Proof.

(i) The function $h : \xi \mapsto e^{-2i\pi\xi x} f(x)$ is infinitely differentiable; furthermore, $h^{(k)}(\xi) = (-2i\pi x)^k e^{-2i\pi\xi x} f(x)$ and $|h^{(k)}(\xi)| \leq 2\pi |x^k f(x)|$. Proposition 14.2.2 applies for $k = 1, 2, \ldots, n$, and

$$\widehat{f}^{(k)}(\xi) = \int e^{-2i\pi\xi x} (-2i\pi x)^k f(x)\, dx.$$

(ii) We prove this for $n = 1$; the result for $n \geq 2$ is obtained by induction. Since $f' \in L^1(\mathbb{R})$, we can compute $\widehat{f'}$ by the formula

$$\widehat{f'}(\xi) = \lim_{a \to +\infty} \int_{-a}^{+a} e^{-2i\pi\xi x} f'(x)\, dx.$$

Integrating by parts shows that

$$\int_{-a}^{+a} e^{-2i\pi\xi x} f'(x)\, dx = \Big[e^{-2i\pi\xi x} f(x) \Big]_{-a}^{+a} + \int_{-a}^{+a} (2i\pi\xi) e^{-2i\pi\xi x} f(x)\, dx. \tag{17.7}$$

Assume for the moment that $f(\pm a)$ has a limit as $a \to +\infty$. Since f is integrable, this limit must be zero. As $a \to +\infty$, (17.7) becomes

$$\int e^{-2i\pi\xi x} f'(x)\, dx = \int (2i\pi\xi) e^{-2i\pi\xi x} f(x)\, dx,$$

which is formula (17.6) for $k = 1$.