17.1.3 Theorem (Riemann–Lebesgue) *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ satisfies the following conditions:*

(i) $\mathscr{F}\ f$ *is continuous and bounded on* $\mathbb{R}$.

(ii) $\mathscr{F}$ *is a continuous linear operator from* $L^1(\mathbb{R})$ *to* $L^\infty(\mathbb{R})$, *and*

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \tag{17.3}$$

(iii) $\lim_{|\xi| \to +\infty} |\widehat{f}(\xi)| = 0.$

Proof.

(i) The continuity of $\widehat{f}$ follows directly from the continuity of the integral (17.1) with respect to the parameter ξ. The function $\xi \mapsto e^{-2i\pi\xi x} f(x)$ is continuous on $\mathbb{R}$ and is dominated by $|f(x)|$, which is in $L^1(\mathbb{R})$. Proposition 14.2.1 applies.

(ii) For all $\xi \in \mathbb{R}$ we have $|\widehat{f}(\xi)| \leq \int |f(x)|\, dx = \|f\|_1$. Thus $\widehat{f}$ is bounded, and $\mathscr{F}$ is continuous from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$.

(iii) For $f = \chi_{[a,b]}$ we have $|\widehat{f}(\xi)| \leq 1/\pi|\xi|$ for $\xi \neq 0$ (Section 17.1.2). Thus $\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0$; clearly this is true for all simple functions. Now take f in $L^1(\mathbb{R})$. Since the simple functions are dense in $L^1(\mathbb{R})$, there exists a sequence g_n of simple functions such that $\lim_{n \to \infty} \|f - g_n\|_1 = 0$ and, for each fixed n, $\lim_{|\xi| \to \infty} |\widehat{g}_n(\xi)| = 0$. From (17.3), $|\widehat{f}(\xi) - \widehat{g}_n(\xi)| \leq \|f - g_n\|_1$ uniformly in $\xi \in \mathbb{R}$ for each fixed n. It follows that $\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0$. □