Rayleigh Benard Convection*

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1 Introduction

The phenomenon we are interested in modelling is the so called "Rayleigh Benard Convection". We consider a layer of fluid heated from below in a motionless state. The fluid on the bottom will be lighter than the fluid on the top therefore it will have the tendency to rise to the top of the layer. The gravity and the viscosity of the fluid itself will contrast such motion but, when the temperature gradient is higher than a certain threshold, we will be able to observe convective motions in the fluid. We will observe that indeed is not the temperature gradient to determine the insurgence of convective motions but a critical a-dimensional parameter called "Rayleigh number" which links: the temperature gradient, the height of the layer, the kinematic viscosity coefficient, the thermal diffusivity coefficient and the volumetric coefficient of thermal expansion.

2 Modelling the onset of convection in a fluid heated from below

The modelling framework for our convection problem is the Navier-Stokes model for Newtonian, viscous, homogeneous and incompressible fluids. The evolution over time of the velocity field $\mathbf{v}(\mathbf{x}, t)$ and the pressure

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field $\mathbf{p}(\mathbf{x}, t)$ of an isotherm fluid is given by the well known Navier-Stokes equations:

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \mathbf{v}_t + (\nabla \mathbf{v})\mathbf{v} = -\frac{1}{\rho}\nabla \rho + \nu \Delta \mathbf{v} + \mathbf{f} \end{cases} \tag{1}$$

In order to study the impact of heating on the layer of fluid we combine the Navier-Stokes equations with the conservation of energy equation under the assumption that the sole external force interacting in our problem is weight and it appears in the momentum equation, while in the Energy equation no external force is considered. Under some assumptions we will be able to simplify the problem in order to find an analytical solution for the linearised system and the numerical solution for the original one.

2.1 Oberbeck- Bussinesq approximation

The well known Oberbeck- Bussinesq approximation allows us to simplify the system of Navier-Stokes equations under the following assumptions:

- the only force applied to the fluid is gravity;
- velocity gradients are small enough to allow us to neglect their impact on temperature;
- density variations are due to temperature variations but not to pressure (no compression);
- all transport coefficients are assumed constant and positive;
- all fluid accelerations are assumed to be much smaller than gravity.

Given the third assumption we end up with the following relation linking density variations to the sole impact of temperature variations:

$$\rho - \rho_0 = -\rho_0 \alpha (T - T_0)$$

where α is the volumetric coefficient of thermal expansion and ρ_0 is the reference density of the fluid at the reference temperature T_0. Assuming that the fluid is slightly compressible allows us to treat the density as constant everywhere except for the term related to external forces, in our case the sole gravity force. We also assume that the fluid is originally in a state of motionless equilibrium such that the unperturbed temperature should be a linear function of the vertical coordinate z: $T_s = T_1 - \beta z$ where $\beta = \nabla T_s$ and T_1 is the temperature at the bottom layer.

Given the original unperturbed state of the fluid, under the above assumptions, we rewrite the system of equations for the perturbation quantities $\mathbf{v}$,p' and θ respectively of velocity, pressure and temperature as follows:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p'}{\rho_0} - \mathbf{g}\alpha\theta + \nu \Delta \mathbf{v} \\ \frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{v}(T_s + \theta) = \chi \Delta \theta \\ \nabla \cdot \mathbf{v} = 0 \end{cases} \tag{2}$$

where χ is the coefficient of thermal diffusivity and ν the coefficient of kinematic viscosity.

The first equation is the momentum equation; we observe that the variations in velocity are caused by variations in pressure, by the gravity force through its impact on density, and heat dispersion. The second equation is the energy balance equation and the last one is the mass equation which assumes this specific formulation under the assumption of incompressible fluid.

Finally, we should make some considerations upon the validity of the Oberbeck-Bussinesq approximation. In general, we can say that the approximation is valid if:

$$h \ll \min\{D_\rho, D_p, D_T\}$$

where h is the thickness of the fluid layer and the quantity D_f is given by:

$$D_f = \left| \frac{1}{f_o} \frac{\mathrm{d} f_s}{\mathrm{d} z} \right|^{-1}$$

2.2 The Rayleigh-Benard Problem

We consider a layer of incompressible fluid of height h such that the z coordinate is $0 \leq z \leq h$ and it is oriented in such way that the gravity acceleration is parallel to the z axis. As mentioned in the introduction, we assume to uniformly heat from below the fluid layer in such way that the temperature of the layer top and bottom boundaries are:

$$\begin{cases} T = T_1 & z = 0 \\ T = T_1 - \Delta T = T_1 - \beta h & z = h \end{cases} \tag{3}$$

We assume the temperature at the boundaries to be fixed and thus $\theta = 0$ on the boundaries. For what concerns the boundary conditions for the velocity field we face multiple options: we can assume rigid or free boundaries. In the first case we assume the *no-slip* condition while in the second case we assume the *stress-free* condition:

$$\begin{cases} \mathbf{v} = 0 & \text{on a rigid boundary} \\ v_z = 0, \frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = 0 & \text{on a free boundary} \end{cases} \tag{4}$$

We can consider both cases of rigid- rigid, free-free boundaries as well as hybrid rigid-free boundaries.

Given the formulation of the problem we want to study its stability. We know that there exist a state of motionless equilibrium which can be easily found by solving the system assuming: $\mathbf{v}_s = \mathbf{0}$, $T_s = T_s(z)$, $p_s = ps_s(z)$.

$$\begin{cases} \mathbf{v}_s = \mathbf{0} \\ T_s(z) = T_1 - \beta z \\ p_s(z) = -\rho_0 g(1 - \alpha(T_1 - T_0))z - \frac{\rho_0 g \alpha \beta}{2} z^2 \end{cases} \tag{5}$$

In order to study the stability of such equilibrium state it is useful to rescale the problem in order to deal with non dimensional variables. We take as length unit the height of the layer, as temperature unit ΔT and finally as time unit the time of vertical diffusion of heat $\tau_v = \frac{h^2}{\chi}$. Therefore the system can be written in the following non-dimensional form:

$$\begin{cases} \frac{1}{P}\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = -\nabla\bar{\omega} - \hat{\mathbf{z}}R\theta + \Delta\mathbf{v} \\ \frac{\partial \theta}{\partial t} - v_z + \mathbf{v} \cdot \nabla\theta = \Delta\theta \\ \nabla \cdot \mathbf{v} = 0 \end{cases} \tag{6}$$

where:

$$R = \frac{\alpha g \Delta T h^3}{\nu\chi} \quad \text{and} \quad P = \frac{\nu}{\chi}$$

are respectively the Rayleigh and Prandtl number, $\hat{\mathbf{z}}$ is the unit vector in the z-direction and $\bar{\omega}$ is a non dimensional form of the quantity $\frac{p'}{\rho_0}$.

The Prandtl number is the ratio of the time scales of the two diffusive processes involved in convection, heat diffusion and momentum diffusion. The Rayleigh number is proportional to the temperature difference across the fluid layer, and relates the strength of the driving mechanism to dissipative processes. It is the control parameter in a convection experiment.

2.3 Bifurcation analysis

The analysis of the non-linear regimes of the Rayleigh-Benard problem brings us to face a typical example of supercritical pitchfork bifurcation, which indeed often characterises convection problems. The Rayleigh number constitutes the discriminant between the so called subcritical and supercritical regimes.

The system is characterised by an initial state of unperturbed motionless equilibrium which is stable as long as the Rayleigh number is below a critical threshold. As soon as the Rayleigh number increases above its critical value the system converges towards three new possible regimes: an unstable motionless state and two new supercritical regimes consisting in the presence of vortices within the fluid. The two new regimes present the same structure and differentiate only by the sign of the velocity and of the temperature perturbation.

We can find the critical values of the Rayleigh number through the linearisation of the system of the study of its stability; it is important to highlight that different boundary conditions will result in different Rayleigh numbers.

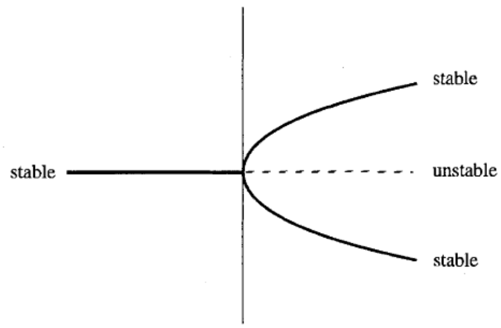

Figure 1: Typical representation of a supercritical pitchfork bifurcation

3 Linear stability analysis and Critical Rayleigh number

The linearized equations for Benard convection should satisfy the principle of exchange of stabilities even when the fluid viscosity depends analytically on temperature and pressure. To compute it we do as follow:

We assume v and θ to be infinitesimal, linearize Equations (6) with respect to these variables. So we can easily eliminate the pressure P and the dependent variables v by applying the operator $\vec{rot}(\vec{rot})$ (well-known formula that is for example used to establish the equation driving an electromagnetic field combining the Maxwell-Faraday and the Maxwell-Ampere equations) to the first Equation (6) , and make use of the second (6). The system then reduces to two equations for v_z and θ. On eliminating θ, we fix a horizontal wavevector $\vec{k}$ =$\{k_x, k_y, 0\}$. Since the coefficients in after taking the $\vec{rot}(\vec{rot})$ equations depend only on z, the equations admit solutions which depend on x, y and t exponentially. We consider therefore solutions of the form (in the form of normal modes) v_z:

$$v_z \approx \exp(\lambda t) w(x) f(z) \tag{7}$$

Here λ is the growth rate, $x = \{x, y, 0\}$, The λ may be complex, i.e. $\lambda = \lambda_r + i\lambda_i$. Such a wave is stable if $\lambda_r < 0$, unstable if $\lambda_r > 0$, and neutrally stable if $\lambda_r = 0$ and $w(x)$ is a spatially periodic solution of the two-dimensional equation $\Delta w + k^2 w = 0$. We can represent it as a linear combination:

$$w(x) = \sum_{j=-N, j\neq 0}^{N} c_j \exp(ik_j x) \tag{8}$$

in which it is understood that the real parts of these expressions must be taken to obtain physical quantities and where the vectors k_j differ only in their orientation: $\|k_j\|$ = k; in addition, $k_-j = -k_j$ and $c_-j = c_j^*$ (the asterisk denotes complex conjugation; the last two equalities are necessary for w to be real). As a result, we obtain the following equation for f:

$$(D^2 - k^2 - \lambda)(D^2 - k^2 - \frac{1}{P}\lambda)(D^2 - k^2) f = -Rk^2 f \tag{9}$$

where $D = d/dz$ and

$$k^2 = k_x^2 + k_y^2 \tag{10}$$

with the new equations (6) and expanding with the normal modes we can reduce Equations (2.2) to a set of conditions for the variable v_z(or f) :

$$f = Df = (D^2 - 2k^2 - \frac{1}{P}\lambda)D^2 f = 0 \tag{11}$$

on a rigid boundary

$$f = D^2 f = D^4 f = 0 \tag{12}$$

on a free boundary.
The previous equation together with the boundary conditions constitute an eigenvalue problem for the growth rates λ and the eigenfunctions $f(z)$. If both layer boundaries are stress-free, this problem can be solved extremely simply [Getling] and leads to the following explicit expression for the eigenvalues A_n corresponding to the eigenfunctions $f_n = \sin(n\pi z)$ $(n = 1, 2, ...)$:

$$\lambda_n = \frac{-P-1}{2}(n^2\pi^2 + k^2) \pm \sqrt{(\frac{P-1}{2})^2(n^2\pi^2 + k^2)^2 + \frac{RPk^2}{n^2\pi^2 + k^2}} \tag{13}$$

It can be immediately seen from this expression that for any R ¿ 0 both existing values of $\lambda_n(P, k, R)$ are real. One of them is always negative while the other one is positive if

$$R > R_n(k) = \frac{(n^2\pi^2 + k^2)^3}{k^2} \tag{14}$$

and negative if $R < R_n(k)$.

The following analysis is taken from [Getling]:
"If both layer surfaces are rigid or one surface is rigid while the other one is free, the calculations are more complicated, but the results are qualitatively the same (the eigenfunctions being different). In the case $R < 0$, if $\|R\|$ exceeds a certain value [depending on P and reaching its maximum (zero) at $P = 1$], the growth rate λ_n has two conjugate complex values. Then the corresponding eigenfunctions describe decaying oscillations which are associated with internal gravity waves. We see that infinitesimal perturbations with a given wavenumber k can grow (i.e., instability is possible) only provided that $R > 0$, and their growth is monotonic. As Reλ_n -the maximum of the real parts of the growth rates ?passes through zero, increasing with R, the corresponding imaginary part also becomes zero. The linear analysis thus indicates that convection sets in at a certain R as steady motion. In other words, a new steady state replaces the stable motionless state of the fluid. This property of Rayleigh-Benard convection is called the principle of exchange of stabilities. It can be shown that the validity of this principle, as well as other above-listed properties of λ_n, does not depend upon the boundary conditions.
Each function $R_n()k$ has a minimum. The line R = Rik) in the plane (k, R) delimits the region where all infinitesimal perturbations decay and the region where the lowest perturbation mode n = 1 grows (see Fig). Obviously, if the motionless state of the fluid in the layer is stable with respect to infinitesimal perturbations; thee quantities R_c and k_c are termed, respectively, the critical Rayleigh number and the critical wavenumber. The critical (neutral) regime ($R = R_c$) corresponds to the onset of steady-state motion with an infinitesimal amplitude and with a unique wavenumber $k = k_c$. If $R > R_c$ (supercritical regime), the layer is convectively unstable, and those perturbations can grow which have wavenumbers lying between the two roots of the equation $R = R(k)$."

Results: For two stress-free boundaries:

$$R_c = \frac{27}{4}\pi^4 = 657.511; k_c = \frac{\pi}{\sqrt{2}} = 2,2221 \tag{15}$$

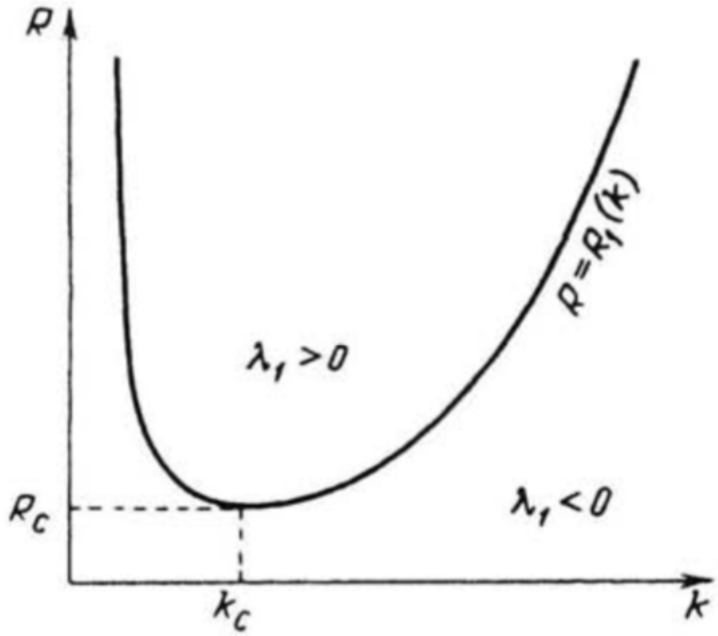

Figure 2: The neutral-stability curve for a layer of motionless fluid. The region of stable states is below the curve and the region of unstable states is above the curve

for two rigid boundaries:

$$R_c = 1707.702; k_c = 3.117 \tag{16}$$

for one rigid and one stress-free condition:

$$R_c = 1100.657; k_c = 2.682 \tag{17}$$

4 Numerical simulation of the onset of convection

In order to replicate numerically the analytical results we consider the original Bussinesq system:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p'}{\rho_0} - \mathbf{g}\alpha\theta + \nu\Delta\mathbf{v} \\ \frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{v}(T_s + \theta) = \chi\Delta\theta \\ \nabla \cdot \mathbf{v} = 0 \end{cases} \tag{18}$$

The integration of the above system of partial differential equations require both a discretisation in time and in space.

4.1 Discretization in space

First, we need to remember that it is impossible to reproduce numerically a layer of fluid whose extension is infinite along the x and y dimensions. Therefore the first level of approximation concerns the limits of the layer along such directions and the relative boundary conditions. We assume a finite layer of fluid, whose extension along the x and y axis is considerably larger (six times larger) compared to the height of the layer.

For simplicity we consider the bi-dimensional case. The layer is split into N rectangles which will determine the points in which we will perform the computation. We use the finite difference method to compute the derivatives with respect to space in each of the small squares resulting from the discretization of the domain. In this way we are able to write all differential operators in the form of matrices.

4.2 Discretization in time

We divide the temporal interval $[0, T]$ in N subintervals of length Δt.Given the suggested discretization we are left with the choice of the numerical scheme of integration. We can choose between explicit Euler, Implicit Euler or other semi-implicit schemes. The usual pros and cons of the suggested schemes apply also in this case: implicit Euler is a better scheme for the conservation of energy but it is more costly from a computational point of view.

4.3 Integration algorithm

One of the main problems we have to face is whether we should use the partial derivatives with respect to space evaluated at time $n + 1$ or at time n. The main assumption that it is made is that the temperature is more dependent on the velocity than the velocity is on the temperature. This allows us to avoid the simultaneity of the equations: we compute at each time step independently the new velocity field and then apply the new velocity field to the transport equation of the temperature. Therefore after computing the velocity field at time $t + 1$ we plug this value into the equation for the temperature and compute the new value of temperature itself. A good way to control that the integration proceeds effectively is to use the divergence as control variable: in order for us to be satisfied with the solution the divergence at time $n + 1$ must remain within a restricted window of very small values.

4.4 Numerical results

By performing the numerical integration of the Bussinesq equations we were able to observe the setting of the different regimes consistently with the relative position with respect to the critical Rayleigh number.

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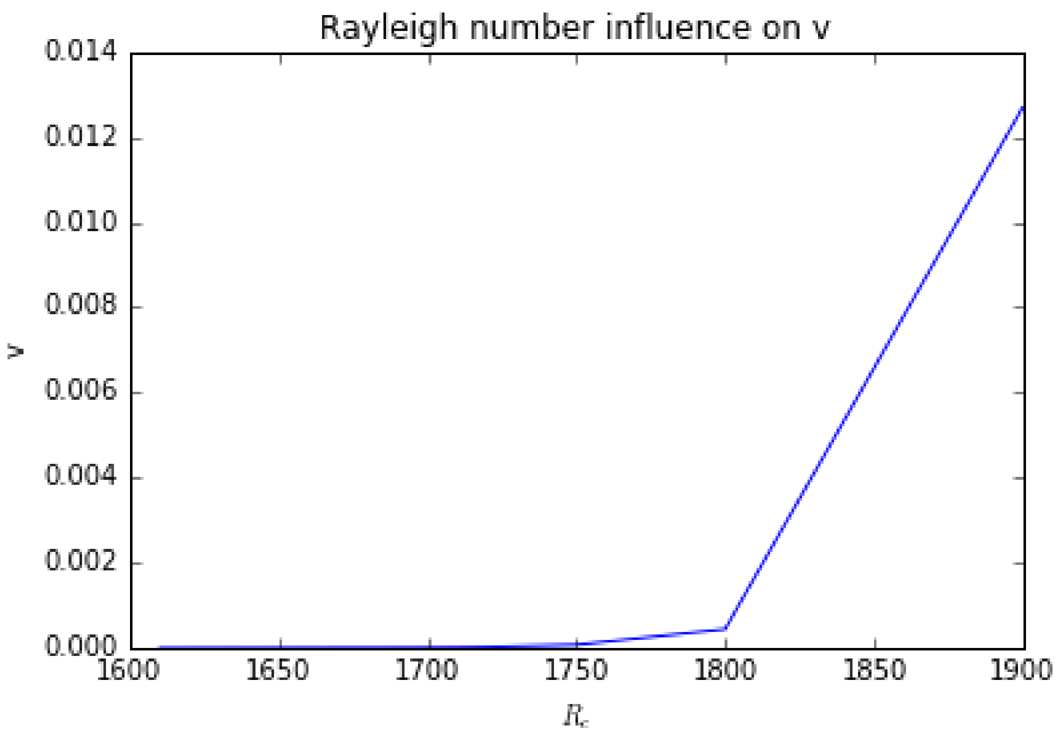

Figure 3: 2 rigid boundaries. We see that the numerical critical Rayleigh number is near the theoretical one

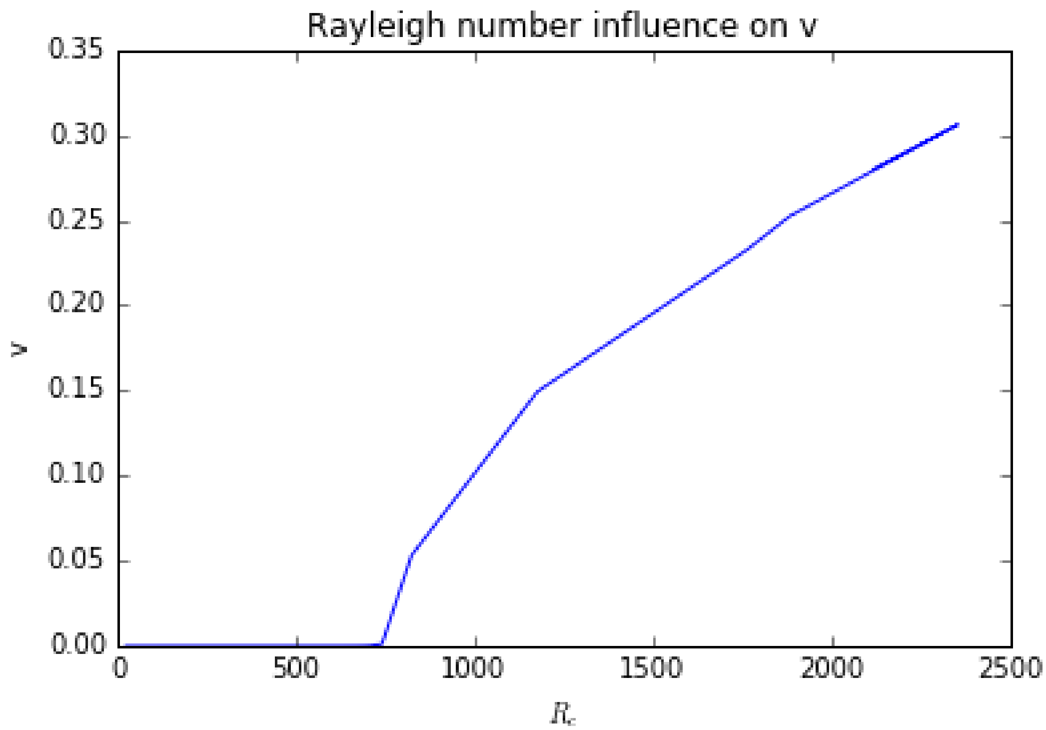

Figure 4: 2 stress free boundaries. We see that the numerical critical Rayleigh number is near the theoretical one