Chapter VII

Analog Filters

Lesson 24

Applications to Analog Filters Governed by a Differential Equation

The tools we have just developed (convolution and the Fourier transform for functions) are going to be used to study analog filters that are governed by a linear differential equation with constant coefficients,

$$\sum_{k=0}^{q} b_k g^{(k)} = \sum_{j=0}^{p} a_j f^{(j)}, \quad a_p \cdot b_q \neq 0, \tag{24.1}$$

where f is the input and $g = A(f)$ is the output. Other conditions must be given to eliminate ambiguity among the possible solutions of (24.1).

24.1 The case where the input and output are in $\mathscr{S}$

This case is very special. The input has no reason to be so regular, but we will see that this is a step toward more general cases.

We assume that $f \in \mathscr{S}$ and look for a solution g in $\mathscr{S}$. If such a g exists, we can take the Fourier transform of both sides of (24.1). Thus

$$\sum_{k=0}^{q} b_k (2i\pi\lambda)^k \widehat{g}(\lambda) = \sum_{j=0}^{p} a_j (2i\pi\lambda)^j \widehat{f}(\lambda). \tag{24.2}$$

Consider the two polynomials

$$P(x) = \sum_{j=0}^{p} a_j x^j \quad \text{and} \quad Q(x) = \sum_{k=0}^{q} b_k x^k$$

and assume that the rational function $P(x)/Q(x)$ has no poles on the imaginary axis. Then $P(2i\pi\lambda)/Q(2i\pi\lambda)$ has no poles for real λ, and (24.2) is equivalent to

$$\widehat{g}(\lambda) = \frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)} \widehat{f}(\lambda). \tag{24.3}$$

This equality completely determines g in $\mathscr{S}$, if it exists, and thus proves the uniqueness of a solution of (24.1) in $\mathscr{S}$. The existence of a solution also follows from (24.3), since the function

$$G(\lambda) = \frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)}\widehat{f}(\lambda)$$

is in $\mathscr{S}$ whenever f is in $\mathscr{S}$. By applying Theorem 19.3.1, we see that $g = \mathscr{F}^{-1}(G)$ is a solution of (24.1) in $\mathscr{S}$.

24.1.1 Proposition *If $P(x)/Q(x)$ has no poles on the imaginary axis and if f is in $\mathscr{S}$, then (24.1) has a unique solution $g \in \mathscr{S}$. In this case, the system*

$$\begin{aligned} A : \mathscr{S} &\to \mathscr{S}, \\ f &\mapsto g \end{aligned}$$

is a filter.

Proof. We have proved the first part of the result and thus need only to show that A is a filter on $\mathscr{S}$. The linearity and invariance present no difficulty. To prove continuity in the topology of $\mathscr{S}$, suppose that a sequence f_n tends to 0 in $\mathscr{S}$. Then $\widehat{f_n}$ tends to 0 in $\mathscr{S}$, as does $\widehat{g}_n$ given by (24.3). Thus g_n tends to 0 by Theorem 19.3.1. □

The differential equation (24.1) has a unique solution without initial conditions being specified. This is because we require the solution g to be in $\mathscr{S}$, which means that g and all of its derivatives vanish at infinity.

We assume in what follows that P/Q has no poles on the imaginary axis. Also, note that $P \not\equiv 0$, since we assume that $a_p \neq 0$.

24.1.2 The output expressed as a convolution $(p < q)$

If we assume that $\deg P < \deg Q$, then the transfer function

$$H(\lambda) = \frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)} \tag{24.4}$$

is in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By decomposing this rational function into partial fractions, we see from Sections 18.2.2 and 22.2.2 that it has an inverse Fourier transform $h = \mathscr{F}^{-1}H$ that is bounded, rapidly decreasing, continuous except perhaps at the origin, and satisfies (24.3),

$$\widehat{g} = \widehat{h} \cdot \widehat{f},$$

which by Proposition 23.2.1(i) implies that

$$g = h * f. \tag{24.5}$$

This is the same kind of formula that we obtained in Section 2.4 for the *RC* filter. The response is the convolution of the input with a fixed function h that is called the *impulse response.* Note that if $\deg P \geq \deg Q$, the computations we have just made no longer make sense.

24.2 Generalized solutions of the differential equation

The formula $g = h * f$, obtained when f is in $\mathscr{S}$, makes sense in the following more general cases.

24.2.1 If f is in $L^1(\mathbb{R})$, then g is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (Propositions 20.2.1, 20.3.1, and 20.3.1) and

$$\begin{aligned} \|g\|_1 &\leq \|h\|_1 \ \|f\|_1, \\ \|g\|_2 &\leq \|h\|_2 \ \|f\|_1, \\ \|g\|_\infty &\leq \|h\|_\infty \|f\|_1. \end{aligned} \tag{24.6}$$

24.2.2 If f is in $L^2(\mathbb{R})$, then g is in $L^2(\mathbb{R})$, it is bounded and continuous (Proposition 20.3.1), it tends to 0 at infinity (Proposition 23.2.1(i)), and

$$\begin{aligned} \|g\|_2 &\leq \|h\|_1 \|f\|_2, \\ \|g\|_\infty &\leq \|h\|_2 \|f\|_2. \end{aligned} \tag{24.7}$$

24.2.3 If f is in $L^\infty(\mathbb{R})$, then g is also bounded and (proposition 20.3.1)

$$\|g\|_\infty \leq \|h\|_1 \|f\|_\infty. \tag{24.8}$$

The system A defined in Proposition 24.1.1 in continuous from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$, and thus it is a filter. Similarly, (24.6) and (24.7) show that A is continuous from $L^1(\mathbb{R})$ to $L^p(\mathbb{R})$ ($p = 1, 2, \infty$), and from $L^2(\mathbb{R})$ to $L^q(\mathbb{R})$ ($q = 2, \infty$).

24.2.4 Definition The response of a filter to the unit step function is called the step response of the filter. This response, h_1, is well-defined as a generalized solution of (24.1). It is bounded by (24.8) and is given by

$$h_1(t) = h * u(t) = \int_{-\infty}^t h(s)\, ds. \tag{24.9}$$

24.3 The impulse response when $\deg P < \deg Q$

The impulse response $h = \mathscr{F}^{-1} H$ is computed by decomposing H into partial fractions. The poles of P/Q are assumed to lie off the imaginary axis. There are two cases to consider: P/Q has only simple poles or P/Q has multiple poles.

24.3.1 The case where $P(x)/Q(x)$ has only simple poles

In this case, H can be decomposed in the form

$$H(\lambda) = \sum_{k=0}^{q} \frac{\beta_k}{2i\pi\lambda - z_k}, \tag{24.10}$$

where $z_1, \ldots, z_q$ are the poles. From Section 22.2.2, read for $\mathscr{F}^{-1}$, we conclude that

$$h(t) = \Big(\sum_{k \in K_-} \beta_k e^{z_k t} \Big) u(t) - \Big(\sum_{k \in K_+} \beta_k e^{z_k t} \Big) u(-t), \tag{24.11}$$

where we have defined

$$K_- = \big\{ k \in \{1, 2, \ldots, q\} \mid \operatorname{Re}(z_k) < 0 \big\},$$

$$K_+ = \big\{ k \in \{1, 2, \ldots, q\} \mid \operatorname{Re}(z_k) > 0 \big\}.$$

24.3.2 The case where $P(x)/Q(x)$ has multiple poles

Let $z_1, z_2, \ldots, z_l$ the poles and let $m_1, m_2, \ldots, m_l$ be their multiplicities. Then we can write H as

$$H(\lambda) = \sum_{k=1}^{l} \sum_{m=1}^{m_k} \frac{\beta_{k,m}}{(2i\pi\lambda - z_k)^m}. \tag{24.12}$$

By using the results in Section 17.3.4, we see that

$$h(t) = \Big(\sum_{k \in K_-} P_k(t) e^{z_k t} \Big) u(t) - \Big(\sum_{k \in K_+} P_k(t) e^{z_k t} \Big) u(-t), \tag{24.13}$$

where

$$P_k(t) = \sum_{m=1}^{m_k} \beta_{k,m} \frac{t^{m-1}}{(m-1)!}.$$

24.3.3 The case of purely imaginary poles

What we have done so far does not allow us to treat an equation like

$$g'' + \omega^2 g = f,$$

where $P(x)/Q(x) = 1/(x^2 + \omega^2)$ has two poles are on the imaginary axis. In this case h is a sinusoid and the Fourier transform of H (when H is considered to be a function) is no longer defined. This problem will be resolved in Section 35.2.3 in the context of distributions.

24.3.4 The case where $\deg P = \deg Q$

Take for example the equation

$$g'' - \omega^2 g = f''.$$

Again, what we have done so far does not apply. Nevertheless, we can still manage to solve the equation. Changing the unknown function to $g_0 = g - f$ lowers the order of the right-hand side:

$$g_0'' - \omega^2 g_0 = \omega^2 f.$$

Then we have $g_0 = h_0 * f$ and $g = f + h_0 * f$. This is no longer a convolution like (24.5), but it will serve the same purpose. On the other hand, it is clear that we can obtain g as

$$g = h_1 * f' \quad \text{or} \quad g = h_2 * f''.$$

In the general case, we change the unknown function to $g_0 = g + \lambda f$ and find that

$$\sum_{k=0}^{q} b_k g_0^{(k)} = \sum_{k=0}^{q} (a_k - \lambda b_k) f^{(k)}.$$

Taking $\lambda = a_q / b_q$ reduces the degree of the right-hand side and brings us back to the case $p < q$. We can then write

$$g = \lambda f + h_0 * f. \tag{24.14}$$

(Note that it is possible that $h_0 \equiv 0$; this happens when $P(x) = \lambda Q(x)$ for all x (see Exercise 24.2).) The representation (24.14) leads to estimates like those given in Section 24.2. In Section 35.2 we will give an expression for g as a convolution without the condition $p < q$, but in this case h will be a distribution.

24.3.5 Summary

When P/Q has no poles on the imaginary axis and $\deg P \le \deg Q$, a unique generalized solution of (24.1) can be defined under the sole condition that $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$. $A(f) = g$ is a filter that we will call the *generalized filter* A associated with (24.1). The output g is given by $g = h * f$ or possibly by a formula like (24.14).

24.4 Stability

24.4.1 Definition An analog system $A : X \to Y$ is said to be stable if there exists an $M > 0$ such that $\|Af\|_\infty \le M \|f\|_\infty$ for all $f \in L^\infty(\mathbb{R}) \cap X$.

By (24.8), the generalized filter A is stable when $\deg P < \deg Q$. If $\deg P = \deg Q$, the system is still stable from what we have seen in Section 24.3.4.

24.4.2 Theorem *The generalized filter governed by equation (24.1), whose output g is defined by (24.5) or (24.14), is stable when* $\deg P \leq \deg Q$ *and the poles of $P(x)/Q(x)$ are not on the imaginary axis.*

$$\left.\begin{array}{l}\deg P \leq \deg Q \text{ and } P/Q \text{ has no}\\ \text{poles on the imaginary axis.}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\text{The generalized filter}\\ A \text{ is stable.}\end{array}\right.$$

24.5 Realizable systems

24.5.1 Definition A system is said to be realizable (or causal) if the equality of two input signals for $t < t_0$ implies the equality of the two output signals for $t < t_0$ (see Section 2.1.2).

For a filter, which is by definition linear and invariant, this condition becomes the following: For all $t_0 \in \mathbb{R}$,

$$f(t) = 0 \text{ for } t < t_0 \quad \Longrightarrow \quad Af(t) = 0 \text{ for } t < t_0.$$

We will see that the realizability of the filter defined in Section 24.3.5 depends simply on its impulse response or on the position of the poles.

Assume that $\deg P \leq \deg Q$.

$$\begin{array}{c}\text{The generalized filter}\\ A \text{ is realizable.}\end{array} \Longleftrightarrow \operatorname{supp}(h) \subset [0, +\infty).$$

If $\operatorname{supp}(h) \subset [0, +\infty)$, the output

$$g(t) = \int_0^{+\infty} h(s) f(t-s)\, ds$$

is 0 for $t < t_0$ when $f(t) = 0$ for $t < t_0$. We prove the other direction by contradiction. Thus suppose that there is a $t_1 < 0$ such that $h(t_1) > 0$. Since h is continuous at t_1, there is an interval (a, b) such that $b < 0$ and $a < t_1 < b$ implies that $h(t) > 0$. For the causal input signal

$$f(t) = \chi_{[0, b-a]}(t),$$

we have an output signal

$$g(t) = \int_{t-b+a}^{t} h(s)\, ds$$

with $g(b) > 0$. This contradicts the fact that A is causal. This is the proof when $\deg P < \deg Q$. In case $\deg P = \deg Q$, one uses the trick introduced in Section 24.3.4.

From formulas (24.11) and (24.13) we see that $\operatorname{supp}(h) \subset [0, +\infty)$ if and only if K_+ is empty. Thus if $\deg P \leq \deg Q$, we have the following result:

$$\boxed{\left.\begin{array}{l}\text{The generalized filter}\\ A \text{ is realizable.}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{The poles of P/Q are located to}\\ \text{the left of the imaginary axis.}\end{array}\right.}$$

24.5.2 Theorem *For the generalized filter defined in Section 24.3.5 with* $\deg P \leq \deg Q$ *to be realizable, it is necessary and sufficient that all the poles of* P/Q *have strictly negative real parts.*

For $\deg P = \deg Q$, the property results from the fact that the output can be written as $g = \lambda f + h_0 * f$, $\lambda \in \mathbb{C}$. In summary, if $\deg P \leq \deg Q$, we have the following result:

$$\boxed{\left.\begin{array}{l}\text{The real parts of all the}\\ \text{poles of } P/Q \text{ are negative.}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{The generalized filter } A\\ \text{is realizable and stable.}\end{array}\right.}$$

24.6 Gain and response time

The gain of a filter of the type described in Section 24.3.5 is defined to be the constant

$$K = H(0).$$

From (24.9) we see that

$$K = \widehat{h}(0) = \lim_{t \to +\infty} h_1(t),$$

which is the ratio between the asymptotic value of the step response and the height of the input step function. The response time is defined to be the time it takes the step response to reach and maintain a certain percentage of its limit, in general 95%:

$$t_r = \min \left\{ t \;\middle|\; \left| \frac{h_1(t) - K}{K} \right| \leq \frac{5}{100} \text{ for all } t > t_r \right\}.$$

24.7 The Routh criterion

The stability of a system depends on the location of the roots of the characteristic equation $Q(x) = 0$ in the complex plane. We note that it is not necessary to compute the roots of this equation to determine whether all their real parts are negative. It is possible to use the *Routh criterion*: The roots of the equation

$$a_0 x^p + a_1 x^{p-1} + \cdots + a_{p-1} x + a_p = 0$$

with real coefficients will all have strictly negative real parts if and only if the elements of the first column of the following array all have the same sign:

$$\begin{bmatrix} a_0 & a_2 & a_4 & a_6 & \cdots \\ a_1 & a_3 & a_5 & a_7 & \cdots \\ b_1^1 & b_2^1 & b_3^1 & b_4^1 & \cdots \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

with

$$b_k^1 = \frac{a_1 a_{2k} - a_0 a_{2k+1}}{a_1},$$
$$b_k^2 = \frac{b_1^1 a_{2k+1} - a_1 b_{k+1}^1}{b_1^1},$$

etc., for $k = 1, 2, \ldots$.

EXAMPLES

(a) $Q(x) = x^4 + 3x^3 + 6x^2 + 9x + 12$.

The Routh matrix is

$$\begin{bmatrix} 1 & 6 & 12 \\ 3 & 9 & 0 \\ 3 & 12 & \\ -3 & & \end{bmatrix},$$

and thus the real parts of the roots are not all negative.

(b) $Q(x) = x^3 + (2k+1)x^2 + (k+1)^2 x + k^2 + 1 = 0$.

The Routh matrix is

$$\begin{bmatrix} 1 & (k+1)^2 & 0 \\ 2k+1 & k^2+1 & 0 \\ \dfrac{2k(k^2+2k+2)}{2k+1} & 0 & \\ k^2+1 & & \end{bmatrix}.$$

For the elements in the first column all to have the same sign, we must have $2k + 1 > 0$ and $2k > 0$. Thus the real parts of the roots of Q are strictly negative if and only if $k > 0$.

24.8 Exercises

Exercise 24.1 Compute explicitly the output g of the generalized filter defined by

$$g' - ag = f, \quad a > 0,$$

and show that it is stable. Is it realizable? Compute the step response.

Exercise 24.2 Let $a, b \in \mathbb{R}$. We wish to study the differential equation

$$g' - ag = f' - bf.$$

In which cases ($a = b$ and $a \neq b$) can one define a generalized filter? Discuss stability and causality.

Exercise 24.3 Consider the generalized filter determined by

$$g'' + 2ag' + bg = f$$

given $a, b \in \mathbb{R}$.

(a) Determine the regions of the (a, b)-plane where the poles of Q are not on the imaginary axis.

(b) Determine the regions corresponding to a realizable filter.

(c) Show that the filter is unstable if $b = 0$.

Exercise 24.4 Does (24.5) define a function when f is slowly increasing?
Hint: See Exercise 20.6.

Exercise 24.5 Compute the transfer function and the impulse response of the generalized filter

$$g''' + g = f'' + f.$$

Is the filter stable? Realizable?

Lesson 25

Examples of Analog Filters

25.1 Revisiting the RC filter

The RC filter was studied in Section 2.4. The equation is

$$RCg' + g = f,$$

and

$$\frac{P(x)}{Q(x)} = \frac{1}{1 + RCx}, \quad z_1 = -\frac{1}{RC}.$$

The filter is stable and realizable (fortunately!). Formula (24.11) shows that

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

and

$$g(t) = \frac{1}{RC} \int_{-\infty}^{t} e^{-\frac{t-s}{RC}} f(s)\, ds.$$

By taking $f = u$, we obtain the step response

$$h_1(t) = (1 - e^{-\frac{t}{RC}})u(t).$$

The gain is $K = 1$. At the times $t = RC$ and $t = 3RC$,

$$h_1(RC) = 1 - e^{-1} \approx 0.63,$$
$$h_1(3RC) = 1 - e^{-3} \approx 0.95.$$

The response time is $t_r = 3RC$. The number RC is called the *time constant* of the filter, or *RC-constant*; it provides a good characterization of the time it takes the filter to respond to an abrupt change in the input. In this sense, it characterizes the system's dynamics. The impulse response and step response are illustrated, respectively, in Figures 25.1 and 25.2

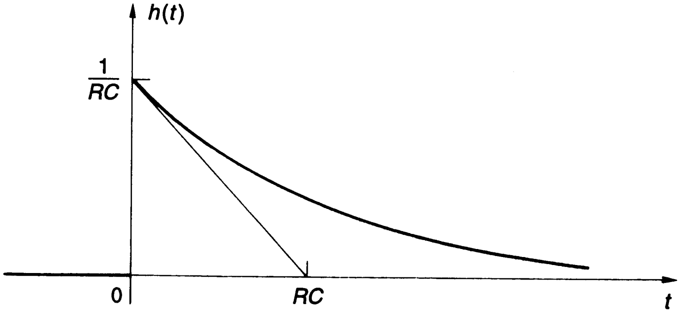


FIGURE 25.1. Impulse response of the RC filter.

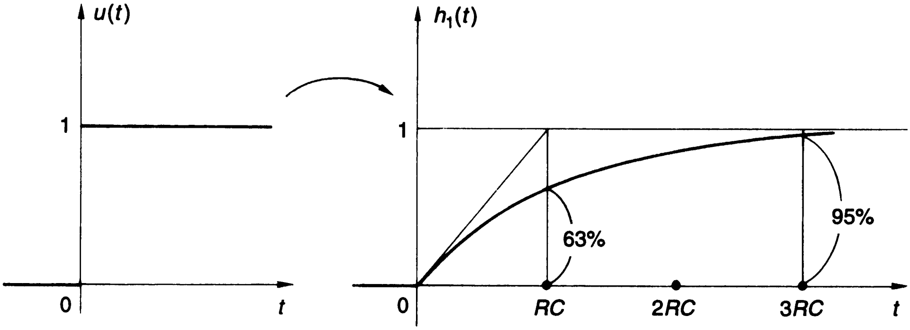


FIGURE 25.2. Step response of the RC filter.

25.2 The *RLC* circuit

If v is the voltage across the capacitance and f is the applied voltage, by Ohm's law,

$$LCv'' + RCv' + v = f,$$

which defines a second-order filter (Figure 25.3).

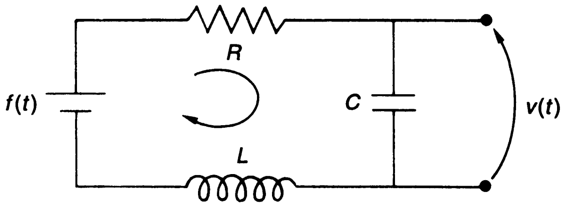


FIGURE 25.3. RLC circuit.

Thus

$$\frac{P(x)}{Q(x)} = \frac{1}{LCx^2 + RCx + 1},$$

and there are three cases to consider that depend on the sign of

$$\Delta = C^2\left(R^2 - 4\frac{L}{C}\right).$$

First case: $\Delta < 0$ $\left(R < 2\sqrt{\frac{L}{C}}\right)$.

Let

$$\omega = \sqrt{4\frac{L}{C} - R^2}, \quad \alpha = \frac{R}{2L}, \quad \beta = \frac{\omega}{2L}.$$

The two poles are complex conjugates and have negative real parts:

$$z = -\alpha + i\beta \quad \text{and} \quad \overline{z} = -\alpha - i\beta.$$

The partial fraction representation of H is

$$H(\lambda) = \frac{1}{i\omega C}\left[\frac{1}{2i\pi\lambda - z} - \frac{1}{2i\pi\lambda - \overline{z}}\right],$$

and we have the representation of h from (24.11) (Figure 25.4):

$$h(t) = \frac{2}{\omega C}\, e^{-\frac{R}{2L}t} \sin\left(\frac{\omega}{2L}t\right) \cdot u(t). \tag{25.1}$$

The response to the input f is thus

$$v(t) = \frac{2}{\omega C}\int_{-\infty}^{t} e^{-\alpha(t-x)} \sin\beta(t-x) f(x)\, dx$$

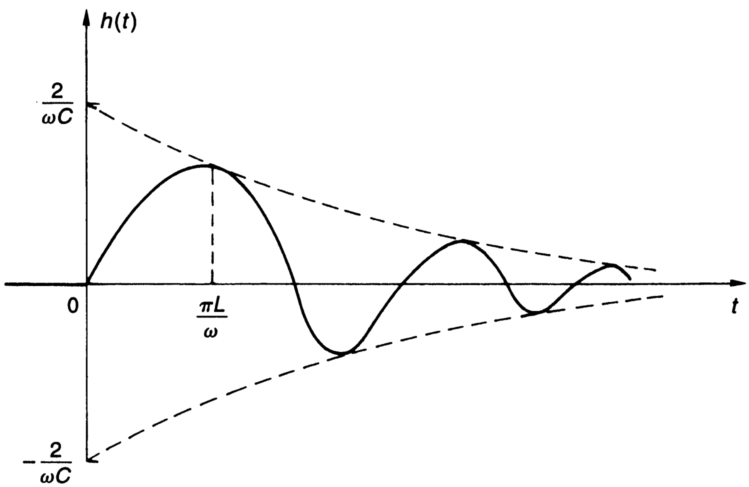


FIGURE 25.4. Impulse response of the RLC circuit (R small).

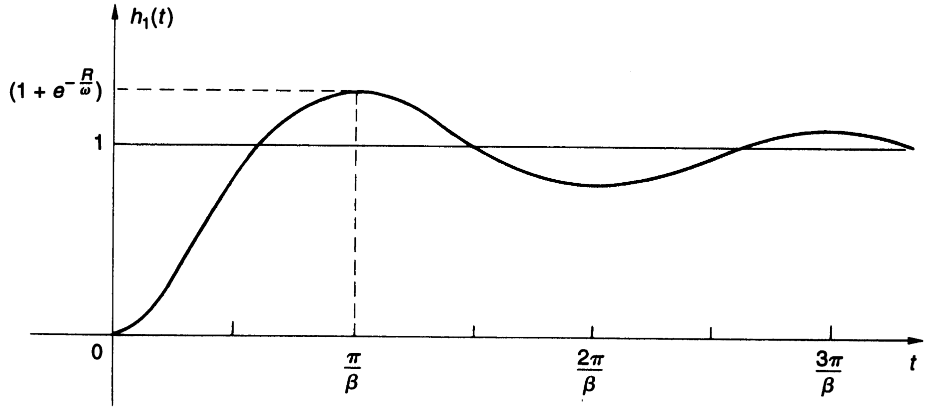


FIGURE 25.5. Step response of the RLC circuit (R small).

and the step response is

$$h_1(t) = \frac{2}{\omega C}\left(\int_0^t e^{-\alpha x}\sin\beta x\,dx\right)u(t).$$

This integral is evaluated by integrating by parts two times:

$$h_1(t) = \left[1 - e^{-\alpha t}(\cos\beta t + \frac{\alpha}{\beta}\sin\beta t)\right]u(t).$$

The step response oscillates around the limit value $K = 1$ (Figure 25.5).

Second case: $\Delta = 0$ $\left(R = 2\sqrt{\frac{L}{C}}\right)$.

In this case,

$$\frac{P(x)}{Q(x)} = \frac{1}{LC\left(x + \frac{R}{2L}\right)^2}$$

has a double real negative pole:

$$z = -\frac{R}{2L}.$$

The impulse response is

$$h(t) = \frac{1}{LC}te^{-\frac{R}{2L}t}u(t), \tag{25.2}$$

and

$$v(t) = \frac{1}{LC}\int_{-\infty}^t (t-s)e^{-\frac{R}{2L}(t-s)}f(s)\,ds.$$

The step response is

$$h_1(t) = \left[1 - \left(1 + \frac{R}{2L}t\right)e^{-\frac{R}{2L}t}\right]u(t).$$

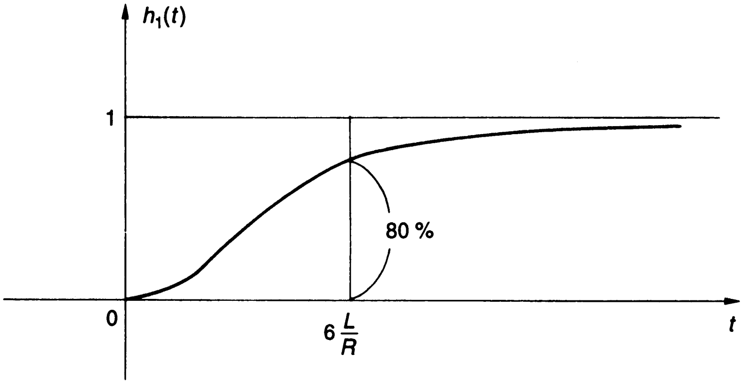


FIGURE 25.6. Step response of the *RLC* circuit in the critical case.

This response no longer oscillates around its asymptote (Figure 25.6).

Third case: $\Delta > 0$ $\left(R > 2\sqrt{\frac{L}{C}}\right)$.

Here we have

$$H(\lambda) = \frac{1}{LC(2i\pi\lambda - z_1)(2i\pi\lambda - z_2)},$$

and $P(x)/Q(x)$ has two real negative poles:

$$z_1 = -\frac{R+\omega}{2L}, \quad z_2 = -\frac{R-\omega}{2L}, \quad \text{where} \quad \omega = \sqrt{R^2 - 4\frac{L}{C}}.$$

H is decomposed as

$$H(\lambda) = -\frac{1}{\omega C}\left[\frac{1}{2i\pi\lambda - z_1} - \frac{1}{2i\pi\lambda - z_2}\right],$$

and

$$h(t) = \frac{-1}{\omega C}\left[e^{z_1 t} - e^{z_2 t}\right] u(t). \tag{25.3}$$

The step response is (Figure 25.7)

$$h_1(t) = \left[1 + \frac{2L}{C\omega(R+\omega)}e^{z_1 t} - \frac{2L}{C\omega(R-\omega)}e^{z_2 t}\right] u(t).$$

The response is slower than in the critical case $\Delta = 0$. The gain is 1 in all three cases. The *RLC* filter is stable and realizable.

25.3 Another second-order filter: $-\frac{1}{\omega}g'' + g = f$

In this example,

$$\frac{P(x)}{Q(x)} = \frac{-\omega^2}{x^2 - \omega^2}, \quad \omega > 0,$$

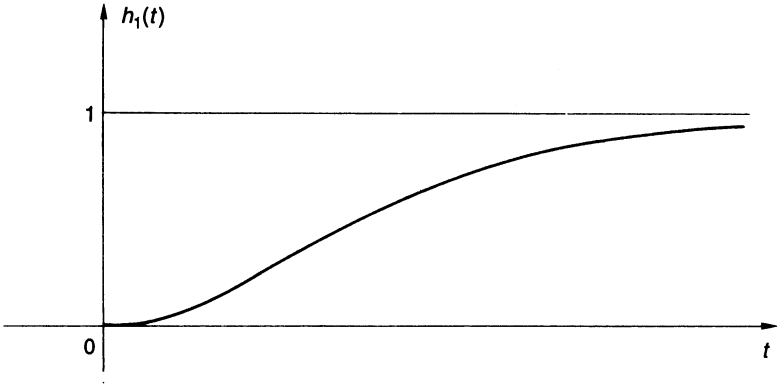


FIGURE 25.7. Step response of the RLC filter (R large).

so that

$$H(\lambda) = \frac{\omega^2}{4\pi^2\lambda^2 + \omega^2}.$$

From Section 18.2.2, the impulse response is (Figure 25.8)

$$h(t) = \frac{1}{2}\omega e^{-\omega|t|}.$$

Thus the output is

$$g(t) = \frac{1}{2}\omega \int_{\mathbb{R}} e^{-\omega|t-s|} f(s)\, ds,$$

and the step response is (Figure 25.9)

$$h_1(t) = \begin{cases} \frac{1}{2}e^{\omega t} & \text{if} \quad t \leq 0, \\ \frac{1}{2}(2 - e^{-\omega t}) & \text{if} \quad t \geq 0. \end{cases}$$

The filter is stable but not realizable.

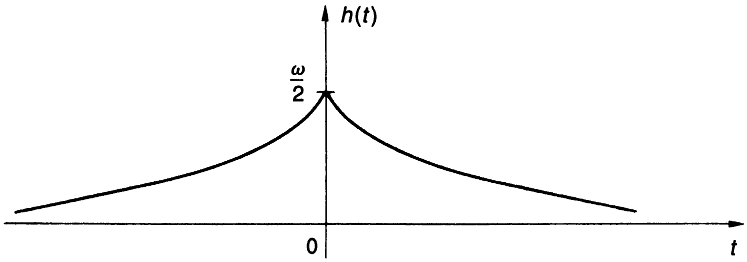


FIGURE 25.8. Impulse response of the filter $-\frac{1}{\omega^2}g'' + g = f$.

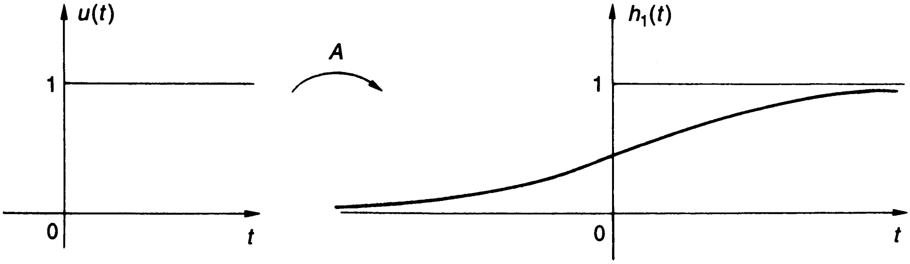


FIGURE 25.9. Step response of the filter $-\frac{1}{\omega^2}g'' + g = f$.

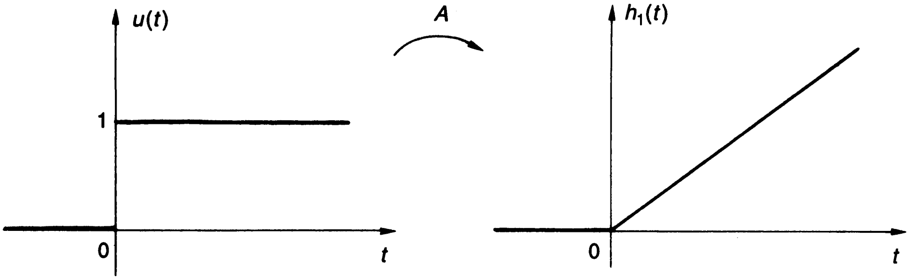


FIGURE 25.10. Step response of the integrator.

25.4 Integrator and differentiator filters

25.4.1 The integrator $g' = f$

In this case, we have

$$\frac{P(x)}{Q(x)} = \frac{1}{x}$$

There is one pole at the origin, and the results of Lesson 24 do not apply. If f is in $\mathscr{S}$, one cannot in general find a g in $\mathscr{S}$. It is easy to study this directly: g is a primitive of f, and if we limit the search to causal signals, then g is determined by having to be causal. In this case,

$$g(t) = \int_{-\infty}^{t} f(s)\, ds,$$

which can be written in terms of the Heaviside function:

$$g = u * f.$$

This is a convolution system whose impulse response is the unit step function. The step response is defined and is the ramp $h_1(t) = tu(t)$ (see Figure 5.10). The gain is infinite; the system is unstable but realizable if limited to causal signals.

25.4.2 The differentiator $g = f'$

Here we have

$$\frac{P(x)}{Q(x)} = x \quad \text{and} \quad \deg P > \deg Q.$$

This filter is clearly realizable but unstable. Neither the impulse response nor the step response can be defined with the tools developed so far. These will be defined later in the context of distributions.

25.5 The ideal low-pass filter

It is customary to describe a filter by the way it modifies the frequencies of the input signal. This is just to say that a filter is described by its transfer function H, since the frequencies of the input and output are related by

$$\widehat{g}(\lambda) = H(\lambda)\widehat{f}\,(\lambda). \tag{25.4}$$

The ideal low-pass filter does not change the frequencies λ for $|\lambda| < \lambda_c$ (λ_c is the cutoff frequency) and completely suppresses the others. Thus the transfer function of the ideal filter is

$$H(\lambda) = \begin{cases} 1 & \text{if} \quad |\lambda| < \lambda_c, \\ 0 & \text{otherwise.} \end{cases}$$

From Section 22.2.2, the h in $L^2(\mathbb{R})$ for which $\widehat{h} = H$ is

$$h(t) = \frac{\sin 2\pi\lambda_c t}{\pi t}.$$

If we consider only input signals with finite energy, then f, h, and H are in $L^2(\mathbb{R})$, and (25.4) can be expressed as (Proposition 23.2.1(i))

$$g = h * f.$$

We know that g is continuous, bounded, and zero at infinity. The right-hand side of (25.4) is in $L^2(\mathbb{R})$ because $\widehat{f}$ is in $L^2(\mathbb{R})$. Thus $\widehat{g}$ and g are in $L^2(\mathbb{R})$. The important issue here is the form of h; it tells us that *the ideal low-pass filter is not realizable.* This is indeed troublesome, but not at all surprising. Faced with the impossibility of having an ideal low-pass filter, the best we can expect is to find realizable filters whose transfer functions approximate that of the ideal filter. In general, the transfer functions of these "real filters" will have a bell-shaped amplitude (see Figure 2.1) and unbounded support. These ideas are illustrated in Figure 25.11.

The better $|H(\lambda)|$ approximates the centered rectangular window, the better will be the performance of the realizable filter. In Section 2.4 we saw that the RC filter acts as a crude low-pass filter. We will see in the next section that the Butterworth filters provide better realizable approximations to the ideal low-pass filter.

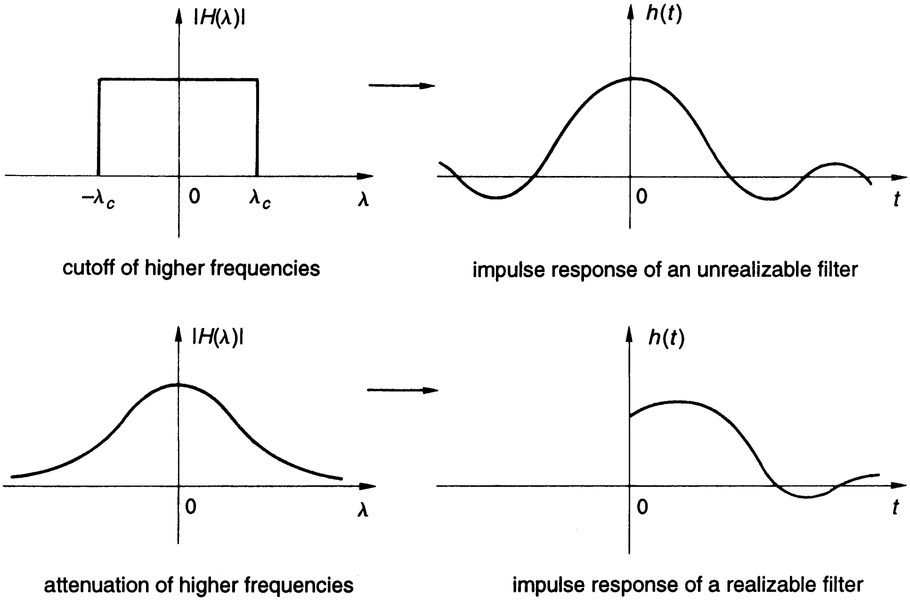


FIGURE 25.11. A real low-pass filter can only attenuate higher frequencies.

25.6 The Butterworth filters

The Butterworth filters are the filters whose energy spectra have the form

$$|H(\lambda)|^2 = \frac{1}{1+\left(\frac{\lambda}{\lambda_c}\right)^{2n}}, \quad \lambda_c > 0. \tag{25.5}$$

For $n = 1$ we have the RC filter with $\lambda_c = 1/(2\pi RC)$. The motivation for increasing n is to produce a cleaner cutoff around λ_c.

As n increases, frequencies in the pass band $|\lambda| < \lambda_c$ are less attenuated, and frequencies in the attenuation band $|\lambda| > \lambda_c$ are more suppressed (see Figure 25.12). Since we have some freedom to choose the phase (only the modulus has been given), we will determine $H(\lambda)$ to obtain a stable and realizable filter. If we require h to be real, then

$$|H(\lambda)|^2 = H(\lambda)\overline{H(\lambda)} = H(\lambda)H(-\lambda). \tag{25.6}$$

The poles of $|H(\lambda)|^2$ are the complex numbers

$$p_k = \lambda_c e^{i\frac{\pi}{2n}(2k+1)}, \quad k = 0, 1, \ldots, 2n-1,$$

and they occur in conjugate pairs. We want $H(\lambda)$ to be a rational function

$$H(\lambda) = \frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)} = F(2i\pi\lambda),$$

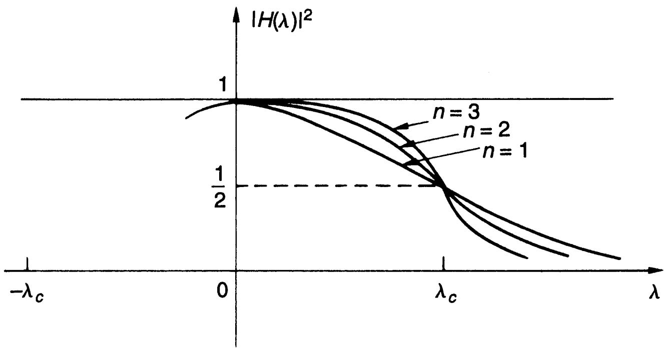


FIGURE 25.12. Energy spectra of the Butterworth filters.

and furthermore, we want the poles

$$z_k = \frac{p_k}{2i\pi}$$

of F to lie to the left of the imaginary axis. This means that the poles p_k must lie above the real axis. Thus, for the poles of $H(\lambda)$ we select those p_k whose imaginary parts are positive. The remaining p_k (the conjugates of the ones selected) are the poles of $H(-\lambda)$. Here are two examples.

Case $n = 2$:

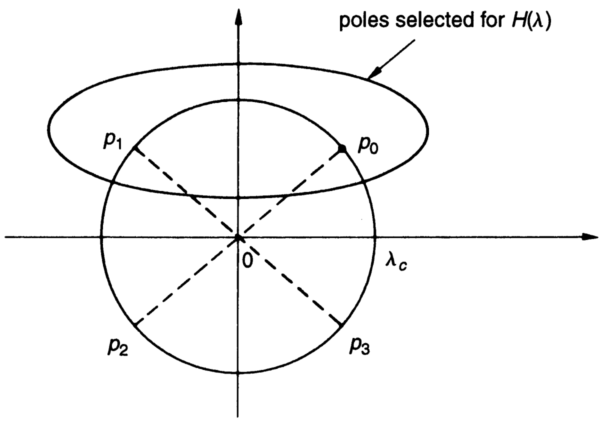


FIGURE 25.13. Butterworth filter of order 2.

In this case (Figure 25.13),

$$H(\lambda) = \frac{p_0 p_1}{(\lambda - p_0)(\lambda - p_1)}.$$

Case $n = 3$:

Here we have (Figure 25.14)

$$H(\lambda) = \frac{-p_0 p_1 p_2}{(\lambda - p_0)(\lambda - p_1)(\lambda - p_2)}.$$

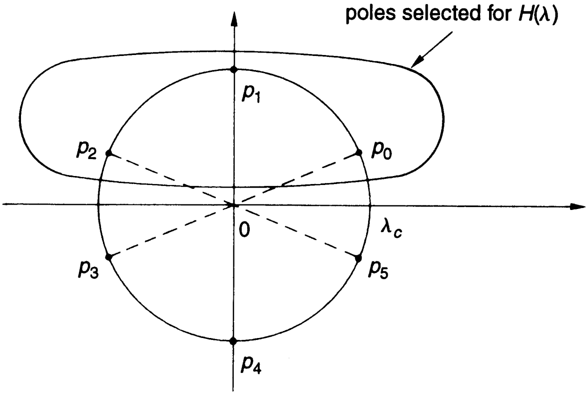


FIGURE 25.14. Butterworth filter of order 3.

We will compute the impulse response for the case $n = 2$. Thus

$$p_0 = p = \frac{\lambda_c}{\sqrt{2}}(1+i) \quad \text{and} \quad p_1 = -\overline{p}.$$

If we let $a = \pi\lambda_c\sqrt{2}$, we have

$$H(\lambda) = -\frac{2i\pi \mid p \mid^2}{p+\overline{p}}\left[\frac{1}{2i\pi\lambda+\alpha} - \frac{1}{2i\pi\lambda+\overline{\alpha}}\right],$$

where $\alpha = a(1-i)$. Referring to Section 22.2.2,

$$h(t) = -ia(e^{-\alpha t} - e^{-\overline{\alpha}t})u(t) = 2ae^{-at}\sin at \cdot u(t).$$

This impulse response has the same form as that of the *RLC* circuit, which is equation (25.1).

25.7 The general approximation problem

There are many ways to approximate the ideal low-pass filter with stable, realizable filters. The Butterworth filters belong to the class of polynomial filters ($P(x) = 1$). The *Chebyshev filters* are also in this class. These are obtained by letting

$$|H(\lambda)|^2 = \frac{1}{1+a^2T_n^2(\lambda)},$$

where $T_n(\lambda)$ is the Chebyshev polynomial of degree n and a is a parameter that determines the amplitude of the oscillations in the pass band. We also mention the *elliptic filters*: $|H(\lambda)|^2$ has the same form as above, but $T_n(\lambda)$ is replaced by a rational function. For an account of this we refer to [BL80].

The general approximation problem, given the frequency specifications, amounts to looking for a rational function that falls within a predetermined template (see Figure 25.15).

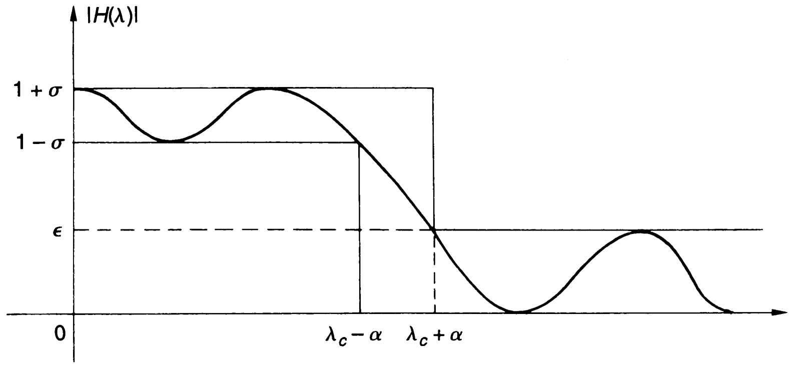


FIGURE 25.15. Approximation of the ideal low-pass filter with given frequency specifications.

25.8 Exercises

Exercise 25.1 Show that it is possible to choose the constants R, L, and C such that the RLC circuit is a Butterworth filter of order 2.

Hint: Take $R = \sqrt{2L/C}$ and compute $|H(\lambda)|^2$ as in Section 25.2, First case. One finds that

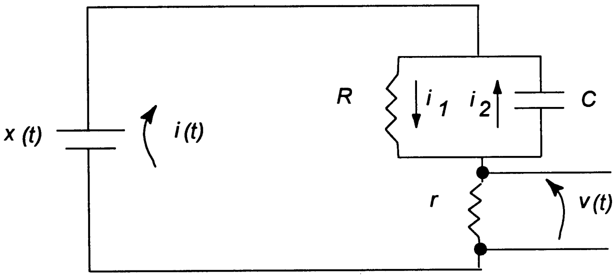
$$|H(\lambda)|^2 = \frac{1}{1 + \left(\frac{\lambda}{\lambda_C}\right)^4} \quad \text{with} \quad \lambda_C = \frac{1}{2\pi\sqrt{LC}}.$$

Exercise 25.2 Discuss the stability of the generalized system

$$g^{(4)} + 6g^{(3)} + 11g'' + 6g' + kg = f$$

as a function of k.

Exercise 25.3 Consider the following electric filter



where x is the input and where the voltage v across the resistance r is the output.

(a) Show that x and v are related by $RrCv' + (R + r)v = rx + RrCx'$.

(b) Compute the transfer function and the step response.

(c) Assume that r is small with respect to R. What is the role of this filter?