19.1.3 Proposition *If $f \in L^1(\mathbb{R})$ decays rapidly, then $\widehat{f}$ is infinitely differentiable.*

Proof. $x^p f(x)$ is in $L^1(\mathbb{R})$ for all $p \in \mathbb{N}$ by Proposition 19.1.2. This implies by Proposition 17.2.1(i) that $\widehat{f}$ is in C^∞. $\square$

Conversely, what does $\widehat{f} \in C^\infty$ imply about f? A partial answer is given by the next result.

19.1.4 Proposition *Assume that f is in C^∞. If $f^{(k)}$ is in $L^1(\mathbb{R})$ for all $k \in \mathbb{N}$, then $\widehat{f}$ decays rapidly.*

Proof. $\widehat{f^{(k)}}(\xi) = (2i\pi\xi)^k \widehat{f}(\xi)$ for all $k \in \mathbb{N}$ by Proposition 17.2.1(ii). By the Riemann–Lebesgue theorem, $\lim_{|\xi| \to +\infty} |\xi|^k |\widehat{f}(\xi)| = 0$. $\square$

Said another way, we have just proved the following results:

(i) The faster f decreases at infinity, the greater the regularity of $\widehat{f}$.

(ii) The more regular f is, the faster $\widehat{f}$ decays.

In particular, if $f \in C^\infty(\mathbb{R})$ and decreases rapidly, the same is true for $\widehat{f}$. Note the similarity of this result and Proposition 5.3.4 about the Fourier coefficients of a periodic function in $C^\infty(\mathbb{R})$.