alternately from the above expression) that $y(n)$ is third-order ergodic iff $S_1 = S_2$. This is a slight generalization of the result given in [1]. Moreover, by making use of Assumption 2, a straightforward computation shows that the variance of the estimate is given by

$$\begin{aligned}
&\mathrm{Var}\{C_y^{(1,N)}(\tau,\rho)\} \\
&= 2 \sum_{\substack{(j,k,i)\in S_1\setminus S_2 \\ j<k}} (B_iB_jB_k)^2(1+\cos\{(\Omega_j-\Omega_k)(\tau-\rho)\}) \\
&\quad + \sum_{(j,j,i)\in S_1\setminus S_2} (B_iB_j^2)^2 \\
&\quad + 2 \sum_{\substack{(j,k,i)\notin S_1 \\ j<k}} \left[(1+\cos\{(\Omega_j-\Omega_k)(\tau-\rho)\})(B_iB_jB_k)^2\right. \\
&\quad \left. \cdot \frac{\mathrm{Sin}^2[N(\Omega_j+\Omega_k-\Omega_i)/2]}{N^2\,\mathrm{Sin}^2[(\Omega_j+\Omega_k-\Omega_i)/2]}\right] \\
&\quad + \sum_{(j,j,i)\notin S_1} \left[(B_iB_j^2)^2 \frac{\mathrm{Sin}^2[N(2\Omega_j-\Omega_i)/2]}{N^2\,\mathrm{Sin}^2[(2\Omega_j-\Omega_i)/2]}\right] \qquad (15)
\end{aligned}$$

from which we get the limiting variance as

$$\begin{aligned}
&\lim_{N\to\infty} \mathrm{Var}\{C_y^{(1,N)}(\tau,\rho)\} \\
&= 2 \sum_{\substack{(j,k,i)\in S_1\setminus S_2 \\ j<k}} (B_iB_jB_k)^2(1+\cos(\{(\Omega_j-\Omega_k)(\tau-\rho)\}) \\
&\quad + \sum_{(j,j,i)\in S_1\setminus S_2} (B_iB_j^2)^2. \qquad (16)
\end{aligned}$$

From (15) and (16), we can infer the following. The limiting variance given by (16) is the "nonergodic part" of the variance of the finite sample estimate $C_y^{(1,N)}(\tau,\rho)$. This limiting variance comprises the first two terms in (15). It does not vary with N and vanishes iff. $S_1 = S_2$, i.e., iff $y(n)$ is third-order ergodic. The other two terms in (15) vary as $1/N^2$ and thus do not contribute to the limiting variance. They comprise the "ergodic part" of the variance. As mentioned in [1], in case $y(n)$ is not third-order ergodic, an averaging over independent realizations (records) in addition to time averaging is required to obtain consistent estimates of $C_y(\tau,\rho)$. In other words, we deal with the estimates $C_y^{(K,N)}(\tau,\rho)$. The nonergodic part of the variance then decreases as $1/K$ while the ergodic part as $1/KN^2$.

Acknowledgment

The authors wish to thank two anonymous reviewers for their valuable suggestions, especially one of the reviewers who pointed out the possibility of generalizing the results to include nonzero biphase and random amplitudes.

References

[1] M. R. Raghuveer and C. L. Nikias, "Bispectrum estimation: A parametric approach," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 33, pp. 1213–1230, Oct. 1985.

[2] ——, "Bispectrum estimation: A digital signal processing framework," *Proc. IEEE*, vol. 75, pp. 869–891, July 1987.

[3] P. J. Huber, B. Kleiner, T. Gasser, and G. Dumermuth, "Statistical methods for investigating phase relations in stationary stochastic processes," *IEEE Trans. Audio Electroacoust.*, vol. 2, pp. 78–86, Mar. 1971.

[4] Y. C. Kim, J. M. Beall, E. J. Powers, and R. W. Miksad, "Bispectrum and nonlinear wave coupling," *Phys. Fluids*, vol. 23, pp. 258–263, Feb. 1980.

[5] R. Brillinger and M. Rosenblatt, "Asymptotic theory of estimates of kth order spectra," in B. Harris, Ed., *Spectral Analysis of Time Series*. New York: Wiley, 1967, pp. 153–188.

[6] ——, "Computation and interpretation of kth order spectra," in B. Harris, Ed., *Spectral Analysis of Time Series*. New York: Wiley, 1967, pp. 189–232.

[7] V. Chandran and S. L. Elgar, "Mean and variance of estimates of the bispectrum of a harmonic random process—An analysis including leakage effects," *IEEE Trans. Signal Processing*, vol. 39, pp. 2640–2651, Dec. 1991.

A Method for Time-Frequency Analysis

Ljubiša Stanković

***Abstract*—A method for time-frequency signal analysis is presented. The proposed method belongs to the general class of smoothed pseudo Wigner distributions. It is derived from the analysis of the Wigner distribution defined in the frequency domain. This method provides some substantial advantages over the Wigner distribution. The well-known cross term effects are reduced or completely removed. The oversampling of signal is not necessary. In addition, the computation time can be significantly shorter. The results are demonstrated on two numerical examples with frequency modulated signals.**

I. Introduction

Time-frequency analysis of signals and systems is an intensively studied area, especially in the last decade. Many papers concerning the theory and application of this analysis have been published. Here, we will mention three excellent review papers [1]–[3]. The oldest technique for time-frequency analysis is the spectrogram via short time Fourier transform. Recently, the most popular techniques are based on the Wigner distribution (WD) or its variation—the generalized Wigner distribution [4]. Some of the important topics from this literature, which will be addressed in this paper, include the following:

1) efficient algorithms for the WD calculation or implementation [5]–[8]
2) aliasing problems [9], [10]
3) suppressing cross terms [3].

The third problem was very effectively resolved by the Choi–Williams method [11] with perservation of marginal properties, but in a computationally very intensive way.

In this note, a computationally efficient method for approximative time-frequency analysis, without need for oversampling, with cross term reduction (or removal), is proposed. It is based on the WD definition in the frequency domain and its relation to the spectrogram.

The method is illustrated by two numerical examples: one with two linear frequency modulated signals and the other with linear and sinusoidal frequecy modulated signals.

Manuscript received April 17, 1992; revised March 22, 1993. The associate editor coordinating the review of this paper and approving it for publication was Prof. Mysore Raghuveer.

The author is with Elektrotehnicki fakultet, Cetinjski put bb, 81000 Podgorica, Montenegro, Yugoslavia.

IEEE Log Number 9213301.

1053–587X/94$04.00 © 1994 IEEE

Authorized licensed use limited to: INRIA. Downloaded on June 01,2022 at 09:04:01 UTC from IEEE Xplore. Restrictions apply.

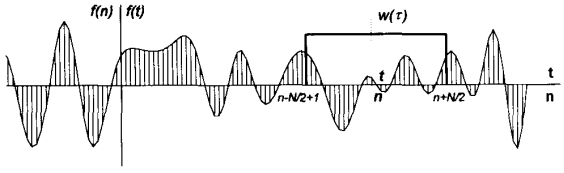

Fig. 1. Analog and discrete signal with windows.

II. Theory

A. Analog Signals

The oldest method for time-frequency analysis is the short time Fourier transform (STFT). This transform is also called the running Fourier transform [12]. Let us consider a long signal $f(t)$ or its samples $f(n)$ (Fig. 1).

The spectral components around time t can be obtained using the STFT in the form of a spectrogram:

$$S(t,\omega) = \left|F(t,\omega)\right|^2 = \left|\int_{-\infty}^{\infty} w(\tau) f(t+\tau) e^{-j\omega\tau}\, d\tau\right|^2 \quad (1)$$

where $w(\tau)$ is a window, whose width is T. This method is very dependent on the window shape.

In the last decade, commonly used methods for time-frequency analysis are based on the WD and its variations. The definition of the WD or the pseudo WD (as it is used for long signals) is

$$W(t,\omega) = \int_{-\infty}^{\infty} w\left(\frac{\tau}{2}\right) w^*\left(-\frac{\tau}{2}\right) \cdot f(t+\tau/2) f^*(t-\tau/2) e^{-j\omega\tau}\, d\tau. \quad (2)$$

The relation between $W(t,\omega)$ and $F(t,\omega)$ is

$$W(t,\omega) = \frac{1}{\pi} F(t,2\omega) \circledcirc F^*(t,2\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t,\omega+\theta/2) F^*(t,\omega-\theta/2)\, d\theta \quad (3)$$

where $\circledcirc$ denotes a convolution in frequency.

If we introduce a narrow window $P(\theta)$ we get

$$\mathrm{SPW}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\theta) F(t,\omega+\theta/2) F^*(t,\omega-\theta/2)\, d\theta \quad \frac{1}{\pi}\int_{-\infty}^{\infty} P(2\theta) F(t,\omega+\theta) F^*(t,\omega-\theta)\, d\theta. \quad (4)$$

Although this formula may be understood as a modification of the smoothed pseudo WD, it shows very interesting effects and leads to a computationally very efficient method. As it will be shown, this method is qualitatively and numerically quite different than the WD smoothing in time domain, using time-finite windows [3], [13].

We will first consider some useful effects which can be gained by an appropriate choice of the window $P(\theta)$. Two special cases are:

1) If $P(\theta) = 2\pi\delta(\theta)$, then the spectrogram, i.e., SPW $(t,\omega) = S(t,\omega)$, is obtained.
2) If $P(\theta) \equiv 1$, for all θ, then the pseudo WD is obtained.

These two special cases suggest the idea of a transform which will be "between" a spectrogram and the WD and which will combine the good properties of both. It is known that the spectrogram does not suffer from cross terms between signals separated in the time-frequency plane. The presence of the cross terms is an annoying trait of the WD. The spectrogram has a significant leakage due to window usage, which is much less exhibited in the case of the WD [14]. For a spectrogram the sampling interval has to be taken by the sampling theorem, while for the WD a signal has to be oversampled by factor 2.

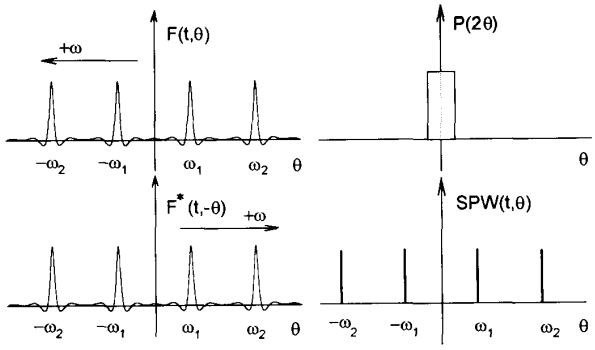

Fig. 2. Cross terms reduction (removal) by windowing.

Windowing the product in convolution (4) through narrow window $P(\theta)$, the cross terms will be reduced or even completely removed. This is easy to see from Fig. 2, where it is supposed that, at the instant t, the signal has the instantaneous frequencies $\pm\omega_1$ and $\pm\omega_2$. The convolution is, after windowing through $P(\theta)$, free of cross terms, which are present if $P(\theta)$ is wide (i.e., when the WD is used).

On the other hand, if the time window $w(\tau)$ is such that the components of STFT are not far from the instantaneous frequencies, the convergence inside $P(\theta)$ is fast. The obtained distribution has the quality of representation almost as the WD, but without cross terms.

In some cases, cross terms will be completely removed. If we have a sum of frequency modulated signals then the cross terms will appear (at the moment t) only if the distance between the instantaneous frequencies is less than the window $P(2\theta)$ width extended by the auto term width. By choosing an appropriate window $P(\theta)$, the sharpness of the WD can be preserved and the cross terms will be avoided. The cross terms will appear only between two very close instantaneous frequencies and will not disturb any other part of time-frequency distribution. An analytical treatment of the above effects may be found in [15].

B. Discrete Signals

The discrete form of the spectrogram is

$$DS(n,k) = |F(n,k)|^2 = \left|\sum_{i=-N/2+1}^{N/2} w(i) f(n+i) \exp\left(-j\frac{2\pi}{N} ik\right)\right|^2 = \left|\sum_{i=0}^{N-1} f_n(i) W_N^{ik}\right|^2. \quad (5)$$

The meaning of $f_n(i)$ and W_N in (5) is obvious.

The discrete WD in the time domain is

$$DW(n,k) = 2\sum_{m=-N+1}^{N} w(m) w^*(-m) f(n+m) f^*(n-m) W_{2N}^{-2mk} \quad (6)$$

where the discrete signal and window in (6) are sampled with half of the sampling interval assumed in (5).

The discretization of the SPW (4) produces

$$\mathrm{DSPW}(n,k) = \sum_{i=-L}^{L} Pd(i) F(n,k+i) F^*(n,k-i) \quad (7)$$

Authorized licensed use limited to: INRIA. Downloaded on June 01,2022 at 09:04:01 UTC from IEEE Xplore. Restrictions apply.

where $2L+1$ is the width of discrete window $Pd(i)$. We see that if $Pd(i) = \delta(i)$ then DSPW (n,k) = DS (n,k). Noting that

$$F(n,k+i)F^*(n,k-i) + F(n,k-i)F^*(n,k+i) = 2\,\mathrm{Real}\,\{F(n,k+i)F^*(n,k-i)\} \tag{8}$$

and assuming $Pd(i)$ is a rectangular window, we have

$$\mathrm{DSPW}\,(n,k) = |F(n,k)|^2 + 2\sum_{i=1}^{L}\mathrm{Real} \cdot \{F(n,k+i)F^*(n,k-i)\}. \tag{9}$$

For the WD calculation, the sampling interval has to be less than one half of the sampling interval specified by the sampling theorem. In the frequency domain, this means that the calculation of the convolution can be performed using the FFT after an appropriate zero padding. For the DSPW zero padding in the frequency domain (i.e., oversampling in time domain) is not necessary, because the aliasing components will be removed in the same way as the cross terms, Fig. 2. If we assume that $F(n,k) = F(n,k+N)$, the aliasing may occur only in the marginal intervals whose width is equal to the width of $Pd(i)$. But, this is not a necessary assumption, because (9) allows a direct calculation. The terms containing the values of $F(n,k \pm i)$ outside the basic period, can simply be omitted. The worst case, the last marginal values of the DSPW, will be always equal to the values of the spectrogram at these points.

The alternative, commonly used, way to avoid oversampling for the WD is in using the analytic signal.

To calculate the DSPW distribution we need to calculate the Fourier transform at the time instant n. This can be done by a recursive formula, from the previous values of the Fourier Transform at the time instant $n-1$, [12]:

$$F(n+1,k) = [f_n(N) - f_n(0) + F(n,k)]\exp\left(j\frac{2\pi}{N}k\right). \tag{10}$$

The initial Fourier Transform calculation has to be performed using the FFT routine. All the subsequent calculations can be done according to (10).[1]

Equation (10) gives the Fourier coefficients when the rectangular window $w(n)$ is applied. If we use, for example, the raised cosine window, then the coefficients should be modified by:

$$F_R(n,k) = \tfrac{1}{2}\{F(n,k) + \tfrac{1}{2}[F(n,k-1) + F(n,k+1)]\}. \tag{11}$$

C. Calculation Complexity

In this subsection, we will compare the proposed method (9)–(11) with the standard WD, regarding the number of required arithmetic operations. We will suppose that inside the window $w(\tau)$ there are N samples. To avoid aliasing in the WD calculation we have to use the analytic signal or to oversample the signal, i.e., to take $M = 2N$ samples inside the same window.

For correct analysis it should be noticed that the product $f(n+m)f^*(n-m)$ is Hermitian symmetric, so only $M/2 = N$ multiplication are needed. Also, the WD is real, so by one FFT routine, two WD can be calculated at the same time.

The multiplications by -1, 0 or by a power of two are not considered neither in the WD nor in the DPSW, because the time needed for the accompanying digital implementations is much shorter than the time needed for other operations.

[1] The recursive formulae are sensitive to the quantization error. The variance of quantization error introduced in one iteration is, in this case, N times less than the variance of quantization error introduced by the standard FFT routine. It means that we can allow as many as N iterations to keep the accumulated error less than the error introduced by the FFT. After each, say N iterations, the Fourier transform can be regenerated by the direct FFT calculation. The time needed for one FFT calculation per N iterations is negligible in the numerical calculation comparison.

TABLE I
THE NUMBERS OF COMPLEX ADDITIONS AND COMPLEX MULTIPLICATIONS FOR DIFFERENT METHODS. THE REAL AND MIXED OPERATIONS ARE EQUIVALENTED BY THE APPROPRIATE NUMBER OF COMPLEX ONES.

	Wigner Distribution with oversampling	*Wigner Distribution of analytic signal*	*DSPW with raised cosine window*
Complex multiplications	$\frac{N}{2}(4+\log_2 N)$	$\frac{N}{4}(3+5\log_2 N)$	$\frac{N}{2}(3+L)$
Complex additions	$N\log_2 2N$	$\frac{5N}{2}\log_2 N$	$\frac{N}{2}(6+L)$

TABLE II
THE NUMBER OF COMPLEX ADDITIONS AND MULTIPLICATIONS FOR DIFFERENT N. I—WIGNER DISTRIBUTION WITH OVERSAMPLING; II—WIGNER DISTRIBUTION OF ANALYTIC SIGNAL; III—SPECTROGRAM (DSPW WITH $L=0$); IV—DSPW WITH $L=1$; V—DSPW WITH $L=2$; VI—DSPW WITH $L=3$

N		*I*	*II*	*III*	*IV*	*V*	*VI*
64	*Additions*	*320*	*528*	*96*	*128*	*160*	*192*
	Multiplicat.	*448*	*960*	*192*	*224*	*256*	*288*
128	*Additions*	*704*	*1216*	*192*	*256*	*320*	*384*
	Multiplicat.	*1024*	*2240*	*384*	*448*	*512*	*576*
256	*Additions*	*1536*	*2752*	*384*	*512*	*640*	*768*
	Multiplicat.	*2304*	*5120*	*768*	*896*	*1024*	*1152*
512	*Additions*	*3328*	*6144*	*768*	*1024*	*1280*	*1536*
	Multiplicat.	*5120*	*11520*	*1536*	*1792*	*2048*	*2304*
1024	*Additions*	*7168*	*13568*	*1536*	*2048*	*2560*	*3072*
	Multiplicat.	*11264*	*25600*	*3072*	*3584*	*4096*	*4608*

The numbers of mathematical operations are given in Table I. The specific values, for different number of samples, are given in Table II. The complex signals and the application of raised cosine window are assumed. If a signal is real, the results are to be changed, but the ratios remain of the same order.

It is obvious that for short $Pd(i)$ a significant saving in computational time may be achieved. It means that the calculation of the DSPW can be more efficient than the calculation of the WD.

For the WD calculation the standard FFT routines, with $M/2\log_2 M$ complex multiplications and $M\log_2 M$ complex additions, are used. The WD is calculated according to (6), using the FFT routines. The analytic signal is calculated in the following order: 1) Find the FFT; 2) equate to zero the values corresponding to the negative frequencies and multiply by 2 the values corresponding to the positive frequencies; and 3) find the inverse FFT.

III. NUMERICAL EXAMPLES

The theory is illustrated by two numerical examples. First, we have a signal in the form of a real valued linear frequency modulated signal

$$f(t) = \cos[a(t+b)^2]. \tag{12}$$

The spectrogram and the WD, as well as results obtained by the proposed method, are given in Fig. 3(a)–(c) respectively. In Fig. 3(d) the WD of an analytic signal is presented.

The second example was taken with a signal in the form of a sum of linear and sinusoidal frequency modulated components:

$$f(t) = A\exp[-ja(t-b)^2 + \exp\{j\{c\cos[d\pi(t+e)] - g\pi t\}\}. \tag{13}$$

The improvement of the time-frequency signal presentation, using the proposed method, may be observed in Fig. 4.

For the selected number of samples ($N = 64$ for DSPW and $2N = 128$ for WD) and the window $Pd(i)$ width $L = 3$, results

Authorized licensed use limited to: INRIA. Downloaded on June 01,2022 at 09:04:01 UTC from IEEE Xplore. Restrictions apply.

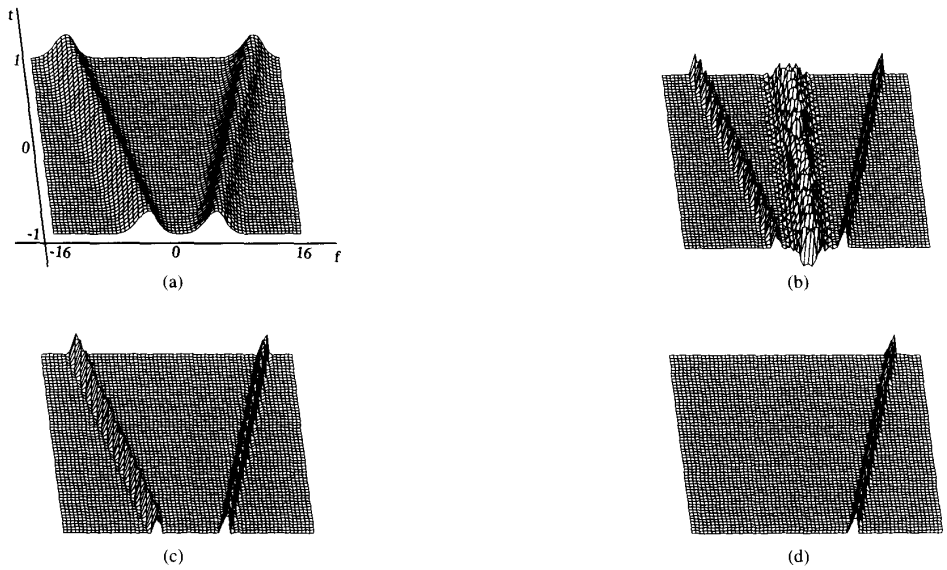

Fig. 3. Time-frequency presentation of real linear frequency modulated signal. (a) Spectrogram. (b) pseudo Wigner distribution. (c) proposed method. (d) Wigner distribution of an analytic signal. Sampling interval $\Delta t = 1/32$ (for the WD $\Delta t/2$): raised cosine window $w(\tau)$ whose width is $T = 2$; window $Pd(i)$ is rectangular with $L = 3$; $a = 4\pi; b = 2$.

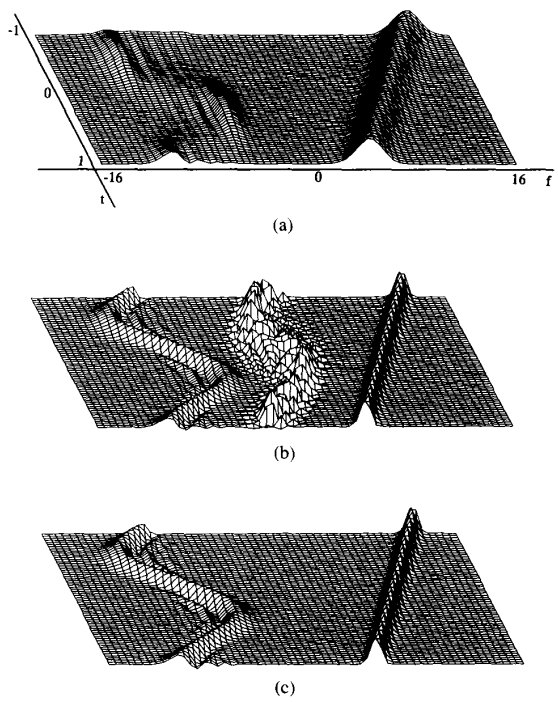

Fig. 4. Time-frequency presentation of linear and sinusoidal frequency modulated signals: a) Spectrogram; b) pseudo Wigner distribution; c) proposed method. Sampling interval $\Delta t = 1/32$ (for WD the $\Delta t/2$); raised cosine window $w(\tau)$ whose width is $T = 2$; window $Pd(i)$ is rectangular with $L = 3; A = 1.5; a = 4\pi; b = 2; c = 6; d = 1.2; e = 1; g = 15$.

are obtained (in both cases) with the number of arithmetic operation given in Table II. The computational savings are evident.

IV. Conclusion

A method for time frequency analysis, based on the smoothed pseudo Wigner distribution and spectrogram, is proposed. This method reduces (or completely removes) the cross term effects. In addition, it is computationally very efficient. The theory is illustrated by the numerical examples.

References

[1] L. Cohen, "Time-frequency distributions—A review," *Proc. IEEE*, vol. 77, pp. 941–981, July 1989.

[2] T. A. C. M. Claasen and W. F. G. Mecklenbräuker, "The Wigner distribution—A tool for time frequency signal analysis, Part III: Relations with other time frequency signal transformations," *Philips J. Res.*, vol. 35, no. 6, pp. 372–389, 1980.

[3] Y. M. Zhu, F. Peyrin, R. et Goutte, "Transformation de Wigner–Ville: Description d'un nouvel outil de traitement du signal et des images," *Annales Telecomm.*, vol. 42, nos. 3–4, pp. 105–117, 1987.

[4] F. Hlawatsch, "Duality and classification of bilinear time-frequency signal representation," *IEEE Trans. Signal Processing*, vol. 39, pp. 1564–1574, July 1991.

[5] B. Boashash and J. B. Black, "An efficient real time implementation of the Wigner–Ville distribution," *IEEE Trans. Acoust. Speech Signal Processing*, vol. ASSP-35, pp. 1611–1618, Nov. 1987.

[6] M. Sun, C. C. Li, L. N. Sekhar, and R. J. Sclabassi, "Efficient computation of the discrete pseudo Wigner distribution," *IEEE Trans. Acoust. Speech Signal Processing*, vol. 37, pp. 1735–1741, Nov. 1989.

[7] W. A. Porter, "Computational aspects of quadratic signal processing," *IEEE Trans. Acoust. Speech Signal Processing*, vol. 38, pp. 137–144, Jan. 1990.

[8] D. Chester, "Discrete Wigner implementations," in *Proc. ISCS* (San Jose, CA), May 1986, pp. 38–41.

[9] T. A. C. M. Claasen and W. F. G. Mecklenbräucker, "The aliasing problem in discrete time Wigner distributions," *IEEE Trans. Acoust. Speech Signal Processing*, vol. ASSP-31, pp. 1067–1072, Oct. 1983.

[10] F. Peyrin and R. Prost, "A unified definition for the discrete-time, discrete frequency, and discrete time/frequency Wigner distributions," *IEEE Trans. Acoust. Speech Signal Processing*, vol. ASSP-34, pp. 858–867, Aug. 1986.

[11] H. I. Choi and W. J. Williams, "Improved time-frequency representation of multicomponent signals using exponential kernels," *IEEE Trans. Acoust. Speech Signal Processing*, vol. 37, pp. 862–871, June 1989.

Authorized licensed use limited to: INRIA. Downloaded on June 01,2022 at 09:04:01 UTC from IEEE Xplore. Restrictions apply.

[12] A. Papoulis, *Signal Analysis*. New York: McGraw Hill, 1977, pp. 174–178.
[13] LJ. Stanković and S. Stanković, "Wigner distribution of noisy signals," *IEEE Trans. Signal Processing*, vol. 41, pp. 956–960, Feb. 1993.
[14] ——, "An analysis of instantaneous frequency representation using time frequency distributions—Generalized Wigner distribution," *IEEE Trans. on Signal Processing*, to be published.
[15] LJ. Stanković, "Wigner higher order spectra of multicomponent signals: A method for higher order time-frequency analysis," in *Proc. Int. Conf. DSP* (Nicosia, Cyprus), July 1993, pp. 100–105.

A General Procedure for the Derivation of Principal Domains of Higher-Order Spectra

Vinod Chandran and Steve Elgar

***Abstract*—A general procedure to determine the principal domain (i.e., nonredundant region of computation) of any higher-order spectrum is presented, using the bispectrum as an example. The procedure is then applied to derive the principal domain of the trispectrum of a real-valued, stationary time series. These results are easily extended to compute the principal domains of other higher-order spectra.**

I. Introduction

The primary purpose of this study is to provide a general procedure for deriving the principal domains (i.e., nonredundant regions) of higher-order spectra and use it to derive the nonredundant region of computation of the trispectrum. Higher-order spectra or polyspectra were introduced for studying nonlinearities and deviations from Gaussianity in stationary random processes. They are defined as the Fourier transforms of higher-order moments or cumulants of a random process. The idea of a spectral representation for higher-order moments of a time series appears in [1], and was further developed in [2]. A spectral representation for cumulants (attributed to Kolmogorov) appears in [2]. Higher-order spectra are derived from first principles in [3], [4]. For a single time series, the first higher-order spectrum is the (auto) power spectrum. The 2nd and 3rd higher-order spectra are the (auto) bispectrum and the (auto) trispectrum, and are defined as the Fourier transforms of the 3rd and 4th cumulants, respectively. Although cross higher-order spectra can be defined for multiple time series, the present study is restricted to a single time series and the prefix auto will be dropped. Further, it is not necessary to define higher-order spectra in terms of cumulants here. Instead, an alternative form involving products of Fourier coefficients of realizations of a random process will be used. This form can be derived from the cumulant based definition using Stieljes Integrals [4].

Manuscript received May 29, 1992; revised January 14, 1993. The associate editor coordinating the review of this paper and approving it for publication was Prof. Jose A. R. Fonollosa. This work was supported by the Office of Naval Research.

The authors have been with the School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752. V. Chandran is now with the Signal Processing Research Centre, School of Electrical and Electronic Systems Engineering, Q.U.T., Brisbane, Queensland, Australia.

IEEE Log Number 9213287.

Let $x[t]$ be a real-valued, stationary random process. The bispectrum, $B(f_1, f_2)$, of the process can be expressed as

$$B(f_1, f_2) = E[X(f_1)X(f_2)X(-f_1 - f_2)] \tag{1}$$

where

$$x[t] = \int e^{j2\pi ft} dX(f)$$

is the Fourier decomposition of a realization of the process, * denotes complex conjugation, $j = \sqrt{-1}$, and $E[\ ]$ is the expectation operator. This form of definition appears in [5] in an application of the bispectrum which precedes the theoretical development of cumulant spectra [3], [4]. Similarly, the trispectrum, $T(f_1, f_2, f_3)$, of the random process may be expressed as

$$T(f_1, f_2, f_3) = E[X(f_1)X(f_2)X(f_3)X(-f_1 - f_2 - f_3)]. \tag{2}$$

The bispectrum has been used in many applications including the study of quadratic interactions, signal reconstruction, system identification, and pattern recognition [7]. The trispectrum [4], [6], [8]–[11] has not enjoyed the same popularity as the bispectrum, partially because of the increased complexity in its computation and interpretation. Dalle Molle and Hinich [10] discuss part of the principal domain of the trispectrum. A procedure for the derivation of the complete principal domain is described in Section II, and applied to the bispectrum in Section III.

II. The Procedure

Let f denote the frequency normalized by the Nyquist frequency, such that $0 \le f \le 1$. If $x[t]$ is a real-valued time series, then the Fourier transform, $X(f)$, is conjugate symmetric, and hence $X^*(f) = X(-f)$. The bispectrum, $B(f_1, f_2)$, is then also given by

$$B(f_1, f_2) = E[X(f_1)X(f_2)X^*(f_1 + f_2)]. \tag{3}$$

The trispectrum or any other higher-order spectrum can also be expressed as the expected value of a product as in (3). The bispectrum (or other higher-order spectrum) possesses redundancy in bifrequency (or polyfrequency) space arising from

1) the interchangeability of any pair of frequencies in its definition (1),
2) redundancy of the negative half of some of the frequency axes owing to the conjugate symmetry property, and
3) periodicity of the Fourier transform at intervals of the sampling frequency for discrete-time processes.

This redundancy can be systematically exploited and eliminated to derive the principal domain or nonredundant region of computation of the bispectrum (or other higher-order spectrum). Thus, $B(f_1, f_2)$ need only be computed for a subset (derived here to illustrate the procedure) of all possible values of bifrequency (f_1, f_2).

Owing to property 1 the bispectrum is symmetric about the lines $f_1 = f_2$; $f_1 = -f_1 - f_2$ (or $2f_1 + f_2 = 0$); $f_2 = -f_1 - f_2$ (or $f_1 + 2f_2 = 0$) as shown in Fig. 1. The kth order polyspectrum will have $k(k+1)/2$ hyperplanes of symmetry in k-frequency space, analogous to the lines of symmetry above. The bispectrum need be computed only on one side of either of these lines, and therefore only for the subset

$$S_1^2 = \{f_1 \ge f_2\} \cap \{f_1 + 2f_2 \ge 0\} \tag{4}$$

1053-587X/94$04.00 © 1994 IEEE

Authorized licensed use limited to: INRIA. Downloaded on June 01,2022 at 09:04:01 UTC from IEEE Xplore. Restrictions apply.