19.2.3 Theorem *The space $\mathscr{S}$ is invariant under the Fourier transform; that is, $f \in \mathscr{S} \Rightarrow \widehat{f} \in \mathscr{S}$.*

Proof. Assume $f \in \mathscr{S}$. Then f is in $L^1(\mathbb{R})$ and decays rapidly. Thus $\widehat{f} \in C^\infty$ by Proposition 19.1.3. Since $f^{(k)}$ is rapidly decreasing for all $k \in \mathbb{N}$, it is integrable for all k by Proposition 19.1.2. We deduce from Proposition 19.1.4 that $\widehat{f}$ is rapidly decreasing. We need to show that the derivatives of $\widehat{f}$ are all rapidly decreasing. Since $\left((-2i\pi x)^q f(x)\right)^{(p)}$ is integrable, we see from (17.5) and (17.6) that

$$\frac{1}{(2i\pi)^p}\mathscr{F}\left(\left((-2i\pi x)^q f(x)\right)^{(p)}\right)(\xi) = \xi^p \mathscr{F}\left((-2i\pi x)^q f(x)\right)(\xi) = \xi^p \widehat{f}^{\,(q)}(\xi). \tag{19.1}$$

The right-hand term is the Fourier transform of an integrable function, so by the Riemann–Lebesgue theorem $\lim_{|\xi|\to\infty} |\xi^p \widehat{f}^{\,(q)}(\xi)| = 0$. $\square$