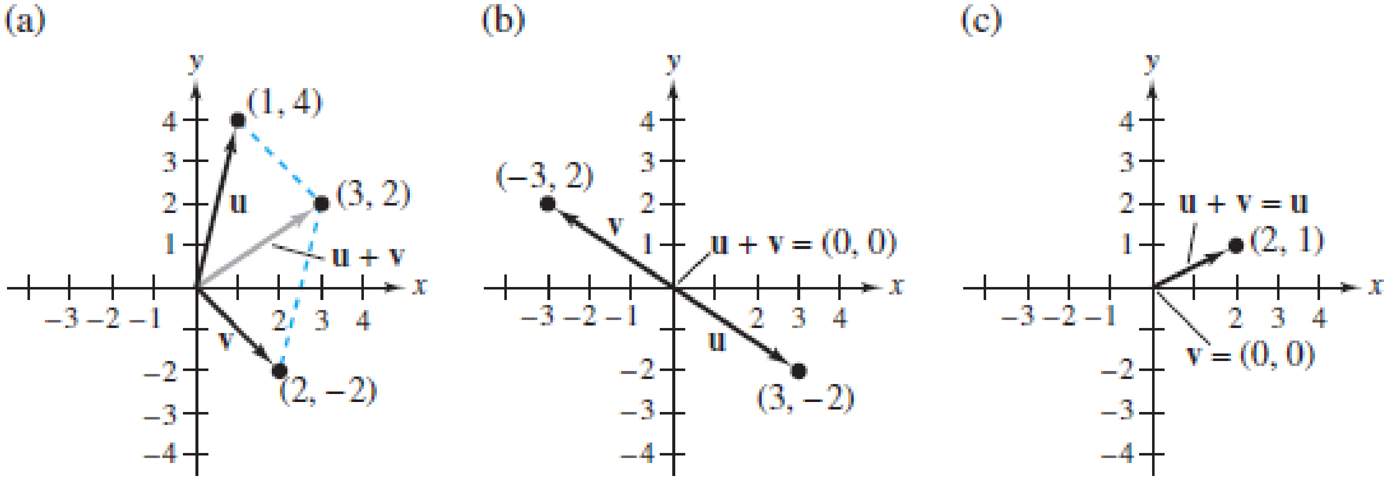


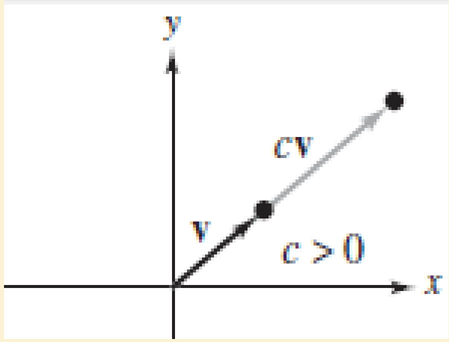
Figure 4.4

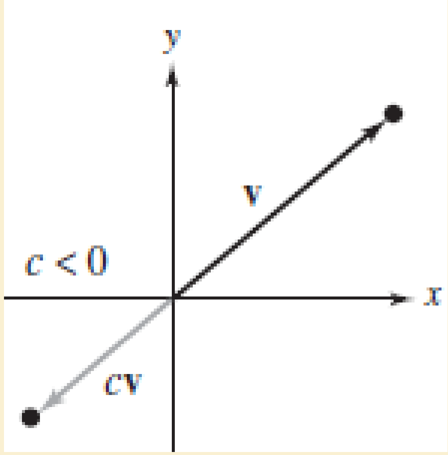
4.1 Vectors in R^n

- Vectors in the Plane

The second basic vector operation is called **scalar multiplication.** To multiply a vector $\mathbf{v}$ by a scalar c, multiply each of the components of $\mathbf{v}$ by c. That is,

$$c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2).$$





The vector $-\mathbf{v}$ is called the **negative** of $\mathbf{v}$. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}),$$

EXAMPLE 3 Operations with Vectors in the Plane

Provided with $\mathbf{v} = (-2, 5)$ and $\mathbf{u} = (3, 4)$, find each vector.

(a) $\frac{1}{2}\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\frac{1}{2}\mathbf{v} + \mathbf{u}$

SOLUTION (a) Because $\mathbf{v} = (-2, 5)$, you have

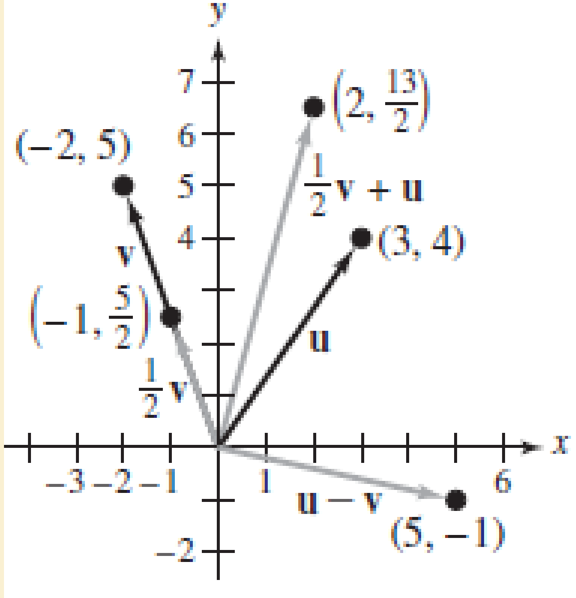
$$\tfrac{1}{2}\mathbf{v} = \left(\tfrac{1}{2}(-2), \tfrac{1}{2}(5)\right) = \left(-1, \tfrac{5}{2}\right).$$

(b) By the definition of vector subtraction, you have

$$\mathbf{u} - \mathbf{v} = (3 - (-2), 4 - 5) = (5, -1).$$

(c) Using the result of part(a), you have

$$\tfrac{1}{2}\mathbf{v} + \mathbf{u} = \left(-1, \tfrac{5}{2}\right) + (3, 4) = \left(-1 + 3, \tfrac{5}{2} + 4\right) = \left(2, \tfrac{13}{2}\right).$$



4.1 Vectors in R^n

- Properties of Vector Addition and Scalar Multiplication in the Plane

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u}+\mathbf{v}$ is a vector in the plane.	Closure under addition
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	Commutative property of addition
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	Associative property of addition
4. $\mathbf{u}+\mathbf{0}=\mathbf{u}$	Additive identity property
5. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in the plane.	Closure under scalar multiplication
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$	Distributive property
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u})=(cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u})=\mathbf{u}$	Multiplicative identity property

4.1 Vectors in R^n

- Vectors in R^n

The discussion of vectors in the plane can now be extended to a discussion of vectors in n-space. A vector in n-space is represented by an **ordered n-tuple.** For instance, an ordered triple has the form (x_1, x_2, x_3), an ordered quadruple has the form (x_1, x_2, x_3, x_4), and a general ordered n-tuple has the form $(x_1, x_2, x_3, \ldots, x_n)$. The set of all n-tuples is called **n-space** and is denoted by R^n.

$$\begin{aligned}
R^1 &= \text{1-space} = \text{set of all real numbers} \\
R^2 &= \text{2-space} = \text{set of all ordered pairs of real numbers} \\
R^3 &= \text{3-space} = \text{set of all ordered triples of real numbers} \\
R^4 &= \text{4-space} = \text{set of all ordered quadruples of real numbers} \\
&\vdots \\
R^n &= n\text{-space} = \text{set of all ordered } n\text{-tuples of real numbers}
\end{aligned}$$

The practice of using an ordered pair to represent either a point or a vector in R^2 continues in R^n. That is, an n-tuple $(x_1, x_2, x_3, \ldots, x_n)$ can be viewed as a **point** in R^n with the x_i's as its coordinates or as a **vector**

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \qquad \text{Vector in } R^n$$

4.1 Vectors in R^n

- Vectors in R^n

Definitions of Vector Addition and Scalar Multiplication in R^n

Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in R^n and let c be a real number. Then the sum of $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n),$$

and the scalar multiple of $\mathbf{u}$ by c is defined as the vector

$$c\mathbf{u} = (cu_1, cu_2, cu_3, \ldots, cu_n).$$

EXAMPLE 4 Vector Operations in R^3

Provided that $\mathbf{u} = (-1, 0, 1)$ and $\mathbf{v} = (2, -1, 5)$ in R^3, find each vector.

(a) $\mathbf{u} + \mathbf{v}$ (b) $2\mathbf{u}$ (c) $\mathbf{v} - 2\mathbf{u}$

SOLUTION (a) To add two vectors, add their corresponding components, as follows.

$$\mathbf{u} + \mathbf{v} = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$$

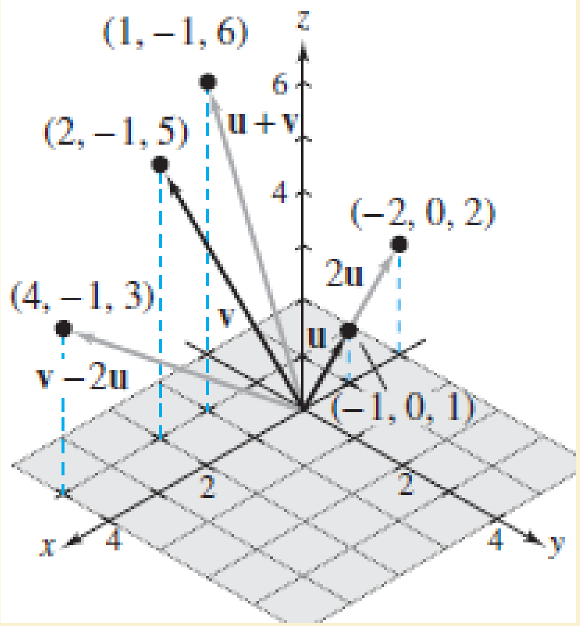
(b) To multiply a vector by a scalar, multiply each component by the scalar, as follows.

$$2\mathbf{u} = 2(-1, 0, 1) = (-2, 0, 2)$$

(c) Using the result of part (b), you have

$$\mathbf{v} - 2\mathbf{u} = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3).$$

Figure 4.7 gives a graphical representation of these vector operations in R^3.



4.1 Vectors in R^n

- Properties of Vector Addition and Scalar Multiplication in R^n

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

EXAMPLE 5 Vector Operations in R^4

Let $\mathbf{u} = (2, -1, 5, 0)$, $\mathbf{v} = (4, 3, 1, -1)$, and $\mathbf{w} = (-6, 2, 0, 3)$ be vectors in R^4. Solve for $\mathbf{x}$.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$ (b) $3(\mathbf{x} + \mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

SOLUTION (a) Using the properties listed in Theorem 4.2, you have

$$\begin{aligned}
\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\
&= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
&= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\
&= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\
&= (18, -11, 9, -8).
\end{aligned}$$

(b) Begin by solving for $\mathbf{x}$ as follows.

$$\begin{aligned}
3(\mathbf{x} + \mathbf{w}) &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} + 3\mathbf{w} &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} - \mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
\mathbf{x} &= \tfrac{1}{2}(2\mathbf{u} - \mathbf{v} - 3\mathbf{w})
\end{aligned}$$

Using the result of part (a) produces

$$\begin{aligned}
\mathbf{x} &= \tfrac{1}{2}(18, -11, 9, -8) \\
&= \left(9, -\tfrac{11}{2}, \tfrac{9}{2}, -4\right).
\end{aligned}$$

4.1 Vectors in R^n

- Properties of Vector Addition and Scalar Multiplication in R^n

THEOREM 4.3 Properties of Additive Identity and Additive Inverse

Let $\mathbf{v}$ be a vector in R^n, and let c be a scalar. Then the following properties are true.

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

EXAMPLE 6 **Writing a Vector as a Linear Combination of Other Vectors**

Provided that $\mathbf{x} = (-1, -2, -2)$, $\mathbf{u} = (0, 1, 4)$, $\mathbf{v} = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$ in R^3, find scalars a, b, and c such that

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

SOLUTION By writing

$$\begin{aligned}\overbrace{(-1, -2, -2)}^{\mathbf{x}} &= a\overbrace{(0, 1, 4)}^{\mathbf{u}} + b\overbrace{(-1, 1, 2)}^{\mathbf{v}} + c\overbrace{(3, 1, 2)}^{\mathbf{w}}\\ &= (-b + 3c, a + b + c, 4a + 2b + 2c),\end{aligned}$$

you can equate corresponding components so that they form the system of three linear equations in a, b, and c shown below.

$$\begin{aligned} -b + 3c &= -1 &&\text{Equation from first component}\\ a + b + c &= -2 &&\text{Equation from second component}\\ 4a + 2b + 2c &= -2 &&\text{Equation from third component}\end{aligned}$$

Using the techniques of Chapter 1, solve for a, b, and c to get

$$a = 1, \quad b = -2, \quad \text{and} \quad c = -1.$$

$\mathbf{x}$ can be written as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$$

Try using vector addition and scalar multiplication to check this result.

4.1 Vectors in R^n

- Properties of Vector Addition and Scalar Multiplication in R^n

Discovery

Is the vector (1, 1) a linear combination of the vectors (1, 2) and (−2, −4)? Graph these vectors in the plane and explain your answer geometrically. Similarly, determine whether the vector (1, 1) is a linear combination of the vectors (1, 2) and (2, 1). What is the geometric significance of these two questions? Is every vector in R^2 a linear combination of the vectors (1, 2) and (2, 1)? Give a geometric explanation for your answer.

4.1 Vectors in R^n

- Exercises

In Exercises 11–16, find the vector **v** and illustrate the indicated vector operations geometrically, where $\mathbf{u} = (-2, 3)$ and $\mathbf{w} = (-3, -2)$.

11. $\mathbf{v} = \frac{3}{2}\mathbf{u}$ **12.** $\mathbf{v} = \mathbf{u} + \mathbf{w}$ **13.** $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$

14. $\mathbf{v} = -\mathbf{u} + \mathbf{w}$ **15.** $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$ **16.** $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

17. Given the vector $\mathbf{v} = (2, 1)$, sketch (a) $2\mathbf{v}$, (b) $-3\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

18. Given the vector $\mathbf{v} = (3, -2)$, sketch (a) $4\mathbf{v}$, (b) $-\frac{1}{2}\mathbf{v}$, and (c) $0\mathbf{v}$.

In Exercises 19–24, let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (2, 2, -1)$, and $\mathbf{w} = (4, 0, -4)$.

19. Find $\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. **20.** Find $\mathbf{u} - \mathbf{v} + 2\mathbf{w}$.

21. Find $2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$. **22.** Find $5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$.

23. Find $\mathbf{z}$, where $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$.

24. Find $\mathbf{z}$, where $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$.

4.1 Vectors in R^n

- Exercises

25. Given the vector $\mathbf{v} = (1, 2, 2)$, sketch (a) $2\mathbf{v}$, (b) $-\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

26. Given the vector $\mathbf{v} = (2, 0, 1)$, sketch (a) $-\mathbf{v}$, (b) $2\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

In Exercises 47–50, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, if possible.

47. $\mathbf{u}_1 = (2, 3, 5)$, $\mathbf{u}_2 = (1, 2, 4)$, $\mathbf{u}_3 = (-2, 2, 3)$,
$\mathbf{v} = (10, 1, 4)$

48. $\mathbf{u}_1 = (1, 3, 5)$, $\mathbf{u}_2 = (2, -1, 3)$, $\mathbf{u}_3 = (-3, 2, -4)$,
$\mathbf{v} = (-1, 7, 2)$

49. $\mathbf{u}_1 = (1, 1, 2, 2)$, $\mathbf{u}_2 = (2, 3, 5, 6)$, $\mathbf{u}_3 = (-3, 1, -4, 2)$,
$\mathbf{v} = (0, 5, 3, 0)$

50. $\mathbf{u}_1 = (1, 3, 2, 1)$, $\mathbf{u}_2 = (2, -2, -5, 4)$, $\mathbf{u}_3 = (2, -1, 3, 6)$,
$\mathbf{v} = (2, 5, -4, 0)$

4.1 Vectors in R^n

- Exercises

True or False? In Exercises 55 and 56, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

55. (a) Two vectors in R^n are equal if and only if their corresponding components are equal.

(b) For a nonzero scalar c, the vector $c\mathbf{v}$ is c times as long as $\mathbf{v}$ and has the same direction as $\mathbf{v}$ if $c > 0$ and the opposite direction if $c < 0$.

56. (a) To add two vectors in R^n, add their corresponding components.

(b) The zero vector $\mathbf{0}$ in R^n is defined as the additive inverse of a vector.

4.2 Vector Spaces

- Definition of Vector Space

In Theorem 4.2, ten special properties of vector addition and scalar multiplication in R^n were listed. Suitable definitions of addition and scalar multiplication reveal that many other mathematical quantities (such as matrices, polynomials, and functions) also share these ten properties. *Any* set that satisfies these properties (or **axioms**) is called a **vector space,** and the objects in the set are called **vectors.**

It is important to realize that the next definition of vector space is precisely that—a *definition*. You do not need to prove anything because you are simply listing the axioms required of vector spaces. This type of definition is called an **abstraction** because you are abstracting a collection of properties from a particular setting R^n to form the axioms for a more general setting.

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space.**

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V.	**Closure under addition**
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	**Commutative property**
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	**Associative property**
4. V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	**Additive identity**
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	**Additive inverse**

Scalar Multiplication:

6. $c\mathbf{u}$ is in V.	**Closure under scalar multiplication**
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	**Distributive property**
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	**Distributive property**
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	**Associative property**
10. $1(\mathbf{u}) = \mathbf{u}$	**Scalar identity**

It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and two operations. When you refer to a vector space V, be sure all four entities are clearly stated or understood. Unless stated otherwise, assume that the set of scalars is the set of real numbers.

4.2 Vector Spaces

- Definition of Vector Space

EXAMPLE 1 R^2 with the Standard Operations Is a Vector Space

The set of all ordered pairs of real numbers R^2 with the standard operations is a vector space. To verify this, look back at Theorem 4.1. Vectors in this space have the form

$$\mathbf{v} = (v_1, v_2).$$

EXAMPLE 2 R^n with the Standard Operations Is a Vector Space

The set of all ordered n-tuples of real numbers R^n with the standard operations is a vector space. This is verified by Theorem 4.2. Vectors in this space are of the form

$$\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n).$$

REMARK: From Example 2 you can conclude that R^1, the set of real numbers (with the usual operations of addition and multiplication), is a vector space.

4.2 Vector Spaces

- Definition of Vector Space

EXAMPLE 3 The Vector Space of All 2×3 Matrices

Show that the set of all 2×3 matrices with the operations of matrix addition and scalar multiplication is a vector space.

SOLUTION If A and B are 2×3 matrices and c is a scalar, then $A + B$ and cA are also 2×3 matrices. The set is, therefore, closed under matrix addition and scalar multiplication. Moreover, the other eight vector space axioms follow directly from Theorems 2.1 and 2.2 (see Section 2.2). You can conclude that the set is a vector space. Vectors in this space have the form

$$\mathbf{a} = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

REMARK: In the same way you are able to show that the set of all 2×3 matrices is a vector space, you can show that the set of all $m \times n$ matrices, denoted by $M_{m,n}$, is a vector space.

EXAMPLE 5 The Vector Space of Continuous Functions (Calculus)

Let $C(-\infty, \infty)$ be the set of all real-valued continuous functions defined on the entire real line. This set consists of all polynomial functions and all other continuous functions on the entire real line. For instance, $f(x) = \sin x$ and $g(x) = e^x$ are members of this set.

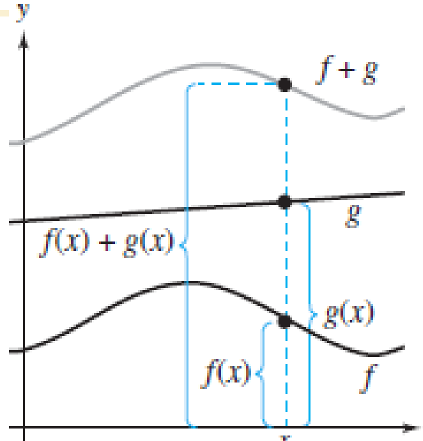
Addition is defined by

$$(f + g)(x) = f(x) + g(x),$$

as shown in Figure 4.8. Scalar multiplication is defined by

$$(cf)(x) = c[f(x)].$$

Show that $C(-\infty, \infty)$ is a vector space.



SOLUTION To verify that the set $C(-\infty, \infty)$ is closed under addition and scalar multiplication, you can use a result from calculus—the sum of two continuous functions is continuous and the product of a scalar and a continuous function is continuous. To verify that the set $C(-\infty, \infty)$ has an additive identity, consider the function f_0 that has a value of zero for all x, meaning that

$$f_0(x) = 0, \quad \text{where } x \text{ is any real number.}$$

This function is continuous on the entire real line (its graph is simply the line $y = 0$), which means that it is in the set $C(-\infty, \infty)$. Moreover, if f is any other function that is continuous on the entire real line, then

$$(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x).$$

This shows that f_0 is the additive identity in $C(-\infty, \infty)$. The verification of the other vector space axioms is left to you.

4.2 Vector Spaces

- Definition of Vector Space

Summary of Important Vector Spaces

$$
\begin{aligned}
R &= \text{set of all real numbers} \\
R^2 &= \text{set of all ordered pairs} \\
R^3 &= \text{set of all ordered triples} \\
R^n &= \text{set of all } n\text{-tuples} \\
C(-\infty, \infty) &= \text{set of all continuous functions defined on the real number line} \\
C[a, b] &= \text{set of all continuous functions defined on a closed interval } [a, b] \\
P &= \text{set of all polynomials} \\
P_n &= \text{set of all polynomials of degree } \le n \\
M_{m,n} &= \text{set of all } m \times n \text{ matrices} \\
M_{n,n} &= \text{set of all } n \times n \text{ square matrices}
\end{aligned}
$$

4.2 Vector Spaces

- Definition of Vector Space

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

1. $0\mathbf{v} = \mathbf{0}$
2. $c\mathbf{0} = \mathbf{0}$
3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$

EXAMPLE 6 The Set of Integers Is Not a Vector Space

The set of all integers (with the standard operations) does not form a vector space because it is not closed under scalar multiplication. For example,

$$\underbrace{\tfrac{1}{2}}_{\text{Scalar}}(\underbrace{1}_{\text{Integer}}) = \underbrace{\tfrac{1}{2}}_{\text{Noninteger}}.$$

4.2 Vector Spaces

- Definition of Vector Space

EXAMPLE 7 The Set of Second-Degree Polynomials Is Not a Vector Space

The set of all second-degree polynomials is not a vector space because it is not closed under addition. To see this, consider the second-degree polynomials

$$p(x) = x^2 \qquad \text{and} \qquad q(x) = -x^2 + x + 1,$$

whose sum is the first-degree polynomial

$$p(x) + q(x) = x + 1.$$

EXAMPLE 8 **A Set That Is Not a Vector Space**

Let $V = R^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the *nonstandard* definition of scalar multiplication listed below.

$$c(x_1, x_2) = (cx_1, 0)$$

Show that V is not a vector space.

SOLUTION In this example, the operation of scalar multiplication is not the standard one. For instance, the product of the scalar 2 and the ordered pair $(3, 4)$ does not equal $(6, 8)$. Instead, the second component of the product is 0,

$$2(3, 4) = (2 \cdot 3, 0) = (6, 0).$$

This example is interesting because it actually satisfies the first nine axioms of the definition of a vector space (try showing this). The tenth axiom is where you get into trouble. In attempting to verify that axiom, the nonstandard definition of scalar multiplication gives you

$$1(1, 1) = (1, 0) \neq (1, 1).$$

The tenth axiom is not verified and the set (together with the two operations) is not a vector space.

4.2 Vector Spaces

- Exercises

True or False? In Exercises 35 and 36, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

35. (a) A vector space consists of four entities: a set of vectors, a set of scalars, and two operations.

(b) The set of all integers with the standard operations is a vector space.

(c) The set of all pairs of real numbers of the form (x, y), where $y \geq 0$, with the standard operations on R^2 is a vector space.

36. (a) To show that a set is not a vector space, it is sufficient to show that just one axiom is not satisfied.

(b) The set of all first-degree polynomials with the standard operations is a vector space.

(c) The set of all pairs of real numbers of the form $(0, y)$, with the standard operations on R^2, is a vector space.

4.3 Subspaces of Vector Spaces

- Definition of Subspace of a Vector Space

In most important applications in linear algebra, vector spaces occur as **subspaces** of larger spaces. For instance, you will see that the solution set of a homogeneous system of linear equations in n variables is a subspace of R^n. (See Theorem 4.16.)

A subset of a vector space is a subspace if it is a vector space (with the *same* operations), as stated in the next definition.

A nonempty subset W of a vector space V is called a **subspace** of V if W is a vector space under the operations of addition and scalar multiplication defined in V.

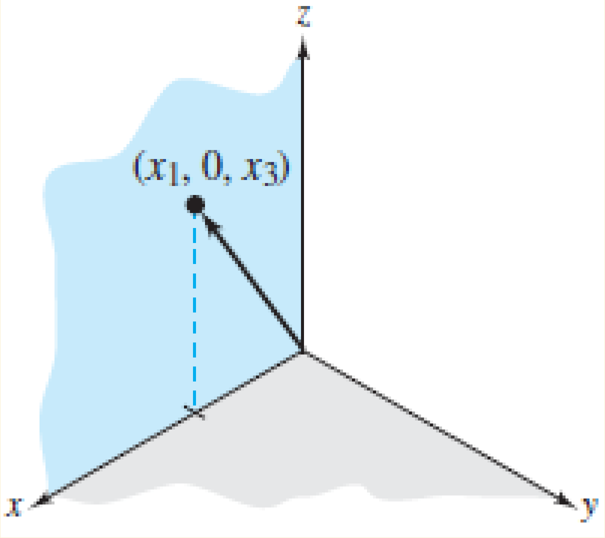
REMARK: Note that if W is a subspace of V, it must be closed under the operations inherited from V.

EXAMPLE 1 **A Subspace of R^3**

Show that the set $W = \{(x_1, 0, x_3): x_1 \text{ and } x_3 \text{ are real numbers}\}$ is a subspace of R^3 with the standard operations.

SOLUTION The set W is nonempty because it contains the zero vector $(0, 0, 0)$.

Graphically, the set W can be interpreted as simply the xz-plane, as shown in Figure 4.9. The set W is closed under addition because the sum of any two vectors in the xz-plane must also lie in the xz-plane. That is, if $(x_1, 0, x_3)$ and $(y_1, 0, y_3)$ are in W, then their sum $(x_1 + y_1, 0, x_3 + y_3)$ is also in W (because the second component is zero). Similarly, to see that W is closed under scalar multiplication, let $(x_1, 0, x_3)$ be in W and let c be a scalar. Then $c(x_1, 0, x_3) = (cx_1, 0, cx_3)$ has zero as its second component and must be in W. The other eight vector space axioms can be verified as well, and these verifications are left to you.



4.3 Subspaces of Vector Spaces

- Test for a Subspace

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

4.3 Subspaces of Vector Spaces

- Test for a Subspace

EXAMPLE 2 The Subspace of $M_{2,2}$

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$, with the standard operations of matrix addition and scalar multiplication.

SOLUTION Recall that a matrix is called *symmetric* if it is equal to its own transpose. Because $M_{2,2}$ is a vector space, you only need to show that W (a subset of $M_{2,2}$) satisfies the conditions of Theorem 4.5. Begin by observing that W is *nonempty*. W is closed under addition because $A_1 = A_1^T$ and $A_2 = A_2^T$, which implies that

$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2.$$

So, if A_1 and A_2 are symmetric matrices of order 2, then so is $A_1 + A_2$. Similarly, W is closed under scalar multiplication because $A = A^T$ implies that $(cA)^T = cA^T = cA$. If A is a symmetric matrix of order 2, then so is cA.

4.3 Subspaces of Vector Spaces

- Test for a Subspace

EXAMPLE 3 The Set of Singular Matrices Is Not a Subspace of $M_{n,n}$

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2,2}$ with the standard operations.

SOLUTION By Theorem 4.5, you can show that a subset W is not a subspace by showing that W is empty, W is not closed under addition, or W is not closed under scalar multiplication. For this particular set, W is nonempty and closed under scalar multiplication, but it is not closed under addition. To see this, let A and B be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then A and B are both singular (noninvertible), but their sum

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is nonsingular (invertible). So W is not closed under addition, and by Theorem 4.5 you can conclude that it is not a subspace of $M_{2,2}$.

EXAMPLE 5 Subspaces of Functions (Calculus)

Let W_5 be the *vector space* of all functions defined on $[0, 1]$, and let W_1, W_2, W_3, and W_4 be defined as follows.

W_1 = set of all polynomial functions defined on the interval $[0, 1]$
W_2 = set of all functions that are differentiable on $[0, 1]$
W_3 = set of all functions that are continuous on $[0, 1]$
W_4 = set of all functions that are integrable on $[0, 1]$

Show that $W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$ and that W_i is a subspace of W_j for $i \leq j$.

SOLUTION From calculus you know that every polynomial function is differentiable on $[0, 1]$. So, $W_1 \subset W_2$. Moreover, $W_2 \subset W_3$ because every differentiable function is continuous, $W_3 \subset W_4$ because every continuous function is integrable, and $W_4 \subset W_5$ because every integrable function is a function. Resulting from the previous remarks, you have $W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$, as shown in Figure 4.10. The verification that W_i is a subspace of W_j for $i \leq j$ is left to you.

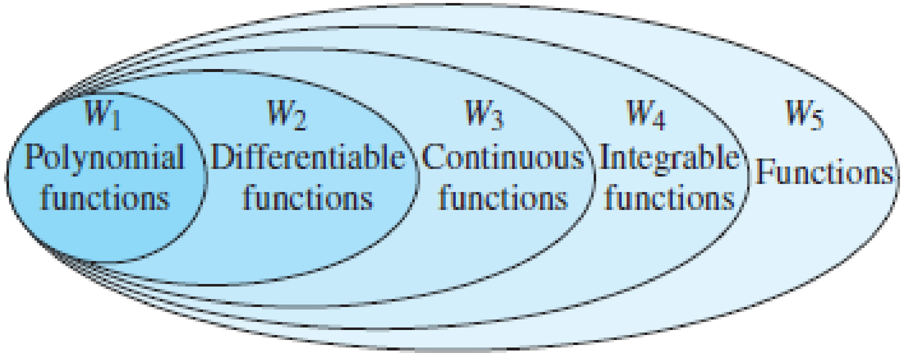


Figure 4.10

4.3 Subspaces of Vector Spaces

- The Intersection of Two Subspaces Is a Subspace

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.

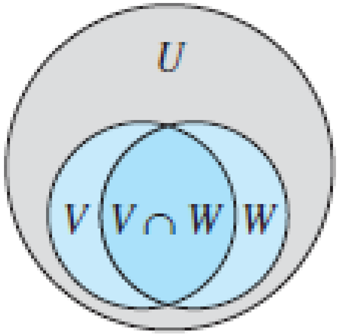


Figure 4.11 The intersection of two subspaces is a subspace.

EXAMPLE 6 **Determining Subspaces of R^2**

Which of these two subsets is a subspace of R^2?

(a) The set of points on the line $x + 2y = 0$

(b) The set of points on the line $x + 2y = 1$

SOLUTION (a) Solving for x, you can see that a point in R^2 is on the line $x + 2y = 0$ if and only if it has the form $(-2t, t)$, where t is any real number. (See Figure 4.12.)

To show that this set is closed under addition, let

$$\mathbf{v}_1 = (-2t_1, t_1) \quad \text{and} \quad \mathbf{v}_2 = (-2t_2, t_2)$$

be any two points on the line. Then you have

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= (-2t_1, t_1) + (-2t_2, t_2) \\ &= (-2(t_1 + t_2), t_1 + t_2) \\ &= (-2t_3, t_3), \end{aligned}$$

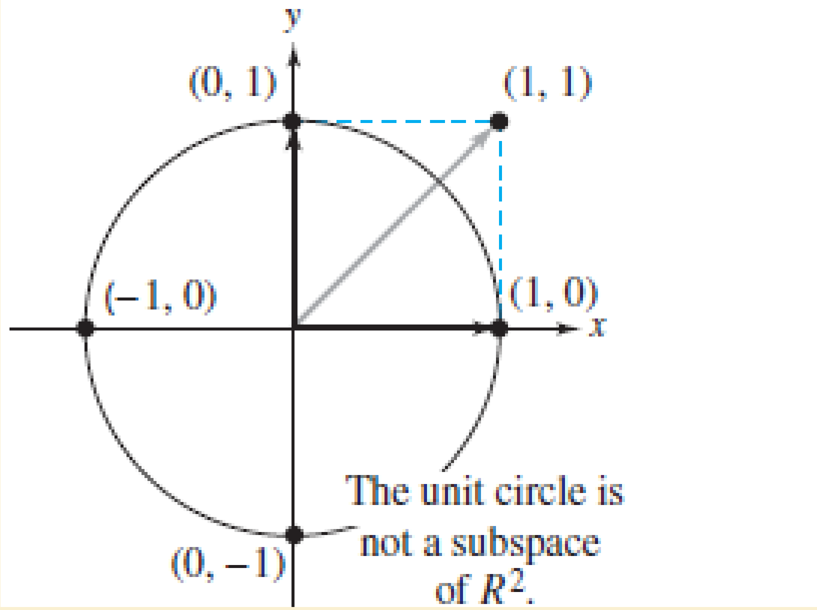
where $t_3 = t_1 + t_2$. $\mathbf{v}_1 + \mathbf{v}_2$ lies on the line, and the set is closed under addition. In a similar way, you can show that the set is closed under scalar multiplication. So, this set is a subspace of R^2.

(b) This subset of R^2 is *not* a subspace of R^2 because every subspace must contain the zero vector, and the zero vector $(0, 0)$ is not on the line. (See Figure 4.12.)

EXAMPLE 7 A Subset of R^2 That Is Not a Subspace

Show that the subset of R^2 consisting of all points on the unit circle $x^2 + y^2 = 1$ is not a subspace.

SOLUTION This subset of R^2 is *not* a subspace because the points (1, 0) and (0, 1) are in the subset, but their sum (1, 1) is not. (See Figure 4.14.) So, this subset is not closed under addition.



4.3 Subspaces of Vector Spaces

- Exercises

True or False? In Exercises 37 and 38, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

37. (a) Every vector space V contains at least one subspace that is the zero subspace.

(b) If V and W are both subspaces of a vector space U, then the intersection of V and W is also a subspace.

(c) If U, V, and W are vector spaces such that W is a subspace of V and U is a subspace of V, then $W = U$.

38. (a) Every vector space V contains two proper subspaces that are the zero subspace and itself.

(b) If W is a subspace of R^2, then W must contain the vector $(0, 0)$.

(c) If W and U are subspaces of a vector space V, then the union of W and U is a subspace of V.

4.4 Spanning Sets and Linear Independence

- Definition of Linear Combination of Vectors

This section begins to develop procedures for representing each vector in a vector space as a **linear combination** of a select number of vectors in the space.

A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars.

EXAMPLE 1 Examples of Linear Combinations

(a) For the set of vectors in R^3,

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ because

$$\begin{aligned}\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 &= 3(0, 1, 2) + (1, 0, -5)\\ &= (1, 3, 1).\end{aligned}$$

(b) For the set of vectors in $M_{2,2}$,

$$S = \left\{\overset{\mathbf{v}_1}{\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}}\right\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ because

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{v}_2 + 2\mathbf{v}_3 - \mathbf{v}_4\\ &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}\\ &= \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}.\end{aligned}$$

EXAMPLE 2 Finding a Linear Combination

Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set S.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-1, 0, 1)}\}$$

SOLUTION You need to find scalars c_1, c_2, and c_3 such that

$$\begin{aligned}(1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)\\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

Equating corresponding components yields the system of linear equations below.

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 1\\ 2c_1 + \ \ c_2 \qquad &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

Using Gauss-Jordan elimination, you can show that this system has an infinite number of solutions, each of the form

$$c_1 = 1 + t, \qquad c_2 = -1 - 2t, \qquad c_3 = t.$$

To obtain one solution, you could let $t = 1$. Then $c_3 = 1$, $c_2 = -3$, and $c_1 = 2$, and you have

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Other choices for t would yield other ways to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

4.4 Spanning Sets and Linear Independence

- Spanning Sets

If every vector in a vector space can be written as a linear combination of vectors in a set S, then S is called a **spanning set** of the vector space.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space V. The set S is called a **spanning set of** V if *every* vector in V can be written as a linear combination of vectors in S. In such cases it is said that S **spans** V.

4.4 Spanning Sets and Linear Independence

- Spanning Sets

EXAMPLE 4 Examples of Spanning Sets

(a) The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

(b) The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial function $p(x) = a + bx + cx^2$ in P_2 can be written as

$$\begin{aligned} p(x) &= a(1) + b(x) + c(x^2) \\ &= a + bx + cx^2. \end{aligned}$$

4.4 Spanning Sets and Linear Independence

- Spanning Sets

EXAMPLE 5 A Spanning Set of R^3

Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3.

SOLUTION Let $\mathbf{u} = (u_1, u_2, u_3)$ be *any* vector in R^3. You need to find scalars c_1, c_2, and c_3 such that

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1)\\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

This vector equation produces the system

$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= u_1\\ 2c_1 + \ c_2 \qquad\quad &= u_2\\ 3c_1 + 2c_2 + \ c_3 &= u_3.\end{aligned}$$

The coefficient matrix for this system has a nonzero determinant, and it follows from the list of equivalent conditions given in Section 3.3 that the system has a unique solution. So, any vector in R^3 can be written as a linear combination of the vectors in S, and you can conclude that the set S spans R^3.

4.4 Spanning Sets and Linear Independence

- Spanning Sets

EXAMPLE 6 A Set That Does Not Span R^3

From Example 3 you know that the set

$$S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$$

does not span R^3 because $\mathbf{w} = (1, -2, 2)$ is in R^3 and cannot be expressed as a linear combination of the vectors in S.

4.4 Spanning Sets and Linear Independence

- Span of a Set

If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the **span of** S is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_k\mathbf{v}_k : c_1, c_2, . . . , c_k \text{ are real numbers}\}.$$

The span of S is denoted by span(S) or span$\{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$. If span($S$) $= V$, it is said that V is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$, or that S **spans** V.

If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$ is a set of vectors in a vector space V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span(S).

4.4 Spanning Sets and Linear Independence

- Linear Dependence and Linear Independence

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$. If there are also nontrivial solutions, then S is called **linearly dependent.**

EXAMPLE 7 Examples of Linearly Dependent Sets

(a) The set $S = \{(1, 2), (2, 4)\}$ in R^2 is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0).$$

(b) The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in R^2 is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

(c) The set $S = \{(0, 0), (1, 2)\}$ in R^2 is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

4.4 Spanning Sets and Linear Independence

- Linear Dependence and Linear Independence

Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or linearly dependent, perform the following steps.

1. From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, write a homogeneous system of linear equations in the variables $c_1, c_2, \ldots,$ and c_k.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

EXAMPLE 9 Testing for Linear Independence

Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

SOLUTION Expanding the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produces

$$c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) = 0 + 0x + 0x^2$$
$$(c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 = 0 + 0x + 0x^2.$$

Equating corresponding coefficients of equal powers of x produces the homogeneous system of linear equations in c_1, c_2, and c_3 shown below.

$$\begin{aligned} c_1 + 2c_2 \phantom{{}+c_3} &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gaussian elimination as follows.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix of this system reduces by Gaussian elimination as follows.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This implies that the system has an infinite number of solutions. So, the system must have nontrivial solutions, and you can conclude that the set S is linearly dependent.

One nontrivial solution is

$$c_1 = 2, \quad c_2 = -1, \quad \text{and} \quad c_3 = 3,$$

which yields the nontrivial linear combination

$$(2)(1 + x - 2x^2) + (-1)(2 + 5x - x^2) + (3)(x + x^2) = 0.$$

EXAMPLE 10 Testing for Linear Independence

Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

SOLUTION From the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

you have

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which produces the system of linear equations in c_1, c_2, and c_3 shown below.

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

Using Gaussian elimination, the augmented matrix of this system reduces as follows.

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system has only the trivial solution and you can conclude that the set S is linearly independent.

4.4 Spanning Sets and Linear Independence

- Linear Dependence and Linear Independence

THEOREM 4.8
A Property of Linearly Dependent Sets

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in S.

THEOREM 4.8
Corollary

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

4.4 Spanning Sets and Linear Independence

- Exercises

In Exercises 1–4, determine whether each vector can be written as a linear combination of the vectors in S.

1. $S = \{(2, -1, 3), (5, 0, 4)\}$

(a) $\mathbf{u} = (1, 1, -1)$ (b) $\mathbf{v} = \left(8, -\frac{1}{4}, \frac{27}{4}\right)$

(c) $\mathbf{w} = (1, -8, 12)$ (d) $\mathbf{z} = (-1, -2, 2)$

2. $S = \{(1, 2, -2), (2, -1, 1)\}$

(a) $\mathbf{u} = (1, -5, -5)$ (b) $\mathbf{v} = (-2, -6, 6)$

(c) $\mathbf{w} = (-1, -22, 22)$ (d) $\mathbf{z} = (-4, -3, 3)$

In Exercises 5–16, determine whether the set S spans R^2. If the set does not span R^2, give a geometric description of the subspace that it does span.

5. $S = \{(2, 1), (-1, 2)\}$ **6.** $S = \{(1, -1), (2, 1)\}$

7. $S = \{(5, 0), (5, -4)\}$ **8.** $S = \{(2, 0), (0, 1)\}$

9. $S = \{(-3, 5)\}$ **10.** $S = \{(1, 1)\}$

11. $S = \{(1, 3), (-2, -6), (4, 12)\}$

4.4 Spanning Sets and Linear Independence

- Exercises

In Exercises 23–34, determine whether the set S is linearly independent or linearly dependent.

23. $S = \{(-2, 2), (3, 5)\}$

24. $S = \{(-2, 4), (1, -2)\}$

25. $S = \{(0, 0), (1, -1)\}$

26. $S = \{(1, 0), (1, 1), (2, -1)\}$

27. $S = \{(1, -4, 1), (6, 3, 2)\}$

28. $S = \{(6, 2, 1), (-1, 3, 2)\}$

29. $S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$

41. Given the matrices

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$$

in $M_{2,2}$, determine which of the matrices listed below are linear combinations of A and B.

(a) $\begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 2 \\ 9 & 11 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 28 \\ 1 & -11 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4.4 Spanning Sets and Linear Independence

- Exercises

True or False? In Exercises 53 and 54, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

53. (a) A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space is called linearly dependent if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has only the trivial solution.

(b) Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

54. (a) A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors.

(b) If a subset S spans a vector space V, then every vector in V can be written as a linear combination of the vectors in S.

4.5 Basis and Dimension

- Definition of Basis

In this section you will continue your study of spanning sets. In particular, you will look at spanning sets (in a vector space) that both are linearly independent *and* span the entire space. Such a set forms a basis for the vector space. (The plural of *basis* is *bases*.)

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a **basis** for V if the following conditions are true.

1. S spans V. 2. S is linearly independent.

REMARK: This definition tells you that a basis has two features. A basis S must have *enough vectors* to span V, but *not so many vectors* that one of them could be written as a linear combination of the other vectors in S.

4.5 Basis and Dimension

- Definition of Basis

EXAMPLE 1 The Standard Basis for R^3

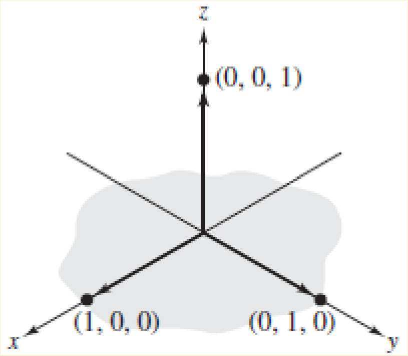
Show that the following set is a basis for R^3.

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

SOLUTION Example 4(a) in Section 4.4 showed that S spans R^3. Furthermore, S is linearly independent because the vector equation

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. (Try verifying this.) So, S is a basis for R^3. (See Figure 4.18.)



4.5 Basis and Dimension

- Definition of Basis

The basis $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is called the **standard basis** for R^3. This result can be generalized to n-space. That is, the vectors

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \ldots, 0) \\ \mathbf{e}_2 &= (0, 1, \ldots, 0) \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \ldots, 1) \end{aligned}$$

form a basis for R^n called the **standard basis** for R^n.

The next two examples describe nonstandard bases for R^2 and R^3.

EXAMPLE 2 The Nonstandard Basis for R^2

Show that the set

$$S = \{\overset{\mathbf{v}_1}{(1, 1)}, \overset{\mathbf{v}_2}{(1, -1)}\}$$

is a basis for R^2.

SOLUTION According to the definition of a basis for a vector space, you must show that S spans R^2 and S is linearly independent.

To verify that S spans R^2, let

$$\mathbf{x} = (x_1, x_2)$$

represent an arbitrary vector in R^2. To show that $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, consider the equation

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{x} \\ c_1(1, 1) + c_2(1, -1) &= (x_1, x_2) \\ (c_1 + c_2, c_1 - c_2) &= (x_1, x_2). \end{aligned}$$

Equating corresponding components yields the system of linear equations shown below.

$$\begin{aligned} c_1 + c_2 &= x_1 \\ c_1 - c_2 &= x_2 \end{aligned}$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has a unique solution. You can now conclude that S spans R^2.

To show that S is linearly independent, consider the linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$
$$c_1(1, 1) + c_2(1, -1) = (0, 0)$$
$$(c_1 + c_2, c_1 - c_2) = (0, 0).$$

Equating corresponding components yields the homogeneous system

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0.$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has only the trivial solution

$$c_1 = c_2 = 0.$$

So, you can conclude that S is linearly independent.

You can conclude that S is a basis for R^2 because it is a linearly independent spanning set for R^2.

4.5 Basis and Dimension

- Definition of Basis

EXAMPLE 3 A Nonstandard Basis for R^3

From Examples 5 and 8 in the preceding section, you know that

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

spans R^3 and is linearly independent. So, S is a basis for R^3.

EXAMPLE 4 A Basis for Polynomials

Show that the vector space P_3 has the basis

$$S = \{1, x, x^2, x^3\}.$$

SOLUTION It is clear that S spans P_3 because the span of S consists of all polynomials of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3, \quad a_0, a_1, a_2, \text{ and } a_3 \text{ are real,}$$

which is precisely the form of all polynomials in P_3.

To verify the linear independence of S, recall that the zero vector $\mathbf{0}$ in P_3 is the polynomial $\mathbf{0}(x) = 0$ for all x. The test for linear independence yields the equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 = \mathbf{0}(x) = 0, \quad \text{for all } x.$$

This third-degree polynomial is said to be *identically equal to zero*. From algebra you know that for a polynomial to be identically equal to zero, all of its coefficients must be zero; that is,

$$a_0 = a_1 = a_2 = a_3 = 0.$$

So, S is linearly independent and is a basis for P_3.

REMARK: The basis $S = \{1, x, x^2, x^3\}$ is called the **standard basis** for P_3. Similarly, the **standard basis** for P_n is

$$S = \{1, x, x^2, \ldots, x^n\}.$$

4.5 Basis and Dimension

- Definition of Basis

EXAMPLE 5 A Basis for $M_{2,2}$

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $M_{2,2}$. This set is called the **standard basis** for $M_{2,2}$. In a similar manner, the standard basis for the vector space $M_{m,n}$ consists of the mn distinct $m \times n$ matrices having a single 1 and all the other entries equal to zero.

4.5 Basis and Dimension

- Definition of Basis

THEOREM 4.9
Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

THEOREM 4.10
Bases and Linear Dependence

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

4.5 Basis and Dimension

- The Dimension of a Vector Space

The discussion of spanning sets, linear independence, and bases leads to an important notion in the study of vector spaces. By Theorem 4.11, you know that if a vector space V has a basis consisting of n vectors, then every other basis for the space also has n vectors. The number n is called the **dimension** of V.

If a vector space V has a basis consisting of n vectors, then the number n is called the **dimension of** V, denoted by $\dim(V) = n$. If V consists of the zero vector alone, the dimension of V is defined as zero.

This definition allows you to observe the characteristics of the dimensions of the familiar vector spaces listed below. In each case, the dimension is determined by simply counting the number of vectors in the standard basis.

1. The dimension of R^n with the standard operations is n.
2. The dimension of P_n with the standard operations is $n + 1$.
3. The dimension of $M_{m,n}$ with the standard operations is mn.

EXAMPLE 9 Finding the Dimension of a Subspace

Determine the dimension of each subspace of R^3.

(a) $W = \{(d, c - d, c): c \text{ and } d \text{ are real numbers}\}$
(b) $W = \{(2b, b, 0): b \text{ is a real number}\}$

SOLUTION The goal in each example is to find a set of linearly independent vectors that spans the subspace.

(a) By writing the representative vector $(d, c - d, c)$ as

$$\begin{aligned}(d, c - d, c) &= (0, c, c) + (d, -d, 0)\\ &= c(0, 1, 1) + d(1, -1, 0),\end{aligned}$$

you can see that W is spanned by the set

$$S = \{(0, 1, 1), (1, -1, 0)\}.$$

Using the techniques described in the preceding section, you can show that this set is linearly independent. So, it is a basis for W, and you can conclude that W is a two-dimensional subspace of R^3.

(b) By writing the representative vector $(2b, b, 0)$ as

$$(2b, b, 0) = b(2, 1, 0),$$

you can see that W is spanned by the set $S = \{(2, 1, 0)\}$. So, W is a one-dimensional subspace of R^3.

EXAMPLE 11 Finding the Dimension of a Subspace

Let W be the subspace of all symmetric matrices in $M_{2,2}$. What is the dimension of W?

SOLUTION Every 2×2 symmetric matrix has the form listed below.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans W. Moreover, S can be shown to be linearly independent, and you can conclude that the dimension of W is 3.

4.5 Basis and Dimension

- The Dimension of a Vector Space

THEOREM 4.12
Basis Tests in an n-Dimensional Space

Let V be a vector space of dimension n.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ spans V, then S is a basis for V.

4.5 Basis and Dimension

- Exercises

Writing In Exercises 7–14, explain why S is not a basis for R^2.

7. $S = \{(1, 2), (1, 0), (0, 1)\}$

8. $S = \{(-1, 2), (1, -2), (2, 4)\}$

9. $S = \{(-4, 5), (0, 0)\}$

10. $S = \{(2, 3), (6, 9)\}$

11. $S = \{(6, -5), (12, -10)\}$

12. $S = \{(4, -3), (8, -6)\}$

13. $S = \{(-3, 2)\}$

14. $S = \{(-1, 2)\}$

Writing In Exercises 25–28, explain why S is not a basis for $M_{2,2}$.

25. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

26. $S = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

27. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix}\right\}$

28. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

4.6 Rank of a Matrix and Systems of Linear Equations

- Some Matrix terminology

Row Vectors of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \begin{matrix} (a_{11}, a_{12}, \ldots, a_{1n}) \\ (a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{matrix}$$

Column Vectors of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

4.6 Rank of a Matrix and Systems of Linear Equations

- Some Matrix terminology

Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace of R^n spanned by the row vectors of A.
2. The **column space** of A is the subspace of R^m spanned by the column vectors of A.

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B, then the row space of A is equal to the row space of B.

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.

4.6 Rank of a Matrix and Systems of Linear Equations

- Finding a Basis

EXAMPLE 3 Finding a Basis for a Subspace

Find a basis for the subspace of R^3 spanned by

$$S = \{\overset{\mathbf{v}_1}{(-1, 2, 5)}, \overset{\mathbf{v}_2}{(3, 0, 3)}, \overset{\mathbf{v}_3}{(5, 1, 8)}\}.$$

SOLUTION Use $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to form the rows of a matrix A. Then write A in row-echelon form.

$$A = \begin{array}{c} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \longrightarrow B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \end{array}$$

So, the nonzero row vectors of B,

$$\mathbf{w}_1 = (1, -2, -5) \quad \text{and} \quad \mathbf{w}_2 = (0, 1, 3),$$

form a basis for the row space of A. That is, they form a basis for the subspace spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

EXAMPLE 4 Finding a Basis for the Column Space of a Matrix

Find a basis for the column space of matrix A from Example 2.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

SOLUTION 1 Take the transpose of A and use elementary row operations to write A^T in row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{array}{cc} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \end{matrix} \end{array}$$

So, $\mathbf{w}_1 = (1, 0, -3, 3, 2)$, $\mathbf{w}_2 = (0, 1, 9, -5, -6)$, and $\mathbf{w}_3 = (0, 0, 1, -1, -1)$ form a basis for the row space of A^T. This is equivalent to saying that the column vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

form a basis for the column space of A.

4.6 Rank of a Matrix and Systems of Linear Equations

- Finding a Basis

If A is an $m \times n$ matrix, then the row space and column space of A have the same dimension.

Definition of the Rank of a Matrix

The dimension of the row (or column) space of a matrix A is called the **rank** of A and is denoted by rank(A).

4.6 Rank of a Matrix and Systems of Linear Equations

- Finding a Basis

EXAMPLE 5 Finding the Rank of a Matrix

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}.$$

SOLUTION Convert to row-echelon form as follows.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Because B has three nonzero rows, the rank of A is 3.

4.6 Rank of a Matrix and Systems of Linear Equations

- The Nullspace of a Matrix