- We know that $f*g$ is defined a.e. It is bounded:

$$|f*g(x)| = \left|\int f(x-t)\,g(t)\,dt\right| \leqslant \left(\int |f(x-t)|^p dt\right)^{1/p} \left(\int |g(t)|^q\right)^{1/q} < \infty$$

(Hölder)

$$\|f*g\|_\infty \leqslant \|f\|_p \|g\|_q.$$

- Continuity:

$$|f*g(x) - f*g(y)| \leqslant \int |f(x-t) - f(y-t)|\,|g(t)|\,dt$$

$$\leqslant \left(\int |f(x-t) - f(y-t)|^p dt\right)^{1/p} \left(\int |g(t)|^q dt\right)^{1/q} \quad \text{(Hölder)}$$

$$\leqslant \|g\|_q \left(\int |f(s) - f(s+y-x)|^p\, ds\right)^{1/p}.$$

$\rightarrow$ continuity of $\tau_a f$ in $L_p(\mathbb{R})$ $\quad \lim_{a\to 0} \|\tau_a f - f\|_p = 0 \quad f \in L^p(\mathbb{R})$

Classical result from integration theory: The set $C_c^0(\mathbb{R})$ (continuous functions with compact support), is dense in $L_p(\mathbb{R})$. $\forall \varepsilon > 0$, $\exists\, h_\varepsilon \in C_c^0(\mathbb{R})$, $\|f - h_\varepsilon\|_p \leq \varepsilon$.

We have: $\|f - \tau_a f\|_p \leqslant \|f - h_\varepsilon\|_p + \|h_\varepsilon - \tau_a h_\varepsilon\|_p + \|\tau_a h_\varepsilon - \tau_a f\|_p$

$$\leqslant 2\|f - h_\varepsilon\|_p + \|h_\varepsilon - \tau_a h_\varepsilon\|_p \leqslant 2\varepsilon + \|h_\varepsilon - \tau_a h_\varepsilon\|_p.$$

Therefore, it suffices to show that for all $h \in C_c^0(\mathbb{R})$, $\|h - \tau_a h\|_p \to 0$ $a \to 0$.

If $h \in C_c^0(\mathbb{R})$ is uniformly continuous, $\|h - \tau_a h\|_\infty \to 0$ $a \to 0$. The proof follows.