



MAT 203

Calculus III

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Review

Review

- » A Review of differentiation, limits, integration, exponentials, and logarithms

Review

Integral Calculus

Anti-Derivatives

An anti-derivative of a function $f(x)$ is a new function $F(x)$ such that

$$\frac{dF(x)}{dx} = f(x)$$

Indefinite and Definite Integrals

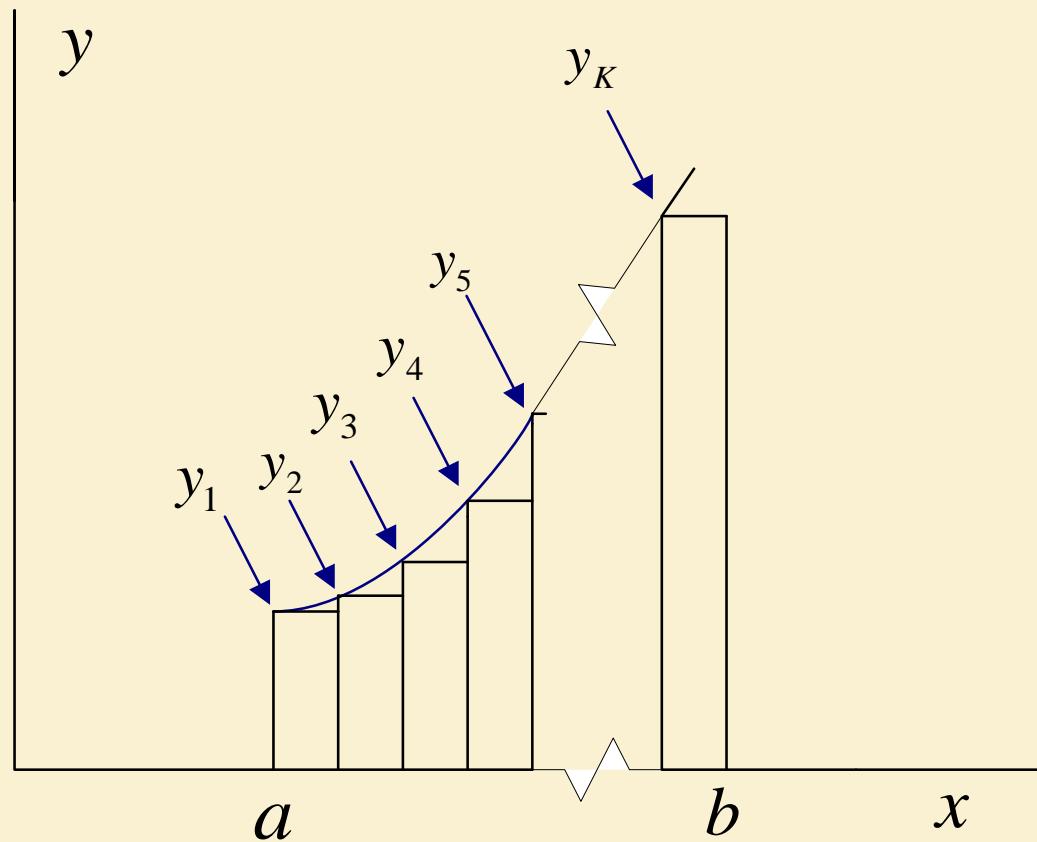
Indefinite

$$\int f(x)dx$$

Definite

$$\int_{x_1}^{x_2} f(x)dx$$

Definite Integral as Area Under the Curve



$$\text{Approximate Area} = \sum_k y_k \Delta x$$

Exact Area as Definite Integral

$$\int_a^b y dx = \lim_{\Delta x \rightarrow dx} \sum_k y_k \Delta x$$

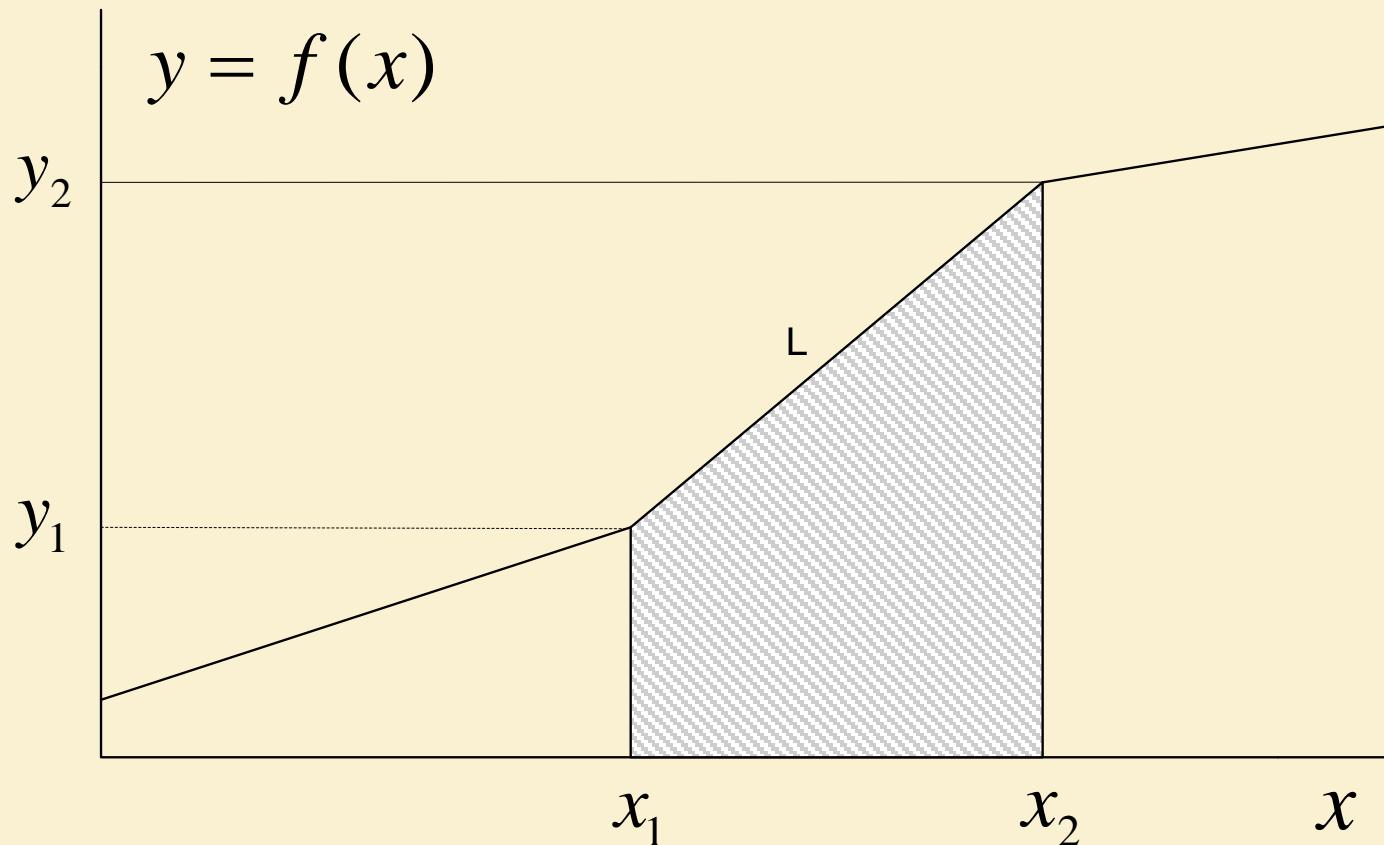
Definite Integral with Variable Upper Limit

$$\int_a^x y dx$$

More “proper” form with “dummy” variable

$$\int_a^x y(u) du$$

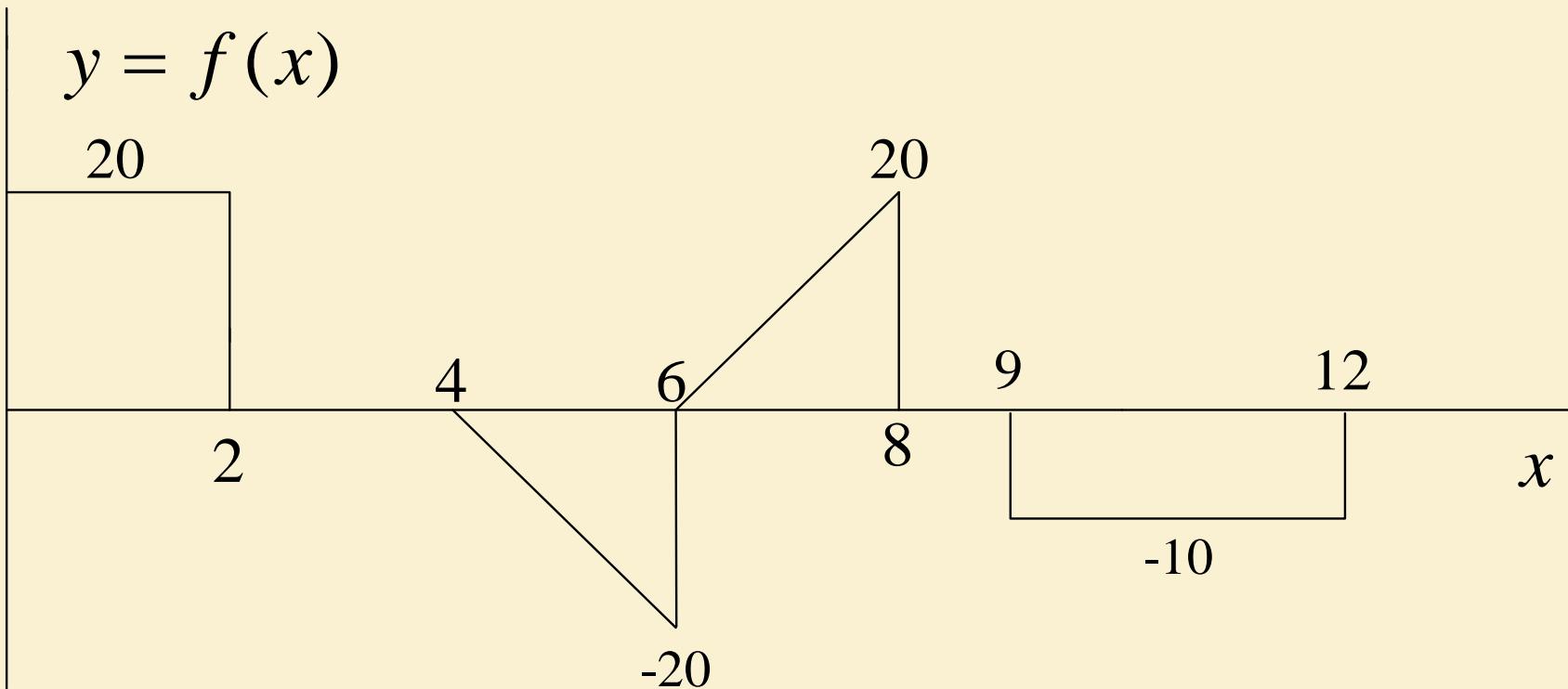
Area Under a Straight-Line Segment



$$\int_{x_1}^{x_2} y dx = \frac{1}{2} (y_2 + y_1)(x_2 - x_1) \quad ??$$

Example; Determine

$$\int_0^{12} y dx$$



Tabulation of Integrals

$$F(x) = \int f(x)dx$$

$$I = \int_a^b f(x)dx$$

$$I = F(x)]_a^b = F(b) - F(a)$$

Common Integrals

$f(x)$	$F(x) = \int f(x)dx$	Integral Number
$af(x)$	$aF(x)$	I-1
$u(x) + v(x)$	$\int u(x)dx + \int v(x)dx$	I-2
a	ax	I-3
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$	I-4
e^{ax}	$\frac{e^{ax}}{a}$	I-5
$\frac{1}{x}$	$\ln x$	I-6
$\sin ax$	$-\frac{1}{a} \cos ax$	I-7
$\cos ax$	$\frac{1}{a} \sin ax$	I-8
$\sin^2 ax$	$\frac{1}{2}x - \frac{1}{4a} \sin 2ax$	I-9

In the three following Examples,
determine the following integral in each
case:

$$z = \int y dx$$

$$y = 12e^{4x}$$

$$y = 12x \sin 2x$$

$$y = 6x^2 + \frac{3}{x}$$

Review

Exponential and Logarithmic Functions

What You Should Learn

- Use the change-of-base formula to rewrite and evaluate logarithmic expressions.
- Use properties of logarithms to evaluate or rewrite logarithmic expressions.
- Use properties of logarithms to expand or condense logarithmic expressions.

Change of Base

- Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e).
- Although common logarithms and natural logarithms are the most frequently used, you may occasionally need to evaluate logarithms with other bases. To do this, you can use the following **change-of-base formula**.

Change-of-Base Formula

Let a , b , and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base b

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Base 10

$$\log_a x = \frac{\log x}{\log a}$$

Base e

$$\log_a x = \frac{\ln x}{\ln a}$$

Properties of Logarithms

- We know that the logarithmic function with base a is the *inverse function* of the exponential function with base a .
- So, it makes sense that the properties of exponents should have corresponding properties involving logarithms.
- For instance, the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

Properties of Logarithms

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true.

Logarithm with Base a

Natural Logarithm

1. Product Property: $\log_a(uv) = \log_a u + \log_a v$ $\ln(uv) = \ln u + \ln v$

2. Quotient Property: $\log_a \frac{u}{v} = \log_a u - \log_a v$ $\ln \frac{u}{v} = \ln u - \ln v$

3. Power Property: $\log_a u^n = n \log_a u$ $\ln u^n = n \ln u$

Example – Using Properties of Logarithms

- **Solution:**

$$\begin{aligned}\bullet \quad a. \ln 6 &= \ln (2 \cdot 3) \\ &= \ln 2 + \ln 3\end{aligned}$$

Rewrite 6 as $2 \cdot 3$.

Product Property

$$\begin{aligned}\bullet \quad b. \ln 2/27 &= \ln 2 - \ln 27 \\&= \ln 2 - \ln 3^3 \\&= \ln 2 - 3 \ln 3\end{aligned}$$

Quotient Property

Rewrite 27 as 3^3 .

Power Property

Rewriting Logarithmic Expressions

- The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra.
- This is true because these properties convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

Example 2 – *Expanding Logarithmic Expressions*

- Expand logarithmic expression.
- $\log_4 5x^3y$

- Solution:

$$\begin{aligned}\log_4 5x^3y &= \log_4 5 + \log_4 x^3 + \log_4 y && \text{Product Property} \\ &= \log_4 5 + 3 \log_4 x + \log_4 y && \text{Power Property}\end{aligned}$$

$$\ln \frac{\sqrt{3x - 5}}{7} = ??$$

Chapter 8

Techniques of

Integration

Chapter 8: Techniques of Integration

- » Integration by parts
- » Trigonometric Integrals
- » Trigonometric Substitutions
- » Integration of Rational Functions by Partial Fractions
- » Integral Tables
- » Numerical Integration
- » Improper Integrals

You will learn how to change unfamiliar integrals into integrals we can recognize, solve, or find in a table. We also extend the idea of the definite integral to improper integrals for which the integrand may be unbounded over the interval of integration.

8.0 Overview

- The following table summarizes the forms of anti-derivatives for many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones.

TABLE 8.1 Basic integration formulas

- | | |
|---|--|
| 1. $\int k \, dx = kx + C$ (any number k) | 12. $\int \tan x \, dx = \ln \sec x + C$ |
| 2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ ($n \neq -1$) | 13. $\int \cot x \, dx = \ln \sin x + C$ |
| 3. $\int \frac{dx}{x} = \ln x + C$ | 14. $\int \sec x \, dx = \ln \sec x + \tan x + C$ |
| 4. $\int e^x \, dx = e^x + C$ | 15. $\int \csc x \, dx = -\ln \csc x + \cot x + C$ |
| 5. $\int a^x \, dx = \frac{a^x}{\ln a} + C$ ($a > 0, a \neq 1$) | 16. $\int \sinh x \, dx = \cosh x + C$ |
| 6. $\int \sin x \, dx = -\cos x + C$ | 17. $\int \cosh x \, dx = \sinh x + C$ |
| 7. $\int \cos x \, dx = \sin x + C$ | 18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$ |
| 8. $\int \sec^2 x \, dx = \tan x + C$ | 19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$ |
| 9. $\int \csc^2 x \, dx = -\cot x + C$ | 20. $\int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left \frac{x}{a} \right + C$ |
| 10. $\int \sec x \tan x \, dx = \sec x + C$ | 21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C$ ($a > 0$) |
| 11. $\int \csc x \cot x \, dx = -\csc x + C$ | 22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left(\frac{x}{a} \right) + C$ ($x > a > 0$) |

8.1 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x) dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x \cos x dx \quad \text{and} \quad \int x^2 e^x dx$$

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx \tag{1}$$

8.1 Integration by Parts

Integration by Parts Formula

$$\int u \, dv = uv - \int v \, du \quad (2)$$

This formula expresses one integral, $\int u \, dv$, in terms of a second integral, $\int v \, du$. With a proper choice of u and v , the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv . The next examples illustrate the technique. To avoid mistakes, we always list our choices for u and dv , then we add to the list our calculated new terms du and v , and finally we apply the formula in Equation (2).

8.1 Integration by Parts

EXAMPLE 1 Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$\begin{aligned} u &= x, & dv &= \cos x \, dx, \\ du &= dx, & v &= \sin x. \end{aligned} \quad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C. \quad \blacksquare$$

8.1 Integration by Parts

$$\int x \cos x \, dx.$$

There are four choices available for u and dv in Example 1:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3 leads to the integral

$$\int (x \cos x - x^2 \sin x) \, dx.$$

8.1 Integration by Parts

The goal of integration by parts is to go from an integral $\int u \, dv$ that we don't see how to evaluate to an integral $\int v \, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx , as you can readily integrate; u is the leftover part. When finding v from dv , any antiderivative will work and we usually pick the simplest one; no arbitrary constant of integration is needed in v because it would simply cancel out of the right-hand side of Equation (2).

8.1 Integration by Parts

EXAMPLE 2 Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = \ln x \quad \text{Simplifies when differentiated}$$

$$dv = dx \quad \text{Easy to integrate}$$

$$du = \frac{1}{x} dx,$$

$$v = x. \quad \text{Simplest antiderivative}$$

Then from Equation (2),

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C. \quad \blacksquare$$

8.1 Integration by Parts

EXAMPLE 3 Evaluate

$$\int x^2 e^x dx.$$

Solution With $u = x^2$, $dv = e^x dx$, $du = 2x dx$, and $v = e^x$, we have

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2 x e^x + 2 e^x + C. \end{aligned}$$

8.1 Integration by Parts

The technique of Example 3 works for any integral $\int x^n e^x dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

$$\int e^x \cos x dx.$$

8.1 Integration by Parts

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x \, dx, \quad v = -\cos x, \quad du = e^x \, dx.$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \end{aligned}$$

8.1 Integration by Parts

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

■

8.1 Integration by Parts

- Evaluating Definite Integrals by Parts

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x) dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) dx \quad (3)$$

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x -axis from $x = 0$ to $x = 4$.

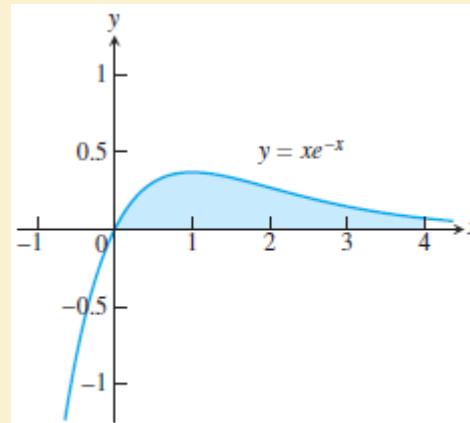


FIGURE 8.1 The region in Example 6.

8.1 Integration by Parts

- Evaluating Definite Integrals by Parts

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x} dx.$$

Let $u = x$, $dv = e^{-x} dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x} dx &= -xe^{-x}]_0^4 - \int_0^4 (-e^{-x}) dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x} dx \\ &= -4e^{-4} - e^{-x}]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$



8.1 Integration by Parts

- Tabular Integration

We have seen that integrals of the form $\int f(x)g(x) dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome; or, you choose substitutions for a repeated integration by parts that just ends up giving back the original integral you were trying to find. In situations like these, there is a way to organize the calculations that prevents these pitfalls and makes the work much easier. It is called **tabular integration** and is illustrated in the following examples.

8.1 Integration by Parts

- Tabular Integration

EXAMPLE 7 Evaluate

$$\int x^2 e^x dx.$$

8.1 Integration by Parts

EXAMPLE 7 Evaluate

$$\int x^2 e^x dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	e^x
$2x$	(-)	e^x
2	(+)	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C.$$

Compare this with the result in Example 3.

8.2 Trigonometric Integrals

- Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines,
- The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

8.2 Trigonometric Integrals

- Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

$$\int \sin^m x \cos^n x \, dx,$$

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

8.2 Trigonometric Integrals

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution This is an example of Case 1.

$$\begin{aligned}
 \int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\
 &= \int (1 - \cos^2 x) \cos^2 x (-d(\cos x)) && \sin x \, dx = -d(\cos x) \\
 &= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
 &= \int (u^4 - u^2) \, du && \text{Multiply terms.} \\
 &= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
 \end{aligned}$$



8.2 Trigonometric Integrals

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\begin{aligned}
 \int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 d(\sin x) && \cos x \, dx = d(\sin x) \\
 &= \int (1 - u^2)^2 \, du && u = \sin x \\
 &= \int (1 - 2u^2 + u^4) \, du && \text{Square } 1 - u^2. \\
 &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C. && \blacksquare
 \end{aligned}$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution This is an example of Case 3.

$$\begin{aligned}
 \int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx \quad m \text{ and } n \text{ both even} \\
 &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\
 &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx \\
 &= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx \right].
 \end{aligned}$$

For the term involving $\cos^2 2x$, we use

$$\begin{aligned}
 \int \cos^2 2x \, dx &= \frac{1}{2} \int (1 + \cos 4x) \, dx \\
 &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right).
 \end{aligned}$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term, we have

$$\begin{aligned}\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx && u = \sin 2x, \\ &= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right). && du = 2 \cos 2x \, dx \\ &&& \text{Again} \\ &&& \text{omitting } C\end{aligned}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C.$$



8.2 Trigonometric Integrals

- Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \quad \int \cos mx \cos nx \, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2} [\cos(m - n)x - \cos(m + n)x], \quad (3)$$

$$\sin mx \cos nx = \frac{1}{2} [\sin(m - n)x + \sin(m + n)x], \quad (4)$$

$$\cos mx \cos nx = \frac{1}{2} [\cos(m - n)x + \cos(m + n)x]. \quad (5)$$

These identities come from the angle sum formulas for the sine and cosine functions. They give functions whose antiderivatives are easily found.

8.2 Trigonometric Integrals

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \end{aligned}$$



8.3 Trigonometric Substitutions

- Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure 8.2.

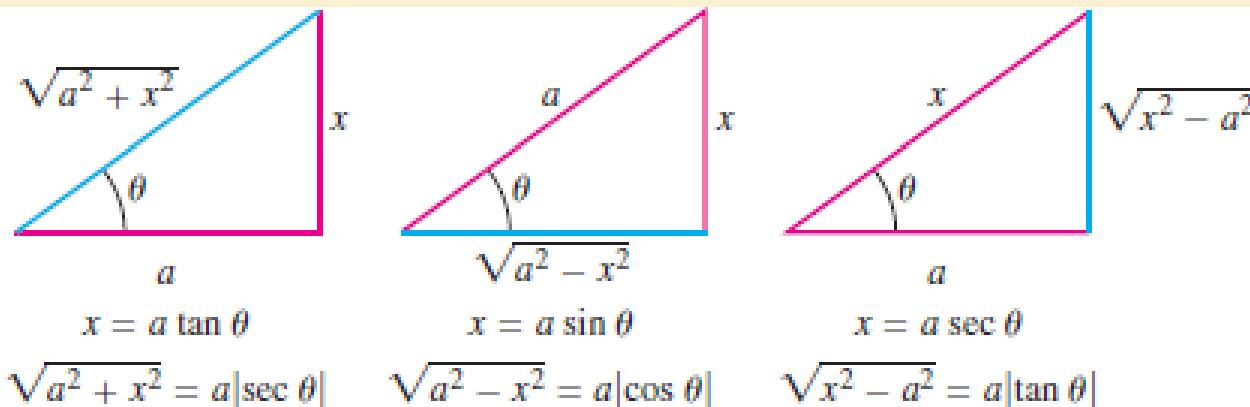


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

8.3 Trigonometric Substitutions

- Trigonometric Substitutions

$$x = a \tan \theta \quad \text{requires} \quad \theta = \tan^{-1} \left(\frac{x}{a} \right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \quad \text{requires} \quad \theta = \sin^{-1} \left(\frac{x}{a} \right) \quad \text{with} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$x = a \sec \theta \quad \text{requires} \quad \theta = \sec^{-1} \left(\frac{x}{a} \right) \quad \text{with} \quad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \geq 1, \\ \frac{\pi}{2} < \theta \leq \pi & \text{if } \frac{x}{a} \leq -1. \end{cases}$$

8.3 Trigonometric Substitutions

Procedure For a Trigonometric Substitution

1. Write down the substitution for x , calculate the differential dx , and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x .

EXAMPLE 2 Evaluate

$$\int \frac{x^2 dx}{\sqrt{9 - x^2}}.$$

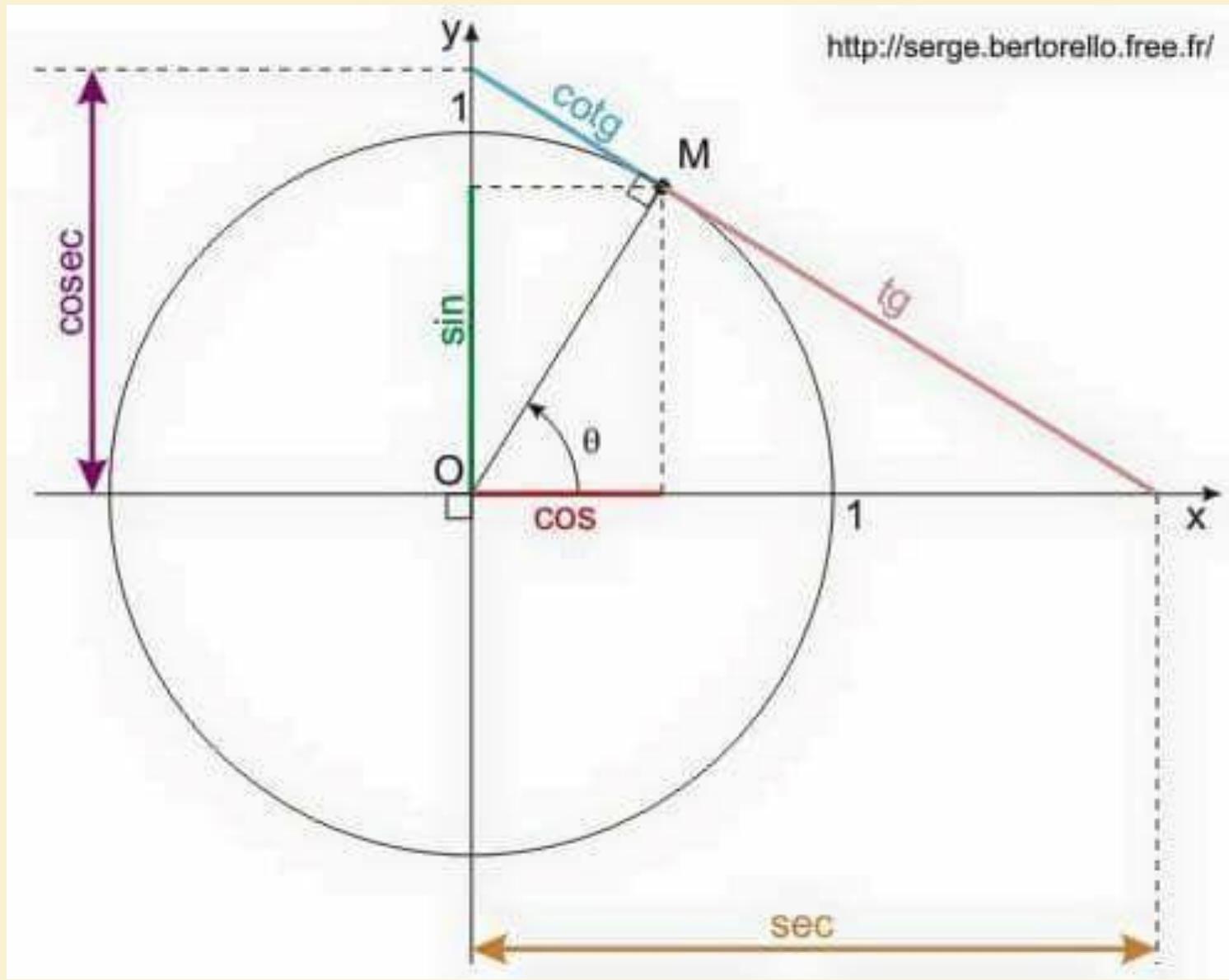
Solution We set

$$x = 3 \sin \theta, \quad dx = 3 \cos \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta.$$

Then

$$\begin{aligned}
 \int \frac{x^2 dx}{\sqrt{9 - x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{|3 \cos \theta|} \\
 &= 9 \int \sin^2 \theta d\theta && \cos \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
 &= 9 \int \frac{1 - \cos 2\theta}{2} d\theta \\
 &= \frac{9}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C \\
 &= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C && \sin 2\theta = 2 \sin \theta \cos \theta \\
 &= \frac{9}{2} \left(\sin^{-1} \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \right) + C && \text{Fig. 8.5} \\
 &= \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2} \sqrt{9 - x^2} + C.
 \end{aligned}$$



8.4 Integration of Rational Functions by Partial Fractions

- This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called partial fractions, which are easily integrated.

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

To find A and B , we first clear Equation (1) of fractions and regroup in powers of x , obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}.$$

$$\begin{aligned}\int \frac{5x - 3}{(x + 1)(x - 3)} dx &= \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx \\ &= 2 \ln|x + 1| + 3 \ln|x - 3| + C.\end{aligned}$$

8.4 Integration of Rational Functions by Partial Fractions

- General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. See Example 3 of this section.
- We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx.$$

8.4 Integration of Rational Functions by Partial Fractions

- General Description of the Method

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is irreducible if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

- Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

continued

8.4 Integration of Rational Functions by Partial Fractions

- General Description of the Method

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

- Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x .
- Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx.$$

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A , B , and C , we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x , obtaining

$$\text{Coefficient of } x^2: \quad A + B + C = 1$$

$$\text{Coefficient of } x^1: \quad 4A + 2B = 4$$

$$\text{Coefficient of } x^0: \quad 3A - 3B - C = 1$$

There are several ways of solving such a system of linear equations for the unknowns A , B , and C , including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned}\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx &= \int \left[\frac{3}{4} \frac{1}{x - 1} + \frac{1}{2} \frac{1}{x + 1} - \frac{1}{4} \frac{1}{x + 3} \right] dx \\ &= \frac{3}{4} \ln |x - 1| + \frac{1}{2} \ln |x + 1| - \frac{1}{4} \ln |x + 3| + K,\end{aligned}$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2

Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2} dx.$$

EXAMPLE 2 Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2} dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

$$\begin{aligned} 6x + 7 &= A(x + 2) + B \\ &= Ax + (2A + B) \end{aligned} \quad \text{Multiply both sides by } (x + 2)^2.$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2} dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) dx \\ &= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} dx \\ &= 6 \ln |x + 2| + 5(x + 2)^{-1} + C. \end{aligned}$$

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{)2x^3 - 4x^2 - x - 3} \\ 2x^3 - 4x^2 - 6x \\ \hline 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx &= \int 2x dx + \int \frac{5x - 3}{x^2 - 2x - 3} dx \\ &= \int 2x dx + \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx \\ &= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C. \end{aligned}$$

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx.$$

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \quad (2)$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Equating coefficients of like terms gives

$$\text{Coefficients of } x^3: \quad 0 = A + C$$

$$\text{Coefficients of } x^2: \quad 0 = -2A + B - C + D$$

$$\text{Coefficients of } x^1: \quad -2 = A - 2B + C$$

$$\text{Coefficients of } x^0: \quad 4 = B - C + D$$

We solve these equations simultaneously to find the values of A , B , C , and D :

$$-4 = -2A, \quad A = 2 \quad \text{Subtract fourth equation from second.}$$

$$C = -A = -2 \quad \text{From the first equation}$$

$$B = (A + C + 2)/2 = 1 \quad \text{From the third equation and } C = -A$$

$$D = 4 - B + C = 1. \quad \text{From the fourth equation}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned} \int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln|x - 1| - \frac{1}{x - 1} + C. \blacksquare \end{aligned}$$

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned} 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\begin{aligned}
 \int \frac{dx}{x(x^2 + 1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right] dx \\
 &= \int \frac{dx}{x} - \int \frac{x \, dx}{x^2 + 1} - \int \frac{x \, dx}{(x^2 + 1)^2} \\
 &= \int \frac{dx}{x} - \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \int \frac{du}{u^2} && u = x^2 + 1, \\
 &\quad du = 2x \, dx \\
 &= \ln |x| - \frac{1}{2} \ln |u| + \frac{1}{2u} + K \\
 &= \ln |x| - \frac{1}{2} \ln (x^2 + 1) + \frac{1}{2(x^2 + 1)} + K \\
 &= \ln \frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K. \quad \blacksquare
 \end{aligned}$$

Other Ways to Determine the Coefficients

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x .

EXAMPLE 8 Find A , B , and C in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution We first clear fractions:

$$x - 1 = A(x + 1)^2 + B(x + 1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x , obtaining

$$1 = 2A(x + 1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x - 1}{(x + 1)^3} = \frac{1}{(x + 1)^2} - \frac{2}{(x + 1)^3}.$$
■

In some problems, assigning small values to x , such as $x = 0, \pm 1, \pm 2$, to get equations in A , B , and C provides a fast alternative to other methods.

EXAMPLE 9 Find A , B , and C in the expression

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}$$

by assigning numerical values to x .

Solution Clear fractions to get

$$x^2 + 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2).$$

Then let $x = 1, 2, 3$ successively to find A , B , and C :

$$x = 1: \quad (1)^2 + 1 = A(-1)(-2) + B(0) + C(0)$$

$$2 = 2A$$

$$A = 1$$

$$x = 2: \quad (2)^2 + 1 = A(0) + B(1)(-1) + C(0)$$

$$5 = -B$$

$$B = -5$$

$$x = 3: \quad (3)^2 + 1 = A(0) + B(0) + C(2)(1)$$

$$10 = 2C$$

$$C = 5.$$

Conclusion:

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{1}{x - 1} - \frac{5}{x - 2} + \frac{5}{x - 3}.$$

Chapter 15

Multiple Integrals

Chapter 15: Multiple Integrals

- » Double and Iterated Integrals over Rectangles
- » Double Integrals over General Regions
- » Area by Double Integration
- » Double Integrals in Polar Form
- » Triple Integrals in Rectangular Coordinates
- » Moments and Centers of Mass
- » Triple Integrals in Cylindrical and Spherical Coordinates
- » Substitutions in Multiple Integrals

In this chapter we consider the integral of a function of two variables $f(x, y)$ over a region in the plane and the integral of a function of three variables $f(x, y, z)$ over a region in space. These integrals are called multiple integrals and are defined as the limit of approximating Riemann sums, much like the single-variable integrals presented in Chapter 5 of the book. We can use multiple integrals to calculate quantities that vary over two or three dimensions, such as the total mass or the angular momentum of an object of varying density and the volumes of solids with general curved boundaries.

15.1 Double and Iterated Integrals over Rectangles

- In this section we define the double integral of a continuous function of two variables $f(x, y)$ over a bounded rectangle R in the plane. The integral is limit of approximating Riemann sums.

The Riemann sums for the integral of a single-variable function $f(x)$ are obtained by partitioning a finite interval into thin subintervals, multiplying the width of each subinterval by the value of f at a point c_k inside that subinterval, and then adding together all the products. A similar method of partitioning, multiplying, and summing is used to construct double integrals.

15.1 Double and Iterated Integrals over Rectangles

- Double Integrals

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region R ,

$$R: \quad a \leq x \leq b, \quad c \leq y \leq d.$$

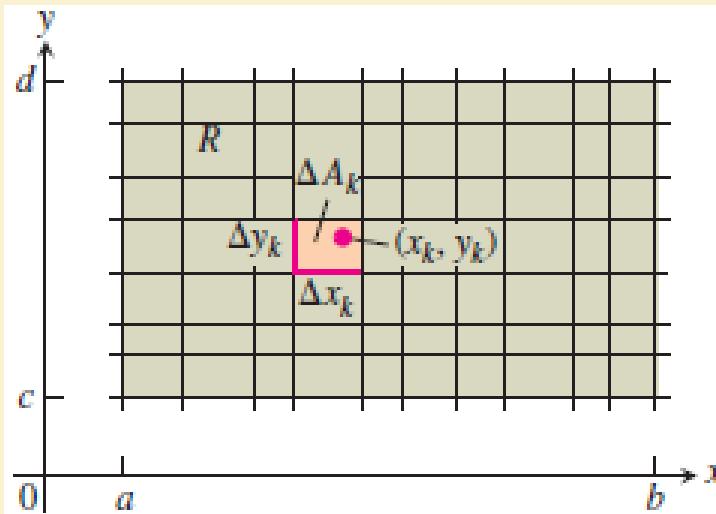


FIGURE 15.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \Delta y_k$.

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k.$$

15.1 Double and Iterated Integrals over Rectangles

- We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of R approach zero.

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k) \Delta A_k,$$

$$\Delta A_k \rightarrow 0, \text{ as } n \rightarrow \infty.$$

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over R , written as

$$\iint_R f(x, y) dA \quad \text{or} \quad \iint_R f(x, y) dx dy.$$

15.1 Double and Iterated Integrals over Rectangles

- Double Integrals as Volumes

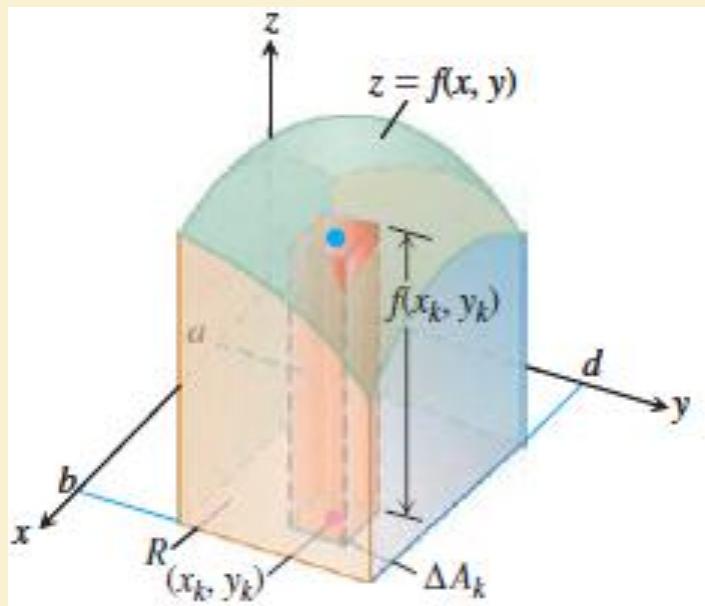


FIGURE 15.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R .

$$\text{Volume} = \lim_{n \rightarrow \infty} S_n = \iint_R f(x, y) \, dA,$$

where $\Delta A_k \rightarrow 0$ as $n \rightarrow \infty$.

When $f(x, y)$ is a positive function over a rectangular region R in the xy -plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy -plane bounded below by R and above by the surface $z = f(x, y)$ (Figure 15.2).

15.1 Double and Iterated Integrals over Rectangles

- Double Integrals as Volumes

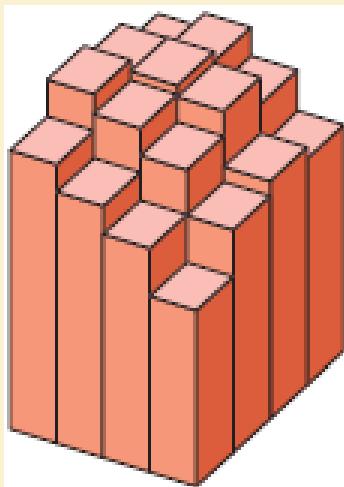
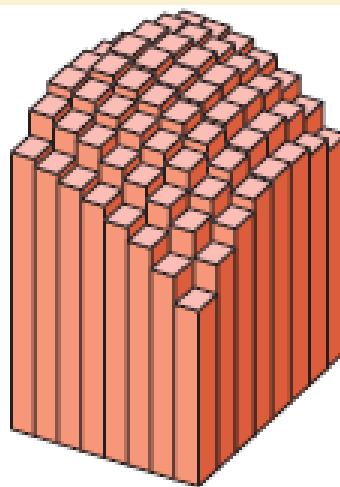
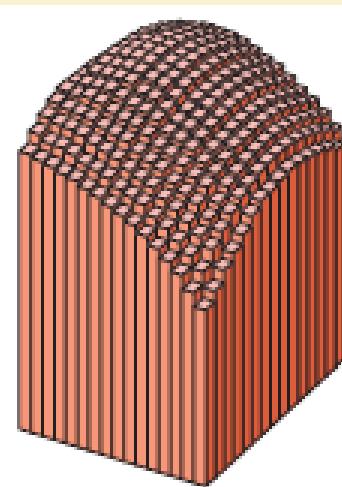
(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

FIGURE 15.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Figure 15.2.

Figure 15.3 shows Riemann sum approximations to the volume becoming more accurate as the number n of boxes increases.

15.1 Double and Iterated Integrals over Rectangles

- Fubini's Theorem for Calculating Double Integrals

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx.$$

Fubini's Theorem says that double integrals over rectangles can be calculated as iterated integrals. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.

Fubini's Theorem also says that we may calculate the double integral by integrating in *either* order, a genuine convenience. When we calculate a volume by slicing, we may use either planes perpendicular to the x -axis or planes perpendicular to the y -axis.

HISTORICAL BIOGRAPHY

Guido Fubini
(1879–1943)

EXAMPLE 1 Calculate $\iint_R f(x, y) dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: 0 \leq x \leq 2, -1 \leq y \leq 1.$$

Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\begin{aligned}\iint_R f(x, y) dA &= \int_{-1}^1 \int_0^2 (100 - 6x^2y) dx dy = \int_{-1}^1 [100x - 2x^3y]_{x=0}^{x=2} dy \\ &= \int_{-1}^1 (200 - 16y) dy = [200y - 8y^2]_{-1}^1 = 400.\end{aligned}$$

Reversing the order of integration gives the same answer:

$$\begin{aligned}\int_0^2 \int_{-1}^1 (100 - 6x^2y) dy dx &= \int_0^2 [100y - 3x^2y^2]_{y=-1}^{y=1} dx \\ &= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)] dx \\ &= \int_0^2 200 dx = 400.\end{aligned}$$

■

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \leq x \leq 1, 0 \leq y \leq 2$.

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$\begin{aligned} V &= \iint_R (10 + x^2 + 3y^2) dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2) dy dx \\ &= \int_0^1 \left[10y + x^2y + y^3 \right]_{y=0}^{y=2} dx \\ &= \int_0^1 (20 + 2x^2 + 8) dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}. \end{aligned}$$
■

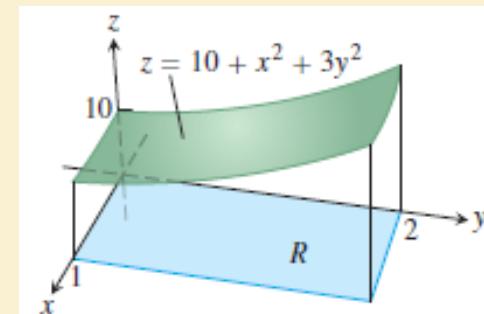


FIGURE 15.7 The double integral $\iint_R f(x, y) dA$ gives the volume under this surface over the rectangular region R (Example 2).

15.2 Double Integrals over General Regions

- Double Integrals over Bounded, Nonrectangular Regions

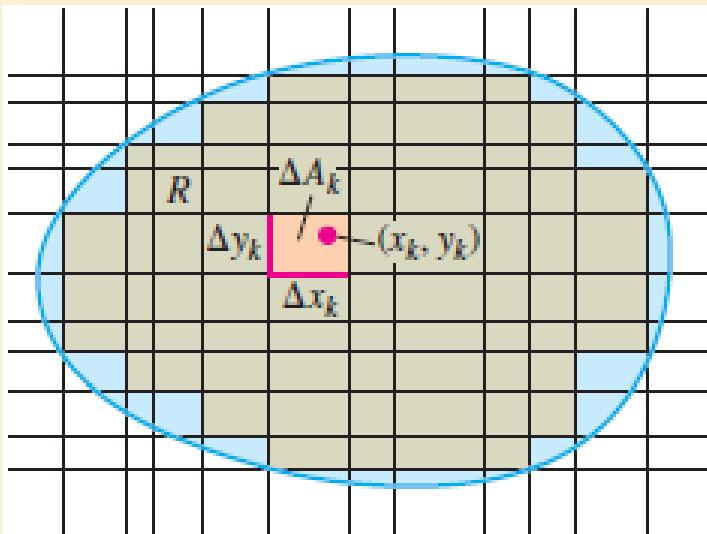


FIGURE 15.8 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k.$$

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k = \iint_R f(x, y) dA.$$

15.2 Double Integrals over General Regions

- Volumes

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy$$

$$V = \int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

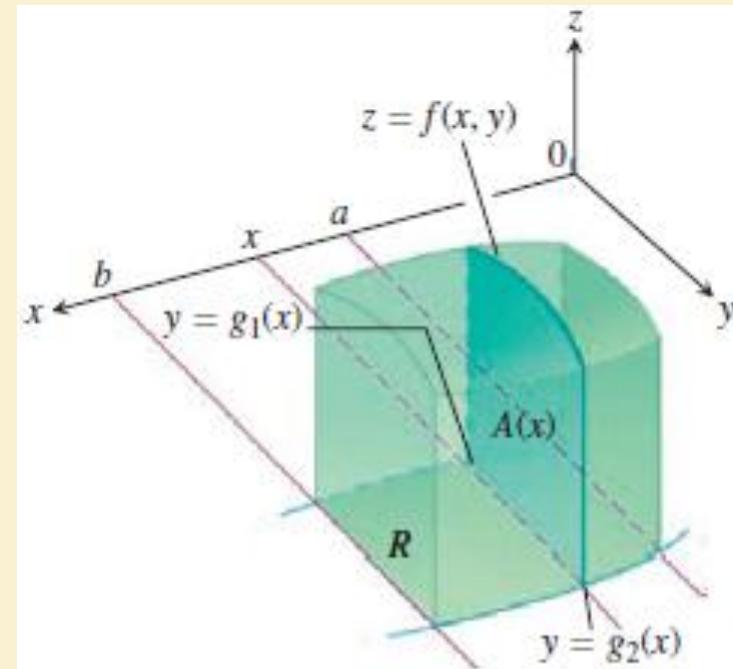


FIGURE 15.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$\int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

15.2 Double Integrals over General Regions

- Volumes

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

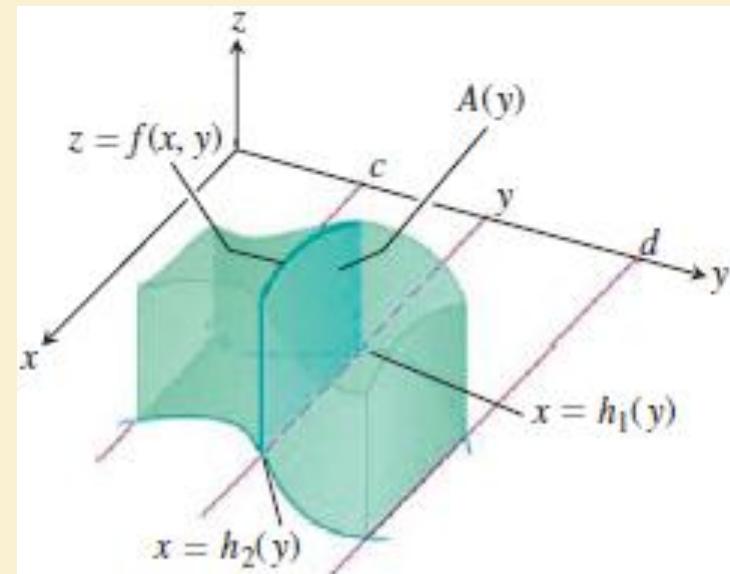


FIGURE 15.11 The volume of the solid shown here is

$$\int_c^d A(y) dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

That the iterated integrals in the last two Equations both give the volume that we defined to be the double integral of f over R is a consequence of the following stronger form of Fubini's Theorem.

For a given solid, Theorem 2 says we can calculate the volume as in Figure 15.10, or in the way shown here. Both calculations have the same result.

15.2 Double Integrals over General Regions

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R .

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

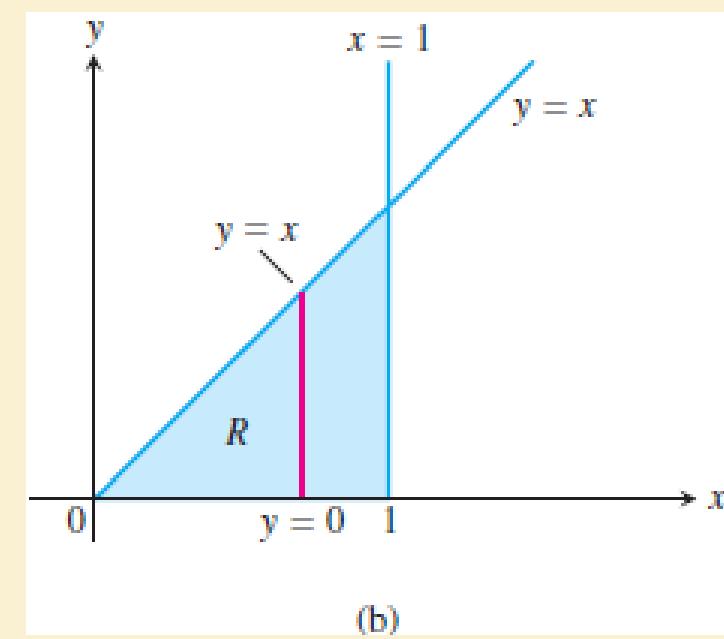
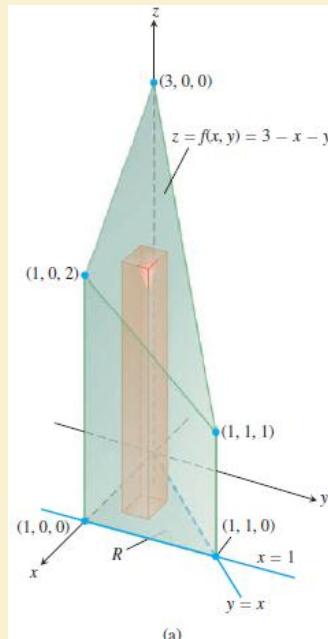
$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

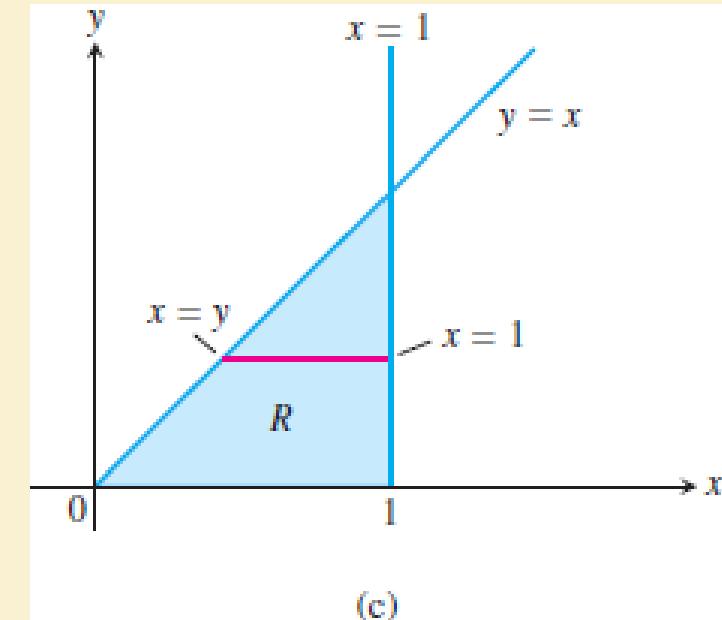
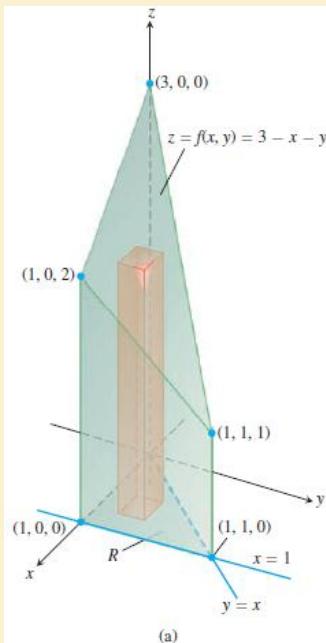
$$\begin{aligned} V &= \int_0^1 \int_0^x (3 - x - y) dy dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1. \end{aligned}$$



When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$\begin{aligned}
 V &= \int_0^1 \int_y^1 (3 - x - y) dx dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy \\
 &= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy \\
 &= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.
 \end{aligned}$$

The two integrals are equal, as they should be. ■



EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x} dA,$$

where R is the triangle in the xy -plane bounded by the x -axis, the line $y = x$, and the line $x = 1$.

Solution The region of integration is shown in Figure 15.13. If we integrate first with respect to y and then with respect to x , we find

$$\begin{aligned}\int_0^1 \left(\int_0^x \frac{\sin y}{y} dy \right) dx &= \int_0^1 \left(y \frac{\sin y}{y} \Big|_{y=0}^{y=x} \right) dx = \int_0^1 \sin x dx \\ &= -\cos(1) + 1 \approx 0.46.\end{aligned}$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy,$$

we run into a problem because $\int ((\sin x)/x) dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we need to use numerical approximations. ■

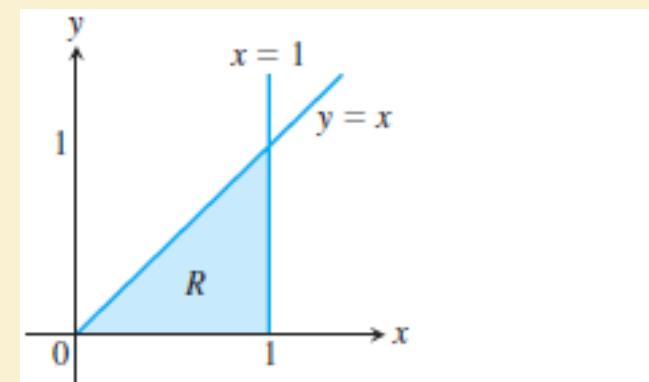


FIGURE 15.13 The region of integration in Example 2.

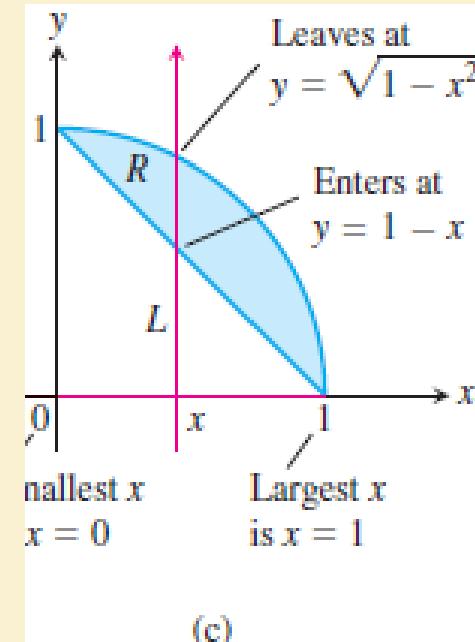
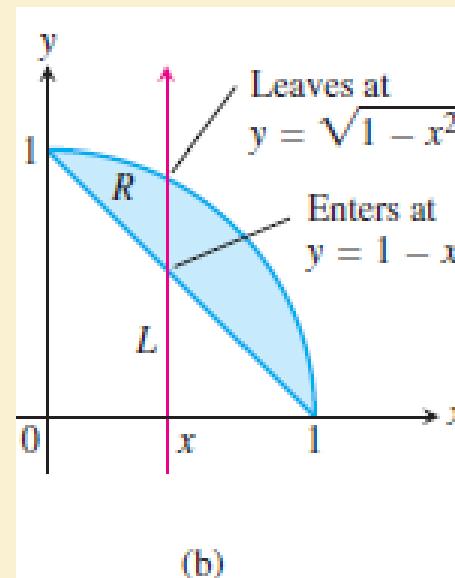
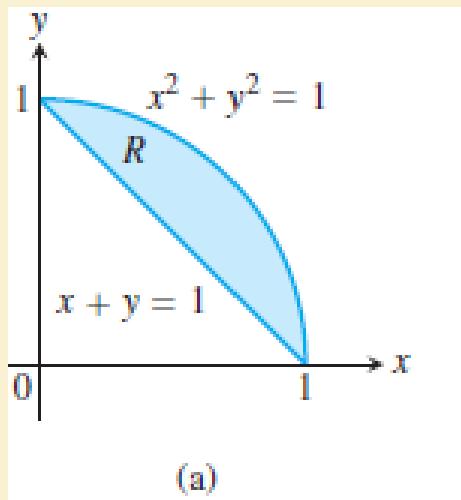
15.2 Double Integrals over General Regions

- Finding Limits of Integration
 - We now give a procedure for finding limits of integration that applies for many regions in the plane.
 - Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Using Vertical Cross-sections When faced with evaluating $\iint_R f(x, y) dA$, integrating first with respect to y and then with respect to x , do the following three steps:

1. *Sketch.* Sketch the region of integration and label the bounding curves (Figure 15.14a).
2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y . Mark the y -values where L enters and leaves. These are the y -limits of integration and are usually functions of x (instead of constants) (Figure 15.14b).
3. *Find the x-limits of integration.* Choose x -limits that include all the vertical lines through R . The integral shown here (see Figure 15.14c) is

$$\iint_R f(x, y) dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y) dy dx.$$



Using Horizontal Cross-sections To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3 (see Figure 15.15). The integral is

$$\iint_R f(x, y) dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y) dx dy.$$

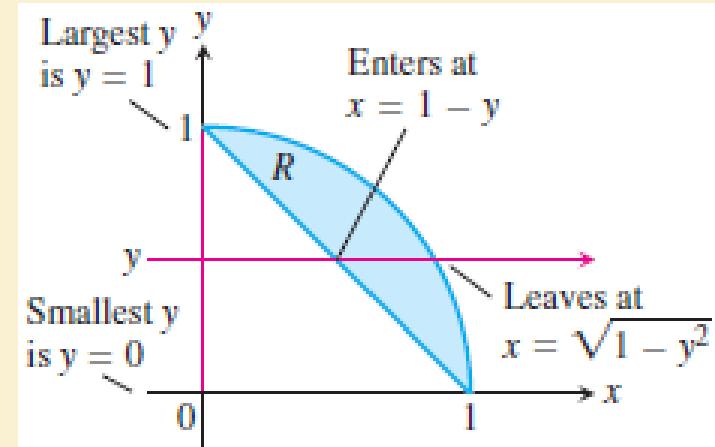


FIGURE 15.15 Finding the limits of integration when integrating first with respect to x and then with respect to y .

15.2 Double Integrals over General Regions

- Properties of Double Integrals
 - Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R , then the following properties hold.

1. *Constant Multiple:* $\iint_R cf(x, y) dA = c \iint_R f(x, y) dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y)) dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$$

15.2 Double Integrals over General Regions

- Properties of Double Integrals

3. *Domination:*

$$(a) \iint_R f(x, y) dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$$

$$(b) \iint_R f(x, y) dA \geq \iint_R g(x, y) dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$$

4. *Additivity:* $\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$

if R is the union of two nonoverlapping regions R_1 and R_2

15.3 Area by Double Integration

- Areas of Bounded Regions in the Plane

DEFINITION

The area of a closed, bounded plane region R is

$$A = \iint_R dA.$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply. To evaluate the integral in the definition of area, we integrate the constant function $f(x, y) = 1$ over R .

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

15.3 Area by Double Integration

Solution We sketch the region (Figure 15.19), noting where the two curves intersect at the origin and $(1, 1)$, and calculate the area as

$$\begin{aligned} A &= \int_0^1 \int_{x^2}^x dy dx = \int_0^1 \left[y \right]_{x^2}^x dx \\ &= \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}. \end{aligned}$$

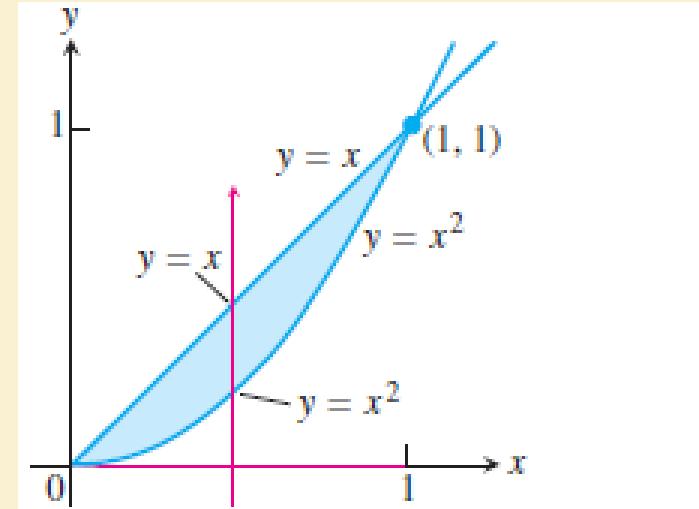
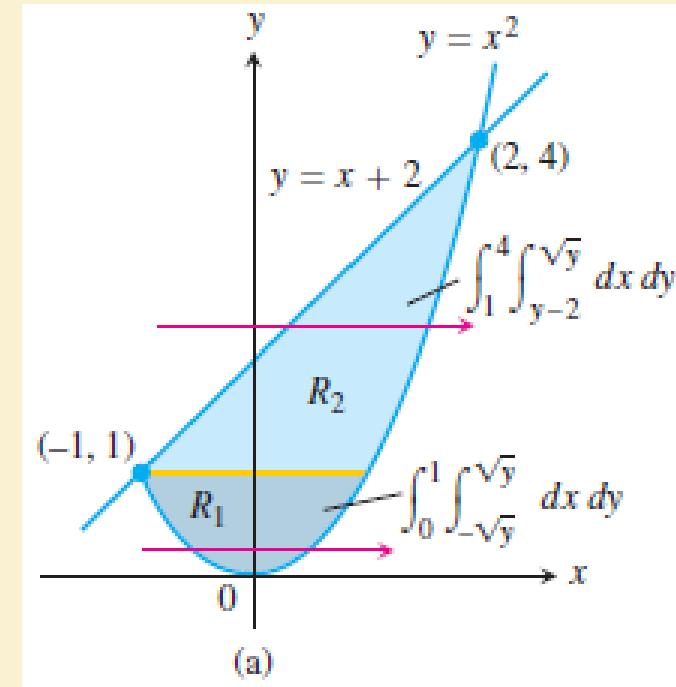


FIGURE 15.19 The region in Example 1. Calculus, 10th Edition, Cengage, ad Ibrahim

EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Figure 15.20a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx dy.$$

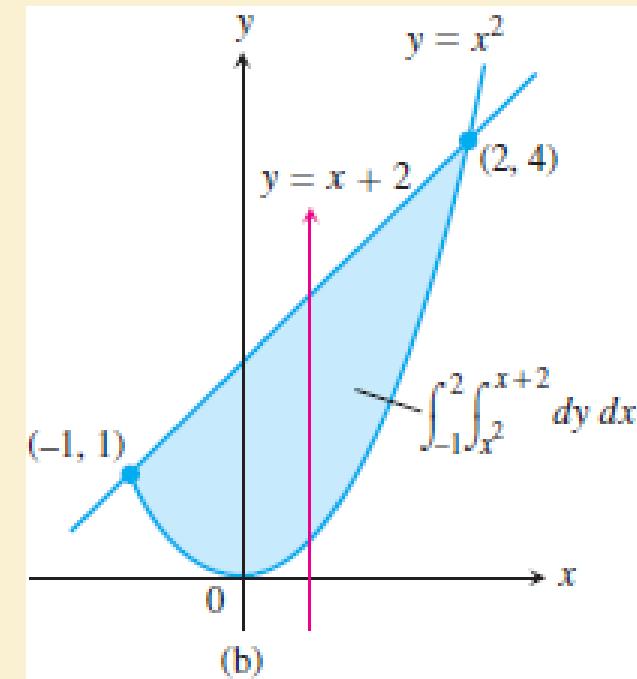


On the other hand, reversing the order of integration (Figure 15.20b) gives

$$A = \int_{-1}^2 \int_{x^2}^{x+2} dy dx.$$

This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^2 \left[y \right]_{x^2}^{x+2} dx = \int_{-1}^2 (x + 2 - x^2) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}. \quad \blacksquare$$



15.3 Area by Double Integration

- Average Value

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA. \quad (3)$$

EXAMPLE 3 Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R: 0 \leq x \leq \pi, 0 \leq y \leq 1$.

Solution The value of the integral of f over R is

$$\begin{aligned} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx &= \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} dx \quad \int x \cos xy \, dy = \sin xy + C \\ &= \int_0^\pi (\sin x - 0) \, dx = -\cos x \Big|_0^\pi = 1 + 1 = 2. \end{aligned}$$

The area of R is π . The average value of f over R is $2/\pi$.

15.4 Double Integrals in Polar Form

- Integrals in Polar Coordinates

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_R f(r, \theta) dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta) r dr d\theta.$$

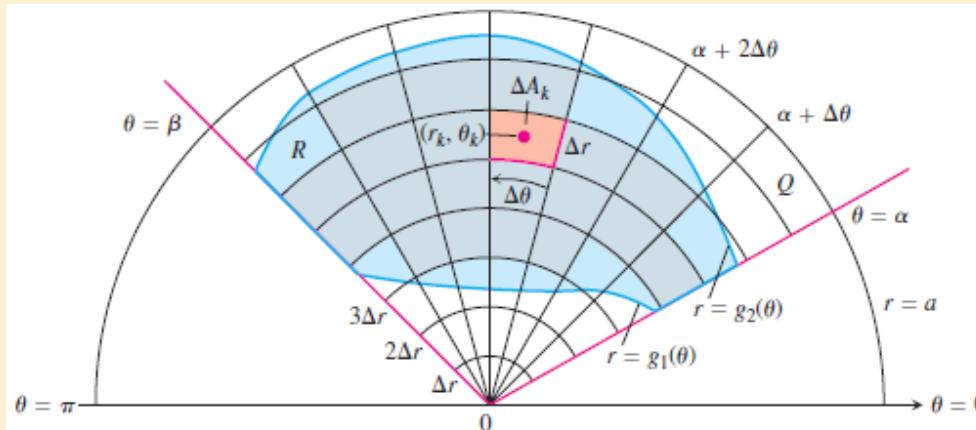


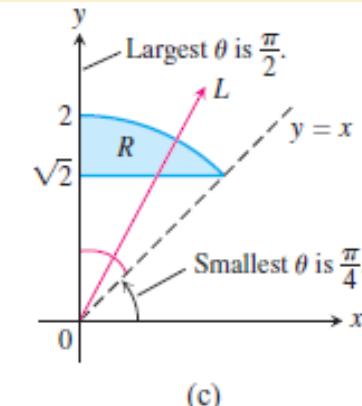
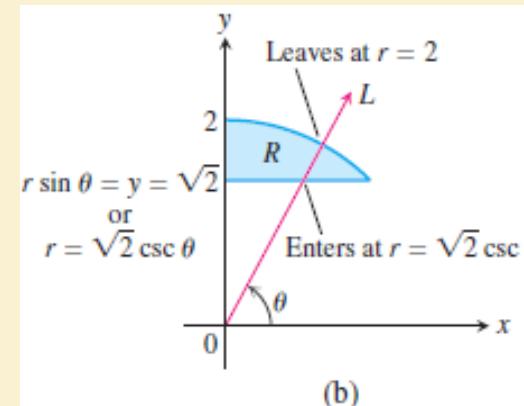
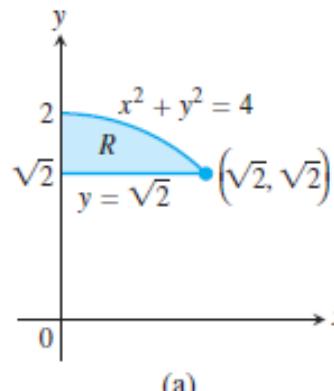
FIGURE 15.21 The region $R: g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta$, is contained in the fan-shaped region $Q: 0 \leq r \leq a, \alpha \leq \theta \leq \beta$. The partition of Q by circular arcs and rays induces a partition of R .

Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta) dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ , take the following steps.

1. *Sketch.* Sketch the region and label the bounding curves (Figure 15.23a).
2. *Find the r -limits of integration.* Imagine a ray L from the origin cutting through R in the direction of increasing r . Mark the r -values where L enters and leaves R . These are the r -limits of integration. They usually depend on the angle θ that L makes with the positive x -axis (Figure 15.23b).
3. *Find the θ -limits of integration.* Find the smallest and largest θ -values that bound R . These are the θ -limits of integration (Figure 15.23c). The polar iterated integral is

$$\iint_R f(r, \theta) dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2} \csc \theta}^{r=2} f(r, \theta) r dr d\theta.$$



EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 15.24).
2. Next we find the r -limits of integration. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos \theta$.
3. Finally we find the θ -limits of integration. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} f(r, \theta) r dr d\theta.$$

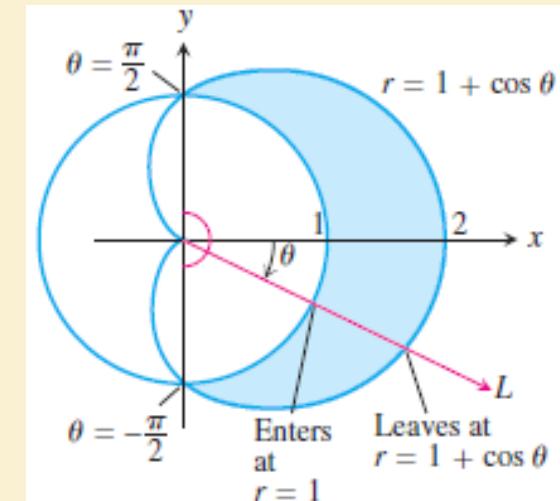


FIGURE 15.24 Finding the limits of integration in polar coordinates for the region in Example 1.

15.4 Double Integrals in Polar Form

- Integrals in Polar Coordinates

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta.$$

Area Differential in Polar Coordinates

$$dA = r \, dr \, d\theta$$

15.4 Double Integrals in Polar Form

- Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y) dx dy$ into a polar integral has two steps. First substitute $x = r \cos \theta$ and $y = r \sin \theta$, and replace $dx dy$ by $r dr d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R . The Cartesian integral then becomes

$$\iint_R f(x, y) dx dy = \iint_G f(r \cos \theta, r \sin \theta) r dr d\theta,$$

where G denotes the same region of integration now described in polar coordinates. This is

EXAMPLE 3 Evaluate

$$\iint_R e^{x^2+y^2} dy dx,$$

where R is the semicircular region bounded by the x -axis and the curve $y = \sqrt{1 - x^2}$ (Figure 15.26).

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y . Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r \cos \theta$, $y = r \sin \theta$ and replacing $dy dx$ by $r dr d\theta$ enables us to evaluate the integral as

$$\begin{aligned}\iint_R e^{x^2+y^2} dy dx &= \int_0^\pi \int_0^1 e^{r^2} r dr d\theta = \int_0^\pi \left[\frac{1}{2} e^{r^2} \right]_0^1 d\theta \\ &= \int_0^\pi \frac{1}{2} (e - 1) d\theta = \frac{\pi}{2} (e - 1).\end{aligned}$$

The r in the $r dr d\theta$ was just what we needed to integrate e^{r^2} . Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■

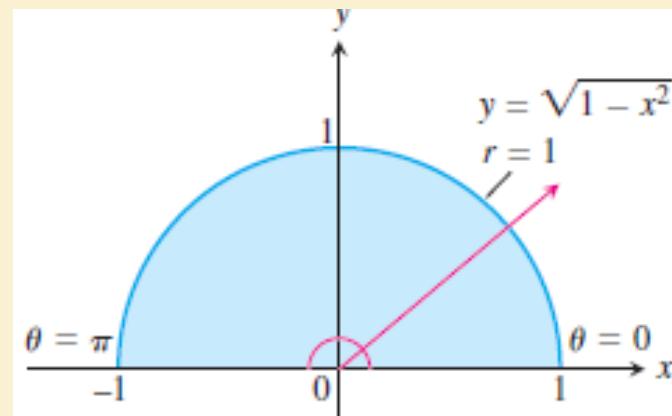


FIGURE 15.26 The semicircular region in Example 3 is the region

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi.$$

15.5 Triple Integrals in Rectangular Coordinates

- We use triple integrals to calculate the volumes of three-dimensional shapes and the average value of a function over a three-dimensional region.

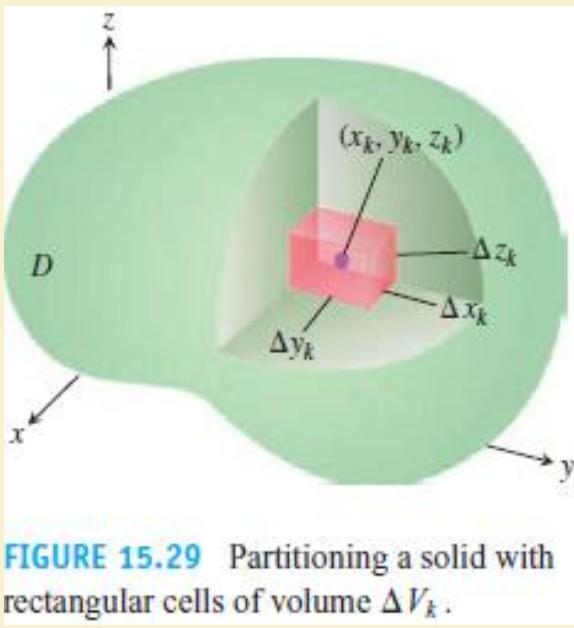


FIGURE 15.29 Partitioning a solid with rectangular cells of volume ΔV_k .

$$S_n = \sum_{k=1}^n F(x_k, y_k, z_k) \Delta V_k$$

$$\lim_{n \rightarrow \infty} S_n = \iiint_D F(x, y, z) dV \quad \text{or} \quad \lim_{\|P\| \rightarrow 0} S_n = \iiint_D F(x, y, z) dx dy dz.$$

15.5 Triple Integrals in Rectangular Coordinates

- Volume of a Region in Space

$$S_n = \sum F(x_k, y_k, z_k) \Delta V_k = \sum 1 \cdot \Delta V_k = \sum \Delta V_k.$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta V_k = \iiint_D dV.$$

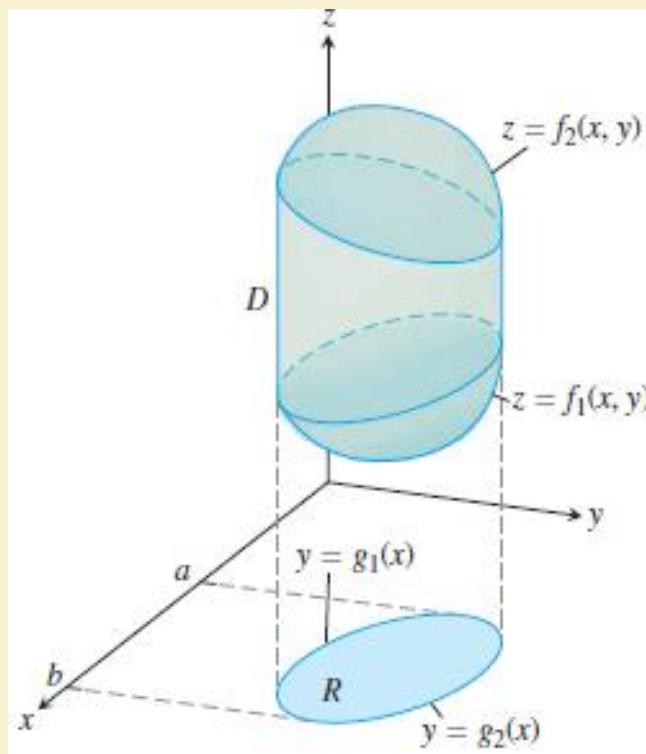
DEFINITION The volume of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

15.5 Triple Integrals in Rectangular Coordinates

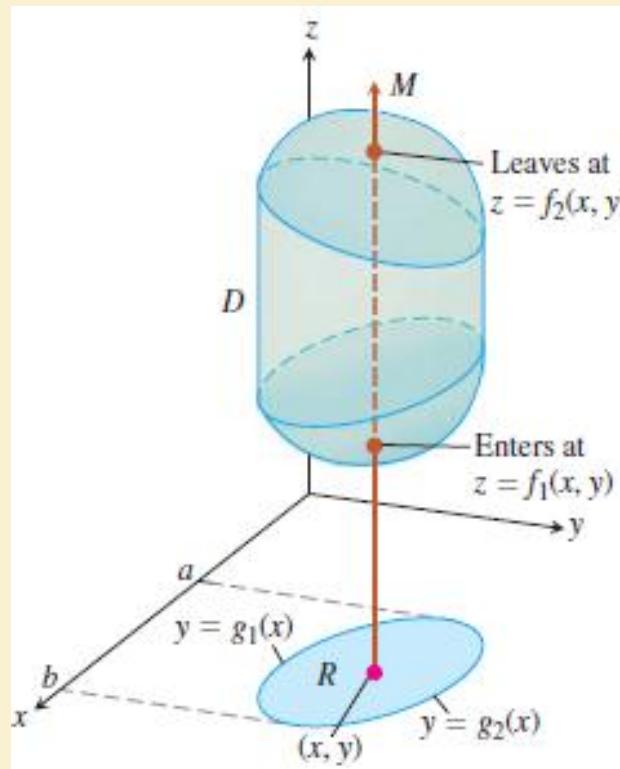
- Finding Limits of Integration in the Order $dz\ dy\ dx$

- Sketch. Sketch the region D along with its “shadow” R (vertical projection) in the xy -plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R .



15.5 Triple Integrals in Rectangular Coordinates

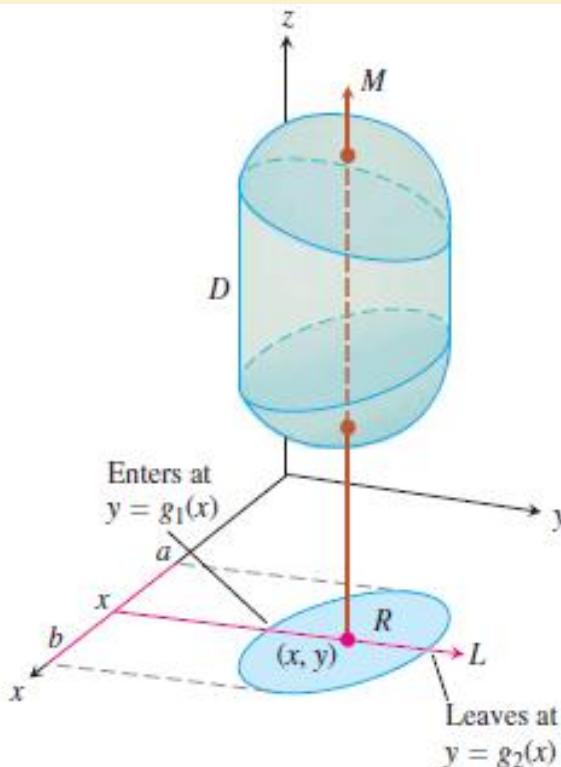
- Finding Limits of Integration in the Order $dz\ dy\ dx$
2. *Find the z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z -axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z -limits of integration.



15.5 Triple Integrals in Rectangular Coordinates

- Finding Limits of Integration in the Order $dz\ dy\ dx$

3. *Find the y-limits of integration.* Draw a line L through (x, y) parallel to the y -axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y -limits of integration.

 z 

15.5 Triple Integrals in Rectangular Coordinates

- Finding Limits of Integration in the Order $dz\ dy\ dx$

4. *Find the x -limits of integration.* Choose x -limits that include all lines through R parallel to the y -axis ($x = a$ and $x = b$ in the preceding figure). These are the x -limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z) \, dz \, dy \, dx.$$

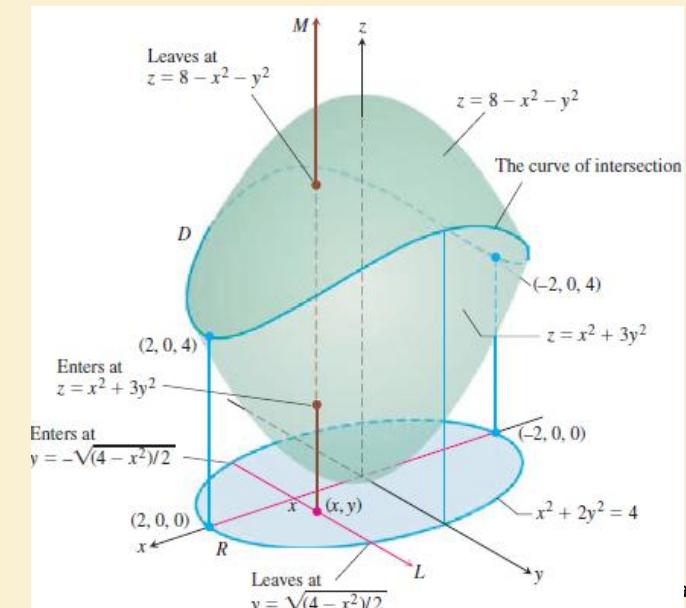
Follow similar procedures if you change the order of integration. The “shadow” of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.

EXAMPLE 1 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

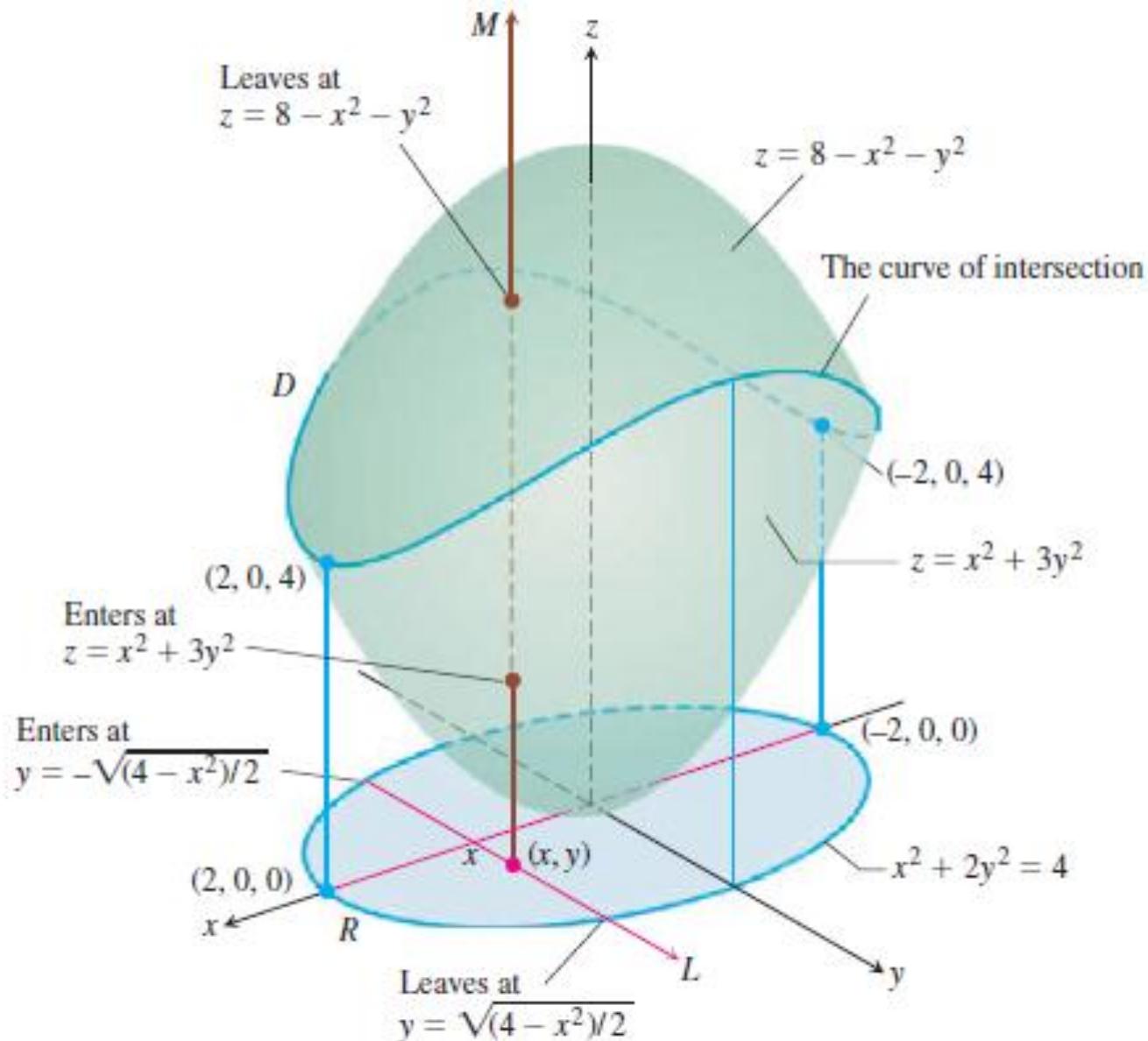
Solution The volume is

$$V = \iiint_D dz dy dx,$$

the integral of $F(x, y, z) = 1$ over D . To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.30) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R , the projection of D onto the xy -plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The “upper” boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.



Now we find the z -limits of integration. The line M passing through a typical point (x, y) in R parallel to the z -axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.



Next we find the y -limits of integration. The line L through (x, y) parallel to the y -axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x -limits of integration. As L sweeps across R , the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

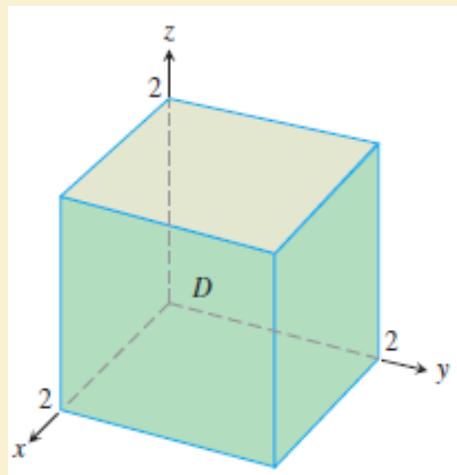
$$\begin{aligned}
 V &= \iiint_D dz dy dx \\
 &= \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx \\
 &= \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2) dy dx \\
 &= \int_{-2}^2 \left[(8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \\
 &= \int_{-2}^2 \left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3} \left(\frac{4 - x^2}{2} \right)^{3/2} \right) dx \\
 &= \int_{-2}^2 \left[8 \left(\frac{4 - x^2}{2} \right)^{3/2} - \frac{8}{3} \left(\frac{4 - x^2}{2} \right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^2 (4 - x^2)^{3/2} dx \\
 &= 8\pi\sqrt{2}. \quad \text{After integration with the substitution } x = 2 \sin u
 \end{aligned}$$

15.5 Triple Integrals in Rectangular Coordinates

- Average Value of a Function in Space

Average value of F over D = $\frac{1}{\text{volume of } D} \iiint_D F \, dV.$ (2)

EXAMPLE 4 Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2, y = 2,$ and $z = 2$ in the first octant.



Chapter 12

Vectors and the

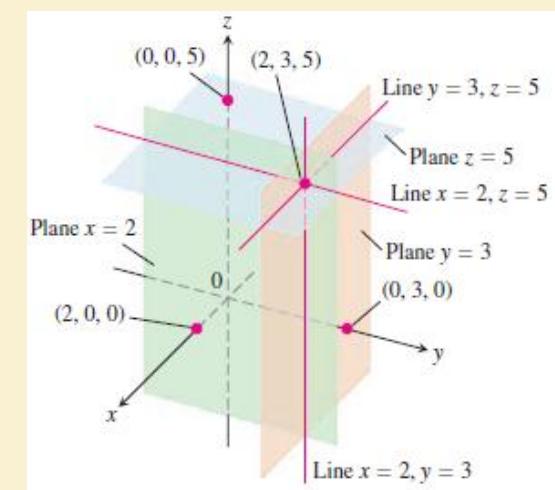
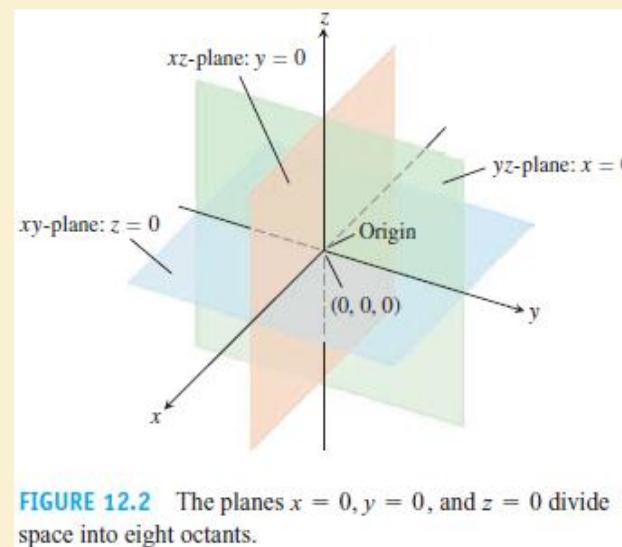
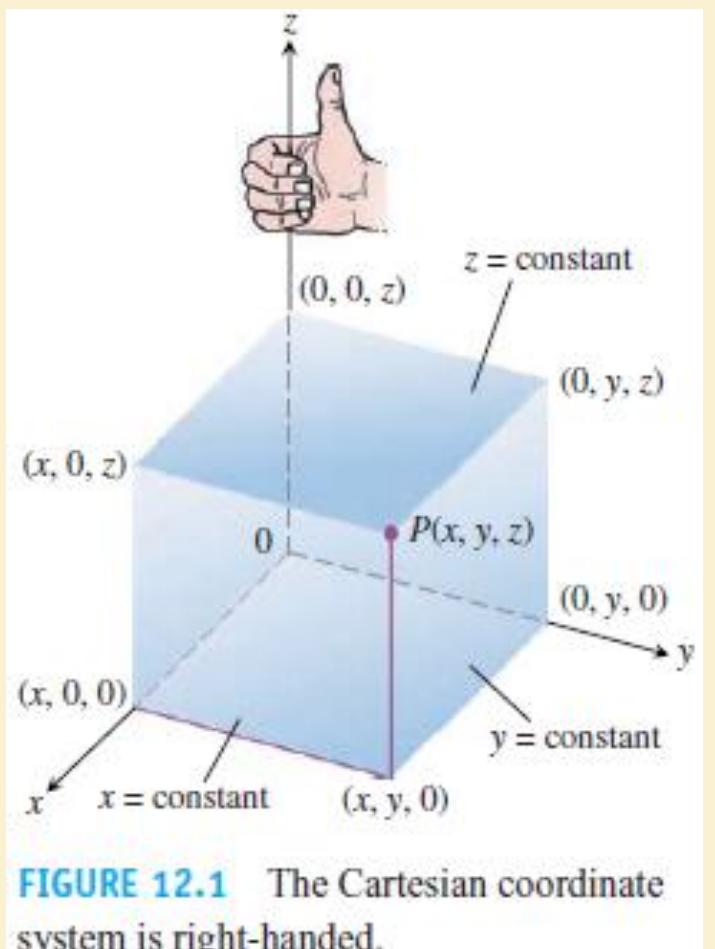
Geometry of Space

Chapter 12: Vectors and the Geometry of Space

- » Three-Dimensional Coordinate Systems
- » Vectors
- » The Dot Product
- » The Cross Product
- » Lines and Planes in Space
- » Cylinders and Quadric Surfaces

To apply calculus in many real-world situations and in higher mathematics, we need a mathematical description of three-dimensional space. In this chapter we introduce three-dimensional coordinate systems and vectors. Building on what we already know about coordinates in the xy -plane, we establish coordinates in space by adding a third axis that measures distance above and below the xy -plane. Vectors are used to study the analytic geometry of space, where they give simple ways to describe lines, planes, surfaces, and curves in space. We use these geometric ideas in the rest of the book to study motion in space and the calculus of functions of several variables, with their many important applications in science, engineering, economics, and higher mathematics.

12.1 Three-Dimensional Coordinate Systems



12.1 Three-Dimensional Coordinate Systems

EXAMPLE 1 We interpret these equations and inequalities geometrically.

(a) $z \geq 0$

The half-space consisting of the points on and above the xy -plane.

(b) $x = -3$

The plane perpendicular to the x -axis at $x = -3$. This plane lies parallel to the yz -plane and 3 units behind it.

(c) $z = 0, x \leq 0, y \geq 0$

The second quadrant of the xy -plane.

(d) $x \geq 0, y \geq 0, z \geq 0$

The first octant.

(e) $-1 \leq y \leq 1$

The slab between the planes $y = -1$ and $y = 1$ (planes included).

(f) $y = -2, z = 2$

The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x -axis. ■

12.1 Three-Dimensional Coordinate Systems

EXAMPLE 2 What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \quad \text{and} \quad z = 3?$$

Solution The points lie in the horizontal plane $z = 3$ and, in this plane, make up the circle $x^2 + y^2 = 4$. We call this set of points “the circle $x^2 + y^2 = 4$ in the plane $z = 3$ ” or, more simply, “the circle $x^2 + y^2 = 4, z = 3$ ” (Figure 12.4). ■

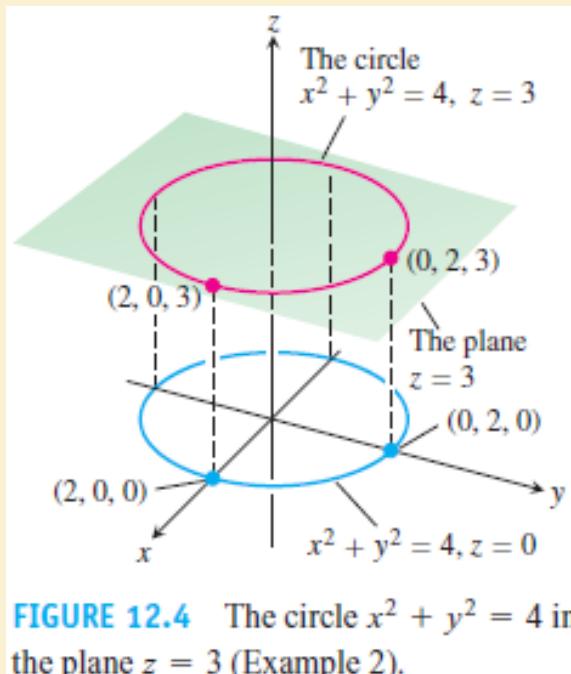


FIGURE 12.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

12.1 Three-Dimensional Coordinate Systems

- Distance and Spheres in Space

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

12.1 Three-Dimensional Coordinate Systems

EXAMPLE 4 Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution We find the center and radius of a sphere the way we find the center and radius of a circle: Complete the squares on the x -, y -, and z -terms as necessary and write each

quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here, we have

$$\begin{aligned} x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\ (x^2 + 3x) + y^2 + (z^2 - 4z) &= -1 \\ \left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) &= -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2 \\ \left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 &= -1 + \frac{9}{4} + 4 = \frac{21}{4}. \end{aligned}$$

From this standard form, we read that $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.



12.2 Vectors

- Component Form

DEFINITIONS The vector represented by the directed line segment \vec{AB} has **initial point** A and **terminal point** B and its **length** is denoted by $|\vec{AB}|$. Two vectors are **equal** if they have the same length and direction.

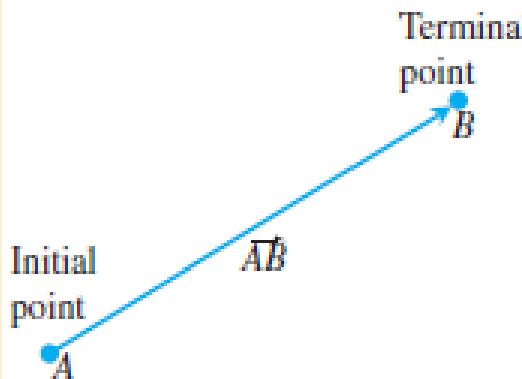


FIGURE 12.7 The directed line segment \vec{AB} is called a vector.

12.2 Vectors

- Component Form

DEFINITION

If \mathbf{v} is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If \mathbf{v} is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

In summary, given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to \overrightarrow{PQ} is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

12.2 Vectors

- Component Form

The magnitude or length of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(see Figure 12.10).

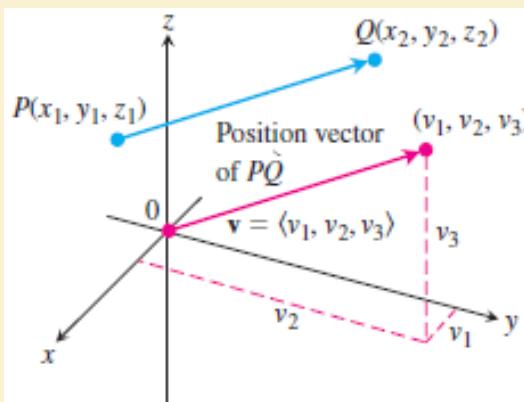


FIGURE 12.10 A vector \overrightarrow{PQ} in standard position has its initial point at the origin. The directed line segments \overrightarrow{PQ} and \mathbf{v} are parallel and have the same length.

12.2 Vectors

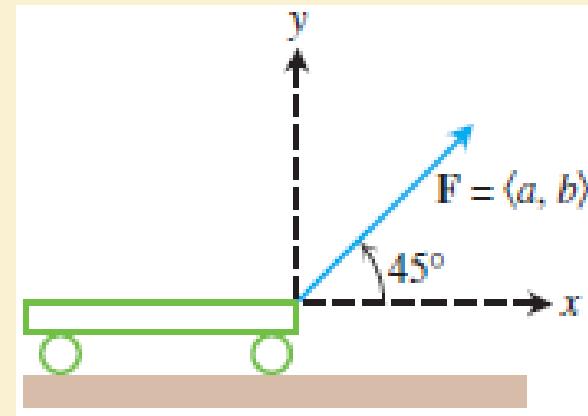
- Component Form

EXAMPLE 2 A small cart is being pulled along a smooth horizontal floor with a 20-lb force \mathbf{F} making a 45° angle to the floor (Figure 12.11). What is the *effective* force moving the cart forward?

Solution The effective force is the horizontal component of $\mathbf{F} = \langle a, b \rangle$, given by

$$a = |\mathbf{F}| \cos 45^\circ = (20) \left(\frac{\sqrt{2}}{2} \right) \approx 14.14 \text{ lb.}$$

Notice that \mathbf{F} is a two-dimensional vector.



12.2 Vectors

- Vector Algebra Operations

DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

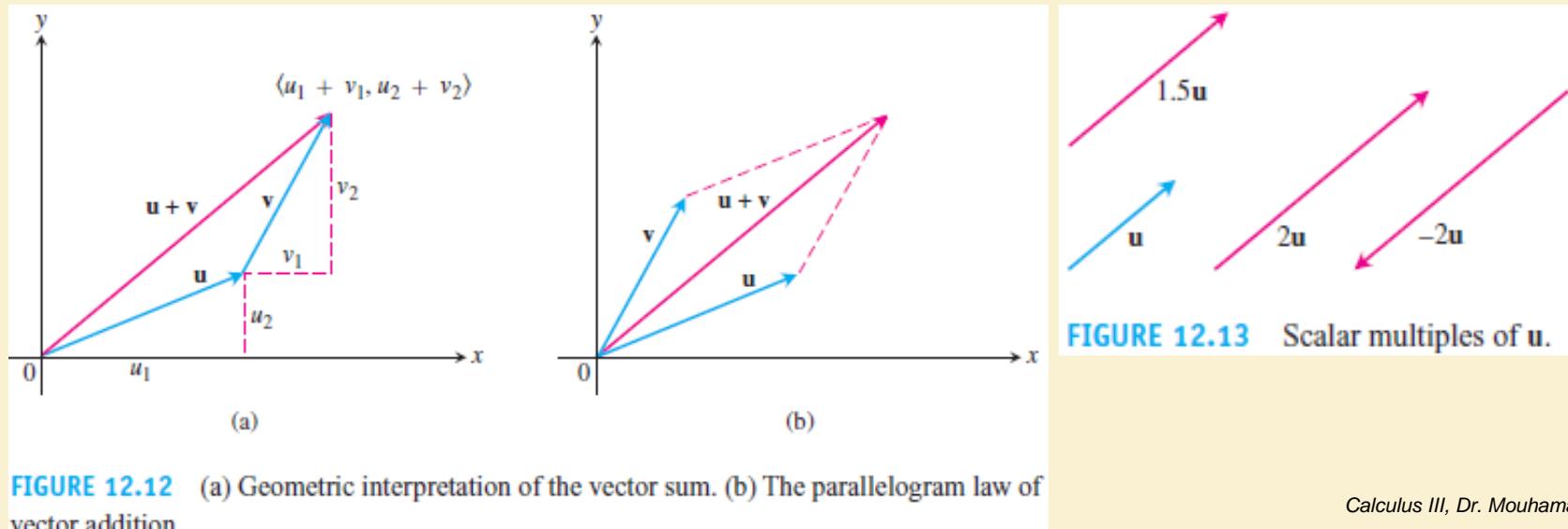


FIGURE 12.12 (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.

12.2 Vectors

- Vector Algebra Operations

Properties of Vector Operations

Let \mathbf{u} , \mathbf{v} , \mathbf{w} be vectors and a , b be scalars.

- | | |
|--|--|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ |
| 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ |
| 5. $0\mathbf{u} = \mathbf{0}$ | 6. $1\mathbf{u} = \mathbf{u}$ |
| 7. $a(b\mathbf{u}) = (ab)\mathbf{u}$ | 8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ |
| 9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ | |

These properties are readily verified using the definitions of vector addition and multiplication by a scalar. For instance, to establish Property 1, we have

$$\begin{aligned}
 \mathbf{u} + \mathbf{v} &= \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle \\
 &= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \\
 &= \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle \\
 &= \langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle \\
 &= \mathbf{v} + \mathbf{u}.
 \end{aligned}$$

12.2 Vectors

- Unit Vectors

A vector \mathbf{v} of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

We call the scalar (or number) v_1 the **i-component** of the vector \mathbf{v} , v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

12.2 Vectors

- Unit Vectors

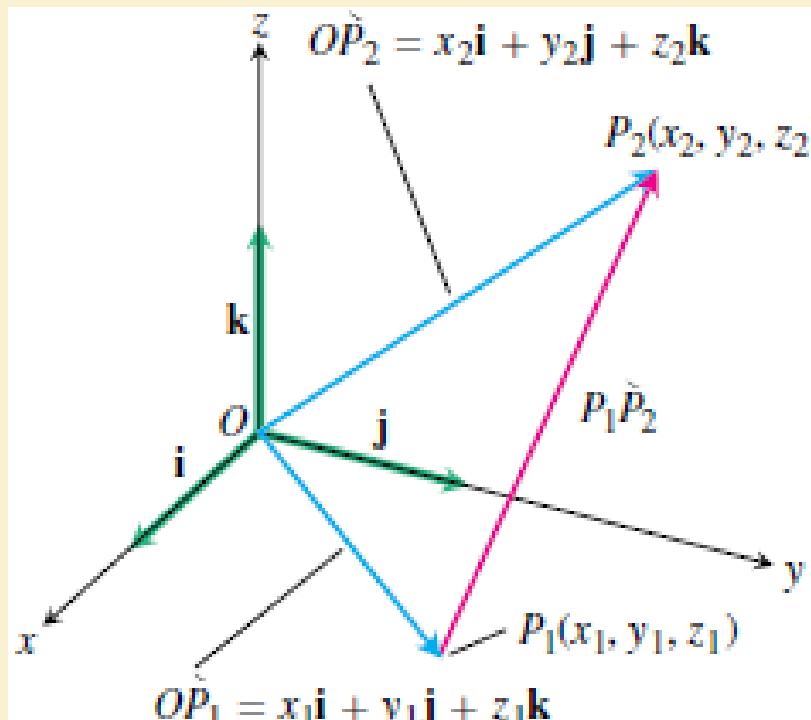


FIGURE 12.15 The vector from P_1 to P_2 is $\vec{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

12.2 Vectors

- Unit Vectors

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of \mathbf{v} ;
2. the equation $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$ expresses \mathbf{v} as its length times its direction.

HISTORICAL BIOGRAPHY

Hermann Grassmann
(1809–1877)

12.2 Vectors

- Unit Vectors

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force \mathbf{F} as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\begin{aligned}\mathbf{F} &= 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6 \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right).\end{aligned}$$



12.2 Vectors

- Midpoint of a Line Segment

The midpoint M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

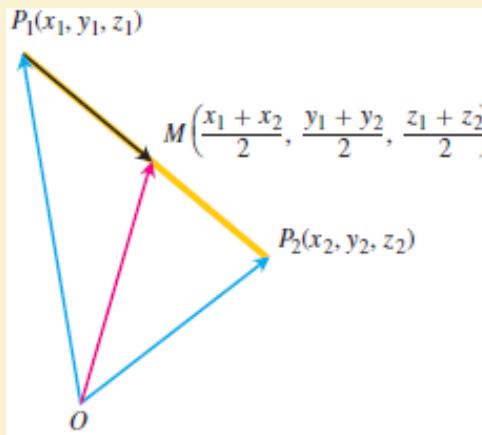


FIGURE 12.16 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2 .



12.2 Vectors

- Applications

Applications

An important application of vectors occurs in navigation.

EXAMPLE 8 A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Applications

An important application of vectors occurs in navigation.

EXAMPLE 8 A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Solution If \mathbf{u} = the velocity of the airplane alone and \mathbf{v} = the velocity of the tailwind, then $|\mathbf{u}| = 500$ and $|\mathbf{v}| = 70$ (Figure 12.17). The velocity of the airplane with respect to the ground is given by the magnitude and direction of the resultant vector $\mathbf{u} + \mathbf{v}$. If we let the positive x -axis represent east and the positive y -axis represent north, then the component forms of \mathbf{u} and \mathbf{v} are

$$\mathbf{u} = \langle 500, 0 \rangle \quad \text{and} \quad \mathbf{v} = \langle 70 \cos 60^\circ, 70 \sin 60^\circ \rangle = \langle 35, 35\sqrt{3} \rangle.$$

Therefore,

$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle = 535\mathbf{i} + 35\sqrt{3}\mathbf{j}$$

$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

and

$$\theta = \tan^{-1} \frac{35\sqrt{3}}{535} \approx 6.5^\circ. \quad \text{Figure 12.17}$$

The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east.

12.3 The Dot Product

- Angle Between Vectors

THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}| |\mathbf{v}|} \right).$$

DEFINITION The dot product $\mathbf{u} \cdot \mathbf{v}$ (“ \mathbf{u} dot \mathbf{v} ”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Proof of Theorem 1 Applying the law of cosines (Equation (8), Section 1.3) to the triangle in Figure 12.21, we find that

$$\begin{aligned} |\mathbf{w}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta && \text{Law of cosines} \\ 2|\mathbf{u}||\mathbf{v}|\cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2. \end{aligned}$$

Because $\mathbf{w} = \mathbf{u} - \mathbf{v}$, the component form of \mathbf{w} is $\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$. So

$$\begin{aligned} |\mathbf{u}|^2 &= (\sqrt{u_1^2 + u_2^2 + u_3^2})^2 = u_1^2 + u_2^2 + u_3^2 \\ |\mathbf{v}|^2 &= (\sqrt{v_1^2 + v_2^2 + v_3^2})^2 = v_1^2 + v_2^2 + v_3^2 \\ |\mathbf{w}|^2 &= (\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2})^2 \\ &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\ &= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2 \end{aligned}$$

and

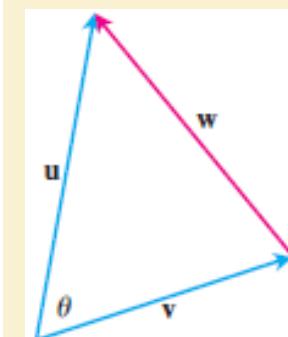
$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Therefore,

$$\begin{aligned} 2|\mathbf{u}||\mathbf{v}|\cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3) \\ |\mathbf{u}||\mathbf{v}|\cos\theta &= u_1v_1 + u_2v_2 + u_3v_3 \\ \cos\theta &= \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}. \end{aligned}$$

Since $0 \leq \theta < \pi$, we have

$$\theta = \cos^{-1} \left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|} \right).$$



12.3 The Dot Product

- Angle Between Vectors

EXAMPLE 2 Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{-4}{(3)(7)} \right) \approx 1.76 \text{ radians.}$$



12.3 The Dot Product

- Perpendicular (Orthogonal) Vectors

Perpendicular (Orthogonal) Vectors

Two nonzero vectors \mathbf{u} and \mathbf{v} are perpendicular or **orthogonal** if the angle between them is $\pi/2$. For such vectors, we have $\mathbf{u} \cdot \mathbf{v} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If \mathbf{u} and \mathbf{v} are nonzero vectors with $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta = 0$, then $\cos \theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$.

DEFINITION Vectors \mathbf{u} and \mathbf{v} are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

12.3 The Dot Product

- Dot Product Properties

Properties of the Dot Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

HISTORICAL BIOGRAPHY

Carl Friedrich Gauss
(1777–1855)

12.3 The Dot Product

- Vector Projections

The vector projection of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}. \quad (1)$$

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$|\mathbf{u}| \cos \theta = \frac{|\mathbf{u}| |\mathbf{v}| \cos \theta}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$

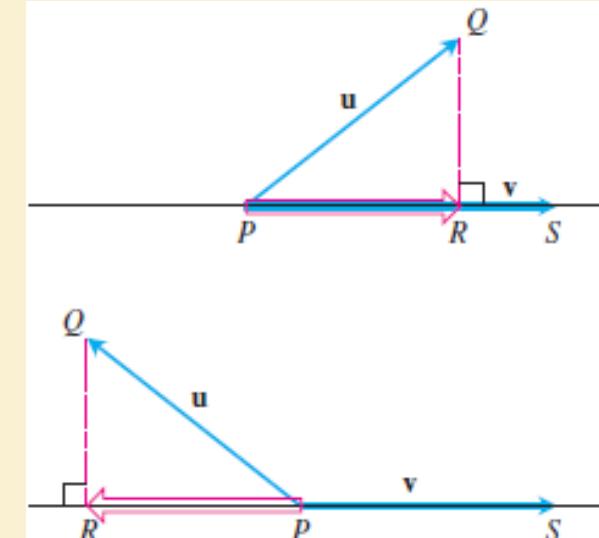


FIGURE 12.23 The vector projection of \mathbf{u} onto \mathbf{v} .

12.3 The Dot Product

- Vector Projections

The scalar component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (2)$$

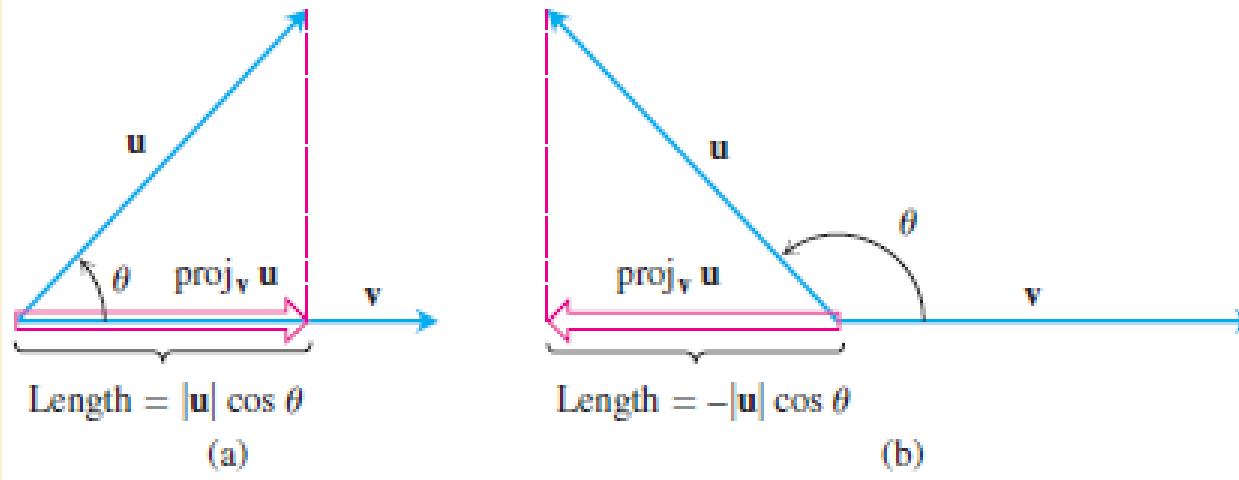


FIGURE 12.25 The length of $\text{proj}_{\mathbf{v}} \mathbf{u}$ is (a) $|\mathbf{u}| \cos \theta$ if $\cos \theta \geq 0$ and (b) $-|\mathbf{u}| \cos \theta$ if $\cos \theta < 0$.

12.3 The Dot Product

- Dot Product Properties and Vector Projections

EXAMPLE 5 Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of \mathbf{u} in the direction of \mathbf{v} .

Solution We find $\text{proj}_{\mathbf{v}} \mathbf{u}$ from Equation (1):

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9} \mathbf{i} + \frac{8}{9} \mathbf{j} + \frac{8}{9} \mathbf{k}.\end{aligned}$$

We find the scalar component of \mathbf{u} in the direction of \mathbf{v} from Equation (2):

$$\begin{aligned}|\mathbf{u}| \cos \theta &= \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.\end{aligned}$$
■

12.3 The Dot Product

- Dot Product Properties and Vector Projections

$$\mathbf{u} = \text{proj}_{\mathbf{v}} \mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) = \underbrace{\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}}_{\text{Parallel to } \mathbf{v}} + \underbrace{\left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \right)}_{\text{Orthogonal to } \mathbf{v}}$$

12.4 The Cross Product

- In studying lines in the plane, when we needed to describe how a line was tilting, we used the notions of slope and angle of inclination.
- In space, we want a way to describe how a plane is tilting. We accomplish this by multiplying two vectors in the plane together to get a third vector perpendicular to the plane.
- The direction of this third vector tells us the “inclination” of the plane. The product we use to multiply the vectors together is the vector or cross product, the second of the two vector multiplication methods.

12.4 The Cross Product

- The Cross Product of Two Vectors in Space

DEFINITION

$$\mathbf{u} \times \mathbf{v} = (\|\mathbf{u}\| \|\mathbf{v}\| \sin \theta) \mathbf{n}$$

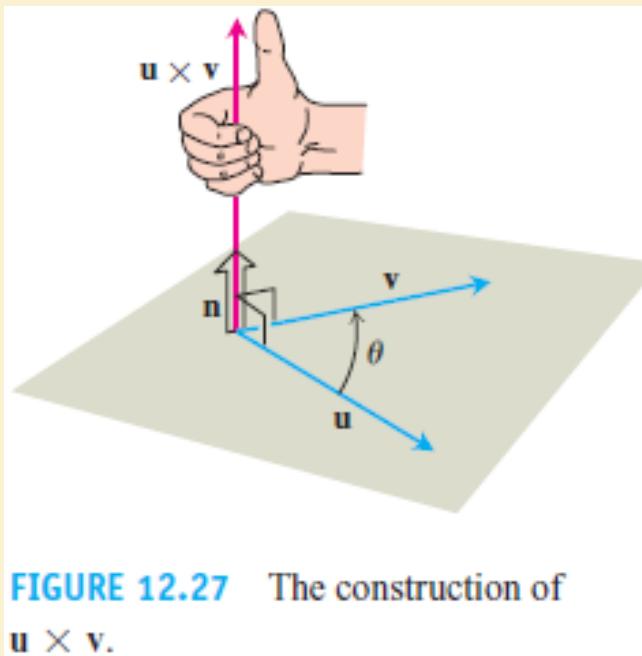


FIGURE 12.27 The construction of $\mathbf{u} \times \mathbf{v}$.

12.4 The Cross Product

- The Cross Product of Two Vectors in Space

Properties of the Cross Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r , s are scalars, then

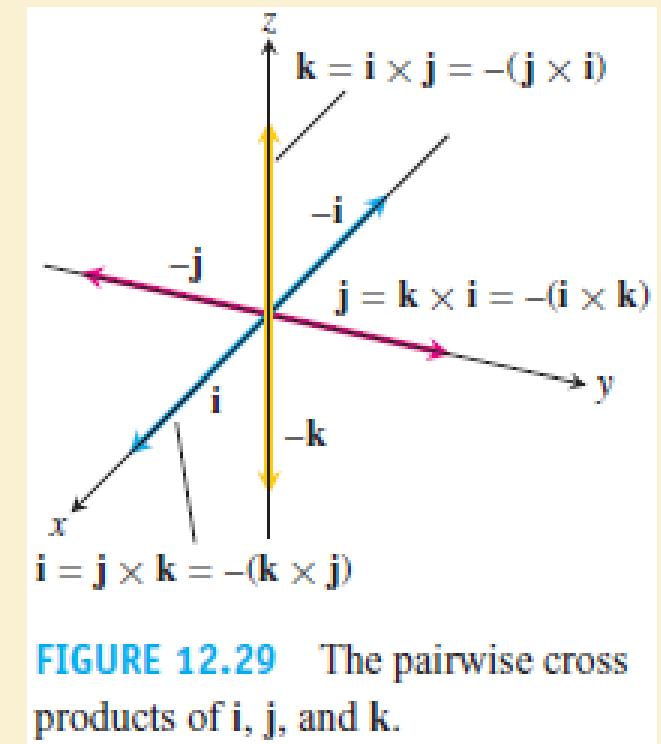
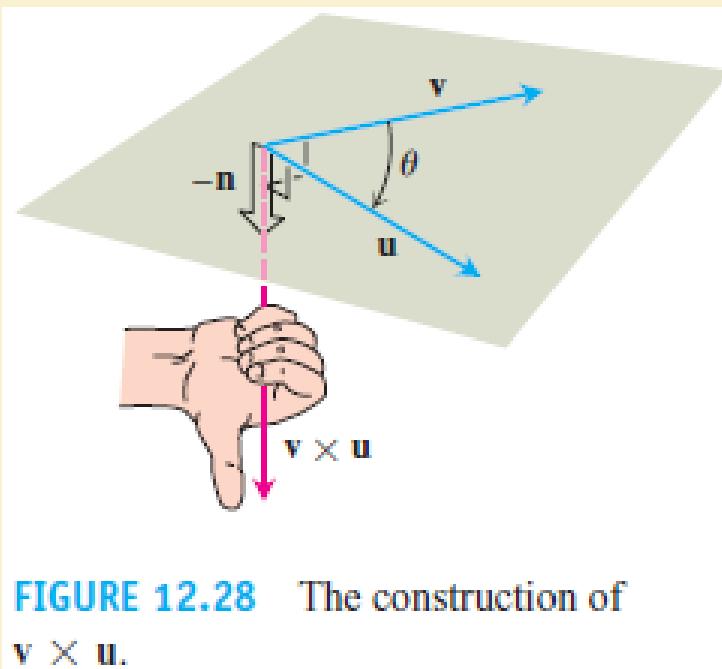
1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

Parallel Vectors

Nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

12.4 The Cross Product

- The Cross Product of Two Vectors in Space



12.4 The Cross Product

$|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

Because \mathbf{n} is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\sin \theta| |\mathbf{n}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

This is the area of the parallelogram determined by \mathbf{u} and \mathbf{v} (Figure 12.30), $|\mathbf{u}|$ being the base of the parallelogram and $|\mathbf{v}| |\sin \theta|$ the height.

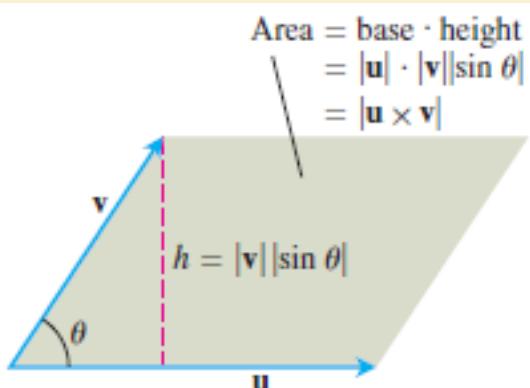


FIGURE 12.30 The parallelogram determined by \mathbf{u} and \mathbf{v} .

12.4 The Cross Product

- Determinant Formula

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Determinants

2×2 and 3×3 determinants are evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE

$$\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = (2)(3) - (1)(-4) \\ = 6 + 4 = 10$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$$

$$- a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

EXAMPLE

$$\begin{vmatrix} -5 & 3 & 1 \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = (-5) \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \\ - (3) \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + (1) \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \\ = -5(1 - 3) - 3(2 + 4) \\ + 1(6 + 4) \\ = 10 - 18 + 10 = 2$$

12.4 The Cross Product

- Determinant Formula

EXAMPLE 2 Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 6\mathbf{k}. \quad \blacksquare\end{aligned}$$

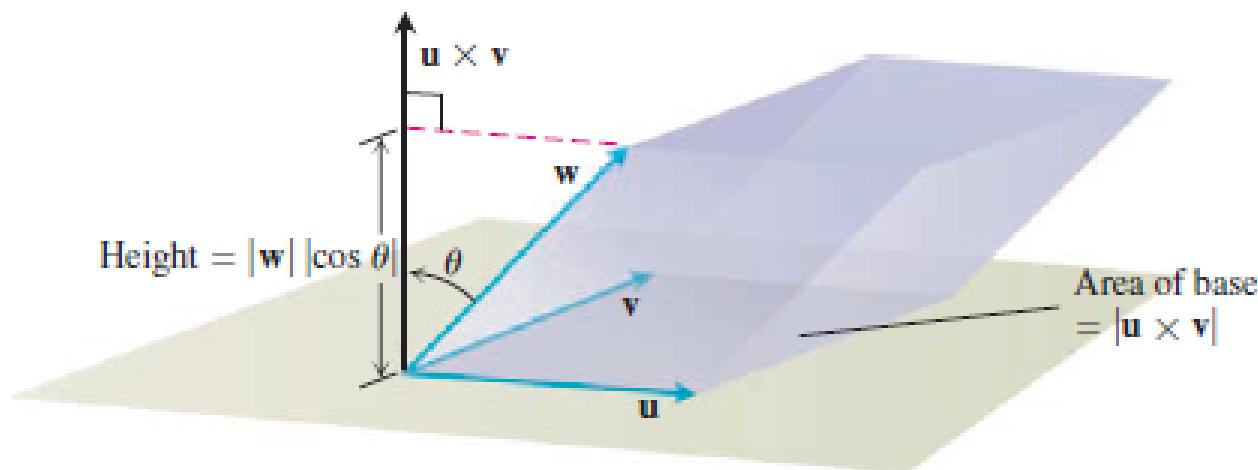
12.4 The Cross Product

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of \mathbf{u} , \mathbf{v} , and \mathbf{w} (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta|,$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} (Figure 12.34). The number $|\mathbf{u} \times \mathbf{v}|$ is the area of the base



$$\begin{aligned}\text{Volume} &= \text{area of base} \cdot \text{height} \\ &= |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta| \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|\end{aligned}$$

12.4 The Cross Product

- Triple Scalar or Box Product

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The dot and cross may be interchanged in a triple scalar product without altering its value.

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

12.5 Lines and Planes in Space

- This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.
- We will use these representations throughout the rest of the book...

12.5 Lines and Planes in Space

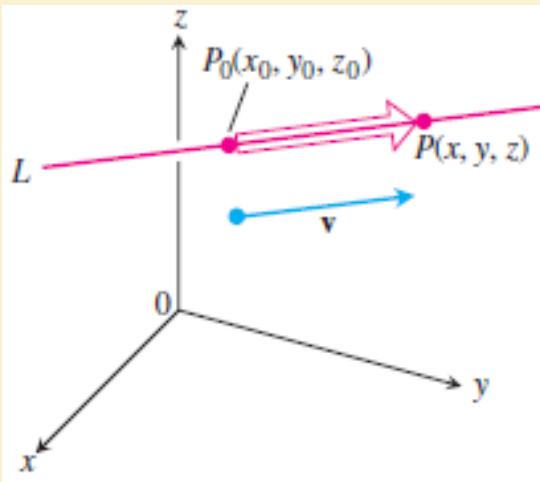
- Lines and Line Segments in Space

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to \mathbf{v} is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty, \quad (2)$$

where \mathbf{r} is the position vector of a point $P(x, y, z)$ on L and \mathbf{r}_0 is the position vector of $P_0(x_0, y_0, z_0)$.



$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \quad (1)$$

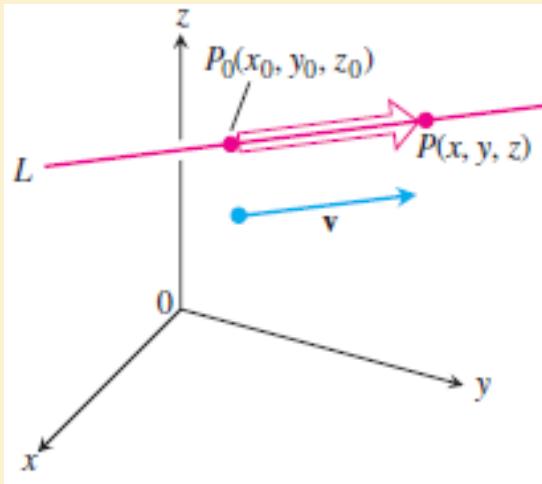
12.5 Lines and Planes in Space

- Lines and Line Segments in Space

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \quad (3)$$



EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.37).

Solution We begin with equations for the line through P and Q , taking them, in this case, from Example 2:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1. \quad \blacksquare$$

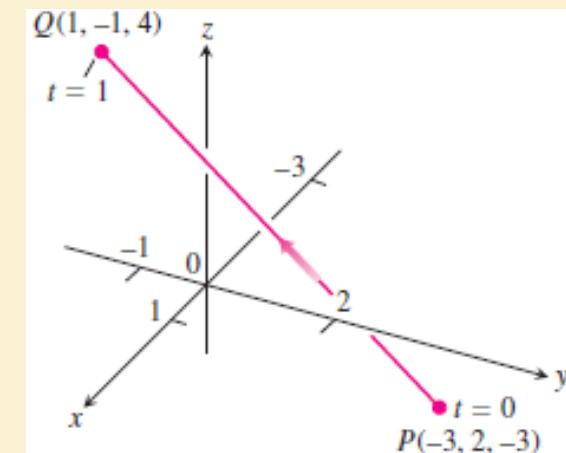


FIGURE 12.37 Example 3 derives a parametrization of line segment PQ . The arrow shows the direction of increasing t .

12.5 Lines and Planes in Space

- Lines and Line Segments in Space

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

??

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60 ft/sec . What is the position of the helicopter after 10 sec ?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10 \text{ sec}$,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle.\end{aligned}$$

After 10 sec of flight from the origin toward $(1, 1, 1)$, the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of $(60 \text{ ft/sec})(10 \text{ sec}) = 600 \text{ ft}$, which is the length of the vector $\mathbf{r}(10)$.

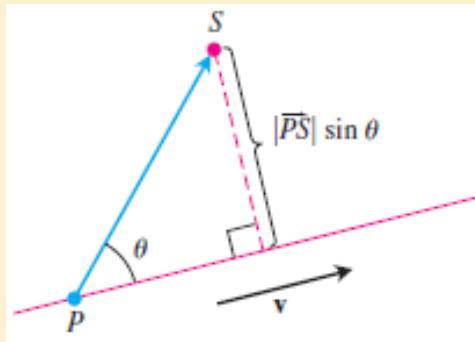
12.5 Lines and Planes in Space

- The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector \mathbf{v} , we find the absolute value of the scalar component of \overrightarrow{PS} in the direction of a vector normal to the line (Figure 12.38). In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}| \sin \theta$, which is $\frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.

Distance from a Point S to a Line Through P Parallel to \mathbf{v}

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad (5)$$



12.5 Lines and Planes in Space

- The Distance from a Point to a Line in Space

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}. \quad \blacksquare$$

12.5 Lines and Planes in Space

- An Equation for a Plane in Space

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:

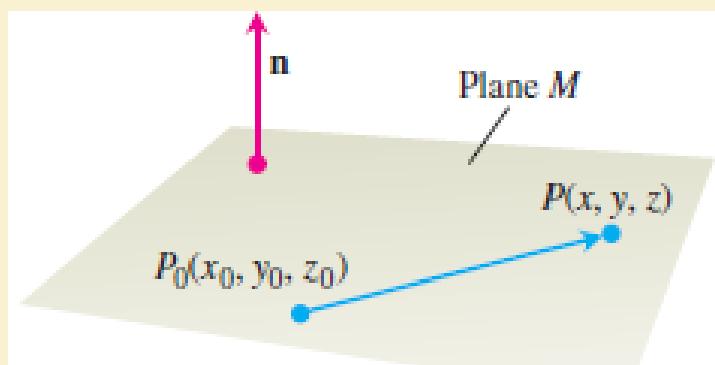
$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

Component equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Component equation simplified: $Ax + By + Cz = D$, where

$$D = Ax_0 + By_0 + Cz_0$$



12.5 Lines and Planes in Space

- An Equation for a Plane in Space

EXAMPLE 6 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22.$$
■

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x , y , and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

12.5 Lines and Planes in Space

- An Equation for a Plane in Space

EXAMPLE 7 Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6.$$



Lines of Intersection

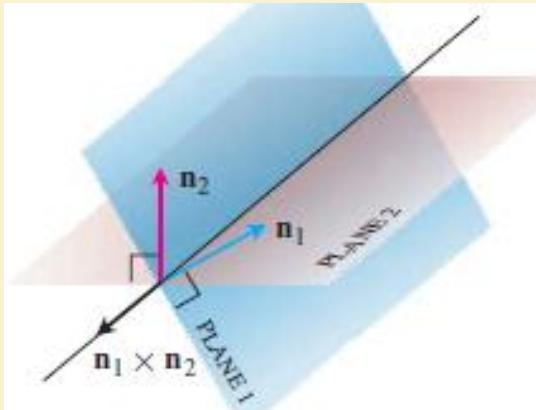
Just as lines are parallel if and only if they have the same direction, two planes are parallel if and only if their normals are parallel, or $\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar k . Two planes that are not parallel intersect in a line.

EXAMPLE 8 Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The line of intersection of two planes is perpendicular to both planes' normal vectors \mathbf{n}_1 and \mathbf{n}_2 (Figure 12.40) and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well. ■



12.5 Lines and Planes in Space

- The Distance from a Point to a Plane

The Distance from a Point to a Plane

If P is a point on a plane with normal $\underline{\mathbf{n}}$, then the distance from any point S to the plane is the length of the vector projection of \overrightarrow{PS} onto \mathbf{n} . That is, the distance from S to the plane is

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \quad (6)$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.

EXAMPLE 11 Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution We find a point P in the plane and calculate the length of the vector projection of \vec{PS} onto a vector \mathbf{n} normal to the plane (Figure 12.41). The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y -intercept $(0, 3, 0)$, then

$$\begin{aligned}\vec{PS} &= (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} \\ &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \\ |\mathbf{n}| &= \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.\end{aligned}$$

The distance from S to the plane is

$$\begin{aligned}d &= \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| && \text{length of proj}_{\mathbf{n}} \vec{PS} \\ &= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right| \\ &= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}.\end{aligned}$$
■

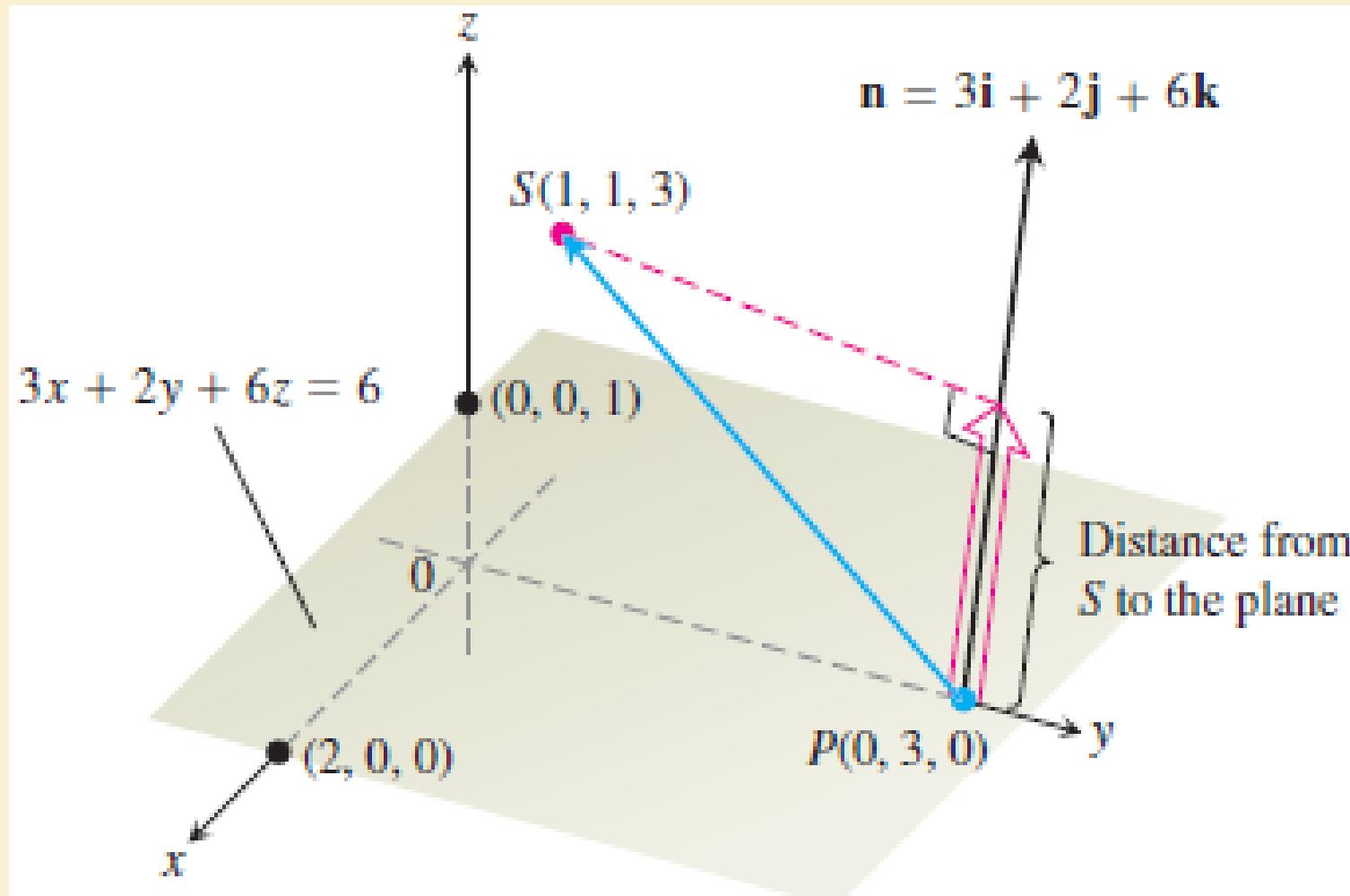


FIGURE 12.41 The distance from S to the plane is the length of the vector projection of \overrightarrow{PS} onto \mathbf{n} (Example 11).

12.5 Lines and Planes in Space

- Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure 12.42).

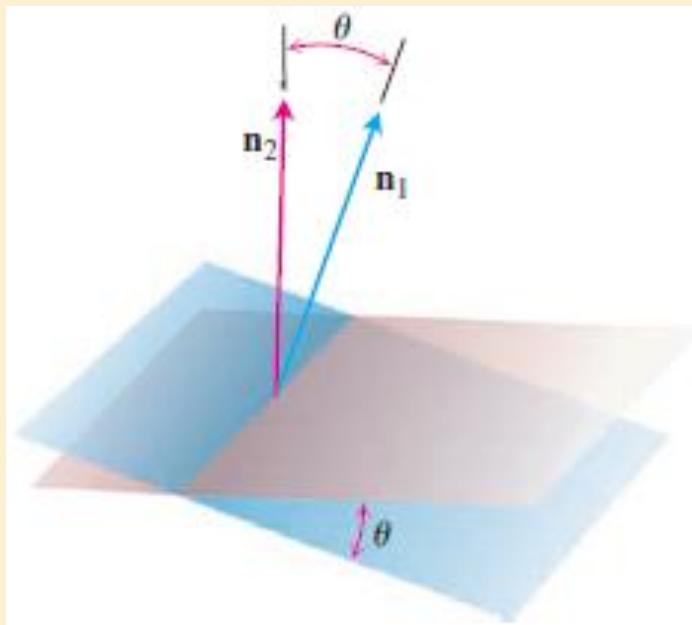


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

12.5 Lines and Planes in Space

- Angles Between Planes

EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) \\ &= \cos^{-1} \left(\frac{4}{21} \right)\end{aligned}$$

≈ 1.38 radians. About 79 deg

