Proof. For each $n > 0$, we introduce the function $g_n(x) = e^{-\frac{2\pi}{n}|x|}$, whose Fourier transform is

$$\widehat{g}_n(\xi) = \frac{1}{\pi}\frac{n}{1+n^2\xi^2}.$$

The functions g_n and $\widehat{g}_n$ are in $\mathrm{L}^1(\mathbb{R})$. We can apply formula (17.4) to the two functions f and $e^{2i\pi tx}g_n(x)$, which in view of Proposition 17.2.4(ii) is

$$\int_{\mathbb{R}} \widehat{f}(x)g_n(x)e^{2i\pi tx}\,dx = \int_{\mathbb{R}} f(u)\widehat{g}_n(u-t)\,du. \tag{18.1}$$

For all $x \in \mathbb{R}$, $|\widehat{f}(x)g_n(x)e^{2i\pi tx}| \le |\widehat{f}(x)|$, and $\lim_{n\to\infty} g_n(x) = 1$. Since $\widehat{f}$ is in $L^1(\mathbb{R})$, we can apply Lebesgue's theorem and pass to the limit under the integral sign. Thus

$$\lim_{n\to\infty}\int_{\mathbb{R}} \widehat{f}(x)g_n(x)e^{2i\pi tx}\,dx = \int_{\mathbb{R}} \widehat{f}(x)e^{2i\pi tx}\,dx = \overline{\mathscr{F}}\,\widehat{f}(t).$$

Assume that f is continuous at t; we need to show that the integral on the right-hand side of (18.1) tends to $f(t)$. Since $\widehat{g}_n \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \widehat{g}_n(\xi)\,d\xi = \lim_{a\to+\infty}\int_{-a}^{+a}\frac{1}{\pi}\frac{n}{1+n^2\xi^2}\,d\xi = 1.$$

Thus we can write

$$\int_{\mathbb{R}} f(u)\widehat{g}_n(u-t)\,du - f(t) = \int_{\mathbb{R}}(f(\xi+t)-f(t))\widehat{g}_n(\xi)\,d\xi. \tag{18.2}$$

Given $\varepsilon > 0$, there exists $\eta > 0$ such that $|y-t| \le \eta$ implies $|f(y)-f(t)| \le \varepsilon$. We decompose (18.2) as follows:

$$\begin{aligned}\int_{\mathbb{R}}(f(\xi+t)-f(t))\widehat{g}_n(\xi)\,d\xi &= \int_{|\xi|\le\eta}(f(\xi+t)-f(t))\widehat{g}_n(\xi)\,d\xi \\ &\quad + \int_{|\xi|>\eta}(f(\xi+t)-f(t))\widehat{g}_n(\xi)\,d\xi.\end{aligned}$$

For all $n > 0$,

$$\int_{|\xi|\le\eta}|f(\xi+t)-f(t)||\widehat{g}_n(\xi)|\,d\xi \le \varepsilon\int_{|\xi|\le\eta}|\widehat{g}_n(\xi)|\,d\xi \le \varepsilon.$$

The last step is to show that $\lim_{n\to\infty}\int_{|\xi|>\eta}(f(t+\xi)-f(t))\widehat{g}_n(\xi)\,d\xi = 0$. For this we have

$$\left|f(t)\int_{|\xi|>\eta}\widehat{g}_n(\xi)\,d\xi\right| = |f(t)|\left(1-\frac{2}{\pi}\arctan n\eta\right), \tag{18.3}$$

and since $\widehat{g}_n$ is even and decreasing on $\mathbb{R}_+$,

$$\left|\int_{|\xi|>\eta} f(t+\xi)\widehat{g}_n(\xi)\,d\xi\right| \le \widehat{g}_n(\eta)\|f\|_1. \tag{18.4}$$

As n tends to $+\infty$, the right-hand sides of (18.3) and (8.4) tend to 0, and this proves the theorem. □