Flocking-Birds
A 2-Dimensional Phase Transition

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Chapter 1

Abstract

This project consists of numerically modeling bird flocking (birds that fly in an organized group). This phenomenon is one example of a more general phenomenon occurring in biological systems known as swarm intelligence. The system is constituted by a population of agents which can move randomly without any central control structure dictating their behavior. Yet a simple set of rules governing the interaction between agents can lead to global organized structures. In this project, we implement a model of agents governed by simple rules, that exhibits the emergence of self-ordered motion. The model known as the Vicsek model [3] is our starting point.
We implement an algorithm to construct the flocking phenomenon and to optimize the different calculus by a space discretization. Self-organization appears after a symmetry breaking associated with a dynamical phase transition, which is identified and partially characterized. Finally, we explain the core of our work through the presentation of results proving a dynamical phase transition in dimension 2 -which is proven to be impossible for 2D-equilibrium system [1]- with critical exponents. We also introduce a model of energy-repulsion that allows us to enter more realistic regimes.

Figure 1.1: flock of starling birds

Contents

Chapter 2

Overview

In a system consisting of many similar units (flocks of birds) the interactions between the units can be simple (attraction/repulsion) or very complex. Under some conditions, transitions can occur during which the birds adopt a pattern of behavior almost completely determined by the collective effects due to the other units in the system. The main feature of phase transitions is that an individual units action is dominated by the influence of the others the unit behaves entirely differently from the way it would behave on its own. In this project, a group of birds randomly oriented will order themselves into an orderly flying flock. Understanding new phenomena (in our case, the transitions in systems of collectively moving units) is usually achieved by relating them to known ones: a more complex system is understood by analyzing its simpler variants. We show that the main features of transitions in this non-equilibrium system are insensitive to the details of the interactions between the objects in the systems

2.1 Statistical physics for flocking birds

2.1.1 Principles

Acknowledgement: This part is inspired by [1]
In a common sense, phase transition is a process, during which a system, consisting of a huge number of interacting particles, undergoes a transition from one phase to another, as a function of one or more external parameters. Phase transitions are defined by the change of one or more specific system variables, called order parameters. This name, order parameter, comes from the observation that phase transitions usually involve an abrupt change in a symmetry property of the system. Mathematically, this value is usually zero in one phase (in the disordered phase) and non-zero in the

other (which is the ordered phase). If the order parameter changes discontinuously during the phase transition, we talk about a first order transition. For example, waters volume changes like this (discontinuously) when it freezes to ice. In contrast, during second order phase transition the order parameter changes continuously, while its derivative with respect to the control parameter, is discontinuous.

2.1.2 Phase transition for flocking birds

In the case of flocking birds[1] the most naturally chosen order parameter is the average normalized velocity v_a -accordingly to [Vicsek]-

$$v_a = \frac{1}{Nv} \left| \sum_{i=1}^{N} \vec{v_i} \right| \tag{2.1}$$

where N is the total number of birds and v_0 is the average velocity (in absolute) of the birds present in the system. If the motion is disordered, the velocities of the individual birds point in random directions and vanishes, whereas for ordered motion the velocities all add up to a vector of absolute velocity close to Nv_0 (thus the order parameter for large N can vary from about zero to about 1).

Critical exponents

As for the definitions used in [1], phenomena associated with a continuous phase transition are often referred to as critical phenomena because of their connection to a critical point at which the phase transition occurs. (Critical, because here the system is extremely sensitive to small changes or perturbations.) Near to the critical point, the behavior of the quantities describing the system (e.g., pressure, density, heat capacity, etc.) are characterized by the so called critical exponents for equilibrium systems. There are reasons and arguments for thinking that the same patterns and critical exponents classification of equilibrium systems apply to the non equilibrium systems-flocking birds here.[Vicsek] For systems of flocking birds, when L goes to ∞, the corresponding equation is:

$$v_a = (\eta_c(\rho) - \eta)^{\beta} \tag{2.2}$$

$$v_a = (\rho - \rho_c(\eta))^{\delta} \tag{2.3}$$

where β and δ are the critical exponents, η the noise and ρ is the density.

[1] a non-equilibrium system !

2.2 Basic models

Modeling of flocks has mainly been considered recently. Perhaps the first widely-known flocking simulation was published by [3], who was primarily motivated by the investigation of the emergence of self-ordered motion in systems of particles with biologically motivated interaction. His model, that we use in this project, consists in a type of dynamics that points out clustering, transport and phase transition in a non-equilibrium system where the velocity of the birds and their position are determined by a simple rule and random fluctuations. The only rule of the model is at each time step a given particle driven with a constant absolute velocity assumes the average direction of motion of the particles in its neighborhood of radius r with some random perturbation added. We show using simulations that, in spite of its simplicity, this model results in a rich, realistic dynamics, including a kinetic phase transition from no transport to finite net transport through spontaneous symmetry breaking of the rotational symmetry.

In this work we consider a model introduced in [3] with a novel type of dynamics in order to investigate clustering, transport, and phase transition in non-equilibrium systems. The only rule of the model is at each time step a given particle driven with a constant absolute velocity assumes the average direction of motion of the particles in its neighborhood of radius r with some random perturbation added. Perturbations can be taken into account in various ways. We represent them by adding a random angle to the angle corresponding to the average direction of motion in the neighborhood of particle i. The angle of the direction of motion $\theta_i(t+\Delta t)$ at time t + Δt, is obtained from

$$< \theta_j(t) >_r = \arctan\left(\frac{< sin(\theta_j)_t >_r}{< cos(\theta_j)_t >_r}\right) + (1 - sign(< cos(\theta_j)_t >_r))\frac{\pi}{2} \tag{2.4}$$

$$\theta_i(t+\Delta t) = < \theta_j(t) >_r + \epsilon \tag{2.5}$$

where ϵ is a random variable uniformly chosen in $[\frac{-\eta}{2}, \frac{\eta}{2}]$

Chapter 3

Vicsek Model

3.1 Construction of the class

In this part, we code the Vicsek model. We would like to simulate the flock of N birds, for N between 100 and 1000. Birds are flocking in a square-shaped cell or size L with periodic boundary conditions. A bird is assimilated to a point particle, with a position $\vec{x_i}$ and a velocity $\vec{v_i}$. The absolute value of $\vec{v_i}$ is constant and equal to v : the birds have the same velocity in norm, they can only change their direction θ. In this model, at every time-step Δt, each bird is adapting its direction θ by copying its neighbours. Neighbours are the birds located in the disk of radius r. To do that, it assumes the average velocity of its neighbours with a random perturbation added η. At each time step, $\vec{x_i}$ and $\vec{v_i}$ are updated as follows :

$$\vec{x_i}(t + \Delta t) = \vec{x_i}(t) + \vec{v_i}\Delta t \tag{3.1}$$

$$\theta_i(t + \Delta t) =< \theta_j(t) >_r +\epsilon \tag{3.2}$$

where ϵ is a random variable uniformly chosen in $[\frac{-\eta}{2}, \frac{\eta}{2}]$ and θ is computed from the velocities as follows :

$$< \theta_j(t) >_r = \arctan\left(\frac{< sin(\theta_j)_t >_r}{< cos(\theta_j)_t >_r}\right) + (1 - sign(< cos(\theta_j)_t >_r))\frac{\pi}{2} \tag{3.3}$$

For a given bird, finding which birds are within a radius r and interact with it has a time-complexity of N which means that each time step has a time-complexity of N^2. In order to improve the complexity, we decided to divide the square cell of size L into smaller squares of size r. The L-sized cell is now divided into $N_{cells} = \frac{L^2}{r^2}$ smaller squares. Each small square is indexed with a couple of integers (i,j). At each time-step, we can compute in which square (i,j) each bird is located by this simple formula :

$$i, j = \lfloor \frac{x_i}{r} \rfloor, \lfloor \frac{y_i}{r} \rfloor \tag{3.4}$$

Thus, we know that birds interacting with a bird in (i,j) are either in the same square (i,j) or in the eight squares around. We only have to check if the birds located in 9 squares out of N_{cells} are indeed closer than r. For $L = 10$ and $r = 1$, the number of birds we have to check is approximately reduced by a factor 10. To represent the configuration of the flock at a given time-step, we created the class Flock and its attributes are :

- the number of birds N,
- the size of the square cell L,
- the noise η,
- the absolute value of the velocities v,
- the interaction radius r,
- the time-step Δt,
- the positions stored in an array (N,2),
- the velocities stored in an array (N,2),
- the number of squares on a side $\frac{L}{r}$
- a matrix called checkering, whose element i,j is a list containing the birds located in the square (i,j).

To construct an instance of the class Flock, the constructor takes as arguments N, L, η, v, Δt, $pos0$ and $vel0$ the initial positions and velocities of the system, that are mostly chosen uniformly distributed. The constructor uses the method localize-birds in order to initialize the matrix checkering.

The method compute-new-velocity(k) takes the bird number k as an argument, and compute the new velocity it will have after the time-step. In order to assume the average direction of the birds within a radius r, we need to check if birds in the 9 cells around are indeed closer than r which is not that simple in regards to the periodic

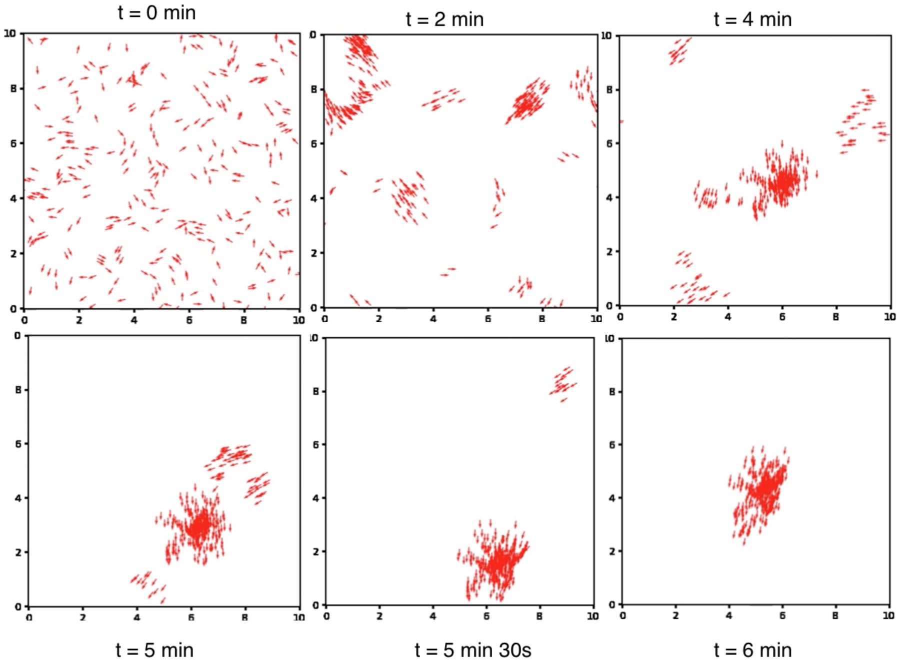

Figure 3.1: Animations for t=0min,t=2min,t=4min,t=5min,t=5min30,t=6min

boundary condition. Therefore, the function dist($\vec{x_1}$,$\vec{x_2}$,L) has been created to compute the distance between 2 points $\vec{x_1}$ and $\vec{x_2}$ with respect to the periodic boundary conditions.
The method make-step() makes the flock evolve by a time-step Δt. For each bird k, it computes the new velocity $v_i^{\vec{new}}$ and stores it in a temporary array. Then, it updates the positions $\vec{x_i}$, the velocities $\vec{v_i}$ and checkering using the method localize-birds().

3.2 Animations

To visualize the flock of birds at a given time-step, we chose to represent a bird by an arrow representing its velocity. Then we obtained an animation with the function make-frame(t). For the animation (Fig 3.1), we chose as parameters $N = 300$,$L = 10$,$\eta = 0,5$,$\Delta t = 0,1$, $v = 1$, $r = 0,5$.

We see that, quite rapidly, birds are gathering into clusters each one having an average direction. Then, when a cluster meets another one, they merge into one. In

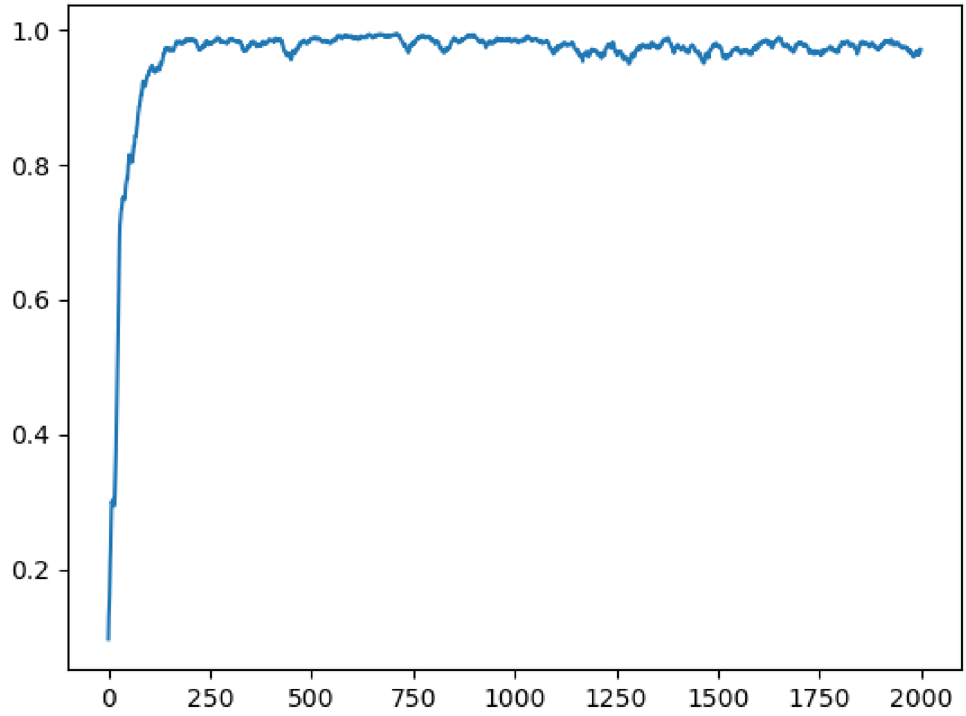

Figure 3.2: v_a versus t

the end, there is only one cluster left which gathers the N birds all moving in the same direction.

To study to what extent the birds tend to flock in the same spontaneously selected direction, we chose to introduce v_a the normalized average velocity :

$$v_a = \frac{1}{Nv} | \sum_{i=1}^{N} \vec{v_i} | \tag{3.5}$$

3.3 Data collection techniques

The class Flock has a method called compute-avr-norm-velocity() that compute v_a at given time-step.

The time it takes for the flock to get an average velocity and to stabilize depends on N, L, r and η. For typical values such as $N = 300$, $L = 10$, $r = 1$, $\Delta t = 1$, $v = 0.1$, $eta = 1$, 150 time-step are enough for the flock to stabilize (Fig 3.2).

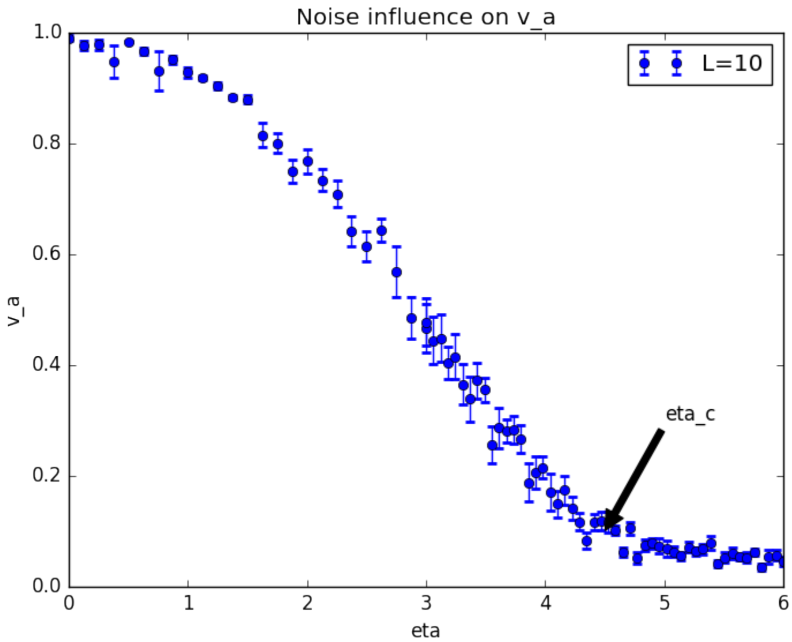

Figure 3.3: The normalized average velocity of the flock v_a after 150 steps versus the noise η. Here, $N = 300$, $L = 10$, $r = 1$, $\Delta t = 1$, $v = 0.1$. η_c can be estimated to 4.3 according to the curve.

3.4 Results

We chose to analyze v_a dependency on the noise η and the density $\rho = \frac{N}{L^2}$ through the variation of L or N.

3.4.1 Noise influence

we chose different values of η in the interval $(0, 6)$, we launched 10 independent simulations (the runs are computed with different initial conditions) for each η and measured the value of v_a. We computed the average and the standard deviation from this 10 runs and we plot it on a graph (Fig 3.3). When the noise η is close to 0, the birds are all flocking in the same direction and v_a is almost equal to 1. When the noise is increased, different birds are still flocking in the same direction (and form different clusters) and the order parameter v_a is positive. When η is above a critical value η_c, the absolute average velocity vanishes and is very close to zero (it would be equal to zero for an infinite system).

In order to determine β, we plotted ln v_a as a function of $\ln(\frac{[\eta_c - \eta]}{\eta_c})$ (Fig3.4).We find that $\beta = 0.68$, which is not expected in theory for an equilibrium system.

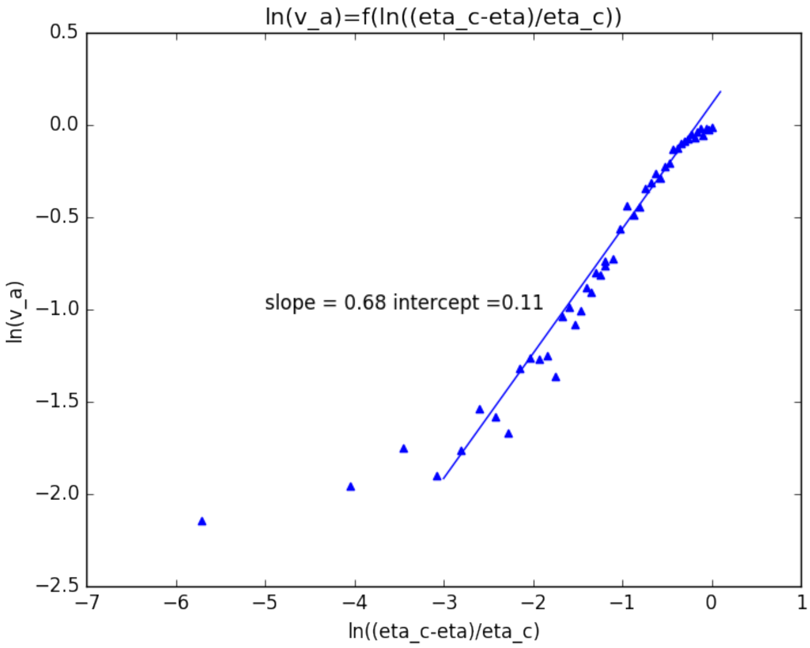

Figure 3.4: Here, $N = 300$, $L = 10$, $r = 1$, $\Delta t = 1$, $v = 0.1$. η_c was estimated to 4.3 according to FIG 3.3.

The curve v_a versus η depends on other parameters, such as the density $\frac{N}{L^2}$. We repeated what we did before for different values of L, the number of birds N being fixed. FIG3.5 shows that the higher is the density, the more ordered is the motion. The best curves (with the lowest standard deviation) are obtained for the highest densities. A flock with a low density takes more time to get ordered, and for $L = 20$ and $L = 15$, 150 time-steps are not enough. One can also notice that η_c depends on the density.

For each curve of FIG3.5, we estimated the value of η_c and we reported it on a graph (FIG3.6). It shows that η_c increases with the density.

η_c function of density

3.4.2 Density

We gradually increased the values of the density ρ and observe a phase transition from a disordered phase with birds moving independently to an ordered phase with the emergence of a collective motion for the different birds. Thus we see [INSERT FIGURE] that for small values of the density ρ, the average velocity v_a vanishes and after a critical value ρ_c it takes positive values and increases. For high values of the density v_a is almost equal to 1. See FIG3.7

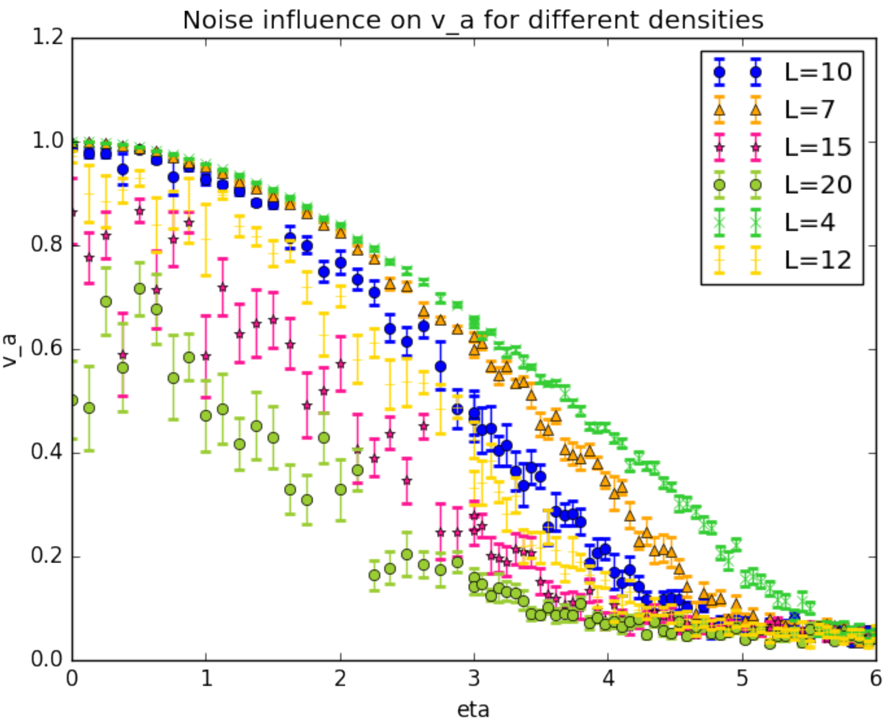

Figure 3.5: Here, $N = 300$, $L = 10$, $r = 1$, $\Delta t = 1$, $v = 0.1$. η_c was estimated to 4.3 according to Fig 3.3.

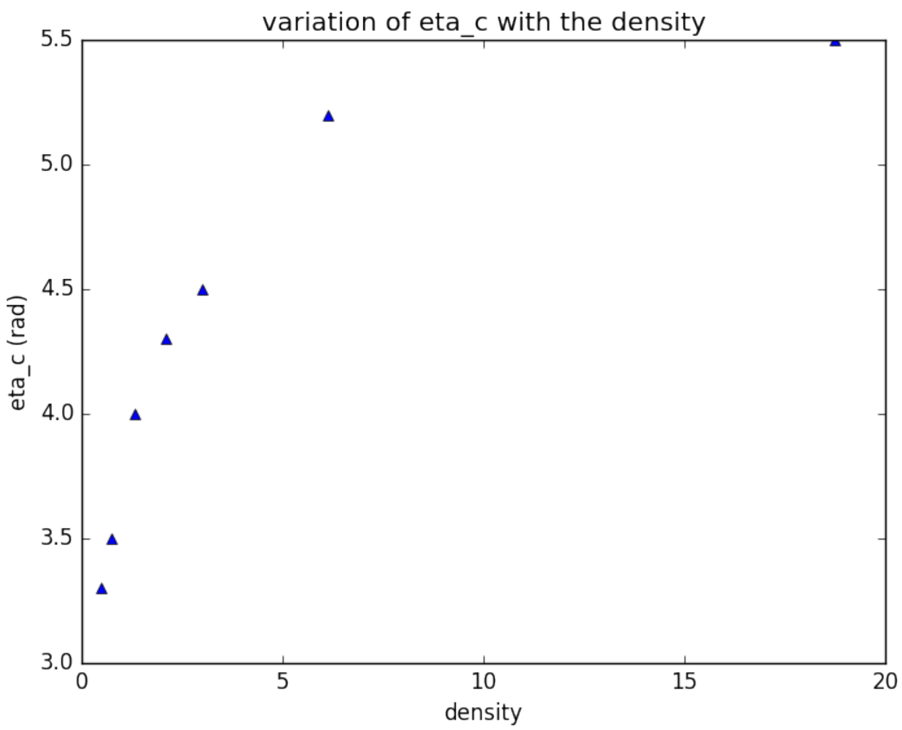

Figure 3.6: variation of η_c with the density

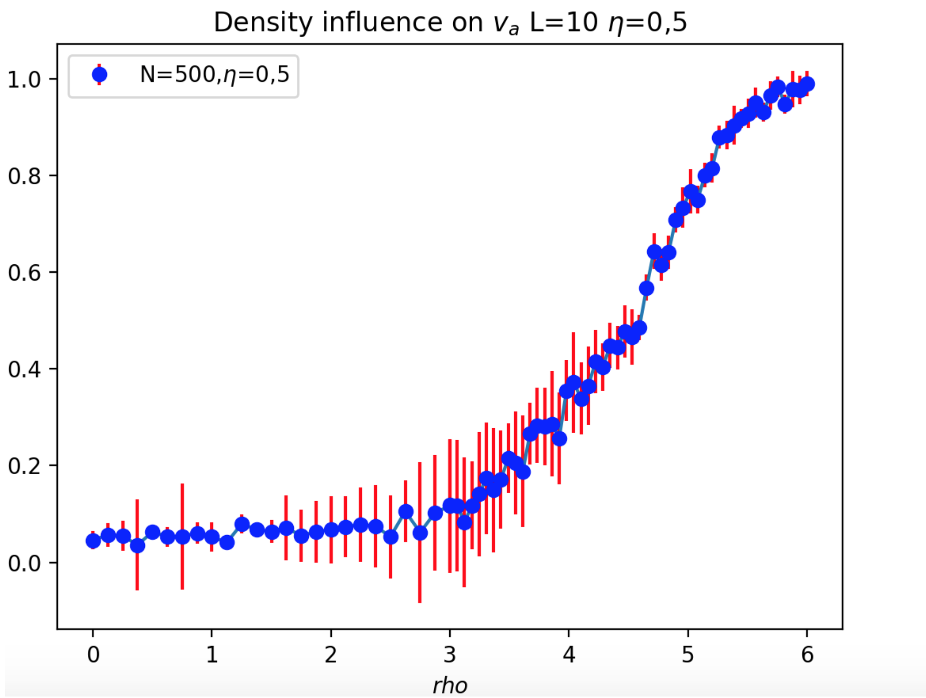

Figure 3.7: The normalized average velocity of the flock v_a after 150 steps versus the density ρ. Here, $N = 500$ and we change the values of L in order to change ρ η_c can be estimated to 4.3 according to the curve

3.5 Model with energy-repulsion

In the Vicsek model there is an explicit alignment rule that brings all the neighbors to finally move in the same direction. The energy-repulsion model is a particular case of an implicit alignment rule where the birds have not to be very closed and also try to minimize their energy H($\{\vec{s_i}\}, \{\vec{v_i}\}$).

$$H = -\frac{1}{2}\sum_{i,j} J_{ij}\vec{s_i}.\vec{s_j} + V(\vec{x_i}) \tag{3.6}$$

where :

- $\vec{s_i}$ are the normalized velocity of bird i : $\vec{v_i} = v\vec{s_i}$
- $J_{i,j}$ is a coupling constant and is equal to 1 in a disk of radius r, 0 outside
- V is the energy due to the repulsion : $V(\{\vec{x_i}\}) = \frac{1}{2}\sum_{i,j} V(\|\vec{x_i} - \vec{x_j}\|)$

In this model, birds are still flocking in a square cell of size L with periodic boundary condition. At each time-step Δt, a bird k is sorted randomly and a new velocity $v_k^{\vec{new}}$ randomly chosen near its current velocity $\vec{v_k}$ is suggested (it deviates by an angle θ sorted in $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$). We compute the variation of energy ΔE induced by this change as follows :

$$\Delta E = \frac{1}{2}\sum_j J_{ij}\vec{s_k}.\vec{s_j} - \frac{1}{2}\sum_j J_{kj}v_k^{\vec{new}}.\vec{s_j}$$

$$-\frac{1}{2}\sum_j V(\vec{x_k} + \vec{v_k}\Delta t, \vec{x_j} + \vec{v_j}\Delta t) + \frac{1}{2}\sum_j V(\vec{x_k} + v_k^{\vec{new}}\Delta t, \vec{x_j} + \vec{v_j}\Delta t)$$

Then, we sort A a uniform variable in [0,1] and the bird accepts the new velocity $v_k^{\vec{new}}$ if $A < arctan(-\beta\Delta E)$. Here, β plays the role of temperature. It means that the more $v_k^{\vec{new}}$ contributes to decrease the energy of the flock, the more it has chance to be accepted.

To represent the flock at given time-step, we create a class just like in the Vicsek's model, but it has some extra instances :

- α
- β

- the repulsive potential V

The method step() makes the flock evolve between two time-steps.

We made an animation with the function update-quiver(t).

This model does not work very well : the flock does not get ordered at all. It can be explained by the fact that birds are changing their velocity every $N\Delta t$ in average, that is to say, very rarely. When they have a velocity, they keep it a long time even though they have completely changed their environment.

A solution could be to have two different time-steps : Δt and δt. Δt would be the time between two changes of velocity for a given bird in average. It is the duration on which the bird estimates ΔE. A bird is sorted every δt. Δt and δt are linked by the formula :

$$\Delta t = N\delta t \tag{3.7}$$

$$v\Delta t \sim r \tag{3.8}$$

This idea is experimented in Energy-model2.ipynb.

Conclusion

The results we have presented support a deep analogy with equilibrium statistical physics. The essential deviation from equilibrium is manifested in the collision rule: since the absolute velocity of the particles is preserved and in most cases an alignment of the direction of motion after interaction is preferred, the total momentum increases both during individual collisions and, as a result, gradually in the whole system of particles as well.
So, is Vicsek's model satisfactory to describe how a flock of birds spontaneously gets direction of motion ? At first sight it is, because the flock of birds indeed becomes ordered. What is quite surprising is that birds spontaneously come very close to each other when they gather in a cluster, despite the lack of any attractive interaction between them. It is even a bit disappointing, because real birds cannot occupy the same place and must keep a reasonable distance between them in order not to be limited in their motion. If in our model birds could not get that close, the average velocity v_a would be lower and we would see more fluctuations in the motion. In Vicsek's model, birds end up so close that they are all interacting together. Secondly, our simulation does not really look like a real flock of starlings, permanently changing its average direction when fluttering in the air. In real flock, birds on the edge of the flock or birds inside the flock may decide to change their direction and induce a change of the direction of the flock. In our model, where there is no leader, when the flock has selected a direction, it does not change anymore. It could be interesting to introduce in the flock a few birds that have independent motion and lead other birds that are interacting with them.

Acknowledgements

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List of Figures